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Pabitra Kumar Maji

Department of Mathematics, Bidhan Chandra College, Asansol 713301, Burdwan (W), West Bengal, India.
email: pabitra_maji@yahoo.com

Harish Garg

School of Mathematics, Thapar Institute of Engineering & Technology, Deemed University, Patiala-147004, Punjab, India
email: harish.garg@thapar.edu

Jianming Zhan

Department of Mathematics, Hubei University for Nationalities, Hubei Province, 445000, P. R. C.
email: zhanjianming@hotmail.com

Surapati Pramanik

Department of Mathematics, Nandalal Ghosh B.T. College, Narayanpur, Dist- North 24 Parganas, West Bengal 743126, India
email: sura_pati@yaho.co.in

Muhammad Irfan Ali

Department of Mathematics, COMSATS Institute of Information Technology Attock, Attock 43600, Pakistan
email: mirfanali13@yahoo.com

Said Broumi

Department of Mathematics, Hassan II Mohammedia-Casablanca University, Kasablanka 20000, Morocco
email: broumisaid78@gmail.com

Mumtaz Ali

University of Southern Queensland, Darling Heights QLD 4350, Australia
email: Mumtaz.Ali@usq.edu.au

Oktay Muhtaroğlu

Department of Mathematics, Tokat Gaziosmanpaşa University, 60250 Tokat, Turkey
email: oktay.muhtaroglu@gop.edu.tr

Ahmed A. Ramadan

Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
email: aramadan58@gmail.com

Sunil Jacob John

Department of Mathematics, National Institute of Technology Calicut, Calicut 673 601 Kerala, India
email: sunil@nitc.ac.in

Aslıhan Sezgin

Department of Statistics, Amasya University, Amasya, Turkey
email: aslihan.sezgin@amasya.edu.tr

Alaa Mohamed Abd El-latif

Department of Mathematics, Faculty of Arts and Science, Northern Border University, Rafha, Saudi Arabia
email: alaa_8560@yahoo.com

Kalyan Mondal

Department of Mathematics, Jadavpur University, Kolkata, West Bengal 700032, India
email: kalyanmathematic@gmail.com

Jun Ye

Department of Electrical and Information Engineering, Shaoxing University, Shaoxing, Zhejiang, P.R. China
email: yehjun@aliyun.com

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, 71516-Assiut, Egypt
email: drshehata2009@gmail.com

İdris Zorlutuna

Department of Mathematics, Cumhuriyet University, Sivas, Turkey
email: izorlu@cumhuriyet.edu.tr

Murat Sarı

Department of Mathematics, Yıldız Technical University, İstanbul, Turkey
email: sarim@yildiz.edu.tr

Daud Mohamad

Faculty of Computer and Mathematical Sciences, University Teknologi Mara, 40450 Shah Alam, Malaysia
email: daud@tmsk.uitm.edu.my

Tanmay Biswas

Research Scientist, Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.- Krishnagar Dist-Nadia, PIN-741101, West Bengal, India
email: tanmaybiswas_math@rediffmail.com

Kadriye Aydemir

Department of Mathematics, Amasya University, Amasya, Turkey
email: kadriye.aydemir@amasya.edu.tr

Ali Boussayoud

LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
email: alboussayoud@gmail.com

Muhammad Riaz

Department of Mathematics, Punjab University, Quaid-e-Azam Campus, Lahore-54590, Pakistan
email: mriaz.math@pu.edu.pk

Contact

Editor-in-Chief
Name: Prof. Dr. Naim Çağman
Email: journalofnewtheory@gmail.com
Phone: +905354092136
Address: Departments of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, Tokat, Turkey

Editors
Name: Assoc. Prof. Dr. Faruk Karaaslan
Email: karaaslan.faruk@gmail.com
Phone: +905058314380
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çankırı Karatekin University, 18200, Çankırı, Turkey

Name: Assoc. Prof. Dr. İrfan Deli
Email: irfandeli@kilis.edu.tr
Phone: +905426732708
Address: M.R. Faculty of Education, Kilis 7 Aralık University, Kilis, Turkey

Name: Asst. Prof. Dr. Serdar Enginoğlu
Email: serdarenginoglu@gmail.com
Phone: +905052241254
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, 17100, Çanakkale, Turkey

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On Some Generalized Open Sets in Ideal Bitopological Spaces

Ibtissam Bukhatwa[1,2], **Sibel Demiralp**[2]

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Abstract — In this article, we introduce and study the concepts of $\gamma_{ij}-I-$sets, $\gamma_{ij}-preI-$open sets, and $\gamma_{ij}-b-I-$open sets by generalizing $(i,j)-I-$open, $(i,j)-preI-$open, and $(i,j)-bI-$ open sets, respectively, in ideal bitopological spaces with an operation $\gamma:\tau\rightarrow P(X)$. Further, we describe and study $(\gamma,\delta)_{ij}-bI-$continuous functions in ideal bitopological spaces through this paper.

Keywords — *Ideal bitopological space, $\gamma_{ij}-I-$ open sets, $\gamma_{ij}-preI-$open sets, $\gamma_{ij}-bI-$open sets, $\gamma_{ij}-I-$continuous functions*

In 1963, Kelly [1] presented the concept of bitopological space (X,τ_1,τ_2) which is a nonempty set X endowed with two topologies τ_1 and τ_2. In 1966, Kuratowski [2] studied and applied the concept of ideals. An ideal on a topological space (X,τ) is a collection of subsets of X having the heredity property (i) if $A\in I$ and $B\subset A$, then $B\in I$ and (ii) if $A\in I$ and $B\in I$, then $A\cup B\in I$. If I is an ideal on X, then (X,τ_1,τ_2,I) is called an ideal bitopological space. In 1979, Kasahara [3] defined an operation $\gamma:\tau\rightarrow P(X)$ which is a mapping on τ such that $U\subseteq\gamma(U)$ for each $U\in\tau$. In 1984, Khedr [4,5], extended the operation γ to bitopological spaces. In 1996, Andrijevic [6] presented the field of topological space the concept of $b-$open sets. In 2007, Al-Hawary and AL-Omari [7] expanded the $b-$open sets to bitopological spaces in addition. Guler and Aslim [8] presented the idea of $bI-$open sets in ideal topological spaces. In 2012, Ekici [9] studied the concept of pre$-I-$open sets, $semi-I-$open sets and $bI-$open sets in ideal topological spaces. In 2011, Rajesh et al. [10] introduced the notion of $pre-I$-open sets in ideal bitopological spaces. In 2015, Ibrahim [11] introduced the concept of $\gamma-pre-I$-open sets in ideal topological spaces and Diganta [12] introduced the notion of $bI-$open sets in ideal bitopological spaces. In 2018, Hussain [13] presented the concept of $\gamma-pre-$open and $\gamma-b-$open sets in the field of topological space. In 2020, Bukhatwa and Demiralp in a similar study introduced and defend the notion of generalized β-open sets in ideal bitopological spaces [14]. For some more significant work in this direction we refer to [10, 15–23].

Introduction

In this article, (X,τ_1,τ_2) always mean bitopological space with no separation axioms are supposed in this space, also (X,τ_1,τ_2,I) be an ideal bitopological space. Let A be a subset of X, by $int_i(A)$ and $cl_i(A)$ we denote respectively the interior and the closure of A with regard to τ_i for $i=1,2$. An operation γ on a bitopological space (X,τ_1,τ_2) is a mapping $\gamma:\tau_1\cup\tau_2\rightarrow P(X)$ such that $U\subseteq U^\gamma$ for all $U\in\tau_1\cup\tau_2$, where U^γ is denotes the value of γ at U. For example the operations $\gamma(U)=U$, $\gamma(U)=cl_i(U)$, $\gamma(U)=int_i(cl_j(U))$ for $U\in\tau_j$ are operations on $\tau_1\cup\tau_2$. If for each $x\in A$, there

[1]i.bukhatwa@gmail.com (Corresponding Author); [2]sdemiralp@kastamonu.edu.tr
[1]Department of Mathematics, Benghazi University, Benghazi, 16063, Libya
[2]Department of Mathematics, Kastamonu University, Kastamonu, 37150, Turkey

exists an $i-$open set U such that $x \in U$ and $U^{\gamma} \subseteq A$, then A is called γ_i-open set. τ_{γ_i} will denote the set of all γ_i-open set in X. Obviously we have $\tau_{\gamma_i} \subseteq \tau_i$. Complement of all γ_i-open sets are called γ_i-closed. Assumed (X, τ_1, τ_2, I) as an ideal bitopological space and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$ named the local function [24] of A with regard to τ_i and I. The definition of local function is giving as: for $A \subset X$, $A_i^*(\tau_i, I) = \{x \in X \mid U \cap A \notin I$, for all $U \in \tau_i(x)\}$ where $\tau_i(x) = \{U \in \tau_i \mid x \in U\}$. Observe additionally that closure operator for $\tau_i^*(I)$, accurate than τ_i, is defined by $Cl_i^*(A) = A \cup A_i^*$. $int_i^*(A)$ called the interior of A in $\tau_i(I)$ and $int_i^*(A_j^*)$ called the interior of A_j^* with regard to topology τ_i, where $A_j^* = \{x \in X \mid U \cap A \notin I$, for every $U \in \tau_j\}$. The $interior_{\gamma_i}$ of A is denoted by $int_{\gamma_i}(A)$ and defined to be the union of all γ_i-open sets of X contained in A and the $closure_{\gamma_i}$ of A is denote by $cl_{\gamma_i}(A)$ and defined to be the intersection of all γ_i-closed sets containing A.

Currently, several definitions from [11, 12, 15, 16] are recalled to be used in this article.

Definition 1.1. A subset A of a bitopological space (X, τ_1, τ_2) with operation γ on $\tau_1 \cup \tau_2$ is named

i. $\gamma_{ij} - pre-$open set if $A \subseteq int_{\gamma_i}(cl_{\gamma_j}(A))$, where $i, j = 1, 2$ and $i \neq j$.

ii. $\gamma_{ij} - b-$open set if $A \subseteq int_{\gamma_i}(cl_{\gamma_j}(A)) \cup cl_{\gamma_j}(int_{\gamma_i}(A))$, where $i, j = 1, 2$ and $i \neq j$.

Definition 1.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space with an operation γ on $\tau_1 \cup \tau_2$. The $\gamma-$local function of A with regard to γ and I is defined as giving: for $A \subset X$, $A_{\gamma_i}^*(\gamma, I) = \{x \in X \mid U \cap A \notin I$, for all $U \in \tau_{\gamma_i}(x)\}$ where $\tau_{\gamma_i}(x) = \{U \in \tau_{\gamma_i} \mid x \in U\}$.

In the case no ambiguity, we will replace $A_{\gamma_i}^*(\gamma, I)$ by $A_{\gamma_i}^*$.

Definition 1.3. *A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is called*

i. $(i, j) - I-$open if $A \subseteq int_i(A_j^*)$, where $i, j = 1, 2$ and $i \neq j$.

ii. $(i, j) - pre - I-$open set if $A \subseteq int_i(cl_j^*(A))$, where $i, j = 1, 2$ and $i \neq j$.

iii. $(i, j) - bI-$open set if $A \subseteq int_i(cl_j^*(A)) \cup cl_j^*(int_i(A))$, where $i, j = 1, 2$ and $i \neq j$.

Definition 1.4. Let (X, τ_1, τ_2, I) be an ideal bitopological space with an operation γ and (Y, σ_1, σ_2) be a bitopological space with an operation β. Then a function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is called pairwise $(\gamma, \beta)_i$-continuous function if $f^{-1}(V)$ is γ_i-open in X for all β_i-open set V in Y.

Definition 1.5. *A* function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is called as an $(i, j) - I-$continuous function (resp. $(i, j) - preI-$continuous, $(i, j) - bI-$continuous) if $f^{-1}(V)$ is $(i, j) - I-$open (resp. $(i, j) - preI-$open, $(i, j) - bI-$open) in X for all σ_i-open set V in Y, where $i, j = 1, 2$ and $i \neq j$.

Throughout the article, we suppose that $i, j = 1, 2$ and $i \neq j$.

$\gamma_{ij} - preI-$Open Sets

In this section, the concept of $\gamma_{ij} - preI-$open sets in ideal bitopological spaces are presented and characterizations of their related notions are given.

Definition 2.1. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be $\gamma_{ij} - I-$open if $A \subseteq int_{\gamma_i}(A_{\gamma_j}^*)$. The set consisting of all $\gamma_{ij} - I-$open sets in X will be denoted by $\gamma_{ij} - IO(X)$.

Example 2.2. Let $X = \{a, b, c, d\}$ be a set and let $\tau_1 = \{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c, d\}\}$, and $I = \{\emptyset, \{a\}\}$ defined an operation $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that $\gamma(U) = Cl_i(A)$ for $U \in \tau_j$. Then we have, γ_{12}-I-open sets are $\{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$.

Definition 2.3. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be $\gamma_{ij} - preI-$open set if $A \subseteq int_{\gamma_i}(cl_{\gamma_j}^*(A))$. The set consisting of all $\gamma_{ij} - preI-$open sets in X will be denoted by $\gamma_{ij} - PIO(X)$.

Definition 2.4. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be γ_{ij}-$preI$-closed set if A^c is a γ_{ij}-$preI$-open set . Equivalently A is said to be γ_{ij}-$preI$-closed set if $A \supseteq cl_{\gamma_i}(int_{\gamma_j}^*(A))$. The set consisting of all γ_{ij}-$preI$-closed sets in X will be denoted by $\gamma_{ij} - PIC(X)$.

Theorem 2.5. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then,

i. Every γ_{ij}-I-open set is γ_{ij}-$preI$-open.

ii. Every γ_{ij}-$preI$-open set is γ_{ij}-pre-open.

Proof. Let A be a subset of X.

i. If A is $\gamma_{ij} - I-$open, then

$$A \subseteq int_{\gamma_i}(A^*_{\gamma_j}) \subseteq int_{\gamma_i}(A^*_{\gamma_j} \cup A) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A))$$

Therefore, A is a $\gamma_{ij} - preI-$open set.

ii. If A is γ_{ij}-$preI$-open, then

$$A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq int_{\gamma_i}(A^*_{\gamma_j} \cup A) \subseteq int_{\gamma_i}(cl_{\gamma_j}(A) \cup A) \subseteq int_{\gamma_i}(cl_{\gamma_j}(A))$$

Therefore, A is a γ_{ij}-pre-open set.

□

But generally the converse of this theorem is not true as giving in the next example.

Example 2.6. From example 2.2, take $A = \{a, b, d\}$. Calculations show that A is γ_{12}-$preI$-open however, it is not γ_{12}-I-open.

Theorem 2.7. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then,

i. The union of any $\gamma_{ij} - preI-$open sets is $\gamma_{ij} - preI-$open set.

ii. The intersection of any $\gamma_{ij} - preI-$closed sets is $\gamma_{ij} - preI-$closed sets.

Proof. i. Let $A_\alpha \in \gamma_{ij} - PIO(X)$ for each $\alpha \in \Lambda$, where Λ is an index set. $A_\alpha \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha))$. Then,

$$\begin{aligned}
\bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \bigcup_{\alpha\in\Lambda}\{int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha))\} \\
\bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \{int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} cl^*_{\gamma_j}(A_\alpha))\} \subseteq \{int_{\gamma_i}((\bigcup_{\alpha\in\Lambda}(A_\alpha)^*_{\gamma_j} \cup A_\alpha)))\} \\
\bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \{int_{\gamma_i}(cl^*_{\gamma_j}(\bigcup_{\alpha\in\Lambda} A_\alpha))\}
\end{aligned}$$

Then, $\bigcup_{\alpha\in\Lambda} A_\alpha$ is $\gamma_{ij} - preI-$open.

ii. The proof follows by using (i) and taking complement.

