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Roughness and Fuzziness Associated with Soft Multisets and Their Application to MADM

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Abstract − In this paper, we tried to hybrid soft sets with multisets. We define some basic properties of soft multiset and present some important results. We define some binary relations, equivalence relations and an indiscernibility relation on soft multiset with examples. The concept of an approximation space associated with each parameter in a soft multiset is discussed and an approximation space associated with the soft multiset is defined. We introduce the novel concepts of roughness and fuzziness associated with soft multiset. We use soft multiset in multi-valued information system. Furthermore, we present an algorithm to cope with uncertainties in multi-attribute decision making (MADM) problems by utilizing soft multisets and related concepts. The efficiency of the algorithm is verified by a case study to find the optimal choice of the real-world problems having uncertainties.

Keywords − Soft multiset, roughness and fuzziness of soft multiset, multi-valued information system, multi-attribute decision-making.

1. Introduction

The rapid development of science has led to an urgent need for the development of modern sets theory. Blizard [1] introduced the multiset theory as a generalization of crisp set theory. Keeping in view the uncertainty element Zadeh [2], in 1965, initiated the idea of fuzzy sets where a membership degree is assigned to each member of the universe of discourse. Molodtsov [3] initiated a novel concept of soft set as a new arithmetical tool for handling uncertainties which traditional arithmetical tools cannot manipulate. Soft set theory and multiset theory has many applications in artificial intelligence, multiple-valued logic, multi-process information fusion, social science, economics, medical science, engineering etc. The advancement in the field of soft set theory has been taking place in a rapid pace due to general nature of parametrization expressed by a soft set, in recent years. Similarly, multiset theory, by assuming that for a given set A an element x occurs a finite number of times, has natural applications in artificial intelligence, multiple-valued logic and decision making problems of the real world problems. Abbas et al. [4] established some generalized operations in soft set theory via relaxed conditions on parameters. Ali et al. [5] introduced some new operations on soft sets. Ali [6] presented some interesting results on on soft sets, rough soft sets and fuzzy soft sets. Ali et al. [7] developed representation of graphs based on neighborhoods and soft sets. Feng et al. [8-13] introduced several interesting results on soft relations applied to semigroups, attribute analysis of information systems, soft sets combined with fuzzy sets, fuzzy soft set, rough set and generalized

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intuitionistic fuzzy soft sets and their applications to multi-attribute decision making. Hayat et al. [14] introduced some new results on type-2 soft sets. Further Maji et al. [15,16] studied the soft set theory and applied this theory to resolve decision-making issues. They also initiated the notion of fuzzy soft set. Aktaş and Çağman [17] presented the concept of soft group. Kong et al. [18] used the soft set hypothetic approach in decision-making issues. Babitha and Sunil [19] introduced the some results on soft set relations, equivalence relations and partitions on soft sets, and soft set function. Babitha and John [20] introduced the idea of soft multiset which is the hybrid structure of multiset and soft set. They shown a relationship between soft multisets and multi-valued information system. They also presented an application of soft multiset in decision-making. Alkhazaleh et al. [21] also introduced some results of soft multiset Applications of soft multiset theory in other fields and real life issues are now capturing momentum. Many researchers have contributed their research work in the field of multiset theory and soft set theory (See [1,5,15,17,18,20-29]).

Liu et al. [30] introduced hesitant IF linguistic operators and presented its application to multi-attribute group decision making (MAGDM) problem. Hashmi et al. [31] introduced the notion of m-polar neutrosophic set and m-polar neutrosophic topology and their applications to multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis. Hashmi and Riaz [32] introduced a novel approach to censuses process by using Pythagorean m-polar fuzzy Dombi's aggregation operators. Naeem et al. [33] introduced Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators. Naeem et al. [34] introduced Pythagorean m-polar Fuzzy Sets and TOPSIS method for the Selection of Advertisement Mode. Riaz et al. [35] introduced N-soft topology and its applications to multi-criteria group decision making (MCGDM). Riaz et al. [36,37] introduced soft rough topology with multi-attribute group decision making problems (MAGDM). Riaz and Hashmi [38] introduced the concept of cubic m-polar fuzzy set and presented multi-attribute group decision making (MAGDM) method for agribusiness in the environment of various cubic m-polar fuzzy averaging aggregation operators. Riaz and Hashmi [39] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its Applications towards multi-attribute decision making problems. Linear Diophantine fuzzy Set (LDFS) is superior than IFSs, PFSs and q-ROFSs. Riaz and Hashmi [40] introduced novel concepts of soft rough Pythagorean m-Polar fuzzy sets and Pythagorean m-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [41-43] established the idea of cubic bipolar fuzzy set and cubic bipolar fuzzy ordered weighted geometric aggregation operators with applications to multi-criteria group decision making (MCGDM). They introduced bipolar fuzzy soft mappings with application to bipolar disorders. Roy and Maji [44] presented a new fuzzy soft set theoretic approach to decision making problems. Şenel [45,46] introduced the relation between soft topological space and soft ditopological space and characterization of soft sets by delta-soft operations. Sezgin and Atagün [47] introduced some new operations of soft sets. Sezgin et al. [48] introduced the idea of soft intersection near-rings with applications. Shabir and Ali [29] established some properties of soft ideals and generalized fuzzy ideals in semigroups. Shabir and Naz [49] introduced the concept of soft topological spaces. Tehrim and Tehrim [50] presented a novel extension of TOPSIS to MCGDM with bipolar neutrosophic soft topology. Wei et al. [51] established hesitant triangular fuzzy operators in MADGDM problems. Xueling et al. [52] introduced decision-making methods based on various hybrid soft sets. Xu and Zhang [53] introduced hesitant fuzzy multi-attribute decision-making based on TOPSIS with incomplete weight information. Xu [54] introduced the concept of intuitionistic fuzzy aggregation operators. Xu and Cai, in their book [55], presented the theory and applications of intuitionistic fuzzy information aggregation. Xu, in his book [56], presented hesitant fuzzy sets theory and various types of hesitant fuzzy aggregation operators. Zhan et al. [57-58] presented the concepts of rough soft hemirings, soft rough covering and its applications to multi-criteria group decision-making (MCGDM) problems. Zhang and Xu [59] presented an extension of TOPSIS in multiple criteria decision making with the help of Pythagorean Fuzzy Sets.

This paper is organized as follows: In Section 2, we present some basic concepts of multiset theory. In Section 3, we discuss some results of soft set theory and soft multiset theory. We also present some new operations on soft multisets. In Section 4, we present some binary relations, equivalence relations and an indiscernibility relations on soft multiset with the help of examples. We also present an application of soft multiset in information system. In Section 5, we present an algorithm to cope with multi-attribute decision making (MADM) problems by utilizing soft multisets and related concepts. This algorithm is also summarized by the flow chart. The efficiency of the algorithm is verified by a case study to find the optimal choice to the various real world problems having uncertainties, imprecisions and vagueness.

2. Preliminaries

In this section, we recall some rudiments of multiset theory.

Definition 2.1. [20] "A multiset over Z is just a pair $< Z, f >$, where $f: Z \to W$ is a function, Z is a crisp set and W is a set of whole numbers.

In order to avoid any confusion we will use square brackets for multisets and braces for sets. Let A be a multiset over crisp set Z with z occurring m times in A. It is denoted by $z \in^m A$. Multi-set A is given by $A =< Z, f >= \left[\frac{k_1}{z_1}, \frac{k_2}{z_2}, \ldots, \frac{k_n}{z_n}\right]$, where z_1 occurring k_1 times, z_2 occurring k_2 times and so on.

Definition 2.2. [60] Let $A =< Z, f >$ and $B =< Z, g >$ be two multisets. Then A is a sub-multiset of B, denoted by $A \subseteq B$ if for all $z \in A$, $f(z) \leq g(z)$.

Definition 2.3. [20] A sub-multiset $A =< Z, f >$ of $B =< Z, g >$ is a whole sub-multiset of B with each element in A having full multiplicity as in B. i.e. $f(z) = g(z)$, for every z in A.

Definition 2.4. [60] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then their union, denoted by $A \cup B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = \max\ (f(z), g(z))$.

Definition 2.5. [60] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then their intersection, denoted by $A \cap B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = \min\ (f(z), g(z))$.

Definition 2.6. [20] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then their sum denoted by $A \oplus B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = f(z) + g(z)$.

Definition 2.7. [60] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then the removal of multiset B from A, denoted by $A \ominus B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = \max\ (f(z) - g(z), 0)$.

Definition 2.8. [20] Let $A =< Z, f >$ be a multiset and $A_1 =< Z, g >$ be a sub-multiset of A. Then the relative compliment of A_1 is given by $A_1^r = A \ominus A_1$.

Definition 2.9. [20] Let $[Z]^n$ denotes the set of all multisets whose elements are in Z such that no element in a multiset appears more than n times. Let $A \in [Z]^n$ be a multiset. The power whole multiset of A denoted by $PW(A)$ is defined as the set of all whole sub-multisets of A. The cardinality of $PW(A)$ is 2^m, where m is the cardinality of the support set (root set) of A".

Example 2.10. Let $M = \left[\frac{2}{g}, \frac{1}{t}, \frac{1}{k}\right]$ be a multiset. Then, by using Definition 2.2, Definition 2.3 and Definition 2.9, the set of all sub-multisets of M is

$$PW(A) = \left\{S_1 = \left[\frac{0}{g}, \frac{0}{t}, \frac{0}{k}\right], S_2 = \left[\frac{0}{g}, \frac{0}{t}, \frac{1}{k}\right], S_3 = \left[\frac{0}{g}, \frac{1}{t}, \frac{0}{k}\right], S_4 = \left[\frac{0}{g}, \frac{1}{t}, \frac{1}{k}\right], S_5 = \left[\frac{1}{g}, \frac{0}{t}, \frac{0}{k}\right], S_6 = \left[\frac{1}{g}, \frac{0}{t}, \frac{1}{k}\right], \right.$$
$$\left. S_7 = \left[\frac{1}{g}, \frac{1}{t}, \frac{0}{k}\right], S_8 = \left[\frac{1}{g}, \frac{1}{t}, \frac{1}{k}\right], S_9 = \left[\frac{2}{g}, \frac{0}{t}, \frac{0}{k}\right], S_{10} = \left[\frac{2}{g}, \frac{0}{t}, \frac{1}{k}\right], S_{11} = \left[\frac{2}{g}, \frac{1}{t}, \frac{0}{k}\right], S_{12} = \left[\frac{2}{g}, \frac{1}{t}, \frac{1}{k}\right]\right\}$$

and $card(P(M)) = (2 + 1)(1 + 1)(1 + 1) = 12$.

Furthermore, the power whole multiset is given by $PW(M) = \{S_1, S_2, S_3, S_4, S_9, S_{10}, S_{11}, S_{12}\}$ and its cardinality is given by $card(PW(M)) = 2^3 = 8$.

3. Some Results on Soft Multi-Sets

In this section, we present some basic notions of soft set and soft multiset along with related properties. We present binary relation, equivalence relation and indiscernibility relation on soft multiset with examples and study important results. We also present an application of soft multiset in information system.

In the sequel, the multiset H represents the initial universe, E is a set of parameters or attributes, $PW(H)$ is a power whole multiset of H and $A \subseteq E$.

Definition 3.1. [3] Let X be the universal set, E be a set of attributes, $P(X)$ be a power set of X and $A \subseteq E$. A pair (λ, A) is called a soft set over X, where $\lambda: A \rightarrow P(X)$ is a set-valued function.

Definition 3.2. [20] A soft multiset σ_A on the universal multiset H is defined by the set of all ordered pairs $\sigma_A = \{(s, \sigma_A(s)): s \in E, \sigma_A(s) \in PW(H)\}$, where $\sigma_A: E \rightarrow PW(H)$ such that $\sigma_A(s) = \emptyset$ if $s \notin A$.

Consider a soft multiset σ_A, where $\mathrm{H} = \left[\frac{\mathrm{k}_1}{\mathrm{h}_1}, \frac{\mathrm{k}_2}{\mathrm{h}_2}, \ldots, \frac{\mathrm{k}_\mathrm{n}}{\mathrm{h}_\mathrm{n}}\right]$, $E = \{s_1, s_2, \ldots, s_m\}$ and $A = E$. Tabular illustration of a soft multiset is most helpful for storing soft multiset in a computer. Here,

$$h_{ij} = \begin{cases} k_i, & \text{if } h_i \in^{k_i} \sigma_A(s_j) \\ 0, & \text{otherwise} \end{cases}$$

Tabular representation of a soft multiset can be written as:

σ_A	s_1	$s_2 \cdots$	s_m
h_1	h_{11}	$h_{12} \cdots$	h_{1m}
h_2	h_{21}	$h_{22} \cdots$	h_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$
h_n	h_{n1}	$h_{n2} \cdots$	h_{nm}

Hereafter, $SM(H)$ denotes the family of all soft multisets over H with attributes from S. Now, we elaborate the definition of soft multiset by the succeeding example.

Example 3.3. Let $H = \left[\frac{k_1}{h_1}, \frac{k_2}{h_2}, \frac{k_3}{h_3}, \frac{k_4}{h_4}, \frac{k_5}{h_5}, \frac{k_6}{h_6}\right]$ be the universal multiset consist of perfumes under consideration, where $\mathrm{h}_1 =$ azzaro, $h_2 =$ coco, $h_3 =$ eternity, $h_4 =$ poison, $h_5 =$ rogue, $\mathrm{h}_6 =$ tresor and k_i denotes the multiplicity of perfume $h_i, i = 1,2, \ldots, 6$.

Let $E = \{\text{expansive, affordable, longlastingfragrance, impressivepackaging}\}$ be the set of all attribues. Let $A = \{\text{expansive, affordable, longlastingfragrance}\} \subseteq E$. Then, the soft multiset σ_A or (σ, A) defined below describe the attractiveness of perfumes under consideration,

$$\sigma_A = (\sigma, A) = \left\{\left(\text{expansive}, \left[\frac{k_2}{h_2}, \frac{k_3}{h_3}\right]\right), \left(\text{affordable}, \left[\frac{k_1}{h_1}, \frac{k_4}{h_4}\right]\right), \left(\text{longlastingfragrance}, \left[\frac{k_5}{h_5}, \frac{k_6}{h_6}\right]\right)\right\}$$

Here the approximation set is multiset. In tabular form, the soft multiset σ_A can be represented as:

σ_A	expensive	affordable	long lasting fragrance
h_1	0	k_1	0
h_2	k_2	0	0
h_3	k_3	0	0
h_4	0	k_4	0
h_5	0	0	k_5
h_6	0	0	k_6

Definition 3.4. [20] "Let $\sigma_A \in SM(H)$. If $\sigma_A(s) = \emptyset$ for all $s \in E$, then σ_A is called an empty or null soft multiset, denoted by σ_ϕ.

Definition 3.5. [20] Let $\sigma_A, \sigma_B \in SM(H)$. Then, σ_A is a soft multi subset of σ_B, denoted by $\sigma_A \,\tilde{\subseteq}\, \sigma_B$, if $\sigma_A(s) \subseteq \sigma_B(s)$, for all $s \in E$".

Example 3.6. Let $H = \left[\frac{k_1}{h_1}, \frac{k_2}{h_2}, \frac{k_3}{h_3}, \frac{k_4}{h_4}, \frac{k_5}{h_5}, \frac{k_6}{h_6}\right]$ be a universal multiset. Let $E = \{s_1, s_2, s_3, s_4\}$ be the set of attributes. Let $A = \{s_1, s_2, s_3\}, B = \{s_1, s_2\} \subseteq E$ and $B \subseteq A$. Consider soft multisets σ_A and σ_B defined on H given as, $\sigma_A = \left\{\left(s_1, \left[\frac{k_2}{h_2}, \frac{k_3}{h_3}\right]\right), \left(s_2, \left[\frac{k_1}{h_1}, \frac{k_4}{h_4}\right]\right), \left(s_3, \left[\frac{k_5}{h_5}, \frac{k_6}{h_6}\right]\right)\right\}$, $\sigma_B = \left\{\left(s_1, \left[\frac{k_2}{h_2}\right]\right), \left(s_2, \left[\frac{k_1}{h_1}, \frac{k_4}{h_4}\right]\right)\right\}$. Then, σ_B is soft multi subset of σ_A.

Definition 3.7. [20] "Let $\sigma_A \in SM(H)$. If $\sigma_A(s) = H$ for all $s \in A$, then σ_A is called A-universal soft multiset, denoted by $\sigma_{\tilde{A}}$. If $A = E$, then A-universal soft multiset is called a universal or absolute soft multiset, denoted by $\sigma_{\tilde{E}}$.

Definition 3.8. [20] Let $\sigma_A, \sigma_B \in SM(H)$. Then, σ_A and σ_B are equal soft multisets, denoted by $\sigma_A \cong \sigma_B$, if and only if $\sigma_A(s) = \sigma_B(s)$, for all $s \in E$.

Definition 3.9. [20] Let $\sigma_A, \sigma_B \in SM(H)$. Then, the union $\sigma_A \,\tilde{\cup}\, \sigma_B$ and the intersection $\sigma_A \,\tilde{\cap}\, \sigma_B$ of σ_A and σ_B is defined by the approximate functions $\sigma_{A\tilde{\cup}B}(s) = \sigma_A(s) \cup \sigma_B(s)$ and $\sigma_{A\tilde{\cap}B}(s) = \sigma_A(s) \cap \sigma_B(s)$ respectively, $\forall s \in E$".

Example 3.10. Let $H = \left[\frac{2}{p_1}, \frac{4}{p_2}, \frac{6}{p_3}, \frac{8}{p_4}, \frac{10}{p_5}, \frac{12}{p_6}\right]$ be a universal multiset consist of smart phones under consideration, where $p_1 =$ Samsung Galaxy Note, $p_2 =$ Nokia lumina(930), $p_3 =$ Huawei Nexus 6p, $p_4 =$ iphone 6, $p_5 =$ Motorola V3i, $p_6 =$ Sony Xperia Z5 Premium, and let $E = \{s_1, s_2, s_3, s_4\}$ be the set of attributes defined as $s_1 =$ long battery timing, $s_2 =$ expensive, $s_3 =$ durable glass screen, and $s_4 =$ metallic body. Let $A = \{s_1, s_2, s_3\}$ and $B = \{s_3, s_4\}$ be subsets of E. Then, we we can write two multisets as follows:

$$\sigma_A = \left\{\left(s_1, \left[\frac{4}{p_2}, \frac{6}{p_3}\right]\right), \left(s_2, \left[\frac{2}{p_1}, \frac{8}{p_4}\right]\right), \left(s_3, \left[\frac{10}{p_5}, \frac{12}{p_6}\right]\right)\right\} \text{ and } \sigma_B = \left\{\left(s_3, \left[\frac{10}{p_5}\right]\right), \left(s_4, \left[\frac{2}{p_1}, \frac{4}{p_2}, \frac{6}{p_3}\right]\right)\right\}$$

Then, their intersection is given by

$$\sigma_A \,\tilde{\cap}\, \sigma_B = \left\{(s_1, \emptyset), (s_2, \emptyset), \left(s_3, \left[\frac{10}{p_5}\right]\right), (s_4, \emptyset)\right\}$$

and their union is given by

$$\sigma_A \,\tilde{\cup}\, \sigma_B = \left\{\left(s_1, \left[\frac{4}{p_2}, \frac{6}{p_3}\right]\right), \left(s_2, \left[\frac{2}{p_1}, \frac{8}{p_4}\right]\right), \left(s_3, \left[\frac{10}{p_5}, \frac{12}{p_6}\right]\right), \left(s_4, \left[\frac{2}{p_1}, \frac{4}{p_2}, \frac{6}{p_3}\right]\right)\right\}$$

Definition 3.11. [20] "Let $A = \{s_1, s_2, \ldots, s_n\}$ be a set of parameters. The NOT set of A denoted by $\neg A$ is defined by $\neg A = \{\neg s_1, \neg s_2, \ldots, \neg s_n\}$, where $\neg s_i = \text{not} s_i, \forall i = 1,2,\ldots,n$.

Definition 3.12. [20] Let $\sigma_A \in SM(H)$. The complement of σ_A over a multiset H is denoted by σ_A^c and is defined by approximate function $\sigma_A^c(\neg s) = H \ominus \sigma_A(s)$ for all $\neg s \in \neg A$".

Here we point out that the law of excluded middle do not hold with respect to complement for soft multiset given in Definition 3.12 defined by Babitha and John [20]. We give the following counter example to explain it more effectively.

Counter Example 3.13. Suppose that $H = \left[\frac{1}{b_1}, \frac{3}{b_2}, \frac{2}{b_3}, \frac{5}{b_4}, \frac{4}{b_5}\right]$ is a universal multiset of bags under consideration. Let $E = \{\text{red}, \text{brown}, \text{black}, \text{grey}, \text{white}\}$ be the set of attributes and $A = \{\text{red}, \text{brown}, \text{black}, \text{grey}\} \subseteq E$. Let $\sigma_A \in SM(H)$. That is,

$$\sigma_A = \left\{\left(\text{red}, \left[\frac{1}{b_1}, \frac{3}{b_2}\right]\right), \left(\text{brown}, \left[\frac{1}{b_1}, \frac{2}{b_3}\right]\right), \left(\text{black}, \left[\frac{5}{b_4}, \frac{4}{b_5}\right]\right)\right\}$$

The NOT set of A is given by $\neg A = \{\text{notred}, \text{notbrown}, \text{notblack}, \text{notgrey}\}$. Then,

$$\sigma_A^c = \left\{\left(\text{notred}, \left[\frac{2}{b_3}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\text{notbrown}, \left[\frac{3}{b_2}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\text{notblack}, \left[\frac{1}{b_1}, \frac{3}{b_2}, \frac{2}{b_3}\right]\right)\right\}$$

Therefore,

$$\sigma_A \tilde{\cup} \sigma_A^c = \left\{\left(red, \left[\frac{1}{b_1}, \frac{3}{b_2}\right]\right), \left(\text{brown}, \left[\frac{1}{b_1}, \frac{2}{b_3}\right]\right), \left(\text{black}, \left[\frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\text{notred}, \left[\frac{2}{b_3}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\text{notbrown}, \left[\frac{3}{b_2}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \text{notblack}\left[\frac{1}{b_1}, \frac{3}{b_2}, \frac{2}{b_3}\right]\right\}$$

$\Rightarrow \sigma_A \tilde{\cup} \sigma_A^c \neq \sigma_{\tilde{E}}$

Now to solve this problem, we modify the definition of complement of soft multiset as given below.

Definition 3.14. Let $\sigma_A \in SM(H)$. Then, the complement σ_A^c of σ_A is defined by the approximate function $\sigma_A^c(s) = H \ominus \sigma_A(s)$, for all $s \in E$. Note that $(\sigma_A^c)^c = \sigma_A$ and $\sigma_\phi^c = \sigma_{\tilde{E}}$.

We see that the law of excluded middle and law of contradiction hold with respect to complement for soft multiset given in Definition 3.14.

Proposition 3.15. Let $\sigma_A \in SM(H)$. Then,

i. Law of excluded middle:

$\sigma_A \tilde{\cup} \sigma_A^c = \sigma_{\tilde{E}}$

ii. Law of contradiction:

$\sigma_A \tilde{\cap} \sigma_A^c = \sigma_\phi$

Proof. The proof is straightforward.

Definition 3.16. Let $\sigma_A, \sigma_B \in SM(H)$. Then, the difference $\sigma_A \tilde{\setminus} \sigma_B$ of σ_A and σ_B is defined by the approximate function $\sigma_{A\tilde{\setminus}B}(s) = \sigma_A(s) \ominus \sigma_B(s)$, $\forall s \in E$.

Definition 3.17. Let $\sigma_A, \sigma_B \in SM(H)$. Then, the symmetric difference or disjunctive union $\sigma_A \tilde{\Delta} \sigma_B$ of σ_A and σ_B is defined by $\sigma_A \tilde{\Delta} \sigma_B = (\sigma_A \tilde{\cup} \sigma_B) \tilde{\setminus} (\sigma_A \tilde{\cap} \sigma_B)$ or $\sigma_A \tilde{\Delta} \sigma_B = (\sigma_A \tilde{\cap} \sigma_B^c) \tilde{\cup} (\sigma_A^c \tilde{\cap} \sigma_B) = (\sigma_A \tilde{\setminus} \sigma_B) \tilde{\cup} (\sigma_B \tilde{\setminus} \sigma_A)$.

Example 3.18. Let $H = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$ be a universal multiset of universities of the world under consideration, and α_i denotes the multiplicity of campuses of university $h_i, i = 1,2,\ldots,6$. The set of facilities which may be provided by these universities is given by $E = \{s_1, s_2, s_3, s_4, s_5\}$ where $s_1 =$ library, $s_2 =$ hostels, $s_3 =$ computer and internet facility, $s_4 =$ international standard course work, and $s_5 =$ best security system. Let $A = \{s_1, s_2, s_3\}$, $B = \{s_3, s_4, s_5\}$ and assume that

$$\sigma_A = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), \left(s_3, \left[\frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right)\right\} \text{ and } \sigma_B = \left\{\left(s_3, \left[\frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right), (s_4, H), \left(s_5, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right)\right\}$$

Then,

$$\sigma_A^c = \left\{\left(s_1, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right), \left(s_2, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right), \left(s_3, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), (s_4, H), (s_5, H)\right\}$$

$$\sigma_B^c = \left\{(s_1, H), (s_2, H), \left(s_3, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4},\right]\right) (s_4, \emptyset), \left(s_5, \left[\frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right)\right\}$$

Now, we observe that

$$\sigma_{A\tilde{\setminus}B}(s_1) = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right], \sigma_{A\tilde{\setminus}B}(s_2) = \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right], \sigma_{A\tilde{\setminus}B}(s_3) = \emptyset, \sigma_{A\tilde{\setminus}B}(s_4) = \emptyset, \text{ and } \sigma_{A\tilde{\setminus}B}(s_5) = \emptyset$$

Thus we have

$$\sigma_A \tilde{\setminus} \sigma_B = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), (s_3, \emptyset), (s_4, \emptyset), (s_5, \emptyset)\right\}$$

Now, we examine that

$$\sigma_{B\tilde{\setminus}A}(s_1) = \emptyset,\ \sigma_{B\tilde{\setminus}A}(s_2) = \emptyset,\ \sigma_{B\tilde{\setminus}A}(s_3) = \emptyset,\ \sigma_{B\tilde{\setminus}A}(s_4) = H, \text{ and } \sigma_{B\tilde{\setminus}A}(s_5) = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]$$

Thus we obtain

$$\sigma_B \tilde{\setminus} \sigma_A = \left\{(s_1, \emptyset), (s_2, \emptyset), (s_3, \emptyset), (s_4, H), \left(s_5, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right)\right\}$$

Therefore,

$$\sigma_A \tilde{\Delta} \sigma_B = (\sigma_A \tilde{\setminus} \sigma_B) \mathbin{\tilde{\cup}} (\sigma_B \tilde{\setminus} \sigma_A)$$

$$\sigma_A \tilde{\Delta} \sigma_B = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4},\right]\right) (s_3, \emptyset), (s_4, H), \left(s_5, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right)\right\}$$

We observe that De-Morgan's laws also hold in soft multiset case.

Proposition 3.19. Let $\sigma_A, \sigma_B \in SM(H)$. Then,

i. $(\sigma_A \mathbin{\tilde{\cup}} \sigma_B)^c = \sigma_A^c \mathbin{\tilde{\cap}} \sigma_B^c$
ii. $(\sigma_A \mathbin{\tilde{\cap}} \sigma_B)^c = \sigma_A^c \mathbin{\tilde{\cup}} \sigma_B^c$

Proof.

i. Let $\sigma_A \mathbin{\tilde{\cup}} \sigma_B = \sigma_D$ where, $\sigma_D(s) = \sigma_A(s) \cup \sigma_B(s), \forall s \in E$. Then,

$$\begin{aligned} \sigma_D^c(s) = (\sigma_A(s) \cup \sigma_B(s))^c &= (\sigma_A(s))^c \cap (\sigma_B(s))^c \\ &= \sigma_A^c(s) \cap \sigma_B^c(s) \\ &= \sigma_A^c(s) \cap \sigma_B^c(s), \forall s \in E \end{aligned}$$

Thus $\sigma_D^c = \sigma_A^c \mathbin{\tilde{\cap}} \sigma_B^c$, i.e. $(\sigma_A \mathbin{\tilde{\cup}} \sigma_B)^c = \sigma_A^c \mathbin{\tilde{\cap}} \sigma_B^c$.

ii. Let $\sigma_A \mathbin{\tilde{\cap}} \sigma_B = \sigma_D$ where, $\sigma_D(s) = \sigma_A(s) \cap \sigma_B(s), \forall s \in E$. Then,

$$\begin{aligned} \sigma_D^c(s) = (\sigma_A(s) \cap \sigma_B(s))^c &= (\sigma_A(s))^c \cup (\sigma_B(s))^c \\ &= \sigma_A^c(s) \cup \sigma_B^c(s) \\ &= \sigma_A^c(s) \cup \sigma_B^c(s), \forall s \in E \end{aligned}$$

Thus $\sigma_D^c = \sigma_A^c \mathbin{\tilde{\cup}} \sigma_B^c$, i.e. $(\sigma_A \mathbin{\tilde{\cap}} \sigma_B)^c = \sigma_A^c \mathbin{\tilde{\cup}} \sigma_B^c$.

Proposition 3.20. Let $\sigma_A \in SM(H)$. Then,

i. $\sigma_A \mathbin{\tilde{\cap}} \sigma_{\tilde{E}} = \sigma_A$ and $\sigma_A \mathbin{\tilde{\cup}} \sigma_{\tilde{E}} = \sigma_{\tilde{E}}$
ii. $\sigma_A \mathbin{\tilde{\cup}} \sigma_A = \sigma_A$ and $\sigma_A \mathbin{\tilde{\cap}} \sigma_A = \sigma_A$
iii. $\sigma_A \mathbin{\tilde{\cup}} \sigma_\phi = \sigma_A$ and $\sigma_A \mathbin{\tilde{\cap}} \sigma_\phi = \sigma_\phi$

Proof. The proof is obvious.

We examine that commutative, associative and distributive laws hold in soft multisets.

Proposition 3.21. Let $\sigma_A, \sigma_B, \sigma_C \in SM(H)$. Then,

i. $\sigma_A \mathbin{\tilde{\cup}} \sigma_B = \sigma_B \mathbin{\tilde{\cup}} \sigma_A$
ii. $\sigma_A \mathbin{\tilde{\cap}} \sigma_B = \sigma_B \mathbin{\tilde{\cap}} \sigma_A$
iii. $(\sigma_A \mathbin{\tilde{\cup}} \sigma_B) \mathbin{\tilde{\cup}} \sigma_C = \sigma_A \mathbin{\tilde{\cup}} (\sigma_B \mathbin{\tilde{\cup}} \sigma_C)$
iv. $(\sigma_A \mathbin{\tilde{\cap}} \sigma_B) \mathbin{\tilde{\cap}} \sigma_C = \sigma_A \mathbin{\tilde{\cap}} (\sigma_B \mathbin{\tilde{\cap}} \sigma_C)$
v. $\sigma_A \mathbin{\tilde{\cup}} (\sigma_B \mathbin{\tilde{\cap}} \sigma_C) = (\sigma_A \mathbin{\tilde{\cup}} \sigma_B) \mathbin{\tilde{\cap}} (\sigma_A \mathbin{\tilde{\cup}} \sigma_C)$
vi. $\sigma_A \mathbin{\tilde{\cap}} (\sigma_B \mathbin{\tilde{\cup}} \sigma_C) = (\sigma_A \mathbin{\tilde{\cap}} \sigma_B) \mathbin{\tilde{\cup}} (\sigma_A \mathbin{\tilde{\cap}} \sigma_C)$

Proof. The proof is obvious.

4. Roughness and Fuzziness Associated with Soft Multi-sets

In [6], Ali has defined soft binary relation, soft equivalence relation and soft indiscernibility relation over soft set. He introduced the idea of approximation space associated with each parameter in a soft set. He also proved that for each soft set over a universe U there is a fuzzy soft set over $P(U)$ which induces a soft equivalence relation over $P(U)$. We extend these ideas to the hybrid soft set with multiset.

Definition 4.1. Let H be a universal multiset and σ_A be a soft multiset over $H \times H$. Then σ_A is called a soft multiset binary relation over H. In fact, σ_A is a parameterized family of binary relations on H, i.e for each parameter or attribute s_i, we have a binary relation $\sigma(s_i)$ on H for each parameter $s_i \in A$.

Definition 4.2. Let σ_A be a soft multiset binary relation over H. If $\sigma_A(s_i) \neq \phi$ is an equivalence relation on H for all $s_i \in A$, then σ_A is called a soft multiset equivalence relation over H.

It is well known that each equivalence relation R on a set partitions the set say H into disjoint classes. Similarly each partition of the set H provides us an equivalence relation R. Therefore a soft multiset equivalence relation over H, provides us a parameterized collection of partitions of H. The following example elaborates this concept more effectively.

Example 4.3. Let $H = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$ and $E = \{s_1, s_2, s_3, s_4, s_5\}$ and $A = \{s_1, s_2, s_3\}$ be a subset of E. Consider a soft multiset given by

$$\sigma_A = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), \left(s_3, \left[\frac{\alpha_5}{h_5}\right]\right)\right\}$$

In tabular form, the soft multiset σ_A is written in Table 1.

Table 1. Tabular representation of soft multiset σ_A

σ_A	s_1	s_2	s_3
h_1	α_1	0	0
h_2	α_2	0	0
h_3	0	α_3	0
h_4	0	α_4	0
h_5	0	0	α_5
h_6	0	0	0

Now we see from the table that each attribute $s_i; i = 1,2,3$ generates an equivalence relation on H. Therefore, we get a soft multiset equivalence relation σ_A over H. For each of the equivalence relation, we have the following equivalence classes.

The equivalence classes for $\sigma_A(s_1)$ are $\left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right], \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$.

The equivalence classes for $\sigma_A(s_2)$ are $\left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right], \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$.

The equivalence classes for $\sigma_A(s_3)$ are $\left[\frac{\alpha_5}{h_5}\right], \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_6}{h_6}\right]$.

We observe that soft multiset σ_A defines an indiscernibility relation. Now we define an indiscernibility relation in the next definition.

Definition 4.4. Let σ_A be a soft multiset. An indiscernibility relation defined by σ_A is attained by the intersection of all the equivalence relations generated by attributes s_i and is denoted by $\text{IND}(\sigma_A)$. i.e

$$\text{IND}(\sigma_A) = \cap_{s_i \in A} \sigma_A(s_i)$$

Example 4.5. Consider Example 4.3 then the partition obtained by the indiscernibility relation $\mathrm{IND}(\sigma_A)$ is

$$\left[\frac{\alpha_1}{h_1},\frac{\alpha_2}{h_2}\right],\left[\frac{\alpha_3}{h_3},\frac{\alpha_4}{h_4}\right],\left[\frac{\alpha_5}{h_5}\right],\left[\frac{\alpha_6}{h_6}\right]$$

It is evident that for each attribute s_i, where $i = 1,2,3$, the soft muli-set $(H, \sigma(s_i))$ give us an approximation spaces in Pawlak's sense [61]. Also (H, σ) is an approximation space.

In the following definition we extend the idea of approximation space for multiset as an extension of approximation space for crisp set given by Chakrabarty et al. in [25].

Definition 4.6. Let H be a multiset called universe and let R be an equivalence relation on H, called indiscernibility relation. The pair (H, R) is called an approximation space.

Definition 4.7. Consider a soft multi set $\mathcal{S} = (\sigma, A)$ over the universe of multiset H and $\mathcal{E}$ be a set of parameters, where $A \subseteq \mathcal{E}$ and σ is a function given as $\sigma: A \to PW(H)$. Then the pair $P = (H, \sigma)$ is called a soft multi approximation space. Following the soft multi approximation space P, we get two approximations to every subset $J \subseteq H$ given by

$$\underline{apr_P}(J) = \left\{\frac{l}{x} \in H : \exists s_i \in A, \left[\frac{l}{x} \in \sigma(s_i) \subseteq J\right]\right\},$$
$$\overline{apr}_P(J) = \left\{\frac{l}{x} \in H : \exists s_i \in A, \left[\frac{l}{x} \in \sigma(s_i) \cap J \neq \emptyset\right]\right\},$$

which we call soft multi P-lower approximation and soft multi P-upper approximation of J. Generally, $\underline{apr_P}(J)$ and $\overline{apr}_P(J)$ are called SMR-approximations of J w.r.t P. If $\underline{apr_P}(J) \neq \overline{apr}_P(J)$ then J is said to be soft multi P-rough set or soft multi rough set (SMR-set) otherwise soft multi P-definable. Also, Soft multi P-positive region set, Soft multi P-negative region set and Soft multi P-boundary region set are defined as follows

$$POS_P(J) = \underline{apr_P}(J)$$
$$NEG_P(J) = -\overline{apr}_P(J)$$
$$BND_P(J) = \overline{apr}_P(J) - \underline{apr_P}(J)$$

Example 4.8. Suppose that $H = \left[\frac{l_1}{x_1},\frac{l_2}{x_2},\frac{l_2}{x_3}\right]$ be universial multiset of dresses under consideration, where l_1, l_2, l_3 are the multiplicity of dress x_1, x_2 and x_3 respectively. Let $E = \{s_1 = \text{modernstyle}, s_2 = \text{reasonableprice}, s_3 = \text{comfortable}, s_4 = \text{durable}, s_5 = \text{digitalpriniting}, s_6 = \text{expensive}\}$ and $A = \{s_1, \ldots, s_5\} \subseteq E$. Let $\mathcal{S} = (\sigma, A)$ be soft multiset over H, where the $\sigma: A \to PW(H)$ mapping describes the attractiveness of dresses under consideration as follows:

$$\sigma(\text{modernstyle}) = \left[\frac{l_1}{x_1}\right]$$
$$\sigma(\text{reasonableprice}) = \left[\frac{l_2}{x_2}\right]$$
$$\sigma(\text{comfortable}) = \left[\frac{l_1}{x_1},\frac{l_2}{x_2}\right]$$
$$\sigma(\text{durable}) = \left[\frac{l_1}{x_1},\frac{l_2}{x_2},\frac{l_3}{x_3}\right]$$
$$\sigma(\text{digital priniting}) = \left[\frac{l_1}{x_1}\right]$$

Thus the soft multiset can be written as

$$\mathcal{S} = (\sigma, A) = \sigma_A = \left\{\left(s_1,\left[\frac{l_1}{x_1}\right]\right),\left(s_2,\left[\frac{l_2}{x_2}\right]\right),\left(s_3,\left[\frac{l_1}{x_1},\frac{l_2}{x_2}\right]\right),(s_4,[H]),\left(s_5,\left[\frac{l_1}{x_1}\right]\right)\right\}$$

The tabular form of soft multi set $\mathcal{S} = (\sigma, A)$ is given in Table 2.

Table 2: Soft multi set (σ, A)

(σ, A)	$\frac{l_1}{x_1}$	$\frac{l_2}{x_2}$	$\frac{l_3}{x_3}$
modernstyle	1	0	0
reasonableprice	0	1	0
comfortable	1	1	0
durable	1	1	1
digital priniting	1	0	0

The soft multiset (σ, A) induces soft multi approximation space $P = (H, \sigma)$.

Equivalence classes for $\sigma(s_1)$ are $\left[\frac{l_1}{x_1}\right]$, $\left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$.

Equivalence classes for $\sigma(s_2)$ are $\left[\frac{l_2}{x_2}\right]$, $\left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right]$.

Equivalence classes for $\sigma(s_3)$ are $\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]$, $\left[\frac{l_3}{x_3}\right]$.

Equivalence classes for $\sigma(s_4)$ are $\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$, $\emptyset$.

Equivalence classes for $\sigma(s_5)$ are $\left[\frac{l_1}{x_1}\right]$, $\left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$.

If we consider $J = \left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right] \subseteq H$, we obtain soft multi P-lower approximation and soft multi P-upper approximation respectively given by

$$\underline{apr_P}(J) = \left[\frac{l_1}{x_1}\right]$$
$$\overline{apr}_P(J) = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$$

Since $\underline{apr_P}(J) \neq \overline{apr}_P(J)$ and $J = \left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right]$ is a soft multi P-rough set or soft multi rough set (SMR-set).

Here $POS_P(J) = \{\left[\frac{l_1}{x_1}\right]\}$, $NEG_P(J) = \emptyset$ and $BND_P(J) = \left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$.

Note that in the case $\underline{apr_P}(J) = \overline{apr}_P(J)$, then J is said to be a soft multi P-definable set.

Remark:

It is clear from above example that the approximations of soft multi rough set are multi sets. So the operations used in soft multi rough set are multiset operations.

Definition 4.9. Let σ_A be a soft multiset over a multiset H. Then $\sigma_A: E \to PW(H)$ is a mapping. Define a map $D_s: PW(H) \to [0,1]$, for all $s \in E$ such that

$$D_s(U) = \begin{cases} \frac{|\sigma_A(s) \cap U|}{|\sigma_A(s)|}, & \text{if } \sigma_A(s) \neq \phi \\ 0, & \text{if } \sigma_A(s) = \phi \end{cases}$$

where $\forall U \in PW(H)$. Obviously for each $s \in A$, D_s is a fuzzy multiset over $PW(H)$. Hence σ_A generates a fuzzy soft multiset over $PW(H)$.

Proposition 4.10. Let σ_A be a soft multiset over a multiset H. Then $D_s(U) = D_s(V)$, for any $s \in A$ if and only if $|\sigma_A(s) \cap U| = |\sigma_A(s) \cap V|$, where $U, V \in PW(H)$.

From Proposition 4.10, it is easy to see that soft multiset σ_A generates a soft multiset binary relation over $PW(H)$. This soft multiset binary relation is denoted by λ_A and is defined as $(U, V) \in \lambda_A(s)$ if and only if $D_s(U) = D_s(V)$, where $U, V \in PW(H), s \in A$.

Theorem 4.11. The soft multiset binary relation λ_A over $PW(H)$ is a soft multiset equivalence relation and each partition $PW(H)/\lambda_A(s)$ preserves a strict order between its equivalence classes for all $s \in A$.

PROOF. We know that for each $s \in A$, $\lambda_A(s)$ is an equivalence relation by using the definition of soft multiset binary relation λ_A. Hence λ_A over $PW(H)$ is a soft multiset equivalence relation. So $PW(H)/\lambda_A(s)$ is a partition of $PW(H)$, for each $s \in A$. If for any $s \in A$, a class in $PW(H)/\lambda_A(s)$ including some element $U \in PW(H)$ is denoted by $[U]_{\lambda_A(s)}$, then for each $V \in [U]_{\lambda_A(s)}$ we get $D_s(U) = D_s(V)$, by definition. This means that a unique real number belonging to [0,1] can be assigned to each class in $PW(H)/\lambda_A(s)$. Let this number be u, for the class $[U]_{\lambda_A(s)}$ and say it characteristic of $[U]_{\lambda_A(a)}$. Hence there is a strict order between the classes because each class in $PW(H)/\lambda_A(s)$ has a unique characteristic from [0,1]. Therefore this order is defined as $[W]_{\lambda_A(s)} \prec [U]_{\lambda_A(s)}$ if and only if $w < u$, where class $[W]_{\lambda_A(s)}$ has w characteristic belonging to [0,1].

Example 4.12. By using Example 4.8, we cosider $H = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$ and soft multiset given by

$$\mathcal{S} = (\sigma, A) = \sigma_A = \left\{\left(s_1, \left[\frac{l_1}{x_1}\right]\right), \left(s_2, \left[\frac{l_2}{x_2}\right]\right), \left(s_3, \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]\right), (s_4, [H]), \left(s_5, \left[\frac{l_1}{x_1}\right]\right)\right\}$$

Let $X = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]$ The power whole sub multisets (sub msets) are

$$\emptyset = \left[\frac{0}{x_1}, \frac{0}{x_2}, \frac{0}{x_3}\right], \left[\frac{l_1}{x_1}\right], \left[\frac{l_2}{x_2}\right], \left[\frac{l_3}{x_3}\right], \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right], \left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right], \text{and } \left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right], H = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$$

As $\left[\frac{l_1}{x_1}\right] \in \sigma(s_1)$ and $D_{s_1}(X) = \frac{|\sigma(s_1) \cap X|}{|\sigma(s_1|} = \frac{\left|\left[\frac{l_1}{x_1}\right]\right|}{\left|\left[\frac{l_1}{x_1}\right]\right|} = \frac{l_1}{l_1} = 1$, so $D_{s_1}\left(\left[\frac{l_1}{x_1}\right]\right) = 1$.

As $\left[\frac{l_2}{x_2}\right] \notin \sigma(s_1)$, so $D_{s_1}\left(\left[\frac{l_2}{x_2}\right]\right) = 0$. Similarly $\left[\frac{l_3}{x_3}\right] \notin \sigma(s_1)$, so $D_{s_1}\left(\left[\frac{l_3}{x_3}\right]\right) = 0$.

Thus the fuzzy multiset λ_1 associated to the parameter s_1 is given by

$$\lambda_1 = \left\{\frac{0}{\emptyset}, \frac{1}{\left[\frac{l_1}{x_1}\right]}, \frac{0}{\left[\frac{l_2}{x_2}\right]}, \frac{0}{\left[\frac{l_3}{x_3}\right]}, \frac{\frac{l_1}{l_1+l_2}}{\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]}, \frac{0}{\left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]}, \frac{\frac{l_1}{l_1+l_3}}{\left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right]}, \frac{\frac{l_1}{l_1+l_2+l_3}}{\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]}\right\}$$

Similarly we can find fuzzy multiset λ_i associated to each parameter s_i, $i = 2,3,4,5$.

It is obvious that degree of membership or rough belongingness is a number from the interval [0,1].

Babitha and John [20] presented the idea of multi-valued information system as given by the following definition.

Definition 4.13. [20] "A multi-valued information system is a quadruple $I = (Z, A, f, U)$ where Z is a non-empty finite set of objects, A is a non-empty finite set of parameters, $U = \bigcup_{a \in A} U_a$, where U is the domain set (value set) of attribute a which has multi-value ($|U_a| \geq 3$) and $f: H \times A \to U$ is a total function such that $f(h, a) \in U_a$ for each $(h, a) \in Z \times A$".

Proposition 4.14. [20] If σ_A is a soft multiset over H, then σ_A is a multi-valued information system.

Example 4.15. Let $H = \left\{\frac{n_1}{h_1}, \frac{n_2}{h_2}, \frac{n_3}{h_3}, \frac{n_4}{h_4}, \frac{n_5}{h_5}, \frac{n_6}{h_6}, \frac{n_7}{h_7}, \frac{n_8}{h_8}, \frac{n_9}{h_9}, \frac{n_{10}}{h_{10}}, \frac{n_{11}}{h_{11}}, \frac{n_{12}}{h_{12}}, \frac{n_{13}}{h_{13}}, \frac{n_{14}}{h_{14}}, \frac{n_{15}}{h_{15}}\right\}$ be a multiset of some brands of shoes under consideration, where $h_i; i = 1,2,3,\ldots,15$. Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}\}$ be the set of attributes defined as $\alpha_1 =$ leather, $\alpha_2 =$ comfortable, $\alpha_3 =$ stylish, $\alpha_4 =$ perfect fine quality, $\alpha_5 =$ relaxable for feet, $\alpha_6 =$ softer sole, $\alpha_7 =$ better grip, $\alpha_8 =$ longer life shoe, $\alpha_9 =$ discount, and $\alpha_{10} =$ cheap. Then, the soft multiset σ_A describes attractiveness of shoes under consideration as given below:

$$\sigma_A = \{(\alpha_1, \sigma_A(\alpha_1)), (\alpha_2, \sigma_A(\alpha_2)), (\alpha_3, \sigma_A(\alpha_3)), (\alpha_4, \sigma_A(\alpha_4)), (\alpha_5, \sigma_A(\alpha_5)), (\alpha_6, \sigma_A(\alpha_6)), (\alpha_7, \sigma_A(\alpha_7)), (\alpha_8, \sigma_A(\alpha_8)), (\alpha_9, \sigma_A(\alpha_9)), (\alpha_{10}, \sigma_A(\alpha_{10}))\}$$

where

$$\sigma_A(\alpha_1) = \left[\frac{n_1}{h_1}, \frac{n_2}{h_2}, \frac{n_4}{h_4}\right], \sigma_A(\alpha_2) = \left[\frac{n_2}{h_2}, \frac{n_4}{h_4}, \frac{n_6}{h_6}, \frac{n_8}{h_8}, \frac{n_{10}}{h_{10}}, \frac{n_{12}}{h_{12}}, \frac{n_{14}}{h_{14}}\right], \sigma_A(\alpha_3) = \left[\frac{n_3}{h_3}, \frac{n_6}{h_6}, \frac{n_9}{h_9}, \frac{n_{12}}{h_{12}}, \frac{n_{15}}{h_{15}}\right],$$

$$\sigma_A(\alpha_4) = \left[\frac{n_4}{h_4}, \frac{n_8}{h_8}, \frac{n_{12}}{h_{12}}\right], \sigma_A(\alpha_5) = \left[\frac{n_5}{h_5}, \frac{n_{10}}{h_{10}}, \frac{n_{15}}{h_{15}}\right], \sigma_A(\alpha_6) = \left[\frac{n_6}{h_6}, \frac{n_{12}}{h_{12}}\right], \sigma_A(\alpha_7) = \left[\frac{n_7}{h_7}, \frac{n_{14}}{h_{14}}\right],$$

$$\sigma_A(\alpha_8) = \left[\frac{n_1}{h_1}, \frac{n_8}{h_8}\right], \sigma_A(\alpha_9) = \left[\frac{n_1}{h_1}, \frac{n_2}{h_2}\right] \text{ and } \sigma_A(\alpha_{10}) = \left[\frac{n_1}{h_1}, \frac{n_2}{h_2}, \frac{n_3}{h_3}\right]$$

Then the quadruple $I = (Z, A, f, U)$ corresponding to the soft multiset given above is a multi-valued information system. Here $Z = H$ and A is the same set of parameters as in soft multiset and $U_{\alpha_1} = \{n_1, n_2, n_4\}$, $U_{\alpha_2} = \{n_2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}\}$, $U_{\alpha_3} = \{n_3, n_6, n_9, n_{12}, n_{15}\}$. For the pair (h_1, α_1), we have $f(h_1, \alpha_1) = n_1$. Similarly we obtain the value of other pairs. Now we construct an information table representing soft multiset σ_A given as:

Table 3. Multi-valued information systems

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
h_1	n_1	0	0	0	0	0	0	n_1	n_1	n_1
h_2	n_2	n_2	0	0	0	0	0	0	n_2	n_2
h_3	0	0	n_3	0	0	0	0	0	0	n_3
h_4	n_4	n_4	0	n_4	0	0	0	0	0	0
h_5	0	0	0	0	n_5	0	0	0	0	0
h_6	0	n_6	n_6	0	0	n_6	0	0	0	0
h_7	0	0	0	0	0	0	n_7	0	0	0
h_8	0	n_8	0	n_8	0	0	0	n_8	0	0
h_9	0	0	n_9	0	0	0	0	0	0	0
h_{10}	0	n_{10}	0	0	n_{10}	0	0	0	0	0
h_{11}	0	0	0	0	0	0	0	0	0	0
h_{12}	0	n_{12}	n_{12}	n_{12}	0	n_{12}	0	0	0	0
h_{13}	0	0	0	0	0	0	0	0	0	0
h_{14}	0	n_{14}	0	0	0	0	n_{14}	0	0	0
h_{15}	0	0	n_{15}	0	n_{15}	0	0	0	0	0

Thus according to Proposition 4.14, it is seen that soft multisets are multi-valued information systems. However, it is clear that multi-valued information systems are not necessarily soft multisets.