□

Definition 2.8. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $\gamma_{ij} - preI-$closure of A is represented by $pI - cl_{\gamma_{ij}}(A)$, is defined as the intersection of each $\gamma_{ij} - preI-$closed subset of X containing A. That is

$$pI - cl_{\gamma_{ij}}(A) = \bigcap\{U \mid U \supseteq A \text{ with } U^c \in \gamma_{ij} - PIO(X)\}.$$

ii. $\gamma_{ij} - preI-$interior of A is represented by $pI - int_{\gamma_{ij}}(A)$, is defined as the union of all $\gamma_{ij} - preI-$open subset of X contained in A. That is

$$pI - int_{\gamma_{ij}}(A) = \bigcup\{U \mid U \subseteq A \text{ with } U \in \gamma_{ij} - PIO(X)\}.$$

Theorem 2.9. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $pI - cl_{\gamma_{ij}}(A^c) = (pI - int_{\gamma_{ij}}(A))^c$

ii. $pI - int_{\gamma_{ij}}(A^c) = (pI - cl_{\gamma_{ij}}(A))^c$

iii. A is $\gamma_{ij} - preI-$open$\Leftrightarrow A = pI - int_{\gamma_{ij}}(A)$

iv. A is $\gamma_{ij} - preI-$closed$\Leftrightarrow A = pI - cl_{\gamma_{ij}}(A)$

The proof can be obtained directly by the definition 2.8 and omitted.

Lemma 2.10. Let X be a space and $A \subset X$. Then,

i. $cl_{\gamma_{ij}}(A) \cap U \subseteq cl_{\gamma_{ij}}(A \cap U)$, for any γ_i-open set U in X.

ii. $int_{\gamma_{ij}}(A \cap V) \subseteq int_{ij}(A) \cap V$, for any γ_i-closed set V in X.

Theorem 2.11. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $pI\text{-}int_{\gamma_{ij}}(A) = A \cap int_{\gamma_i}(cl^*_{\gamma_j}(A))$

ii. $pI\text{-}cl_{\gamma_{ij}}(A) = A \cup cl_{\gamma_i}(int^*_{\gamma_j}(A))$

PROOF. i. We have $pI\text{-}int_{\gamma_{ij}}(A) \subseteq A$ and since $pI\text{-}int_{\gamma_{ij}}(A)$ is γ_{ij}-$preI$-open, then

$$pI - int_{\gamma_{ij}}(A) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(pI - int_{\gamma_{ij}}(A)))$$

Therefore, $pI\text{-}int_{\gamma_{ij}}(A) \subseteq A \cap int_{\gamma_i}(cl^*_{\gamma_j}(A))$. Also,

$$int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq cl^*_{\gamma_j}(int_{\gamma_i}(cl^*_{\gamma_j}(A)))$$

Then we have,

$$int_{\gamma_i}(int_{\gamma_i}(cl^*_{\gamma_j}(A))) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(int_{\gamma_i}(cl^*_{\gamma_j}(A))))$$
$$int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(int_{\gamma_i}(cl^*_{\gamma_j}(A))))$$

Therefore, $int_{\gamma_i}(cl^*_{\gamma_j}(A))$ is γ_{ij}-$preI$-open. Thus,

$$A \cap (int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq pI - int_{\gamma_{ij}}(A)$$

ii. The proof follows from (i) and by taking complement.

□

Theorem 2.12. Let (X,τ_1,τ_2,I) be an ideal bitopological space with an operation γ on τ and $A \subset X$. Then,

i. If $I = \{\varnothing\}$, then A is $\gamma_{ij} - preI-$open if and only if A is $\gamma_{ij} - pre-$open.

ii. If $I = P(X)$ then A is γ_{ij}-$preI$-open if and only if A is γ_i-open.

PROOF. i. We have just to show that if $I = \{\varnothing\}$ and A is $\gamma_{ij} - pre-$open, then A is $\gamma_{ij} - preI-$open. If $I = \{\varnothing\}$, then $A^*_{\gamma_i} = cl_{\gamma_i}(A)$ for every subset A of X. Assumed A to be $\gamma_{ij} - pre-$open set, then

$$A \subseteq int_{\gamma_i}(cl_{\gamma_j}(A)) \subseteq int_{\gamma_i}(A^*_{\gamma_j}) \subseteq int_{\gamma_i}(A^*_{\gamma_j} \cup A) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A))$$

Therefore, A is $\gamma_{ij} - preI-$open.

ii. Let $I = P(X)$, then $A^*_{\gamma_j} = \varnothing$ for all subset A of X. Let A be any $\gamma_{ij} - preI-$open set, then $A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) = int_{\gamma_i}(A \cup A^*_{\gamma_j}) = int_{\gamma_i}(A)$. Therefore, A is γ_i-open. Opposite is obvious.

□

$\gamma_{ij} - bI-$Open Sets

In this section, the concept of $\gamma_{ij} - bI-$open sets in ideal bitopological spaces are presented and characterizations of their related notions are given.

Definition 3.1. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be $\gamma_{ij} - bI-$open set if $A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))$. The set consisting of all $\gamma_{ij} - bI-$open sets in X will be symbolized by $\gamma_{ij} - BIO(X)$.

Definition 3.2. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is called $\gamma_{ij} - bI-$closed set if A^c is an $\gamma_{ij} - bI-$open set. Equivalently A is called $\gamma_{ij} - bI-$closed set if $A \supseteq cl_{\gamma_i}(int^*_{\gamma_j}(A)) \cap int^*_{\gamma_j}(cl_{\gamma_i}(A))$. The set consisting of all $\gamma_{ij} - bI-$closed sets in X will be symbolized by $\gamma_{ij}-$BIC(X).

Theorem 3.3. Let (X, τ_1, τ_2, I) be an ideal bitopological space and A be a subset of X. Then,

i. A is $\gamma_{ij} - bI-$open if and only if A^c is $\gamma_{ij} - bI-$closed.

ii. If A is $\gamma_{ij} - preI-$open then A is $\gamma_{ij} - bI-$open.

iii. If A is γ_{ij}-I-open then A is γ_{ij}-bI-open.

iv. If A is γ_{ij}-bI-open then A is γ_{ij}-b-open.

Proof. i. This can be obtained directly by the ii and taking complement.

ii. Let A be $\gamma_{ij} - preI-$open set. Then,

$$A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))$$

iii. By theorem 2.5.

iv. The prove will be obtained directly by using the fact that $\tau^*(I)$ is accurate than τ. But generally, the converse of this theorem is not true as giving in the next examples.

□

Example 3.4. Let $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}\}$, and $I = \{\emptyset, \{a\}\}$. For $U \in \tau_i$, let

$$\gamma(U) = \begin{cases} int_i(Cl_j(U)), & a \notin U \\ U & , \quad a \in U \end{cases}$$

Take $A = \{b, c\}$ then A is γ_{12}-bI-open in X but neighter γ_{12}-I-open nor γ_{12}-$preI$-open in X.

Example 3.5. Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a\}\}$, $\tau_2 = \{\varnothing, X, \{b\}\}$, and $I = \{\varnothing, \{b\}\}$. Let define an operation $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that $\gamma(U) = U$ for all $U \in \tau_i$. Then we have, $\gamma_{12} - bI-$open sets are $\{\varnothing, X, \{a\}, \{a, c\}, \{a, b\}, \{b, c\}\}$. Take $A = \{b\}$. Calculations show that A is $\gamma_{12} - b-$open but not $\gamma_{12} - bI-$open.

Theorem 3.6. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Let A be a subset of X. If A is $\gamma_{ij} - bI-$open with $int_{\gamma_i}(A) = \varnothing$, then A is $\gamma_{ij} - preI-$open.

Proof. Let A be $\gamma_{ij} - bI-$open set, then

$$A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(\varnothing) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A))$$

□

Theorem 3.7. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then,

i. The union of any $\gamma_{ij} - bI-$open sets is a $\gamma_{ij} - bI-$open set.

ii. The intersection of any $\gamma_{ij} - bI-$closed sets is a $\gamma_{ij} - bI-$closed set.

PROOF. i. Let $A_\alpha \in \gamma_{ij} - BIO(X)$ for each $\alpha \in \Lambda$, where Λ is an index set.

$$A_\alpha \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A_\alpha))$$

Thus,

$$\begin{aligned}
\bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \bigcup_{\alpha\in\Lambda}\{int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A_\alpha))\} \\
&\subseteq \bigcup_{\alpha\in\Lambda}\{int_{\gamma_i}(A_\alpha \cup (A_\alpha)^*_{\gamma_j}) \cup (int_{\gamma_i}(A_\alpha)) \cup (int_{\gamma_i}(A_\alpha))^*_{\gamma_j}\} \\
&\subseteq \{int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} A_\alpha) \cup (\bigcup_{\alpha\in\Lambda}(A_\alpha)^*_{\gamma_j}\ )\} \\
&\quad \cup\, int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} A_\alpha) \cup int_{\gamma_i}(\bigcup_{\alpha\in\Lambda}(A_\alpha)^*_{\gamma_j}) \\
&\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(\bigcup_{\alpha\in\Lambda} A_\alpha)) \cup\ cl^*_{\gamma_j}(int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} A_\alpha))
\end{aligned}$$

Then, $\bigcup_{\alpha\in\Lambda} A_\alpha$ is $\gamma_{ij} - bI-$open.

ii. The proof follows by using (i) and taking complement.

□

Theorem 3.8. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Let A and U be subsets of X. If A is $\gamma_{ij} - bI-$open and $U \in \tau_1 \cap \tau_2$,then $A \cap U$ is $\gamma_{ij} - bI-$open.

PROOF. Let A be $\gamma_{ij} - bI-$open set, then $A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))$.

$$\begin{aligned}
A \cap U &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \cap U \\
&\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A) \cap U) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A) \cap U) \\
&\subseteq int_{\gamma_i}((A \cup A^*_{\gamma_j}) \cap U) \cup ((int_{\gamma_i}(A) \cup int(A)^*_{\gamma_j}) \cap U) \\
&\subseteq int_{\gamma_i}((A \cap U) \cup (A \cap U)^*_{\gamma_j}) \cup ((int_{\gamma_i} A \cap U) \cup int(A \cap U)^*_{\gamma_j}) \\
A \cap U &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A \cap U)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A \cap U))
\end{aligned}$$

□

Definition 3.9. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $\gamma_{ij} - bI-$closure of A, symbolized by $bI - cl_{\gamma_{ij}}(A)$, is defined as the intersection of all $\gamma_{ij} - bI-$closed subset of X containing A. That is

$$bI - cl_{\gamma_{ij}}(A) = \bigcap\{U \mid U \supseteq A \text{ with } U^c \in \gamma_{ij} - BIO(X)\}.$$

ii. $\gamma_{ij} - bI-$interior of A, symbolized by $bI - int_{\gamma_{ij}}(A)$, is defined as the union of each $\gamma_{ij} - bI-$open subset of X contained in A. That is

$$bI - int_{(\gamma_{ij})}(A) = \bigcup\{U \mid U \subseteq A \text{ with } U \in \gamma_{ij} - BIO(X)\}.$$

Theorem 3.10. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $bI - cl_{\gamma_{ij}}(A) = A \cup \{cl_{\gamma_i}(int^*_{\gamma_j}(A)) \cap int^*_{\gamma_j}(cl_{\gamma_i}(A))\}$

ii. $bI - int_{\gamma_{ij}}(A) = A \cap \{int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))\}$

The proof follows from Theorem 2.11.

Theorem 3.11. For an ideal bitopological space (X, τ_1, τ_2, I) with an operation γ on τ and $A \subset X$, we have, *i.* If $I = \{\varnothing\}$, then A is $\gamma_{ij} - bI-$open if and only if A is $\gamma_{ij} - b-$open.

ii. If $I = P(X)$, then A is γ_i-open.

PROOF. i. We have just to show that if $I = \{\varnothing\}$ and A is $\gamma_{ij} - b-$open, then A is $\gamma_{ij} - bI-$open. If $I = \{\varnothing\}$, then $A^*_{\gamma_j} = cl_{\gamma_i}(A)$ for every subset A of X . Let A be $\gamma_{ij} - b-$open set. Then,

$$\begin{aligned} A &\subseteq int_{\gamma_i}(cl_{\gamma_j}(A)) \cup cl_{\gamma_j}(int_{\gamma_i}(A)) \\ &\subseteq int_{\gamma_i}(A^*_{\gamma_j}) \cup (int_{\gamma_i}(A))^*_{\gamma_j} \\ &\subseteq int_{\gamma_i}(A \cup A^*_{\gamma_j}) \cup int_{\gamma_i}(A) \cup (int_{\gamma_i}(A))^*_{\gamma_j} \\ &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \end{aligned}$$

So A is $\gamma_{ij} - bI-$open.

ii. Let $I = P(X)$, then$A^*_{\gamma_j} = \varnothing$ for every subset A of X. Let A be any $\gamma_{ij} - bI-$open set. Then,

$$\begin{aligned} A &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \\ &= int_{\gamma_i}(A \cup A^*_{\gamma_j}) \cup int_{\gamma_i}(A) \cup (int_{\gamma_i}(A))^*_{\gamma_j} \\ &= int_{\gamma_i}(A) \cup int_{\gamma_i}(A) \end{aligned}$$

So A is γ_i-open. Opposite is obvious.

□

$(\gamma,\beta)_{ij} - I-$Continuous Functions

In this section, the concept of $(\gamma,\beta)_{ij} - I-$continuous functions in ideal bitopological spaces are introduced and characterizations of their related notions are given.

Throughout this section, (X, τ_1, τ_2, I) be an ideal bitopological space with an operation γ and (Y, σ_1,σ_2) be a bitopological space with an operation β.

Definition 4.1. A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\gamma, \beta)_{ij} - I-$continuous function (resp. $(\gamma,\beta)_{ij} - preI-$continuous, $(\gamma, \beta)_{ij} - bI-$continuous) if $f^{-1}(V)$ is $\gamma_{ij} - I-$open (resp. $\gamma_{ij} - preI-$open, $\gamma_{ij} - bI-$open) in X for every β_i-open set V in Y, for $i, j = 1, 2$ and $i \neq j$.

It is clear that every $(\gamma,\beta)_{ij} - I-$continuous functions is $(\gamma,\beta)_{ij} - preI-$continuous and $(\gamma,\beta)_{ij} - bI-$continuous but the converse is not true as shown in the example.

Example 4.2. Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 = \{\varnothing, X, \{a\}\}$, $I = \{\varnothing, \{a\}\}$, and let for $U \in \tau_i$,

$$\gamma(U) = \begin{cases} int_i(Cl_j(U)), & a \notin U \\ U \quad , & a \in U \end{cases}$$

Let $\sigma_1 = \{\varnothing, X, \{b\}, \{a, b\}\}, \sigma_2 = \{\varnothing, X, \{c\}\}, \beta(V) = V$ for $V \in \sigma_i$. Let define a function $f : (X, \tau_1, \tau_2, I) \to (X, \sigma_1, \sigma_2)$ such that $f(a) = c, f(b) = b, f(c) = a$. Then f is $(\gamma, \beta)_{12}$-b-I-continuous but neither $(\gamma, \beta)_{12}$-I-continuous nor $(\gamma, \beta)_{12}$-$preI$-continuous because $\{a, b\}$ is σ_1-open set and $f^{-1}(\{a, b\}) = \{b, c\}$ which is γ_{12}-b-I-open in X but neither γ_{12}-I-open nor γ_{12}-$preI$-open in X.

Definition 4.3. A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1,\sigma_2)$ is said to be pairwise $(\gamma,\beta)_i - I-$continuous function if $f^{-1}(V)$ is $\gamma_i - I-$open in X for every β_i-open set V in Y.

Note that the concept of pairwise $(\gamma,\beta)_i - I-$continuous and $(\gamma, \beta)_{ij} - I-$continuous are independent.

Example 4.4. Let $X = \{a,b,c\}$ be a set and $\tau_1 = \{\varnothing, X, \{b\}, \{b, c\}\}$, $\tau_2 = \{\varnothing, X, \{b, c\}\}$, $I = \{\varnothing, \{a\}\}$, with operation $\gamma(U) = U$ for $U \in \tau_i$ and $\sigma_1 = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}, \sigma_2 = \{\varnothing, X, \{b, c\}\}$, with operation

$$\beta(V) = \begin{cases} Cl_j(V), & c \notin V \\ V \quad , & c \in V \end{cases}$$

for $V \in \sigma_i$ and define $f : (X, \tau_1, \tau_2, I) \to (X, \sigma_1, \sigma_2)$ such that $f(a) = b, f(b) = c$ and $f(c) = a$. Then f is $(\gamma, \beta)_{12}$-I-continuous function but it is not $(\gamma, \beta)_1$-I-continuous because $\{a\}$ is σ_1-open set and $f^{-1}(\{a\}) = \{c\}$ which is γ_{12}-I-open but not γ_1-I-open in X.

Theorem 4.5. For the function $f:(X,\tau_1,\tau_2,I)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent.

i. f is $(\gamma,\beta)_{ij}-I-$continuous,

ii. For all x in X and each β_i-open set V in Y containing $f(x)$, there exists a $\gamma_{ij}-I-$open set U of X containing x such that $f(U)\subset V$.

Proof. (i⇒ii) Let V be a β_i-open set in Y such that $f(x)\in V$. Since f is $(\gamma,\beta)_{ij}-I-$continuous, $f^{-1}(V)$ is $\gamma_{ij}-I-$open set in X. Let $U=f^{-1}(V)$, then $f(x)\in f(U)\subset V$.

(ii⇒i) Let V be a β_i-open set in Y and let $x\in f^{-1}(V)$. Then we have $f(x)\in V$. By (ii), there exists an $\gamma_{ij}-I-$open set U in X containing x such that $f(U)\subset V$. Therefore, $x\in U\in f^{-1}(V)$. Hence $f^{-1}(V)$ is $\gamma_{ij}-I-$open set in X, so f is $(\gamma,\beta)_{ij}-I-$continuous.

□

Theorem 4.6. Let $f:(X,\tau_1,\tau_2,I)\to(Y,\sigma_1,\sigma_2)$ be a $(\gamma,\beta)_{ij}-I-$continuous function. Then, the following statements are equivalent:

i. The inverse image of every β_i-closed set in Y is $\gamma_{ij}-I-$closed set in X.

ii. For each subset U of X, $f(I-cl_{\gamma_{ij}}(U))\subset cl_{\beta_i}(f(U))$.

iii. For each subset V of Y, $I-cl_{\gamma_{ij}}(f^{-1}(V))\subset f^{-1}(cl_{\beta_i}(V))$.

Proof. (i⇒ii) Let $U\subset X$. Since $cl_{\beta_i}(f(U))$ is β_i-closed set in Y, therefore by (i), we have $f^{-1}(cl_{\sigma_i}(f(U)))$ is $\gamma_{ij}-I-$closed set in X. Also $U\subset f^{-1}\ (cl_{\beta_i}(f(U)))$ and $I-cl_{\beta_i}(U)$ is the smallest set $\gamma_{ij}-I-$closed set containing U. Therefore,

$$I-cl_{\beta_i}(U)\subset f^{-1}\ (cl_{\beta_i}(f(U)))$$

This implies that $f(I-cl_{\gamma_{ij}}(U))\subset cl_{\beta_i}(f(U))$.

(ii⇒iii) Let $V\subset Y$. Then $f^{-1}(V)\subset X$ by (ii), that

$$f(I-cl_{\gamma_{ij}}(f^{-1}(V)))\subset cl_{\beta_i}(f(f^{-1}(V)))\subset cl_{\beta_i}(V)$$

Hence $I-cl_{\gamma_{ij}}\ (f^{-1}(V))\subset f^{-1}(cl_{\beta_i}(V))$.

(iii⇒i) Let V be a β_i-closed set in Y. By (iii),

$$I-cl_{\gamma_{ij}}\ (f^{-1}(V))\subset f^{-1}(cl_{\beta_i}(V))=f^{-1}(V)$$

Therefore, $f^{-1}(V)=I-cl_{\gamma_{ij}}\ (f^{-1}(V))$ and so $f^{-1}(V)$ is $\gamma_{ij}-I-$closed set in X. □

Theorem 4.7. The function $f:(X,\tau_1,\tau_2,I)\to(Y,\sigma_1,\sigma_2)$ is $(\gamma,\beta)_{ij}-I-$ continuous function if and only if $f^{-1}(int_{\beta_i}(V))\subset I-\ int_{\gamma_{ij}}(f^{-1}(V))$ for every subset V of Y.

Proof. (⇒) Let V be a β_i-open set in Y and f is a $(\gamma,\beta)_{ij}-I-$ continuous function. Then, $f^{-1}(int_{\beta_i}(V)$ is $\gamma_{ij}-I$-open set in X and we have

$$f^{-1}(int_{\beta_i}(V))\subset I-int_{\gamma_{ij}}(f^{-1}(int_{\beta_i}(V)))\subset I-\ int_{\gamma_{ij}}(f^{-1}(V))$$

(⇐) Let V be a β_i-open set in Y, then $int_{\beta_i}(V)=V$. Therefore,

$$f^{-1}(V)\subset f^{-1}(int_{\beta_i}(V))\subset I-\ int_{\gamma_{ij}}(f^{-1}(V))$$

Then, $f^{-1}(V)=I-int_{\gamma_{ij}}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $\gamma_{ij}-I-$open set in X and so f is a $(\gamma,\beta)_{ij}-I-$ continuous function. □

Note that,in generally the composition of two $(\gamma,\beta)_{ij}-I-$ continuous functions need not to be $(\gamma,\beta)_{ij}-I-$ continuous function, as giving in the next examples for that defined (Z,ℓ_1,ℓ_2) to be a bitopological space with an operation δ.

Example 4.8. Let (X, τ_1, τ_2, I) and $(X, \sigma_1, \sigma_2, J)$ be two ideal bitopological spaces such that $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a, b\}\}$, $\tau_2 = \{\varnothing, X, \{a\}, \{a, b\}\}$, and $I = \{\varnothing, \{b\}\}$

$$\gamma(U) = \begin{cases} cl_j(U) & , \text{if } b \notin U \\ U & , \text{if } b \in U \end{cases}$$

for $U \in \tau_i$. Let $\sigma_2 = \{\varnothing, X, \{b\}, \{b, c\}\}$, $\sigma_2 = \{\varnothing, X, \{b, c\}\}$, $J = \{\varnothing, \{a\}\}$, $\beta(V) = V$ for $V \in \sigma_i$. Let define a function $f : (X, \tau_1, \tau_2, I) \to (X,\sigma_1,\sigma_2)$ such that $f(a) = b, f(b) = a$ and $f(c) = c$ and let (X, ℓ_1, ℓ_2) be a bitoplogical space such that $\ell_1 = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}, \ell_2 = \{\varnothing, X, \{b, c\}\}$,

$$\delta(W) = \begin{cases} cl_j(W) & , \text{if } c \notin W \\ W & , \text{if } c \in W \end{cases}$$

for $W \in \ell_i$ and define $g : (X, \sigma_1, \sigma_2, J) \to (X, \ell_1, \ell_2)$ such that $g(a) = b$, $g(b) = a$ and $g(c) = a$. Then f is $(\gamma, \beta)_{12}-I-$continuous function and g is $(\beta, \delta)_{12}-I-$continuous function but the composition $g \circ f$ is not $(\gamma, \delta)_{12}-I-$continuous function because $\{a\}$ is δ_i-open set and $(g \circ f)^{-1}(\{a\}) = \{c\} \notin \gamma_{12}-IO(X)$.

Theorem 4.9. Let f be a function from (X, τ_1, τ_2, I) to (Y, σ_1, σ_2) and g from $(Y, \sigma_1, \sigma_2, J)$ to (Z, ℓ_1, ℓ_2). Then $g \circ f$ is $(\gamma, \delta)_{ij} - I-$continuous if f is $(\gamma,\beta)_{ij} - I-$continuous and g is pairwise $(\beta, \delta)_i - I-$continuous.

Proof. Let w be any δ_i-open set in Z. Since g is pairwise $(\beta, \delta)_i$-continuous, then $g^{-1}(w)$ is a β_i-open set in Y. On the other hand, since f is $(\gamma,\delta)_{ij}$-I-continuous, we have $f^{-1}(g^{-1}(w)) \in \gamma_{ij} - IO(X)$. Therefore $g \circ f$ is $(\gamma, \delta)_{ij} - I-$continuous. □

Conclusion

In this study, we defend the notion of $\gamma_{ij} - I-$open sets, $\gamma_{ij} - perI-$open sets and $\gamma_{ij} - bI$open by generalizing by $(i, j) - I-$open, $(i, j) - preI-$open, $(i, j) - bI-$open sets respectively, in ideal bitopological spaces with an operation $\gamma : \tau \to P(X)$. We show that every $\gamma_{ij} - I-$open set is a $\gamma_{ij} - preI-$open sets and $\gamma_{ij} - bI-$open sets but the converse is nt always true.

Consequently the following diagrams are true:

$$\gamma_{ij} - I - open \to \gamma_{ij} - preI - open \to \gamma_{ij} - bI - open$$

$$\gamma_{ij} - preI - open(\gamma_{ij} - bI - open) \leftrightarrow \gamma_{ij}pre - open(\gamma_{ij} - b - open)(I = \{\emptyset\})$$

$$(\gamma, \delta)_{ij} - I - continuous \to (\gamma, \delta)_{ij} - preI - Continuous \to (\gamma, \delta)_{ij} - bI - Continuous$$

These notations, defined in this study, can be extended to other practicable researched of topology such as fuzzy topology, soft topology, intuitionistic topology and so on. Also generalized separation axioms can be introduced by the concept of generalized $I-$open set, $pre-$open sets and $b-$open sets.

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References

[1] J. C. Kelly, *Bitopological Spaces*, Proceedings of the London Mathematical Society 3(1) (1963) 71-89.

[2] K. Kuratowski, *Topology Vols 1 and 2*, Academic Press (1966).

[3] S. Kasahara, *Operation-Compact Spaces*, Mathematica Japonica 24(1) (1979) 97-105.

[4] F. H. Khedr, *Operation on Bitopologies*, Delta Journal Of Science 8(1) (1984) 309-320.

[5] F. H. Khedr, K. H. Abdelhakiem, *Operations on Bitopological Spaces*, Fasciculi Mathematici 45 (2010) 47-57.

[6] D. Andrijević, *On b-open Sets*, Matematički Vesnik 205 (1996) 59-64.

[7] T. Al-Hawary, A. Al-Omari, *b-open and b−continuity in Bitopological Spaces*, Al-Manarah 13(3) (2007) 89-101.

[8] A. C. Guler, G. Aslim, *bI-open Sets and Decomposition of Continuity via Idealization*, Proceedings of Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan 22 (2005) 27-32.

[9] E. Ekici, *On Pre-I-open sets, Semi-I-open sets and b-I-open sets in Ideal Topological Spaces*, Acta Universitatis Apulensis 30 (2012) 293-303.

[10] N. Rajesh, M. Caldas, S. Jafari, *Preopen Sets in Ideal Bitopological Spaces*, Boletim da Sociedade Paranaense de Matemática 29(2) (2011) 61-68.

[11] H. Z. Ibrahim, *On Pre*$-\gamma - I-$*Open Sets In Ideal Topological Spaces*, arxiv preprint arxiv: 1501.02494. 2015 Jan 11.

[12] S. D. Jyoti, *bI-open Sets in Ideal Bitopological Space*, International Journal of Pure and Applied Mathematics 105(1) (2015) 7-18.

[13] S. Hussain, *On Generalized Open Sets*, Hacettepe Journal of Mathematics and Statistics 47(6) (2018) 1438-1446.

[14] I. Bukhatwa, S. Demiralp, *On Generalized* β*-Open Sets in Ideal Bitopological Space*, European Journal of Pure and Applied Mathematics 13(2) (2020) 269-279.

[15] M. Ilkhan, M. Akyigit, E. E. Kara, *On New Types of Sets via* $\gamma-$*open Sets in Bitopological Spaces*, Communications Series A1 Mathematics & Statistics 67(1) (2017) 225-234.

[16] M. Kar, S. S. Thakur, S. S. Rana, J. K. Maitra, *I-Continuous Functions in Ideal Bitopological Spaces*, American Journal of Engineering Research 3(3) (2014) 51-55.

[17] M. Akdağ, *On bI-open Sets and bI-continuous Functions*, International Journal of Mathematics and Mathematical Sciences Article ID 75721 (2007) 13 pages.

[18] M. Caldas, S. Jafari, N. Rajesh, *Some Fundamental Properties of Beta-open Sets in Ideal Bitopological Spaces*, European Journal of Pure and Applied Mathematics 6(2) (2013) 247-255.

[19] B. C. Tripathy, D. J. Sarma, *Generalized b-closed Sets in Bitopological Spaces*, Proyecciones (Antofagasta) 33(3) (2014) 315-324.

[20] J. K. Maitra, H. K. Tripathi, *Local Function in Generalized Ideal Topological Spaces*, VISLESANA 11(1) (2014) 191-195.

[21] H. Ogata, *Operations on Topological Spaces and Associated Topology*, Mathematica Japonica 36 (1991) 175-184.

[22] G. Şenel, N. Çağman, *Soft Closed Sets on Soft Bitopological Space*, Journal of New Results in Science 3(5) (2014) 57-66.

[23] G. Şenel, *On Bitopological Spaces*, Master's Thesis, Ege University (2009) İzmir, Turkey.

[24] R. Vaidyanathaswamy, The Localisation Theory in Set Topology, Proceedings of the Academic of Sciences, 20 (1945) 51-61.