5. A Soft Multi-Set Approach to Multi-Attribute Decision-Making

In this section, we discuss the fuzzy whole sub-multisets of $PW(H)$, affiliated with each attribute of a soft multiset σ_A over H. These fuzzy sub-multisets of $PW(H)$ generate equivalence relations. These fuzzy sub-multisets and equivalence relations perform a vital job in decision making. We extend some results and algorithm which as used for soft set given in [6] to soft multiset. Thus we deduce that a soft multiset over a multiset H induces a fuzzy soft multiset over $PW(H)$ which yields a soft multiset equivalence relation on $PW(H)$. First we present an algorithm to the multi-attribute decision-making (MADM) for the selection of a best fertilizer.

Algorithm:

Step 1. Input suitable parameter set E and universal multiset H.

Step 2. Input a soft multiset σ_A over H.

Step 3. Compute fuzzy multiset D_{s_i} over $PW(H)$ corresponding to parameter s_i, $\forall s_i \in A$.

Step 4. Construct equivalence relation $\lambda_A(s_i)$, $\forall s_i \in A$ such that an equivalence class $C_j(s_i)$ of $\delta(s_i)$ contains those submultisets of $PW(H)$ that have same degree of membership.

Step 5. Arrange equivalence classes $C_j(s_i)$ of $\lambda_A(s_i)$, $\forall s_i \in A$ in order.

Step 6. Choose the equivalence class represented by $h(s_i)$ that has highest order.

Step 7. Compute the disjunctive union $T(s_i)$ such that

$$T(s_i) = \begin{cases} \Delta k(s_i), & \text{if } |h(s_i)| > 1 \\ k(s_i), & \text{if } |h(s_i)| = 1 \end{cases}$$

where $k(s_i) \in h(s_i)$.

Step 8. Find $P(\frac{n}{x}) = \{s_i : \frac{n}{x} \in T(s_i)\}$.

Step 10. Choose that $\frac{n}{x}$ that has $\max\{|P(\frac{n}{x})|\}$.

Numerical Example 5.1. Let us suppose that a farmer wants to buy a particular type of fertilizer for his fields satisfying his demand. Let $U = [20/u, 30/a, 25/c, 35/p]$ be the universal multiset of fertilizers under consideration, where u denotes "Ammonium Phosphate", a denotes "Urea", c denotes "Calcium Nitrate", p denotes "Ammonium Nitrate" and multiplicity of fertilizer denotes the number of sacks of fertilizer that are required to fertilize the fields. Let $E = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of attributes defined as s_1 = rapid plant growth, s_2 = maintain green color in plants, s_3 = improve alkaline soils, s_4 = bacteria killer, s_5 = activator for enzymes in plants. Consider a soft multiset describing the different types of fertilizers under consideration and is given by

$$\sigma_A = \left\{\left(s_1, \left[\frac{20}{u}\right]\right), \left(s_2, \left[\frac{25}{c}, \frac{35}{p}\right]\right), \left(s_3, \left[\frac{30}{a}\right]\right), \left(s_4, \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right), \left(s_5, \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]\right)\right\}$$

For the attribute s_1, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_1} = \left\{\frac{0}{\left[\frac{0}{\phi}\right]}, \frac{1}{\left[\frac{20}{u}\right]}, \frac{0}{\left[\frac{30}{a}\right]}, \frac{0}{\left[\frac{25}{c}\right]}, \frac{0}{\left[\frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_1)$ generated by D_{s_1} are

$$C_1(s_1) = \left\{\phi, \left[\frac{30}{a}\right], \left[\frac{25}{c}\right], \left[\frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}\right], \left[\frac{30}{a}, \frac{35}{p}\right], \left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

$$C_2(s_1) = \left\{\left[\frac{20}{u}\right], \left[\frac{20}{u}, \frac{30}{a}\right], \left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by $C_2(s_1) \succ C_1(s_1)$.

Since $C_2(s_1)$ has highest order, hence $h(s_1) = C_2(s_1)$.

Then, $T(s_1) = \Delta k(s_1) = (\cup\, k(s_1)) \cap (\cap\, k(s_1))^c$, where $k(s_1) \in h(s_1)$.

$T(s_1) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{20}{u}\right]^c.$

$T(s_1) = \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right].$

For the attribute s_2, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_2} = \left\{\frac{0}{\phi}, \frac{0}{\left[\frac{20}{u}\right]}, \frac{0}{\left[\frac{30}{a}\right]}, \frac{0.4}{\left[\frac{25}{c}\right]}, \frac{0.6}{\left[\frac{35}{p}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0.4}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0.6}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0.4}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0.6}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{0.4}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{0.6}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_2)$ generated by D_{s_2} are

$$C_1(s_2) = \left\{\phi, \left[\frac{20}{u}\right], \left[\frac{30}{a}\right], \left[\frac{20}{u}, \frac{30}{a}\right]\right\}, C_2(s_2) = \left\{\left[\frac{25}{c}\right], \left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{30}{a}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]\right\},$$

$$C_3(s_2) = \left\{\left[\frac{35}{p}\right], \left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]\right\}, C_4(s_2) = \left\{\left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by $C_4(s_2) \succ C_3(s_2) \succ C_2(s_2) \succ C_1(s_2)$.

Since $C_4(s_2)$ has highest order, hence $h(s_2) = C_4(s_2)$.

Then, $T(s_2) = \Delta k(s_2) = (\cup\, k(s_2)) \cap (\cap\, k(s_2))^c$, where $k(s_2) \in h(s_2)$.

$T(s_2) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{25}{c}, \frac{35}{p}\right]^c.$

$T(s_2) = \left[\frac{20}{u}, \frac{30}{a}\right].$

For the attribute s_3, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_3} = \left\{\frac{0}{\phi}, \frac{0}{\left[\frac{20}{u}\right]}, \frac{1}{\left[\frac{30}{a}\right]}, \frac{0}{\left[\frac{25}{c}\right]}, \frac{0}{\left[\frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_3)$ generated by D_{s_3} are

$$C_1(s_3) = \left\{\phi, \left[\frac{20}{u}\right], \left[\frac{25}{c}\right], \left[\frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]\right\},$$

$$C_2(s_3) = \left\{\left[\frac{30}{a}\right], \left[\frac{20}{u}, \frac{30}{a}\right], \left[\frac{30}{a}, \frac{25}{c}\right], \left[\frac{30}{a}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by $C_2(s_3) \succ C_1(s_3)$.

Since $C_2(s_3)$ has highest order, hence $h(s_3) = C_2(s_3)$.

Then, $T(s_3) = \Delta k(s_3) = (\cup\, k(s_3)) \cap (\cap\, k(s_3))^c$, where $k(s_3) \in h(s_3)$.

$T(s_3) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{30}{a}\right]^c.$

$T(s_3) = \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]$.

For the attribute s_4, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_4} = \left\{\frac{0}{\phi}, \frac{0.18}{\left[\frac{20}{u}\right]}, \frac{0.27}{\left[\frac{30}{a}\right]}, \frac{0.23}{\left[\frac{25}{c}\right]}, \frac{0.32}{\left[\frac{35}{p}\right]}, \frac{0.45}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0.41}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0.5}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0.5}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0.59}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0.55}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{0.68}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{0.77}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{0.73}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{0.82}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_4)$ generated by D_{s_4} are

$$C_1(s_4) = \{\phi\}, C_2(s_4) = \left\{\left[\frac{20}{u}\right]\right\}, C_3(s_4) = \left\{\left[\frac{30}{a}\right]\right\}, C_4(s_4) = \left\{\left[\frac{25}{c}\right]\right\}, C_5(s_4) = \left\{\left[\frac{35}{p}\right]\right\},$$

$$C_6(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}\right]\right\}, C_7(s_4) = \left\{\left[\frac{20}{u}, \frac{25}{c}\right]\right\}, C_8(s_4) = \left\{\left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}\right]\right\},$$

$$C_9(s_4) = \left\{\left[\frac{30}{a}, \frac{35}{p}\right]\right\}, C_{10}(s_4) = \left\{\left[\frac{25}{c}, \frac{35}{p}\right]\right\}, C_{11}(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]\right\}, C_{12}(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]\right\},$$

$$C_{13}(s_4) = \left\{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]\right\}, C_{14}(s_4) = \left\{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}, C_{15}(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by

$C_{15}(s_4) \succ C_{14}(s_4) \succ C_{12}(s_4) \succ C_{13}(s_4) \succ C_{11}(s_4) \succ C_9(s_4) \succ C_{10}(s_4) \succ C_8(s_4) \succ C_6(s_4) \succ C_7(s_4) \succ C_5(s_4) \succ C_3(s_4) \succ C_4(s_4) \succ C_2(s_4) \succ C_1(s_4)$.

Since $C_{15}(s_4)$ has highest order, hence $h(s_4) = C_{15}(s_4)$.

Then, $T(s_4) = k(s_4) = [\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}]$, since $|h(s_4)| = 1$.

For the attribute s_5, we have a fuzzy submultiset of PW(H) given as:

$$D_{s_5} = \left\{\frac{0}{\phi}, \frac{0.25}{\left[\frac{20}{u}\right]}, \frac{0}{\left[\frac{30}{a}\right]}, \frac{0.3125}{\left[\frac{25}{c}\right]}, \frac{0.4375}{\left[\frac{35}{p}\right]}, \frac{0.25}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0.5625}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0.6875}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0.3125}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0.4375}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0.75}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{0.5625}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{0.6875}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{0.75}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_5)$ generated by D_{s_5} are

$$C_1(s_5) = \left\{\phi, \left[\frac{30}{a}\right]\right\}, C_2(s_5) = \left\{\left[\frac{20}{u}\right], \left[\frac{20}{u}, \frac{30}{a}\right]\right\}, C_3(s_5) = \left\{\left[\frac{25}{c}\right], \left[\frac{30}{a}, \frac{25}{c}\right]\right\}, C_4(s_5) = \left\{\left[\frac{35}{p}\right], \left[\frac{30}{a}, \frac{35}{p}\right]\right\},$$

$$C_5(s_5) = \left\{\left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]\right\}, C_6(s_5) = \left\{\left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]\right\},$$

$$C_7(s_5) = \left\{\left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}, C_8(s_5) = \left\{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by

$C_8(s_5) \succ C_7(s_5) \succ C_6(s_5) \succ C_5(s_5) \succ C_4(s_5) \succ C_3(s_5) \succ C_2(s_5) \succ C_1(s_5)$.

Since $C_8(s_5)$ has highest order, hence $h(s_5) = C_8(s_5)$.

Then, $T(s_5) = \Delta k(s_5) = (\cup k(s_5)) \cap (\cap k(s_5))^c$, where $k(s_5) \in h(s_5)$.

$T(s_5) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]^c = \left[\frac{30}{a}\right]$.

Now $P(\frac{20}{u}) = \{s_2, s_3, s_4\} \Rightarrow |P(\frac{20}{u})| = 3,$

$P(\frac{30}{a}) = \{s_1, s_2, s_4, s_5\} \Rightarrow |P(\frac{30}{a})| = 4,$

$P(\frac{25}{c}) = \{s_1, s_3, s_4\} \Rightarrow |P(\frac{25}{c})| = 3$ and

$P(\frac{35}{p}) = \{s_1, s_3, s_4\} \Rightarrow |P(\frac{35}{p})| = 3.$

Therefore, $\max\{|P(\frac{n}{x})|\} = 4, \forall \frac{n}{x} \in H.$

Thus 30 sacks of "Urea" are selected to fertilize the fields because "Urea" fertilizer has maximum qualities.

6. Conclusion

We introduced some fundamental properties of soft multisets and related results. We defined binary relation, equivalence relation and indiscernibility relation on soft multisets with the help of illustrations. We introduced the concept of an approximation space associated with each parameter in a soft multiset and an approximation space associated with the soft multiset. We presented the novel concepts of roughness and fuzziness associated with soft multiset with the help of illustrations. We studied soft multisets in multi-valued information system. Furthermore, We presented an Algorithm to cope with uncertainties in the multi-attribute decision making problems by utilizing soft multiset theory. The proposed Algorithm is also summarized by the flow chart. The effectiveness of the Algorithm has verified by a case study to make the best selection of fertilizer.

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References

[1] W. D. Blizard, *Multiset Theory*, Notre Dame Journal of Formal Logic 30 (1989) 36-65.

[2] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.

[3] D. Molodtsov, *Soft Set Theory-First Results*, Computers and Mathematics with Applications 37(4-5) (1999) 19-31.

[4] M. Abbas, M. I. Ali, S. Romaguerac, *Generalized Operations in Soft Set Theory via Relaxed Conditions on Parameters*, Filomat 31(19) (2017) 5955-5964. https://doi.org/10.2298/FIL1719955A

[5] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, *On Some New Operations in Soft Set Theory*, Computers and Mathematics with Applications 57 (2009) 1547-1553.

[6] M. I. Ali, *A Note on Soft Sets, Rough Soft Sets and Fuzzy Soft Sets*, Applied Soft Computing 11 (2011) 3329-3332.

[7] M. I. Ali, M. Shabir, F. Feng, *Representation of Graphs Based on Neighborhoods and Soft Sets*, International Journal of Machine Learning and Cybernetics 8(5) (2017) 1525-1534. https://doi.org/10.1007/s13042-016-0525-z

[8] F. Feng, M. Irfan Ali, M. Shabir, *Soft Relations Applied to Semigroups*, Filomat 27(7) (2013) 1183-1196. https://doi.org/10.2298/FIL1307183F

[9] F. Feng, M. Akram, B. Davvaz, V. L. Fotea, *Attribute Analysis of Information Systems Based on Elementary Soft Implications*, Knowledge-Based Systems 70 (2014) 281-292.

[10] F. Feng, Y.B. Jun, X. Liu, L. Li, *An Adjustable Approach to Fuzzy Soft Set Based Decision Making*, Journal of Computational and Applied Mathematics 234(1) (2010) 10-20.

[11] F. Feng, C. Li, B. Davvaz, M.I. Ali, *Soft Sets Combined with Fuzzy Sets and Rough Sets, A Tentative Approach*, Soft Computing 14(9) (2010) 899-911.

[12] F. Feng, X. Y. Liu, V. Leoreanu-Fotea, Y. B. Jun, *Soft Sets and Soft Rough Sets*, Information Sciences 181(6) (2011) 1125-1137.

[13] F. Feng, H. Fujita, M. I. Ali, R. R. Yager, X. Liu, *Another View on Generalized Intuitionistic Fuzzy Soft Sets and Related Multiattribute Decision Making Methods*, IEEE Transactions on Fuzzy Systems 27(3) (2019) 474-488.

[14] K. Hayat, M. I. Ali, BY. Cao, F. Karaaslan, *New Results on Type-2 Soft Sets*, Hacettepe Journal of Mathematics and Statistics 47(4) (2018) 855-876.

[15] P. K. Maji, A. R. Roy, R. Biswas, *An Application of Soft Sets in A Decision Making Problem*, Computers and Mathematics with Applications 44(8-9) (2002) 1077-1083.

[16] P. K. Maji, A. R. Roy, R. Biswas, *Soft Set Theory*, Computers and Mathematics with Applications 45(4-5) (2003) 555-562.

[17] H. Aktaş, N. Çağman, *Soft Sets and Soft Group*, Information Sciences 1(77) (2007) 2726-2735.

[18] Z. Kong, L. Gaoa, L. Wang, S. Li, *The Normal Parameter Reduction of Soft Sets and Its Algorithm*, Computers and Mathematics with Applications 56 (2008) 3029-3037.

[19] K. V. Babitha, J. J. Sunil, *Soft Set Relations and Functions*, Computers and Mathematics with Applications 60 (2010) 1840-1849.

[20] K. V. Babitha, S. J. John, *On Soft Multi Sets*, Annals of Fuzzy Mathematics and Informatics 5(1) (2013) 35-44.

[21] S. Alkhazaleh, A. R. Salleh, N. Hassan, *Soft Multisets Theory*, Applied Mathematical Sciences 5(72) (2011) 3561-3573.

[22] A. Adeel, M. Akram, Ali N. A. Koam, *Group Decision-Making Based on m-Polar Fuzzy Linguistic TOPSIS Method*, Symmetry 11(735) (2019) 1-20. https://doi.org/10.3390/sym11060735

[23] M. Akram, M. Arshad, *A Novel Trapezoidal Bipolar Fuzzy TOPSIS Method for Group Decision-Making*, Group Decision and Negotiation 28 (2018) 565-584. https://doi.org/10.1007/s10726-018-9606-6

[24] N. Çağman, S. Karataş, S. Enginoglu, *Soft Topology*, Computers and Mathematics with Applications 62 (2011) 351-358.

[25] K. Chakrabarty, R. Biswas, S. Nanda, *Fuzziness in Rough Sets*, Fuzzy Sets and Systems 110(2) (2000) 247-251.

[26] K. P. Girsh, S. J. John, *General Relations between Partially Ordered Multisets and Their Chains and Antichains*, Mathematical Communications14(2) (2009) 193-206.

[27] M. B. Gorzalzany, *A Method of Inference in Approximate Reasoning Based on Interval-Valued Fuzzy Sets*, Fuzzy Sets and Systems 21 (1987) 1-17.

[28] D. Pie, D. Miao, *From Soft Sets to Information System*, IEEE International Conference on Granular Computing (2005) 617-621 Beijing, China.

[29] M. Shabir, M. I. Ali, *Soft Ideals and Generalized Fuzzy Ideals in Semigroups*, New Mathematics and Natural Computation 5(3) (2009) 599-615.

[30] X. Liu, Y. Ju, S. Yang, *Hesitant Intuitionistic Fuzzy Linguistic Aggregation Operators and Their Applications to Multi Attribute Decision Making*, Journal of Intelligent and Fuzzy Systems 26(3) (2014) 1187-1201.

[31] M. R. Hashmi, M. Riaz, F. Smarandache, *m-polar Neutrosophic Topology with Applications to Multi-Criteria Decision-Making in Medical Diagnosis and Clustering Analysis*, International Journal of Fuzzy Systems 22(1) (2020), 273-292. https://doi.org/10.1007/s40815-019-00763-2

[32] M. R. Hashmi, M. Riaz, *A Novel Approach to Censuses Process by using Pythagorean m-polar Fuzzy Dombi's Aggregation Operators*, Journal of Intelligent and Fuzzy Systems 38(2) (2020) 1977-1995. https://doi.org/10.3233/JIFS-190613

[33] K. Naeem, M. Riaz, X. D. Peng, D. Afzal, *Pythagorean Fuzzy Soft MCGDM Methods Based on TOPSIS, VIKOR and Aggregation Operators*, Journal of Intelligent and Fuzzy Systems 37(5) (2019) 6937-6957. https://doi.org/10.3233/JIFS-190905

[34] K. Naeem, M. Riaz, D. Afzal, *Pythagorean m-polar Fuzzy Sets and TOPSIS method for the Selection of Advertisement Mode*, Journal of Intelligent and Fuzzy Systems 37(6) (2019) 8441-8458. https://doi.org/10.3233/JIFS-191087

[35] M. Riaz, N. Çağman, I. Zareef, M. Aslam, *N-Soft Topology and its Applications to Multi-Criteria Group Decision Making*, Journal of Intelligent and Fuzzy Systems 36(6) (2019) 6521-6536. https://doi.org/10.3233/JIFS-182919

[36] M. Riaz, F. Samrandache, A. Firdous, A. Fakhar, *On Soft Rough Topology with Multi-Attribute Group Decision Making*, Mathematics **7**(67) (2019) 1-18. https://doi.org/10.3390/math7010067

[37] M. Riaz, B. Davvaz, A. Firdous, A. Fakhar, *Novel Concepts of Soft Rough Set Topology with Applications*, Journal of Intelligent and Fuzzy Systems 36(4) (2019) 3579-3590.

[38] M. Riaz, M. R. Hashmi, *MAGDM for Agribusiness in the Environment of Various Cubic m-polar Fuzzy Averaging Aggregation Operators*, Journal of Intelligent and Fuzzy Systems 37(3) (2019) 3671-3691. https://doi.org/10.3233/JIFS-182809

[39] M. Riaz, M. R. Hashmi, *Linear Diophantine Fuzzy Set and Its Applications Towards Multi-Attribute Decision Making Problems*, Journal of Intelligent and Fuzzy Systems 37(4) (2019) 5417-5439. https://doi.org/10.3233/JIFS-190550

[40] M. Riaz and M. R. Hashmi, *Soft Rough Pythagorean m-Polar Fuzzy Sets and Pythagorean m-Polar Fuzzy Soft Rough Sets with Application to Decision-Making*, Computational and Applied Mathematics (2019). https://doi.org/10.1007/s40314-019-0989-z

[41] M. Riaz, S. T. Tehrim, *Cubic Bipolar Fuzzy Ordered Weighted Geometric Aggregation Operators and Their Application Using Internal and External Cubic Bipolar Fuzzy Data*, Computational and Applied Mathematics, 38(2) (2019) 1-25.

[42] M. Riaz, S. T. Tehrim, *Multi-Attribute Group Decision Making Based Cubic Bipolar Fuzzy Information Using Averaging Aggregation Operators*, Journal of Intelligent and Fuzzy Systems 37(2) (2019) 2473-2494.

[43] M. Riaz, S. T. Tehrim, *Bipolar Fuzzy Soft Mappings with Application to Bipolar Disorders*, International Journal of Biomathematics 12(7) (2019). https://doi.org/10.1142/S1793524519500803

[44] R. Roy, P. K. Maji, *A Fuzzy Soft Set Theoretic Approach to Decision Making Problems*, Journal of Computational and Applied Mathematics 203(2) (2007) 412-418.

[45] G. Şenel, *The Relation Between Soft Topological Space and Soft Ditopological Space*, Communications Series A1: Mathematics and Statistics 67(2) (2018) 209-219. https://doi.org/10.1501/Commua1-0000000875

[46] G. Şenel, *Characterization of Soft Sets by delta-Soft Operations*, Journal of Polytechnic 21(3) (2018) 707-713. https://doi.org/10.2339/politeknik.389631

[47] A. Sezgin, A. O. Atagün, *On Operations of Soft Sets*, Computers and Mathematics with Applications 61(5) (2011) 1457-1467.

[48] A. Sezgin, A. O. Atagün, N. Çağman, *Soft Intersection Near-Rings with Applications*, Neural Computing and Applications 21(1) (2012) 221-229.

[49] M. Shabir, M. Naz, *On Soft Topological Spaces*, Computers and Mathematics with Applications 61 (2011) 1786-1799.

[50] S. T. Tehrim, M. Riaz, *A Novel Extension of TOPSIS to MCGDM with Bipolar Neutrosophic Soft Topology*, Journal of Intelligent and Fuzzy Systems 37(4) (2019) 5531-5549. https://doi.org/10.3233/JIFS-190668

[51] G. Wei, H. Wang, X. Zhao, R. Lin, *Hesitant Triangular Fuzzy Information Aggregation In Multiple Attribute Decision Making*, Journal of Intelligence and Fuzzy Systems 26(3) (2014) 1201-1209.
[52] Xueling Ma, Q. Liu, J. Zhan, *A Survey of Decision Making Methods Based on Certain Hybrid Soft Set Models*, Artificial Intelligence Review 47 (2017) 507-530.
[53] Z. S. Xu, X. L. Zhang, *Hesitant Fuzzy Multi-Attribute Decision-Making Based on TOPSIS with Incomplete Weight Information*, Knowledge-Based Systems 52 (2013) 53-64.
[54] Z. S. Xu, *Intuitionistic Fuzzy Aggregation Operators*, IEEE Transactions on Fuzzy Systems 15(6) (2007) 1179-1187.
[55] Z. S. Xu, X.Q. Cai, *Intuitionistic Fuzzy Information Aggregation: Theory and Applications*, Science Press Beijing and Springer-Verlag Berlin Heidelberg, 2012.
[56] Z. S. Xu, *Studies in Fuzziness and Soft Computing: Hesitant Fuzzy Sets Theory*, Springer International Publishing Switzerland, 2014.
[57] J. Zhan, Q. Liu, B. Davvaz, *A New Rough Set Theory: Rough Soft Hemirings*, Journal of Intelligence and Fuzzy Systems 28(4) (2015) 1687-1697.
[58] J. Zhan, J. C. R. Alcantud, *A Novel Type of Soft Rough Covering and Its Application to Multicriteria Group Decision Making*, Artificial Intelligence Review 52 (2019) 2381-2410. https://doi.org/10.1007/s10462-018-9617-3
[59] X. L. Zhang, Z. S. Xu, *Extension of TOPSIS to Multiple Criteria Decision Making with Pythagorean Fuzzy Sets*, International Journal of Intelligent Systems 29(12) (2014) 1061-1078.
[60] A. Syropoulos, *Mathematics of Multisets*, Multiset Processing (2001) 347–358.
[61] Z. Pawlak, *Rough Sets*, International Journal of Computer Science 11(1982) 341-356.

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Some Variations of Janowski Functions Associated with Srivastava-Attiya Operator

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Abstract − In this paper, we consider some new subclasses of analytic functions with bounded boundary and bounded radius rotation associated with Attia-Srivastava operator. The coefficient bounds, integral representations, convolution properties belong to theses classes are investigated.

Keywords − *Srivastava- Attia operator, Janowski functions, subordination, convolution, starlike convex functions*

1. Introduction

Let A be the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk U, where

$$U = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For two functions $F(z)$ and $G(z)$ analytic in U , we say that $F(z)$ is subordinate to $G(z)$, denoted by

$$F \prec G \quad \text{or} \quad F(z) \prec G(z),$$

if there exists an analytic function $w(z)$ with

$$|w(z)| \leq |z| \quad \text{such that} \quad F(z) = G(w(z)).$$

Furthermore if the function $G(z)$ is univalent in U then we have the following equivalence [1–3]

$$F(z) \prec G(z) \iff F(0) = G(0) \text{ and } F(U) \subset G(U).$$

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For two analytic functions $f(z)$ given by (1) and $g(z)$

$$g(z) = z + \sum_{n=2}^{\infty} e_n z^n, \quad (z \in U),$$

their Convolution or Hadamard product is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n e_n z^n, \quad (z \in U).$$

For arbitrary fixed numbers A, B, α and β satisfying $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $P_\beta [A, B, \alpha]$ denote the family of functions

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots,$$

regular in U and such that $h(z)$ is in $P_\beta [A, B, \alpha]$ if and only if

$$h(z) \prec (1 - \alpha) \left(\frac{1 + Az}{1 + Bz} \right)^\beta + \alpha. \tag{2}$$

Therefore, $h(z)$ is in $P_\beta [A, B, \alpha]$ if and only if

$$h(z) = \frac{(1 - \alpha) (1 + Aw(z))^\beta + \alpha (1 + Bw(z))^\beta}{(1 + Bw(z))^\beta}, \tag{3}$$

for some $w(z)$ with $|w(z)| \leq |z|$. By taking $\beta = 1$, then the class $P_\beta [A, B, \alpha]$, reduces to $P [A, B, \alpha]$, defined by Polatoglu in [4], if we take $\alpha = 0$, $\beta = 1$, then the class $P_\beta [A, B, \alpha]$, reduces to the well known class $P [A, B]$, defined and studied by Janowski in [5] and setting $\alpha = 0$, $\beta = 1$, $A = 1$, $B = -1$, the class $P_\beta [A, B, \alpha]$, reduces to the class P of functions with positive real part. For more details see [6–15].

One can easily verify that $p \in P_\beta [A, B, \alpha]$, if and only if, there exists $g \in P [A, B]$, such that

$$p(z) = (1 - \alpha) g(z) + \alpha.$$

The Herglotz representation of the functions of the class $P_\beta [A, B, \alpha]$, is given by

$$h(z) = \alpha + \frac{1 - \alpha}{2} \int_0^{2\pi} \left(\frac{1 + Aze^{-i\theta}}{1 + Bze^{-i\theta}} \right)^\beta \mathrm{d}\mu(\theta),$$

where μ is non decreasing function in $[0, 2\pi]$ such that $\int_0^{2\pi} \mathrm{d}\mu(\theta) = 2$.

For $A = 1, B = -1$, the class $P_\beta [A, B, \alpha]$, reduces to the class $P_\beta (\alpha)$, presented by Dziok recently [16, Th.3] and further by setting $\alpha = 0, \beta = 1, A = 1, B = -1$, we obtain the class P of analytic functions with real part greater than zero.

Now we define the subclass $P_{m,\beta} [A, B, \alpha]$, of analytic functions as follows;

Definition 1.1. A function $p(z)$ analytic in U belongs to the class $P_{m,\beta} [A, B, \alpha]$, if and only if

$$p(z) = \alpha + \frac{1 - \alpha}{2} \int_0^{2\pi} \left(\frac{1 + Aze^{-i\theta}}{1 + Bze^{-i\theta}} \right)^\beta \mathrm{d}\mu(\theta), \tag{4}$$

where $\mu(\theta)$ is non decreasing function in $[0, 2\pi]$ with

$$\int_0^{2\pi} \mathrm{d}\mu(\theta) = 2 \quad \text{and} \quad \int_0^{2\pi} |\mathrm{d}\mu(\theta)| \leq m,$$

where, $m \geq 2$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

Now using Horglotz-Stieltjes formula for the functions in the class $P_{m,\beta} [A, B, \alpha]$, given in (4), we obtain

$$p(z) = \left(\frac{m}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2} \right) p_2(z),$$

where $p_1, p_2 \in P_\beta [A, B, \alpha]$ see ([16], Theorem 3).

For $\beta = 1$, the class $P_{m,\beta} [A, B, \alpha] = P_m [A, B, \alpha]$ [33] and for $\alpha = 0, \beta = 1, A = 1, B = -1, P_{m,1} [1, -1, 0] = P_m$ [17].

We consider the function

$$\phi (z; s, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^s}, \tag{5}$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $s \in \mathbb{C}$. The function $\phi (z; s, b)$ contain many well known functions as a special case such as Riemann and Hurwitz Zeta functions for more details, see [18, 19].

Using the technique of convolution and the function $\phi (z; s, b)$ Srivastava and Attiya given in [20]. In addition see also ([21, 22]) introduced and studied the linear operator

$$J_{s,b} f : A \to A,$$

defined, in terms of the Hadamard product (or convolution), by

$$J_{s,b}(f)(z) = \phi(z; s, b) * f(z), \qquad f \in A, \ (z \in U), \tag{6}$$

where $*$ denotes the convolution and

$$\psi (z; s, b) = (1 + b)^s \left(\phi(z; s, b) - b^{-s}\right) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s z^n, \quad (z \in U). \tag{7}$$

Therefore, using (6) and (7), we have

$$J_{s,b}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s a_n z^n, \quad (z \in U).$$

For special values of b and s the operator contain many known operators, see [23, 24].

With the help of the class $P_{m,\beta} [A, B, \alpha]$, along with generalized Srivastava and Attiya operator given in [20] , we now define the following subclass of analytic functions;

Definition 1.2. A function $f \in A$, is in the class $R^{s,b}_{m,\beta} [A, B, \alpha]$, if and only if

$$\frac{z\left(J_{s,b} f(z)\right)'}{J_{s,b} f(z)} \in P_{m,\beta} [A, B, \alpha], \quad (z \in U).$$

Definition 1.3. A function $f \in A$, is in the class $V^{s,b}_{m,\beta} [A, B, \alpha]$, if and only if

$$1 + \frac{z\left(J_{s,b} f(z)\right)''}{\left(J_{s,b} f(z)\right)'} \in P_{m,\beta} [A, B, \alpha], \quad (z \in U).$$

where $m \geq 2$, $b \in \mathbb{C} \setminus \left(\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}\right)$, $s \in \mathbb{C}$, $-1 \leq B < A \leq 1, 0 \leq \alpha < 1$, $0 < \beta \leq 1$. We also note that

$$f(z) \in V^{s,b}_m [A, B, \alpha, \beta] \Leftrightarrow z f(z)' \in R^{s,b}_{m,\beta} [A, B, \alpha]. \tag{8}$$

Remarks:

(i) $R^{0,b}_{m,1} [A, B, 0] = R_m [A, B], V^{0,b}_{m,1} [A, B, 0] = V_m [A, B]$, the well known classes presented and studied in [25] and [26].

(ii) $R^{0,b}_{m,1} [1, -1, 0] = R_m, V^{0,b}_{m,1} [A, B, 0] = V_m$, we have the well known class introduced and studied in [17] and [27].

(iii) $R^{0,b}_{m,1} [2\zeta - 1, -1, 0], V^{0,b}_{m,1} [2\zeta - 1, -1, 0]$, were presented and studied in [28].

To avoid repetition, it is admitted once that $m \geq 2$, $b \in \mathbb{C} \setminus \left(\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}\right)$, $s \in \mathbb{C}$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

2. Preliminary Lemma

We need the following Lemma which will be used in our main results.

Lemma 2.1. [29] Let $f(z)$ be subordinate to $g(z)$, with

$$f(z)=1+\sum_{n=1}^{\infty}a_nz^n,\quad g(z)=1+\sum_{n=1}^{\infty}b_nz^n.$$

If $g(z)$ is univalent in U and $g(U)$ is convex, then $|a_n|\leq|b_1|$.

Lemma 2.2. Let $p(z)\in P_{m,\beta}\left[A,B,\alpha\right]$, be of the form (1). Then

$$|q_n|\leq\beta\left(A-B\right)|1-\alpha|\,.$$

The proof is immediate by using Lemma 2.1.

Lemma 2.3. Let $p(z)\in P_{m,\beta}\left[A,B,\alpha\right]$, be of the form (1). Then

$$|q_n|\leq\frac{m}{2}\beta\left(A-B\right)|1-\alpha|\,.$$

The proof is immediate by using Lemma 2.2.

Lemma 2.4. Let $p(z)\in P_{m,\beta}\left[A,B,\alpha\right]$, be of the form (1). Then

$$\begin{aligned}&\frac{(1-\alpha)}{4}\left[(m+2)\left(\frac{1-Ar}{1-Br}\right)^{\beta}-(m-2)\left(\frac{1+Ar}{1+Br}\right)^{\beta}\right]+\alpha\\ \leq\ &\mathfrak{Re}p(z)\leq|p(z)|\leq\frac{(1-\alpha)}{4}\left[(m+2)\left(\frac{1+Ar}{1+Br}\right)^{\beta}-(m-2)\left(\frac{1-Ar}{1-Br}\right)^{\beta}\right]+\alpha.\end{aligned}$$

This results is sharp.
The proof is immediate by using Lemma 2.3.

Lemma 2.5. [30] Let ψ be convex and let g be starlike in U. Then for F analytic in U with $F(0)=1$, $\frac{\psi*Fg}{\psi*g}$ is contained in the convex hull of $F(U)$.

3. Main Results

Theorem 3.1. Let $p(z)\in P_{m,\beta}\left[A,B,\alpha\right]$, with $m\geq2$. Then, for $|z|=r<1$,

$$\left|zp'(z)\right|\leq\frac{(A-B)\,\beta r\left[(m+2)\,\frac{(1+Ar)^{\beta-1}}{(1+Br)^{\beta+1}}+(m-2)\,\frac{(1-Ar)^{\beta-1}}{(1-Br)^{\beta+1}}\right]\mathfrak{Re}p(z)}{\left[(m+2)\left(\frac{1+Ar}{1+Br}\right)^{\beta}-(m-2)\left(\frac{1-Ar}{1-Br}\right)^{\beta}\right]+\frac{4\alpha}{1-\alpha}}.$$

Proof. The proof is immediate by using Lemma 2.4. □

Putting $\alpha=0,\beta=1$ in Theorem 3.1, we can obtain Corollary 3.2, below which is comparable to the result obtained by Noor and Malik [31].

Corollary 3.2. Let $p(z)\in P_{m,\beta}\left[A,B,\alpha\right]$, with $m\geq2$. Then, for $|z|=r<1$,

$$\left|zp'(z)\right|\leq\frac{(A-B)\,r\left\{m-4Br+mB^2r^2\right\}\mathfrak{Re}p(z)}{(1-Br^2)\,(2+mr\,(A-B)-2ABr^2)}.$$

Theorem 3.3. Let $f(z)\in R_{m,\beta}^{s,b}\left[A,B,\alpha\right]$. Then

$$|a_n|\leq\frac{(b+n)^s\left(\frac{m}{2}\beta\left(A-B\right)|1-\alpha|\right)_{n-1}}{(b+1)^s\,(n-1)!}. \tag{9}$$

This result is sharp.

Proof. Let

$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} = p(z), \qquad (z \in U), \tag{10}$$

where $p(z) \in P_{m,\beta}[A,B,\alpha]$ and $p(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$.

Then from the definition we have

$$J_{s,b}f(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{11}$$

where

$$b_n = \left(\frac{b+1}{b+n}\right)^s a_n. \tag{12}$$

From (10) and (11), we have

$$\begin{aligned} z + \sum_{n=2}^{\infty} n b_n z^n &= \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=1}^{\infty} q_n z^n\right) \\ &= \left(\sum_{n=1}^{\infty} b_n z^n\right)\left(1 + \sum_{n=1}^{\infty} q_n z^n\right), \quad b_1 = 1 \\ &= \sum_{n=1}^{\infty} b_n z^n + \left(\sum_{n=1}^{\infty} b_n z^n\right)\left(\sum_{n=1}^{\infty} q_n z^n\right). \end{aligned}$$

By using the Cauchy's product formula [32], for the power series we have

$$z + \sum_{n=2}^{\infty} n b_n z^n = \sum_{n=1}^{\infty} b_n z^n + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n-1} b_j q_{n-j}\right) z^n.$$

Equating the coefficient of z^n, we have

$$n b_n = b_n + \sum_{j=1}^{n-1} b_j q_{n-j}.$$

By using induction on n, and Lemma 2.3, we obtain

$$b_n = \frac{\left(\frac{m}{2}\beta\left(A-B\right)|1-\alpha|\right)_{n-1}}{(n-1)!}.$$

Using the value of b_n, we obtain (9).

Sharpness is given for the functions $f_1 \in A$ such that

$$\begin{aligned} \frac{z\left(J_{s,b}f_1(z)\right)'}{J_{s,b}f_1(z)} &= \left(\frac{m}{2}+\frac{1}{2}\right)\left((1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\beta} + \alpha\right) \\ &\quad - \left(\frac{m}{2}-\frac{1}{2}\right)\left((1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\beta} + \alpha\right). \end{aligned}$$

This complete the proof of Theorem 3.3. □

Putting $s = 0, \beta = 1$ in Theorem 3.3, we can obtained the following Corollary.

Corollary 3.4. Let $f(z) \in R_{m,1}^{0,b}[A,B,\alpha]$. Then

$$|a_n| \le \frac{\left(\frac{m}{2}\left(A-B\right)|1-\alpha|\right)_{n-1}}{(n-1)!}.$$

This result is sharp.

Putting $s=0,\beta=1,A=1,B=-1$ in Theorem 3.3, we can obtain Corollary 3.5, below which is comparable to the result obtained by Noor [33].

Corollary 3.5. Let $f(z)\in R_{m,1}^{0,b}[1,-1,\alpha]=R_m(\alpha)$. Then

$$|a_n|\le\frac{(m|1-\alpha|)_{n-1}}{(n-1)!},\quad \text{for all } n\ge 2.$$

This result is sharp.

Theorem 3.6. Let $f(z)\in V_{m,\beta}^{s,b}[A,B,\alpha]$. Then

$$|a_n|\le\frac{(b+n)^s\left(\frac{m}{2}\beta(A-B)|1-\alpha|\right)_{n-1}}{(b+1)^s\,n!}. \tag{13}$$

This result is sharp.

Proof. The proof of Theorem 3.6 is similar to that of Theorem 3.3, so the details are omitted. □

For an analytic functions $f(z)$, we consider the operator

$$F(z)=I_c(f(z))=\frac{1+c}{z^c}\int_0^z t^{c-1}f(t)\,\mathrm{d}t,\quad c>-1. \tag{14}$$

The operator I_c, when $c\in\mathbb{N}$, was introduced by Bernardi [24]. The operator I_1, was studied by Libera [34] and Livingston [35].

Theorem 3.7. If $f(z)$ is of the form of (1) , belongs to $R_{m,\beta}^{s,a}[A,B,\alpha]$ and $F(z)=z+\sum_{n=2}^{\infty}d_nz^n$, where $F(z)$, is an integral operator given by (14). Then

$$|d_n|\le\frac{(1+c)(b+n)^s\left(\frac{m}{2}\beta(A-B)|1-\alpha|\right)_{n-1}}{(n+c)(b+1)^s(n-1)!}.$$

Proof. From (14), we can easily write

$$(1+c)f(z)=cF(z)+zF'(z),$$

or equivalently,

$$(1+c)z+\sum_{n=2}^{\infty}(1+c)a_nz^n=cz+\sum_{n=2}^{\infty}cd_nz^n+z+\sum_{n=2}^{\infty}nd_nz^n.$$

Thus we have,

$$(n+c)d_n=(1+c)a_n,$$

using the estimate from Theorem 3.3, we have

$$|d_n|\le\frac{(1+c)(b+n)^s\left(\frac{m}{2}\beta(A-B)|1-\alpha|\right)_{n-1}}{(n+c)(b+1)^s(n-1)!}.$$

we obtain the required result. □

Putting $s=0,\beta=1,$ in Theorem 3.7, we can obtained the following Corollary.

Corollary 3.8. If $f(z)$ is of the form of (1), belongs to $R_{m,1}^{0,a}[1,-1,\alpha]$ and $F(z)=z+\sum_{n=2}^{\infty}d_nz^n$, where $F(z)$, is an integral operator given by (14), then

$$|d_n|\le\frac{(1+c)\left(\frac{m}{2}(A-B)|1-\alpha|\right)_{n-1}}{(n+c)\,n!}.$$

Putting $s=0,\beta=1,A=1,B=-1,$ in Theorem 3.7, we can obtained the following Corollary.

Corollary 3.9. If $f(z)$ is of the form of (1), belongs to $R^{s,a}_{m,\beta}[A,B,\alpha]$ and $F(z)=z+\sum\limits_{n=2}^{\infty}d_nz^n$, where $F(z)$, is an integral operator given by (14), then

$$|d_n|\le\frac{(1+c)\,(m\,|1-\alpha|)_{n-1}}{(n+c)\,n!}.$$

Theorem 3.10. If $f(z)$ is of the form of (1) , belongs to $R^{s,a}_{m,\beta}[A,B,\alpha]$ and $F(z)=z+\sum\limits_{n=2}^{\infty}d_nz^n$, where $F(z)$, is an integral operator given by (14), then

$$|d_n|\le\frac{(1+c)\,(b+n)^s\left(\frac{m}{2}\beta\,(A-B)\,|1-\alpha|\right)_{n-1}}{(n+c)\,(b+1)^s\,n!}.$$

Proof. The proof of Theorem 3.10 is similar to that of Theorem 3.7 so the details are omitted. □

Theorem 3.11. If $f(z)$ is of the form of (1) , belongs to $R^{s,b}_{2,\beta}[A,B,\alpha]$ if and only if

$$\frac{1}{z}\left\{f*\left\{\begin{array}{c}\left(z+\sum\limits_{n=2}^{\infty}nb_nz^n\right)\left(1+B(e^{i\theta})\right)^{\beta}-\left(z+\sum\limits_{n=2}^{\infty}b_nz^n\right)\\ \times\left((1-\alpha)\left(1+A(e^{i\theta})\right)^{\beta}-\alpha\left(1+B(e^{i\theta})\right)^{\beta}\right)\end{array}\right\}\right\}\neq0,\tag{15}$$

where b_n is given by (12) and $0\le\theta<2\pi$.

Proof. Assume that $f(z)\in R^{s,b}_{2,\beta}[A,B,\alpha]$, then, we have

$$\frac{z\,(J_{s,b}f(z))'}{J_{s,b}f(z)}\prec(1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\beta}+\alpha,$$

if and only if

$$\frac{z\,(J_{s,b}f(z))'}{J_{s,b}f(z)}\neq(1-\alpha)\left(\frac{1+A(e^{i\theta})}{1+B(e^{i\theta})}\right)^{\beta}+\alpha,\tag{16}$$

for all $z\in U$, and $0\le\theta<2\pi$. The condition (16) can be written as

$$z\,(J_{s,b}f(z))'\left(1+B(e^{i\theta})\right)^{\beta}-J_{s,b}f(z)\left((1-\alpha)\left(1+A(e^{i\theta})\right)^{\beta}-\alpha\left(1+B(e^{i\theta})\right)^{\beta}\right)\neq0.\tag{17}$$

On the other hand we know that

$$z\,(J_{s,b}f(z))'=z+\sum_{n=2}^{\infty}nb_nz^n.\tag{18}$$

Combining (5), (6), (18) and (17) we get the convolution property (15) asserted by Theorem 3.11. □

Putting $s=0,\alpha=0,m=2$ and $\beta=1$ in Theorem 3.11, we can obtain Corollary 3.12, below which is comparable to the result obtained by Silverman and Silvia [36].

Corollary 3.12. A function f defined by (1) is in the class $S[A,B]$, if and only if

$$\frac{1}{z}\left\{f(z)*\frac{z-Lz^2}{(1-z)^2}\right\}\neq0,\quad(z\in U),\tag{19}$$

for all $L=L_\theta=\frac{e^{-i\theta}+A}{A-B}$ and also $L=1$.

Putting $s=0,\alpha=0,m=2,\beta=1,A=1-2\sigma$ and $B=-1$ in Theorem 3.11, we can obtain Corollary 3.13, below which is comparable to the result obtained by Silverman and Silvia [37].

Corollary 3.13. A function f defined by (1) is in the class $S^*(\alpha)$, if and only if

$$\frac{1}{z}\left\{f(z)*\frac{z-Mz^2}{(1-z)^2}\right\}\neq 0, \quad (z\in U), \tag{20}$$

for all $M=M_\theta=\frac{e^{-i\theta}+1-2\sigma}{2(1-\sigma)}$, $(0\leq\sigma<1)$ and also $M=1$.

Theorem 3.14. A function $f(z)\in V^{s,b}_{2,\beta}[A,B,\alpha]$ if and only if

$$\frac{1}{z}\left\{f*\left\{\begin{array}{c}\left(1+\sum\limits_{n=2}^{\infty}n^2b_nz^{n-1}\right)\left(1+B\left(e^{i\theta}\right)\right)^{\beta}-\left(1+\sum\limits_{n=2}^{\infty}nb_nz^{n-1}\right)\\ \times\left((1-\alpha)\left(1+A(e^{i\theta})\right)^{\beta}-\alpha\left(1+B(e^{i\theta})\right)^{\beta}\right)\end{array}\right\}\right\}\neq 0. \tag{21}$$

for all $b_n=\left(\frac{1+b}{n+b}\right)^s a_n$ and $0\leq\theta<2\pi$.

Proof. The proof of Theorem 3.14 is similar to that of Theorem 3.11 so the details are omitted. □

Theorem 3.15. Let $f(z)\in R^{s,b}_{2,\beta}[A,B,\alpha]$. Then

$$f(z)=\left(z\cdot\exp\left((1-\alpha)\int_0^z\frac{\left((1+Aw(t))^{\beta}-(1+Bw(t))^{\beta}\right)}{t\,(1+Bw(t))^{\beta}}\mathrm{d}t\right)\right)*\left(\sum_{n=0}^{\infty}(b+n)^s z^n\right), \tag{22}$$

where $\omega(z)$ is analytic in U, with $\omega(0)=0$ and $|\omega(z)|<1$.

Proof. For $f(z)\in R^{s,b}_{2,\beta}[A,B,\alpha]$, then from definition of subordination we can have

$$\frac{z\,(J_{s,b}f(z))'}{J_{s,b}f(z)}=(1-\alpha)\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\beta}+\alpha, \tag{23}$$

where $w(z)$ analytic in U, with $w(0)=0$ and $|w(z)|<1$.

$$\frac{(J_{s,b}f(z))'}{J_{s,b}f(z)}-\frac{1}{z}=\frac{(1-\alpha)\left((1+Aw(z))^{\beta}-(1+Bw(z))^{\beta}\right)}{z\,(1+Bw(z))^{\beta}}, \tag{24}$$

which, upon integration, yield

$$\log\frac{J_{s,b}f(z)}{z}=(1-\alpha)\int_0^z\frac{\left((1+Aw(t))^{\beta}-(1+Bw(t))^{\beta}\right)}{t\,(1+Bw(t))^{\beta}}\mathrm{d}t. \tag{25}$$

From (5) and (6), we obtain

$$f(z)*\left(\sum_{n=0}^{\infty}\frac{z^n}{(b+n)^s}\right)=z\cdot\exp\left((1-\alpha)\int_0^z\frac{\left((1+Aw(t))^{\beta}-(1+Bw(t))^{\beta}\right)}{t\,(1+Bw(t))^{\beta}}\mathrm{d}t\right), \tag{26}$$

and our assertion follows immediately. □

Putting $\alpha=0,\beta=1$ and $m=2$ in Theorem 3.15, we can obtain the following Corollary

Corollary 3.16. Let $f(z)\in R^{s,b}_{2,1}[A,B,0]$. Then

$$f(z)=\left(z\cdot\exp\left((A-B)\int_0^z\frac{w(t)}{t\,(1+Bw(t))}\mathrm{d}t\right)\right)*\left(\sum_{n=0}^{\infty}z^n\right),$$

where $\omega(z)$ is analytic in U, $\omega(0)=0$ and $|\omega(z)|<1$.

Putting $s = 0, \alpha = 0, \beta = 1, A = 1$ and $B = -1$ in Theorem 3.15, we can obtain the following Corollary.

Corollary 3.17. Let $f(z) \in R_{2,1}^{s,b}[1,-1,0]$. Then

$$f(z) = z \cdot \exp\left(2\int_0^z \frac{w(t)}{t(1-w(t))}\mathrm{d}t\right)\left(\sum_{n=0}^{\infty} z^n\right),$$

where $\omega(z)$ is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$.

Putting $\alpha = 0, \beta = 1, A = 1, B = -1$ and $m = 2$ in Theorem 3.15, we can obtain the following Corollary.

Corollary 3.18. Let $f(z) \in R_{2,1}^{s,b}[1,-1,0]$. Then

$$f(z) = z \cdot \exp\left(2\int_0^z \frac{w(t)}{t(1-w(t))}\mathrm{d}t\right)\left(\sum_{n=0}^{\infty} z^n\right),$$

where $\omega(z)$ is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$.

Theorem 3.19. Let $f(z) \in V_{2,\beta}^{s,b}[A, B, \alpha]$. Then

$$\begin{aligned} f(z) &= \left(\int_0^z \exp\left((1-\alpha)\int_0^\zeta \frac{\left((1+Aw(t))^\beta - (1+Bw(t))^\beta\right)}{t(1+Bw(t))^\beta}\mathrm{d}t\right)d\zeta\right) \\ &\quad * \left(\sum_{n=0}^{\infty}(n+b)^s a_n z^n\right), \end{aligned}$$

where $\omega(z)$ is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof. The proof of Theorem 3.19 is similar to that of Theorem 3.15 so the details are omitted. □

Theorem 3.20. Let $\psi \in C$ and $f(z) \in R_{2,\beta}^{s,b}[A, B, \alpha]$. Then $\psi * f \in R_{2,\beta}^{s,b}[A, B, \alpha]$.