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A Novel Hybrid Sharing Economy Based Blockchain Model (Proof of Meet)

Selçuk Topal[1]

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Abstract − This paper is the first part of the Proof of Meet (PoM) work. This part focuses on the need for the model and its system (roughly) rather than giving technical details. In the second part, full technical details and architecture of the model will be discussed. The model focuses on consensus-based sharing (meeting) economy model. The sharing (meeting) describes the relationship between what is already somewhere and what goes there. The consensus is built on active clients (where the client can be anything capable of a change of location) on a social activity (location and action). There are two parts of clients; the first who/which go somewhere for a purpose (C_1) and the second who/which are regularly somewhere for a purpose (C_2). C_1 is in the domain of at least two clients and C_2 is in at least one. To build a decentralised blockchain with no energy-consuming, the paper proposes a consensus system, PoM, on the top of Proof of Stake (PoS) and Blockchain-Based Proof of Location (BBPL). The sharing economy model is one of the applications of the proposed blockchain model. The blockchain system called HOX is available to use very large area distributed ledger systems from a dynamic transportation system to dynamic big data. HOX is an instance of PoM application. Consequently, the proposed model provides the opportunity to record taxes, earn a fair share of labour and use blockchain actively in daily life within the legal framework.

Keywords − *Proof of meet, proof of stake, blockchain, consensus, location-based blockchain, dynamic big data*

1. Introduction

A blockchain is a recording system consisting mainly of data blocks and created for decentralisation based on a distributed ledger system. The system starts with a genesis block, and subsequent operations are recorded in blocks, respectively. Records cannot be deleted or changed. When a certain number of approved data is entered from the blocks, the next block is passed. The data is encrypted cryptographically with hash, and the transactions made with these decryptions are confirmed. Nodes, called miners, compete for the decryptions to make these confirmations. This contest is sometimes held in PoW system depending on the processing power of the devices and sometimes the amount of coins that have been staked in the PoS system. Therefore, it is a struggle where the equipment is stronger, or the amount of coins is higher in the race. The PoM system in this article proposes a more meeting-based structure. A system is mentioned in which the people who meet more and more and spend more time are included in the dynamics of the system. Since the idea of blockchain and decentralisation came into our lives practically, we continue to experience different entropy for a new digital adventure. While some scientists, human societies, and technology lovers are supporters of this issue, on the other hand, the efforts of ordinary people and most administrations to escape (without being distant but indifferent) continue with increasing momentum. Of course, the issue is not only finding meaning in a

[1]s.topal@beu.edu.tr (Corresponding Author),
[1]Department of Mathematics, Bitlis Eren University, Bitlis, Turkey

decentralised system but also in this environment where pandemic days are experienced, a safe and practical digitalisation has started Proof of Meet to find a lot of discussions. By digitalisation, the intent is to ensure that digital payment systems are fully integrated into everyday life, minimising fraud, and third-party control to even zero levels. One of the other and most important issues is the transfer time of digital assets, and the blocks swelling and getting heavier, causing more and more data occupation. For example, the Bitcoin blockchain [1] has 100 GB data records and its enamelling causes a lot of electricity to be consumed. On the other hand, although the emphasis on decentralisation has come to the fore, the power of the devices making enamel in the mining system determines the strongest in the system. The Proof of Stake concept has been introduced, and many applications have been developed for the solution of this type of problem to minimise energy consumption. Currently, Ethereum [2] system is in preparation for PoS. In today's PoS-based applications, stake power, that is, the dominance of the number of digital assets in the total digital asset, in other words, the dominant, block maker and block approvals of the system with the total number of coins in the total system have led the system to move away from decentralisation. This problem remains an obvious problem as there is a predominance. However, an attempt is made to deal with the random selection of blockers and transaction validators (used for stake) from those with the largest number of coins. Considered by the country's governments, blockchain systems constitute a corrosive pressure factor for taxation and asset tracking. The main reason is that digital assets that are not under the control of the bank cannot be registered with the daily life economy. On the other hand, how much share do social enterprises get benefits from this much developments? Of course, it is near to zero. Because payments made through digital payment systems (especially blockchain-based) continue to remain an informal economic situation for governments, as such, of course, this digital money and asset systems remain minimal to contribute to the real economy of businesses and the real economy itself. The model proposed by this study aims to minimise the informal economy that everyone in the model ecosystem will gain. The model proposes an improved system using PoS and BBPL models. On the other hand, a fairer and shared attitude will be developed with the activation of block validators and constructors in the PoS system, real labour factors and C_1-C_2s rather than the number of coins.

2. Proof of Meet (PoM): HOX Instance

Amoretti et al. [3] proposed a reliable LBS system. They focused on the network, which has mobile nodes connected to the Internet via the Wi-Fi or cellular network interface and can exchange information with neighbouring nodes through short-range communication technologies such as Bluetooth. It is a node that asks its proving neighbours to collect location information. A Witness for a Prover the node that provides the Prover with the proof of position. The following flow provides the fictive system in the PoM system:
C_1s meet in one place (at least two C_1s) and on mobile (they match with a Bluetooth type function via the app, but since Bluetooth will drain the battery of the mobile device, location-based distance in Wi-Fi can be controlled in a certain time interval.). Wi-Fi or cellular network unit is activated, indicating that the C_1s that meet are there, and position information is obtained. The information that they are in the place is confirmed by C_2 with C_1's mobile fingerprint feature. Then, the total time spent at the venue is processed on the application, and C_2 sends the data to be validated to the blockchain for approval after the approval of the meeting with fingerprints and payment by included in the tax payment system.

PoM (Restricted to HOX) aims to the blockchain as follows:

1. Constructing a position and meet system to protect existing execution from deception. The system where trysters (clients who/which meet) are rewarded with coins both in today's payment system and on the blockchain. Benefits from rewards are provided in the following ways:
    a. Clients who/which meet the most. (minting method (mm))
    b. Clients who/which meet with the most clients (for C_1 clients). (mm)
    c. Having the longest meeting period. (mm)
    d. To be included in the tax payment through the application by paying with a PayPal or similar system at a discount in the shopping area. (taxed method (tm))

e. By approving the start and end of the meeting of C_2 (on the commercial basis), gains customer loyalty and intensity and the payments are recorded in tax. (mm+tm)

2. The creation of blocks includes information such as a meeting id, start-end time, id of the clients in the meeting (clients in group C_1), id of C_2 (also coded with the Wi-Fi mac address). With this feature, location, time and stakeholders are included in the system pool.
3. In the model, a customised PoM is used to store proofs of meet. Two parts of proof validators and block validators play a role equally weighted: C_1 and C_2. The ones with the highest power are randomly selected from one set at a time. (mm)
4. The consensus algorithm for public blockchain (PoM) that depend on a validator's stake power ($P_1(c_i)$) in the network is calculated as follows for C_1:
   The stake power of ci which is a client in C_1 is denoted by $P_1(c_i)$.

   $SC_1(c_i)$: The staked coin amount of c_i.
   $TC_1(c_i)$: The staked time period (seconds) of coins of c_i.
   $CMN(c_i)$: The total number of clients c_i meets up to the present time.
   $MMN(c_i)$: The total number of meetings of c_i up to the present time.
   $TMN(c_i)$: The total time spent (seconds) from all meetings of c_i up to the present time.

   $P_1(c_i) = w_1SC_1(c_i) + w_2TC_1(c_i) + w_3CMN(c_i) + w_4MMN(c_i) + w_5TMN(c_i)$
   $\sum_{h=1}^{5} w_h = 1$ where $0.1 < w_h < 1$.

5. The consensus algorithm for public blockchain (PoM) that depend on a validator's stake power $P_2(c_i)$ in the network are calculated for C_2 as follows:
   The stake power of c_i which is a client in C_2 is denoted by $P_2(c_i)$.

   $SC_2(c_i)$: The staked coin amount of c_i.
   $TC_2(c_i)$: The staked time period (seconds) of coins of c_i.
   $CHN(c_i)$: The total number of clients c_i hosted to the present time.
   $THN(c_i)$: The total time spent (seconds) from all hosts up to the present time.
   $DHN(c_i)$: The total discount amount as % up to the present time.

   $P_2(c_i) = k_1SC_2(c_i) + k_2TC_2(c_i) + k_3CHN(c_i) + k_4THN(c_i) + k_5DHN(c_i)$
   $\sum_{h=1}^{5} k_h = 1$ where $0.1 < k_h < 1$.

Remark 2.1. In the system, the coin is mined, and discounts are earned dependent on the number of people met and the duration of the meeting. With these contributions, clients also become validators of the system, that is, they also get block rewards at the rate of P_1 and P_2.

Remark 2.2. A C_1 and a C_2 cannot meet at the venue of C_2. A C_1 and a C_2 can only meet in a place they do not own.

Remark 2.3. The application can register the amount paid with the payment system it will provide. Thus, a fraud about discount can be prevented easily.

Remark 2.4. If a C_2 tries to mislead the system for discounts to gain more stake power, it is thrown out of the system and loses all staked coins. If the C_2 client goes this way; naturally, the client would have deceived the tax system.

3. Dynamic Big Data Equipped with Location (DBDL)

Of course, when PoM is used for transportation systems, it should be an advantage to have less time spent at the destination. In this sense, the PoM system can be adapted to a reward increasing system for the shorter time interval with a reward rate change process. Due to the difficulty of having more than two C_1s at the meeting, another hybrid structure can be used by using the system in [3]. Thus, witness and neighbourhood features can be used since there is one client in C_1 and C_2. For more complex data systems, especially dynamic data (transactional data) including the transportation process, require considering time dimension and location data. Such data sets and their analysis relates to financial (orders, invoices, payments), work (plans, activity records), and logistics (deliveries, storage records, travel records). Events between clients, clients' spatial, quantitative and qualitative data changes are subject to non-static evaluations for data analysis. With a blockchain record of these non-static processes, it is possible to reward the employees of the company and improve the quality of the work done by equipping the business development processes of the companies with quality control techniques. It can contribute to the development of not only companies but also companies that do business within the framework of quality control systems. PoM can be a decentralised and fair approach to business development in terms of strategy determination, labour-earning rate and incentive rewards, especially in business lines where use and evaluation of temporal (time-wise) and spatial changes in data sets is essential.

Remark 3.1. Ethereum system charges a fee for the mint rewards that customers received from their staking, meeting and confirmation. To prevent this, an integration on Quorum Blockchain [4] can be provided easily. Positively, Quorum has a PoM-compliant structure with the Raft consensus protocol running on the blockchain.

Remark 3.2. PoM, Avalanche [5] and Quorum integration can be achieved easily without requiring a new blockchain construction. Besides, by integrating with the RaidaTech [6] infrastructure, all transactions can be zero fees.

4. Conclusion

It is well known that within the framework of blockchain studies and discussions, a negative argument against the state and banking systems is set out, and an attempt is made to create a mental conflict environment. A hybrid model that can be called a middle ground is introduced with PoM. Because the hybrid space of the current legal payment systems with the applications in PoM will ensure that the coins become a legal payment instrument within the system. Although the services offered by PoM are not a direct payment instrument, the services provide DeFi advantages in blockchain and payment agent in terms of being included in the registered tax system. The system needs to be developed in terms of both scalability and data size management and undergoes a good testing process. With the Proof of Meet system, many applications that include dynamic data can be solved decentrally and fairly such as workloads, data tracking, job-worker rewards, location data frauds and tax auditing. However, many projects related to technical details in the project in the frame of logic, cryptographic, mathematics and computer science can be revealed. DAG-based transaction equipment with homotopy type theory will surely open new gates for novel structures to extend blockchain with ontology-based logic tensor vector.

References

[1] S. Nakamoto, *Bitcoin: A Peer-to-Peer Electronic Cash System*, https://bitcoin.org/bitcoin.pdf (2008) 1-9.

[2] G. Wood, *Ethereum: A Secure Decentralised Generalised Transaction Ledger*, Ethereum Project Yellow Paper 151 (2014) 1-32.

[3] M. Amoretti, G. Brambilla, F. Medioli, F. Zanichelli, *Blockchain-Based Proof of Location*, IEEE International Conference on Software Quality, Reliability and Security Companion (QRS-C) (2018).

[4] https://www.goquorum.com, 14.08.2020.

[5] https://www.avalabs.org, 14.08.2020.

[6] https://raidatech.com, 14.08.2020.

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Measurement of Radiation Dose Emitted by 4×4 cm² and 10×10 cm² Aperture of Linear Accelerator[1]

Elif Sarıkaya[2], Melek Gokbulut[3], Gulhan Guler Avcı[4], Halis Bozkurt[5], İbrahim Yigitoglu[6], Arzu Coskun[7]

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Abstract − The amount of the beam emitted from a linear accelerator's head can be adjusted by changing the jaw intervals of multi-leaf collimator (MLC) system. One of the parameters that are effective in delivering the prescribed dose to the patient is also the gantry angle. In this study, the jaw intervals of the linear accelerator in the Oncology Unit of the Tokat Gaziosmanpaşa University Faculty of Medicine have been determined as 4×4 cm^2 and 10×10 cm^2. The radiation dose emitted for both conditions has measured for 6 MV and 15 MV photon beams at different gantry angles. The measurements are taken at gantry angles of 0^o, 90^o, 180^o, and 270^o. The highest dose rate value is obtained in 10×10 cm^2 jaw interval for 15 MV photon energy at 0^o gantry angle. Then, the percentages of change between dose rate values measured at different portal angles are calculated. The obtained results are evaluated, considering other studies in the literature.

Keywords − *Linear accelerators, Gantry angles, Jaw intervals, Radiation dose*

1. Introduction

Various types of cancer rank second among the causes of death worldwide. For this reason, ways to deal with the disease are being explored. Today, methods of dealing with cancer develop over time. Radiotherapy, which plays a very important role in cancer treatment, is one of the comprehensive and fastest developing treatment methods. In radiotherapy, kilovoltage x-ray beams are useful in the treatment of skin lesions and shallow tumours. At the same time, they are not sufficient for tumours with deep localisation and are limited to high skin dose. Megavoltage beams not only more penetrating but is also of great benefit in delivering the maximum dose below the skin surface. In this context, one of the most advanced methods used to destroy cancer tissue is linear accelerators [1].

The principal radiation method for the treatment of deeply located tumours is x-rays of very high energy with penetrating power. In this regard, radiotherapy is applied through a radioactive source, typically Cobalt-60, or x-rays produced by stopping the electron beam with a tungsten target in a linear accelerator where electrons are accelerated. Alternatively, the electrons themselves may be used directly to treat more

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[2]elifoksuzzz@gmail.com; [3]melek.kgb@gmail.com (Corresponding Author); [4]gulhan.guleravci@gop.edu.tr; [6]iyigitoglu@yahoo.com; [7]arzucos@hotmail.com
[2,3,6]Department of Physics, Faculty of Arts and Sciences, Tokat Gaziosmanpasa University, Tokat, Turkey
[4,5]Department of Radiation Oncology, Faculty of Medicine, Tokat Gaziosmanpasa University, Tokat, Turkey
[7]Department of Physics, Health Sciences Institute, Mersin Toros Universty, Mersin, Turkey

superficial cancers. Electron linear accelerator (linac) that accelerates charged particles in a straight line instead of circular or race track orbits such as cyclotron and synchrotron is mounted in a gantry that rotates on a stand containing electronic and other systems. For use in treatment, the linac can be rotated into position about the horizontal gantry axis, and the radiation beam emerging from the collimator is always directed through and centred on the gantry axis. The point where the centre of the beam intersects with the gantry axis in the space is called isocenter. The couch, which includes three linear movements along the isocenter and rotation around one or more axes, is centred such that the patient's tumour is at the isocenter. In the isocentric system, precise and reproducible treatment is provided by using multiple fields directed at the tumour from different gantry angles [2]. The beam spot displacement, collimator asymmetry, and movement of the collimator or slide rotation axis, which may cause misalignment of the x-ray beam, have been investigated by developing a test method [3] sensitive to these problems. Also, detailed information on developments in accelerator structures, variable energy connections, microtrons, beam transport systems and head design were examined, and numerical data related to them were evaluated [4]. Irradiation of healthy tissue with irradiation in the radiotherapy technique is an important limitation of radiotherapy treatment. Many organs that are sensitive to radiation damage (the spinal cord, salivary glands, lungs, and the eyes are common examples) should be especially considered during radiotherapy treatment planning. Advances in technology have led to design modifications in the collimating systems of modern accelerators.