Proof. Let $F(z) = \psi * f$. Then by using some properties of convolution we have

$$\begin{aligned} \frac{z(J_{s,b}F(z))'}{J_{s,b}F(z)} &= \frac{\psi * z(J_{s,b}f(z))'}{\psi * J_{s,b}f(z)} \\ &= \frac{\psi * \frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} J_{s,b}f(z)}{\psi * J_{s,b}f(z)} \\ &= \frac{\psi * p(z)J_{s,b}f(z)}{\psi * J_{s,b}f(z)}, \end{aligned}$$

where $p(z) = \frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}$. Since $f(z) \in R_{2,\beta}^{s,b}[A, B, \alpha]$, therefore $\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} \in P_\beta[A, B, \alpha] \subset P[A, B, \alpha] \subset P$ [38] and hence.$J_{s,b}f(z) \in S^*$. Then by Lemma 2.5, $F(z)$, lies in the convex hull of $p(z)$ and consequently, $F \in R_{2,\beta}^{s,b}[A, B, \alpha]$. □

Theorem 3.21. Let $f \in V_{m,\beta}^{s,b}[A, B, \alpha]$ and $h \in R_{m,\beta}^{s,b}[A, B, \alpha]$. Let $H(z)$ be defined as

$$J_{s,b}H(z) = \int_0^z \left[(J_{s,b}f(t))'\right]^{\lambda_1}\left[\frac{J_{s,b}h(t)}{z}\right]^{\lambda_2}\mathrm{d}t, \tag{27}$$

where λ_1 and λ_2 are positive real numbers with $\lambda_1 + \lambda_2 = 1$. Then $H \in V_{m,\beta}^{s,b}[A, B, \alpha]$.

Proof. Suppose $f(z) \in V^{s,b}_{m,\beta}[A,B,\alpha]$, and $h(z) \in R^{s,b}_{m,\beta}[A,B,\alpha]$.

From (27), we have

$$J_{s,b}H(z) = \left[(J_{s,b}f(z))'\right]^{\lambda_1} \left[\frac{J_{s,b}h(z)}{z}\right]^{\lambda_2}. \tag{28}$$

Logarithmic differentiation implies that

$$\frac{z\,(J_{s,b}H(z))'}{J_{s,b}H(z)} = \frac{\left(z\,(J_{s,b}f)'\right)'}{(J_{s,b}f)'} + \frac{z\,(J_{s,b}h)'}{J_{s,b}h} \tag{29}$$

$$= \lambda_1 p_1(z) + \lambda_2 p_2(z), \tag{30}$$

for all $p_1, p_2 \in P_{m,\beta}[A,B,\alpha]$. Using the fact that the class $P_{m,\beta}[A,B,\alpha]$, is convex set. Therefore $\lambda_1 p_1(z) + \lambda_2 p_2(z) \in P_{m,\beta}[A,B,\alpha]$. Hence

$$\frac{z\,(J_{s,b}H(z))'}{J_{s,b}H(z)} \in P_{m,\beta}[A,B,\alpha],$$

and consequently $H \in V^{s,b}_{m,\beta}[A,B,\alpha]$. □

References

[1] A. Aral, V. Gupta, *On q-Baskakov Type Operators*, Demonstratio Mathematica 1(42) (2009) 109–122.

[2] T. Bulboaca, *Differential Subordinations and Superordinations*, Recent Results, House of Scientic Book Publ. Cluj-Napoca, 2005.

[3] S. S. Miller, P. T. Mocanu, *Subordinates of Differential Superordinations*, Complex Variables 48(10) (2003) 815–826.

[4] Y. Polatoglu, M. Bolcal, A. Sen, E. Yavuz, A *Study on The Generalization of Janowski Functions in The Unit Disc*, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 22 (2006) 27–31.

[5] W. Janowski, *Some Extremal Problems for Certain Families of Analytic Functions*, Annales Polonici Mathematici 28 (1973) 297–326.

[6] N. Khan, B. Khan, Q. Z. Ahmad, S. Ahmad, *Some Convolution Properties of Multivalent Analytic Functions*, AIMS Mathematics 2(2) (2017) 260–268.

[7] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, I. Ali, *Upper Bound of The Third Hankel Determinant for a Subclass of q-Starlike Functions*, Symmetry 11 (2019) Article ID 347 1–13.

[8] S. Mahmood, Q. Z. Ahmad, H. M. Srivastava, N. Khan, B. Khan, M. Tahir, *A Certain Subclass of Meromorphically q-Starlike Functions Associated with The Janowski Functions*, Journal of Inequalities and Applications 2019 (2019) Article ID 88 1–11.

[9] K. I. Noor, N. Khan, M. Darus, Q. Z. Ahmad, B. Khan, *Some Properties of Analytic Functions Associated with Conic Type Regions*, International Journal of Analysis and Applications 16(5) (2018) 689–701.

[10] K. I. Noor, N. Khan, K. Piejko, *Alpha Convex Functions Associated with Conic Domain*, International Journal of Analysis and Applications 11(2) (2016) 70–80.

[11] K. I. Noor, N. Khan, *Some Variations of Janowski Functions Associated with m-Symmetric Points*, Journal of New Theory 11 (2016) 16–28.

[12] K. I. Noor, N. Khan, Q. Z. Ahmad, N. Khan, Y. L. Chung, *On Certain Subclass of Analytic Functions*, Armenian Journal of Mathematics 10(11) (2018) 1–15.

[13] H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan, B. Khan, *Hankel and Toeplitz Determinants for A Subclass of q-Starlike Functions Associated with A General Conic Domain*, Mathematics 7 (2019) Article ID 181 1–15.

[14] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, *Some General Classes of q-Starlike Functions Associated with The Janowski Functions*, Symmetry 11 (2019) Article ID 292 1–14.

[15] M. Sabil, Q. Z. Ahmad, B. Khan, M. Tahir, N. Khan, *Generalisation of Certain Subclasses of Analytic and bi-univalent Functions*, Maejo International Journal of Science and Technology 13(01) (2019) 1–9.

[16] J. Dziok, *Meromorphic Functions with Bounded Boundary Rotation*, Acta Mathematica Scientia 34(2) (2014) 466–472.

[17] B. Pinchuk, *Functions of Bounded Boundary Rotation*, Israel Journal of Mathematics 10 (1971) 6–16.

[18] H. M. Srivastava, J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, The Netherlands 2001.

[19] H. M. Srivastava, A. A. Attiya, *An Integral Operator Associated with The Hurwitz-Lerch Zeta Function and Differential Subordination*, Integral Transforms and Special Functions 18 (2007) 207–216.

[20] J. L. Liu, *Subordinations for Certain Multivalent Analytic Functions Associated with The Generalized Srivastava-Attiya Operator*, Integral Transforms and Special Functions 19 (2008) 893–901.

[21] S. D. Lin, H. M. Srivastava, *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Applied Mathematics and Computation 154 (2004) 725–733.

[22] D. Raducanu, H. M Srivastava, *A New Class of Analytic Functions Defined by means of A Convolution Operator Involving The Hurwitz-Lerch Zeta Function*, Integral Transforms and Special Functions 18 (2007) 933–943.

[23] J. W. Alexander, *Functions Which Map The Interior of The Unit Circle upon Simple Regions*, Annals of Mathematics Second Series 17(1) (1915) 12–22.

[24] S. D. Bernardi, *Convex and Starlike Univalent Functions*, Transactions of the American Mathematical Society 135 (1969) 429–446.

[25] K. I. Noor, K. Yousaf, *On Classes of Analytic Functions Related with Generalized Janowski Functions*, World Applied Sciences Journal 13 (2011) 40–47.

[26] K. I. Noor, M. Arif, *Mapping Properties of An Integral Operator*, Applied Mathematics Letters 25 (2012) 1826–1829.

[27] V. Paatero, *Uber Die Konforme Abbildung Von Gebieten, Deren Ränder Vonbeschränkter Drehung Sind*, Annales Academiae Scientiarum Fennicae: Series A. 33(9) (1931) page 77.

[28] K. S. Padmanabhan, R. Parvatham, *Properties of A Class of Functions with Bounded Boundary Rotation*, Annales Polonici Mathematici 31 (1975) 311–323.

[29] W. Rogosinski, *On The Coefficients of Subordinate Functions*, Proceedings of the London Mathematical Society 48(2) (1943) 48–82.

[30] S. Ruscheweyh, T. Shiel-small, *Hadamard Product of Schlicht Functions and Polya-Schoenberg Conjecture*, Commentarii Mathematici Helvetici 48 (1973) 119–135.

[31] K. I. Noor, S. N. Malik, M. Arif, M. Raza, *On Bounded Boundary and Bounded Radius Rotation Related with Janowski Function*, World Applied Sciences Journal 12 (6) (2011) 895–902.

[32] A. W. Goodman, *Univalent Functions*, Vol. I & II, polygonal Publishing House, Washington, New Jersey, 1983.

[33] K. I. Noor, *Higher Order Close-to-Convex Functions*, Math. Japonica 37(1) (1992) 1–8.

[34] R. J. Libera, *Some Classes of Regular Univalent Functions*, Proceedings of the American Mathematical Society 16 (1965) 755–758.

[35] A. E. Livingston, *On The Radius Of Univalence Of Certain Analytic Functions*, Proceedings of the American Mathematical Society 17 (1996) 352–357.

[36] H. Silverman, E. M. Silvia, *Subclasses of Starlike Functions Subordinate to Convex Functions*, Canadian Journal of Mathematics 1 (1985) 48–61.

[37] H. Silverman, E. M. Silvia, D. Telage, *Convolution Conditions for Convexity, Starlikeness and Spiral-Likeness*, Mathematische Zeitschrift 162 (1978) 125–130.

[38] S. Hussain, M. Arif, S. N. Malik, *Higher Order Close-to-Convex Functions Associated with Attiya-Srivastava Operator*, Bulletin of the Iranian Mathematical Society 40(4) (2014) 911–920.

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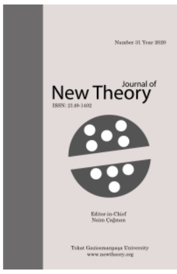

TOPSIS for Multi Criteria Decision Making in Octagonal Intuitionistic Fuzzy Environment by Using Accuracy Function

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Abstract − Multi Criteria Decision Making (MCDM) enables a strong valid platform in domains where choosing the best of the best among various attributes is quite complicated. This paper provides a suitable methodology for solving MCDM problems in Intuitionistic Fuzzy region. In this paper we shall be dealing with the environment of octagonal intuitionistic fuzzy numbers. These numbers are more suitable to deal with uncertainties than other generalized form of fuzzy numbers. There are ways to solve MCDM in IF environment. Many have used ∝-cuts of numbers which are complicated calculations usually ending up with deviation from the results. Despite of solving the problem using ∝-cuts, we propose a new ranking technique in the procedure. This ranking technique is called an accuracy function for octagonal intuitionistic fuzzy numbers. Octagonal Intuitionistic fuzzy numbers are introduced along with its membership and non-membership values. For application, a numerical example is solved at the end of this paper.

Keywords *− Fuzzy Numbers (FN's), Intuitionistic Fuzzy Numbers (IFN's), Octagonal Intuitionistic Fuzzy Number (OIFN's), Accuracy Function (AF), TOPSIS*

1. Introduction

Multi Criteria Decision Making is based upon formation and designing decision and outlining problems composed of complex multi pattern. The whole purpose is to give decision makers a feasible solution to such problems. Predictably, there does not exist an exclusive optimal answer for such matter and it is mandatory to utilize the choice maker's performance to evaluate and characterize between solutions. MCDM is a dynamic region of research since the 1960's. Different approach has been proposed by distinct scholars to solve the MCDM problems.

The TOPSIS (Technique for order of preference by Similarity to Ideal Solution) is a Multi- Criteria Decision analysis method proposed by [1] which was further extended by [2]. TOPSIS is set upon the concept that the selected alternative should have the minimum distance from Positive Ideal Solution (PIS) and maximum distance from Negative Ideal Solution (NIS).

Fuzzy set theory was proposed by [3] to represent non exact information into a better form. Later, [4-5] gave the idea of Intuitionistic fuzzy set (IFS) as more compact and precise form of fuzzy set. Different types of fuzzy numbers and various actions on them were researched by many researchers. They investigated on various properties and fluctuations of intuitionistic fuzzy numbers and the first property of correlation between these numbers. Intuitionistic fuzzy sets are already proven to be commodious deal with vagueness

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and perplexity. Both the degree of membership and non-membership functions in IFS combined by the sum is less than one. Many researchers used fuzzy numbers in decision making by considering new parameters and present their précised application in MCGDM, consisting of medical and smart phone selections [6-7]. Prediction of games is very curious topic and fuzzy numbers can be used to predict sports is proposed by [8-9]. Soft sets are considered more precise in vague and hesitate environment. Many researchers discussed the applications, considering MCDM problems but in recent, using accuracy function in uncertain and vague environment a generalized TOPSIS is proposed [10]. But still there are some problems which are solved by fuzzy numbers due to their graphical representations. Ranking of optimal solution using octagonal numbers, is also proposed by [11-12]. Fuzzy numbers are used in the problems having fluctuations. Triangular, Trapezoidal, pentagonal numbers are used in uncertain environment to deal with the fluctuations [13-18]. Development of fuzzy to intuitionistic and into neutrosophic and then further divisions of numbers are done by [19-23]. Ye. Worked in intuitionistic environment and developed a new theory to tackle the problems having uncertain environment [24-26]. Nowadays researchers are also focused on the development of new theories to solve MCDM problems. Recently many researches are done in the field of fuzzy numbers but still there was a gap of octagonal numbers [27-31].

Here in this paper, we shall be working on octagonal intuitionistic fuzzy numbers and its Accuracy Function to solve the TOPSIS. Initially the rating of choice is represented as octagonal intuitionistic fuzzy numbers. The Accuracy Function is developed for the decision making applied to TOPSIS method with octagonal intuitionistic fuzzy numbers.

2. Preliminaries

Definition 2.1: Fuzzy Number [FN]

A fuzzy number is generalized form of a real number. It doesn't represent a single value, instead a group of values, where each entity has its membership value between $[0,1]$. Fuzzy number $\bar{S}$ is a fuzzy set in R if it satisfies the given conditions.

- $\exists$ relatively one $y \in R$ with $\mu_{\bar{S}}(y) = 1$.
- $\mu_{\bar{S}}(y)$ is piecewise continuous.
- $\bar{S}$ should be convex and normal.

Definition 2.2: Triangular Fuzzy Number [TFN]

A Triangle fuzzy number $\bar{S}$ is denoted by tuples, $\bar{S}(x) = (\omega_1, \omega_2, \omega_3)$, where ω_1 , ω_2, and ω_3 are real numbers and $\omega_1 \leq \omega_2 \leq \omega_3$ with membership function defined as,

$$\mu_{\bar{S}}(x) = \begin{cases} \dfrac{x-\omega_1}{\omega_2-\omega_1}, & \omega_1 \leq x \leq \omega_2 \\ \dfrac{\omega_3-x}{\omega_3-\omega_2}, & \omega_2 \leq x \leq \omega_3 \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.3: Octagonal Fuzzy Number [OFN]

A fuzzy number S is an octagonal fuzzy number denoted by $\bar{S} = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8)$ where ω_1 , ω_2 , ω_3 , ω_4 , ω_5, ω_6, ω_7, and ω_8 are real numbers and $\omega_1 \leq \omega_2 \leq \omega_3 \leq \omega_4 \leq \omega_5 \leq \omega_6 \leq \omega_7 \leq \omega_8$ with membership function defined as,

$$\mu_{\bar{S}}(x)=\begin{cases} 0, & x<\omega_1 \\ k\left(\dfrac{x-\omega_1}{\omega_2-\omega_1}\right), & \omega_1\le x\le\omega_2 \\ k, & \omega_2\le x\le\omega_3 \\ k+(1-k)\left(\dfrac{x-\omega_3}{\omega_4-\omega_3}\right), & \omega_3\le x\le\omega_4 \\ 1, & \omega_4\le x\le\omega_5 \\ k+(1-k)\left(\dfrac{\omega_6-x}{\omega_6-\omega_5}\right), & \omega_5\le x\le\omega_6 \\ k, & \omega_6\le x\le\omega_7 \\ k\left(\dfrac{\omega_8-x}{\omega_8-\omega_7}\right), & \omega_7\le x\le\omega_8 \\ 0, & x>\omega_8 \end{cases}$$

with $0<k<1$.

3. Material and Method

Fuzzy numbers are very helpful in problem solving like MCDM and MCGDM. These numbers are proposed here along with accuracy function (AF). An Intuitionistic fuzzy set $\bar{S}^I(x)$ of S is defined as set of ordered triples as,

$$\bar{S}^I(x)=\left\{<x,\mu_{\bar{S}^I}(x),\vartheta_{\bar{S}^I}(x)>\mid x\in S\right\}$$

Where, $\mu_{\bar{S}^I}(x)$, $\vartheta_{\bar{S}^I}(x)$ are considered as MF's non- MF's such that $\mu_{\bar{S}^I}(x),\vartheta_{\bar{S}^I}(x): S\rightarrow[0,1]$, and $0\le\mu_{\bar{S}^I}(x)+\vartheta_{\bar{S}^I}(x)\le 1,\forall x\in S$.

For every IF set $\hat{A}$ in S, if $\pi_{\hat{A}}(S)=1-\mu_{\hat{A}}(S)-\nu_{\hat{A}}(S)$, then $\pi_{\hat{A}}(x)$ is called the indeterminacy degree [0,1], or hesitancy degree of S to $\hat{A}$.

4. Calculations

Definition 4.1: Octagonal Intuitionistic Fuzzy Number [OIFN]

A Fuzzy Number denoted by:

$\overline{S}^I=\{(\omega_{\mathbf{1}},\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8),(\omega'_{\mathbf{1}},\omega'_2,\omega'_3,\omega_4,\omega_5,\omega'_6,\omega'_7,\omega'_8)\}$ where $\omega_{\mathbf{1}}$, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7, ω_8, $\omega'_{\mathbf{1}}$, ω'_2, ω'_3, ω'_4, ω'_5 ω'_6, ω'_7, and ω'_8 are real numbers with $\omega_{\mathbf{1}}\le\omega_2\le\omega_3\le\omega_4\le\omega_5\le\omega_6\le\omega_7\le\omega_8$ and $\omega'_{\mathbf{1}}\le\omega'_2\le\omega'_3\le\omega'_4\le\omega'_5\le\omega'_6\le\omega'_7\le\omega'_8$. Its membership and non-membership functions are given by;

$$\mu_{\bar{S}}(x)=\begin{cases} 0, & x<\omega_1 \\ k\left(\dfrac{x-\omega_1}{\omega_2-\omega_1}\right), & \omega_1\le x\le\omega_2 \\ k, & \omega_2\le x\le\omega_3 \\ k+(1-k)\left(\dfrac{x-\omega_3}{\omega_4-\omega_3}\right), & \omega_3\le x\le\omega_4 \\ 1, & \omega_4\le x\le\omega_5 \\ k\ +\ (1-k)\left(\dfrac{\omega_6-x}{\omega_6-\omega_5}\right), & \omega_5\le x\le\omega_6 \\ k, & \omega_6\le x\le\omega_7 \\ k\left(\dfrac{\omega_8-x}{\omega_8-\omega_7}\right), & \omega_7\le x\le\omega_8 \\ 0, & x>\omega_8 \end{cases}$$

with $0<k<1$.

$$\vartheta_{\bar{S}^I}(x)=\begin{cases}1, & x<\omega'_1\\ 1+(1-k)\left(\dfrac{\omega'_1-x}{\omega'_2-\omega'_1}\right), & \omega'_1\le x\le\omega'_2\\ k, & \omega'_2\le x\le\omega'_3\\ k+k\left(\dfrac{\omega'_3-x}{\omega'_4-\omega'_3}\right), & \omega'_3\le x\le\omega'_4\\ 0, & \omega'_4\le x\le\omega'_5\\ k\left(\dfrac{x-\omega'_5}{\omega'_6-\omega'_5}\right), & \omega'_5\le x\le\omega'_6\\ k, & \omega'_6\le x\le\omega'_7\\ k+(1-k)\left(\dfrac{x-\omega'_7}{\omega'_8-\omega'_7}\right), & \omega'_7\le x\le\omega'_8\\ 1, & x>\omega'_8\end{cases}$$

where $0<k<1$.

Definition 4.2: Accuracy Function

Let $\mathfrak{z}=(\mu_0,\vartheta_0)$ be an intuitionistic fuzzy number, then $H(\mathfrak{z})$ is the Accuracy Function of $\mathfrak{z}$ given by

$$H(\mathfrak{z})=\mu_0+\vartheta_0$$

4.2.1. Accuracy Function of an Octagonal Intuitionistic Fuzzy Number

Let $O_c=\{(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8);(\omega'_1,\omega'_2,\omega'_3,\omega_4,\omega_5,\omega'_6,\omega'_7,\omega'_8)\}$ be an octagonal intuitionistic fuzzy number. Then its Accuracy Function $H(O_c)$ is given by,

$$H(O_c)=\frac{\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6+\omega_7+\omega_8}{8}+\frac{\omega'_1+\omega'_2+\omega'_3+\omega_4+\omega_5+\omega'_6+\omega'_7+\omega'_8}{8}$$

6. TOPSIS Algorithm

Step 1. Construction of Decision Matrix

First of all, a decision matrix $D_M=\left[X_{ij}\right]_{m\times n}$,where $i=1,2,3,\cdots,m$ and $j=1,2,3,\cdots,n$ comprising of "m" alternatives and "n" criterions is designed as

$$D_M=\begin{bmatrix}X_{11} & X_{12} & \dots & X_{1n}\\ X_{21} & X_{22} & \dots & X_{2n}\\ \vdots & & \vdots & \vdots\\ X_{m1} & X_{m2} & \dots & X_{mn}\end{bmatrix} \quad (1)$$

$\left[X_{ij}\right]_{m\times n}$ represents score of the i_{th} alternative regarding the j_{th} criteria. Our environment is intuitionistic fuzzy, so the initial decision matrix would be in IFN.

Whatever the intuitionistic fuzzy score, it can be reduced to crisp value using the accuracy formula for intuitionistic fuzzy number. The final decision matrix obtained in this step would now be in crisp environment after the application of Accuracy Function formula.

Step 2. Normalization

Decision Matrix is then normalized to form a normalized decision matrix $R=\left[r_{ij}\right]_{m\times n}$ by

$$r_{ij}=\frac{X_{ij}}{\sqrt{\sum_{i=1}^{m}X_{ij}^2}} \quad (2)$$

where $j = 1, 2, 3, \cdots, n$. "R" is the normalized score of decision matrix.

Normalized Decision Matrix thus obtained is of the form

$$R = \begin{bmatrix} r_{\mathbf{11}} & \cdots & r_{\mathbf{1}n} \\ \vdots & \ddots & \vdots \\ r_{m\mathbf{1}} & \cdots & r_{mn} \end{bmatrix} \tag{3}$$

Step 3. Computation of Weight Matrix

The weights accredited by the decision makers to the criteria are taken as a weight matrix, W. These weights are then used in step 4 for the calculation of WNDM.

$$W = [w_1 \quad w_2 \quad \cdots \quad w_j \quad \cdots \quad w_n]^T \tag{4}$$

where, $\sum w_j = 1$.

Step 4. Computation of Weighted Normalized Decision Matrix

In this step, WNDM $R' = [r'_{ij}]_{m\times n}$ is calculated by substituting the values of r_{ij} from matrix 3 and the weights from weight matrix 4 in below equation

$$R' = [r'_{ij}]_{m\times n} = W_j \times R_{ij} \tag{5}$$

Hence, WNDM is given by

$$R' = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_{\mathrm{n}} \end{bmatrix} = \begin{bmatrix} w_1 \cdot r_{11} & \cdots & w_n \cdot r_{1n} \\ \vdots & \ddots & \vdots \\ w_1 \cdot r_{m1} & \cdots & w_n \cdot r_{mn} \end{bmatrix} \tag{6}$$

Step 5. Calculation of PIS and NIS

Positive Ideal Solution:

$O_j^+ = S_{\mathbf{1}}^+, S_2^+, S_3^+, \cdots, S_n^+$

where S_i^+= {max (S_{ij}); $j \in J^+$, min(S_{ij}); j ∈ J^-}

Negative Ideal Solution:

$O_j^- = S_{\mathbf{1}}^-, S_2^-, \cdots, S_n^-$

where S_i^- ={min (S_{ij}); $j \in J^-$, max(S_{ij}); j ∈ J^-}

Here,

$J^+ = \{ j = 1, 2, 3, \cdots, n;\ j$ linked with benefit criteria}

$J^- = \{ j = 1, 2, 3, \cdots, n;\ j$ linked with cost criteria}

Step 6. Determination of separation measure for each Alternative

Separation Measure of each alternative is to be measured from PIS and NIS respectively.

$$T_i^+ = \sqrt{\sum_{j=\mathbf{1}}^{n} (S_j^+ - S_{ij})^2}, \qquad i = 1, 2, 3 \cdots m \tag{7}$$

$$T_i^- = \sqrt{\sum_{j=1}^{n}(S_i^- - S_{ij})^2}, \qquad i = 1,2,3 \cdots m \tag{8}$$

Step 7. Computation of Relative Closeness to Ideal Solution C_i

For each alternative, Closeness coefficient is calculated by

$$C_i = \frac{T_i^-}{T_i^+ + T_i^-} \tag{9}$$

Where $0 \leq C_i \leq 1$; $i = 1,2,3,m$

Step 8. Result

Alternatives get ranked depending upon the closeness coefficient from most beneficial to least value. The alternative possessing highest value of closeness coefficient is then taken into account.

7. Numerical Analysis

Statemen 7.1: The problem is to exalt safety by mitigating liabilities and focusing on enhancing the safety of system. Here we have to inspect the use and application of MCDM in Safety Assessment. We have three companies ($A_i = 1,2,3$). Four Criterions are $C_1 =$ Detailed information about crew members and their behavior; C_2 =planning, preview and scenarios of risk management; $C_3 =$ Comparison with industry and $C_4 =$ Cost Control. The following data form 7.1 is constructed.

Table1. Initial decision matrix in octagonal intuitionistic fuzzy environment

	C_1	C_2	C_3	C_4
A_1	{(1,2,3,4,6,8,9,10); (0,2,3,4,6,7,11,13)}	{(4,6,8,10,11,12,14,15); (2,3,4,10,11,13,14,16)}	{(7,8,10,11,13,14,16,18); (5,6,8,11,13,15,17,19)}	{(6,8,10,14,16,17,19,20); (5,7,9,10,14,16,18,19)}
A_2	{(3,4,6,8,10,12,14,16); (2,5,7,8,10,11,13,15)}	{(5,6,9,12,15,17,18,20); (3,7,10,12,15,16,19,20)	{(8,10,12,14,16,18,19,20); (6,7,9,14,16,17,18,19)}	{(9,10,12,13,15,16,18,19); (7,9,11,13,15,16,17,18)}
A_3	{(2,6,7,8,9,10,11,12); (1,2,6,8,9,13,14,15)}	{(6,7,9,10,12,13,15,17); (4,5,6,10,12,14,16,18)}	{(1,2,3,4,5,6,7,8); (0,1,2,4,5,6,7,8)}	{(4,7,10,13,16,18,19,20); (3,6,9,13,16,17,18,20)}

Step1. By the use of Accuracy Function, we defuzzified the above values into crisp notation given by Table 1.

Table 2. Defuzzified decision matrix

■	C_1	C_2	C_3	C_4
A_1	11.125	19.125	23.875	26
A_2	18	25.5	27.875	27.25
A_3	16.625	19.625	8.625	26.125

Step2. Normalized Decision Matrix is given by Table 3.

Table 3. Normalized decision matrix

■	C_1	C_2	C_3	C_4
A_1	0.413	0.511	0.633	0.567
A_2	0.669	0.681	0.739	0.594
A_3	0.618	0.524	0.229	0.570
W	0.3	0.4	0.1	0.2

Step 3. Weights assigned by DM's to the criteria are given by the matrix;

$$w = \begin{pmatrix} 0.25 \\ 0.15 \\ 0.25 \\ 0.35 \end{pmatrix}^T \tag{10}$$

Step 4. Weighted Normalized Decision Matrix is given by Table 4.

Table 4. Weighted normalized decision matrix

■	C_1	C_2	C_3	C_4
A_1	0.124	0.204	0.0631	0.113
A_2	0.201	0.272	0.074	0.119
A_3	0.185	0.210	0.023	0.114

Step 5. Calculation of PIS (Positive Ideal Solution) and NIS (Negative Ideal Solution):

A+ = {0.201, 0.272, 0.074, 0.113} and A- = {0.124, 0.204, 0.023, 0.119}

Step 6. Determination of Separation Measure is given by Table 5 and Table 6.

Table 5. Separation measure T_i^+

■	T_i^+
A_1	0.1033
A_2	0.006
A_3	0.08

Table 6. Separation Measure T_i^-

■	T_i^-
A_1	0.04
A_2	0.114
A_3	0.061

Step 7. Determination of RCC to ideal solution C_i^*

$$A_1 = 0.279$$

$$A_2 = 0.950$$

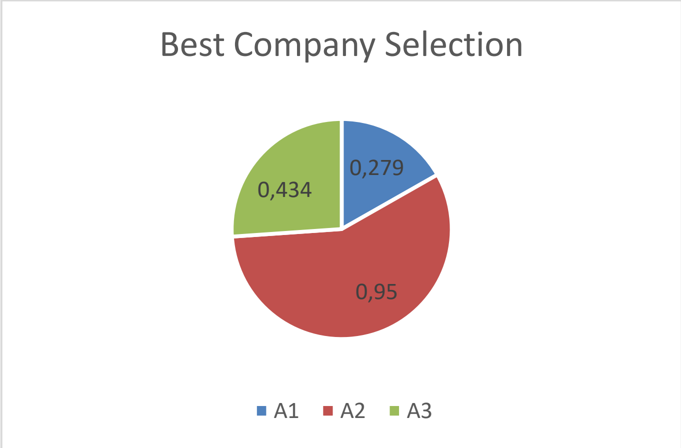
$$A_3 = 0.434$$

Result: $A_2 > A_3 > A_1$

Hence it is concluded that the second company is the best choice.

8. Result Discussion

The proposed method and algorithm are applied on MCDM type problem. The problem is to exalt safety by mitigating liabilities and focusing on enhancing the safety of system. Here we have the inspected results as shown below,



Hence it is concluded that the second company is the best choice.

9. Conclusion

This research focuses on Multi Criteria Decision Making issues in intuitionistic fuzzy region in which the assessment of choices is represented as Octagonal intuitionistic fuzzy numbers. The Accuracy Function is made for Multi Criteria Decision Making as an alternate to alpha cuts of intuitionistic fuzzy numbers which sum up to complicated calculations. Accuracy Function is applied to TOPSIS technique with OIFNs which reduces the complexity of the environment from complex intuitionistic fuzzy to crisp. The derived results help us conclude that customer can have the safety measures by using their various choice factors.

References

[1] C. Hwang, K. Yoon, *Multiple Attribute Decision Making: Methods and Applications*, *A State-of-the-Art Survey*, Springer (1981).

[2] X. Zhang, P. Liu, *Method for Aggregating Triangular Fuzzy Intuitionistic Fuzzy Information and Its Application to Decision Making*, Technological and Economic Development of Economy 16(2) (2010) 280-290.

[3] L. A. Zadeh, *Fuzzy Sets*, Information and Computation 8(3) (1965) 338-353.

[4] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems 20 (1986) 87-96.

[5] K. T. Atanassov, *More on Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems 33 (1989) 37-46.

[6] M. Saqlain, N. Jafar, A. Riffat, *Smart Phone Selection by Consumers' İn Pakistan: FMCGDM Fuzzy Multiple Criteria Group Decision Making Approach*, Gomal University Journal of Research 34(1) (2018) 27-31.

[7] M. N. Jafar, K. Muniba, A. Saeed, S. Abbas, I. Bibi, *Application of Sanchez's Approach to Disease Identification Using Trapezoidal Fuzzy Numbers*, International Journal of Latest Engineering Research and Applications 4(9) (2019) 51-57.

[8] M. Saeed, M. Saqlain, M. Riaz, *Application of Generalized Fuzzy TOPSIS in Decision Making for Neutrosophic Soft set to Predict the Champion of FIFA 2018: A Mathematical Analysis*, Punjab University Journal of Mathematics 51(8) (2019) 111-126.

[9] M. Saqlain, N. Jafar, R. Hamid, A. Shahzad, *Prediction of Cricket World Cup 2019 by TOPSIS Technique of MCDM-A Mathematical Analysis*, International Journal of Scientific & Engineering Research 10(2) (2019) 789-792.

[10] M. Saqlain, M. Saeed, M. R. Ahmad, F. Smarandache, *Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application*, Neutrosophic Sets and Systems 27 (2019) 131-137.

[11] P. Rajarajeswari, G. Menaka, *A New Approach for Ranking of Octagonal Intuitionistic Fuzzy Numbers*, International Journal of Fuzzy Logic Systems (2017), http://doi.org/ 10.5072/zenodo

[12] F. E. Boran, S. Genç, M. Kurt, D. Akay, *A Multi-Criteria Intuitionistic Fuzzy Group Decision Making for Supplier Selection with TOPSIS Method*, Expert System and Applications 36(8) (2009) 11363-11368.

[13] D. Bakbak, V. Uluçay, *Multicriteria Decision-Making Method Using the Cosine Vector Similarity Measure Under Intuitionistic Trapezoidal Fuzzy Multi-Numbers in Architecture*, 6th International Multidisciplinary Studies Congress (Multicongress'19) (2019), Gaziantep, Türkiye.

[14] D. Bakbak, V. Uluçay, M. Şahin, *Intuitionistic Trapezoidal Fuzzy Multi-Numbers and Some Arithmetic Averaging Operators with Their Application in Architecture*, 6th International Multidisciplinary Studies Congress (Multicongress'19) (2019), Gaziantep, Türkiye.

[15] V. Uluçay, I. Deli, M. Şahin, *Intuitionistic Trapezoidal Fuzzy Multi-Numbers and Its Application to Multi-Criteria Decision-Making Problems*, Complex & Intelligent Systems 5(1) (2019) 65-78.

[16] V. Uluçay, I. Deli, M. Şahin, *Trapezoidal Fuzzy Multi-Number and Its Application to Multi-Criteria Decision-Making Problems*, Neural Computing and Applications 30 (2018) 1469-1478.

[17] S. Broumi, F. Smarandache, *Intuitionistic Fuzzy Soft Expert Sets and its Application in Decision Making*, Journal of New Theory (1) (2015) 89-105.

[18] M. Saqlain, F. Smarandache, *Octagonal Neutrosophic Number: Its Different Representations, Properties, Graphs and De-Neutrosophication*, Neutrosophic Sets and Systems (2020) (Accepted).

[19] M. Saqlain, A. Hamza, S. Farooq, *Linear and Non-Linear Octagonal Neutrosophic Number: Its Representation, $\alpha - Cut$ and Applications*, International Journal of Neutrosophic Science 3 (2020) 1-17.

[20] A. Chakraborty, S. P. Mondal, A. Ahmadian, N. Senu, S. Alam, S. Salahshour, *Different Forms of Triangular Neutrosophic Numbers, De-Neutrosophication Techniques, and their Applications*, Symmetry 10 (8) (2018) 1-28. https://doi.org/10.3390/sym10080327

[21] A. Chakraborty, S. P. Mondal, A. Mahata, S. Alam, *Different Linear and Non-Linear Form of Trapezoidal Neutrosophic Numbers, De-Neutrosophication Techniques and Its Application in Time Cost Optimization Technique, Sequencing Problem*, Operation Research 11 (2018). https://doi.org/10.1051/ro/2019090

[22] A. Chakraborty, S. P. Mondal, S. Alam, A. Ahmadian, N. Senu., D. De, S. Salahshour, *The Pentagonal Fuzzy Number: Its Different Representations, Properties, Ranking, Defuzzification and Application in Game Problems*, Symmetry 11(2) (2019).

[23] S. Narayanamoorthy, S. Maheswari, *The Intelligence of Octagonal Fuzzy Number to Determine the Fuzzy Critical Path: A New Ranking Method*, Scientific Programming, 2016 (2016), Article ID 6158208, 8 pages.

[24] J. Ye, *Prioritized Aggregation Operators of Trapezoidal Intuitionistic Fuzzy Sets and Their Application to Multi Criteria Decision Making*, Neural Computing and Applications 25(6) (2014) 1447-1454.

[25] K. K. Yen, S. Ghoshray, G. Roig, G., *A Linear Regression Model Using Triangular Fuzzy Number Coefficients*, Fuzzy Sets and System 106(2) (1999) 167-177.

[26] J. Ye, *Trapezoidal Neutrosophic Set and Its Application to Multiple Attribute Decision-Making*, Neural Computing and Applications 26 (2015) 1157-1166.

[27] M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, F. Smarandache, *Aggregate Operators of Neutrosophic Hypersoft Set*, Neutrosophic Sets and Systems 32 (2020) 294-306. https://doi.org/10.5281/zenodo.3723155

[28] M. Saqlain, M. N. Jafar, M. Riaz, *A New Approach of Neutrosophic Soft Set with Generalized Fuzzy TOPSIS in Application of Smart Phone Selection*, Neutrosophic Sets and Systems 32 (2020) 307-316 2020. https://doi.org/10.5281/zenodo.3723161

[29] M. Saqlain, N. Jafar, S. Moin, M. Saeed, S. Broumi, *Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets*, Neutrosophic Sets and Systems 32 (2020) 317-329. https://doi.org/10.5281/zenodo.3723165

[30] E. Sulukan, N. Çağman, T. Aydın, *Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets and Their Application to a Performance-Based Value Assignment Problem*, Journal of New Theory (29) (2019) 79-88. https://dergipark.org.tr/tr/download/article-file/906764

[31] F. Smarandache, *Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set, Spherical Fuzzy Set, and q-Rung Ortho pair Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision* (revisited), Journal of New Theory 29 (2019) 1-31. https://dergipark.org.tr/tr/download/article-file/906372

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Fuzzy Orbit Irresolute Mappings

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Abstract − Fuzzy orbit topological space is a new structure very recently given by [1]. This new space is based on the notion of open fuzzy orbit sets. The aim of this paper is to provide applications of open fuzzy orbit sets. We introduce the notions of fuzzy orbit irresolute mappings and fuzzy orbit open (resp. irresolute open) mappings and studied some of their properties.

Keywords − *Fuzzy orbit, fuzzy orbit topology, fuzzy orbit closure, fuzzy orbit neighbourhood, fuzzy orbit irresolute mapping*

1. Introduction

The fuzzy set theory introduced by Zadeh [2] provides natural bases for building new branches of fuzzy mathematics. As a generalization of topological space in fuzzy setting, the concept of fuzzy topological space introduced by Chang [3] and studied further by many topologists (cf. [4, 5, 6, 7, 8, 9]). Malathi and Uma [10] in 2017 introduced the notions of the orbit of a fuzzy set under a mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ and an open fuzzy orbit set in a fuzzy topological space$(\mathcal{P}, \sigma)$. Very recently, Majeed and El-Sheikh [1] studied the behavior of the collection of open fuzzy orbit sets and discussed the conditions on the mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ to obtain a fixed orbit of these fuzzy sets. Majeed and El-Sheikh proved that the collection of all open fuzzy orbit sets under the mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ construct a fuzzy topology, denoted by σ_{FO}, which is coarser than σ.Our purpose, in this work, is to define a new class of mappings between fuzzy topological spaces by using open fuzzy orbit sets. That is, we define the class of fuzzy orbit irresolute mappings on fuzzy topological spaces. This notion is independent from the notion of fuzzy continuous in the sense of Chang (see Examples 4.1 and 4.2). Also, we define and study fuzzy orbit open (resp. irresolute open) mappings.

2. Preliminaries

Throughout this paper, $\mathcal{P}$ will refer to the initial universe, $I = [0,1]$, $I_0 = (0,1]$, and $I^{\mathcal{P}}$ is the family of all fuzzy sets of $\mathcal{P}$. For $x \in \mathcal{P}$ and $t \in I_0$, a fuzzy point ($\mathcal{F}$-point, for short) x_t is defined as t if $x = y$ and 0 otherwise, $\forall\, y \in \mathcal{P}$. A $\mathcal{F}$ -point x_t is said to be belongs to a fuzzy set ω, denoted $x_t \in \omega$, if and only if $\omega(x) \geq t$. For $\delta, \omega \in I^{\mathcal{P}}$, δ is called quasi-coincident with ω, denoted by $\delta\, q \omega$ if $\delta(x) + \omega(x) > 1$ for some $x \in \mathcal{P}$, otherwise we write $\delta\, \bar{q}\, \omega$. And $\delta q \omega$ if and only if $\exists x_t;\ x_t \in \delta,\ x_t\, q\, \omega$.

Next, we list some definitions and basic properties about the notions of the orbit of fuzzy set and fuzzy orbit topological spaces and other related concepts.

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Definition 2.1. [10] Let $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ be a mapping and $\omega \in I^{\mathcal{P}}$. Then,

i. The fuzzy orbit ($fo.$, for short) of ω under $\mathcal{J}$, denoted by $O_{\mathcal{J}}(\omega)$ is defined as $O_{\mathcal{J}}(\omega) = \{\omega, \mathcal{J}(\omega), \mathcal{J}^2(\omega), \dots\}$.

ii. The Fuzzy orbit set (fo.s, for short) of ω under $\mathcal{J}$ is defined as $FO_{\mathcal{J}}(\omega) = \omega \wedge \mathcal{J}(\omega) \wedge \mathcal{J}^2(\omega) \wedge \dots$ the intersection of all members of $O_{\mathcal{J}}(\omega)$.

iii. If $(\mathcal{P}, \sigma)$ is a fuzzy topological space (*fts*, for short) and $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$, then the *fo.s* under $\mathcal{J}$ which belongs to σ is called an open fuzzy orbit set under $\mathcal{J}$ (open-fo.s, for short). The complement of an open-fo.s is called a closed fuzzy orbit set under $\mathcal{J}$ (closed-fo.s, for short).

Definition 2.2. [10] Let $(\mathcal{P}, \sigma)$and $(\mathcal{Q}, \sigma^*)$ be two fts's. Let $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$. A mapping $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ is called fo.continuous, if the inverse image of every open fuzzy set (open-fs, for short) in $\mathcal{Q}$ is an open-fo.s in $\mathcal{P}$.

Definition 2.3. [3] Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ be two fts's. A mapping $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ is called a fuzzy continuous (f.continuous, for short) if and only if the inverse image of each open-fs in $\mathcal{Q}$ is an open-fs in $\mathcal{P}$.

Majeed and El-Sheikh studied the collection of open-fo.s's and introduced some properties of these sets. They determined the cases on the mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ becomes fixed open-fo.s (i.e., $I(\delta) = \delta$) for each open-fo.s δ, where $\mathcal{P}$ is a nonempty countable set. The following theorem explains that.

Theorem 2.1. [1] Let $(\mathcal{P}, \sigma)$be a fts and δ be an open-fo.s under the mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$. Then, $\mathcal{J}(\delta) = \delta$ whenever $\mathcal{J}$ is either bijective or constant mapping.

Remark 2.1. From now on, any mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ will be considered as a bijective or constant mapping on a nonempty countable set $\mathcal{P}$.

Theorem 2.2. [1] Let $(\mathcal{P}, \sigma)$ be a fts and let σ_{FO} denotes the set of all open-fo.s's under the mapping $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$. Then,$\sigma_{FO}$ constructs a fuzzy topology on $\mathcal{P}$ coarser than σ. The pair $(\mathcal{P}, \sigma_{FO})$ is called fuzzy orbit topological space (fo.ts, for short) associated with $(\mathcal{P}, \sigma)$.

Definition 2.4. [1] Let $(\mathcal{P}, \sigma_{FO})$ be a fo.ts and $\omega \in I^{\mathcal{P}}$. Then,

i. The fo.closure of ω, denoted by $cl_{FO}(\omega)$, is defined as,

$$cl_{FO}(\omega) = \wedge\{\delta \in I^{\mathcal{P}}: \delta \text{ } is \text{ } a \text{ } closed - fo.s \text{ } and \text{ } \omega \leq \delta\}$$

ii. The fo.interior of ω, denoted by $Int_{FO}(\omega)$, is defined as,

$$Int_{FO}(\omega) = \vee\{\delta \in I^{\mathcal{P}}: \delta \text{ } is \text{ } an \text{ } open - fo.s \text{ } and \text{ } \delta \leq \omega\}$$

Proposition 2.1. [1] Let $(\mathcal{P}, \sigma_{FO})$ be a fo.ts and $\omega \in I^{\mathcal{P}}$. Then,

$$Int_{FO}(\omega) \leq Int(\omega) \leq \omega \leq cl(\omega) \leq cl_{FO}(\omega).$$

Proposition 2.2. [1] Let $(\mathcal{P}, \sigma_{FO})$ be a fo.ts and $\omega, \delta \in I^{\mathcal{P}}$. Then,

i. $cl_{FO}(\overline{0}) = \overline{0}$ *(resp.* $Int_{FO}(\overline{0}) = \overline{0}$*)* and $cl_{FO}(\overline{1}) = \overline{1}$ *(resp.* $Int_{FO}(\overline{1}) = \overline{1}$*).*
ii. $\omega \leq cl_{FO}(\omega)$ $(resp. Int_{FO}(\omega) \leq \omega)$.
iii. $cl_{FO}(\omega \vee \delta) = cl_{FO}(\omega) \vee cl_{FO}(\delta)$ $(resp. Int_{FO}(\omega \wedge \delta) = Int_{FO}(\omega) \wedge Int_{FO}(\delta))$.
iv. $If \ \omega \leq \delta, then \ cl_{FO}(\omega) \leq cl_{FO}(\delta)$ $(resp. Int_{FO}(\omega) \leq Int_{FO}(\delta))$.
v. $cl_{FO}(cl_{FO}(\omega)) = cl_{FO}(\omega)$ $(resp. Int_{FO}(Int_{FO}(\omega)) = Int_{FO}(\omega))$.
vi. $\omega \ is \ closed - fo. (resp. open) iff \ \omega = cl_{FO}(\omega) \big(resp. \omega = Int_{FO}(\omega)\big)$.

Theorem 2.3. [1] Let $(\mathcal{P}, \sigma_{FO})$ be a fo.ts and $\omega \ \ I^{\mathcal{P}}$.Then,

i. $\bar{1} - Int_{FO}(\omega) = cl_{FO}(\bar{1} - \omega)$.
ii. $\bar{1} - cl_{FO}(\omega) = Int_{FO}(\bar{1} - \omega)$.

3. Fuzzy Orbit Neighbourhood

Definition 3.1. A fuzzy set ω in a fts $(\mathcal{P}, \sigma)$is said to be a fuzzy orbit neighbourhood (fo.nbhd, for short) of a $\mathcal{F}$-point x_t if and only if there exists an open-fo.s δ such that $x_t \in \delta \leq \omega$.

Theorem 3.1. Let $(\mathcal{P}, \sigma)$ be a fts and $\omega \ \in I^{\mathcal{P}}$. Then, ω is an open-fo.s if and only if ω is a fo.nbhd for any $\mathcal{F}$-point $x_t \in$ ω.

Proof. Suppose ω is an open-fo.s and let $x_t \in$ ω. Since ω ≤ ω and ω is an open-fo.s, then ω is a fo.nbhd of x_t.

Conversely, since for all $x_t \in$ ω, there exists an open-fo.s δ_k such that $x_t \in \delta_k \leq \omega$. Then, $\vee x_t \leq \vee_{k \in \omega} \delta_k \leq \omega$. Since every fuzzy set can be represented by the union of its $\mathcal{F}$-points, then $\vee x_t = \omega$. Also, by Theorem 2.2, $\vee_{k \in \omega} \delta_k$ is an open-fo.s. Thus, ω is an open-fo.s.

Definition 3.2. A fuzzy set ω in a fts $(\mathcal{P}, \sigma)$is said to be a fuzzy orbit Q-neighbourhood (fo.Q-nbhd, for short) of a $\mathcal{F}$-point x_t if ∃ an open-fo.s δ such that $x_t q \delta \ \leq \omega$.

Definition 3.3. Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma')$ be any two fts's. Let $\mathcal{I}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{I}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ be any two mappings. A mapping $\psi: (\mathcal{P}, \sigma_{FO}) \longrightarrow (\mathcal{Q}, \sigma'_{FO})$ is said to be f.continuous, if the inverse image of any open-fo.s under the mapping $\mathcal{I}_2$ in $(\mathcal{Q}, \sigma')$ is an open-fo.s under the mapping $\mathcal{I}_1$ in $(\mathcal{P}, \sigma)$.

Theorem 3.2. Let $\psi: (\mathcal{P}, \sigma_{FO}) \longrightarrow (\mathcal{Q}, \sigma'_{FO})$ and $g: (\mathcal{Q}, \sigma'_{FO}) \longrightarrow (Z, \sigma''_{FO})$ be two mappings. Then, $go\psi$ is f. continuous mapping if ψ and g are f.continuous.

Proof. The proof is straightforward.

4. Fuzzy Orbit Irresolute (Irresolute Open) Mappings

Our goal here is to introduce and study the concept of irresolute (resp. irresolute open) mappings in fst's by using the concepts of open-fo.s's.

Definition 4.1. Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ be any two fts's. Let $\mathcal{I}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{I}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ be any two mappings. A mapping $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ is said to be fuzzy orbit irresolute (fo.irresolute, for short), if the inverse image of every open-fo.s under the mapping $\mathcal{I}_2$ in $(\mathcal{Q}, \sigma^*)$ is an open-fo.s under the mapping $\mathcal{I}_1$ in $(\mathcal{P}, \sigma)$.

The concept of f.continuous in the sense of Chang and fo.irresolute are independent. The next two examples explain that.

Example 4.1. Let $\mathcal{P} = \{k_1, k_2, k_3\}$ and $\mathcal{Q} = \{s_1, s_2, s_3\}$. Define $\sigma = \{\bar{0}, \bar{1}, \omega\}$ and $\sigma^* = \{\bar{0}, \bar{1}, \delta_1, \delta_2\}$ where $\omega \in I^{\mathcal{P}}$and $\delta_1, \delta_2 \in I^{\mathcal{Q}}$ such that $\omega = \{(k_1, 0.2), (k_2, 0.3), (k_3, 0.3)\}$, $\delta_1 = \{(s_1, 0.2), (s_2, 0.3), (s_3, 0.3)\}$ and $\delta_2 = \{(s_1, 0.6), (s_2, 0.5), (s_3, 0.7)\}$. Clearly, $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ are fts's.

Define $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$, $\mathcal{I}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{I}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ as $\psi(k_1) = s_1, \psi(k_2) = s_3, \psi(k_3) = s_2, \mathcal{I}_1(k_1) = k_1, \mathcal{I}_1(k_2) = k_3,\ \mathcal{I}_1(k_3) = k_2$and $\mathcal{I}_2(s_1) = s_1, \mathcal{I}_2(s_2) = s_3,\ \mathcal{I}_2(s_3) = s_2$. Then, ψ is fo.irresolute but not f.continuous mapping, since δ_2 is an open-fs in $\mathcal{Q}$, however $\psi^{-1}(\delta_2) \notin \sigma$.

Example 4.2. Let $\mathcal{P} = \{k_1, k_2, k_3\}$ and $\mathcal{Q} = \{s_1, s_2, s_3\}$. Define $\sigma = \{\bar{0}, \bar{1}, \omega\}$ and $\sigma^* = \{\bar{0}, \bar{1}, \delta\}$ where $\omega \in I^{\mathcal{P}}$ and $\delta \in I^{\mathcal{Q}}$ such that $\omega = \{(k_1, 0.4), (k_2, 0.4), (k_3, 0.7)\}$, $\delta = \{(s_1, 0.7), (s_2, 0.4), (s_3, 0.4)\}$. Clearly, $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ are fts's.

Define $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$, $\mathcal{J}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{J}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ as $\psi(k_1) = s_2, \psi(k_2) = s_3, \psi(k_3) = s_1$, $\mathcal{J}_1(k_1) = k_1, \mathcal{J}_1(k_2) = k_3$, $\mathcal{J}_1(k_3) = k_2$ and $\mathcal{J}_2(s_1) = s_1, \mathcal{J}_2(s_2) = s_3$, $\mathcal{J}_2(s_3) = s_2$. Then, ψ is f.continuous but not fo.irresolute mapping, since δ is an open-fo.s under $\mathcal{J}_2$ in $\mathcal{Q}$, however $\psi^{-1}(\delta) = \omega$ is not an open-fo.s under $\mathcal{J}_1$ in $\mathcal{P}$.

Theorem 4.1. Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ be any two fts's, let $(\mathcal{P}, \sigma_{FO})$ and $(\mathcal{Q}, \sigma^*_{FO})$ be its associative fo.ts's with $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ respectively. Let $\mathcal{J}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{J}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ be any two mappings. Then $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ is fo.irresolute mapping iff $\psi: (\mathcal{P}, \sigma_{FO}) \longrightarrow (\mathcal{Q}, \sigma^*_{FO})$ is f.continuous mapping.

Proof. Straightforward.

Theorem 4.2. Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ be any two fts's. Let $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ be any mapping. If ψ is fo.continuous mapping, then ψ is fo.irresolute mapping.

Proof. The proof is straightforward by Definition 4.1 and Definition 2.2.

The inverse direction of Theorem 4.2 may not be held, In Example 4.1, ψ *is* fo.irresolute mapping, however, it is not fo.continuous since δ_2 is an open-fs in $\mathcal{Q}$, but its inverse image is not open-fo.s in $\mathcal{P}$.

Some characterizations of fo.irresolute mapping are given next.

Theorem 4.3. Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ be any two fts's. Let $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ be any mapping. Then, the following statements are equivalent:

(a) ψ is fo.irresolute mapping,

(b) For every $\mathcal{F}$-point x_t of $\mathcal{P}$ and every open-fo.s δ in $\mathcal{Q}$ such that $\psi(x_t) \in \delta$, there is an open-fo.s ω in $\mathcal{P}$ such that $x_t \in \omega$ and $\psi(\omega) \leq \delta$,

(c) For every closed-fo. ν in $\mathcal{Q}$, $\psi^{-1}(\nu)$ is closed-fo.s in $\mathcal{P}$,

(d) For every $\mathcal{F}$-point x_t of $\mathcal{P}$ and every fo.nbhd δ in $\mathcal{Q}$ of $\psi(x_t)$, $\psi^{-1}(\delta)$ is a fo.nbhd of x_t in $\mathcal{P}$,

(e) For every $\mathcal{F}$*-point* x_t of $\mathcal{P}$ and every *fo.nbhd* δ *in* $\mathcal{Q}$ *of* $\psi(x_t)$, *there is a fo.nbhd* ω in $\mathcal{P}$ of x_t such that $\psi(\omega) \leq \delta$,

(f) For every $\mathcal{F}$*-point* x_t of $\mathcal{P}$ and every open-fo.s δ in $\mathcal{Q}$ such that $\psi(x_t) q\, \delta$, there is an open-fo.s ω in $\mathcal{P}$ such that $x_t q \omega$ and $\psi(\omega) \leq \delta$,

(g) For every $\mathcal{F}$-point x_t of $\mathcal{P}$ and every fo.Q-nbhd δ in $\mathcal{Q}$ of $\psi(x_t)$, $\psi^{-1}(\delta)$ is fo.Q-nbhd of x_t in $\mathcal{P}$,

(h) For every $\mathcal{F}$-point x_t of $\mathcal{P}$ and every fo.Q-nbhd δ in $\mathcal{Q}$ of $\psi(x_t)$, there is a fo.Q-nbhd ω of x_t such that $\psi(\omega) \leq \delta$,

(i) $\psi(cl_{FO}(\omega)) \leq cl_{FO}(\psi(\omega))$, for every fuzzy set ω of $\mathcal{P}$,

(j) $cl_{FO}(\psi^{-1}(\delta)) \leq \psi^{-1}(cl_{FO}(\delta))$, for every fuzzy set δ of $\mathcal{Q}$,

(k) $\psi^{-1}(Int_{FO}(\delta)) \leq Int_{FO}(\psi^{-1}(\delta))$, for every fuzzy set δ of $\mathcal{Q}$.

PROOF.

(a)⟹(b) Let x_t *be a* $\mathcal{F}$*-point* of $\mathcal{P}$ and δ be an *open-fo.s* in $\mathcal{Q}$ under the mapping $\mathcal{I}_2$ such that $\psi(x_t) \in \delta$. Put $\omega = \psi^{-1}(\delta)$. Then, by (a) ω is an *open-fo.s* in $\mathcal{P}$ under the mapping $\mathcal{I}_1$ such that $x_t \in \omega$ and $\psi(\omega) = \psi(\psi^{-1}(\delta)) \leq \delta$. Hence, $\psi(\omega) \leq \delta$.

(b)⟹(a) Let δ be an open-fo.s in $\mathcal{Q}$. Let $x_t \in \psi^{-1}(\delta)$. Then, $\psi(x_t) \in$ δ. Now by *(b)* there is an open-fo.s ω in $\mathcal{P}$ such that $x_t \in \omega$ and $\psi(\omega) \leq$ δ. Then, $x_t \in$ ω $\leq \psi^{-1}(\delta)$. Hence by Theorem 3.1 $\psi^{-1}(\delta)$ is an open-fo.s in $\mathcal{P}$. Thus, ψ is fo.irresolute mapping.

(a)⇔(c) Obvious.

(a)⟹(d) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and let δ be a *fo.nbhd* of $\psi(x_t)$. Then, there is an *open-fo.s* ν in $\mathcal{Q}$ such that $\psi(x_t) \in \nu \leq \delta$. *Now* $\psi^{-1}(\nu)$ *is an open-fo.s in* $\mathcal{P}$, because ψ *is a fo.*irresolute mapping and $x_t \in \psi^{-1}(\nu) \leq \psi^{-1}(\delta)$. *Thus,* $\psi^{-1}(\delta)$ is a *fo.nbhd* of x_t in $\mathcal{P}$.

(d)⟹(e) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and let δ be a *fo.nbhd* of $\psi(x_t)$. Then, by hypothesis $\omega = \psi^{-1}(\delta)$ is a *fo.nbhd* of x_t and $\psi(\omega) = \psi(\psi^{-1}(\delta)) \leq \delta$. Hence, $\psi(\omega) \leq \delta$.

(e) ⟹(b) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and let δ be an open-fo.s in $\mathcal{Q}$ containing $\psi(x_t)$. Then, δ is a *fo.nbhd* of $\psi(x_t)$, so there is a *fo.nbhd* ω of x_t of $\mathcal{P}$ such that $x_t \in \omega$ and $\psi(\omega) \leq \delta$. Therefore, there exists an open-fo.s ω' in $\mathcal{P}$ such that $x_t \in \omega' \leq \omega$. Clearly, $\psi(\omega') \leq \psi(\omega) \leq \delta$.

(a)⟹(f) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and δ be an *open-fo.s* in $\mathcal{Q}$ such that $\psi(x_t)q\delta$. *Let* $\omega = \psi^{-1}(\delta)$, *then* ω *is an open-fo.s in P and* $x_t q\ \omega$ *and* $\psi(\omega) = \psi(\psi^{-1}(\delta)) \leq \delta$.

(f)⟹(g) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and δ be a *fo.Q-nbhd* of $\psi(x_t)$ in $\mathcal{Q}$. Then, there exists an *open-fo.s* ν in $\mathcal{Q}$ such that $\psi(x_t)q\nu \leq \delta$. By hypothesis there is an *open-fo.* ω in $\mathcal{P}$ such that $x_t q\omega$ and $\psi(\omega) \leq \nu$. Thus $x_t q\omega \leq \psi^{-1}(\nu) \leq \psi^{-1}(\delta)$. Hence, $\psi^{-1}(\delta)$ is a *fo.Q-nbhd* of x_t.

(g)⟹(h) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and δ be a *fo.Q-nbhd* of $\psi(x_t)$ *in* $\mathcal{Q}$. Then, $\omega = \psi^{-1}(\delta)$ is a *fo.Q-nbhd* of x_t and $\psi(\omega) \leq \psi(\psi^{-1}(\delta)) \leq \delta$.

(h)⟹(f) Let x_t be a $\mathcal{F}$*-point* of $\mathcal{P}$ and δ be an open-fo.s in $\mathcal{Q}$ such that $\psi(x_t)q\delta$. *Then,* δ is a *fo.Q-nbhd* of $\psi(x_t)$. So, there is a *fo.Q-nbhd* ω of x_t such that $\psi(\omega) \leq \delta$. Therefore, there exists an *open-fo.s* ν in $\mathcal{P}$ such that $x_t q\nu \leq \omega$. Hence, $x_t q\nu$ and $\psi(\nu) \leq \psi(\omega) \leq \delta$.

(f)⟹(a) Let η be an *open-fo.s* in $\mathcal{Q}$ and $x_t \in \psi^{-1}(\eta)$. Clearly, $\psi(x_t) \in \eta$. Choose the $\mathcal{F}$*-point* $\overline{1} - x_t$. Then, $\psi(\overline{1} - x_t)q\eta$. And so by *(f)* there exists an *open-fo.s* ω such that $\overline{1} - x_t q\omega$ and $\psi(\omega) \leq \eta$. Now, $\overline{1} - x_t q\omega$ this implies $x_t \in \omega$. Thus, $x_t \in \omega \leq \psi^{-1}(\eta)$. Hence, by Theorem 3.1, $\psi^{-1}(\eta)$ is an *open-fo.s* in $\mathcal{P}$.

(i)⟹(c) Let δ be any closed-fo.s in $\mathcal{Q}$. Then, from *(i)*, $\psi(cl_{FO}(\psi^{-1}(\delta))) \leq cl_{FO}(\psi(\psi^{-1}(\delta)) \leq cl_{FO}(\delta) = \delta$. By taking the inverse of the equality we get $cl_{FO}(\psi^{-1}(\delta)) \leq \psi^{-1}(\delta)$. Since $\psi^{-1}(\delta) \leq cl_{FO}(\psi^{-1}(\delta))$. Then, we have $\psi^{-1}(\delta) = cl_{FO}(\psi^{-1}(\delta))$. Hence, $\psi^{-1}(\delta)$ is a closed- fo.s in $\mathcal{P}$.

(c)⟹(i) Suppose that *(c)* holds. Let ω be a fuzzy set of $\mathcal{P}$. Since $\omega \leq \psi^{-1}(\psi(\omega))$, then $\omega \leq \psi^{-1}(\psi(cl_{FO}(\omega)))$. Now, $cl_{FO}(\psi(\omega))$ is a closed-fo.s contains ω. Consequently, $cl_{FO}(\omega) \leq cl_{FO}(\psi^{-1}(cl_{FO}(\psi(\omega)))) = \psi^{-1}(cl_{FO}(\psi(\omega)))$ and so $\psi(cl_{FO}(\omega)) \leq cl_{FO}(\psi(\omega))$.

(i)⟹(j) Let δ be a fuzzy set of Q. Then, $\psi^{-1}(\delta)$ is a fuzzy set of $\mathcal{P}$. Therefore by *(i)*, $\psi(cl_{FO}(\psi^{-1}(\delta))) \leq cl_{FO}(\psi(\psi^{-1}(\delta))) \leq cl_{FO}(\delta)$. Hence, $cl_{FO}(\psi^{-1}(\delta)) \leq \psi^{-1}(cl_{FO}(\delta))$.

(j)⟹(i) Let $\delta = \psi(\omega)$ where ω is a fuzzy set of $\mathcal{P}$, and we know that $\omega \leq \psi^{-1}(\delta)$ which implies $cl_{FO}(\omega) \leq cl_{FO}(\psi^{-1}(\delta))$. Thus, $cl_{FO}(\omega) \leq cl_{FO}(\psi^{-1}(\delta)) \leq \psi^{-1}(cl_{FO}(\delta)) \leq \psi^{-1}(cl_{FO}(\psi(\omega)))$. Therefore, $\psi(cl_{FO}(\omega)) \leq cl_{FO}(\psi(\omega))$.

(a)⟹(k) Let δ be an *open-fo.s* in $\mathcal{Q}$. Clearly $\psi^{-1}(Int_{FO}(\delta))$ is an *open-fo.s* in $\mathcal{P}$ and we have $\psi^{-1}(Int_{FO}(\delta)) \leq Int_{FO}(\psi^{-1}(Int_{FO}(\delta))) \leq Int_{FO}(\psi^{-1}(\delta))$.

(k)⟹(a) Let δ be an open-fo.s in $\mathcal{Q}$. Then, $Int_{FO}(\delta) = \delta$ and $\psi^{-1}(\delta) = \psi^{-1}(Int_{FO}(\delta)) \leq Int_{FO}(\psi^{-1}(\delta))$. Hence, we have $\psi^{-1}(\delta) = Int_{FO}(\psi^{-1}(\delta))$. This means that $\psi^{-1}(\delta)$ is an *open-fo.s* in $\mathcal{P}$. Hence, the proof is complete.

Theorem 4.4. Let $(\mathcal{P},\sigma)$, $(\mathcal{Q},\sigma^*)$ and (Z,σ^{**}) be fts's. Let $\psi: (\mathcal{P},\sigma) \longrightarrow (\mathcal{Q},\sigma^*)$ and $g: (\mathcal{Q},\sigma^*) \longrightarrow (Z,\sigma^{**})$ be two mappings. Then, $go\psi$ is

i. fo.irresolute mapping if ψ and g are fo.irresolute,
ii. fo.continuous if ψ is fo.irresolute and g is fo.continuous.

Proof. From Definition 4.1 and Definition 2.2 we can obtain the result.

Definition 4.2 Let $(\mathcal{P},\sigma)$ and $(\mathcal{Q},\sigma^*)$ be any two *fts's.* Let $\mathcal{I}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{I}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ be any two mappings. A mapping $\psi: (\mathcal{P},\sigma) \longrightarrow (\mathcal{Q},\sigma^*)$ is said to be

i. fuzzy orbit open (resp. closed) mapping (fo.open (resp. closed)) mapping, if the image of every open-(resp. closed-)fs in $\mathcal{P}$ is an open-(resp. closed-) fo.s in Q.

ii. fuzzy orbit irresolute open (resp. closed) mapping (fo.irresolute open (resp. irresolute closed), for short) mapping, if the image of every open-(resp. closed-) fo.s in $\mathcal{P}$ is an open-(resp. closed-) fo.s in Q.

The relationship between fo.open mappings and fo.irresolute open mappings is given in the following theorem.

Theorem 4.5. Let $(\mathcal{P},\sigma)$ and $(\mathcal{Q},\sigma^*)$ be any two *fts's.* If $\psi: (\mathcal{P},\sigma) \longrightarrow (\mathcal{Q},\sigma^*)$ *is* fo.open mapping, then ψ is fo.irresolute open mapping.

Proof. The proof uses only the fact every open-fo.s is an open-fs and the hypothesis.

The converse of Theorem 4.5 does not hold. We show that in the following example.

Example 4.3. Let $\mathcal{P} = \{k_1,k_2,k_3\}$ and $\mathcal{Q} = \{s_1,s_2,s_3\}$. Define $\sigma = \{\bar{0},\bar{1},\omega_1,\omega_2\}$ and $\sigma^* = \{\bar{0},\bar{1},\delta_1,\delta_2\}$ where $\omega_1,\omega_2 \in I^{\mathcal{P}}$ and $\delta_1,\delta_2 \in I^{\mathcal{Q}}$ such that

$$\omega_1 = \{(k_1,0.9),(k_2,0.5),(k_3,0.6)\},\ \omega_2 = \{(k_1,0.2),(k_2,0.2),(k_3,0.2)\},$$

$$\delta_1 = \{(s_1,0.5),(s_2,0.6),(s_3,0.9)\},\ \delta_2 = \{(s_1,0.2),(s_2,0.2),(s_3,0.2)\}$$

Clearly, $(\mathcal{P},\sigma)$ and $(\mathcal{Q},\sigma^*)$ are fts's. Define $\psi: (\mathcal{P},\sigma) \longrightarrow (\mathcal{Q},\sigma^*)$, $\mathcal{I}_1: \mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{I}_2: \mathcal{Q} \longrightarrow \mathcal{Q}$ as $\psi(k_1) = s_3, \psi(k_2) = s_1, \psi(k_3) = s_2, \mathcal{I}_1(k_1) = k_3, \mathcal{I}_1(k_2) = k_2, \mathcal{I}_1(k_3) = k_1$ and $\mathcal{I}_2(s_1) = s_1, \mathcal{I}_2(s_2) = s_2, \mathcal{I}_2(s_3) = s_3$. Then, ω_2 is an open-fo.s in $\mathcal{P}$ and $\psi(\omega_2) = \delta_2$ which is also open-fo.s in Q, so ψ is fo.irresolute open. But ψ does not fo.open mapping, since there is an open-fs ω_1 in $\mathcal{P}$ and $\psi(\omega_1) = \delta_1$ is not open-fo.s in Q.

Theorem 4.6. Let $(\mathcal{P},\sigma)$ and $(\mathcal{Q},\sigma^*)$ be any two *fts's* and let $\psi: (\mathcal{P},\sigma) \longrightarrow (\mathcal{Q},\sigma^*)$ is fo.irresolute open. If δ is a fuzzy set of Q and ω is a closed-fo.s in $\mathcal{P}$ containing $\psi^{-1}(\delta)$, then there exists a closed-fo.s η of Q containing δ such that $\psi^{-1}(\eta) \leq \omega$.

Proof. Let δ be a fuzzy set of Q and ω be a closed-fo.s in $\mathcal{P}$ such that $\psi^{-1}(\delta) \leq \omega$. Then, $\bar{1} - \omega$ is an open-fo.s in $\mathcal{P}$. By hypothesis $\psi(\bar{1} - \omega)$ is an open-fo.s in Q. Let $\eta = \bar{1} - \psi(\bar{1} - \omega)$ (i.e., η is a closed-fo.s in Q). Since $\psi^{-1}(\delta) \leq \omega$, we have $\bar{1} - \omega \leq \bar{1} - \psi^{-1}(\delta)$, implies $\psi(\bar{1} - \omega) \leq \psi(\bar{1} - \psi^{-1}(\delta)) = \bar{1} - \psi(\psi^{-1}(\delta)) \leq \bar{1} - \delta$. Hence, $\delta \leq \bar{1} - \psi(\bar{1} - \omega) = \eta$. Since ψ is fo.irresolute open, then η is a closed-fo.in Q and $\psi^{-1}(\eta) = \psi^{-1}(\bar{1} - \psi(\bar{1} - \omega)) = \bar{1} - \psi^{-1}(\psi(\bar{1} - \omega)) \leq \omega$. Consequently, $\psi^{-1}(\eta) \leq \omega$.

Theorem 4.7. Let $(\mathcal{P}, \sigma)$ and $(\mathcal{Q}, \sigma^*)$ be any two *fts's.* A mapping $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ is fo.irresolute open iff $\psi(Int_{FO}(\omega)) \leq Int_{FO}(\psi(\omega))$, for every fuzzy set ω of $\mathcal{P}$.

Proof. Suppose ψ is fo.irresolute open. Then, $\psi(Int_{FO}(\omega))$ is an open-fo.s in Q. Hence, $\psi(Int_{FO}(\omega)) = Int_{FO}(\psi(Int_{FO}(\omega))) \leq Int_{FO}(\psi(\omega))$.

Sufficiency, let ω be an open-fo.s in $\mathcal{P}$, then by hypothesis $\psi(Int_{FO}(\omega)) \leq Int_{FO}(\psi(\omega))$. Hence, $\psi(\omega)$ is an open-fo.s in Q.

Theorem 4.8. Let $(\mathcal{P}, \sigma)$, $(\mathcal{Q}, \sigma^*)$ and (Z, σ^{**}) be *fts's.* Let $\psi: (\mathcal{P}, \sigma) \longrightarrow (\mathcal{Q}, \sigma^*)$ and $g: (\mathcal{Q}, \sigma^*) \longrightarrow (Z, \sigma^{**})$ be fo.irresolute open mappings. Then, $go\psi$ is fo.irresolute open.

Proof. Straightforward from Definition 4.2.

References

[1] R. N. Majeed, S. A. El-Sheikh, *Fuzzy Orbit Topological Spaces*, IOP Conf. Series: Materials Science and Engineering 571 (2019) 012026 1-11.

[2] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.

[3] C. L. Chang, *Fuzzy Topological Spaces*, Journal of Mathematical Analysis and Applications 24 (1968) 182-190.

[4] S. A. Ghour, A. Fora, *On CDH Fuzzy Spaces*, Journal of Intelligent and Fuzzy Systems 30(2) (2016) 935-941.

[5] K. K. Azad, *Fuzzy Hausdorff Spaces and Fuzzy Perfect Mappings*, Journal of Mathematical Analysis and Applications 82 (1981) 297-305.

[6] H. C. Chamuah, B. C. Chetia, *Application of Fuzzy Topological Relation in Flood Prediction*, International Journal of Computer Applications 122(7) (2015) 8-13.

[7] D. N. Georgiou, B. K. Papadopoulos, *On Fuzzy Compactness*, Journal of Mathematical Analysis and Applications 233 (1999) 86-101.

[8] O. Tantawy, S. A. El-Sheikh, R. Naser, *Fuzzy Pairwise Separation Axioms in Fuzzy Bitopological Spaces*, Jökull Journal 63(12) (2013) 243-260.

[9] O. A. Tantawy, S.A. El-sheikh, R. N. Majeed, *A Note on "Separation Axioms in Fuzzy Bitopological Spaces*", Journal of Intelligent and Fuzzy Systems 28 (2015) 2243-2244.

[10] R. Malathi, M. K. Uma, *Fuzzy Orbit* Continuous Mappings*, Annals of Fuzzy Mathematics and Informatics 13(4) (2017) 465-474.

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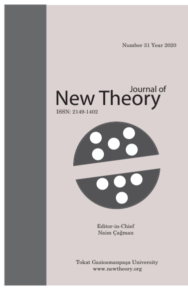

Hesitant Fuzzy h-ideals of Γ-hemirings

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Abstract — The purpose of this paper is to introduce and study hesitant fuzzy h-ideals (h-bi-ideals, h-quasi-ideals) of a Γ-hemiring. We investigate several properties these ideals. We show that hesitant fuzzy ideals are closed under intersection, cartesian product and composition. We also obtain some inter-relations between these ideals and characterizations of h-regular Γ-hemiring.

Keywords — *Fuzzy, hesitant, ideal, cartesian product, intersection, regular, Γ-hemiring.*

1. Introduction

Semiring, introduced by Vandiver [1] in 1934 with two associative binary operations where one distributes over the other. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups may hold in semirings, since semiring is a generalization of ring. Also, semirings has some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics. Ideals of semiring play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To ammend this gap Henriksen [2] defined a more restricted class of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings are given by Iizuka [3], which are called h-ideals. Torre [4], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to amend the gap between ring ideals and semiring ideals. The notion of Γ-semiring was introduced by Rao [5] as a generalization of Γ-ring as well as of semiring. Γ-semirings also includes ternary semirings and provide algebraic home to nonpositives cones of totally ordered rings.

The theory of fuzzy sets, proposed by Zadeh [6], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Since then several extensions and generalizations of fuzzy sets have been introduced in the literature, for example, intuitionistic fuzzy sets [7], interval valued fuzzy sets [8], fuzzy multisets [9] etc.. As an important generalization of these notions, in 2010, Torra [10] introduced the hesitant fuzzy set which permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1. The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy set theory has been applied to several practical problems, see [11–18]. Jun et al. [19] applied notion of hesitant fuzzy sets to semigroups and

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investigated several properties. Since then many researchers developed this ideas.
The main aim of this paper is to study some properties of ideals of Γ-hemiring using hesitant fuzzy set. We also obtain some characterizations.

2. Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-hemiring if there exists a mapping
$S \times \Gamma \times S \to S$ ((a,α,b) ↦ aαb) satisfying the following conditions:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi)$a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

A subset A of a Γ-hemiring S is called a left(resp. right) ideal of S if A is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset A of a Γ-hemiring S is called an ideal if it is both left and right ideal of S.
A subset A of a Γ-hemiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$.
A subset A of a Γ-hemiring S is called a bi-ideal if A is closed under addition and $A\Gamma S\Gamma A \subseteq A$.
A left ideal A of S is called a left h-ideal if $x, z \in S$, $a, b \in A$ and $x + a + z = b + z$ implies $x \in A$. A right h-ideal is defined analogously.

Definition 2.2. A fuzzy subset of a non-empty set S is defined as a function $\mu : S \to [0, 1]$.

Definition 2.3. Hesitant fuzzy set on S in terms of a function H that when applied to S returns a subset of [0, 1].

Throughout this paper unless otherwise mentioned S denotes the Γ-hemiring and for any two set P and Q, we use the following notation:

$$\cap(P, Q) = P \cap Q \text{ and } \cup(P, Q) = P \cup Q.$$

3. Hesitant fuzzy h-ideals

In this section, the notions of hesitant fuzzy ideals in Γ-hemiring are introduced and some of their basic properties are investigated.

Definition 3.1. Let H be a non empty hesitant fuzzy subset of a Γ-hemiring S. Then H is called a hesitant fuzzy left ideal [hesitant fuzzy right ideal] of S if

(i) $H(x + y) \supseteq \cap\{H(x), H(y)\}$

(ii) $H(x\gamma y) \supseteq H(y)$ [respectively $H(x\gamma y) \supseteq H(x)$].

for all $x, y \in S$ and $\gamma \in \Gamma$.
A hesitant fuzzy ideal of a Γ-hemiring S is a non empty hesitant fuzzy subset of S which is a hesitant fuzzy left ideal as well as a hesitant fuzzy right ideal of S.
Note that if H is a hesitant fuzzy left or right ideal of a Γ-hemiring S, then $H(0) \supseteq H(x)$ for all $x \in S$.

Definition 3.2. A hesitant fuzzy left ideal H of a Γ-hemiring S is called a hesitant fuzzy left h-ideal if for all $a, b, x, z \in S$, $x + a + z = b + z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$.

A hesitant fuzzy right h-ideal is defined similarly.

Example 3.3. Let $S = \Gamma$ =the set of non-positive integers. Then S forms a Γ-hemiring with usual addition and multiplication of integers. Define H be a hesitant fuzzy subset of S as follows

$$\begin{aligned} H(x) &= [0,1] \quad \text{if } \ x = 0 \\ &= 0.2 \cup (0.3, 0.8] \quad \text{if x is even} \\ &= [0.5, 0.7) \quad \text{if x is odd} \end{aligned}$$

The hesitant fuzzy subset H of S is a hesitant fuzzy ideal S.

Throughout this section, we prove results only for hesitant fuzzy left ideals. Similar results can be obtained for hesitant fuzzy right ideals and hesitant fuzzy ideals.

Definition 3.4. The characteristic hesitant fuzzy set of H of a set A is defined as

$$H_{\chi_A}(x) = \begin{cases} [0,1], & if \ x \in A; \\ \phi, & if \ x \notin A. \end{cases}$$

Definition 3.5. Let H_1 and H_2 be any two hesitant fuzzy sets of a Γ-hemiring S. Define intersection of H_1 and H_2 by

$$(H_1 \cap H_2)(x) = \cap(H_1(x), H_2(x))$$

for all $x \in S$.

Proposition 3.6. Intersection of a non-empty collection of hesitant fuzzy left h-ideals is a hesitant fuzzy left h-ideal of S.

Proof. Let $\{H_i : i \in I\}$ be a non-empty family of ideals of S. Let $x, y \in S$ and $\gamma \in \Gamma$.
Then

$$\begin{aligned} (\underset{i\in I}{\cap} H_i)(x+y) &= \underset{i\in I}{\cap}\{H_i(x+y)\} \supseteq \underset{i\in I}{\cap}\{\cap\{H_i(x), H_i(y)\}\} \\ &= \cap\{\underset{i\in I}{\cap} H_i(x), \underset{i\in I}{\cap} H_i(y)\} = \cap\{(\underset{i\in I}{\cap} H_i)(x), (\underset{i\in I}{\cap} H_i)(y)\}. \end{aligned}$$

Again

$$(\underset{i\in I}{\cap} H_i)(x\gamma y) = \underset{i\in I}{\cap}\{H_i(x\gamma y)\} \supseteq \underset{i\in I}{\cap}\{H_i(y)\} = (\underset{i\in I}{\cap} H_i)(y).$$

Hence $\underset{i\in I}{\cap} H_i$ is a hesitant fuzzy left ideal of S.

Suppose $x \in S$ be such that $x + a + z = b + z$, for $z, a, b \in S$. Then

$$\begin{aligned} (\underset{i\in I}{\cap} H_i)(x) &= \underset{x\in I}{\cap}\{\mu_i(x)\} \supseteq \underset{i\in I}{\cap}\{\cap\{H_i(a), H_i(b)\}\} \\ &= \cap\{\underset{i\in I}{\cap} H_i(a), \underset{i\in I}{\cap} H_i(b)\} = \cap\{(\underset{i\in I}{\cap} H_i)(a), (\underset{i\in I}{\cap} H_i)(b)\}. \end{aligned}$$

Therefore $\underset{i\in I}{\cap} H_i$ is a hesitant fuzzy left h-ideal of S. □

Proposition 3.7. Let $f : R \to S$ be a morphism of Γ-hemirings (see, [20])and H be a hesitant fuzzy left h-ideal of S, then $f^{-1}(H)$ is a hesitant fuzzy left h-ideal of R where $f^{-1}(H)(x) = H(f(x))$ for $x \in S$.

Proof. Let $f : R \to S$ be a morphism of Γ-hemirings.
Let H be a hesitant fuzzy left ideal of S and $r, s \in R$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} f^{-1}(H)(r+s) &= H(f(r+s)) = H(f(r) + f(s)) \\ &\supseteq \cap\{H(f(r)), H(f(s))\} = \cap\{f^{-1}(H)(r), f^{-1}(H)(s)\} \end{aligned}$$

Again $(f^{-1}(H))(r\gamma s) = H(f(r\gamma s)) = H(f(r)\gamma f(s)) \supseteq H(f(s)) = (f^{-1}(H))(s)$.

Thus $f^{-1}(H)$ is a hesitant fuzzy left ideal of R.

Suppose $x, a, b, z \in R$ be such that $x + a + z = b + z$. Then $f(x) + f(a) + f(z) = f(b) + f(z)$.
$(f^{-1}(H))(x) = H(f(x)) \supseteq \cap\{H(f(a)), H(f(b))\} = \cap\{f^{-1}(H)(a), f^{-1}(H)(b)\}$.
Therefore $f^{-1}(H)(x)$ is a hesitant fuzzy left h-ideal of R.

□

Definition 3.8. Let H_1 and H_2 be hesitant fuzzy subsets of X. The cartesian product of H_1 and H_2 is defined by

$$(H_1 \times H_2)(x, y) = \cap(H_1(x), H_2(y))$$

for all $x, y \in X$.

Theorem 3.9. Let H_1 and H_2 be two hesitant fuzzy left h-ideals of a Γ-hemiring S. Then $H_1 \times H_2$ is a hesitant fuzzy left h-ideal of the Γ-hemiring $S \times S$.

PROOF. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}(H_1 \times H_2)((x_1, x_2) + (y_1, y_2)) &= (H_1 \times H_2)(x_1 + y_1, x_2 + y_2) \\ &= \cap\{H_1(x_1 + y_1), H_2(x_2 + y_2)\} \\ &\supseteq \cap\{\cap\{H_1(x_1), H_1(y_1)\}, \cap\{H_2(x_2), H_2(y_2)\}\} \\ &= \cap\{\cap\{H_1(x_1), H_2(x_2)\}, \cap\{H_1(y_1), H_2(y_2)\}\} \\ &= \cap\{(H_1 \times H_2)(x_1, x_2), (H_1 \times H_2)(y_1, y_2)\}\end{aligned}$$

and

$$\begin{aligned}(H_1 \times H_2)((x_1, x_2)\gamma(y_1, y_2)) &= (H_1 \times H_2)(x_1\gamma y_1, x_2\gamma y_2) = \cap\{H_1(x_1\gamma y_1), H_2(x_2\gamma y_2)\} \\ &\supseteq \cap\{H_1(y_1), H_2(y_2)\} = (H_1 \times H_2)(y_1, y_2).\end{aligned}$$

Hence $H_1 \times H_2$ is a hesitant fuzzy left ideal of $S \times S$.
Now, let $(a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S \times S$ be such that $(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2)$ i.e., $(x_1 + a_1 + z_1, x_2 + a_2 + z_2) = (b_1 + z_1, b_2 + z_2)$. Then $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$ so that

$$\begin{aligned}(H_1 \times H_2)(x_1, x_2) &= \cap\{H_1(x_1), H_2(x_2)\} \\ &\supseteq \cap\{\cap\{H_1(a_1), H_1(b_1)\}, \cap\{H_2(a_2), H_2(b_2)\}\} \\ &= \cap\{\cap\{H_1(a_1), H_2(a_2)\}, \cap\{H_1(b_1), H_2(b_2)\}\} \\ &= \cap\{(H_1 \times H_2)(a_1, a_2), (H_1 \times H_2)(b_1, b_2)\}.\end{aligned}$$

Therefore $H_1 \times H_2$ is a hesitant fuzzy left h-ideal of $S \times S$. □

4. Hesitant fuzzy h-bi-ideals and h-quasi-ideals

Definition 4.1. Let H_1 and H_2 be two hesitant fuzzy sets of a Γ-hemiring S. Define composition of H_1 and H_2 by

$$\begin{aligned}H_1 o H_2(x) &= \bigcup_{x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H_1(a_i), H_1(c_i), H_2(b_i),\ H_2(d_i)\}\}] \\ &= \phi, \text{if x cannot be expressed as above}\end{aligned}$$

where $x, z, a_i, b_i, c_i, d_i \in S$, $\gamma_i, \delta_i \in \Gamma$ and i=1,...,n.

Lemma 4.2. Let H_1 and H_2 be two hesitant fuzzy h-ideal of a Γ-hemiring S. Then $H_1 o H_2 \subseteq H_1 \cap H_2 \subseteq H_1, H_2$.

PROOF. Suppose H_1 and H_2 be two hesitant fuzzy h-ideal of a Γ-hemiring S. Then

$$\begin{aligned}(H_1 o H_2)(x) &= \bigcup_{x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}\{\bigcap_i\{\cap\{H_1(a_i), H_1(c_i), H_2(b_i),\ H_2(d_i)\}\}\} \\ &\qquad \text{where } x, a_i, b_i, c_i, d_i \in S, \gamma_i, \delta_i \in \Gamma \text{ and } i = 1, ..., n. \\ &\subseteq \cup\{\bigcap_i\{H_1(a_i), H_1(c_i)\}\} \\ &\subseteq \bigcup_{x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}\{\cap\{H_1(\sum_{i=1}^{n} a_i\gamma_i b_i), H_1(\sum_{i=1}^{n} c_i\delta_i d_i)\}\} = H_1(x)\end{aligned}$$

Since this is true for every representation of x, $H_1oH_2 \subseteq H_1$.
Similarly we can prove that $H_1oH_2 \subseteq H_2$.
Therefore $H_1oH_2 \subseteq H_1 \cap H_2 \subseteq H_1, H_2$.
Hence the lemma. □

Definition 4.3. A hesitant fuzzy subset H of a Γ-hemiring S is called hesitant fuzzy h-bi-ideal if for all $a, b, x, y, z \in S$ and $\alpha, \beta \in \Gamma$ we have
(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$
(ii) $H(x\alpha y) \supseteq \cap\{H(x), H(y)\}$
(iii) $H(x\alpha y\beta z) \supseteq \cap\{H(x), H(z)\}$
(iv) $x+a+z = b+z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$

Definition 4.4. A hesitant fuzzy subset H of a Γ-hemiring S is called hesitant fuzzy h-quasi-ideal if for all $a, b, x, y, z \in S$ we have
(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$
(ii) $(HoH_{\chi_S}) \cap (H_{\chi_S}oH) \subseteq H$
(iii) $x+a+z = b+z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$

Theorem 4.5. A hesitant fuzzy subset H of a Γ-hemiring S is a hesitant fuzzy left h-ideal of S if and only if for all $a, b, x, y, z \in S$, we have
(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$
(ii) $H_{\chi_S}oH \subseteq H$.
(iii) $x+a+z = b+z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$.

Proof. Assume that H is a hesitant fuzzy left h-ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If x can be expressed as $x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z$, for $a_i, b_i, c_i, d_i \in S$, $\gamma_i, \delta_i \in \Gamma$ and i=1,...,n, then we have

$$\begin{aligned}(H_{\chi_S}oH)(x) &= \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H_{\chi_S}(a_i), H_{\chi_S}(c_i), H(b_i), H(d_i)\}\}] \\ &\subseteq \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H(a_i\gamma_i b_i), H(c_i\delta_i d_i)\}\}] \\ &\subseteq \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\cap\{H(\sum_{i=1}^{n} a_i\gamma_i b_i), H(\sum_{i=1}^{n} c_i\delta_i d_i)\}] = H(x).\end{aligned}$$

This implies that $H_{\chi_S}oH \subseteq H$.
Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of hesitant fuzzy left h-ideal. Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}H(x\gamma y) &\supseteq (H_{\chi_S}oH)(x\gamma y) = \bigcup_{x\gamma y+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H_{\chi_S}(a_i), H_{\chi_S}(c_i), H(b_i), H(d_i)\}\}] \\ &\supseteq H(y)(\text{since } x\gamma y + 0 + 0 = x\gamma y + 0).\end{aligned}$$

Hence H is a hesitant fuzzy left h-ideal of S. □

Theorem 4.6. Let H_1 and H_2 be a hesitant fuzzy right h-ideal and a hesitant fuzzy left h-ideal of a Γ-hemiring S, respectively. Then $H_1 \cap H_2$ is a hesitant fuzzy h-quasi-ideal of S.

Proof. Let x, y be any two elements of S. Then

$$\begin{aligned}(H_1 \cap H_2)(x+y) &= \cap\{H_1(x+y), H_2(x+y)\}\\ &\supseteq \cap\{\cap\{H_1(x), H_1(y)\}, \cap\{H_2(x), H_2(y)\}\}\\ &= \cap\{\cap\{H_1(x), H_2(x)\}, \cap\{H_1(y), H_2(y)\}\}\\ &= \cap\{(H_1 \cap H_2)(x), (H_1 \cap H_2)(y)\}.\end{aligned}$$

On the other hand, we have

$$((H_1 \cap H_2)oH_{\chi_S}) \cap (H_{\chi_S}o(H_1 \cap H_2)) \subseteq (H_1oH_{\chi_S}) \cap (H_{\chi_S}oH_2) \subseteq (H_1 \cap H_2).$$

Now let $a, b, x, z \in S$ such that $x + a + z = b + z$. Then

$$\begin{aligned}(H_1 \cap H_2)(x) &= \cap(H_1(x), H_2(x))\\ &\supseteq \cap(\cap(H_1(a), H_1(b)), \cap(H_2(a), H_2(b)))\\ &= \cap(\cap(H_1(a), H_2(a)), \cap(H_1(b), H_2(b)))\\ &= \cap((H_1 \cap H_2)(a), (H_1 \cap H_2)(b))\end{aligned}$$

This completes the proof. □

Lemma 4.7. Any hesitant fuzzy h-quasi-ideal of S is a hesitant fuzzy h-bi-ideal of S.

Proof. Let H be any hesitant fuzzy h-quasi-ideal of S. It is sufficient to show that $H(x\alpha y\beta z) \supseteq \cap\{H(x), H(z)\}$ and $H(x\alpha y) \supseteq \cap\{H(x), H(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.
In fact, by the assumption, we have

$$\begin{aligned}H(x\alpha y\beta z) &\supseteq ((HoH_{\chi_S}) \cap (H_{\chi_S}oH))(x\alpha y\beta z)\\ &= \cap\{(HoH_{\chi_S})(x\alpha y\beta z), (H_{\chi_S}oH)(x\alpha y\beta z)\}\\ &= \cap\{\underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i + p = \sum_{i=1}^{n} c_i\delta_i d_i + p}{\cup(\cap(H(a_i), H(c_i)), \cup(\cap(H(b_i), H(d_i))}\}\\ &\supseteq \cap\{H(x), H(z)\} \text{ since } x\alpha y\beta z + 0 + 0 = x\alpha y\beta z + 0.\end{aligned}$$

Similarly, we can show that $H(x\alpha y) \supseteq \cap\{H(x), H(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$.

□

Definition 4.8. A Γ-hemiring S is said to be h-hemiregular if for each $x \in S$, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

Theorem 4.9. Let S be a h-hemiregular Γ-hemiring. Then for any hesitant fuzzy right h-ideal H_1 and any hesitant fuzzy left h-ideal H_2 of S we have $H_1oH_2 = H_1 \cap H_2$.

Proof. Let S be a h-hemiregular Γ-hemiring. By Lemma 4.2, we have $H_1oH_2 \subseteq H_1 \cap H_2$.
For any $a \in S$, there exist $x, y, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a + a\alpha x\beta a + z = a\gamma y\delta a + z$.
Then

$$\begin{aligned}(H_1oH_2)(a) &= \underset{a+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}{\cup\{\cap\{H_1(a_i), H_1(c_i), H_2(b_i), H_2(d_i)\}\}}\\ &\supseteq \cap\{H_1(a\alpha x), H_1(a\gamma y), H_2(a)\}\\ &\supseteq \cap\{H_1(a), H_2(a)\} = (H_1 \cap H_2)(a).\end{aligned}$$

Therefore $(H_1 \cap H_2) \subseteq (H_1oH_2)$.
Hence $H_1oH_2 = H_1 \cap H_2$. □

Theorem 4.10. Let S be a h-hemiregular Γ-hemiring. Then

(i) $H \subseteq HoH_{\chi_S}oH$ for every hesitant fuzzy h-bi-ideal H of S.

(ii) $H \subseteq HoH_{\chi_S}oH$ for every hesitant fuzzy h-quasi-ideal H of S.

PROOF. (i) Suppose that H be any hesitant fuzzy h-bi-ideal of S and x be any element of S. Since S is h-hemiregular there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$. Now

$$\begin{aligned}(HoH_{\chi_S}oH)(x) &= \cup(\cap\{(HoH_{\chi_S})(a_i), (HoH_{\chi_S})(c_i), H(b_i), H(d_i)\}) \\ &\qquad x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z \\ &\supseteq \cap\{(HoH_{\chi_S})(x\alpha a), (HoH_{\chi_S})(x\gamma b)H(x)\} \\ &\supseteq \cap\{H(x), H(x)\} \\ &\qquad (since\ x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a, \\ &\qquad\quad x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b). \\ &= H(x)\end{aligned}$$

This implies that $H \subseteq HoH_{\chi_S}oH$.

(ii) This is straight forward from Lemma 4.7 □

Theorem 4.11. Let S be a h-hemiregular Γ-hemiring. Then

(i) $H_1 \cap H_2 \subseteq H_1oH_2oH_1$ for every hesitant fuzzy h-bi-ideal H_1 and every hesitant fuzzy h-ideal H_2 of S.

(ii) $H_1 \cap H_2 \subseteq H_1oH_2oH_1$ for every hesitant fuzzy h-quasi-ideal H_1 and every hesitant fuzzy h-ideal H_2 of S.

PROOF. (i) Suppose S is a h-hemiregular Γ-hemiring and H_1, H_2 be any hesitant fuzzy h-bi-ideal and hesitant fuzzy h-ideal of S, respectively and x be any element of S. Since S is h-hemiregular, there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

$$\begin{aligned}(H_1oH_2oH_1)(x) &= \cup(\cap\{(H_1oH_2)(a_i), (H_1oH_2)(c_i), H_1(b_i), H_1(d_i)\}) \\ &\qquad x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z \\ &\supseteq \cap\{(H_1oH_2)(xa), H_1oH_2)(xb), H_1(x)\} \\ &\supseteq \cap\{\cap\{H_1(x), H_2(a\beta x\alpha a), H_2(b\delta x\alpha a), H_2(a\beta x\gamma b), H_2(b\delta x\gamma b), H_1(x)\} \\ &\quad (since\ x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a, \\ &\qquad x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b) \\ &\supseteq \cap\{H_1(x), H_2(x)\} = (H_1 \cap H_2)(x).\end{aligned}$$

(ii) follows from Lemma 4.7. □

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References

[1] H.S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. 40 (1934) 914 - 920.

[2] M. Henriksen, *Ideals in semiringss with commutative addition*, Amer. Math. Soc. Notices, 6(1958) 321.

[3] K. Iizuka, *On the Jacobson radical of semiring*, Tohoku Math.J., 11(2) (1959)409-421

[4] D.R.La Torre, *On h-ideals and k-ideals in hemirings*,Publ. Math. Debrecen 12(1965), 219-226.

[5] M.M.K.Rao, *Γ-semirings-1*, Southeast Asian Bull. of Math., 19(1995), 49-54

[6] L.A.Zadeh, *Fuzzy Sets*, Information and Control, 8(1965),338 - 353.

[7] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1983) 87 - 96.

[8] P. Sarkar and S. Kar, *Interval-Valued Primary Fuzzy Ideal of Non-commutative Semigroup*, International Journal of Applied and Computational Mathematics, Volume 3, Issue 4,(2017) 3945 - 3960.

[9] J. Casasnovas andF Rossello, S*calar and fuzzy cardinalities of crisp and fuzzy multisets*, International Journal of Intellegent Systems, Vol. 24, Issue 6, (2009), 587 - 623.

[10] V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst. 25 (2010), 529 - 539.

[11] Y. B. Jun and S. Z. Song, *Hesitant fuzzy set theory applied to filters in MTL-algebras*, Honam Math. J. 36 (2014), no. 4, 813 - 830.

[12] R. M. Rodriguez, Luis Martinez and Francisco Herrera, *Hesitant fuzzy linguistic term sets for decision making*, IEEE Trans. Fuzzy Syst. 20, no. 1, (2012) 109 - 119.

[13] V. Torra and Y. Narukawa, *On hesitant fuzzy sets and decision*, in: The 18th IEEE International Conference on Fuzzy Systems, pp. 1378 - 1382, Jeju Island, Korea, 2009.

[14] G. Wei, *Hesitant fuzzy prioritized operators and their application to multiple attribute decision making*, Knowledge-Based Systems, 31 (2012), 176 - 182.

[15] M. Xia and Z. S. Xu, *Hesitant fuzzy information aggregation in decision making*, Internat. J. Approx. Reason., 52 (2011), no. 3, 395 - 407.

[16] M. Xia, Z. S. Xu, and N. Chen, *Some hesitant fuzzy aggregation operators with their application in group decision making*, Group Decision Negotiation, 22 (2013), 259 - 279.

[17] Z. S. Xu and M. Xia, *Distance and similarity measures for hesitant fuzzy sets*, Inform. Sci., 181 (2011), no. 11, 2128 - 2138.

[18] B. Zhu, Z. Xu, and M. Xia, *Hesitant fuzzy geometric Bonferroni means*, Inform. Sci., 205 (2012), 72 - 85.

[19] Y. B. Jun and M. Khan, *Hesitant fuzzy bi-ideals in semigroups*, Commun. Korean Math. Soc. 30 (2015), No. 3, 143 - 154

[20] T.K. Dutta, S.K. Sardar, *On matrix* Γ*-semirings*, Far East J. Math. Sci.,Vol 7, No. 1 (2002), 17 - 31.

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Novel Methods for Solving the Conformable Wave Equation

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Abstract — In this paper, a two-dimensional conformable fractional wave equation describing a circular membrane undergoing axisymmetric vibrations is formulated. It was found that the analytical solutions of the fractional wave equation using the conformable fractional formulation can be easily and efficiently obtained using separation of variables and double Laplace transform methods. These solutions are compared with the approximate solution obtained using the differential transform method for certain cases.

Keywords — *conformable derivative, wave equation, double Laplace transform, differential transform method*

1. Introduction

The fractional formulation of differential equations is an extension of the fractional calculus that was first introduced in 1695 when L'Hôpital and Leibniz discussed the extension of the integer order derivative to the derivative of order 1/2. Both Euler and Lacroix studied the fractional order derivative and defined the fractional derivative using the expression for the nth derivative of the power function [1]. Several physical and mechanical systems can be modeled more accurately using fractional derivative formulations due to the fact that many systems contain internal damping, which implies that it is impossible to derive equations describing the physical behavior of a non-conservative system using the classical energy based approach. The fractional derivative formulations can be successfully obtained in non-conservative systems by minimizing certain functionals with fractional derivative terms using some techniques from calculus of variations [2]. Several fractional formulations for derivatives and integrals such as Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, and Grünwald-Letnikov have been introduced with applications in science and engineering (refer to [1–6]).

While the classical definitions of fractional derivatives such as Riemann-Liouville and Caputo try to satisfy the fundamental properties of standard derivatives such as the derivatives of constant, product rule, quotient rule, and chain rule. None of the definitions are successful in their attempts other than the shared linear property between all the definitions of fractional derivatives [7]. Khalil et al. [8] put forward a new definition of fractional derivative named conformable fractional derivative as follows:

Definition 1.1. For $0 < \beta \leq 1$, given a function $f : [0, \infty) \to \Re$ such that for all $t > 0$ and $\beta \in (0, 1)$, the βth order conformable fractional derivative (CFD) of f, denoted by $G_\beta(f)(t)$, can be written as:

$$G_\beta(f)(t) = f^{(\beta)}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\beta}) - f(t)}{\varepsilon}. \quad (1)$$

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If f is β-differentiable in some $(0, b)$, $b > 0$, and the limit of $f^{(\beta)}(t)$ exists as t approaches 0^+, then by CFD definition:

$$f^{(\beta)}(0) = \lim_{t \to 0^+} f^{(\beta)}(t). \tag{2}$$

The CFD definition is an extension of the classical derivative that happens naturally and satisfies the properties of standard derivative. The conformable derivative of constant, the product rule, the quotient rule, and the chain rule all satisfy the standard formula of standard limit-based derivative [9]. Various conformable fractional forms have been introduced to many mathematical notions such as Norther's symmetry theorem and Action Principle for particles under frictional forces and have been shown to be much simpler than the ones with classical fractional derivative formulations such as Riemann-Liouville and Caputo [9]. For more applications of conformable fractional derivative, see also [10, 11]. G_β satisfies all the standard derivative properties in the following theorem [7, 8]:

Theorem 1.2. Assume that $0 < \beta \leq 1$, and f , h be β-differentiable at a point t, then:

(i) $G_\beta(mf + wh) = mG_\beta(f) + wG_\beta(h)$, for all $m, w \in \Re$.

(ii) $G_\beta(t^s) = st^{s-\beta}$, for all $s \in \Re$.

(iii) $G_\beta(fh) = fG_\beta(h) + hG_\beta(f)$.

(iv) $G_\beta(\frac{f}{h}) = \frac{hG_\beta(f) - fG_\beta(h)}{h^2}$.