In particular, MLCs are consisting of tungsten alloy leaves that can change shape and move independently according to almost any lesion and organ [5]. In a dynamic multi-leaf collimation system, the leaves are continuously in motion. At the same time, the beam is on to deliver optimum dose distributions which result in greater dose uniformity in the target and lower doses in the surrounding critical organs [6]. With this feature of the collimator system, healthy tissues are protected much better. Jaw intervals of collimators are determined according to the treatment plan. Jaws can be adjusted from 4×4 cm^2 to 40×40 cm^2. Previous studies have shown that effective doses can be given to the tumour with correct leaf positions [7, 8]. Thus, the correct dosage is given to the right volume. At the same time, it is protected by collimation in healthy tissue.

Radiation therapy aims to deliver the maximum dose to the tumour region while protecting the surrounding healthy tissues. To achieve this, various planning techniques have been carried out. In deeply located lesions, 4-6- and 15 MV x-rays, in superficial settlements, electrons with energies of 4-6-9-12-15 and 18 MeV are sufficient for all clinical applications [9]. In linear accelerators, to not to harm the healthy tissue while destroying the cancerous tissue, the treatment plan that contains the leaf positions for all control points (gantry angles) has been developed by changed the photon flux with the return of gantry [10].

Gantry angle is a very important parameter that is effective in dose reconstruction for precise and specific treatment. A simple measurement method and algorithm has been developed to calibrate the portal angle of a linear accelerator and test the reliability of portal angle measurements to prevent healthy tissue exposure to excessive radiation [11, 12]. To realise the mechanical quality assurance of linear accelerators, it is important to determine the actual zero degrees of the gantry angle [13]. The purpose of this work is to measure the scattered dose rate values at different gantry angles, to evaluate the changes between the measured the dose rates and is to determine the highest dose rate value in terms of radiation safety.

2. Materials and methods

In this work, the measurements have been performed using FLUKE Victoreen ASM 990 portable detector at the Nuclear Medicine Department, Medicine Faculty in Tokat Gaziosmanpaşa University. The ASM-990 series are designed to be detected alpha, beta, gamma, neutron, or x-ray radiation within a range of 1 μR/h to 1 R/h, depending on the selected probe, such as Geiger-Muller, neutron, proportional counter, scintillation. These detectors are used as a general survey meter with the proper probe combination [14]. The count rates and the dose rates are generally established through empirical calibration procedures. In this study, beam profile measurements were performed for 6 MV and 15 MV photon beams using the 4×4 cm^2 - 10×10

cm^2 field size applicators. Measurements were made at gantry angles of 0°, 90°, 180°, and 270°. The detector is located in the space between the surprising corridor and the front door in the section where the linear accelerator is located. The measurements were taken as cps (count per second), and the obtained values were converted to μSv/h. Here, the count rate is taken as about five cps per μSv/h [15]. By taking into account the measured time interval, the calculated values were converted to mSv/y.

3. Results

The dose rates values obtained according to the gantry angles varying in different jaw intervals for two different photon beams energies values are presented in Table 1.

Table 1. Dose rates values (mSv/y) according to gantry angle and jaw intervals for 6 MV and15 MV photon beams

	Dose rates for different gantry angels with 4×4 cm^2 and 10×10 cm^2 jaw intervals at 6 MV photon beam		**Dose rates for different gantry angels with 4×4 cm^2 and 10×10 cm^2 jaw intervals at 15 MV photon beam**	
Gantry Angle	**4×4 cm^2**	**10×10 cm^2**	**4×4 cm^2**	**10×10 cm^2**
0°	2.86	4.20	29.7	33.2
90°	1.88	2.43	13.2	16.6
180°	1.95	2.72	11.2	16.5
270°	1.14	2.03	17.7	23.8

The graphic representations of dose rates measured for 6 MV and 15 MV photon beams according to changing gantry angles, in 4×4 cm^2 and 10×10 cm^2 jaw intervals are given in Figure 1 and Figure 2, respectively.

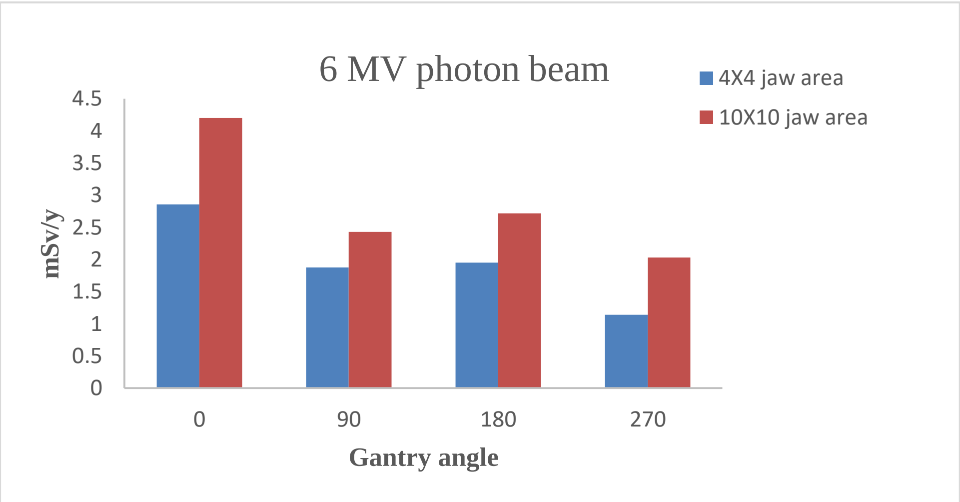

Fig. 1. Change of dose rates values according to gantry angle and jaw intervals in 6 MV photon beam

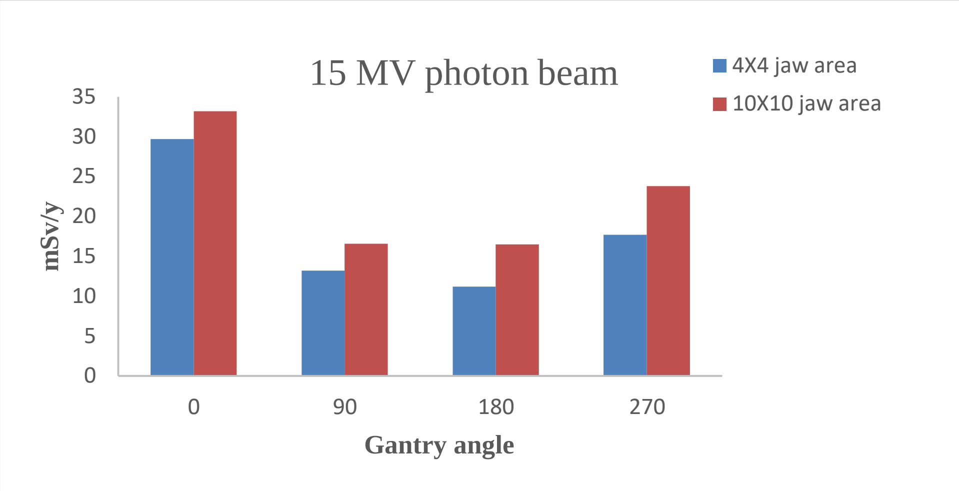

Fig. 2. Change of dose rates values according to gantry angle and jaw intervals in 15 MV photon beam

4. Discussion

In this study, the scattered dose rate values of the measured for the selected gantry angles of 0, 90, 180, and 270 degrees were presented in Table 1. The measurements were taken for each of the two different photon beams values (6 MV and 15 MV) and two different jaw intervals (4 × 4 cm^2 and 10 × 10 cm^2). From Table 1, it was observed that the dose rate increased when the collimator jaw intervals were increased from 4 × 4 cm^2 to 10 × 10 cm^2. The highest dose rates values have been measured for larger jaw interval in constant gantry angle values. An increase in the scattered dose rates with increases jaw interval is expected due to the change in the field size effects [16] the radiation output exiting the linac head. It was also observed that the amount of absorbed surface dose increased with increases jaw interval accordingly [17].

As can be seen in Figs. 1 and 2, when compared with other gantry angles, the highest value is observed where the gantry angle is chosen 0°. Also, at the same jaw interval and energy values, it is seen that the dose rates values measured at 90^o and 180^o gantry angles are close to each other. The maximum change between the dose rate values measured at different gantry angles in both 4 × 4 and 10 × 10 jaw intervals for 6 MV photon beam is taken at 0^o-270^o gantry angles. In contrast, the minimum change is taken at 90^o-180^o gantry angles. The rate of change between the external dose rates values measured at 0^o-270^o gantry angles in the 4 × 4 jaw interval is 60%, and the rate of change between the external dose rates values measured at 90^o-180^o gantry angles is 4%. In the 10 × 10 jaw interval, the rate of change between the external dose rates values measured at 0^o-270^o gantry angles is 52% while the rate of change between the external dose rates values measured at 90^o-180^o gantry angles is 11%.

For 15 MV photon beam, the maximum change between the dose rates values measured at different gantry angles in both 4 × 4 and 10 × 10 jaw intervals is observed at 0^o-180^o gantry angles, and the minimum change is observed at 90^o-180^o gantry angles. In the 4x4 jaw interval, the maximum change is found 55% and the minimum change is found 15%. On the other hand, in the 10 × 10 jaw interval, the maximum change is 51%, while the minimum change is very small and is less than 1%. The rate of change between measured dose rates values is between 1% and 60%. The greatest deviation among these values was observed at 0^o gantry angle.

During the irradiation, the linac is rotated around the horizontal gantry axis. Here the measurements have been performed in standard conditions where the gantry angle is moving in the direction of the clockwise. The beam is opposite in the vertical plane when the gantry is rotated from 0^o to 180^o. At gantry angle of 0^o, the linac head is towards the floor while at gantry angle of 180^o, the linac head is towards the ceiling. At gantry angles of 90^o and 270^o, the linac head is directed towards the opposite left and right walls in the horizontal plane.

In studies investigating the effects of gantry rotations using electronic portal imaging devices [18,19], it has been assumed that unless attenuation and scatter conditions changed, the number of particles per second coming to electronic portal imaging devices would ideally be invariant with gantry angle. However, it has been stated that the images taken at various gantry angles were dissimilar. One of the most important factors causing this difference is reported to be scattered from different environmental structures such as walls, floors and ceilings. It is also stated that the extent of the effect may depend on the construction materials used [18].

In our study, the difference between the measured dose rates values is predicted to could arise due to backscattering from the different surrounding structures depending on the position of the linac head at different gantry angles. Addition to this, for the maximum change at gantry angles of 0◦ and 180◦, the distance between the head of a linac and the floor/ceiling is also predicted to may have been effective on the backscattered radiation until it reaches the detector. For the minimum change, it is considered that the distance of the scattered beam from the detector may be affected. Besides, it is envisaged that the effect of energy on the scattered dose amount can be explained by obtained different maximum change percentage in different energy values. As seen in figure 2, the highest scattering dose rate is 33.2 mSv/y (33200 μSv/y) and is measured for 15 MV photon energy at a gantry angle of 0^o in 10×10 jaw interval. The maximum allowable yearly dose rate is 50000μSv/y [20]. Although the measured scattering dose rate value is small compared to the allowable annual dose value, it is a very high value considering the position of the detector.

5. Conclusion

In our study, scattered dose rate measurements in the clinical linear accelerator were made using the FLUKE Victoreen ASM 990 portable detector placed in the space between the surprising corridor and the front door in the section where the linear accelerator. The obtained dose rates were found to be high, although they were performed behind the confluence corridor. According to the data obtained, it is concluded that the patient is exposed to extra radiation due to secondary rays and radiation pollution occurs in the environment. This has shown that additional precautions can be taken to be protected from secondary radiation. The variation between the dose rates values obtained for different gantry angles was calculated. From the obtained results, it is concluded that deviations in values measured at different gantry angles may result from backscatters from surrounding structures depending on the position of the linac head. Besides, it was concluded that another important effect is the distance between the scattered beam and the detector and photon energy.

References

[1] P. Mayles, A. Nahum, J. C. Rosenwald, *Handbook of Radiotherapy Physics: Theory and Practice*, United States of America: Taylor and Francis Group, LLC, 2007.

[2] C. J. Karzmark, R. J. Morton, *A Primer on Theory and Operation of Linear Accelerators in Radiation Therapy*, Wisconsin: Medical Physics Publishing, Inc.4555 Helgesen Drive, Second edition, 1998.

[3] W. R. Lutz, R. D. Larsen, B. E. Bjarngard, *Beam Alignment Test for Therapy Accelerators*, International Journal of Radiation Oncology Biology Physics 7 (1981) 1727-1731.

[4] C. J. Karzmark, *Advance in Linear Accelerator Design for Radiotherapy*, Medical Physics 11(2) (1984) 105-128.

[5] A. Boyer, P. Biggs, J. Galvin, E. Klein, T. LoSasso, D. Low, K. Mah, C. Yu, *Basic Applications of Multileaf Collimators*, Madison: American Association of Physicists in Medicine by Medical Physics Publishing Report No. 72, 2001.

[6] C. S. Chui, S. Spirou, T. LoSasso, *Testing of Dynamic Multi-Leaf Collimation*, Medical Physics 23(5) (1996) 635-641.

[7] S. V. Spirou, C. S. Chui, *Generation of Arbitrary Intensity Profiles by Dynamic Jaws or Multileaf Collimators*, Medical Physics 21 (1994) 1031-1041.

[8] J. H. Mott, A. R. Hounsell, G. J. Budgell, J. M. Wilkinson, P. C Williams, *Customised Compensation Using Intensity Modulated Beams Delivered by Dynamic Multileaf Collimation*, Radiotherapy and Oncology 53 (1999) 59-65.

[9] G. Aydın, *Temel ve Klinik Radyoterapi*, Temel Radyoterapi Fiziği I, İzmir: Hürriyet Matbaa (2013) 39-42.

[10] M. Bakhtiari, L. Kumaraswamy, D. W. Bailey, S. de Boer, H. K. Malhotra, M. B. Podgorsak, *Using an EPID for Patient-Specific VMAT Quality Assurance*, Medical Physics 38(3) (2011) 1366-1373.

[11] P. Rowshanfarzad, M. Sabet, D. J. Connor, P. M. McCowan, B. M. McCurdy, P. B. Greer, *Gantry Angle Determination During arc IMRT: Evaluation of a Simple EPID-based Technique and two Commercial Inclinometers*, Journal of Applied Clinical Medical Physics 13(6) (2012) 203-214.

[12] K. Higgins, J. Treas, A. Jones, N. A. Fallahian, D. Simpson, *An Improved Method for Calibrating the Gantry Angles of Linear Accelerators*, The Radiation Safety Journal 105(3) (2013) 196-198.

[13] L. Chang, S. Y. Ho, J. M. Wu, C. Y. Yu, C. C. Sung, *Technical Innovation to Calibrate The Gantry Angle Indicators of Linear Accelerators*, Journal of Applied Clinical Medical Physics 2(1) (2001) 54-58.