(v) $G_\beta(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(vi) If f is a differentiable function, then $G_\beta(f)(t) = t^{1-\beta}\frac{df}{dt}$.

For more mathematical examples about each property in theorem 1, we refer to [7, 12]. Çenesiz and Kurt [10] discussed the possibility of applying the CFD definition for solving the two-dimensional and three-dimensional time fractional wave equation in rectangular domain. As a result, Çenesiz and Kurt [10] showed how the conformable fractional derivatives can easily and efficiently transform fractional differential equations into classical usual differential equations without the need for complicated methods to find the analytical solutions for partial fractional differential equations of higher dimensional systems. On the other hand, Tasbozanet et al. [11] discussed how to find the analytical traveling wave solutions in the sense of the conformable derivatives for nonlinear partial differential equations such as Nizhnik-Novikov-Veselov and Klein-Gordon equations by introducing a method consisting of a series of exponential functions, known as exp-function method, to study nonlinear evolution equations.

Recently, numerical and analytical solution methods to the conformable fractional differential equations are attracting attention from all over the world. Yavuz discussed in [13] some novel methods such as Adomian decomposition method and modified homotopy perturbation method for solving the initial boundary value problems in the sense of conformable fractional differentiation. Yavuz and Yaşkıran applied in [14] comfortable derivatives in modeling neuronal dynamics using methods of modified homotopy perturbation and reduced differential transform to solve the conformable fractional cable equation (CFCE). In addition, CFCE has also been solved in [15] using Adomian decomposition method and variational iteration method. In [16], conformable derivative has been successfully applied to solve the Black-Scholes equation of the European call option pricing models using Adomian decomposition method and modified homotopy perturbation method.

The CFD is a type of the local fractional derivative (LFD) [17]. The LFD has been successfully applied in modeling several applications in engineering such as the entropy (function of state) analysis of thermodynamic systems and the control theory of dynamic systems [18]. A new mathematical branch, known as fractal calculus, have been recently introduced in modeling various mathematical and engineering phenomena in hierarchical structures or porous media such as fractal kinetics [19], heat conduction in fractal medium [19], and the porous hairs of polar bear [20]. Research studies showed that there is a relation between the fractional order and the fractional dimension [19]. Several definitions of fractal derivatives have been proposed by researchers such as Chen's fractal derivative and Ji-Huan He's fractal derivative (HFD) [20]. However, some fractional derivatives lacks the physical and geometrical interpretation, therefore, the fractal calculus is very helpful in providing a physical interpretation for many fractional models in fractal media [19]. Both LFD and HFD have been applied extensively in science and engineering due to their accurate mathematical properties, physical insights, and geometrical interpretations [18–20]. The fractal derivative with fractal dimensions can be applied in modeling engineering problems and describing their discontinuous media [21] such as the applications of multi-scale fabrics and wool fibers by modeling their water permeation [20]. LFD and HFD have been defined in [18, 20] on a fractal space as follows:

Definition 1.3. For a fractal dimension, β, where $0 < \beta \leq 1$, given a set of non-differentiable functions with fractal dimension, say $C_\beta(a, b)$ such that for $\Phi(x) \in C_\beta(a, b)$, the βth order local fractional derivative (LFD) of $\Phi(x)$ at $x = x_0$, denoted by $D_x^{(\beta)}\Phi(x_0)$, can be written as:

$$D_x^{(\beta)}\Phi(x_0) = \Phi^{(\beta)}(x_0) = \frac{d^\beta \Phi(x)}{dx^\beta}|_{x=x_0} = \lim_{x\to x_0} \frac{\triangle^\beta(\Phi(x) - \Phi(x_0))}{(x - x_0)^\beta}, \tag{3}$$

where $\triangle^\beta(\Phi(x) - \Phi(x_0)) \cong \Gamma(1 + \beta)\triangle(\Phi(x) - \Phi(x_0))$.

Definition 1.4. Using figure 1 in [21], the fractal geometry describes the distance between two points, say x_a and x_b, in a discontinuous media i.e. porous medium such that M is supposed to be the smallest measure (thickness) in the given fractal media where any discontinuity less than this measure is neglected. Given a fractal dimension, say β, and constant, say ξ, the Ji-Huan He's fractal derivative (HFD) can be written [20, 21] as follows:

$$\frac{D\Phi(t)}{Dx^\beta} = \lim_{\Delta x\to M} \frac{\Phi(x_a) - \Phi(x_b)}{\xi M^\beta} = \Gamma(1 + \beta) \lim_{\Delta x\to M} \frac{\Phi(x_a) - \Phi(x_b)}{(x_a - x_b)^\beta}, \tag{4}$$

where $\Delta x = x_a - x_b$; and Δx tends only to M and it does not tend to 0. By using the fractal gradient [19], $\xi M^\beta = \frac{M^\beta}{\Gamma(1+\beta)}$ such that ξM^β is extremely small, but $\xi M^\beta > M$. For more applications using HFD in applied science and engineering, we refer to [19, 20, 22, 27].

In addition, He's fractional derivative (HFcD) has been applied for modeling several scientific phenomena (see [20, 23]). The physical and geometrical interpretations of the HFcD were discussed in [19, 27]. The following is the definition of HFcD [20, 23]:

Definition 1.5. Assume β to be the fractional dimension of the fractal medium, the He's fractional derivative (HFcD), denoted by $\frac{\partial^\beta}{\partial t^\beta}$, can be written as::

$$\frac{\partial^\beta \Psi}{\partial t^\beta} = \frac{1}{\Gamma(m - \beta)} \frac{d^m}{dt^m} \int_{t_0}^{t} (\xi - t)^{m-\beta-1}[\Psi_0(\xi) - \Psi(\xi)]d\xi, \tag{5}$$

where for a fractional-order problem in fractal media, the continuum partner of problem with the same initial and boundary conditions of the fractal partner has the same solution which is $\Psi_0(x, t)$ [20].

The conformable fractional derivative (CFD) is basically a generalized fractal derivative or q-derivative [24]. The q-derivative is very important in quantum calculus where the derivative is expressed using Leibniz's notation and the spacetime is discontinuous in quantum scales [27] (see also [25]). The generalized q-derivative (fractal derivative) using CFD definition 1 can be written [24] as follows:

Definition 1.6. Using definition 1, given a function $\Psi : [0, \infty) \to \Re$ such that for all $t > 0$ and $\beta \in (0, 1)$, and by assuming $q = 1 + \varepsilon t^{-\beta}$ where q tends to 1 and ε tends to 0, the generalized q-derivative (fractal derivative), denoted by $G_\beta(\Psi)(t)$, is written as:

$$G_\beta(\Psi)(t) = \Psi_q^{(\beta)}(t) = \lim_{q\to 1} \frac{\Psi(qt) - \Psi(t)}{qt^\beta - t^\beta} = \lim_{q\to 1} \frac{\Psi(qt) - \Psi(t)}{(q - 1)t^\beta}. \tag{6}$$

This generalized q-derivative coincides with definition 11 of q-derivative in [27].

Weberszpil and Chen in [26] showed that using the method of change of variables in part (vi) of theorem 1 to transform t to $1 + \frac{x}{l_0}$, the CFD is simply a Hausdorff derivative (HD) which is valid for differential functions. HD is a kind of fractal derivatives [28] that has been applied in various engineering phenomena to describe the physical behaviors and complex mechanics [29]. HD extends the modeling approach used in the classical continuum mechanics to fractal materials using the Hausdorff calculus [28]. Some examples of HD applications in science and engineering are anomalous diffusion, non-Guassian distribution, creep and relaxation in fractal media, and viscosity [28, 29].

CFD is simply a usual Newton derivative multiplied by the term $t^{1-\beta}$ [17]. The term $t^{1-\beta}$ in the definition 1 is basically a type of fractional conformable function (FCF) (see definition 5 in [17]) [17]. CFD combines the properties of usual derivative with the properties of fractional derivatives [30]. Therefore, CFD can be applied to extend and generalize theorems from the classical calculus such as integration by parts, mean value theorem, power series expansion, and Rolle's theorem [30]. From definition 1, the function is differentiable in the sense of conformable derivatives which implies that the Taylor power series expansion (TPSE) exists for CFD, while the other forms of fractional derivatives where functions are not differentiable, TPSE do not exist, when there are infinitely differentiable functions at some points [30]. As a result, several researchers got motivated to explore the CFD and apply it in modeling phenomena in applied science and engineering [30].

The CFD can be physically interpreted as a modified standard limit-based derivative in magnitude and direction [31]. Therefore, CFD is a special case of the well-known directional derivative (DD). The directional derivative is a kind of G$\widehat{a}$teaux derivative (GD). Zhao and Luo proposed in [17] a new generalized form of CFD named the general conformable fractional derivative (GRCFD) by extending and generalizing the definition of G$\widehat{a}$teaux derivative (see definition 2 in [17]) into Extended G$\widehat{a}$teaux derivative (see definition 3 in [17]) and Linear Extended G$\widehat{a}$teaux derivative (see definition 4 in [17]) together with the definition of CFD. The physical and geometrical interpretations of CFD were also discussed in [17] using GRCFD as a special case of CFD. Using definitions 2, 3, 4, and 5 and using $\Re^+$ as a space in [17] and definition 1 in this paper, GRCFD can be defined [17] as follows:

Definition 1.7. For $0 < \beta \leq 1$, given a fractional conformable function, say $\Omega(m, \beta)$, the general conformable fractional derivative (GRCFD) can be written as:

$$D^{\beta}_{\Omega} G_m = \lim_{\varepsilon \to 0} \frac{G(m + \varepsilon\Omega(m, \beta)) - G(m)}{\varepsilon}. \tag{7}$$

For the definition of GRCFD of arbitrary order, we refer to definition 7 in [17]. Since CFD is a modified version of Newton derivative, then the geometrical and physical meaning of CFD can be interpreted [17] as the slope of tangent where the value of the given function in the definition of G$\widehat{a}$teaux derivative in [17] changes as m (independent variable) changes ε, and the magnitude and direction of the velocity of particle are obtained from the ratio limit of the changes in function value. In addition, the Extended G$\widehat{a}$teaux derivative can be interpreted [17] as a special case of velocity of particle where the magnitude and direction of this velocity depends only on $\Omega(m, \varepsilon, \beta)$, while the physical meaning of the Linear Extended G$\widehat{a}$teaux derivative is just a modified version of usual velocity (as a multiple of usual velocity of particle) in magnitude and direction where this derivative can be geometrically represented [17] as the gradient of a given function, G , projected onto $\Omega(m, \beta)$ (we also refer to [32] for new proposed multiplicative (geometric) forms of conformable fractional derivatives and integrals).

In addition, Guzmán et al. [33] proposed a new definition of local fractional derivative known as the non-conformable fractional derivative (NCFD) which is also extended naturally from the usual derivative of a function in a point. NCFD can be defined as [33]: Given a function $\psi : [0, +\infty) \to \Re$. The NCFD, denoted by N-derivative of ψ of order β can be written as: $N^{\beta}_1 \psi(t) = \lim_{\varepsilon \to 0} \frac{\psi(t+\varepsilon e^{t^{-\beta}}) - \psi(t)}{\varepsilon}$, for all $t > 0$ and $\beta \in (0, 1)$. If the function ψ is β-differentiable in some $(0, b)$, and $\lim_{t \to 0^+} N^{(\beta)}_1 \psi(t)$ exists, then we have: $N^{(\beta)}_1 \psi(0) = \lim_{t \to 0^+} N^{(\beta)}_1 \psi(t)$. By comparing both CFD and NCFD, the angle of the tangent line to the curve in NCFD is not conserved, while in CFD is conserved [33]. For more new results about NCFD definition, we refer to [34]. Recently, several research studies with applications have been done using the definition NCFD such as the oscillatory character of liénard's system [36] (see also [35]), Laplace transform [37], and Hermite-Hadamard inequality [38].

Circular vibrating membrane problem (CVMP) has been applied in several applications in engineering such as industrial dynamic filtration modules and vibratory shear enhanced process (VSEP) for wastewater treatment systems [39, 40]. CVMP has been also used extensively in investigating the transverse vibration using a vibrating membrane in a linearly transverse direction and analyzing the modes of transverse vibratory motion [41]. CVMP studies the vibration of membranes (vibration equation) which has many practical applications in industry and bioengineering [42]. Studying the two-dimensional analysis of wave mechanics and propagation in CVMP is very important in building the components of microphones, speakers, and some medical and industrial instruments [42].

In this paper, we formulate the two-dimensional time fractional wave partial differential equation in the sense of conformable fractional derivative for a circular membrane undergoing axisymmetric vibrations, and we solve it using the methods of separation of variables, double Laplace transform, and reduced differential transform. We compare and discuss all obtained approximate solutions using those methods and the error between analytical and approximate solutions.

In Section 2, the conformable fractional wave partial differential equation is solved using the methods of separation of variables, double Laplace transform, and reduced differential transform. In Section 3, we discuss the error between analytical and approximate solutions from section 2, and we compare all results with the classical analytical solution from [43, 44]. In Section 4, the conclusion of this study is presented.

2. Conformable fractional wave equation

In this section, we investigate the conformable fractional mixed initial- boundary value problem of a circular membrane [44] of radius R and constant density ρ_o where the initial vibration conditions are radially symmetric or axisymmetric. Under such conditions, polar coordinates (r, θ) can be introduced such that $m(x, y, t) = M(r, t)$ where the displacement is independent of θ, and the initial displacement and velocity functions can be written as $q(r)$ and $n(r)$, respectively. The laplacian in polar coordinates can be written as:

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right). \tag{8}$$

Since the initial vibration conditions are axisymmetric, they are dependent only on the radial distance r from the center of the circle. Hence, $q(r)$ and $p(r)$ do not depend on θ, and instead they depend only on r, and from equation (3), the term $\frac{\partial^2}{\partial \theta^2} = 0$. Consequently, the governing system of equations for a circular membrane undergoing axisymmetric vibrations can be mathematically modeled by the following two-dimensional wave partial differential equation equation in the sense of CFD:

$$\frac{\partial^{2\beta} M}{\partial t^{2\beta}} = c_o^2\left(\frac{\partial^2 M}{\partial r^2} + \frac{1}{r}\frac{\partial M}{\partial r}\right). \tag{9}$$

$0 < r < R$; $t > 0$; $0 < \beta \leq 1$; and $c_o^2{=}(\tau_o/\rho_o)$ where τ_o is the assumed to be the constant value of the elastic membrane stretch-resisting restorative force per unit length or surface tension. Equation (9) is subjected to the following boundary and initial conditions:

$$M(R, t) = 0;\text{ and } M(r, t) \text{ bounded as } r \to 0 \text{ for } t > 0. \tag{10}$$

$$M(r, 0) = q(r);\text{ and } \frac{\partial^\beta M}{\partial t^\beta}(r, 0) = p(r);\text{ for } 0 < r < R \text{ and } 0 < \beta \leq 1. \tag{11}$$

The problem is divided into two main parts; analytical solution part and approximate solution part:

2.1. The analytical solution by the separation of variables method

By using the separation of variables method, we let $M(r, t) = V(r)G(t)$ to be the solution form of the governing conformable fractional wave partial differential equation and boundary conditions. The following is obtained from substituting the assumed solution form in equation (9):

$$\frac{d^{2\beta}G(t)}{dt^{2\beta}}V(r) = c_o^2\left(\frac{d^2V(r)}{dr^2}G(t) + \frac{1}{r}\frac{dV(r)}{dr}G(t)\right). \tag{12}$$

Dividing both sides of equation (12) by c_o^2, left hand side term of equation (12) by $G(t)$, and the two terms of the right hand side of equation (12) by $V(r)$, we obtain:

$$\frac{d^{2\beta}G(t)}{dt^{2\beta}}\frac{1}{G(t)c_o^2} = c_o^2\left(\frac{d^2V(r)}{dr^2}\frac{1}{V(r)} + \frac{1}{r}\frac{dV(r)}{dr}\frac{1}{V(r)}\right) \equiv -\lambda^2. \tag{13}$$

where λ is the separation constant. As a result, the following two equations are obtained:

$$\frac{d^{2\beta}G(t)}{dt^{2\beta}} + c_o^2 G(t)\lambda^2 = 0. \tag{14}$$

$$\frac{d^2V(r)}{dr^2} + \frac{1}{r}\frac{dV(r)}{dr} + \lambda^2 V(r) = 0. \tag{15}$$

From equation (14), it is necessary to introduce the sequential CFD from [45] as follows:

Definition 2.1. For $0 < \beta < 1$, and $n \in \mathbb{Z}^+$, given a function $f : [0,\infty) \to \Re$, the nth order of sequential CFD can be generally written as:
${}^{(n)}G_\beta f(t)=G_\beta G_\beta G_\beta ... G_\beta f(t)$. Let's consider the $f : [0,\infty) \to \Re$ to be a second continuously differentiable function [45] and $\beta \in (0, 0.5]$, then the 2^{nd} order of sequential CFD is written as:

$$^{(2)}G_\beta f(t) = G_\beta G_\beta f(t) = \begin{cases} (1-\beta)t^{1-2\beta}f^{(1)}(t) + t^{2-2\beta}f^{(2)}(t) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases} \tag{16}$$

By using the sequential CFD definition and property (vi) from theorem (1), equation (14) can be re-written as:

$$(1-\beta)t^{1-2\beta}G^{(1)}(t) + t^{2-2\beta}G^{(2)}(t) + c_o^2 G(t)\lambda^2 = 0. \tag{17}$$

Multiplying both sides equation (15) by r^2 to make calculations simple, we obtain:

$$r^2\frac{d^2V(r)}{dr^2} + r\frac{dV(r)}{dr} + r^2\lambda^2 V(r) = 0. \tag{18}$$

Let's now introduce the change of variables [44]: $s = \lambda r$ for $V(r) = \psi(s)$ such that for $\frac{dV(r)}{dr}$, it is transformed into the following:

$$\frac{dV(r)}{dr} = \frac{d\psi(s)}{dr}(s) = \frac{d\psi(s)}{ds}\frac{ds}{dr}(s) = \lambda\frac{d\psi(s)}{ds}. \tag{19}$$

Similarly, for $\frac{d^2V(r)}{dr^2}$, it is transformed into the following:

$$\frac{d^2V(r)}{dr^2} = \frac{d}{dr}\left(\lambda\frac{d\psi(s)}{ds}(s)\right) = \lambda^2\frac{d^2\psi}{ds^2}(s). \tag{20}$$

Substituting $r = (\frac{s}{\lambda})$ and results from (19) and (20) in equation (18), we obtain the following equation:

$$s^2\frac{d^2\psi(s)}{ds^2} + s\frac{d\psi(s)}{ds} + s^2\psi(s) = 0; \text{ for } 0 < s < \lambda R. \tag{21}$$

From the boundary condition in (10), $\psi(s)$ in equation (21) is also bounded as $s \to 0$, and $\psi(\lambda R) = 0$. By using the results from the eigenvalue problem involving the Bessel function of the first kind of order zero in [43, 44], we have: $V(R) = 0 \to J_o(\lambda R) = 0$ where λR is the root of Bessel function J_o, and $V(r) = J_o(\lambda r)$. Hence, it can be concluded that for $n \in \mathbb{Z}^+$, $\lambda_n = \frac{\xi_n}{R}$, and $V_n(r) = J_o(\frac{\xi_n r}{R})$ is the corresponding solution to equation (10) where J_o has infinitely many positive zeros such that $\xi_1 < \xi_2 < \xi_3 < ... < \xi_n$ where ξ_n is the n^{th} positive zero of the Bessel function J_o.

For equation (17), the WolframAlpha computational intelligence solver is used to obtain the following solution:

$$G_n(t) = E_n cos\left(\frac{c_o\lambda_n t^\beta}{\beta}\right) + K_n sin\left(\frac{c_o\lambda_n t^\beta}{\beta}\right); \text{for } n \in \mathbb{Z}^+. \tag{22}$$

By substituting $\lambda_n = \frac{\xi_n}{R}$ in equation (22), the solution can be re-written as:

$$G_n(t) = E_n cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right) + K_n sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right); \text{for } n \in \mathbb{Z}^+. \tag{23}$$

By using the superposition principle, the general solution for the conformable fractional mixed initial-boundary value can be written as a linear combination of both $V_n(r)$ and $G_n(t)$:

$$\begin{aligned} M(r,t) &= \sum_{n=1}^{\infty} M_n(r,t) = \sum_{n=1}^{\infty} V_n(r)G_n(t) = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) \\ &\times \left[E_n cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right) + K_n sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right]; \text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{24}$$

To find the coefficients, E_n and K_n, from equation (24) so the general solution satisfies the initial conditions in (11), the first condition $M(r,0) = q(r)$ is substituted in equation (24) as follows:

$$\begin{aligned} M(r,0) = q(r) &= \sum_{n=1}^{\infty} M_n(r,0) = \sum_{n=1}^{\infty} V_n(r)G_n(0) = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) \\ &\times \left[E_n cos\left(\frac{c_o\xi_n (0)^\beta}{\beta R}\right) + K_n sin\left(\frac{c_o\xi_n (0)^\beta}{\beta R}\right)\right] \\ = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) &\times \left[E_n cos\,(0) + K_n sin\,(0)\right] = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) E_n; \text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{25}$$

For the second initial condition, $\frac{\partial M}{\partial t}(r,0) = p(r)$, in (11), we first find $\frac{\partial^\beta M}{\partial t^\beta}(r,t)$ from equation (24) using the two examples from [7] where $G_\beta(sin(\frac{t^\beta}{\beta})) = cos(\frac{t^\beta}{\beta})$ and $G_\beta(cos(\frac{t^\beta}{\beta})) = -sin(\frac{t^\beta}{\beta})$, and our previous conclusion $\lambda_n = \frac{\xi_n}{R}$ as follows:

$$\begin{aligned} \frac{\partial^\beta M}{\partial t^\beta}(r,t) = p(r) &= \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)\left(\frac{c_o\xi_n}{R}\right) \\ \times \left[-E_n sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right) \right. &+ \left. K_n cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right]; \\ &\text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{26}$$

We now substitute $\frac{\partial M}{\partial t}(r,0) = p(r)$ in equation (26) as follows:

$$\begin{aligned} \frac{\partial^\beta M}{\partial t^\beta}(r,0) = p(r) &= \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)\left(\frac{c_o\xi_n}{R}\right) \\ \times \left[-E_n sin\left(\frac{c_o\xi_n 0^\beta}{\beta R}\right) \right. &+ \left. K_n cos\left(\frac{c_o\xi_n 0^\beta}{\beta R}\right)\right] \\ = \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)\left(\frac{c_o\xi_n}{R}\right) &\left[-E_n sin\,(0) + K_n cos\,(0)\right] \\ = \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right) &\left(\frac{c_o\xi_n}{R}\right) K_n; \text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{27}$$

Using the orthogonality property of Bessel function $J_0(\frac{\xi_n r}{R})$ and representing the normalization constant in terms of $J_1(\xi_n)$, we obtain the following:

$$E_n = \frac{\left\langle q(r), J_0\left(\frac{\xi_n r}{R}\right)\right\rangle_r}{\left\| J_0\left(\frac{\xi_n r}{R}\right)\right\|_r^2} = \frac{2}{R^2 J_1^2(\xi_n)} \int_0^R r q(r) J_0\left(\frac{\xi_n r}{R}\right) \mathrm{d}r; \text{for } n \in \mathbb{Z}^+. \tag{28}$$

$$K_n = \left(\frac{R}{c_o\xi_n}\right)\frac{\left\langle p(r), J_0\left(\frac{\xi_n r}{R}\right)\right\rangle_r}{\left\|J_0\left(\frac{\xi_n r}{R}\right)\right\|_r^2} = \frac{2}{Rc_o\xi_n J_1^2(\xi_n)}\int_0^R rp(r)J_0\left(\frac{\xi_n r}{R}\right)\,\mathrm{d}r; \quad \text{for } n \in \mathbb{Z}^+. \tag{29}$$

By substituting the results from (28) and (29) in equation (24), the most general solution for the conformable fractional mixed initial-boundary value problem emerging from the separation of variables method can be written as follows:

$$\begin{aligned} M(r,t) = &\sum_{n=1}^{\infty}\left[\left(\frac{2}{R^2J_1^2(\xi_n)}\int_0^R q(r)J_0\left(\frac{\xi_n r}{R}\right)r\,\mathrm{d}r\right)cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right]J_0\left(\frac{\xi_n r}{R}\right)\\ &+\sum_{n=1}^{\infty}\left[\left(\frac{2}{Rc_o\xi_n J_1^2(\xi_n)}\int_0^R p(r)J_0\left(\frac{\xi_n r}{R}\right)r\,\mathrm{d}r\right)sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right]J_0\left(\frac{\xi_n r}{R}\right); \\ &\text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{30}$$

2.2. The analytical solution by the conformable fractional double Laplace transform method

The classical Laplace transform method for a function of single variable has been used extensively in solving ordinary differential equations and partial differential equations. Double Laplace transform and other multiple Laplace transformations were introduced by Estrin and Higgins in [46] to solve partial differential equations. Double Laplace transform (DLT) has been rarely introduced or not at all for certain cases in the literature for solving partial differential equations [47]. Introducing double Laplace transform to solve the fractional differential equations is an open math problem [48]. Eltayeb and Kılıçman in [49] used the DLT and Sumudu transform methods to solve non-fractional one-dimensional wave equation with variable coefficients (see also [50, 51]). There are some recent research studies on solving fractional differential equations such as heat and telegraph equations in the sense of Caputo derivatives [48, 52].

To define the conformable fractional double Laplace transform, let's first define the conformable fractional integral (CFI) [31] as follows:

Definition 2.2. For $0 < \beta \le 1$, given a function $f : [0,\infty) \to \Re$ such that for all $t \ge 0$, the βth order conformable fractional integral (CFI) of f from 0 to t can be written as:

$$I_\beta(f)(t) = \int_0^t f(\psi)d_\beta\psi = \int_0^t f(\psi)\psi^{\beta-1}d\psi. \tag{31}$$

If $\beta = 1$, then $I_\beta(f)(t) = I_{\beta=1}(t^{\beta-1}f)(t)$ which is the classical improper Riemann integral of a function $f(t)$. For $0 < \beta \le 1$, given a continuous function f on $(0,\infty)$, then $G_\beta(f)(t)\,[I_\beta(f)(t)] = f(t)$.

Let's now define the conformable fractional double Laplace transform (CFDLT) as follows:

Definition 2.3. For $0 < \beta \le 1$, given a function $M(r,t) : [0,\infty) \to \Re$ such that for all $r, t > 0$, the βth order conformable fractional double Laplace transform (CFDLT) of $M(r,t)$, denoted by $\ell_\beta^{rt}[M(r,t)]$, starting from 0 can be written as:

$$\begin{aligned} \ell_\beta^{rt}[M(r,t)] = \ell_\beta^{r}\ell_\beta^{t}[M(r,t)] = \underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) &= \int_0^\infty e^{-s_a\frac{r^\beta}{\beta}}\int_0^\infty e^{-s_b\frac{t^\beta}{\beta}}M(r,t)\,d_\beta t\,d_\beta r\\ &= \int_0^\infty\int_0^\infty e^{-(s_a\frac{r^\beta}{\beta}+s_b\frac{t^\beta}{\beta})}M(r,t)\,d_\beta r\,d_\beta t \\ &= \int_0^\infty\int_0^\infty e^{-(s_a\frac{r^\beta}{\beta}+s_b\frac{t^\beta}{\beta})}M(r,t)r^{\beta-1}t^{\beta-1}\,dr\,dt, \end{aligned} \tag{32}$$

where $s_a, s_b \in \mathbb{C}$. The above definition is true provided that the above integral exists. Previously, it is assumed that $M(r,t) = V(r)G(t)$. By using definition (9), the CFDLT can be written [52]:

$$\ell_\beta^{rt}[V(r)G(t)] = \ell_\beta^r \ell_\beta^t [V(r)G(t)] = \underline{\mathrm{V}}_\beta(s_a)\underline{\mathrm{G}}_\beta(s_b) = \ell_\beta^r[V(r)]\ell_\beta^t[G(t)]. \tag{33}$$

Let's show the CFDLT of the second-order conformable fractional partial derivative (CFPD) with respect to t [53] as follows:

$$\begin{aligned} \ell_\beta^{rt}\left[\frac{\partial^{2\beta}}{\partial t^{2\beta}}M(r,t)\right] &= \int_0^\infty \int_0^\infty e^{-s_a\frac{r^\beta}{\beta}} e^{-s_b\frac{t^\beta}{\beta}} \frac{\partial^{2\beta} M}{\partial t^{2\beta}}(r,t)\, d_\beta r\, d_\beta t \\ &= \int_0^\infty e^{-s_a\frac{r^\beta}{\beta}} \int_0^\infty \left\{ e^{-s_b\frac{t^\beta}{\beta}} \frac{\partial^{2\beta} M}{\partial t^{2\beta}}(r,t)\, d_\beta t\right\} d_\beta r. \end{aligned} \tag{34}$$

To find the above inner integral, let's use the theorem 3.1 of conformable fractional integration by parts and lemma 2.8 in [2] in addition to definition (7) to obtain the following:

$$\begin{aligned} &\ell_\beta^{rt}\left[\frac{\partial^{2\beta}}{\partial t^{2\beta}}M(r,t)\right] \\ &= \int_0^\infty e^{-s_a\frac{r^\beta}{\beta}} \left\{ e^{-s_b\frac{t^\beta}{\beta}} M(r,t)|_{t=0}^{\infty} - \int_0^\infty \left(\frac{\partial^{2\beta}}{\partial t^{2\beta}} e^{-s_b\frac{t^\beta}{\beta}}\right) M(r,t)\, d_\beta t\right\} d_\beta r \\ &= \int_0^\infty e^{-s_a\frac{r^\beta}{\beta}} M(r,0) r^{\beta-1} dr \\ &+ \int_0^\infty \int_0^\infty \left[s_b^2 \frac{e^{-s_b\frac{t^\beta}{\beta}}}{t^{\beta-1}} - s_b \frac{e^{-s_b\frac{t^\beta}{\beta}}}{t^\beta}(1-\beta)\right] e^{-s_b\frac{t^\beta}{\beta}} e^{-s_a\frac{r^\beta}{\beta}} r^{\beta-1}\, dt\, dr \\ &= s_b^{2\beta}\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - s_b^{2\beta-1}\underline{\mathrm{M}}_\beta^{rt}(s_a,0) - s_b^{2\beta-2}(\underline{\mathrm{M}}_\beta^{rt})_t(s_a,0). \end{aligned} \tag{35}$$

As a result, The CFDLT of the first-order conformable fractional partial derivative (CFPD) with respect to t can be similarly written as:

$$\ell_\beta^{rt}\left[\frac{\partial^\beta}{\partial t^\beta}M(r,t)\right] = s_b^\beta \underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - s_b^{\beta-1}\underline{\mathrm{M}}_\beta^{rt}(s_a,0). \tag{36}$$

The CFDLT of the first-order conformable fractional partial derivative (CFPD) with respect to t can be also generally written as:

$$\ell_\beta^{rt}\left[\frac{\partial^\beta}{\partial t^\beta}M(r,t)\right] = s_b^\beta \underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - \sum_{\gamma=0}^{\zeta-1} s_b^{\beta-1-\gamma}\ell_r\left[\frac{\partial^\gamma M(r,0)}{\partial t^\gamma}\right]. \tag{37}$$

The double Laplace transform in (37) coincides with the general form of the double Laplace transform of the partial fractional derivatives in the sense of Caputo derivatives in [48, 52]. The complex double integral formula in [47, 52] can be used to write the inverse conformable fractional double Laplace transform, denoted by $(\ell_\beta^{rt})^{-1}[\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)]$, as follows:

Definition 2.4. For $0 < \beta \leq 1$, given an analytic function $\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)$ for all $s_a, s_b \in \mathbb{C}$ such that both s_a and s_b are defined [52] by $Re\{s_a \geq \varrho\}$ and $Re\{s_b \geq \varsigma\}$, where $\varrho, \varsigma \in \Re$, the inverse conformable fractional double Laplace transform (ICFDLT) can be written as follows:

$$\begin{aligned} &(\ell_\beta^{rt})^{-1}[\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)] = (\ell_\beta^r)^{-1}\ell_\beta^t)^{-1}[\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)] = \\ &M(r,t) = \frac{1}{2\pi i}\int_{\varrho-i\infty}^{\varrho+i\infty} e^{s_a r} ds_a \frac{1}{2\pi i}\int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{s_b t}\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) ds_b \\ &= \frac{-1}{4\pi^2}\int_{\varrho-i\infty}^{\varrho+i\infty}\int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{s_a r}e^{s_b t}\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)\, ds_a\, ds_b. \end{aligned} \tag{38}$$

Let's prove the existence and uniqueness of CFDLT in the following theorem:

Theorem 2.5. For $0 < \beta \leq 1$, given a continuous exponential-order function $M(r,t) : [0,\infty) \rightarrow \Re$ such that for some $\varrho, \varsigma \in \Re$ and $s_a, s_b \in \mathbb{C}$ where $Re\{s_a > \varrho\}$ and $Re\{s_b > \varsigma\}$, then there exists a conformable fractional double Laplace transform of $M(r,t)$, denoted by $\underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b)$, for both s_a and s_b.

PROOF. Since $M(r,t)$ is a continuous exponential-order function $M(r,t) : [0,\infty) \rightarrow \Re$ such that for some $\varrho, \varsigma \in \Re$ and $s_a, s_b \in \mathbb{C}$ on the interval $[0,\infty) = \{r,t|0 \leq r,t < \infty\}$, then $\exists L \in \mathbb{Z}_+$ such that $\forall s_a > S_a$ and $s_b > S_b$ [47, 48] as follows:

$$|M(r,t)| \leq L\, e^{\varrho \frac{r^\beta}{\beta} + \varsigma \frac{t^\beta}{\beta}}, \tag{39}$$

$$\text{Examine: } \sup_{r,t>0} \left| \frac{M(r,t)}{e^{\omega \frac{r^\beta}{\beta} + \mu \frac{t^\beta}{\beta}}} \right| < 0, \text{then we have the following:}$$

$$\begin{aligned}
&\lim_{(r,t)\rightarrow\infty} e^{-\omega \frac{r^\beta}{\beta} - \mu \frac{t^\beta}{\beta}} |M(r,t)| = L\, e^{-(\omega-\varrho)r^{\frac{r^\beta}{\beta}}} e^{-(\mu-\varsigma)r^{\frac{t^\beta}{\beta}}} = 0;\ \forall\, \omega > \varrho;\ \mu > \varsigma \\
&\text{Similarly,} |\underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b)| = \left| \int_0^\infty \int_0^\infty e^{-(s_a \frac{r^\beta}{\beta} + s_b \frac{t^\beta}{\beta})} M(r,t)\, d_\beta r\, d_\beta t \right| \\
&= \left| \int_0^\infty \int_0^\infty e^{-(s_a \frac{r^\beta}{\beta} + s_b \frac{t^\beta}{\beta})} M(r,t) r^{\beta-1} t^{\beta-1}\, dr\, dt \right| \\
&\leq L \int_0^\infty \int_0^\infty e^{-((s_a-\varrho) \frac{r^\beta}{\beta} + (s_b-\varsigma) \frac{t^\beta}{\beta})} M(r,t) r^{\beta-1} t^{\beta-1}\, dr\, dt \\
&= \int_0^\infty e^{-(s_a-\varrho)\frac{r^\beta}{\beta}} r^{\beta-1}\, dr \int_0^\infty e^{-(s_b-\varsigma)\frac{t^\beta}{\beta}} t^{\beta-1}\, dt \\
&= \frac{L}{(s_a-\varrho)(s_b-\varsigma)};\ \forall\, Re\{s_a > \varrho\},\ Re\{s_b > \varsigma\}.
\end{aligned} \tag{40}$$

Since the $\lim_{(s_a,s_b)\rightarrow\infty} |\underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b)| = \lim_{(s_a,s_b)\rightarrow\infty} \underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b) = 0$ [47], then the conformable fractional double Laplace transform (CFLT) of $M(r,t)$ exists and can be written as (32) $\forall\, s_a > \varrho,\ s_b > \varsigma$. □

Numerical Experiment 1:

By using the above definitions and theorems of the CFDLT, let's solve the mixed initial-boundary value problem (equation (9)) subject to the following boundary and initial conditions:

$$M(R,t) = 0; \text{ and } M(r,t) \text{ bounded as } r \rightarrow 0 \text{ for } t > 0. \tag{41}$$

$$M(r,0) = 0; \text{ and } \frac{\partial^\beta M}{\partial t^\beta}(r,0) = cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right); \text{ for } 0 \leq r < R \text{ and } 0 < \beta \leq 1. \tag{42}$$

Let's apply the CFDLT method to equation (9), the following is obtained:

$$\begin{aligned}
&s_b^{2\beta} \underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b) - s_b^{2\beta-1} \underline{\mathrm{M}}_{\beta}^{rt}(s_a, 0) - s_b^{2\beta-2} (\underline{\mathrm{M}}_{\beta}^{rt})_t(s_a, 0) \\
&= c_o^2 \left(\frac{\partial^2 \underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b)}{\partial r^2} + \frac{1}{r} \frac{\partial \underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b)}{\partial r} \right).
\end{aligned} \tag{43}$$

Similarly, let's apply the conformable fractional single Laplace transform of the initial conditions in (42):

$$\underline{\mathrm{M}}_{\beta}^{rt}(s_a, 0) = 0; \text{ and } (\underline{\mathrm{M}}_{\beta}^{rt})_t(s_a, 0) = \left(\frac{s_b}{s_b^2 + \left(\frac{1}{\beta}\right)^2} + \left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2 + \left(\frac{1}{\beta}\right)^2} \right) \right). \tag{44}$$

By substituting the initial conditions of (44) in equation (43), we obtain:

$$s_b^{2\beta}\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - s_b^{2\beta-1}(0) - s_b^{2\beta-2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2} + \left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right) = c_o^2\left(\frac{\partial^2\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)}{\partial r^2} + \frac{1}{r}\frac{\partial\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)}{\partial r}\right). \tag{45}$$

Let's simplify (45) to obtain the following:

$$s_b^{2\beta}\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - \frac{s_b^{2\beta}}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2} + \left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right) = c_o^2\left(\frac{\partial^2\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)}{\partial r^2} + \frac{1}{r}\frac{\partial\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)}{\partial r}\right). \tag{46}$$

By taking $s_b^{2\beta}$ as a common factor on the left side of (46) and dividing both sides by c_o^2, we obtain the following:

$$\frac{s_b^{2\beta}}{c_o^2}\left(\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - \frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2} + \left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right) = \left(\frac{\partial^2\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)}{\partial r^2} + \frac{1}{r}\frac{\partial\underline{\mathrm{M}}_\beta^{rt}(s_a,s_b)}{\partial r}\right). \tag{47}$$

Assume that $\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b) = \underline{\mathrm{M}}_\beta^{rt}(s_a,s_b) - \frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2} + \left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right).$ (48)

By applying the assumption in (48) on (47) and combine the left-hand side term with the right-hand side terms together, the following is obtained:

$$\frac{\partial^2\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r^2} + \frac{1}{r}\frac{\partial\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r} - \frac{s_b^{2\beta}}{c_o^2}\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b) = 0. \tag{49}$$

Multiplying all terms in (49) on both sides by r^2, we obtain:

$$r^2\frac{\partial^2\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r^2} + r\frac{\partial\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r} - \frac{s_b^{2\beta}}{c_o^2}r^2\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b) = 0. \tag{50}$$

The WolframAlpha computational intelligence solver is used to obtain the following solution of (50):

$$\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b) = \psi\, J_0\left(\frac{is_b^\beta r}{c_o}\right) + \varphi\, Y_0\left(\frac{-is_b^\beta r}{c_o}\right); \tag{51}$$

where $J_0\left(\frac{is_b^\beta r}{c_o}\right)$ and $Y_0\left(\frac{-is_b^\beta r}{c_o}\right)$.

are the zeroth order Bessel functions of $1st$ and $2nd$ kind, respectively.

From the boundary conditions in (42), $M(R,t) = 0$ and $M(r,t)$ remains bounded as $r \to 0$ for $t > 0$ which means that $\underline{\mathrm{M}}_\beta^{rt}(R,s_b)$ has a finite value. As a result, $\underline{\mathrm{M}}_{\beta*}^{rt}(R,s_b)$ has a finite value, and from the physical point of view for wave equation solution in [43], φ is set to be zero so that the whole term, $Y_0\left(\frac{-is_b^{\beta}r}{c_o}\right)$, is terminated. The solution of (51) becomes as follows:

$$\begin{aligned}&\underline{\mathrm{M}}_{\beta *}^{rt}(s_a,s_b)=\psi\,J_0\left(\frac{is_b^{\beta}r}{c_o}\right);\\&\text{where } J_0\left(\frac{is_b^{\beta}r}{c_o}\right)\\&\text{is the zeroth order Bessel functions of } 1st \text{ kind.}\end{aligned} \tag{52}$$

Similarly, since $M(R,t)=0$ from (42), then $\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)=0$. let's substitute $\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)=0$ and (52) in equation (48) to obtain the following:

$$\underline{\mathrm{M}}_{\beta *}^{rt}(R,s_b)=\psi\,J_0\left(\frac{is_b^{\beta}R}{c_o}\right)=\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)-\frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right). \tag{53}$$

$$\underline{\mathrm{M}}_{\beta *}^{rt}(R,s_b)=\psi\,J_0\left(\frac{is_b^{\beta}R}{c_o}\right)=-\frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right). \tag{54}$$

As a result, ψ can be written as follows:

$$\begin{aligned}\psi&=\left(\frac{-\frac{1}{s_b^2}\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right)\\&=-\left(\frac{\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{s_b^2J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right).\end{aligned} \tag{55}$$

By substituting (55) in equation (52), the following is obtained:

$$\underline{\mathrm{M}}_{\beta *}^{rt}(s_a,s_b)=-\left(\frac{\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{s_b^2J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right)J_0\left(\frac{is_b^{\beta}r}{c_o}\right). \tag{56}$$

By substituting (56) in equation (48), we obtain the following:

$$\underline{M}_{\beta}^{rt}(s_a,s_b) = -\left(\frac{\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{s_b^2 J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right) J_0\left(\frac{is_b^{\beta}r}{c_o}\right) \tag{57}$$
$$+\frac{\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)}{s_b^2}.$$

After simplifications, we obtain:

$$\underline{M}_{\beta}^{rt}(s_a,s_b) = \frac{J_0\left(\frac{is_b^{\beta}R}{c_o}\right)-J_0\left(\frac{is_b^{\beta}r}{c_o}\right)}{s_b^2 J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right). \tag{58}$$

By using the residue theorem of the complex inversion formula and the solution in [54] with a few mathematical simplifications, it is easy to obtain the ICFDLT of equation (58) which is the following approximate analytical solution for equation (9) subject to the boundary and initial conditions in (41) and (42), respectively, using the method of CFDLT:

$$M(r,t) = \sum_{\xi=1}^{\infty}\frac{iJ_0\left(\lambda_{\xi}\frac{r}{R}\right)\left\{cos\left(-\lambda_{\xi}\frac{c_o t^{\beta}}{R\beta}\right)+i\,sin\left(-\lambda_{\xi}\frac{c_o t^{\beta}}{R\beta}\right)\right\}}{\lambda_{\xi}^2\left(\frac{c_o}{R}\right)J_1(\lambda_{\xi})}; \tag{59}$$
$$\text{where } \frac{i\,R\,s_{b_{\xi}}^{\beta}}{c_o}=\lambda_{\xi};\ \frac{i\,r\,s_{b_{\xi}}^{\beta}}{c_o}=\lambda_{\xi}\left(\frac{r}{R}\right)\ [54];\text{ and } 0<\beta\le 1.$$

Figures (1), (2), and (3) show the numerical simulation of the approximate analytical solution (59) for $\beta = 1; 0.75; 0.50$, respectively.

2.3. The approximate analytical solution by the conformable reduced differential transform method

To show the efficiency of CFD, let's obtain an approximate analytical solution to the two-dimensional conformable fractional wave equation. A fractional differential equation (FDE) approximate method is the conformable fractional differential transform method (CFDTM) in the sense of CFD [55]. The differential transform method (DTM) was introduced by Zhou [57] for solving ordinary differential equations by formulating Taylor series [55, 57]. With the introduction of fractional differential equations (FDEs), the fractional differential transform method (FDTM) was developed by Arikoglu and Ozkol in [56] to solve FDEs by formulating power series. Similarly, CFDTM can be used to solve CFD by formulating conformable fractional power series, and can be defined as [55]:

Definition 2.6. For some $0 < \beta \le 1$, given a function $f(t)$ is infinitely β-differentiable function. Then, the conformable fractional differential transform of $f(t)$ can be written as:

$$F_{\beta}(k) = \frac{1}{\beta^k k!}\left[\left(G_{\beta}^{t_o} f\right)^{(k)}(t)\right]_{t=t_o}, \tag{60}$$

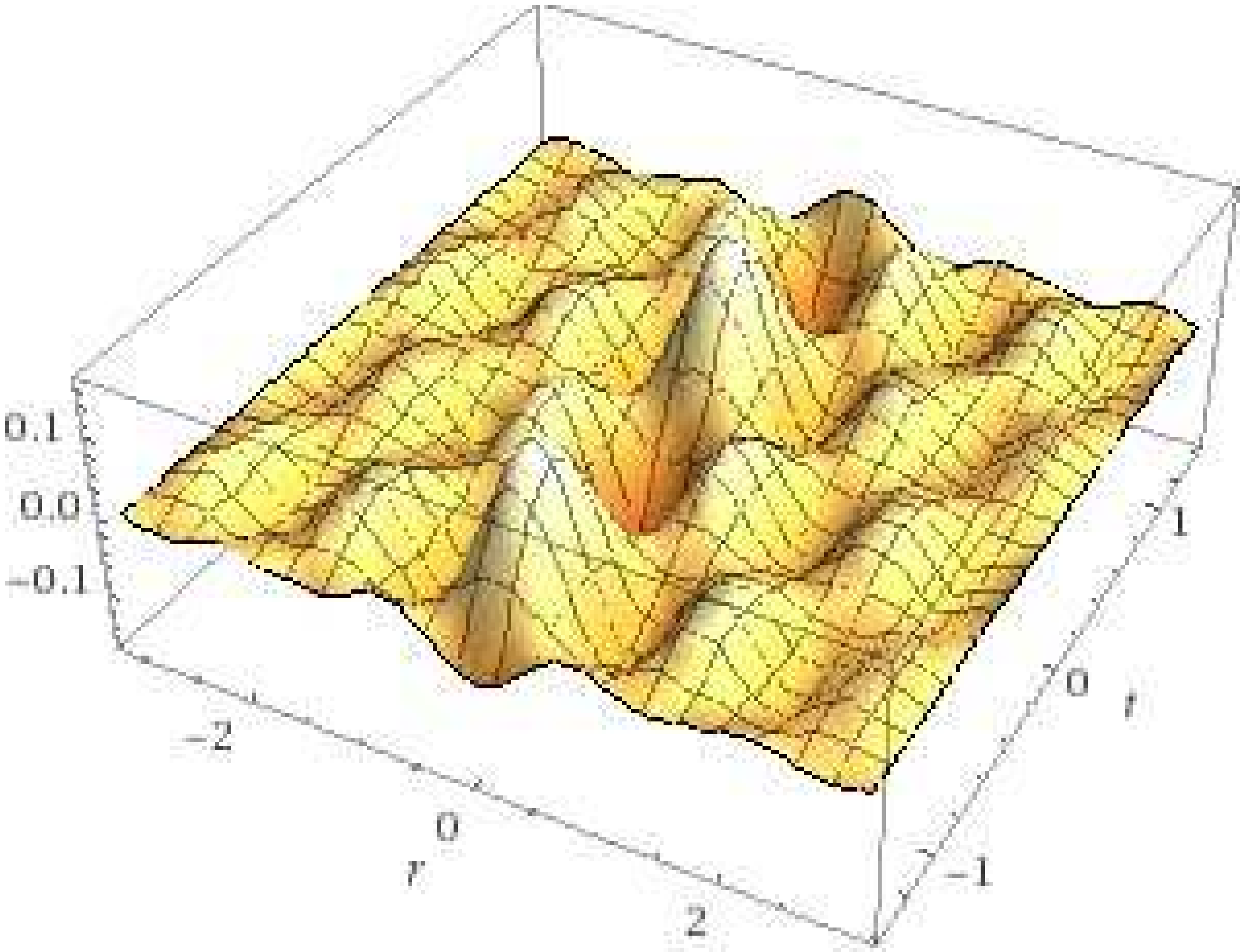

Fig. 1. Approximate Analytical Solution in (59) for $\beta = 1$

where $(G_\beta^{t_o} f)^{(k)}(t)$ is the kth number of CFD application's times, and the conformable fractional differential transform of initial conditions for integer order derivatives can be also written as [55]:

$$F_\beta(k) = \begin{cases} \frac{1}{(\beta k)!}\left[\left(\dfrac{d^{\beta k} f(t)}{dt^{\beta k}}\right)\right]_{t=t_o} \text{ for } k{=}0,1,...,\left(\dfrac{n}{\beta}-1\right) & \text{if } \beta k \in \mathbb{Z}^+ \\ 0 & \text{if } \beta k \notin \mathbb{Z}^+ \end{cases} \tag{61}$$

Definition 2.7. Suppose that $F_\beta(k)$ is the conformable fractional differential transform for $f(t)$ such that the inverse conformable fractional differential transform of $F_\beta(k)$ can be written as [55]:

$$f(t) = \sum_{k=0}^{\infty} F_\beta(k)(t-t_o)^{\beta k} = \sum_{k=0}^{\infty} \frac{1}{\beta^k k!}\left[\left(G_\beta^{t_o} f\right)^{(k)}(t)\right]_{t=t_o}(t-t_o)^{\beta k}. \tag{62}$$

Recently, Acan et al. [58] introduced the reduced differential transform method (RDTM) for solving partial differential equation, and Acan and Baleanu [59] developed a new definition for the conformable reduced differential transform method (CRDTM) as follows:

Definition 2.8. For some $0 < \beta \leq 1$, given a function $m(x,t)$ is analytic continuously β-differentiable function with respect to time t and space x. Then, the conformable reduced differential transform of $m(x,t)$ can be written as:

$$M_k^\beta(x) = \frac{1}{\beta^k k!}\left[\left({}_tG_\beta^{(k)} m\right)\right]_{t=t_o}, \tag{63}$$

where ${}_tG_\beta{}^{(k)} m = (({}_tG_\beta)({}_tG_\beta)...({}_tG_\beta))m(x,t)$, and the conformable reduced differential transform of

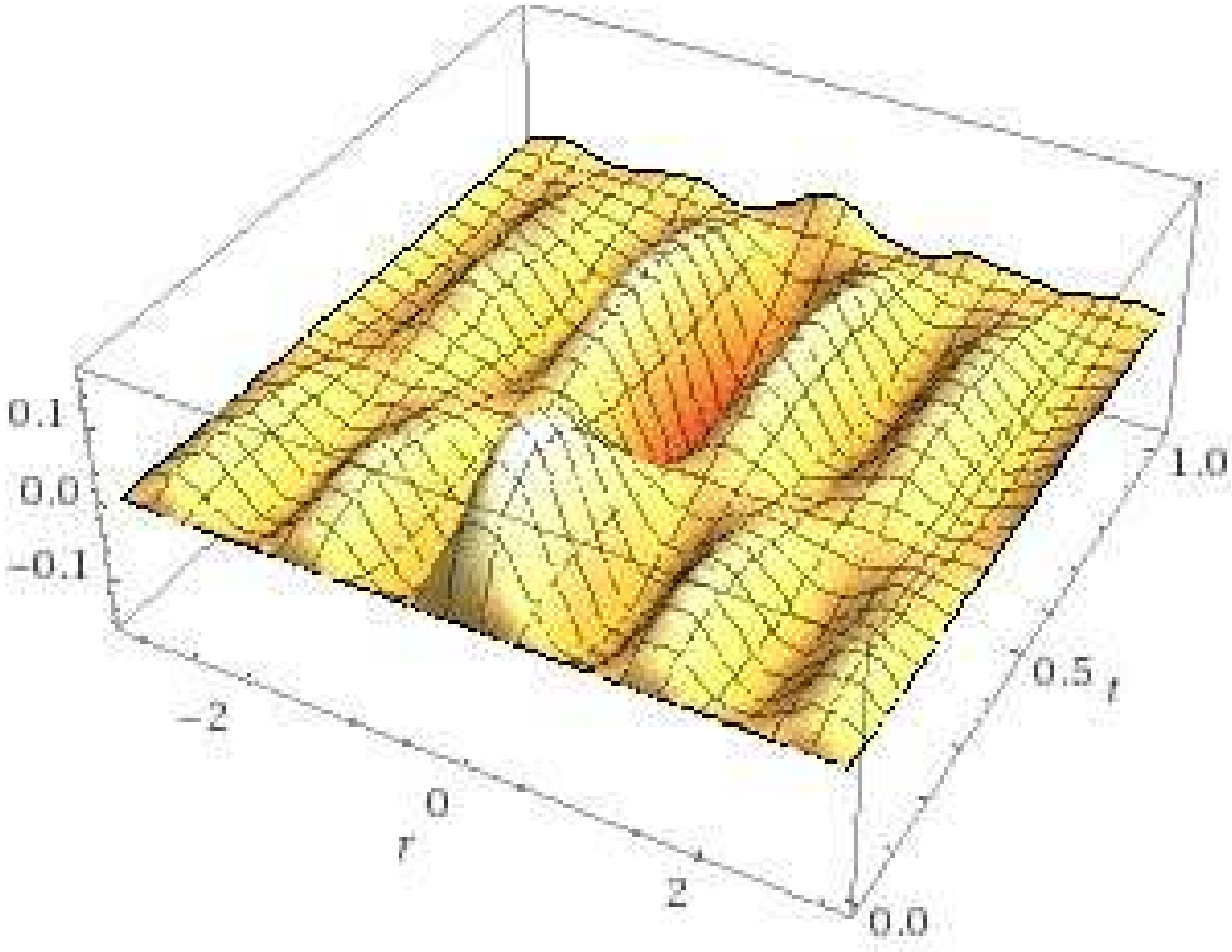

Fig. 2. Approximate Analytical Solution in (59) for $\beta = 0.75$

initial conditions for integer order derivatives can be also written as [58, 59]:

$$F_\beta(k) = \begin{cases} \frac{1}{(\beta k)!}\left[\left(\frac{\partial^{\beta k}}{\partial t^{\beta k}} m(x,t)\right)\right]_{t=t_o} \text{ for } k=0,1,\ldots,\left(\frac{n}{\beta}-1\right) & \text{if } \beta k \in \mathbb{Z}^+ \\ 0 & \text{if } \beta k \notin \mathbb{Z}^+ \end{cases} \tag{64}$$

For (61) and (64), n is the order of conformable differential operator for ordinary differential equation and partial differential equation, respectively.