[14] ASM 990 Series Advanced Survey Meter, https://www.flukebiomedical.com/products/radiation-measurement, 25-March-2020.

[15] G. Chabot, *Instrumentation and Measurements Surveys and Measurements (SM)*, Health Physics Society, http://hps.org/publicinformation/ate/q4148.html, 25-May-2020.

[16] W. P. Mayles, R. Lake, A. McKenzie, E. M. Macaulay, H. M. Morgan, T.J. Jordan, S. K. Powley, *Physics Aspects of Quality Control in Radiotherapy*, York, England: The Institute of Physics and Engineering in Medicine, 1999.

[17] G. Gursoy, E. Eser, I. Yigitoglu, H. Koc, F. C. Kahraman, S. Yamcıcier, *Investigation of The Effects of A Carbon-Fibre Tabletop on The Surface Dose and Attenuation Dose for Megavoltage Photon Beams*, International Journal of Radiation Research 16(2) (2018) 1-7.

[18] P. W. Chin, D. G. Lewis, E. Spezi, *Correction for Dose-Response Variations in A Scanning Liquid Ion Chamber EPID As A Function of Linac Gantry Angle*, Physics in Medicine and Biology 49 (2004) 93-103.

[19] N. Zhao, B. Kim, J. Chin-Snyder, K. Bellon, M. Glide-Hurst, D. Chen, I. J. Chetty, *IMRT and RapidArc Commissioning of A TrueBeam Linear Accelerator using TG-119 Protocol Cases*, Journal of Applied Clinical Medical Physics 15(5) (2014) 74-88.

[20] J. H. Abuqubu, *Conversion Factor of Radioactivity Measurements Counts per Second (cps) into Dose Rates in Aqaba-Jordan Phosphogypsum Disposal Piles*, International Journal Of Scientific & Engineering Research 7(7) (2016) 691-697.

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Q-soft Translation of Q-soft Subgroups

Rasul Rasuli[1]

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Original Article

Abstract — In this study, we introduce the concept Q-soft translations of Q-soft subgroups. Next we investigate the properties of them and we prove that every Q-soft translation of Q-soft subgroup is also Q-soft subgroup. Finally, we consider them under homomorphism and anti-homomorphism of Q-soft subgroups and Q-soft normal subgroups.

Keywords — *Q-soft subsets, group theory, Q-soft subgroups, Q-soft normal subgroups, Q-soft translation, homomorphism*

Introduction

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra and other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. The concept of soft sets was first formulated by Molodtsov [1] as a completely new mathematical tool for solving problems dealing with uncertainties. After then Maji et al. [2] defined the operations of soft sets. The operations of soft sets have also been studied by Ali et al. [3], Çağman et al. [4], Çağman [5], and Sezgin and Atagün [6] in detail. Some researchers have applied soft sets theory to many different areas such as decision making [7–9], algebras [10–13] using these operations. The author investigated soft Lie ideals and anti soft Lie ideals and extension of Q-soft ideals in semigroups [14,15]. In [16,17], the authors introduced the concept of Q-soft subgroups and Q-soft normal subgroups and discussed the characterisations them under homomorphism and anti-homomorphism. In this paper, we define Q-soft translations of Q-soft subgroups and we show some properties of them. Next we prove that every Q-soft translation of Q-soft subgroup is also Q-soft subgroup. Also we obtain between Q-soft translation of Q-soft subgroup of group G and subgroup of group G. Later we prove that soft image and soft pre-image of Q-soft translation of Q-soft subgroup under group homomorphism is also Q-soft subgroup. Finally, we prove that soft image and soft pre-image of Q-soft translation of Q-soft normal subgroup under group homomorphism is also Q-soft subgroup.

[1]rasulirasul@yahoo.com (Corresponding Author)
[1]Department of Mathematics, Payame Noor University (PNU), Tehran, Iran

preliminaries

In this section we recall some of the fundamental concepts and definition, which are necessary for this paper. For details we refer reders to [1,15–18]. Throughout this work, Q is a non-empty set, U refers to an initial universe set, E is a set of parameters and $P(U)$ is the power set of U.

Definition 2.1. For any subset A of E, a Q-soft subset $f_{A\times Q}$ over U is a set, defined by a function $f_{A\times Q}$, representing a mapping $f_{A\times Q} : E \times Q \to P(U)$, such that $f_{A\times Q}(x, q) = \emptyset$ if $x \notin A$. A soft set over U can also be represented by the set of ordered pairs $f_{A\times Q} = \{((x, q), f_{A\times Q}(x, q)) \mid (x, q) \in E \times Q, f_{A\times Q}(x, q) \in P(U)\}$. Note that the set of all Q-soft subsets over U will be denoted by $QS(U)$. From here on, "soft set" will be used without over U.

Definition 2.2. Let $f_{A\times Q}, f_{B\times Q} \in QS(U)$. Then,

i. $f_{A\times Q}$ is called an empty Q-soft subset, denoted by $\Phi_{A\times Q}$, if $f_{A\times Q}(x, q) = \emptyset$ for all $(x, q) \in E \times Q$.

ii. $f_{A\times Q}$ is called a $A\times Q$-universal soft set, denoted by $f_{A\tilde{\times}Q}$, if $f_{A\times Q}(x, q) = U$ for all $(x, q) \in A\times Q$.

iii. $f_{A\times Q}$ is called a universal Q-soft subset, denoted by $f_{E\tilde{\times}Q}$, if $f_{A\times Q}(x, q) = U$ for all $(x, q) \in E\times Q$.

iv. The set $Im(f_{A\times Q}) = \{f_{A\times Q}(x, q) : (x, q) \in A \times Q\}$ is called image of $f_{A\times Q}$ and if $A \times Q = E \times Q$, then $Im(f_{E\times Q})$ is called image of $E \times Q$ under $f_{A\times Q}$.

v. $f_{A\times Q}$ is a Q-soft subset of $f_{B\times Q}$, denoted by $f_{A\times Q}\tilde{\subseteq} f_{B\times Q}$, if $f_{A\times Q}(x, q) \subseteq f_{B\times Q}(x, q)$ for all $(x, q) \in E \times Q$.

vi. $f_{A\times Q}$ and $f_{B\times Q}$ are soft equal, denoted by $f_{A\times Q} = f_{B\times Q}$, if and only if $f_{A\times Q}(x, q) = f_{B\times Q}(x, q)$ for all $(x, q) \in E \times Q$.

vii. The set $(f_{A\times Q}\tilde{\cup} f_{B\times Q})(x, q) = f_{A\times Q}(x, q) \cup f_{B\times Q}(x, q)$ for all $(x, q) \in E \times Q$ is called union of $f_{A\times Q}$ and $f_{B\times Q}$.

viii. The set $(f_{A\times Q}\tilde{\cap} f_{B\times Q})(x, q) = f_{A\times Q}(x, q) \cap f_{B\times Q}(x, q)$ for all $(x, q) \in E \times Q$ is called intersection of $f_{A\times Q}$ and $f_{B\times Q}$.

Example 2.3. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of parameters. Let $Q = \{q\}$, $A = \{x_1, x_2\}$, $B = \{x_2, x_3\}$, $C = \{x_4\}$, $D = \{x_5\}$, and $F = \{x_1, x_2, x_3\}$. Define

$$f_{A\times Q}(x, q) = \begin{cases} \{u_1, u_2, u_3\}, & \text{if } x = x_1 \\ \{u_1, u_5\}, & \text{if } x = x_2 \end{cases}$$

$$f_{B\times Q}(x, q) = \begin{cases} \{u_1, u_2\}, & \text{if } x = x_2 \\ \{u_2, u_4\}, & \text{if } x = x_3 \end{cases}$$

$$f_{F\times Q}(x, q) = \begin{cases} \{u_1, u_2, u_3, u_4\}, & \text{if } x = x_1 \\ \{u_1, u_2, u_5\}, & \text{if } x = x_2 \\ \{u_2, u_4\}, & \text{if } x = x_3 \end{cases}$$

$f_{C\times Q}(x_4, q) = U$ and $f_{D\times Q}(x_5, q) = \{\}$. Then we will have $f_{C\times Q} = f_{C\tilde{\times}Q}$ and $f_{D\times Q} = \Phi_{D\times Q}$. Note that the definition of classical subset is not valid for the soft subset. For example, $f_{A\times Q}\tilde{\subseteq} f_{F\times Q}$ does not imply that every element of $f_{A\times Q}$ is an element of $f_{F\times Q}$. Thus $f_{A\times Q}\tilde{\subseteq} f_{F\times Q}$ but $f_{A\times Q} \nsubseteq f_{F\times Q}$ as classical subset.

Definition 2.4. Let $\varphi : A \to B$ be a function and $f_{A\times Q}, f_{B\times Q} \in QS(U)$. Then soft image $\varphi(f_{A\times Q})$ of $f_{A\times Q}$ under φ is defined by

$$\varphi(f_{A\times Q})(y, q) = \begin{cases} \cup\{f_{A\times Q}(x, q) \mid (x, q) \in A \times Q, \varphi(x) = y\}, & \text{if } \varphi^{-1}(y) \neq \emptyset \\ \emptyset, & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

and soft pre-image (or soft inverse image) of $f_{B\times Q}$ under φ is $\varphi^{-1}(f_{B\times Q})(x, q) = f_{B\times Q}(\varphi(x), q)$ for all $(x, q) \in A \times Q$.

Definition 2.5. Let $(G,.),(H,.)$ be any two groups. The function $f : G \to H$ is called a homomorphism (anti-homomorphism) if $f(xy) = f(x)f(y)(f(xy) = f(y)f(x))$, for all $x, y \in G$.

Proposition 2.6. Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

i. H is a subgroup of G.

ii. $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Definition 2.7. Let $(G,.)$ be a group and $f_{G\times Q} \in QS(U)$. Then, $f_{G\times Q}$ is called a Q-soft subgroup over U if $f_{G\times Q}(xy, q) \supseteq f_{G\times Q}(x, q) \cap f_{G\times Q}(y, q)$ and $f_G(x^{-1}, q) = f_{G\times Q}(x, q)$ for all $x, y \in G, q \in Q$. Throughout this paper, G denotes an arbitrary group with identity element e_G and the set of all Q-soft subgroup with parameter set G over U will be denoted by $S_{G\times Q}(U)$.

Example 2.8. Let $G = \{1,-1\}$ be a group and $U = \{u_1, u_2, u_3, u_4\}, Q = \{q\}$. Let $f_{G\times Q} = \{((1, q), \{u_1, u_2\}), ((-1, q), \{u_1, u_3\})\}$, then $f_{G\times Q} \in S_{G\times Q}(U)$.

Proposition 2.9. $f_{G\times Q} \in S_{G\times Q}(U)$ if and only if $f_{G\times Q}(xy^{-1}, q) \supseteq f_{G\times Q}(x, q) \cap f_{G\times Q}(y, q)$ for all $x, y \in G, q \in Q$.

Definition 2.10. Let $f_{G\times Q} \in S_{G\times Q}(U)$ then $f_{G\times Q}$ is said to be a Q-soft normal subgroup of G if $f_{G\times Q}(xy, q) = f_{G\times Q}(yx, q)$, for all $x, y \in G$ and $q \in Q$. Throughout this paper, G denotes an arbitrary group and the set of all Q-soft normal subgroup with parameter set G over U will be denoted by $NS_{G\times Q}(U)$.

Example 2.11. Let $U = \{u_1, u_2, u_3\}$ be an initial universe set and $(\mathbb{R},+)$ be an additive real group. Define $f_{\mathbb{R}\times Q} : \mathbb{R} \times Q \to P(U)$ as

$$f_{\mathbb{R}\times Q}(x, q) = \begin{cases} \{u_1, u_2\}, & \text{if } x \in \mathbb{R}^{\geq 0} \\ \{u_3\}, & \text{if } x \in \mathbb{R}^{<0} \end{cases}$$

then $f_{\mathbb{R}\times Q} \in NS_{\mathbb{R}\times Q}(U)$.

Q-soft Translation

Definition 3.1. Let $f_{G\times Q} \in QS(U)$ and $\alpha \in P(U)$. Then $T = T_\alpha^{f_{G\times Q}} : G \times Q \to P(U)$ is called a soft translation of $f_{G\times Q}$ if $T(x, q) = f_{G\times Q}(x, q) \cup \alpha$, for all $x \in G, q \in Q$ and $\alpha \in P(U)$.

Proposition 3.2. If T is a Q-soft translation of a Q-soft subgroup $f_{G\times Q}$ of a group G, then $T(x^{-1}, q) = T(x, q)$ and $T(e, q) \supseteq T(x, q)$ for all $x \in G$ and $q \in Q$.

Proof. Let $x \in G$ and $q \in Q$. Then,

$$\begin{aligned} T(x, q) = f_{G\times Q}(x, q) \cup \alpha &= f_{G\times Q}((x^{-1})^{-1}, q) \cup \alpha \\ &\supseteq f_{G\times Q}(x^{-1}, q) \cup \alpha \\ &= T(x^{-1}, q) \\ &= f_{G\times Q}(x^{-1}, q) \cup \alpha \\ &\supseteq f_{G\times Q}(x, q) \cup \alpha \\ &= T(x, q) \end{aligned}$$

Then, $T(x^{-1}, q) = T(x, q)$. Also

$$\begin{aligned} T(e, q) = f_{G\times Q}(e, q) \cup \alpha &= f_{G\times Q}(xx^{-1}, q) \cup \alpha \\ &\supseteq (f_{G\times Q}(x, q) \cap f_{G\times Q}(x^{-1}, q)) \cup \alpha \\ &= f_{G\times Q}(x, q) \cup \alpha \\ &= T(x, q) \end{aligned}$$

Thus, $T(e, q) \supseteq T(x, q)$. □

Proposition 3.3. Let T is a Q-soft translation of a Q-soft subgroup $f_{G\times Q}$ of a group G. If $T(xy^{-1},q) = T(e,q)$, then $T(x,q) = T(y,q)$ for all $x, y \in G$ and $q \in Q$.

PROOF. Let $x, y \in G$ and $q \in Q$. Then,

$$\begin{aligned}
T(x,q) = f_{G\times Q}(x,q) \cup \alpha \quad &= f_{G\times Q}(xy^{-1}y,q) \cup \alpha \\
&\supseteq (f_{G\times Q}(xy^{-1},q) \cap f_{G\times Q}(y,q)) \cup \alpha \\
&= (f_{G\times Q}(xy^{-1},q) \cup \alpha) \cap (f_{G\times Q}(y,q) \cup \alpha) \\
&= T(xy^{-1},q) \cap T(y,q) \\
&= T(e,q) \cap T(y,q) \\
&= T(y,q) \\
&= f_{G\times Q}(y,q) \cup \alpha \\
&= f_{G\times Q}(yx^{-1}x,q) \cup \alpha \\
&\supseteq (f_{G\times Q}(yx^{-1},q) \cap f_{G\times Q}(x,q)) \cup \alpha \\
&= (f_{G\times Q}(yx^{-1},q) \cup \alpha) \cap (f_{G\times Q}(x,q) \cup \alpha) \\
&= T(yx^{-1},q) \cap T(x,q) \\
&= T(e,q) \cap T(x,q) \\
&= T(x,q)
\end{aligned}$$

Therefore, $T(x,q) = T(y,q)$. □

Proposition 3.4. If T is a Q-soft translation of a Q-soft subgroup $f_{G\times Q}$ of a group G, then T is a Q-soft subgroup of G.

PROOF. Let $x, y \in G$ and $q \in Q$. Then,

$$\begin{aligned}
T(xy^{-1},q) = f_{G\times Q}(xy^{-1},q) \cup \alpha \quad &\supseteq (f_{G\times Q}(x,q) \cap f_{G\times Q}(y^{-1},q)) \cup \alpha \\
&\supseteq (f_{G\times Q}(x,q) \cap f_{G\times Q}(y,q)) \cup \alpha \\
&= f_{G\times Q}(x,q) \cup \alpha) \cap (f_{G\times Q}(y,q) \cup \alpha) \\
&= T(x,q) \cap T(y,q)
\end{aligned}$$

Therefore, by Proposition 2.9 we get that T is a Q-soft subgroup of G. □

Proposition 3.5. If T is a Q-soft translation of a Q-soft subgroup $f_{G\times Q}$ of a group G, then $H = \{x \in G : T(x,q) = T(e,q)\}$ is a subgroup of G.

PROOF. Let $x, y \in H$ and $q \in Q$. As $T(x^{-1},q) = T(x,q) = T(e,q)$ so $x^{-1} \in H$. Now

$$\begin{aligned}
T(xy^{-1},q) \supseteq T(x,q) \cap T(y,q) = T(e,q) \cap T(e,q) \quad &= T(e,q) \\
&= T((xy^{-1})(xy^{-1})^{-1},q) \\
&\supseteq T(xy^{-1},q) \cap T(xy^{-1},q) \\
&= T(xy^{-1},q)
\end{aligned}$$

Thus, $T(xy^{-1},q) = T(e,q)$ and then $xy^{-1} \in H$. Therefore, Proposition 2.6 will give us that H is a subgroup of G. □

Proposition 3.6. Let T is a Q-soft translation of a Q-soft subgroup $f_{G\times Q}$ of a group G and $T(xy^{-1},q) = M^*$ such that for all $\alpha \in P(U)$ we have that $\alpha \subset M^*$. Then $T(x,q) = T(y,q)$ for all $x, y \in G$ and $q \in Q$.

PROOF. Let $x, y \in G$ and $q \in Q$. Now

$$\begin{aligned} T(x,q) = T(xy^{-1}y,q) \supseteq T(xy^{-1},q) \cap T(y,q) &= M^* \cap T(y,q) \\ &= T(y,q) \\ &= T(y^{-1},q) \\ &= T(x^{-1}xy^{-1},q) \\ &\supseteq T(x^{-1},q) \cap T(xy^{-1},q) \\ &= T(x,q) \cap T(xy^{-1},q) \\ &= T(x,q) \cap M^* \\ &= T(x,q) \end{aligned}$$

Therefore, $T(x,q) = T(y,q)$. □

Proposition 3.7. Let $\varphi : G \to H$ be a group epihomomorphism and $f_{G\times Q} \in S_{G\times Q}(U)$. If $T_\alpha^{f_{G\times Q}}$ be a Q-soft translation of $f_{G\times Q}$, then $\varphi(T_\alpha^{f_{G\times Q}}) \in S_{H\times Q}(U)$

PROOF. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$\begin{aligned} \varphi(T_\alpha^{f_{G\times Q}})(h_1h_2^{-1},q) &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_1g_2^{-1}) = h_1h_2^{-1}\} \\ &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_1)\varphi(g_2^{-1}) = h_1h_2^{-1}\} \\ &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2^{-1}) = h_2^{-1}\} \\ &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_1) = h_1, \varphi^{-1}(g_2) = h_2^{-1}\} \\ &= \cup\{f_{G\times Q}(g_1g_2^{-1},q) \cup \alpha \mid g_1,g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\ &\supseteq \cup\{[f_{G\times Q}(g_1,q) \cap f_{G\times Q}(g_2,q)] \cup \alpha \mid g_1,g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\ &= \cup\{[f_{G\times Q}(g_1,q) \cup \alpha] \cap [f_{G\times Q}(g_2,q) \cup \alpha] \mid g_1,g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\ &= [\cup\{f_{G\times Q}(g_1,q) \cup \alpha \mid g_1 \in G, \varphi(g_1) = h_1,\}] \\ &\quad \cap[\cup\{f_{G\times Q}(g_2,q) \cup \alpha \mid g_2 \in G, \varphi(g_2) = h_2,\}] \\ &= [\cup\{T_\alpha^{f_{G\times Q}}(g_1,q) \mid g_1 \in G, \varphi(g_1) = h_1,\}] \\ &\quad \cap[\cup\{T_\alpha^{f_{G\times Q}}(g_2,q) \mid g_2 \in G, \varphi(g_2) = h_2,\}] \\ &= \varphi(T_\alpha^{f_{G\times Q}})(h_1,q) \cap \varphi(T_\alpha^{f_{G\times Q}})(h_2,q) \end{aligned}$$

Then, by Proposition 2.9 we get that $\varphi(T_\alpha^{f_{G\times Q}}) \in S_{H\times Q}(U)$. □

Proposition 3.8. Let $\varphi : G \to H$ be a group homomorphism and $f_{H\times Q} \in S_{H\times Q}(U)$. If $T_\alpha^{f_{H\times Q}}$ be a Q-soft translation of $f_{H\times Q}$, then $\varphi^{-1}(T_\alpha^{f_{G\times Q}}) \in S_{G\times Q}(U)$.

PROOF. Let $g_1, g_2 \in G$ and $q \in Q$. Then,

$$\begin{aligned} \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1g_2^{-1},q) &= T_\alpha^{f_{G\times Q}}(\varphi(g_1g_2^{-1}),q) \\ &= T_\alpha^{f_{G\times Q}}(\varphi(g_1)\varphi(g_2^{-1}),q) \\ &= T_\alpha^{f_{G\times Q}}(\varphi(g_1)\varphi^{-1}(g_2),q) \\ &= f_{G\times Q}(\varphi(g_1)\varphi^{-1}(g_2),q) \cup \alpha \\ &\supseteq [f_{G\times Q}(\varphi(g_1),q) \cap f_{G\times Q}(\varphi(g_2),q)] \cup \alpha \\ &= [f_{G\times Q}(\varphi(g_1),q) \cup \alpha] \cap [f_{G\times Q}(\varphi(g_2),q) \cup \alpha] \\ &= T_\alpha^{f_{G\times Q}}(\varphi(g_1),q) \cap T_\alpha^{f_{G\times Q}}(\varphi(g_2),q) \\ &= \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1,q) \cap \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_2,q) \end{aligned}$$

Thus, $\varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1g_2^{-1},q) \supseteq \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1,q) \cap \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_2,q)$. Now Proposition 2.9 gets us that $\varphi^{-1}(T_\alpha^{f_{G\times Q}}) \in S_{G\times Q}(U)$ □

Proposition 3.9. Let $\varphi : G \to H$ be a group anti- epihomomorphism and $f_{G\times Q} \in S_{G\times Q}(U)$. If $T_\alpha^{f_{G\times Q}}$ be a Q-soft translation of $f_{G\times Q}$, then $\varphi(T_\alpha^{f_{G\times Q}}) \in S_{H\times Q}(U)$.

Proof. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$\begin{aligned}
\varphi(T_\alpha^{f_{G\times Q}})(h_1h_2^{-1},q) &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_1g_2^{-1}) = h_1h_2^{-1}\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_2^{-1})\varphi(g_1) = h_1h_2^{-1}\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi(g_2^{-1}) = h_2^{-1}, \varphi(g_1) = h_1\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2^{-1},q) \mid g_1,g_2 \in G, \varphi^{-1}(g_2) = h_2^{-1}, \varphi(g_1) = h_1\} \\
&= \cup\{f_{G\times Q}(g_1g_2^{-1},q) \cup \alpha \mid g_1,g_2 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\} \\
&\supseteq \cup\{[f_{G\times Q}(g_1,q) \cap f_{G\times Q}(g_2,q)] \cup \alpha \mid g_1,g_2 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\} \\
&= \cup\{[f_{G\times Q}(g_1,q) \cup \alpha] \cap [f_{G\times Q}(g_2,q) \cup \alpha] \mid g_1,g_2 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\} \\
&= [\cup\{f_{G\times Q}(g_1,q) \cup \alpha \mid g_1 \in G, \varphi(g_1) = h_1,\}] \\
&\quad \cap [\cup\{f_{G\times Q}(g_2,q) \cup \alpha \mid g_2 \in G, \varphi(g_2) = h_2,\}] \\
&= [\cup\{T_\alpha^{f_{G\times Q}}(g_1,q) \mid g_1 \in G, \varphi(g_1) = h_1,\}] \\
&\quad \cap [\cup\{T_\alpha^{f_{G\times Q}}(g_2,q) \mid g_2 \in G, \varphi(g_2) = h_2,\}] \\
&= \varphi(T_\alpha^{f_{G\times Q}})(h_1,q) \cap \varphi(T_\alpha^{f_{G\times Q}})(h_2,q)
\end{aligned}$$

Then, by Proposition 2.9 we get that $\varphi(T_\alpha^{f_{G\times Q}}) \in S_{H\times Q}(U)$. □

Proposition 3.10. Let $\varphi : G \to H$ be a group anti-homomorphism and $f_{H\times Q} \in S_{H\times Q}(U)$. If $T_\alpha^{f_{H\times Q}}$ be a Q-soft translation of $f_{H\times Q}$, then $\varphi^{-1}(T_\alpha^{f_{G\times Q}}) \in S_{G\times Q}(U)$.

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then,

$$\begin{aligned}
\varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1g_2^{-1},q) &= T_\alpha^{f_{G\times Q}}(\varphi(g_1g_2^{-1}),q) \\
&= T_\alpha^{f_{G\times Q}}(\varphi(g_2^{-1})\varphi(g_1),q) \\
&= T_\alpha^{f_{G\times Q}}(\varphi^{-1}(g_2)\varphi(g_1),q) \\
&= f_{G\times Q}(\varphi^{-1}(g_2)\varphi(g_1),q) \cup \alpha \\
&\supseteq [f_{G\times Q}(\varphi(g_1),q) \cap f_{G\times Q}(\varphi(g_2),q)] \cup \alpha \\
&= [f_{G\times Q}(\varphi(g_1),q) \cup \alpha] \cap [f_{G\times Q}(\varphi(g_2),q) \cup \alpha] \\
&= T_\alpha^{f_{G\times Q}}(\varphi(g_1),q) \cap T_\alpha^{f_{G\times Q}}(\varphi(g_2),q) \\
&= \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1,q) \cap \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_2,q)
\end{aligned}$$

Thus, $\varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1g_2^{-1},q) \supseteq \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_1,q) \cap \varphi^{-1}(T_\alpha^{f_{G\times Q}})(g_2,q)$ and so by Proposition 2.9 we have $\varphi^{-1}(T_\alpha^{f_{G\times Q}}) \in S_{G\times Q}(U)$. □

Proposition 3.11. Let $\varphi : G \to H$ be a group epihomomorphism and $f_{G\times Q} \in NS_{G\times Q}(U)$. If $T_\alpha^{f_{G\times Q}}$ be a Q-soft translation of $f_{G\times Q}$, then $\varphi(T_\alpha^{f_{G\times Q}}) \in NS_{H\times Q}(U)$

Proof. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$\begin{aligned}
\varphi(T_\alpha^{f_{G\times Q}})(h_1h_2, q) &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1g_2) = h_1h_2\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1)\varphi(g_2) = h_1h_2\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\
&= \cup\{f_{G\times Q}(g_1g_2, q) \cup \alpha \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\
&= \cup\{f_{G\times Q}(g_2g_1, q) \cup \alpha \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\
&= \varphi(T_\alpha^{f_{G\times Q}})(h_2h_1, q)
\end{aligned}$$

Then, $\varphi(T_\alpha^{f_{G\times Q}}) \in NS_{H\times Q}(U)$. □

Proposition 3.12. Let $\varphi : G \to H$ be a group homomorphism and $f_{H\times Q} \in NS_{H\times Q}(U)$. If $T_\alpha^{f_{H\times Q}}$ be a Q-soft translation of $f_{H\times Q}$, then $\varphi^{-1}(T_\alpha^{f_{H\times Q}}) \in NS_{G\times Q}(U)$.

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then,

$$\begin{aligned}
\varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_1g_2, q) &= T_\alpha^{f_{H\times Q}}(\varphi(g_1g_2), q) \\
&= T_\alpha^{f_{H\times Q}}(\varphi(g_1)\varphi(g_2), q) \\
&= f_{H\times Q}(\varphi(g_1)\varphi(g_2), q) \cup \alpha \\
&= f_{H\times Q}(\varphi(g_2)\varphi(g_1), q) \cup \alpha \\
&= T_\alpha^{f_{H\times Q}}(\varphi(g_2)\varphi(g_1), q) \\
&= T_\alpha^{f_{H\times Q}}(\varphi(g_2g_1), q) \\
&= \varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_2g_1, q)
\end{aligned}$$

Thus, $\varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_1g_2, q) = \varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_2g_1, q)$ and so $\varphi^{-1}(T_\alpha^{f_{H\times Q}}) \in NS_{G\times Q}(U)$. □

Proposition 3.13. Let $\varphi : G \to H$ be a group anti epihomomorphism and $f_{G\times Q} \in NS_{G\times Q}(U)$. If $T_\alpha^{f_{G\times Q}}$ be a Q-soft translation of $f_{G\times Q}$, then $\varphi(T_\alpha^{f_{G\times Q}}) \in NS_{H\times Q}(U)$

Proof. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$\begin{aligned}
\varphi(T_\alpha^{f_{G\times Q}})(h_1h_2, q) &= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1g_2) = h_1h_2\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_2)\varphi(g_1) = h_1h_2\} \\
&= \cup\{T_\alpha^{f_{G\times Q}}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\} \\
&= \cup\{f_{G\times Q}(g_1g_2, q) \cup \alpha \mid g_1, g_2 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\} \\
&= \cup\{f_{G\times Q}(g_2g_1, q) \cup \alpha \mid g_1, g_2 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\} \\
&= \varphi(T_\alpha^{f_{G\times Q}})(h_2h_1, q)
\end{aligned}$$

Then, $\varphi(T_\alpha^{f_{G\times Q}}) \in NS_{H\times Q}(U)$. □

Proposition 3.14. Let $\varphi : G \to H$ be a group anti-homomorphism and $f_{H\times Q} \in NS_{H\times Q}(U)$. If $T_\alpha^{f_{H\times Q}}$ be a Q-soft translation of $f_{H\times Q}$, then $\varphi^{-1}(T_\alpha^{f_{H\times Q}}) \in NS_{G\times Q}(U)$.

PROOF. Let $g_1, g_2 \in G$ and $q \in Q$. Then,

$$\begin{aligned}\varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_1g_2,q) &= T_\alpha^{f_{H\times Q}}(\varphi(g_1g_2),q)\\ &= T_\alpha^{f_{H\times Q}}(\varphi(g_2)\varphi(g_1),q)\\ &= f_{H\times Q}(\varphi(g_2)\varphi(g_1),q)\cup\alpha\\ &= f_{H\times Q}(\varphi(g_1)\varphi(g_2),q)\cup\alpha\\ &= T_\alpha^{f_{H\times Q}}(\varphi(g_1)\varphi(g_2),q)\\ &= T_\alpha^{f_{H\times Q}}(\varphi(g_2g_1),q)\\ &= \varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_2g_1,q)\end{aligned}$$

Thus, $\varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_1g_2,q) = \varphi^{-1}(T_\alpha^{f_{H\times Q}})(g_2g_1,q)$ and so $\varphi^{-1}(T_\alpha^{f_{H\times Q}}) \in NS_{G\times Q}(U)$. □

Conclusion

In this paper, we defined the concept Q-soft translations of Q-soft subgroups and investigated the properties of them and showed that every Q-soft translation of Q-soft subgroup is also Q-soft subgroup. Also, we considered them under homomorphism and anti-homomorphism of Q-soft subgroups and Q-soft normal subgroups. Now one can define the isomorphisms of them and it is can be as open problem.