Definition 2.9. Suppose that $M_k^\beta(x)$ is the conformable reduced differential transform for $m(x,t)$ such that the inverse conformable reduced differential transform of $M_k^\beta(x)$ can be written as [58, 59]:

$$m(x,t) = \sum_{k=0}^{\infty} M_k^\beta(x)(t-t_o)^{\beta k} = \sum_{k=0}^{\infty} \frac{1}{\beta^k k!}\left[\left({}_tG_\beta^{(k)} m\right)\right]_{t=t_o}(t-t_o)^{\beta k}. \tag{65}$$

For theorems and basic operations about both DTM and CRDTM, we refer to [55, 59].

Numerical Experiment 2:

By using the basic operations of CRDTM in [59], CRDTM is applied to solve the mixed initial-boundary value problem (see equation (9)) as follows:

$$\beta(k+1)(k+2)M_{k+2}^\beta(r) = \left[\frac{\partial^2}{\partial r^2}M_k^\beta(r) + \sum_{j=0}^{k} M_{k-j}^\beta(r)\frac{\partial}{\partial r}M_j^\beta(r)\right]; \tag{66}$$

c_o^2 is assumed to be equal 1 for simplicity

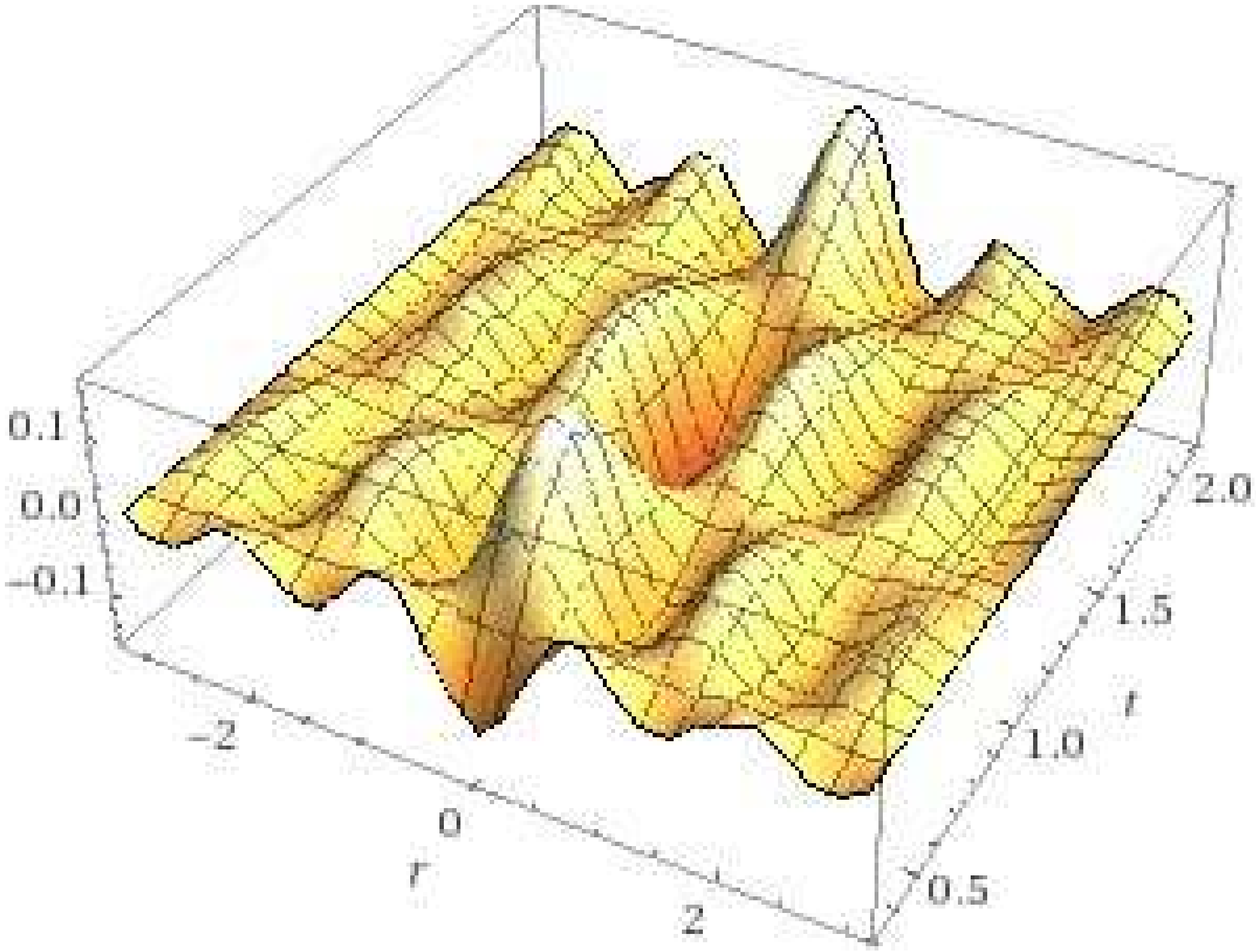

Fig. 3. Approximate Analytical Solution in (59) for $\beta = 0.50$

Hence, the recurrence relation of equation (66) can be written as follows:

$$M_{k+2}^{\beta}(r) = \left[\frac{\frac{\partial^2}{\partial r^2} M_k^{\beta}(r) + \sum_{j=0}^{k} M_{k-j}^{\beta}(r) \frac{\partial}{\partial r} M_j^{\beta}(r)}{\beta(k+1)(k+2)} \right], \tag{67}$$

where $M_k^{\beta}(r)$ is the conformable reduced differential function. For the initial conditions in (11), we assume that $q(r) = cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right)$ and $p(r) = 2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right)$. By applying CRDTM to the assumed initial conditions, we obtain the following:

$$\begin{aligned} M_0^{\beta}(r) &= cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right). \\ M_1^{\beta}(r) &= 2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right). \end{aligned} \tag{68}$$

By substituting (68) in equation (67), the following $M_k^{\beta}(r)$ values are obtained as follows:

$$
\begin{aligned}
M_2^{\beta}(r) &= \frac{-cos\left(\frac{r}{\beta}\right) - sin\left(\frac{r}{\beta}\right)}{2!\beta^2},\\
M_3^{\beta}(r) &= \frac{-2cos\left(\frac{r}{\beta}\right) - 2sin\left(\frac{r}{\beta}\right)}{3!\beta^3},\\
M_4^{\beta}(r) &= \frac{cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right)}{4!\beta^4},\\
M_5^{\beta}(r) &= \frac{2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right)}{5!\beta^5},\\
M_6^{\beta}(r) &= \frac{-cos\left(\frac{r}{\beta}\right) - sin\left(\frac{r}{\beta}\right)}{6!\beta^6}.
\end{aligned}
\tag{69}
$$

Consequently, the set of values $\{M_k^{\beta}(r)\}_{k=0}^{n}$ provides the following approximate solution:

$$
\begin{aligned}
\widetilde{m}_w(r,t) &= \sum_{k=0}^{w} M_k^{\beta}(r)t^{\beta k}\\
&= \begin{cases}
\sum_{k=0}^{w} \frac{(-1)^{\frac{3k}{2}}}{k!\beta^k}\left[cos\left(\frac{rt^{\beta k}}{\beta}\right) + sin\left(\frac{rt^{\beta k}}{\beta}\right)\right]; & \text{if } k \text{ is even}\\
2\left[cos\left(\frac{rt^{\beta k}}{\beta}\right) + sin\left(\frac{rt^{\beta k}}{\beta}\right)\right] & \\
+\sum_{k=3}^{w} \frac{(-1)^{\frac{2k}{3}+(k-(k-3))}}{k!\beta^k}\left[2cos\left(\frac{rt^{\beta k}}{\beta}\right) + 2sin\left(\frac{rt^{\beta k}}{\beta}\right)\right]; & \text{if } k \text{ is odd}
\end{cases}
\end{aligned}
\tag{70}
$$

3. Comparison of results and Discussion

Consequently, after trying to solve this particular two-dimensional wave equation using the classical definitions of fractional derivatives such as Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, and Grünwald-Letnikov, the analytical solution is very complicated to obtain or even may impossible to obtain due to the fact that the classical fractional derivatives are nonlocal differential operators represented using convolution integrals with a weakly singular kernels [60]. To show a simple example of how complicated to obtain analytical solution using classical fractional derivatives for this particular problem, we refer to the general solution obtained in [61] for the time fractional wave equation for a vibrating string. We also refer to a method used in solving classical fractional differential equations (FDEs) in [62], but it cannot find analytical solutions to some examples and cases of FDEs. As a result, the conformable fractional derivatives (CFD) are local operator and can be implemented successfully and easily in various case studies arising from science and engineering in comparison to classical fractional derivatives. CFD can also be used very efficiently in constructing mathematical models for complex problems in physics and engineering.

Due to the difficulty of analytical solutions using classical fractional derivatives, several research studies have developed approximate methods to approximate analytical solutions for the fractional differential equations in the calculus of variations. Approximate methods for FDEs have been introduced successfully in [63–66].

To discuss the error between the analytical and approximate solutions from using all three methods in sections 2.1, 2.2, and 2.3, let's do a numerical test for various values of β and t with various initial conditions from the suggested numerical experiments in this paper and using example 2 in section 4.2 of [43] to discuss the accuracy, reliability, and applicability of the three proposed methods in sections 2. All numerical data of the obtained approximate solutions in table 1, 3, and 3 have been calculated and approximated for the first three terms using an online computer software, known as Keisan Online Calculator service, developed by CASIO COMPUTER CO., LTD.

Numerical Example 1:

By using the mixed initial-boundary value problem in (9) and (10), and example 2 in section 4.2 of [43], the initial conditions in (11) can be written as:

$$M(r,0) = q(r) = 1 - r^2; \text{ and } \frac{\partial^\beta M}{\partial t^\beta}(r,0) = p(r) = 0; \text{ for } 0 < r < R \text{ and } 0 < \beta \leq 1, \quad R = c_o = 1. \tag{71}$$

The above example represents a circular membrane with axisymmetric initial shape [43]. By using the conformable separation of variables method (CSVM) in section 2.1, the approximate analytical solution can be written as:

$$M_{approximate}(r,t) = \sum_{n=1}^{\infty} \left[\frac{8}{\xi_n^3 J_1(\xi_n)} cos \left(\frac{\xi_n t^\beta}{\beta} \right) J_0(\xi_n r) \right]. \tag{72}$$

Similarly, the analytical solution in [43] using the separation of variables method (SVM) can be also written as:

$$M_{analytical}(r,t) = \sum_{n=1}^{\infty} \left[\frac{8}{\xi_n^3 J_1(\xi_n)} cos\left(\xi_n t\right) J_0(\xi_n r) \right]. \tag{73}$$

Table 1 shows the numerical data for both analytical and approximate analytical solutions from using CSVM and SVM for different values of r, t, and β. The absolute error between the analytical and approximate analytical solutions, written as $Error = |M_{approximate}(r,t) - M_{analytical}(r,t)|$, has also been recorded in table 1. From Table 1, it is obvious that at various values of r and t, when β values are getting close to 1, absolute error values become very small. At $\beta = 1$, the obtained approximate analytical solution from CSVM becomes equivalent to the analytical solution from SVM. Figure 4 shows the approximate solutions for different values of t and β at a fixed $r = 0.5$. From Figure 4, at $\beta = 0.75$ the obtained approximate solution by CSVM are closer to the analytical solution using the SVM for integer-order derivatives. Therefore, the behavior of membrane's displacement with respect to time at various β values at a fixed value of membrane radii [42] in Figure 4 can be described as the value of β increases in the conformable formulation (CSVM), the approximate solution from CSVM becomes closer to the analytical solution using the integer-order SVM, and the absolute error value between analytical and approximate solutions becomes small.

Table 1. Comparison of the Analytical and Approximate Solutions from using SVM and CSVM

(r,t)	SVM	β	CSVM	Error
(0.1,0.1)	1.0003	0.25	0.9963	4E-3
		0.75	1.0002	1E-4
		1	1.0003	0
(0.3,0.3)	0.9051	0.25	0.9005	4.6E-3
		0.75	0.9050	1E-4
		1	0.9051	0
(0.5,0.5)	0.7496	0.25	0.7432	6.4E-3
		0.75	0.7494	2E-4
		1	0.7496	0
(0.7,0.7)	0.5152	0.25	0.5056	9.6E-3
		0.75	0.5148	4E-4
		1	0.5152	0
(0.9,0.9)	0.1803	0.25	0.1752	5.1E-3
		0.75	0.1800	3E-4
		1	0.1803	0

Numerical Example 2:

By using the numerical experiment 1 in section 2.2, the approximate solution in (59) can be written with only the real part which satisfies the mixed initial-boundary value problem in (9), (10), and (11) as follows: [54]

$$M(r,t) = \sum_{\xi=1}^{\infty} \frac{J_0\left(\lambda_\xi \frac{r}{R}\right) sin\left(\lambda_\xi \frac{c_o t^\beta}{R\beta}\right)}{\lambda_\xi^2 \left(\frac{c_o}{R}\right) J_1(\lambda_\xi)}; \quad \text{where } 0 < \beta \le 1. \tag{74}$$

The above equation (74) represents the approximate solution with a real part only using the conformable double Laplace transform method (DLTM). Let's also assume $R = c_o = 1$. So, equation (74) can be simplified as follows:

$$M(r,t) = \sum_{\xi=1}^{\infty} \frac{J_0(\lambda_\xi r)\, sin\left(\lambda_\xi \frac{t^\beta}{\beta}\right)}{\lambda_\xi^2 J_1(\lambda_\xi)}; \quad \text{where } 0 < \beta \le 1. \tag{75}$$

To compare the above approximate solution with approximate analytical solution, let's use the proposed mixed initial-boundary value problem in (41) and (42) to find the approximate analytical solution in the sense of conformable derivative. Since $M(r,0) = q(r) = 0$, then $E_n = 0$ in (28). K_n in (29) can be found as follows:

$$\begin{aligned} K_n &= \frac{2}{R c_o \xi_n J_1^2(\xi_n)} \int_0^R p(r) J_0\left(\frac{\xi_n r}{R}\right) r \, \mathrm{d}r \\ &= \frac{2}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0(\xi_n r)\, r \, \mathrm{d}r. \end{aligned} \tag{76}$$

By using integration by parts for (76) and the identity (11) in [43], we have the following: $u = cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right)$; $du = \left(-sin\left(\frac{r}{\beta}\right) + cos\left(\frac{r}{\beta}\right)\right) dr$; $dv = J_0(\xi_n r)\, r \, \mathrm{d}r$; and $v = \frac{1}{\xi_n} J_1(\xi_n)$. Let

$\left(-sin\left(\frac{r}{\beta}\right)+cos\left(\frac{r}{\beta}\right)\right)=\omega$; we have the following:

$$\begin{aligned}
&\int_0^1 cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \,\mathrm{d}r \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)-\int_0^1 \frac{J_1\left(\xi_n\right)}{\xi_n} \omega \,\mathrm{d}r \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\int_0^1 \frac{\left(-cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)\right)}{\xi_n} J_1\left(\xi_n\right) \,\mathrm{d}r \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\frac{J_1\left(\xi_n\right)}{\xi_n}\left[\left(-\frac{sin\left(\frac{r}{\beta}\right)}{\frac{1}{\beta}}-\frac{cos\left(\frac{r}{\beta}\right)}{\frac{1}{\beta}}\right)\right]_{r=0}^{r=1} \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\frac{J_1\left(\xi_n\right)}{\xi_n}\left[-\beta sin\left(\frac{1}{\beta}\right)-\beta cos\left(\frac{1}{\beta}\right)+\frac{1}{\beta}\right] \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\frac{J_1\left(\xi_n\right)}{\xi_n}\left[-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)\right] \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)}{\xi_n} J_1\left(\xi_n\right).
\end{aligned} \tag{77}$$

By substituting (77) in (76), we obtain K_n as follows:

$$\begin{aligned}
&\frac{2}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \,\mathrm{d}r \\
&=\frac{2}{\xi_n J_1^2(\xi_n)} \frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)}{\xi_n} J_1\left(\xi_n\right) \\
&=\frac{2}{\xi_n^2 J_1(\xi_n)}\left[cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)\right].
\end{aligned} \tag{78}$$

To obtain the approximate analytical solution using CSVM, let's substitute (78) in (30) as follows:

$$\begin{aligned}
&M(r,t)=\sum_{n=1}^{\infty} \frac{2}{\xi_n^2 J_1(\xi_n)}\left[cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)\right] \\
&\times\left[sin\left(\frac{\xi_n t^{\beta}}{\beta}\right) J_0\left(\xi_n r\right)\right]; \\
&\text{for } n\in\mathbb{Z}^{+} \text{ and } c_o=R=1.
\end{aligned} \tag{79}$$

Table 2 shows the numerical data for approximate solutions from using CSVM and DLTM for different values of r, t and β. The absolute error between approximate solutions using CSVM and DLTM has been recorded in table 2. At $r=t=0.9$ and $\beta=0.50$, both approximate solutions using CSVM and DLTM in Table 2 are equivalent to each other with no absolute error between them. When $\beta=0.75$ or $\beta=1$ for $r=t=0.1; 0.3; 0.5; 0.7; 0.9$, the absolute error values become very small. Figure 5 shows the approximate solutions for different values of t and β at a fixed $r=0.5$. Between $t=0.1$ and $t=0.2$ at a fixed value ($r=0.5$) of membrane radii in figure 5, the behavior of membrane's displacement with respect to time shows that the numerical values of approximate solution DLTM and CSVM at various β values are very close to each other and the absolute error values between them small. In table 2, it is also clear that numerical value of approximate solutions using CSVM at $\beta=1$ and DLTM at $\beta=0.50$ are close to each other and the error between them is very small. When the time is very small i.e. $t = 0.1$, both approximate solutions from using CSVM and DLTM are very close to each other in value and the absolute error values between them become smaller than other numerical values of the same approximate solutions at larger time periods.

Table 2. Comparison of the Analytical and Approximate Solutions from using CSVM and DLTM

(r,t)	β	CSVM	DLTM	Error
(0.1,0.1)	0.50	0.0017	0.0071	5.4E-3
	0.75	6.275E-4	0.0027	2.073E-3
	1	2.643E-4	0.0011	8.357E-4
(0.3,0.3)	0.50	0.0190	0.0083	0.0107
	0.75	0.0093	0.0041	5.2E-3
	1	0.0052	0.0023	2.9E-3
(0.5,0.5)	0.50	0.0356	0.0117	0.0239
	0.75	0.0198	0.0066	0.0132
	1	0.0125	0.0041	8.4E-3
(0.7,0.7)	0.50	0.0291	0.0178	0.0113
	0.75	0.0176	0.0109	6.7E-3
	1	0.0121	0.0075	4.6E-3
(0.9,0.9)	0.50	0.0098	0.0098	0
	0.75	0.0063	0.0064	1E-4
	1	0.0046	0.0047	1E-4

Numerical Example 3:

To compare the approximate solutions from using CSVM and conformable reduced differential transform method (CRDTM), let's use the approximate solution in (70) from the numerical experiment 2 in section 2.3. Similarly, we need to find the approximate analytical solution in the sense of conformable derivative using CSVM for the mixed initial-boundary value problem in the numerical experiment 2 as we did in numerical example 2.We choose $R = c_o = 1$ in this example. Since $q(r) = cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)$ and $p(r) = 2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right)$, then let's find E_n in (28) and K_n in (29) as follows:

$$\begin{aligned} E_n &= \frac{2}{R^2 J_1^2(\xi_n)} \int_0^R rq(r) J_0\left(\frac{\xi_n r}{R}\right) \mathrm{d}r \\ &= \frac{2}{J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \, \mathrm{d}r. \end{aligned} \tag{80}$$

$$\begin{aligned} K_n &= \frac{2}{R c_o \xi_n J_1^2(\xi_n)} \int_0^R p(r) J_0\left(\frac{\xi_n r}{R}\right) r \, \mathrm{d}r \\ &= \frac{4}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \, \mathrm{d}r. \end{aligned} \tag{81}$$

From the result in (77), E_n and K_n can be written as follows:

$$\begin{aligned} E_n &= \frac{2}{J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \, \mathrm{d}r \\ &= \frac{2}{\xi_n J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right]. \end{aligned} \tag{82}$$

$$\begin{aligned} K_n &= \frac{4}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \, \mathrm{d}r \\ &= \frac{4}{\xi_n^2 J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right]. \end{aligned} \tag{83}$$

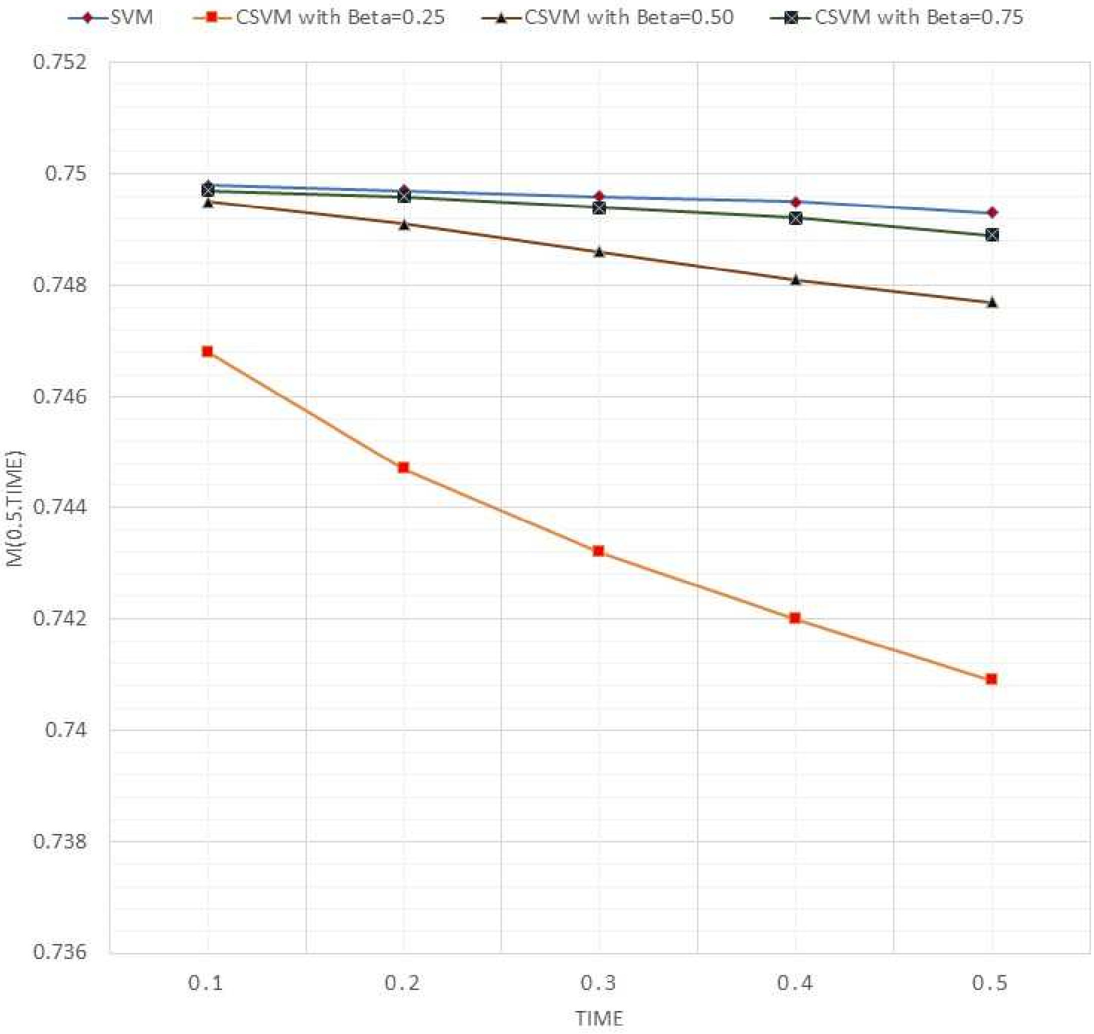

Fig. 4. Comparison of Analytical and Approximate Solutions in (73) and (72) for different values of β at a fixed $r = 0.5$

By substituting both (82) and (83) in (30), we obtain the following approximate analytical solution using CSVM:

$$\begin{aligned} M(r,t) = &\sum_{n=1}^{\infty} \frac{2}{\xi_n J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right] \\ &\times \left[cos\left(\frac{\xi_n t^{\beta}}{\beta}\right) J_0\left(\xi_n r\right)\right] \\ &+ \sum_{n=1}^{\infty} \frac{4}{\xi_n^2 J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right] \\ &\times \left[sin\left(\frac{\xi_n t^{\beta}}{\beta}\right) J_0\left(\xi_n r\right)\right]; \end{aligned} \tag{84}$$

for $n \in \mathbb{Z}^+$ and $c_o = R = 1$.

The numerical data for approximate solutions from using CSVM and CRDTM have been recorded in Table 3 for various values of r, t and β. Table 2 shows also the absolute error between approximate solutions using CSVM and CRDTM. The absolute error value is the smallest at $r = t = 0.7$ and $\beta = 1$ in Table 3 which implies that both approximate solutions using CSVM and CRDTM are very close in numerical value to each other. From Table 3, it is very clear that at $\beta = 0.85$ and $\beta = 1$ at various values of r and t, most of the absolute error values between approximate solutions from using CSVM and CRDTM are smaller than absolute error values at $\beta = 0.75$. Figure 6 shows the approximate solutions for different values of t and β at a fixed value ($r = 0.5$) of membrane radii. The behavior of

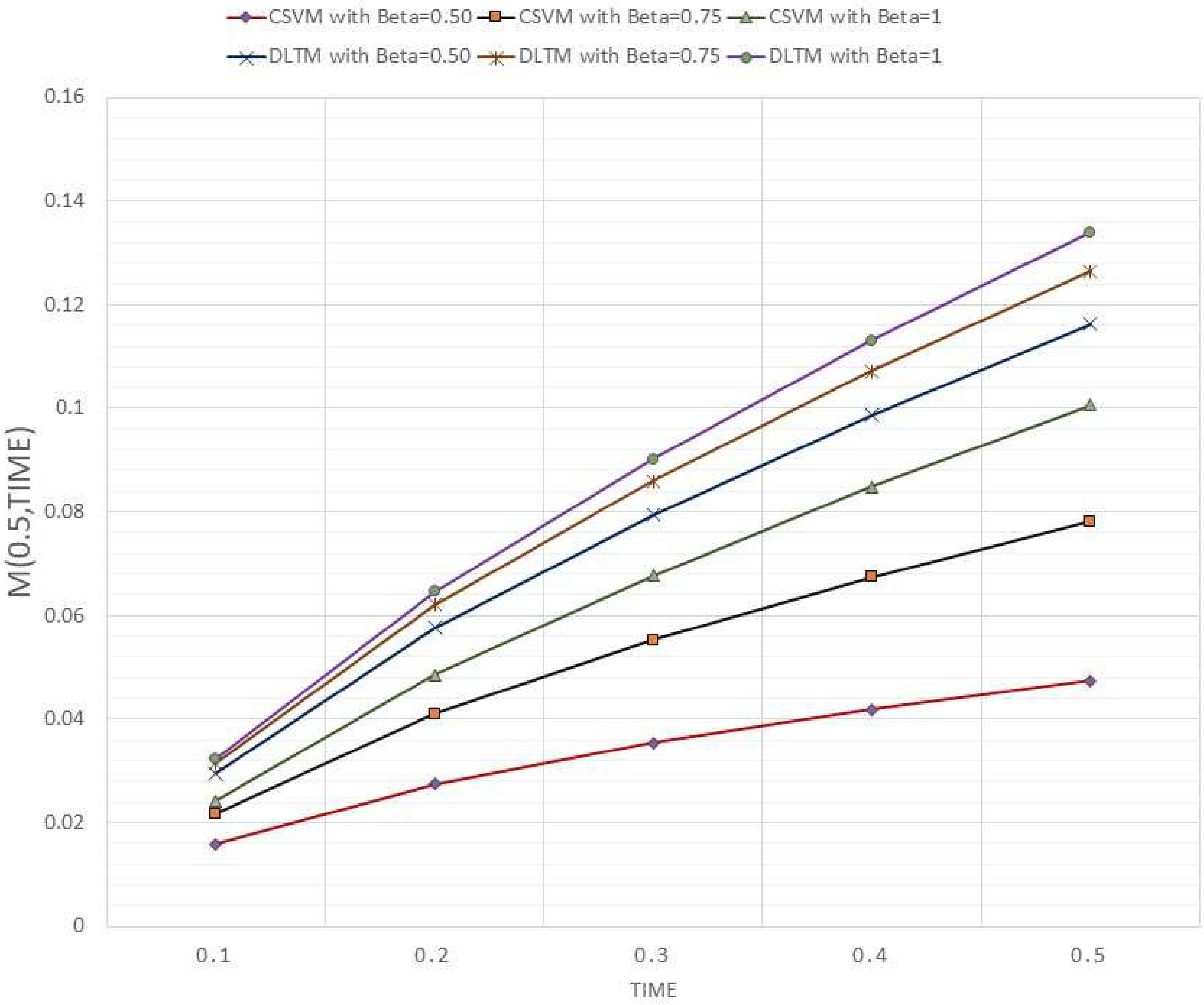

Fig. 5. Comparison of Approximate Solutions in (75) and (79) for different values of β at a fixed $r = 0.5$

membrane's displacement with respect to time in figure 6 shows that using CSVM at $\beta = 1$ is closer in numerical value to the numerical values using CRDTM at $\beta = 0.75; 0.85; 1$. Using CSVM at $\beta = 0.75$ and $\beta = 1$, the numerical values of approximate solutions are close to each other, but both solutions farther in value comparing to the approximate solution using CSVM at $\beta = 1$.

Table 3. Comparison of the Analytical and Approximate Solutions from using CSVM and CRDTM

(r,t)	β	CSVM	CRDTM	Error
(0.1,0.1)	0.75	1.2840	0.3207	0.9633
	0.85	1.2806	0.7651	0.5155
	1	1.2828	1.1667	0.1161
(0.3,0.3)	0.75	0.8764	0.3232	0.5532
	0.85	0.8719	0.7677	0.1042
	1	0.8672	1.1689	0.3017
(0.5,0.5)	0.75	0.9710	0.3292	0.6418
	0.85	0.9647	0.7734	0.1913
	1	1.4054	1.1739	0.2315
(0.7,0.7)	0.75	1.2609	0.3380	0.9229
	0.85	1.2533	0.7819	0.4714
	1	1.2445	1.1817	0.0628
(0.9,0.9)	0.75	0.6170	0.3495	0.2675
	0.85	0.6135	0.7931	0.1796
	1	0.6092	1.1922	0.5830

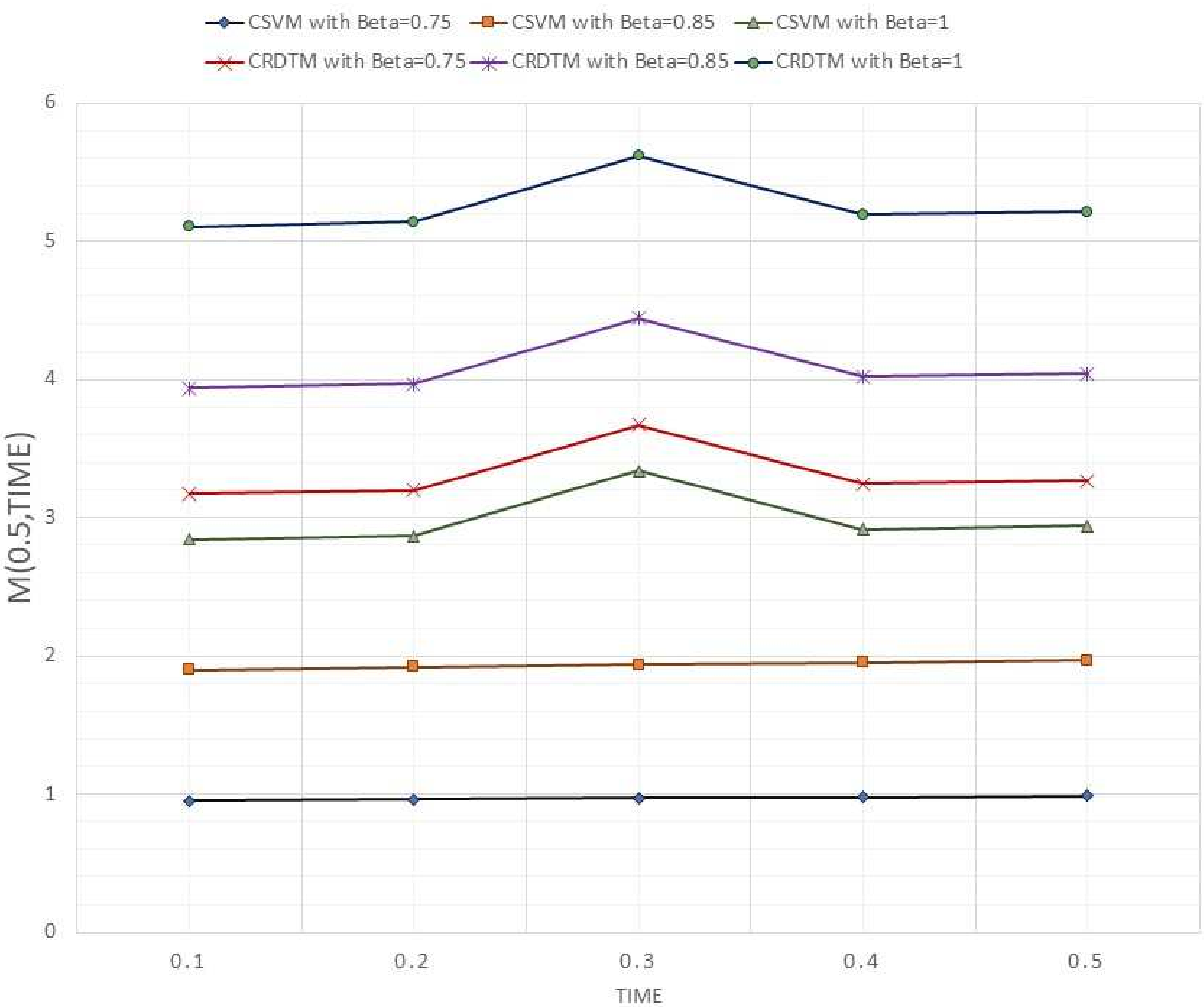

Fig. 6. Comparison of Approximate Solutions in (84) and (70) for different values of β at a fixed $r = 0.5$

By comparing the analytical and approximate solutions in (30), (59) and (70), with the classical non fractional standard analytical solution in [43, 44], we obtain the same analytical solution provided by [43, 44] by substituting $\beta = 1$ in equations (30), (59) and (70) since $0 < \beta \leq 1$. Figures (7), (8), and (9) show the comparison of analytical and approximate solutions in (30) and (70) graphically for various values of $\beta = 1; 0.75; 0.25$.

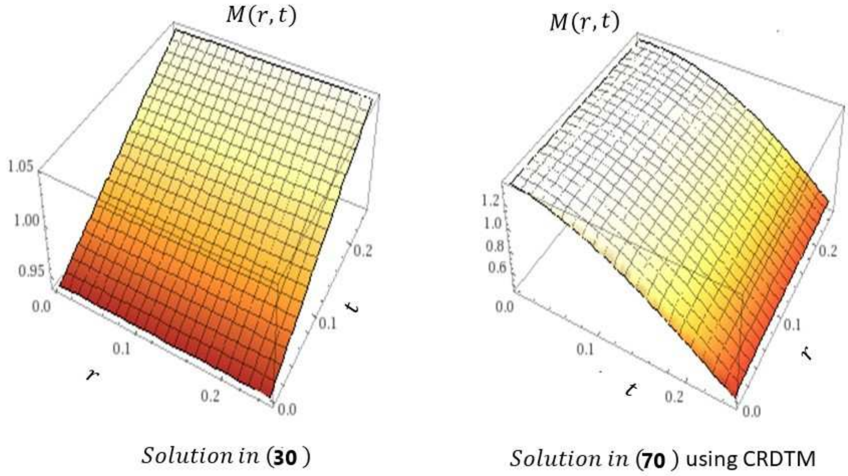

Fig. 7. Comparison of Solutions in (30) and (70) for $\beta = 1$

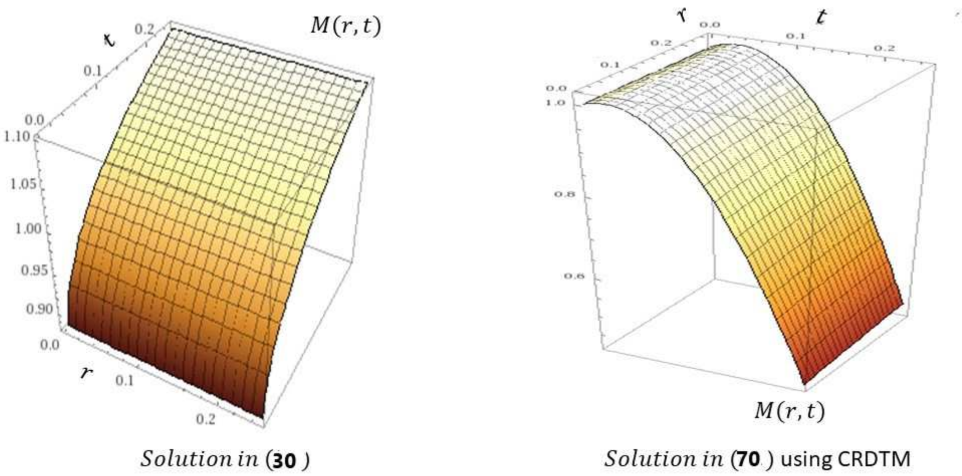

Fig. 8. Comparison of Solutions in (30) and (70) for $\beta = 0.75$

Thus, the CFD formulation is a simple fractional definition to obtain analytical solutions for fractional partial differential equations in comparison to the complicated classical fractional formulations that require various theorems, generalizations, or mathematical extensions to obtain analytical solution or even in some cases can not be obtained at all without introducing numerical and approximate

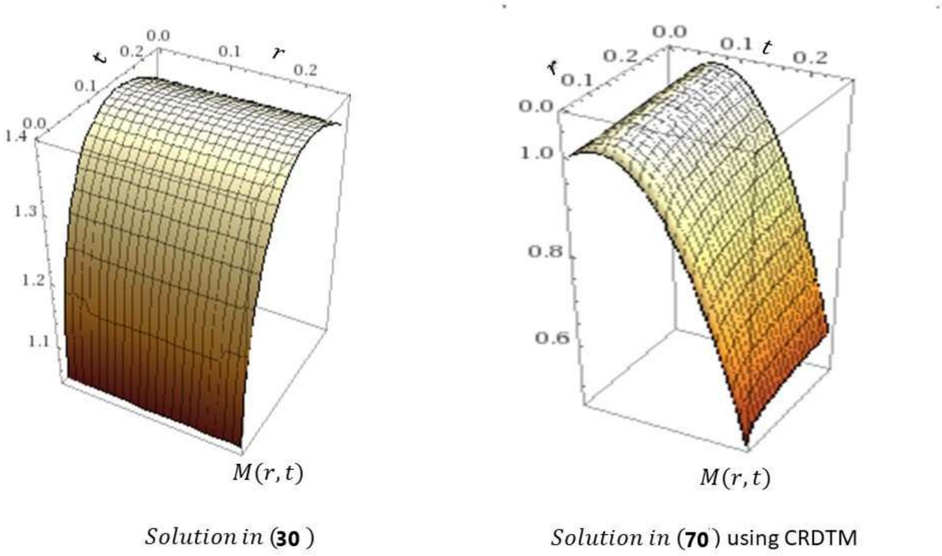

Fig. 9. Comparison of Solutions in (30) and (70) for $\beta = 0.25$

methods. The analytical solutions provided in this paper can be extended to solve higher order fractional PDEs more efficiently than nonlocal classical fractional derivatives formulations.

4. Conclusion

Fractional differential equations have been undergoing major developments due to the importance of understanding the physical and dynamical behavior of problems arising from physics and engineering applications. This article sheds the light on the importance of the conformable fractional derivatives (CFD) and the fact that the CFD can provide efficient analytical and approximate analytical solutions for the two-dimensional fractional wave equation using novel methods such as conformable separation of variables, conformable double Laplace transform, and conformable reduced differential transform methods. We believe that the conformable fractional formulation can be applied effectively in modeling various PDEs problems.

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References

[1] R. Almeida, D. Tavares, and D. F. Torres, *The Variable-Order Fractional Calculus of Variations*, arXiv preprint:1805.00720l, Springer International Publishing, 2018.

[2] O. P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*,Journal of Mathematical Analysis and Applications. **272** (2002), no. 1, 368–379 .

[3] M. Klimek, *Fractional sequential mechanics-models with symmetric fractional derivative*,Czechoslovak Journal of Physics. **51** (2001), no.12, 1348–1354.

[4] A. B. Malinowska and D. F. Torres, *Fractional variational calculus in terms of a combined Caputo derivative*,arXiv preprint:1007.0743, 2010.

[5] S. G. Samko and B. Ross, *Integration and differentiation to a variable fractional order*, Integral Transforms and Special Functions. **1** (1993), no.4, 277–300.

[6] Z. Yi, *Fractional differential equations of motion in terms of combined Riemann-Liouville derivatives*,Chinese Physics B. **21** (2012), no. 8, 084502.

[7] M. Abu Hammad and R. Khalil, *Conformable fractional heat differential equation*, Int. J. Pure Appl. Math. **94** (2014), no.2, 215–221.

[8] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics. **264** (2014), 65–70.

[9] M. J. Lazo and D. F. Torres, *Variational calculus with conformable fractional derivatives*, IEEE/CAA Journal of Automatica Sinica. **4** (2017), no.2, 340–352.

[10] Y. Çenesiz and A. Kurt, *The new solution of time fractional wave equation with conformable fractional derivative definition*, Journal of New Theory. **7** (2015), 79–85

[11] O. Tasbozan, Y. Çenesiz, A. Kurt, and D. Baleanu, *New analytical solutions for conformable fractional PDEs arising in mathematical physics by exp-function method*, Open Physics. **15** (2017), no.1, 647–651.

[12] R. Khalil and H. Abu-Shaab, *Solution of some conformable fractional differential equations*, Int. J. Pure Appl. Math. **103** (2015), no.4, 667–673.

[13] M. Yavuz, *Novel solution methods for initial boundary value problems of fractional order with conformable differentiation*, An International Journal of Optimization and Control: Theories & Applications (IJOCTA) **8** (2017), no.1, 1–7.

[14] M. Yavuz and B. Yaşkıran, *Conformable Derivative Operator in Modelling Neuronal Dynamics*, Applications & Applied Mathematics **13** (2018), no.2, 803–817.

[15] M. Yavuz and B. Yaşkıran, *Approximate-analytical solutions of cable equation using conformable fractional operator*, New Trends in Mathematical Sciences **5** (2017), no.4, 209–219.

[16] M. Yavuz and N. Özdemir, *A different approach to the European option pricing model with new fractional operator*, Mathematical Modelling of Natural Phenomena **13** (2018), no.1, 12.

[17] D. Zhao and M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo **54** (2017), no.3, 903–917.

[18] O. Acan, D. Baleanu, M. M. A. Qurashi, and M. G. Sakar, *Analytical approximate solutions of (n+1)-dimensional fractal heat-like and wave-like equations*, Entropy **19** (2017), no.7, 296.

[19] J. He, *Fractal calculus and its geometrical explanation*, Results in Physics **10** (2018), 272–276.

[20] J. He, Z. Li, and Q. Wang, *A new fractional derivative and its application to explanation of polar bear hairs*, Journal of King Saud University-Science **28** (2016), no.2, 190–192.

[21] J. He, *A New Fractal Derivation*, Thermal Science **15** (2011), no.1, 145–147.

[22] Y. Hu and J. He, *On fractal space-time and fractional calculus*, Thermal Science **20** (2016), no.3, 773–777.

[23] H. Liu, Z. Li, and Y. Yao, *A fractional nonlinear system for release oscillation of silver ions from hollow fibers*, Journal of Low Frequency Noise, Vibration and Active Control **38** (2019), no.1, 88–92.

[24] A. Atangana, D. Baleanu, and A. Alsaedi, *New properties of conformable derivative*, Open Mathematics **13** (2015), no.1, 889–898.

[25] J. E. N. Nápoles Valdés, J. A. C. Medina, P. M. Guzmán, and L. M. Lugo, *A New Local Fractional Derivative of q-uniform Type*, Discontinuity, Nonlinearity, and Complexity **8** (2019), no.1, 101–109.

[26] J. Weberszpil and W. Chen, *Generalized Maxwell relations in thermodynamics with metric derivatives*, Entropy **19** (2017), no.8, 407.

[27] J. He, *A tutorial review on fractal spacetime and fractional calculus*, International Journal of Theoretical Physics **53** (2014), no.11, 3698–3718.

[28] Y. Liang, W. Chen, and W. Cai, *Hausdorff Calculus: Applications to Fractal Systems*, Walter de Gruyter GmbH & Co KG, **6**, 2019.

[29] W. Chen, F. Wang, B. Zheng, and W. Cai, *Non-Euclidean distance fundamental solution of Hausdorff derivative partial differential equations*, Engineering Analysis with Boundary Elements **84** (2017), 213–219.

[30] S. Meng and Y. Cui, *The Extremal Solution To Conformable Fractional Differential Equations Involving Integral Boundary Condition*, Mathematics **7** (2019), no.2, 186.

[31] F. Silva, D. M. Moreira, and M. A. Moret, *Conformable Laplace Transform of Fractional Differential Equations*, Axioms, **7** (2018), no. 3, 55.

[32] T. Abdeljawad and M. Grossman, *On geometric fractional calculus*, Journal of Semigroup Theory and Applications **2** (2016), 2051-2937.

[33] P. M. Guzmán, G. Langton, L. M. Lugo, J. Medina, and J. E. N. Nápoles Valdés, *A new definition of a fractional derivative of local type*, Journal of Mathematical Analysis **9** (2018), no.2, 88–98.

[34] J. E. N. Nápoles Valdés, P. M. Guzmán, and L. M. Lugo, *Some New Results on Nonconformable Fractional Calculus*, Advances in Dynamical Systems and Applications **13** (2018), no.2, 167–175.

[35] A. F. Imbert, J. Mendez-Bermudez, J. E. N. Nápoles Valdés, and J. M. Sigarreta Almira, *On fractional Liénard-type systems*, Revista Mexicana de Fisic **65** (2019), no.6, 618–625.

[36] P. M. Guzmán and J. E. N. Nápoles Valdés, *A NOTE ON THE OSCILLATORY CHARACTER OF SOME NON CONFORMABLE GENERALIZED LIÉNARD SYSTEM*, Advanced Mathematical Models & Applications **4** (2019), no.2, 127–133.

[37] F. MARTÍNEZ, P. O. MOHAMMED, and J. E. N. Nápoles VALDÉS, *NON-CONFORMABLE FRACTIONAL LAPLACE TRANSFORM*, Kragujevac Journal of Mathematics **46** (2022), no.3, 341–354.

[38] J. E. N. Nápoles Valdés, J. M. Rodríguez, and J. M. Sigarreta, *New Hermite–Hadamard Type Inequalities Involving Non-Conformable Integral Operators*, Symmetry **11** (2019), no.9, 1108.

[39] M. Y. Jaffrin, *Dynamic filtration with rotating disks, and rotating and vibrating membranes: an update*, Current Opinion in Chemical Engineering, **1** (2012), no.2, 171–177.

41

[40] A. Kola, Y. Ye, P. Le-Clech, and V. Chen, *Transverse vibration as novel membrane fouling mitigation strategy in anaerobic membrane bioreactor applications*, Journal of membrane science, **455** (2014), 320–329.

[41] A. Kola, Y. Ye, A. Ho, P. Le-Clech, and V. Chen, *Application of low frequency transverse vibration on fouling limitation in submerged hollow fibre membranes*, Journal of membrane science, **409** (2012), 54–65.

[42] H. Singh, H. M. Srivastava, and D. Kumar, *A reliable numerical algorithm for the fractional vibration equation*, Chaos, Solitons & Fractals **103** (2017), 131–138.

[43] N. H. Asmar, *Partial differential equations with Fourier series and boundary value problems*, Courier Dover Publications, 2016.

[44] D. J. Wollkind and B. J. Dichone, *Comprehensive Applied Mathematical Modeling in the Natural and Engineering Sciences*, Springer International Publishing, 2017.

[45] T. Abdeljawad, *On conformable fractional calculus*, Journal of computational and Applied Mathematics. **279** (2015), 57–66.

[46] T. A. Estrin and T. J. Higgins, *The solution of boundary value problems by multiple Laplace transformations*, Journal of the Franklin Institute, **252** (1951), no. 2, 153–167.

[47] L. Debnath, *The double Laplace transforms and their properties with applications to functional, integral and partial differential equations*, International Journal of Applied and Computational Mathematics, **2** (2016), no. 2, 223–241.

[48] A. M. O. Anwar, F. Jarad, D. Baleanu, and F. Ayaz, *Fractional Caputo heat equation within the double Laplace transform*, Rom. Journ. Phys., **58** (2013), 15–22.

[49] H. Eltayeb and A. Kılıçman, *On double Sumudu transform and double Laplace transform*, Malaysian journal of mathematical sciences, **4** (2010), no. 1, 17–30.

[50] H. Eltayeb and A. Kılıçman, *A note on solutions of wave, Laplace's and heat equations with convolution terms by using a double Laplace transform*, Applied Mathematics Letters, **21** (2008), no. 12, 1324–1329.

[51] A. Kılıçman and H. E. Gadain, *On the applications of Laplace and Sumudu transforms*, Journal of the Franklin Institute, **347** (2010), no. 5, 848–862.

[52] R. R. Dhunde and G. L. Waghmare, *Double Laplace transform method for solving space and time fractional telegraph equations*, International Journal of Mathematics and Mathematical Sciences, **2016** (2016), 1414595.

[53] M. Omran and A. Kiliçman, *Fractional double Laplace transform and its properties*, AIP Conference Proceedings, **1795** (2017), no. 1, 020021.

[54] O. Korfalı and İ. B. Parlak, *Vibrations of a Circular Membrane Subjected To a Pulse*, Vibration Problems ICOVP 2005, Springer International Publishing, 2006.

[55] E. Ünal and A. Gökdoğan, *Solution of conformable fractional ordinary differential equations via differential transform method*, Optik-International Journal for Light and Electron Optics. **128** (2017), 264–273.

[56] A. Arikoglu and I. Ozkol, *Solution of fractional differential equations by using differential transform method*, Chaos, Solitons & Fractals. **34** (2007), no.5, 1473–1481.

[57] J. K. Zhou, *Differential transformation and its application for electrical circuits*, Huazhong University Press, Wuhan, China, 1986.

[58] O. Acan, O. Firat, Y. Keskin, and G. Oturanc, *Solution of conformable fractional partial differential equations by reduced differential transform method*, Selcuk Journal of Applied Mathematics, 2016.

[59] O. Acan and D. Baleanu, *A New Numerical Technique for Solving Fractional Partial Differential Equations*, Miskolc Mathematical Notes, **19** (2018), no.1, 3–18.

[60] D. Avcı, B. Eroglu, and N. Ozdemir, *Conformable heat problem in a cylinder*,Proceedings, International Conference on Fractional Differentiation and its Applications. (2016), 572–581.

[61] T. Sandev and Ž. Tomovski, *The general time fractional wave equation for a vibrating string*, Journal of Physics A: Mathematical and Theoretical, **43** (2010), no.5, 055204.

[62] E. Demirci and N. Ozalp, *A method for solving differential equations of fractional order*, Journal of Computational and Applied Mathematics, **236** (2012), no.11, 2754–2762.

[63] T. Blaszczyk and M. Ciesielski, *Numerical solution of fractional Sturm-Liouville equation in integral form*, Fractional Calculus and Applied Analysis, **17** (2014), no.2, 307–320.

[64] C. Çelik and M. Duman, *Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative*, Journal of computational physics, **231** (2012), no.4, 1743–1750.

[65] S. Momani and Z. Odibat, *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos, Solitons & Fractals, **31** (2007), no.5, 1248–1255.

[66] N. A. Pirim and F. Ayaz, *Hermite collocation method for fractional order differential equations*, An International Journal of Optimization and Control: Theories & Applications (IJOCTA), **8** (2018), no.2, 228–236.

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Decompositions of Soft $\alpha-$continuity and Soft $A-$continuity

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Abstract − In this paper, we introduce the concepts of soft $\alpha A-$set, soft $\alpha B-$set, soft $\alpha C-$set and soft $\alpha LC-$set in soft topological spaces which are defined over an initial universe with a fixed set of parameters and discuss their relationships with each other and other weaker forms of soft open sets with counterexamples. We also investigate soft $\alpha A-$continuity, soft $\alpha B-$continuity, soft $\alpha C-$continuity and soft $\alpha LC-$continuity. Finally, we obtain two decompositions of soft α-continuity and a decomposition of soft A-continuity.

Keywords − *Soft set, Soft topological space, Soft $\alpha A-$set, Soft $\alpha B-$set, Soft $\alpha C-$set, Soft $\alpha LC-$set, Soft $\alpha A-$continuous function, Soft $\alpha B-$continuous function, Soft $\alpha C-$continuous function, Soft $\alpha LC-$continuous function.*

1. Introduction

The concept of soft sets was initiated by Molodtsov [1] in 1999 as a completely new approach for modeling vagueness and uncertainty. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Later Maji et al. [2] presented some new definitions on soft sets such as a subset, the complement of a soft set. Research works on soft sets are progressing rapidly in recent years.

Shabir and Naz [3] introduced the soft topological spaces which are defined over an initial universe with a fixed set of parameters. Later Aygünoğlu and Aygün [4], Min [5], Zorlutuna et al. [6] and Hussain and Ahmad [7] continued to study the properties of soft topological spaces. They got many important results in soft topological spaces. Recently, weak forms of soft open sets have been studied [8–19] in soft topological spaces.

The purpose of this paper is to introduce the notions of soft $\alpha A-$set, soft $\alpha B-$set, soft $\alpha C-$set and soft $\alpha LC-$set in soft topological spaces. We study the relations between these different types of subsets in soft topological spaces. We also introduce soft $\alpha A-$continuous, soft $\alpha B-$continuous, soft $\alpha C-$continuous and soft $\alpha LC-$continuous functions. Finally, we obtain some decompositions.

2. Preliminary

In this section, we present the basic definitions and results of soft set theory which may be found in earlier studies.

Definition 2.1. [1] Let X be an initial universe set and E be the set of all possible parameters with respect to X. Let $P(X)$ denote the power set of X. A pair (F, A) is called a soft set over X where $A \subseteq E$ and $F : A \to P(X)$ is a set valued mapping.

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The set of all soft sets over X is denoted by $SS(X)_E$.

Definition 2.2. [2] A soft set (F, A) over X is said to be a null soft set denoted by Φ if for all $e \in A$, $F(e) = \emptyset$. A soft set (F, A) over X is said to be an absolute soft set denoted by $\widetilde{A}$ if for all $e \in A$, $F(e) = X$.

Definition 2.3. [3] Let Y be a nonempty subset of X, then $\widetilde{Y}$ denotes the soft set (Y, E) over X for which $Y(e) = Y$, for all $e \in E$. In particular, (X, E) will be denoted by $\widetilde{X}$.

Definition 2.4. [2] For two soft sets (F, A) and (G, B) over X, we say that (F, A) is a soft subset of (G, B) if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. We write $(F, A) \sqsubseteq (G, B)$. (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(G, B) \sqsubseteq (F, A)$. Then (F, A) and (G, B) are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.5. [2] The union of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$, $H(e) = F(e)$ if $e \in A \backslash B$, $H(e) = G(e)$ if $e \in B \backslash A$, $H(e) = F(e) \cup G(e)$ if $e \in A \cap B$. We write $(F, A) \sqcup (G, B) = (H, C)$.

Definition 2.6. [20] The intersection (H, C) of two soft sets (F, A) and (G, B) over X, denoted by $(F, A) \sqcap (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

Definition 2.7. [3] The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \backslash (G, E)$, is defined as $H(e) = F(e) \backslash G(e)$ for all $e \in E$.

Definition 2.8. [3] The relative complement of a soft set (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c = (F^c, E)$ where $F^c : E \longrightarrow P(X)$ is a mapping given by $F^c(e) = X \backslash F(e)$ for all $e \in E$.

Definition 2.9. [3] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if

(1) Φ, $\widetilde{X} \in \tau$
(2) If (F, E), $(G, E) \in \tau$, then $(F, E) \sqcap (G, E) \in \tau$
(3) If $\{(F_i, E)\}_{i \in I} \in \tau$, $\forall i \in I$, then $\sqcup_{i \in I}(F_i, E) \in \tau$.

The triplet (X, τ, E) is called a soft topological space over X. Every member of τ is called a soft open set in X. A soft set (F, E) over X is called a soft closed set in X if its relative complement $(F, E)^c$ belongs to τ. We will denote the family of all soft open sets (resp., soft closed sets) of a soft topological space (X, τ, E) by SOS(X, τ, E) (resp., SCS(X, τ, E)).

Definition 2.10. Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X.

(1) [3] The soft closure of (F, E) is the soft set $cl(F, E) = \sqcap\{(G, E) :\ (G, E)$ is soft closed and $(F, E) \sqsubseteq (G, E)\}$.

(2) [6] The soft interior of (F, E) is the soft set $int(F, E) = \sqcup\{(H, E) : (H, E)$ is soft open and $(H, E) \sqsubseteq (F, E)\}$.

Clearly, $cl(F, E)$ is the smallest soft closed set over X which contains (F, E) and $int(F, E)$ is the largest soft open set over X which is contained in (F, E).

Throughout the paper, the spaces X and Y (or (X, τ, E) and (Y, ν, K)) stand for soft topological spaces assumed unless stated otherwise.

Definition 2.11. A soft set (F, E) is called

(i) soft semi-open [8] in a soft topological space X if $(F, E) \sqsubseteq cl(int(F, E))$.
(ii) soft pre-open [9] in a soft topological space X if $(F, E) \sqsubseteq int(cl(F, E))$.
(iii) soft α-open [10] in a soft topological space X if $(F, E) \sqsubseteq int(cl(int(F, E)))$.

The relative complement of a soft semi-open (resp., soft pre-open, soft α-open) set is called a soft semi-closed (resp., soft pre-closed, soft α-closed) set.

We will denote the family of all soft semi-open (resp., soft pre-open and soft α-open) sets of a soft topological space (X, τ, E) by SSOS(X, τ, E) (resp., SPOS(X, τ, E) and SαOS(X, τ, E)).

Definition 2.12. [8] Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X. The soft semi-closure of (F, E) is the soft set $cl_s(F, E) = \sqcap\{(H, E) :\ (H, E)$ is soft semi-closed and $(F, E) \sqsubseteq (H, E)\}$ and $cl_s(F, E)$ is soft semi-closed.

Theorem 2.13. [8] Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X. We have $(F, E) \sqsubseteq cl_s(F, E) \sqsubseteq cl(F, E)$.

Definition 2.14. *Let (X, τ, E) be a soft topological space. A soft set (F, E) is called*

(1) *a soft regular closed set [19] in X if* $(F, E) = cl(int(F, E))$.
(2) *a soft $A-$set [17] in X if $(F, E) = (G, E) \sqcap (H, E)$, where (G, E) is a soft open set and (H, E) is a soft regular closed set in X.*
(3) a soft t-set [17] in X if $int(cl(F, E)) = int(F, E)$.
(4) a soft $B-$set [17] in X if $(F, E) = (G, E) \sqcap (H, E)$, where (G, E) is a soft open set and (H, E) is a soft t-set in X.
(5) a soft α^*-set [16] in X if $int(cl(int(F, E))) \sqsubseteq (F, E)$.
(6) a soft $C-$set [16] in X if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft open and (H, E) is a soft α^*-set in X.
(7) a soft locally closed set (briefly; soft $LC-$set) [14] in X if $(F, E) = (G, E) \sqcap (H, E)$, where (G, E) is soft open and (H, E) is soft closed in X.

We will denote the family of all *soft regular closed sets* (resp., soft $A-$sets, soft $B-$sets, soft $C-$sets and soft $LC-$sets) of a soft topological space X by SRCS(X) (resp., SAS(X), SBS(X), SCS(X) and SLCS(X)).

Remark 2.15. In a soft topological space (X, τ, E);

(1) every soft open set is soft α-open [10],
(2) every soft α-open set is soft pre-open and soft semi-open [10],
(3) every soft regular closed set is soft closed [19],
(4) every soft open set is a soft $A-$set [17],
(5) every soft $A-$set is soft semi-open [17],
(6) every soft $A-$set is a soft $LC-$set [14],
(7) every soft $LC-$set is a soft $B-$set [14],
(8) every soft $B-$set is a soft $C-$set [16].

Definition 2.16. [21] Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X. (F, E) is called (1) a soft dense set if $cl(F, E) = \widetilde{X}$, (2) a soft nowhere dense set if $int(cl(F, E)) = \Phi$.

Definition 2.17. [22] Let $SS(X)_E$ and $SS(Y)_K$ be families of soft sets, $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then the mapping $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ is defined as:

(1) Let $(F, E) \in SS(X)_E$. The image of (F, E) under f_{pu}, written as $f_{pu}(F, E) = (f_{pu}(F), p(E))$, is a soft set in $SS(Y)_K$ such that

$$f_{pu}(F)(y) = \begin{cases} \cup_{x \in p^{-1}(y) \cap A} u(F(x)) & , p^{-1}(y) \cap A \neq \emptyset \\ \emptyset & , otherwise \end{cases}$$

for all $y \in K$.

(2) Let $(G, K) \in SS(Y)_K$. The inverse image of (G, K) under f_{pu}, written as $f_{pu}^{-1}(G, K) = (f_{pu}^{-1}(G), p^{-1}(K))$, is a soft set in $SS(X)_E$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))) & , p(x) \in K \\ \emptyset & , otherwise \end{cases}$$

for all $x \in E$.

Definition 2.18. [6] Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called a soft continuous function if for each $(G, K) \in \upsilon$ we have $f_{pu}^{-1}(G, K) \in \tau$.

Definition 2.19. Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called

(1) a soft semi-continuous function [15] if for each $(G, K) \in SOS(Y)$ we have $f_{pu}^{-1}(G, K) \in SSOS(X)$.
(2) a soft α-continuous function [10] if for each $(G, K) \in SOS(Y)$ we have $f_{pu}^{-1}(G, K) \in S\alpha OS(X)$.
(3) a soft pre-continuous function [10] if for each $(G, K) \in SOS(Y)$ we have $f_{pu}^{-1}(G, K) \in SPOS(X)$.
(4) a soft $A-$continuous function [17] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft $A-$set in X.
(5) a soft $B-$continuous function [17] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft $B-$set in X.
(6) a soft $C-$continuous function [16] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft $C-$set in X.
(7) a soft $LC-$continuous function [14] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft $LC-$set in X.

Remark 2.20. Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then,

(1) every soft continuous function is soft α-continuous [10],
(2) every soft α-continuous function is soft semi-continuous and soft pre-continuous [10],
(3) every soft continuous function is soft $A-$continuous [17],
(4) every soft $A-$continuous function is soft semi-continuous [17],
(5) every soft $A-$continuous function is soft $LC-$continuous [14],
(6) every soft $LC-$continuous function is soft $B-$continuous [14],
(6) every soft $B-$continuous function is soft $C-$continuous [16].

3. Soft $\alpha A-$sets, Soft $\alpha B-$sets, Soft $\alpha C-$sets and Soft $\alpha LC-$sets

Definition 3.1. A soft set (F, E) in a soft topological space (X, τ, E) is called

1) a soft $\alpha A-$set if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft $\alpha-$open and (H, E) is soft regular closed.
2) a soft $\alpha B-$set if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft $\alpha-$open and (H, E) is a soft $t-$set in X.
3) a soft $\alpha C-$set if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft $\alpha-$open and $int(cl(int(H, E))) \sqsubseteq (H, E)$.
4) a soft $\alpha LC-$set if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft $\alpha-$open and (H, E) is soft closed.

We will denote the family of all soft $\alpha A-$sets (resp.,soft $\alpha B-$sets, soft $\alpha C-$sets and soft $\alpha LC-$sets) of (X, τ, E) by $S\alpha AS(X)$ (resp., $S\alpha BS(X)$, $S\alpha CS(X)$ and $S\alpha LCS(X)$).

Theorem 3.2. For a soft topological space (X, τ, E), the following hold:

1) Every soft $A-$set is a soft $\alpha A-$set.
2) Every soft $B-$set is a soft $\alpha B-$set.
3) Every soft $C-$set is a soft $\alpha C-$set.
4) Every soft $LC-$set is a soft $\alpha LC-$set.

PROOF. The proofs are obvious since every soft open set is soft $\alpha-$open. □

Example 3.3. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F, E)\}$ such that

$$(F, E) = \{(e_1, \{x_1\}),\ (e_2, \{x_2\})\}.$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X [17]. Then $(G, E) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_2\})\}$ is a soft α-open set in X but not soft open. Since $(G, E) = (G, E) \sqcap \tilde{X}$ and $\tilde{X}$ is a soft $t-$set, (G, E) is a soft $\alpha B-$set but not a soft $B-$set.

Example 3.4. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology τ on X and the soft set $(G, E) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_2\})\})\}$ in Example 3.3. (G, E) is a soft $\alpha C-$set but not a soft $C-$set.

Example 3.5. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology τ on X and the soft set $(G, E) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_2\})\})\}$ in Example 3.3. (G, E) is a soft $\alpha LC-$set but not a soft $LC-$set.

Proposition 3.6. In a soft topological space (X, τ, E), every soft $\alpha A-$set is a soft $\alpha LC-$set.

Proof. The proof is obvious since every soft reguler closed set is soft closed. □

Example 3.7. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \tilde{X}, (F_1, E), ..., (F_{11}, E)\}$ such that

$$\begin{aligned}
(F_1, E) &= \{(e_1, \{x_1\}), (e_2, \{x_1\})\}, \\
(F_2, E) &= \{(e_1, \{x_2\}), (e_2, \{x_2\})\}, \\
(F_3, E) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}, \\
(F_4, E) &= \{(e_1, \{x_1, x_2, x_3\}), (e_2, \{x_1, x_3\})\}, \\
(F_5, E) &= \{(e_1, \{x_1, x_2, x_4\}), (e_2, \{x_1, x_2, x_3\})\}, \\
(F_6, E) &= \{(e_1, \{x_2\}), (e_2, \varnothing)\}, \\
(F_7, E) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1\})\}, \\
(F_8, E) &= \{(e_1, \{x_1, x_2, x_3\}), (e_2, \{x_1, x_2, x_3\})\}, \\
(F_9, E) &= \{(e_1, X), (e_2, \{x_1, x_2, x_3\})\}, \\
(F_{10}, E) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2, x_3\})\}, \\
(F_{11}, E) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_3\})\}.
\end{aligned}$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X [19]. Let us take a soft set $(G, E) = (F_9, E) \sqcap (F_{11}, E)^c = \{(e_1, \{x_3, x_4\}), (e_2, \{x_2\})\}$ on X. Since (G, E) is a soft $LC-$set, (G, E) is a soft $\alpha LC-$set. But (G, E) is not a soft $\alpha A-$ set.

Proposition 3.8. In a soft topological space (X, τ, E), every soft $\alpha B-$set is a soft $\alpha C-$set.

Proof. Let (F, E) be a soft $\alpha B-$set, so $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft $\alpha-$open and (H, E) is a soft $t-$set. Since (H, E) is a soft $t-$set, $int(cl(H, E)) = int(H, E)$. Then $int(cl(int(H, E))) \sqsubseteq int(cl(H, E)) = int(H, E) \sqsubseteq (H, E)$. Hence we obtain (F, E) is a soft $\alpha C-$set. □

Example 3.9. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ such that

$$(F_1, E) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\},$$

$$(F_2, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\},$$

$$(F_3, E) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}.$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X. $(G, E) = \{(e_1, \{x_3\}), (e_2, \{x_1, x_3\})\}$ is a soft α^*-set since $int(cl(int(G, E))) = int(G, E)$. So it is a soft $C-$set and a soft $\alpha C-$set. But (G, E) is not a soft $\alpha B-$set.

Proposition 3.10. In a soft topological space (X, τ, E), every soft $\alpha LC-$set is a soft $\alpha B-$set.

Proof. Let $(G, E) \sqcap (H, E)$ be a soft $\alpha LC-$set such that (G, E) is soft $\alpha-$open and $cl(H, E) = (H, E)$. Since $int(cl(H, E)) = int(H, E)$ then the proof is obvious. □

Example 3.11. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ such that

$$(F_1, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\},$$

$$(F_2, E) = \{(e_1, \{x_3\}), (e_2, \{x_3\})\},$$

$$(F_3, E) = \{(e_1, \{x_2, x_3\}), (e_2, \{x_2, x_3\})\}.$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X. Then $(G, E) = \{(e_1, \{x_2\}), (e_2, \{x_1, x_2\})\}$ is a soft $\alpha B-$set, but it is not a soft $\alpha LC-$set.

Lemma 3.12. [12] Let (X, τ, E) be a soft topological space, $(F, E) \in S\alpha OS(X)$ and $(G, E) \in SSOS(X)$. Then $(F, E) \sqcap (G, E) \in SSOS(X)$.

Proposition 3.13. In a soft topological space (X, τ, E), every soft $\alpha A-$set is soft semi-open.

Proof. Let $(G, E) \sqcap (H, E)$ be a soft $\alpha A-$set, (G, E) is soft $\alpha-$open and $cl(int(H, E)) = (H, E)$. Hence (H, E) is soft semi-open. Using Lemma 3.12 we have that $(G, E) \sqcap (H, E)$ is soft semi-open. □

Theorem 3.14. [14] For a soft topological space (X, τ, E), we have $SAS(X) = SSOS(X) \cap SLCS(X)$.

Theorem 3.15. For a soft topological space (X, τ, E), we have $SAS(X) = S\alpha AS(X) \cap SLCS(X)$.

Proof. Every soft $\alpha A-$set is soft semi-open by Proposition 3.13 and $SAS(X) = SSOS(X) \cap SLCS(X)$ by Theorem 3.14. Hence $S\alpha AS(X) \cap SLCS(X) \subseteq SSOS(X) \cap SLCS(X) = SAS(X)$. So we obtain $S\alpha AS(X) \cap SLCS(X) \subseteq SAS(X)$.
By Theorem 3.2, we have that every soft $A-$set is a soft $\alpha A-$set. Also, since $SAS(X) = SSOS(X) \cap SLCS(X)$ then $SAS(X) \subseteq SLCS(X)$ and thus $SAS(X) \subseteq S\alpha AS(X) \cap SLCS(X)$. □

Proposition 3.16. Let (X, τ, E) be a soft topological space. A soft set (F, E) over X is a soft $\alpha-$open set iff $(F, E) = (G, E) \setminus (H, E)$ where (G, E) is soft open and (H, E) is soft nowhere dense.

Proof. If (F, E) is a soft $\alpha-$open set we have

$$(F, E) = int(cl(int(F, E))) \setminus ((int(cl(int(F, E)))) \setminus (F, E))$$

where $int(cl(int(F, E))) \setminus (F, E)$ clearly is soft nowhere dense.
Conversely, if $(F, E) = (G, E) \sqsubseteq (H, E)$, (G, E) is soft open, (H, E) is soft nowhere dense, we can see that $(G, E) \sqsubset cl(int(F, E))$ and consequently

$$int(cl(int(F, E))) \sqsupseteq (G, E) \sqsupseteq (F, E)$$

So the proof is complete. □

Proposition 3.17. Let (X, τ, E) be a soft topological space. (F, E) is a soft $\alpha B - set$ if and only if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft $B-$set and $int(H, E)$ is soft dense.

Proof. Let (F, E) be a soft $\alpha B-$set, we have $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is a soft $\alpha-$open set and (H, E) is a soft $t-$set. By Proposition 3.16, we write $(G, E) = (G_1, E) \sqcap (G_2, E)$ where (G_1, E) is a soft open $B-$set and $int(G_2, E)$ is soft dense. Hence $(F, E) = (G, E) \sqcap (H, E) = ((G_1, E) \sqcap (G_2, E)) \sqcap (H, E) = ((G_1, E) \sqcap (H, E)) \sqcap (G_2, E)$ where $(G_1, E) \sqcap (H, E)$ is a soft $B-$set and $int(G_2, E)$ is soft dense.
Conversely, let $(F, E) = (H, E) \sqcap (G_2, E)$ such that (H, E) is a soft $B-$set and $int(G_2, E)$ is soft dense. Then we have $(H, E) = (G_1, E) \sqcap (H_1, E)$ where (G_1, E) is soft open and (H_1, E) is a soft $t-$set. Thus $(F, E) = (H, E) \sqcap (G_2, E) = ((G_1, E) \sqcap (H_1, E)) \sqcap (G_2, E) = ((G_1, E) \sqcap (G_2, E)) \sqcap (H, E)$ is soft $\alpha-$open from Proposition 3.16 and (H, E) is a soft $t-$set. Thus (F, E) is a soft $\alpha B-$set. □

Theorem 3.18. [8] A soft subset (F, E) in a soft topological space (X, τ, E) is soft semi-cosed iff $int(cl(F, E)) \sqsubseteq (F, E)$.

Theorem 3.19. [12] Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X, E)$. Then (F, E) is soft semi-closed iff $(F, E) = (F, E) \sqcup int(cl(F, E))$.

Another description of soft $\alpha B-$sets is given in the next result.

Proposition 3.20. Let (X, τ, E) be a soft topological space. (F, E) is a soft $\alpha B - set$ iff $(F, E) = (G, E) \sqcap cl_s(F, E)$ for some $(G, E) \in S\alpha OS(X)$.

PROOF. Let (F,E) be a soft $\alpha B-$set, we have $(F,E)=(G,E)\sqcap(H,E)$ where (G,E) is a soft $\alpha-$open set and (H,E) is a soft $t-$set. Now $(F,E)\sqsubseteq(G,E)$ and $(F,E)\sqsubseteq(H,E)$, so $cl_s(F,E)\sqsubseteq cl_s(H,E)=(H,E)\sqcup int(cl(H,E))$ by Theorem 3.19. Since (H,E) is a soft $t-$set, $int(cl(H,E))=int(H,E)$. Hence we obtain

$$(F,E)\sqsubseteq(G,E)\sqcap cl_s(F,E)\sqsubseteq(G,E)\sqcap cl_s(H,E)=(G,E)\sqcap((H,E)\sqcup int(cl((H,E))))=$$

$$=(G,E)\sqcap((H,E)\sqcup int(H,E))=(G,E)\sqcap(H,E)=(F,E).$$

Conversely, assume that $(F,E)=(G,E)\sqcap cl_s(F,E)$ for some $(G,E)\in S\alpha OS(X)$. Put $(H,E)=cl_s(F,E)$. Then (H,E) is soft semi-closed and we have $int(cl(F,E))\sqsubseteq(H,E)$ by Theorem 3.18. Hence $int(cl(H,E))=int(H,E)$ and (H,E) is a soft $t-$set. Therefore (F,E) is a soft $\alpha B-$set. □

Theorem 3.21. For a soft topological space (X,τ,E) we have

$$S\alpha OS(X)=SPOS(X)\cap S\alpha BS(X).$$

PROOF. It is clear that $S\alpha OS(X)\subseteq SPOS(X)\cap S\alpha BS(X)$. For the converse, let $(F,E)\in SPOS(X)\cap S\alpha BS(X)$. From $(F,E)\in SPOS(X)$ we have $(F,E)\sqsubseteq int(cl(F,E))$. Since $(F,E)\in S\alpha BS(X)$, we have that $(F,E)=(G,E)\sqcap cl_s(F,E)$ for some $(G,E)\in S\alpha OS(X)$ by Proposition 3.20. Also $cl_s(F,E)=(F,E)\sqcup int(cl(F,E))=int(cl(F,E))$ from Theorem 3.19. Thus $(F,E)=(G,E)\sqcap int(cl(F,E))$ where $(G,E)\in S\alpha OS(X)$ and $int(cl(F,E))\in SOS(X)\subseteq S\alpha OS(X)$. Therefore $(F,E)=(G,E)\sqcap int(cl(F,E))\in S\alpha OS(X)$. □

Theorem 3.22. For a soft topological space (X,τ,E) the following hold:

$$S\alpha OS(X)=SPOS(X)\cap S\alpha LCS(X).$$

PROOF. Since every soft $\alpha-$open set is soft pre-open and every soft $\alpha-$open set is soft $\alpha LC-$set, we obtain $S\alpha OS(X)\subseteq SPOS(X)\cap S\alpha LCS(X)$.
From Theorem 3.21, $S\alpha OS(X)=SPOS(X)\cap S\alpha BS(X)$. Also, since every soft $\alpha LC-$set is a soft $\alpha B-$ set we obtain $SPOS(X)=SPOS(X)\cap S\alpha LCS(X)\subseteq SPOS(X)=SPOS(X)\cap S\alpha BS(X)=S\alpha OS(X)$. □

4. Decompositions of Soft $\alpha-$continuity and Soft $A-$continuity

In this section, two new decompositions of soft $\alpha-$continuity are given. Also we obtain a decomposition of soft $A-$continuity.

Definition 4.1. Let (X,τ,E) and (Y,ϑ,K) be soft topological spaces. Let $u:X\to Y$ and $p:E\to K$ be mappings and $f_{pu}:SS(X)_E\to SS(Y)_K$ be a function. Then the function f_{pu} is called soft $\alpha A-$continuous (resp., soft $\alpha B-$continuous, soft $\alpha C-$continuous, soft $\alpha LC-$continuous) if for each $(G,K)\in SOS(Y)$, $f_{pu}^{-1}(G,K)$ is a soft $\alpha A-$set (resp., soft $\alpha B-$set, soft $\alpha C-$set, soft $\alpha LC-$set) in X.

Theorem 4.2. Let (X,τ,E) and (Y,ϑ,K) be soft topological spaces and $f_{pu}:SS(X)_E\to SS(Y)_K$ be a function. Then the following hold:

1) If f_{pu} is soft $A-$continuous, then it is soft $\alpha A-$continuous.
2) If f_{pu} is soft $B-$continuous, then it is soft $\alpha B-$continuous.
3) If f_{pu} is soft $C-$continuous, then it is soft $\alpha C-$continuous.
4) If f_{pu} is soft $LC-$continuous, then it is soft $\alpha LC-$continuous.

PROOF. The proof is obvious from Theorem 3.2. □

Theorem 4.3. Let (X,τ,E) and (Y,ϑ,K) be soft topological spaces and $f_{pu}:SS(X)_E\to SS(Y)_K$ be a function. Then every soft $\alpha A-$continuous function is soft semi-continuous.

PROOF. The proof is obvious from Proposition 3.13. □

Theorem 4.4. Let (X, τ, E) and (Y, ϑ, K) be soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a function.Then f_{pu} is soft $A-$continuous and soft $\alpha A-$continuous, then it is soft $LC-$continuous.

PROOF. This is a direct consequence of Theorem 3.15. □

Theorem 4.5. Let (X, τ, E) and (Y, ϑ, K) be soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a function.Then f_{pu} is soft $A-$continuous and soft $\alpha LC-$continuous, then it is soft $\alpha B-$continuous.

PROOF. The proof is obvious from Proposition 3.10. □

Theorem 4.6. Let (X, τ, E) and (Y, ϑ, K) be soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a function.Then f_{pu} is soft $A-$continuous iff it is soft $\alpha A-$continuous and soft $LC-$continuous.

PROOF. This follows immediately from Theorem 3.15. □

Theorem 4.7. Let (X, τ, E) and (Y, ϑ, K) be soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a function.Then f_{pu} is soft $\alpha-$continuous iff it is soft pre-continuous and soft $\alpha B-$continuous.

PROOF. This follows immediately from Theorem 3.21. □

Theorem 4.8. Let (X, τ, E) and (Y, ϑ, K) be soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a function.Then f_{pu} is soft $\alpha-$continuous iff it is soft pre-continuous and soft $\alpha LC-$continuous.

PROOF. This follows immediately from Theorem 3.22. □

References

[1] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl. 37 (4–5) (1999) 19–31.

[2] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Comput. Math. Appl. 45 (4-5) (2003) 555–562.

[3] M. Shabir, M. Naz, *On soft topological spaces*, Comput. Math. Appl. 61 (7) (2011) 1786–1799.

[4] A. Aygünoğlu, H. Aygün, *Some notes on soft topological spaces*, Neural Computing and Applications, 21 (1) (2012) 113–119.

[5] W. K. Min, *A note on soft topological spaces*, Comput. Math. Appl. 62 (9) (2011) 3524–3528.

[6] I. Zorlutuna, M. Akdağ, W. K. Min, S. Atmaca, *Remarks on soft topological spaces*, Ann. Fuzzy Math. Inform. 3 (2) (2012) 171–185.

[7] S. Hussain, B. Ahmad, *Some properties of soft topological spaces*, Comput. Math. Appl. 62 (11) (2011) 4058–4067.

[8] B. Chen, *Soft semi-open sets and related properties in soft topological spaces*, Appl. Math. Inf. Sci. 7 (1) (2013) 287–294.

[9] I. Arockiarani, A. Arokialancy, *Generalized soft $g\beta$-closed sets and soft $gs\beta$-closed sets in soft topological spaces*, International Journal of Mathematical Archive 4 (2) (2013) 1–7.

[10] M. Akdağ, A. Özkan, *Soft α-Open Sets and Soft α-Continuous Functions*, Abstr. Appl. Anal. http://dx.doi.org/10.1155/2014/891341, (2014).

[11] Z. Güzel Ergül, Ş. Yüksel, N. Tozlu, *On soft generalized preregular closed and open sets in soft topological spaces*, Applied Mathematical Sciences 8 (158) (2014) 7875–7884.

[12] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, A. M. Abd El-Latif, *γ-operation and decompositions of some forms of soft continuity in soft topological spaces*, Ann. Fuzzy Math. Inform. 7 (2) (2014) 181–196.

[13] K. Kannan, *Soft generalized closed sets in soft topological spaces*, Journal of Theoretical and Appl. Inform. Technology 37 (1) (2012) 17–21.

[14] A.H. Kocaman, N. Tozlu, *Soft locally closed sets and decompositions of soft continuity*, Annals of Fuzzy Mathematics and Informatics 11 (2) (2016) 173–181.

[15] J. Mahanta, P. K. Das, *On soft topological space via semi-open and semi-closed soft sets*, Cornell University Library, 2012.

[16] N. Tozlu, Ş. Yüksel and Z. Güzel Ergül, *Soft C-Sets and a Decomposition of Soft Continuity in soft Topological Spaces*, International Journal of Mathematics Trends and Technology 16 (1) (2014) 58–69.

[17] N. Tozlu and Ş. Yüksel, *Soft A-sets and soft B-sets in Soft Topological spaces*, Mathematical Sciences and Applications E-Notes 5 (2) (2017) 17–25.

[18] S. Yuksel, N. Tozlu, Z. Guzel Ergul, *On soft generalized closed sets in soft topological spaces*, Journal of Theoretical and Appl. Inform. Technology 55 (2) (2013) 17–21.

[19] Ş. Yüksel, N. Tozlu and Z. Güzel Ergül, *Soft regular generalized closed sets in soft topological spaces*, International Journal of Mathematical Analysis 8 (8) (2014) 355–367.

[20] F. Feng, Y. B. Jun, X. Z. Zhao, *Soft semirings*, Comput. Math. Appl. 56 (2008) 2621–2628.

[21] C. Sekar, J. Subhashinin, *Related properties of soft dense and soft pre-open sets in soft topological spaces*, International Journal of Innovative and Applied Research 2 (9) (2014) 34–48.

[22] A. Kharal, B. Ahmad, *Mappings on Soft Classes*, New Math. Nat. Comput. 7 (3) (2011) 471–481.

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On Generalized Contraction Principles over $S-$metric Spaces with Application to Homotopy

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Abstract — In the present paper, we introduce the concept of a class of generalized contraction mappings called A-contraction on S-metric space and investigate the existence of fixed points over such spaces. Analogue result has been formulated in integral setting over such an S-metric space. Moreover, the result is applied to homotopy theory.

Keywords — *Fixed point, A-contraction, S-metric space.*

1. Introduction and Preliminaries

In sixties, attempts were initiated through the study of 2-metric spaces by S.Gähler [1,2] to generalize the metric space. However, Ha et al. [3] have pointed out that the results over 2-metrics spaces are independent, rather than generalizations, of the corresponding results in metric spaces. Another such generalization is D-metric space introduced by Dhage [4] in 1992 where he proved some results on fixed points of contraction mappings over complete and bounded D-metric spaces. But in 2006, Mustafa and Sims [5] pointed out that Dhage's notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid. They introduced a more appropriate and robust version of a generalized metric space namely G-metric space in 2006. Sedghi et al. [6,7] improved and modified D-metric space and thus introduced D^*-metric space. They proved some basic properties of D^*-metric spaces and some fixed point theorems on it. In continuation with untiring attempts to find a most appropriate one, Sedghi et al. [8,9] recently introduced and characterized the concept of S-metric space which modifies D-metric and G-metric spaces.

Definition 1.1. (S-metric space) Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,
(S1) $S(x, y, z) = 0$, if and only if $x = y = z$,
(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called an S-metric space.

According to Sedghi et al. [8], some of the examples of such S-metric spaces are:
(1) Let $X = \mathbb{R}^n$ and $\|.\|$ be a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $\|.\|$ be a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

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(3) Let X be a nonempty set, d be a metric on X, then $S(x,y,z)=d(x,z)+d(y,z)$ is an S-metric on X.
(4) [intuitive geometric example for S-metric] Let $X=\mathbb{R}^2$, d be a metric on X, therefore, $S(x,y,z)=d(x,y)+d(x,z)+d(y,z)$ is an S-metric on X. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a within the triangle, then the inequality $S(x,y,z)\leq S(x,x,a)+S(y,y,a)+S(z,z,a)$ holds.
(5) Let $\mathbb{R}$ be the real line. Then $S(x,y,z)=|x-z|+|y-z|$ for all $x,y,z\in\mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.

Definition 1.2. [8] Let (X,S) be an S-metric space and $A\subset X$.
(1) A subset A of X is called S-bounded if there exists $r>0$ such that $S(x,x,y)<r$ for all $x,y\in A$.
(2) A sequence $\{x_n\}$ in X converges to $x\in X$ if and only if $S(x_n,x_n,x)\to 0$ as $n\to\infty$. That is for every $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that $S(x_n,x_n,x)<\epsilon$ whenever $n\geq n_0$. We denote this by $\lim_{n\to\infty}x_n=x$ or $\lim_{n\to\infty}S(x_n,x_n,x)=0$.
(3) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$. That is for every $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $S(x_n,x_n,x_m)<\epsilon$ whenever $n,m\geq n_0$.
(4) The S-metric space (X,S) is called complete if every Cauchy sequence is convergent to an element of X.

Lemma 1.3. [8] For a S-metric space X, we have $S(x,x,y)=S(y,y,x)\ \forall x,y\in X$.

Lemma 1.4. [9] Let (X,S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$, then $S(x_n,x_n,y_n)\to S(x,x,y)$ as $n\to\infty$.

Definition 1.5. [9] Let $T:X\to Y$ be a map from an S-metric space X to an S-metric space Y. Then T is continuous at $x\in X$ if and only if $Tx_n\to Tx$ in Y whenever $x_n\to x$ in X.

A mapping T is continuous at X if and only if it is continuous at all $x\in X$.

Theorem 1.6. [8] Let (X,S) be a complete S-metric space and let $F:X\to X$ be a contraction i.e

$$S\left(F(x),F(x),F(y)\right)\leq LS\left(x,x,y\right)\quad\text{for all } x,y\in X$$

where $0\leq L<1$. Then F has a unique fixed point $u\in X$. Furthermore, for any $x\in X$ we have $\lim_{n\to\infty}F^n(x)=u$ with

$$S\left(F^n(x),F^n(x),u\right)\leq\frac{2L^n}{1-L}S\left(x,x,F(x)\right).$$

Theorem 1.7. [8] Let (X,S) be a compact S-metric space and let $F:X\to X$ satisfying

$$S\left(F(x),F(x),F(y)\right)<S\left(x,x,y\right)\quad\text{for all } x,y\in X \text{ and } x\neq y.$$

Then F has a unique fixed point in X.

2. A-contraction and fixed point

Akram et al. [10, 11] have defined A-contractions as follows: Let a nonempty set A consisting of all functions $\alpha:\mathbb{R}_+^3\to\mathbb{R}_+$ satisfying
(A_1) : α is continuous on the set $\mathbb{R}_+^3$ of all triplets of nonnegative reals (with respect to the Euclidean metric on $\mathbb{R}^3$).
(A_2) : $a\leq kb$ for some $k\in[0,1)$ whenever $a\leq\alpha\left(a,b,b\right)$ or $a\leq\alpha\left(b,a,b\right)$ or $a\leq\alpha\left(b,b,a\right)$, for all $a,b\in\mathbb{R}_+$.

Definition 2.1. [10] A self map T on a metric space X is said to be A-contraction if it satisfies the condition

$$d\left(Tx,Ty\right)\leq\alpha\left(d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right)\right)$$

for all $x,y\in X$ and for some α in A.

Following the definition of A-contraction mappping on a metric space (see [10]- [11]) and over a 2-metric space (see [12]), we now define A-contractions on an S-metric space and prove fixed point theorem on it.

Definition 2.2. A self map T on an S-metric space X is said to be A-contraction if it satisfies the condition

$$S(Tx, Tx, Ty) \leq \alpha (S(x, x, y), S(x, x, Tx), S(y, y, Ty))$$

for all $x, y \in X$ and for some α in A.

Now we state our main theorem.

Theorem 2.3. Let (X, S) be a complete S-metric space and let T be A-contraction mapping on X. Then, T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary element of X and consider the sequence $\{x_n\}$ of iterates $x_{n+1} = Tx_n$, $n \in N$. Now

$$S(x_1, x_1, x_2) = S(Tx_0, Tx_0, Tx_1) \leq \alpha (S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_1, x_1, x_2))$$

implies

$$S(x_1, x_1, x_2) \leq kS(x_0, x_0, x_1) \tag{1}$$

for some $k \in [0, 1)$ because $\alpha \in A$. By easy iteration one can check that

$$S(x_n, x_n, x_{n+1}) \leq k^n S(x_0, x_0, x_1). \tag{2}$$

For all $m > n$ and by using Lemma 1.3 and (S2) we get

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2\sum_{i=n}^{m-2} k^i S(x_0, x_0, x_1) + k^{m-1} S(x_0, x_0, x_1) \\ &\leq 2[k^n + k^{n+1} + ...k^{m-1}] S(x_0, x_0, x_1) \\ &\leq \frac{2k^n}{1-k} S(x_0, x_0, x_1). \end{aligned}$$

Taking limit as $m, n \to \infty$ we get $S(x_n, x_n, x_m) \to 0$. This proves that the sequence $\{x_n\}$ is Cauchy and by completeness of X, $x_n \to z$ for some $z \in X$ as $n \to \infty$. Now,

$$\begin{aligned} S(z, z, Tz) &\leq 2S(z, z, x_{n+1}) + S(Tz, Tz, x_{n+1}) \\ &= 2S(x_{n+1}, x_{n+1}, z) + S(Tx_n, Tx_n, Tz) \\ &\leq 2S(x_{n+1}, x_{n+1}, z) + \alpha(S(x_n, x_n, z), S(x_n, x_n, Tx_n), S(z, z, Tz)) \\ &= 2S(x_{n+1}, x_{n+1}, z) + \alpha(S(x_n, x_n, z), S(x_n, x_n, x_{n+1}), S(z, z, Tz)). \end{aligned}$$

Therefore by taking limit as $n \to \infty$ we get $S(z, z, Tz) \leq \alpha(0, 0, S(z, z, Tz))$, which implies that $S(z, z, Tz) = 0$. So z is a fixed point of T. For uniqueness, let $u, v \in X$ be two distinct fixed points of T. So by definition of A-contraction,

$$\begin{aligned} S(u, u, v) = S(Tu, Tu, Tv) &\leq \alpha (S(u, u, v), S(u, u, Tu), S(v, v, Tv)) \\ &= \alpha (S(u, u, v), S(u, u, u), S(v, v, v)) \\ &= \alpha (S(u, u, v), 0, 0). \end{aligned}$$

Then by axiom A_2 of α we have $u = v$ and so the fixed point is unique. □

Now we give an example in support of the Theorem 2.3.

Example 2.4. First we take a function $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ as $\alpha(x,y,z) = \beta.(y+z)$, where $0 < \beta < \frac{1}{2}$, which satisfies the property (A_1) obviously. Now

$$a \le \alpha(a,b,b) = \beta.(b+b) = 2\beta.b \text{ implies } a \le k.b \text{ where } k = 2\beta < 1,$$

$$a \le \alpha(b,a,b) = \beta.(a+b) \text{ implies } a \le k.b \text{ where } k = \frac{\beta}{1-\beta} < 1 \text{ and also}$$

$$a \le \alpha(b,b,a) = \beta.(b+a) = \text{ implies } a \le k.b \text{ where } k = \frac{\beta}{1-\beta} < 1.$$

So α satisfies the property (A_2). Now Let $X = [0,1]$ and $S(x,y,z) = |x-z| + |y-z|$. Clearly (X,S) is a complete S metric space. Let $T : X \to X$ be given by

$$T(x) = \begin{cases} \frac{x}{4}, & \text{for } x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

One can check that T is an A-contraction on $X = [0,1]$ and satisfies all the conditions of the Theorem 2.3. Also T has a unique fixed point at $x = 0$.

Now we show that the above Theorem 2.3 holds for A-contraction mapping, in absence of which, the map T fails to produce any fixed point in the underlying space though other conditions remain invariant.

Example 2.5. Let $X = [0,1] \subset \mathbb{R}$ and $S(x,y,z) = |x-z| + |y-z|$. Then, (X,S) is a complete S metric space. Take a function α as defined in the previous Example 2.4. Then, α satisfies the properties (A_1) and (A_2). If we assume $T : X \to X$ as

$$T(x) = \begin{cases} 1, & \text{for } x \in [0,1) \\ \frac{1}{3}, & \text{for } x = 1. \end{cases}$$

Then T is a self mapping on a complete S-metric space $[0,1]$. Next let $x = \frac{1}{2}$ and $y = 1$, then it is easy to check that $\beta > \frac{1}{2}$, which leads to the conclusion, that T is not an A-contraction mapping. Also, T has no fixed point in X though other conditions of the Theorem 2.3 are being satisfied.

Theorem 2.6. Let (X,S) be a complete S-metric space and let T_1 and T_2 satisfy

$$S(T_1x, T_1x, T_2y) \le \alpha\left(S(x,x,y), S(x,x,T_1x), S(y,y,T_2y)\right)$$

for all $x, y \in X$ and for some α in A. Then T_1 and T_2 have a unique common fixed point in X.

Proof. Let us construct the following sequence in X.

$$x_n = \begin{cases} T_1x_{n-1}, & \text{whenever } n \in \mathbb{N} \text{ is odd and} \\ T_2x_{n-1}, & \text{whenever } n \in \mathbb{N} \text{ is even} \end{cases}$$

Then

$$\begin{aligned} S(x_1,x_1,x_2) &= S(T_1x_0,T_1x_0,T_2x_1) \\ &\le \alpha(S(x_0,x_0,x_1), S(x_0,x_0,T_1x_0), S(x_1,x_1,T_2x_1)) \\ &= \alpha(S(x_0,x_0,x_1), S(x_0,x_0,x_1), S(x_1,x_1,x_2)) \end{aligned} \tag{3}$$

and therefore from the property of α we have $S(x_1,x_1,x_2) \le kS(x_0,x_0,x_1)$. Also, we see that

$$\begin{aligned} S(x_2,x_2,x_3) = S(x_3,x_3,x_2) &= S(T_1x_2,T_1x_2,T_2x_1) \\ &\le \alpha(S(x_2,x_2,x_1), S(x_2,x_2,T_1x_2), S(x_1,x_1,T_2x_1)) \\ &= \alpha(S(x_1,x_1,x_2), S(x_2,x_2,x_3), S(x_1,x_1,x_2)) \end{aligned} \tag{4}$$

and we get from the property of α that $S(x_2,x_2,x_3) \leq kS(x_1,x_1,x_2) \leq k^2S(x_0,x_0,x_1)$. Proceeding in a similar fashion, we see that $S(x_n,x_n,x_{n+1}) \leq kS(x_{n-1},x_{n-1},x_n) \leq k^nS(x_0,x_0,x_1)$ for all $n \in \mathbb{N}$. Then it is a routine calculation to check that $\{x_n\}$ is Cauchy and since X is complete, there exists some $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now,

$$\begin{aligned} S(z,z,T_1z) &\leq 2S(z,z,x_{2n}) + S(T_1z,T_1z,x_{2n}) \\ &= 2S(z,z,x_{2n}) + S(T_1z,T_1z,T_2x_{2n-1}) \\ &\leq 2S(z,z,x_{2n}) + \alpha(S(x_{2n-1},x_{2n-1},z),S(z,z,T_1z), \\ &\quad S(x_{2n-1},x_{2n-1},x_{2n})). \end{aligned} \tag{5}$$

Since α is continuous, taking n tending to infinity we get $S(z,z,T_1z) \leq \alpha(0,S(z,z,T_1z),0)$ implying that $S(z,z,T_1z) = 0$ i.e. $T_1z = z$. In a similar way we can show that $T_2z = z$ and therefore z is a common fixed point of T_1 and T_2. Uniqueness of fixed point is obvious.

□

3. Result in integral setting

In 2002, Branciari [13] first analyzed the existence of fixed point of a contractive mapping of integral type defined over a complete meric space (X,d).

Theorem 3.1. [13] Let (X,d) be a complete metric space, $c \in (0,1)$ and let $f : X \to X$ be a mapping such that for each $x,y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c\int_0^{d(x,y)} \varphi(t)dt \tag{6}$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^nx = a$.

Rhoades [15] extended the result of Branciari by replacing the condition (6) by the following

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c\int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{[d(x,fy)+d(y,fx)]}{2}\}} \varphi(t)dt. \tag{7}$$

Since then numerous generalizations have been made in this direction (see [15], [14] for details). Motivated by these results we apply and prove the analogue of A-contraction mapping over a complete S-metric space.

An important definition is needed to state our theorem in this section.

Definition 1.2. (Sub additivity)
$u : [0,+\infty) \to [0,+\infty)$ is sub additive on each $[a,b] \subset [0,+\infty)$ if

$$\int_0^{a+b} u(t)dt \leq \int_0^a u(t)dt + \int_0^b u(t)dt. \tag{8}$$

Now we state our result as following.

Theorem 3.2. Let T be a self-mapping of a complete S-metric space (X,S) satisfying the following condition:

$$\int_0^{S(Tx,Tx,Ty)} \varphi(t)dt \leq \alpha\left(\int_0^{S(x,x,y)} \varphi(t)dt, \int_0^{S(x,x,Tx)} \varphi(t)dt, \int_0^{S(y,y,Ty)} \varphi(t)dt\right) \tag{9}$$

for each $x,y \in X$ with some $\alpha \in A$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral), sub additive on each $[a,b] \subset [0,+\infty)$, nonnegative, and such that

$$\text{for each } \epsilon > 0, \quad \int_0^\epsilon \varphi(t)dt > 0. \tag{10}$$

Then T has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_n T^nx = z$.

PROOF. Let x_0 be an arbitrary element of X and, for brevity, consider $x_{n+1} = Tx_n$. then for each integer $n \geq 1$, from (9) we get,

$$\begin{aligned}
&\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \\
= &\int_0^{S(Tx_{n-1},Tx_{n-1},Tx_n)} \varphi(t)dt \\
\leq &\ \alpha\left(\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt, \int_0^{S(x_{n-1},x_{n-1},Tx_{n-1})} \varphi(t)dt, \int_0^{S(x_n,x_n,Tx_n)} \varphi(t)dt\right) \\
\leq &\ \alpha\left(\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt, \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt, \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt\right).
\end{aligned}$$

Then by the axiom A_2 of function α,

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \leq k\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \tag{11}$$

for some $k \in [0,1)$ as $\alpha \in A$.
In similar fashion, one can obtain

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt &\leq k\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\leq k^2\int_0^{S(x_{n-2},x_{n-2},x_{n-1})} \varphi(t)dt \\
&\leq ... \\
&\leq k^n\int_0^{S(x_0,x_0,x_1)} \varphi(t)dt.
\end{aligned} \tag{12}$$

Now for $m > n$,

$$\begin{aligned}
S(x_n,x_n,x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i,x_i,x_{i+1}) + S(x_{m-1},x_{m-1},x_m) \\
&\leq 2\sum_{i=n}^{m-1} S(x_i,x_i,x_{i+1}).
\end{aligned}$$

Now applying subadditivity of $\varphi(t)$

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_m)} \varphi(t)dt &\leq \int_0^{2S(x_n,x_n,x_{n+1})} \varphi(t)dt + \int_0^{2S(x_{n+1},x_{n+1},x_{n+2})} \varphi(t)dt + ... \\
&+ \int_0^{2S(x_{m-2},x_{m-2},x_{m-1})} \varphi(t)dt + \int_0^{2S(x_{m-1},x_{m-1},x_m)} \varphi(t)dt \\
&\leq [k^n + k^{n+1} + ... + k^{m-2} + k^{m-1}]\int_0^{2S(x_0,x_0,x_1)} \varphi(t)dt \\
&= k^n[1 + k + ... + k^{m-n-2} + k^{m-n-1}]\int_0^{2S(x_0,x_0,x_1)} \varphi(t)dt \\
&\leq \frac{k^n}{1-k}\int_0^{2S(x_0,x_0,x_1)} \varphi(t)dt.
\end{aligned}$$

Now taking limit as $m,n \to \infty$, we get $\lim_{m,n\to\infty}\int_0^{S(x_n,x_n,x_m)} \varphi(t)dt = 0$ which, from (10) implies that

$$\lim_{m,n} S(x_n,x_n,x_m) = 0.$$

Therefore, $\{x_n\}$ is Cauchy, hence convergent. Call the limit z.
From (9) we get

$$\begin{aligned}\int_0^{S(Tz,Tz,x_{n+1})}\varphi(t)dt &= \int_0^{S(Tz,Tz,Tx_n)}\varphi(t)dt \\ &\leq \alpha\left(\int_0^{S(z,z,x_n)}\varphi(t)dt, \int_0^{S(z,z,Tz)}\varphi(t)dt, \int_0^{S(x_n,x_n,x_{n+1})}\varphi(t)dt\right).\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{S(Tz,Tz,z)}\varphi(t)dt \leq \alpha\left(0, \int_0^{S(z,z,Tz)}\varphi(t)dt, 0\right).$$

So by the axiom A_2 of function α,

$$\int_0^{S(Tz,Tz,z)}\varphi(t)dt = k.0 = 0$$

which, from (10), implies that $S(Tz, Tz, z) = 0$ or, $Tz = z$.
Next suppose that $w(\neq z)$ be another fixed point of T. Then from (9) we have

$$\begin{aligned}\int_0^{S(z,z,w)}\varphi(t)dt &= \int_0^{S(Tz,Tz,Tw)}\varphi(t)dt \\ &\leq \alpha\left(\int_0^{S(z,z,w)}\varphi(t)dt, \int_0^{S(z,z,Tz)}\varphi(t)dt, \int_0^{S(w,w,Tw)}\varphi(t)dt\right) \\ &= \alpha\left(\int_0^{S(z,z,w)}\varphi(t)dt, \int_0^{S(z,z,z)}\varphi(t)dt, \int_0^{S(w,w,w)}\varphi(t)dt\right) \\ &= \alpha\left(\int_0^{S(z,z,w)}\varphi(t)dt, 0, 0\right).\end{aligned}$$

So by the axiom A_2 of function α,

$$\int_0^{S(z,z,w)}\varphi(t)dt = 0$$

which, from (10), implies that $S(z, z, w) = 0$ or, $z = w$ and so the fixed point is unique. □

Remark 3.3. On setting $\varphi(t) = 1$ over $\mathbb{R}^+$, the contractive condition of integral type transforms into a general contractive condition not involving integrals.

4. An application to homotopy

In this section, we obtain a homotopy result as an application of Theorem 2.3. For this purpose first we give the definition of homotopy between two functions.

Definition 4.1. [16] Let X, Y be two topological spaces, and let $G, S : X \to Y$ be two continuous mappings. Then, a homotopy from G to S is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = Gx$ and $H(x, 1) = Sx$, for all $x \in X$. Also, G and S are called homotopic mappings.

Theorem 4.2. Let X be a complete $S-$metric space and U be an open and V be a closed subset of X with $U \subset V$. Let the operator $F : V \times [0, 1] \to X$ satisfies the following conditions:
1) $x \neq F(x, t)$ for every $x \in V \backslash U$ and for any $t \in [0, 1]$,
2) There exists some $\alpha \in A$ such that

$$S(F(x,t), F(x,t), F(y,t)) \leq \alpha(S(x,x,y), S(x,x,F(x,t)), S(y,y,F(y,t))) \quad (13)$$

for all $t \in [0,1]$ and $x, y \in V$,
3) There exists a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$S(F(x,t), F(x,t), F(x,s))) \leq |f(t) - f(s)| \tag{14}$$

$\forall\, t, s \in [0,1]$ and for every $x \in V$,
4) For any $r > 0$ we have $\alpha(a, b, 0) \leq \delta r < r$ whenever $a \leq r$ or $b \leq r$, where $0 < \delta < 1$.
Then $F(.,0)$ has a fixed point if and only if $F(.,1)$ has a fixed point.

Proof. Let us define $G = \{t \in [0,1] : F(x,t) = x \text{ for some } x \in U\}$.
First let us assume that $F(.,0)$ has a fixed point. Then $F(x,0) = x$ for some $x \in U$ since (1) holds. Then $0 \in G$ and thus G is non-empty. We will show that G is a clopen subset of $[0,1]$, then from connectedness of $[0,1]$ we can easily say that $G = [0,1]$.
First we prove that G is open. let $t_0 \in G$ then there exists $x_0 \in U$ such that $F(x_0, t_0) = x_0$ [as (1) holds]. Therefore there exists $r > 0$ such that $B(x_0, r) \subset U$, where $B(x_0, r) = \{x \in X : S(x, x, x_0) < r\}$. Now let, $x \in \overline{B(x_0, r)} = \{x \in X : S(x, x, x_0) \leq r\}$ and we choose

$$\epsilon = \frac{1}{2}\left[r - \sup_{x \in \overline{B(x_0,r)}} \alpha\left(S(x,x,x_0), S(x,x,F(x,t_0)), 0\right)\right].$$

Therefore $\epsilon > 0$ by condition (4). Since f is continuous on $[0,1]$, there exists $\eta(\epsilon) > 0$ such that $|f(t) - f(t_0)| < \epsilon$ whenever $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon)) \subset [0,1]$. Now,

$$\begin{aligned} S(F(x,t), F(x,t), x_0) &= S(F(x,t), F(x,t), F(x_0,t_0)) \\ &\leq 2S(F(x,t), F(x,t), F(x,t_0)) + S(F(x_0,t_0), F(x_0,t_0), \\ &\quad F(x,t_0)) \\ &= 2S(F(x,t), F(x,t), F(x,t_0)) + S(F(x,t_0), F(x,t_0), \\ &\quad F(x_0,t_0)) \\ &\leq 2|f(t) - f(t_0)| + \alpha(S(x,x,x_0), S(x,x,F(x,t_0)), \\ &\quad S(x_0,x_0,F(x_0,t_0))) \\ &= 2|f(t) - f(t_0)| + \alpha(S(x,x,x_0), S(x,x,F(x,t_0)), 0). \end{aligned} \tag{15}$$

Therefore, whenever $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon)) \subset [0,1]$, we get $S(F(x,t), F(x,t), x_0) \leq r$ implying that $F(x,t) \in \overline{B(x_0,r)}$. Therefore $F(.,t) : \overline{B(x_0,r)} \to \overline{B(x_0,r)}$ for every fixed $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon))$. Now since $F(.,t)$ satisfies all the conditions of Theorem 2.3 we have, $F(.,t)$ has a fixed point in $\overline{B(x_0,r)} \subset V$, but it must be in U as condition (1) holds. Therefore $t \in G$ for every $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon))$. Hence $(t_0 - \eta(\epsilon), t_0 + \eta(\epsilon)) \subset G$. So G is open in $[0,1]$.
Now we show that G is closed also. Let $\{t_n\} \subset G$ such that $t_n \to t^* \in [0,1]$ as $n \to \infty$. Then there exists $x_n \in U$ such that $x_n = F(x_n, t_n)$ for all $n \in \mathbb{N}$. Moreover we have,

$$\begin{aligned} S(x_n, x_n, x_m) &= S(F(x_n,t_n), F(x_n,t_n), F(x_m,t_m)) \\ &\leq 2S(F(x_n,t_n), F(x_n,t_n), F(x_n,t_m)) + S(F(x_n,t_m), F(x_n,t_m), \\ &\quad F(x_m,t_m)) \\ &\leq 2|f(t_n) - f(t_m)| + \alpha(S(x_n,x_n,x_m), S(x_n,x_n,F(x_n,t_m)), \\ &\quad S(x_m,x_m,F(x_m,t_m)) \\ &= 2|f(t_n) - f(t_m)| + \alpha(S(x_n,x_n,x_m), S(x_n,x_n,F(x_n,t_m)), 0) \\ &\leq 2|f(t_n) - f(t_m)| + \delta S(x_n,x_n,x_m) \end{aligned} \tag{16}$$

which implies $S(x_n, x_n, x_m) \leq \frac{2}{1-\delta}|f(t_n) - f(t_m)| \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is Cauchy in X and since X is complete thus it converges to some $x^* \in V$. Now we show that $F(x^*, t^*) = x^*$. Here we see that,

$$\begin{aligned} S(x_n, x_n, F(x^*,t^*)) &= S(F(x_n,t_n), F(x_n,t_n), F(x^*,t^*)) \\ &\leq 2S(F(x_n,t_n), F(x_n,t_n), F(x_n,t^*)) \\ &\quad + S(F(x_n,t^*), F(x_n,t^*), F(x^*,t^*)) \\ &\leq 2|f(t_n) - f(t^*)| + \alpha(S(x_n,x_n,x^*), S(x_n,x_n,F(x_n,t^*)), \\ &\quad S(x^*,x^*,F(x^*,t^*))). \end{aligned}$$

Now $S(x_n, x_n, F(x_n, t^*)) \leq |f(t_n) - f(t^*)| \to 0$ as $n \to \infty$. Thus using continuity of α we get,

$$S(x^*, x^*, F(x^*, t^*)) \leq \alpha(0, 0, S(x^*, x^*, F(x^*, t^*))) \quad (17)$$

and therefore by the property of α we have, $S(x^*, x^*, F(x^*, t^*)) \leq k.0 = 0$ implying that $S(x^*, x^*, F(x^*, t^*)) = 0$ that is $F(x^*, t^*) = x^*$. Therefore by condition (1) we get $x^* \in U$ and so $t^* \in G$. Hence G is closed also and so $G = [0, 1]$ that is $F(., 1)$ has also a fixed point. The converse part can be shown in a similar way. □

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References

[1] S.Gähler, *2-metric Raume and ihre topologische strucktur,* Math. Nachr. 26 (1963) 115-148.

[2] S.Gähler, *Lineare 2-normietre Raume*, Math. Nachr. 28 (1965) 1-43.

[3] K. S. Ha, Y. J. Cho, and A. White, *Strictly convex and strictly 2-convex 2-normed spaces*, Mathematica Japonica 33(3) 1988 375-384.

[4] B.C. Dhage, *Generalized metric spaces mappings with fixed point*, Bull. Calcutta Math. Soc. 84 (1992) 329-336.

[5] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear Convex Anal. 7 (2006) 289-297.

[6] S. Sedghi, K.P.R. Rao, N. Shobe, *Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces,* Internat J. Math. Math. Sci. 6 (2007) 225-237.

[7] S. Sedghi, N. Shobe, H. Zhou, *A common fixed point theorem in D^*-metric spaces, Fixed Point Theory Appl.* (2007) Article ID 27906 13 pages.

[8] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces,* Mat. Vesnik 64(3) (2012) 258-266.

[9] S. Sedghi and N. V. Dung, *Fixed point theorems on S-metric spaces,* Mat. Vesnik 66(1) (2014) 113-124.

[10] M. Akram, A. A. Siddiqui, *A fixed point theorem for A-contractions on a class of generalised metric spaces,* Korean J. Math. Sciences 10(2) (2003) 1-5.

[11] M. Akram, A. A. Zafar, A. A. Siddiqui, *A general class of contractions: A- contractions,* Novi Sad J. Math. 38(1) (2008) 25-33.

[12] M. Saha, D. Dey, *Fixed point theorems for a class of A-contractions on a 2-metric space*, Novi Sad J. Math. 40(1) (2010) 3-8.

[13] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type,* Int. J. Math. Math. Sci 29(9) (2002) 531-536.

[14] D. Dey, A. Ganguly, M. Saha, *Fixed point theorems for mappings under general contractive condition of integral type*, Bull. Math. Anal. Appl. 3(1) (2011) 27-34.

[15] B.E . Rhoades, *Two fixed point theorems for mappings satisfying a general contractive condition of integral type,* International Journal of Mathematics and Mathematical Sciences 63 (2003) 4007-4013.

[16] C. Vetro, F. Vetro, *A Homotopy Fixed Point Theorem in 0-Complete Partial Metric Space,* Filomat 29(9) (2015) 2037-2048.

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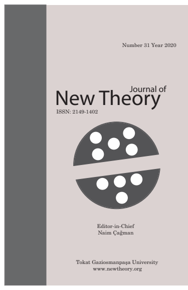

Another Decomposition of Nano Continuity Using Ng^*-Closed Sets

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Abstract − There are various types of nano generalization of continuous function in the development of nano topology. In this paper, we obtain a decomposition of nano continuity using a nano generalized continuity called nano g^*-continuity in nano topology.

Keywords − *Ng^*-closed set, $Nglc^*$-set, nano g^*-continuous function, $Nglc^*$-continuous function*

1. Introduction

Different types of nano generalizations of continuous function were introduced and studied by various authors in the recent development of nano topology. The decomposition of nano continuity is one of the many problems in nano topology. Recently, Ganesan et. al. [2] obtained on some decomposition of nano continuity. In this paper, we obtain a decomposition of nano continuity in nano topological spaces using nano g^*-continuity in nano topological spaces.

2. Preliminary

Definition 2.1. [3] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$.
   i.e., $L_R(X) = \bigcup_{x \in U}\{R(X) : R(X) \subseteq X\}$ where $R(x)$ denotes the equivalence class determined by X.

2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$.
   i.e., $U_R(X) = \bigcup_{x \in U}\{R(X) : R(X) \cap \mathrm{X} \neq \emptyset\}$

3. The boundary region of X with respect to R is the set of all objects, which can be neither in nor as not-X with respect to R and it is denoted by $B_R(X)$.
   i.e., $B_R(X) = U_R(X) - L_R(X)$.

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Proposition 2.2. [3] If (U, R) is an approximation space and $X, Y \subseteq U$, then

1. $L_R(X) \subseteq$X$\subseteq U_R(X)$.
2. $L_R(\emptyset) \ = U_R(\emptyset) \ = \emptyset, L_R(U) = U_R(U) = U$.
3. $U_R(X \cup Y) \ = U_R(X) \cup U_R(Y)$.
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$.
5. $L_R(X \cup Y) \ \supseteq L_R(X) \cup L_R(Y)$.
6. $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$.
7. $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$.
8. $U_R(X^c) = [L_R((X)]^c$ and $L_R(X^c = [U_R(X)]^c$.
9. $U_R(U_R(X)) = L_R(U_R(X)) = U_R(X)$.
10. $L_R(L_R(X)) = U_R(L_R(X)) = L_R(X)$.

Definition 2.3. [3] Let U be an universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$, where $X \subseteq U$. Then, by proposition 2.2, $\tau_R(X)$ satisfies the following axioms

1. $U, \emptyset \in \tau_R(X)$.
2. The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

   Then, $\tau_R(X)$ is called the nano topology on U with respect to X.

   The space $(K, \tau_R(X))$ is the nano topological space. The elements of are called nano open sets.

Definition 2.4. [3] If $(U, \tau_R(X))$ is the nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then

1. The nano interior of the set M is defined as the union of all nano open subsets contained in A and it is denoted by $NInt(A)$. That is, $NInt(A)$ is the largest nano open subset of A.
2. The nano closure of the set A is defined as the intersection of all nano closed sets containing A and it is denoted by $NCl(A)$. That is, $NCl(A)$ is the smallest nano closed set containing A.

Definition 2.5. Let $(U, \tau_R(X))$ be a nano topological space. A subset A of $(U, \tau_R(X))$ is called

1. Nano generalised closed (briefly, Ng-closed) set [1] if $Ncl(A) \subseteq V$ whenever $A \subseteq V$ and V is nano open in $(U, \tau_R(X))$. The complement of Ng-closed set is called Ng-open.
2. Nano generalised star closed (briefly, Ng^*-closed) set [5] Ncl(A) $\subseteq V$ whenever $A \subseteq V$ and V is Ng-open in $(U, \tau_R(X))$. The complement of Ng^*-closed set is called Ng^*-open.

Definition 2.6. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau'_R(Y))$ is called:

1. nano continuous [4] if the inverse image of every nano closed set in V is nano closed in U.
2. nano g^*-continuous [6] if the inverse image of every nano closed set in V is Ng^*-closed in U.

Proposition 2.7. [5] Every nano closed set is Ng^*-closed set but not conversely.

Proposition 2.8. [6] Every nano continuous function is nano g^*-continuous but not conversely.

3. Decomposition of nano continuity

In this section, we obtain a decomposition of nano continuity in nano topological spaces by using nano g^*-continuity.

To obtain a decomposition of nano continuity, we first introduce the notion of $Nglc^*$-continuous function in nano topological spaces and prove that a function is nano continuous if and only if it is both nano g^*-continuous and $Nglc^*$-continuous.

Definition 3.1. A subset A of a space $(U, \tau_R(X))$ is said to be $Nglc^*$-set if $A = M \cap O$, where M is Ng-open set and O is nano closed in $(U, \tau_R(X))$.

Example 3.2. Let $U = \{a, b, c\}$, with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, the nano topology $\tau_R(X) = \{U, \emptyset, \{a\}\}$. Then, nano closed are U, $\emptyset$, and $\{b, c\}$. Then, $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Here, the set $\{c\}$ is $Nglc^*$-set in $(U, \tau_R(X))$.

Proposition 3.3. Every nano closed set is $Nglc^*$-set but not conversely.

PROOF. It is follows from Definition 3.1.

Example 3.4. Let $U = \{a, b, c\}$ with $U/R = \{\{b\}, \{a, c\}\}$ and $X = \{b\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{b\}\}$. The nano closed sets are U, $\emptyset$, and $\{a, c\}$. Then, $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Here, the set $\{a, b\}$ is $Nglc^*$-set but not nano closed in $(U, \tau_R(X))$.

Remark 3.5. Ng^*-closed sets and $Nglc^*$-sets are independent of each other.

Example 3.6. Let $U = \{a, b, c\}$ with $U/R = \{\{c\}, \{a, b\}, \{b, a\}\}$ and $X = \{a, b\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{a, b\}\}$. The nano closed sets are U, $\emptyset$, and $\{c\}$. Then, $Ng^*(U, \tau_R(X)) = \{U, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ and $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}\}$. Here, the set $\{a, c\}$ is an Ng^*-closed but not $Nglc^*$-set in $(U, \tau_R(X))$.

Example 3.7. Let $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a, c\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}\}$. The nano closed sets are U, $\emptyset$, $\{a\}$, and $\{b, c\}$. Then, $Ng^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b, c\}\}$ and $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Here, the set $\{a, b\}$ is an $Nglc^*$-set but not Ng^*-closed in $(U, \tau_R(X))$.

Proposition 3.8. Let $(U, \tau_R(X))$ be a nano topological space. Then, a subset A of $(U, \tau_R(X))$ is nano closed if and only if it is both Ng^*-closed and $Nglc^*$-set.

PROOF. Necessity is trivial. To prove the sufficiency, assume that A is both Ng^*-closed and $Nglc^*$-set. Then, $A = M \cap O$, where M is Ng-open set and O is nano closed set in $(U, \tau_R(X)))$. Therefore, $A \subseteq M$ and $A \subseteq O$ and so by hypothesis, $N_{cl}(A) \subseteq M$ and $N_{cl}(A) \subseteq O$. Thus, $N_{cl}(A) \subseteq M \cap O = A$ and hence $Ncl(A) = A$ i.e., A is nano closed set in $(U, \tau_R(X))$.

Definition 3.9. Let $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is called $Nglc^*$-continuous if for each nano closed set B of $(V, \tau'_R(Y))$, $f^{-1}(B)$ is $Nglc^*$-set of $(U, \tau_R(X))$.

Example 3.10. Let $U = \{a, b, c\}$ with $U/R = \{\{c\}, \{a, b\}\}$ and $X = \{c\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{c\}\}$. Then, nano closed sets are U, $\emptyset$, and $\{a, b\}$. Then, $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $V = \{a, b, v\}$ with $V/R' = \{\{b\}, \{a, c\}\}$ and $Y = \{a, b\}$. Then, the nano topology $\tau'_R(Y) = \{V, \emptyset, \{b\}, \{a, c\}\}$. The nano closed sets are V, $\emptyset$, $\{b\}$, and $\{a, c\}$. Let $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ be the identity function. Then, f is $Nglc^*$-continuous function. Since for the nano closed set $\{a, c\}$ in $(V, \tau'_R(Y))$, $f^{-1}(\{a, c\}) = \{a, c\}$, which is $Nglc^*$ set in $(U, \tau_R(X))$.

Proposition 3.11. Every nano continuous function is $Nglc^*$-continuous but not conversely..

Example 3.12. Let $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{a\}\}$. The nano closed sets are U, $\emptyset$, and $\{b, c\}$. Then, $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $V = \{a, b, c\}$ with $V/R' = \{\{a\}, \{b, c\}\}$ and $Y = \{a, c\}$. Then, nano topology $\tau'_R(Y) = \{V, \emptyset, \{a\}, \{b, c\}\}$. The nano closed sets are V, $\emptyset$, $\{a\}$, and $\{b, c\}$. Let $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ be the identity function. Then, f is $Nglc^*$-continuous function. Since for the nano closed set $\{a\}$ in $(V, \tau'_R(Y))$, $f^{-1}(\{a\}) = \{a\}$, which is not nano closed in $(U, \tau_R(X))$, f is not nano continuous.

Remark 3.13. Nano g^*-continuity and $Nglc^*$-continuity are independent of each other.

Example 3.14. Let $U = \{a, b, c\}$ with $U/R = \{\{c\}, \{a, b\}, \{b, a\}\}$ and $X = \{a, b\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{a, b\}\}$. The nano closed sets are U, $\emptyset$, and $\{c\}$. Then, $Ng^*(U, \tau_R(X)) = \{U, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ and $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}\}$. Let $V = \{a, b, c\}$ with $V/R' = \{\{a\}, \{b, c\}\}$ and $Y = \{a\}$. Then, nano topology $\tau'_R(Y) = \{V, \emptyset, \{a\}\}$. The nano closed sets are V, $\emptyset$, and $\{b, c\}$. Let $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ be the identity function. Then, f is nano g^*-continuous function. Since for the nano closed set $\{b, c\}$ in $(V, \tau'_R(Y))$, $f^{-1}(\{b, c\}) = \{b, c\}$, which is not $Nglc^*$-set in $(U, \tau_R(X))$, f is not $Nglc^*$-continuous.

Example 3.15. Let $U = \{a, b, c\}$ with $U/R = \{\{b\}, \{a, c\}\}$ and $X = \{b\}$. Then, nano topology $\tau_R(X) = \{U, \emptyset, \{b\}\}$. The nano closed sets are U, $\emptyset$, and $\{a, c\}$. Then, $Ng^*(U, \tau_R(X)) = \{U, \emptyset, \{a, c\}\}$ and $Nglc^*(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $V = \{a, b, c\}$ with $V/R' = \{\{a\}, \{b, c\}, \{c, b\}\}$ and $Y = \{b, c\}$. Then, nano topology $\tau'_R(Y) = \{V, \emptyset, \{b, c\}\}$. The nano closed sets are V, $\emptyset$, and $\{a\}$. Let $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ be the identity function. Then, f is $Nglc^*$-continuous function. Since for the nano closed set $\{a\}$ in $(V, \tau'_R(Y))$, $f^{-1}(\{a\}) = \{a\}$, which is not Ng^*-closed set in $(U, \tau_R(X))$, f is not nano g^*-continuous.

We have the following decomposition for nano continuity

Theorem 3.16. A function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is nano continuous if and only if it is both nano g^*-continuous and $Nglc^*$-continuous.

PROOF. Assume that f is nano continuous. Then, by Proposition 2.8 and Proposition 3.11, f is both nano g^*-continuous and $Nglc^*$-continuous.

Conversely, assume that f is both nano g^*-continuous and $Nglc^*$-continuous. Let B be a nano closed subset of $(V, \tau'_R(Y))$. Then, $f^{-1}(B)$ is both Ng^*-closed and $Nglc^*$-set. By Proposition 3.8, $f^{-1}(B)$ is a nano closed set in $(U, \tau_R(X))$ and so f is nano continuous.

References

[1] K. Bhuvaneshwariand and K. Mythili Gnanapriya, Nano generalized closed sets in nano topological spaces, International Journal of Scientific and Research Publications 4(5) (2014) 01-03.

[2] S. Ganesan, M. Meena and R. Selva Vinayagam, On decomposition of nano continuity, Journal of Applied Science and Computations 5(11) (2018) 1270- 1287.

[3] M. LellisThivagar and Carmel Richard, On nano forms of weakly open sets, International Journal of Mathematics and Statistics Invention 1(1) (2013) 31-37.

[4] M. LellisThivagar and Carmel Richard, On nano continuity, Mathematical Theory And Modeling 3(7) (2013) 32-37.

[5] V. Rajendran P. Sathishmohan and K. Indirani, On nano generalized star closed sets in nano topological spaces, International Journal of Applied Research 1(9) (2015) 04-07.

[6] V. Rajendran P. Sathishmohan and R. Nithyakala, On new class of continuous functions in nano topological spaces, Malaya Journal of Matematic 6(2) (2018) 385-389.

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A New Class of Closed Set in Digital Topology

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Abstract — The purpose of this paper is to introduce a new class of closed set called g*$\omega\alpha$-closed sets in digital topology. We establish a relationship between closed and g*$\omega\alpha$-closed sets in digital topology. Also, we obtained the properties of g*$\omega\alpha$-closed sets in digital plane.

Keywords — *g*$\omega\alpha$-closed sets, g*$\omega\alpha$-open sets, digital plane.*

1. Introduction

In the literature, the concept of Digital Topology was first introduced and studied in the late 60's by the computer image analysis researcher Azriel Rosenfeld [1]. The digital line, the digital plane and the three dimensional digital spaces are of great importance in the study of applications of point set topology to computer graphics. Digital Topology consist in providing algorithmic tools for pattern recognition, image analysis and image processing using a discrete formalism for geometrical objects and it is applied in image processing.

First we recall the related definitions and some properties of the digital plane. The digital line or called Khalimsky Line is the set of integers Z, equipped with the topology K having 2n+1, 2n, 2n-1 : n $\in$ Z as a subbas and is denoted by (Z, K). Thus, a subset U is open in (Z, K) if and only if, whenever x $\in$ U is an even integer, then x-1, x+1 $\in$ U. Let (Z^2, K^2) be the topological product of two digital lines (Z, K), where Z^2 = Z $\times$ Z and K^2 = K $\times$ K. This space is called the digital plane ([2], [3], [4], [5], [6], [7]). For each point x $\in$ Z^2, there exists a smallest open set containing x say U(x). For the case of x = (2n+1, 2m+1), U(x) = 2n+1 $\times$ 2m+1 ; for the case of x = (2n, 2m), U(x) = 2n-1, 2n, 2n+1 $\times$ 2m-1, 2m, 2m+1 ; for the case of x = (2n, 2m+1), U(x) = 2n-1, 2n, 2n+1 $\times$ 2m+1 ; for the case of x = (2n+1, 2m), U(x) = 2n+1 $\times$ 2m-1, 2m, 2m+1 where n, m $\in$ Z.

For a subset E of (Z^2, K^2), we have the following three subsets as follows:

E_F = x $\in$ E: x is closed in (Z^2, K^2) ; E_{K^2} = x $\in$ E: x is open in (Z^2, K^2) ; E_{mix} = E $\setminus$ (E_F $\cup$ E_{K^2}). Then it is shown that E_F = (2n, 2m) $\in$ E: n, m $\in$ Z ; E_{K^2} = (2n+1, 2m+1) $\in$ E: n, m $\in$ Z and E_{mix} = (2n, 2m+1) $\in$ E: n, m $\in$ Z $\cup$ (2n+1, 2m) $\in$ E : n, m $\in$ Z.

In the digital plane if the corner points of a digital plane are even then it is called closed set. If the corner points of a digital plane are odd it is called an open set.

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2. Preliminaries

Definition 2.1. A subset A of a topological space X is called a

(i) g*$\omega\alpha$-closed [8] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open in X.

(ii) g*$\omega\alpha$-open [8] if U $\subseteq$ int(A) whenever U $\subseteq$ A and U is $\omega\alpha$-closed in X.

(iii) $\omega\alpha$-closed [9] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is ω-open in X.

Definition 2.2. A subset A of a topological space X is called a

(i) T_g*$\omega\alpha$-space [10] if every g*$\omega\alpha$-closed set is closed.

(ii) g$\omega\alpha T_g\omega\alpha$-space [10] if every g$\omega\alpha$-closed set is g*$\omega\alpha$-closed.

(iii) $T_{g\omega\alpha}$-space [11] if every g$\omega\alpha$-closed set is closed.

3. g*$\omega\alpha$-Closed Sets in Digital Plane

Lemma 3.1. [4] Let (Z^2, K^2) be a digital plane. Then the following properties hold:

(i) if m is even point, that is m = (2n, 2m), then cl(2n, 2m) = 2n, 2m

(ii) if m is odd point, that is m = (2n+1, 2m+1), then
cl(2n+1, 2m+1) = 2n, 2n+1, 2n+2 $\times$ 2m, 2m+1, 2m+2

(iii) if m is mixed point, that is m = (2n+1, 2m) or (2n, 2m+1), then
cl(2n, 2m+1) = 2n $\times$ 2m ,2m+1, 2m+2
cl(2n+1, 2m) = 2n, 2n+1, 2n+2 $\times$ 2m

Theorem 3.2. Every closed in (Z^2, K^2) is g*$\omega\alpha$-closed in (Z^2, K^2).

PROOF. Let A be a subset of (Z^2, K^2). Let us consider the following three cases:

(i) The set A contains all even points ($E_F \subseteq$ A)
that is A = (2n, 2m), then U(A) = { 2n-1, 2n, 2n+1 } $\times$ { 2m-1, 2m, 2m+1 }. Let A $\subseteq$ U and U is $\omega\alpha$-open (Z^2, K^2). Then by lemma 3.1, cl(A) = cl({ 2n, 2m }) = { 2n, 2m } = A,
that is cl(A) = A.
This implies cl(A) = A $\subseteq$ U. Hence A is g*$\omega\alpha$-closed in (Z^2, K^2).

(ii) The set A contains all even, odd and mixed points ($E_{mix} \cup E_F \cup E_{K^2} \subseteq$ A)
Let A ={ 2n, 2n$\pm$1, 2n$\pm$2 ... 2n$\pm$q $\pm$ 2m, 2m$\pm$1, 2m$\pm$2 ... 2m$\pm$q} where n, m and q are even integer.
Then U = {2n, 2n$\pm$1, 2n$\pm$2 . . . 2n$\pm$q, 2n$\pm$q$\pm$1 } $\times$ { 2m, 2m$\pm$1, 2m$\pm$2 ... 2m$\pm$q, 2m$\pm$q$\pm$1}.
Let A $\subseteq$ U, where U is $\omega\alpha$-open in (Z^2, K^2). Since A is closed, cl(A) = A. Therefore cl(A) = A $\subseteq$ U. Therefore A is g*$\omega\alpha$-closed in (Z^2, K^2).

(iii) A contains all even and mixed points ($E_F \cup E_{mix} \subseteq$ A) Let A = { 2n } $\times$ { 2m, 2m$\pm$1, 2m$\pm$2 ... 2m$\pm$q }, where n, m, q are even integers. Then U = { 2n, 2n$\pm$1, 2n$\pm$2 ... 2n$\pm$q, 2n$\pm$q$\pm$1 } $\times$ { 2m, 2m$\pm$1, 2m$\pm$2 ... 2m$\pm$q, 2m$\pm$q$\pm$1 } for any n, m, q are even integers. Let A $\subseteq$ U, where U is $\omega\alpha$-open in (Z^2, K^2). Since A is closed, cl(A) = A $\subseteq$ U, that is cl(A) $\subseteq$ U. Therefore A is g*$\omega\alpha$-closed set in (Z^2, K^2). Similarly, A is g*$\omega\alpha$-closed by considering A = { 2n, 2n$\pm$1, 2n$\pm$2 ... 2n$\pm$q } $\times$ {2m }.

□

Remark 3.3. The following example shows that the converse is not true in general.

Example 3.4. Let A = (2,2), (2,3), (2,4), (3,3), (3,4), (4,4) and U = {1, 2, 3, 4, 5} $\times$ {1, 2, 3, 4, 5} Then cl(A) = {2, 3, 4} $\times$ {2, 3, 4} , that is cl(A) $\subseteq$ U. Therefore A is g*$\omega\alpha$-closed in (Z^2, K^2). But cl(A) = A, hence A is not closed in (Z^2, K^2).

Theorem 3.5. Every open set in (Z^2, K^2) is $g^*\omega\alpha$-open in (Z^2, K^2).

PROOF. Let us consider the following three cases:

Case (i): A contains all odd points ($E_{K^2} \subseteq A$). Let A = (2n+1, 2m+1) and U = φ. Assume that U$\subseteq$ A, where A is $\omega\alpha$-closed in (Z^2, K^2). Then U $\subseteq$ A = int(A), as A is open in (Z^2, K^2). Therefore U $\subseteq$ int(A) and U is $\omega\alpha$-closed in (Z^2, K^2). Hence A is $g^*\omega\alpha$-open in (Z^2, K^2).

Case (ii): A contains all even, odd and mixed points ($E_{mix} \cup E_{K^2} \cup E_F \subseteq A$) Let A = { 2n, 2n±1, 2n±2 … 2n±q } × { 2m, 2m±1, 2m±2 … 2m±q }, for any n, m, q is odd integers and U = { 2n, 2n±1, 2n±2 … 2n±q, 2n±(q-1) } × { 2m, 2m±1, 2m±2 … 2m±q, 2m±(q-1) }. Let A $\subseteq$ U, where U is $\omega\alpha$-closed in (Z^2, K^2), that is U $\subseteq$ int(A) = A, as A is open in (Z^2, K^2). Therefore U $\subseteq$ int(A) and hence A is $g^*\omega\alpha$-open in (Z^2, K^2).

Case (iii): A contains odd and mixed points ($E_{mix} \cup E_{K^2} \cup E_F \subseteq A$) A contains odd and mixed points ($E_{K^2} \cup E_{mix} \subseteq A$) Let A = {2n+1} × { 2m, 2m±1, 2m±2 … 2m±q} and U = φ. Let U $\subseteq$ A, where U is $\omega\alpha$-closed in (Z^2, K^2). That is U $\subseteq$ A = int(A), because A is open in (Z^2, K^2). Therefore U $\subseteq$ int(A) and hence A is $g^*\omega\alpha$-open in (Z^2, K^2).

□

Example 3.6. The converse of the above theorem is not true follows from the example 3.4.

Theorem 3.7. If A is $g^*\omega\alpha$-closed in (Z^2, K^2), then it does not contain all odd points ($E_{K^2} \not\subset A$).

PROOF. Let A be any set in (Z^2, K^2) which contains all odd points. Let A = (2n+1, 2m+1) and U = (2n+1, 2m+1) be $\omega\alpha$-open set in (Z^2, K^2). Then cl(A) = {2n, 2n+1, 2n+2} × {2m, 2m+1, 2m+2} , we get cl(A) U, which is contradiction to the assumption. Hence A does not contain all odd points. □

Theorem 3.8. A $g^*\omega\alpha$-open set A in (Z^2, K^2) does not contain all even points ($E_F \not\subset A$).

PROOF. Let A be any set in (Z^2, K^2) which contain all even points. Let A = (2n, 2m) and U = (2n, 2m) be any $\omega\alpha$-closed in (Z^2, K^2). Let us assume that U $\subseteq$ A, where U is $\omega\alpha$-closed in (Z^2, K^2). Then U int(A), since int(A) = φ, if A is even, which is contradiction to the fact that A contains all even points. Hence A does not contain all even points. □

Remark 3.9. Union of an open and $g^*\omega\alpha$-open set is again a $g^*\omega\alpha$-open.

Theorem 3.10. Let A and E be subsets of (Z^2, K^2)

(i) if E is non empty $g^*\omega\alpha$-closed, then $E_F \neq \varphi$.

(ii) if E is $\omega\alpha$-closed and E $\subseteq$ $B_{mix} \cup B_{K^2}$ holds for some subset B of (Z^2, K^2), then E = φ.

(iii) The set U(A_F) $\cup$ A_{mi} $\cup$ A_{k^2} is $g^*\omega\alpha$-open containing A.

PROOF. .

Case (i) Let y be any point in E, then y $\in$ cl(E) = E, as E is closed. Let us consider the following three cases: Let y $\in$ E_F , then $E_F \neq \varphi$. Let y $\in$ E_{K^2} , that is y = (2n+1, 2m+1) where n, m $\in$ Z. Then cl({ y }) = { 2n, 2n+1, 2n+2 } × { 2m, 2m+1, 2m+2 } $\subseteq$ E. Thus, there exists a point x = (2n, 2m) such that x $\in$ E_F. Therefore $E_F \neq \varphi$. Let y $\in$ E_{mix}, that is y = (2n+1, 2m). Then cl({y}) = { 2n, 2n+1, 2n+2 } × { 2m} $\subseteq$ E. Thus there exists a point x = (2n, 2m) $\subseteq$ E_F such that $E_F \neq \varphi$. Similarly, $E_F \neq \varphi$ for x = (2n, 2m + 1). Therefore in all the three cases we have $E_F \neq \varphi$.

Case (ii) Suppose on the contrary E $\neq \varphi$. From case (i), $E_F \neq \varphi$. By hypothesis $E_F \subseteq (B_{mix} \cup B_{K^2})_F$ = φ, which is contradiction to the assumption. Hence E = φ.

Case (iii) We have to prove that U(A_F) ∪ A_{mix} ∪ A_{K^2} is g*$\omega\alpha$-open containing A. We know that U(A_F) is an open set in (Z^2, K^2). Then we have to show that A_{mix} ∪ A_{K^2} is g*$\omega\alpha$-open in (Z^2,K^2). Let F be any non-empty $\omega\alpha$-closed set such that F ⊆ A_{mix} ∪ A_{K^2} . But from case (ii), F = φ, implies F ⊆ (A_{mix} ∪ A_{K^2}). Thus, A_{mix} ∪ A_{K^2} is g*$\omega\alpha$-open in (Z^2, K^2) and U(A_F) is an open set in (Z^2, K^2). Therefore U(A_F) ∪ Amix ∪ A_{K^2} is g*$\omega\alpha$-open set in (Z^2, K^2) by remark 3.9. Therefore A ⊆ U(A_F) ∪ Amix ∪ A_{K^2}. Therefore U(A_F) ∪ A_{mix} ∪ A_{K^2} is g*$\omega\alpha$-open set containing A.

□

Remark 3.11. [10] If X is a T_g*$\omega\alpha$-space, then every singleton set {x} is either open or $\omega\alpha$-closed.

Theorem 3.12. The digital plane (Z , K) is a T_g*$\omega\alpha$-space.

Proof. Let {x} be any point in (Z^2, K^2).
Let us consider the following three cases:

Case (i) if {x} is odd, that is x = (2n+1, 2m+1), then {x} is open in (Z^2, K^2).

Case (ii) if {x} is even, that is x = (2n, 2m). Then {x} is closed in (Z^2, K^2).

Case (iii) if {x} is a mixed point, that is x = (2n, 2m+1) or x = (2n+1, 2m).
Let U be any α-open set containing { x }. Then αcl({ x }) = { x } ∪ cl(int(cl({ x }))) ={ x } ∪ cl(int([{ 2n } × { 2m, 2m+1, 2m+2 }])) = { x } ∪ cl(φ) = { x } ⊆ U. Therefore { x } is $\omega\alpha$-closed in (Z^2, K^2). Similarly, {x} is $\omega\alpha$-closed in (Z^2, K^2) for x = (2n+1, 2m). Thus, we have, {x} is either open or $\omega\alpha$-closed in all the cases. Therefore from remark 3.11, we have (Z^2, K^2) is T_g*$\omega\alpha$-space. Hence (Z^3, K^2) is T_g*$\omega\alpha$-space.

□

Corollary 3.13. The digital plane (Z^2, K^2) is a $T_{g\omega\alpha}$-space.

Theorem 3.14. The digital plane (Z^2, K^2) is ${}_{g\omega\alpha}T_g{}^*{}_{\omega\alpha}$-space.

Proof. Let A be g$\omega\alpha$-closed in (Z^2, K^2). From corollary 3.13, (Z^2, K^2) is $T_{g\omega\alpha}$-space, so A is closed. From [8], every closed set in g*$\omega\alpha$-closed. Hence A is g*$\omega\alpha$-closed in (Z^2, K^2). Hence (Z^2, K^2) is ${}_{g\omega\alpha}T_g$*$\omega\alpha$-space. □

Theorem 3.15. Let B be a non empty subset of (Z^2, K^2). If B = φ, then B is g*$\omega\alpha$-open in (Z^2,K^2).

Proof. Let F be an $\omega\alpha$-closed in (Z^2, K^2) such that F ⊆ B. From hypothesis, B_E = φ, then B = B$_mix$ ∪ B_{K^2}. Then from Theorem 3.10 (ii), we have F = φ. Thus, we say that, whenever F is $\omega\alpha$-closed and F ⊆ B, F = φ ⊆ int(B). This implies F ⊆ int(B). Thus B is g*$\omega\alpha$-open in (Z^2, K^2). □

Remark 3.16. [8] A topological space X is said to be g*$\omega\alpha$-closed if and only if cl(A) \ A does not contain any non empty $\omega\alpha$-closed sets.

Theorem 3.17. Let A be a subset of (Z^2, K^2) and x be a point of (Z^2, K^2). If A is g*$\omega\alpha$-closed in (Z^2, K^2) and x ∈ A_{mix}, then cl({x}) \ {x} ⊆ A and hence cl({x}) ⊆ A.

Proof. From thy hypothesis, we have x ∈ Amix, that is x = (2n, 2m+1) or x = (2n+1, 2m).
Let x = (2n, 2m+1). Then cl({ x }) = cl({ 2n, 2m+1 }) = { 2n } × { 2m, 2m+1, 2m+2 } = {(2n, 2m) (2n, 2m+1) (2n, 2m+2)} = { x_1, x , x_2 }, where x_1 = (2n, 2m), x = (2n, 2m+1) and x_2 = (2n, 2m+2).
Let x = (2n+1, 2m), then cl({ x }) = cl({ 2n+1, 2m }) = { 2n, 2n+1, 2n+2 } × {2m } = { (2n, 2m), (2n+1, 2m), (2n+2, 2m) } = { x_1, x, x_2 }, where x_1 = (2n, 2m), x = (2n+1, 2m) and x_2 = (2n+2, 2m).
Thus, cl({ x }) \ { x } = { x_1, x, x_2 } \ { x } = { x_1, x_2 }. It should be noted that {x_1} and {x_2} are $\omega\alpha$-closed singleton sets in (Z^2, K^2).
Let us prove: x_1 ∈ A or x_2 ∈ A.
Consider x_1 ∉ A, then x_1 ∈ cl({ x }) ⊆ cl(A), implies that x_1 ∈ cl(A) \ A. Thus cl(A) \ A contains a $\omega\alpha$-closed set { x_1 }, which is contradiction to the remark 3.16.

Consider $x_2 \notin$ A, then $x_2 \in$ cl({ x }) $\subseteq$ cl(A), implies that $x_2 \in$ cl(A) \ A contains a $\omega\alpha$-closed set {x_2}, which is again contradiction to the remark 3.16. Therefore, $x_1 \in$ A or $x_2 \in$ A. Hence cl({ x }) $\subset$ A, because x $\in A_{mix} \subset$ A. □

Theorem 3.18. The following properties holds for any singleton set {x} in (Z^2, K^2):

(i) if x $\in (Z^2)_{K^2}$, then { x } is g*$\omega\alpha$-open, but not g*$\omega\alpha$-closed in (Z^2, K^2).

(ii) if x $\in (Z^2)_F$, then { x } is g*$\omega\alpha$-closed, but not g*$\omega\alpha$-open in $(Z^2)_{K^2}$.

(iii) if x $\in (Z^2)_{mix}$, then { x } is not g*$\omega\alpha$-closed, it is g*$\omega\alpha$-open in (Z^2, K^2).

Proof. (i) We know that { x } is open in (Z^2, K^2). Then { x } is g*$\omega\alpha$-open in (Z^2,K^2) [8]. Let x = (2n+1, 2m+1) $\in (Z)_{K^2}$, that is x = (2n+1, 2m+1) $\subseteq$ U = (2n+1, 2m+1), where U is $\omega\alpha$ open set in (Z^2, K^2). Then cl({ x }) = { 2n, 2n+1, 2n+2 } × { 2m, 2m+1, 2m+2 }. But cl({x}) U, this implies {x} is not g*$\omega\alpha$-closed in (Z^2, K^2).

(ii) Let x $\in (Z^2)_F$, that is {x}is closed in (Z^2, K^2). From [8], {x} is g*$\omega\alpha$-closed in (Z^2,K^2). Let, F = (2n, 2m) $\subseteq$ {x} =(2n, 2m), where F is $\omega\alpha$-closed in (Z^2, K^2). Then F $\subsetneq$ int({x}) , because int({x}) = φ, if x is even. This shows that {x} is not g*$\omega\alpha$-open in (Z^2, K^2).

(iii) Let x $\in (Z^2)_{mix}$, that is x = (2n+1, 2m) or x = (2n, 2m+1).
The cl({ x }) = { 2n, 2n+1, 2n+2 } × { 2m } $\not\subset$ { x } = U, where U is $\omega\alpha$-open in (Z^2, K^2). Therefore {x} is not g*$\omega\alpha$-closed in (Z^2, K^2).
Similarly, we can prove that {x} is not g*$\omega\alpha$-closed by taking x = (2n, 2m+1).

□

Theorem 3.19. Let B be a non empty subset of (Z^2, K^2). For a subset $B_F \neq \varphi$, if B is g*$\omega\alpha$-open in (Z^2, K^2), then (U({ x }))$K_2 \subset$ B holds for each point x $\in B_F$.

Proof. Let x $\in B_F$. Since {x} is closed, {x} is g*$\omega\alpha$-closed [8] and {x} $\subset$ B. As B is g*$\omega\alpha$-open, {x} $\subset$ int(B). This shows that {x} is the interior point of the set B. Thus, for the smallest open set U(X) containing x, U(X) $\subset$ B. That is int({ x }) = int(int(B)) = int(B) and int({x}) =U(x). Therefore U(x) $\subset$ int(B).
Let us consider x = (2n, 2m). Since U(2n, 2m) = { 2n-1, 2n, 2n+1 } × { 2m-1, 2m, 2m+1 }. Then $(U(\{ x \}))_{K^2}$ = { (x_1, x_2) $\in$ U(x): x_1 and x_2 are odd } = { y_1, y_2, y_3, y_4 }, where y_1 = (2n-1, 2m-1), y_2 = (2n-1, 2m+1), y_3 = (2n+1, 2m-1) and y_4 = (2n+1, 2m+1). Thus for each point P_i ($1 \leq i \leq 4$), we have $P_i \in$ B and $P_i \cap B \neq \varphi$. Therefore $(U(\{ x \}))_{K^2} \subset$ B. □

References

[1] A. Rosenfeld, *Digital Topology*, Amer. Math. Monthly, 86 (1979) 621–630.

[2] R, Devi, K. Bhuvaneshwari and H. Maki, *Weak Form of gρ-closed Sets, where ρ ∈ α, α*, α** and The Digital Plane*, Mem. Fac. Sci. Kochi Univ. Math.25, (2004) 37–54.

[3] R. Devi, S. N. Rajapriya, K. Muthukumarswamy and H. Maki, *ξ-Closed Sets in Topological Spaces and Digital Planes*, Scientiae Mathematicae Japanicae, Online e-2006 615–631.

[4] R. Devi and M. Vigneshwaram, *gα-Closed Sets in the Digital Plane*, Int. Jl. of General Topology, 4(1-2) (2011) 91–95.

[5] R. Devi and M. Vigneshwaran, *On GαO-Kernel in Digital Plane*, Int. Jl. of Mathematical Archive, 3(6) (2012) 2358–2373.

[6] E. D. Khalimsky, R. Kopperman and P. R. Meyer, *Computer Graphics and Connected Topologies in Finite Ordered Set*, Topology Applications, 36 (1990) 1–17.

[7] T. Y. Kong, R. Kopperman and P. R. Meyer, *A Topologies Approach to Digital Topology*, Amer. Math. Monthly, 98 (1991) 901–907.

[8] P. G. Patil, S. S. Benchalli and Pallavi S. Mirajakar, *Generalized Star $\omega\alpha$-closed Sets in Topological Spaces*, Jl. of New Results in Science, 9 (2015) 37–45.

[9] S. S. Benchalli, P. G. Patil and P. M. Nalwad, *$\omega\alpha$-closed sets in topological spaces*, The Global J. Appl. Maths Math. Sciences, 2(1-2) (2009) 53–63.

[10] P. G. Patil, S. S. Benchalli and Pallavi S. Mirajakar, *Generalized Star $\omega\alpha$-spaces in Topological Spaces*, Int. Jl. of Scientific and Innovative Mathematical Research, 3 Special Issue 1 (2015) 399–391.

[11] S. S. Benchalli, P. G. Patil and P. M. Nalwad, *some weaker forms of continuous functions in topological spaces*, Jl. of Advanced Studies in Topology, 7(2) (2016) 101–109.