References

[1] D. A. Molodtsov, *Soft Set Theory-First Results*, Computers and Mathematics With Applications 37(4) (1999) 19–31.

[2] P. K. Maji, R. Biswas, A. R. Roy, *Soft Set Theory*, Computers and Mathematics with Applications 45(4-5) (2003) 555–562.

[3] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, *On Some New Operations in Soft Set Theory*, Computers and Mathematics with Applications 57(9) (2008) 2621–2628.

[4] N. Çağman, S. Enginoğlu, *Soft Matrix Theory and Its Decision Making*, Computers and Mathematics with Applications 59 (2010) 3308–3314.

[5] N. Çağman, *Contributions to The Theory of Soft Sets*, Journal of New Results in Science 4 (2014) 33–41.

[6] A. Sezgin, A. O. Atagün, *On Operations of Soft Sets*, Comp. Math. Appl., 61 (2011) 1457–1467.

[7] N. Çağman, S. Karataş, *Intuitionistic Fuzzy Soft Set Theory and Its Decision Making*, Journal of Intelligent and Fuzzy Systems 24(4) (2013) 829–836.

[8] N. Çağman, S. Enginoğlu, *Soft Set Theory and uni-int Decision Making*, European Journal of Operational Research 207 (2010) 848–855.

[9] F. Feng, Y. Li and N. Çağman, *Generalized uni-int Decision Making Schemes Based on Choice Value Soft Sets*, European Journal of Operational Research 220 (2012) 162–170.

[10] H. Aktaş, N. Çağman, *Soft Sets and Soft Groups*, Information Sciences 177(13) (2007) 2726–2735.

[11] A. O. Atagün, A. Sezgin, *Soft Substructures of Rings, Fields and Modules*, Computers and Mathematics with Applications 61(3) (2011) 592–601.

[12] N. Çağman, F. Çıtak, H. Aktaş, *Soft int-group and Its Applications to Group Theory*, Neural Computing and Applications 21 (2012) 151–158.

[13] F. Feng, Y. B. Jun, X. Zhao, *Soft Semirings*, Computers and Mathematics with Applications 56 (2008) 2621–2628.

[14] R. Rasuli, *Soft Lie Ideals and Anti Soft Lie Ideals*, The Journal of Fuzzy Mathematics Los Angeles 26(1) (2018) 193–202.

[15] R. Rasuli, *Extension of Q-soft Ideals in Semigroups*, International Journal of Open Problems in Computer Science and Mathematics 10(2) (2017) 6–13.

[16] F. Hassani, R. Rasuli, *Q-soft Subgroups and Anti-Q-soft Subgroups in Universal Algebra*, The Journal of Fuzzy Mathematics Los Angeles 26(1) (2018) 139–152.

[17] R. Rasuli, *Q-Soft Normal Subgroups*, Journal of New Theory (26) (2019) 13–22.

[18] T. Hungerford, *Algebra*, Graduate Texts in Mathematics. Springer (2003).

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Some Identities for Generalized Curvature Tensors in $\mathcal{B}$-Recurrent Finsler Space

Adel Mohammed Ali Al-Qashbari[1]

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Abstract − The purpose of the present paper is to consider and study a certain identities for some generalized curvature tensors in $\mathcal{B}$-recurrent Finsler space F_n in which Cartan's second curvature tensor P^i_{jkh} satisfies the generalized of recurrence condition with respect to Berwald's connection parameters G^i_{kh} which given by the condition $\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, where $\mathcal{B}_m$ is covariant derivative of first order (Berwald's covariant differential operator) with respect to x^m, it's called a generalized $\mathcal{B}$P-recurrent space. We shall denote it briefly by $G\mathcal{B}P\text{-}RF_n$. We have obtained Berwald's covariant derivative of first order for the h(v)-torsion tensor P^i_{kh}, the deviation tensor P^i_h and the covariant derivative of the tensor $H_{kp.h}$ (in the sense of Berwald), also we find some theorems of the R-Ricci tensor R_{jk} and the curvature vector $R_j$in our space. We obtained the necessary and sufficient condition for Berwald's covariant derivative of Weyl's projective curvature tensor W^i_{jkh} and its torsion tensor W^i_{kh} in our space. Also, we have proved that in $G\mathcal{B}P\text{-}RF_n$, Cartan's second curvature tensor P^i_{jkh} and the v(hv)-torsion tensor P^i_{kh} for $n = 4$.

Keywords − *Finsler space, Cartan's second curvature tensor P^i_{jkh}, Generalized $\mathcal{B}P$-recurrent space, Weyl's projective curvature tensor W^i_{jkh} , Cartan's fourth curvature tensor R^i_{jkh}*

1. Introduction

The generalized recurrent space characterized by different curvature tensors and used the sense of Berwald studied by Pandey et al. [1], and Ahsan and Ali [2], studied the properties of W-curvature tensor and its applications. The concept of the recurrent for different curvature tensors have been discussed by Qasem [3] and Matsumoto [4], they studied the generalized birecurrent of first and second kind, also studied the special birecurrent of first and second kind and W^i_{jkh} generalized birecurrent Finsler space studied by Qasem and Saleem [5]. The generalized birecurrent space was studied by Hadi [6], Qasem and Abdallah [7], Qasem and Saleem [8], Abdallah [9], Qasem and Abdallah [10-12], Qasem and Baleedi [13,14]. The generalized birecurrent Finsler space studied by Qasem [15]. F.Y.A. Qasem et al. [16] studied of GR^h-TRI affinely.

[1]Adel_ma71@yahoo.com (Corresponding Author)
[1]Department of Mathematic, Faculty of Educ. -Aden, University of Aden, Aden, Yemen

Consider an n-dimensional Finsler space, Fig. 1., equipped with the metric function F satisfies the requisite conditions [16]. Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^{i}_{jk} *1 .
These are symmetric in their lower indices.

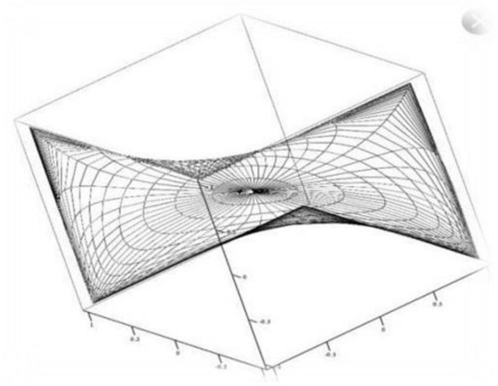

Fig.1. The figure for Finsler Space as a Locally Minkowskian Space

The vectors y_i and y^i satisfy the following relations [16]

$$\text{a) } y_i = g_{ij}\, y^j \text{ and b) } y_i\, y^i = F^2 \#(1.1)$$

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by [16]

$$g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \;, & if \quad i = k \\ 0 \;, & if \quad i \neq k \end{cases} \#(1.2)$$

The tensor C_{ijk} defined by

$$C_{ijk} = \frac{1}{2}\, \dot{\partial}_i\, g_{jk} = \frac{1}{4}\, \dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 \#(1.3)$$

is known as (h) hv - torsion tensor [16].

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by

$$\text{a) } C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0, \qquad \text{b) } y_i\, C^i_{jk} = 0, \qquad \text{c) } C_{ijk}\, y^j = 0$$
$$\text{d) } g_{rj}\, C^r_{ik} = C_{ijk}\,, \qquad \text{e) } C^i_{jk}\, g^{jk} = C^i, \qquad \text{f) } C_{ijk}\, g^{jk} = C_i \qquad (1.4)$$

Berwald's covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor filed T^i_j with respect to x^k is given by

$$\mathcal{B}_k\, T^i_j := \partial_k\, T^i_j - \left(\dot{\partial}_r\, T^i_j\right) G^r_k + T^r_j\, G^i_{rk} - T^i_r\, G^r_{jk} \qquad (1.5)$$

Berwald's covariant derivative of the metric function, the vector y^i and the unit vector l^i vanish identically [16], i.e.

$$\text{a) } \mathcal{B}_k\, y^i = 0, \text{b) } \mathcal{B}_k\, F = 0 \text{ and c) } \mathcal{B}_k\, y_i = 0 \qquad (1.6)$$

But Berwald's covariant derivative of the metric tensor g_{ij} doesn't vanish, i.e. $\mathcal{B}_k\, g_{ij} \neq 0$ and given by

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h\, \mathcal{B}_h\, C_{ijk} \qquad (1.7)$$

Berwald's covariant differential operator with respect to x^h commutes with partial differential operator with respect to y^k, according to [16]

$$\left(\dot{\partial}_k\, \mathcal{B}_h - \mathcal{B}_h\, \dot{\partial}_k\right) T^i_j = T^r_j G^i_{khr} - T^i_r\, G^r_{khj} \qquad (1.8)$$

where T^i_j is any arbitrary tensor field.

*1 The indices $i, j, k, \ldots$ assume positive integral values from 1 to n.

The hv-curvature tensor P^i_{jkh} and the v(hv)–torsion tensor P^i_{kh} satisfy [16]

$$\text{a) } P^i_{jkh}\, y^j = P^i_{kh},\ \text{b) } g_{ir}\, P^r_{jkh} = P_{ijkh}$$
$$\text{c) } g_{rp}\, P^r_{kh} = P_{kph},\ \text{d) } P^i_{jki} = P_{jk} \tag{1.9}$$
$$\text{e) } P^i_{ki} = P_k,\ \text{f) } P^i_{kh}\, y^k = P^i_{kh}\, y^h = 0$$

also the hv-curvature tensor P^i_{jkh} is defined by

$$\text{a) } P^i_{jkh} = \Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \tag{1.10}$$

or equivalent by

$$\text{b) } P^i_{jkh} = \dot{\partial}_h\, \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k}$$

or

$$\text{c) } P^i_{jkh} = C^i_{kh|j} - C_{jkh|r}\, g^{ir} + C^r_{jk}\, P^i_{rh} - P^r_{jh}\, C^i_{rk}$$

where

$$\text{a) } \Gamma^{*i}_{jkh}\, y^j = P^i_{kh}, \qquad \text{b) } \Gamma^{*i}_{jkh}\, y^k = 0 \text{ and} \qquad \text{c) } y_i\, \Gamma^{*i}_{kjh} = -\, P_{kjh} \tag{1.11}$$

The projective curvature tensor W^i_{jkh} is known as (Wely's projective curvature tensor), the projective torsion tensor W^i_{jk} is known as (Wely's torsion tensor) and the projective deviation tensor W^i_j is known as (Wely's deviation tensor) are defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{n+1} H_{[hk]} + \frac{2\,y^i}{n+1} \dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{n^2-1}\left(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right) \tag{1.12}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2\,\{ \frac{\delta^i_{[j}}{n^2-1}\left(n\, H_{k]} - y^r H_{k]\,r}\right) \tag{1.13}$$

and

$$W^i_j = H^i_j - H\, \delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i \tag{1.14}$$

respectively.

The tensors W^i_{jkh}, W^i_{jk} and W^i_k are satisfying the following identities [16]

$$\text{a) } W^i_{jkh}\, y^j = W^i_{kh} \text{ and b) } W^i_{jk}\, y^j = W^i_k \tag{1.15}$$

The projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h.

Cartan's third curvature tensor R^i_{jkh} , Fig.2., and the R-Ricci tensor R_{jk} in sense of Cartan, respectively, given by [16]

$$\text{a) } R^i_{jkh} = \Gamma^{*i}_{hjk} + \left(\Gamma^{*i}_{ljk}\right) G^l_h + C^i_{jm}\left(G^m_{kh} - G^m_{kl}\, G^l_h\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h$$
$$\text{b) } R^i_{jkh}\, y^j = H^i_{kh},\ \text{c) } R_{jk}\, y^j = H_k \tag{1.16}$$
$$\text{d) } R_{jk}\, y^k = R_j,\ \text{e) } R^i_{jki} = R_{jk}$$

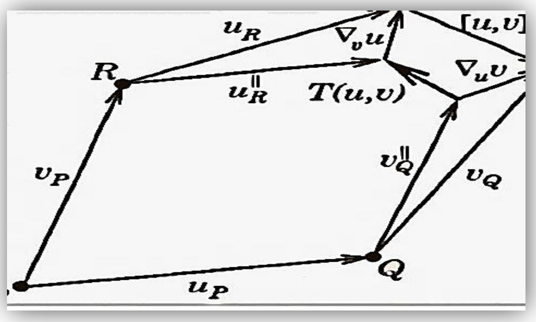

Fig.2. The figure for Covariant Derivative for Cartan's Torsion in Geometrical

Berwald curvature tensor H^i_{jkh} and h(v)–torsion tensor H^i_{kh} form the components of tensors are defined as follow [16]

$$\text{a) } H^i_{jkh} \coloneqq \partial_j G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj} G^r_k - \frac{h}{k} \quad (1.17)$$

and

$$\text{b) } H^i_{kh} \coloneqq \partial_h G^i_k + G^r_k C^i_{rh} - h/k$$

They are also related by [16]

$$\text{a) } H^i_{jkh} y^j = H^i_{kh}, \text{ b) } H^i_{jkh} = \dot{\partial}_j H^i_{kh} \text{ and c) } H^i_{jk} = \dot{\partial}_j H^i_k \quad (1.18)$$

These tensors were constructed initially by means of the tensor H^i_h , called the deviation tensor, given by

$$\text{a) } H^i_h \coloneqq 2\, \partial_h G^i - \partial_r G^i_h y^r + 2\, G^i_{hs} G^s - G^i_s G^s_h \quad (1.19)$$

where

$$\text{b) } \dot{\partial}_k G^i_h = G^i_{kh}$$

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.18) and (1.19), we have the following:

$$\text{a) } H^i_{jk} y^j = H^i_k \text{ , b) } g_{ip} H^i_{jk} = H_{jp.k} \text{ and c) } H_i y^i = (n-1)H \quad (1.20)$$

2. A Generalized $\mathcal{BR}$-Recurrent Space

Cartan's second curvature tensor P^i_{jkh} satisfies the condition

$$\mathcal{B}_n P^i_{jkh} = \lambda_n P^i_{jkh} \text{ , } P^i_{jkh} \neq 0 \quad (2.1)$$

is called a recurrent Finsler space, where λ_n is non-zero covariant vectors field.
A Finsler space F_n whose the curvature tensor P^i_{jkh} satisfies the condition

$$\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}), P^i_{jkh} \neq 0 \quad (2.2)$$

where $\mathcal{B}_m$ is covariant derivative of first order (Berwald's covariant differential operator) with respect to x^m, the quantities λ_m and μ_m are non-null covariant vectors field. It is called such space as *a generalized $\mathcal{B}P$-recurrent space*, he denoted it briefly by $G\mathcal{B}P\text{-}RF_n$.

Definition 2.1. A Finsler space F_n whose Cartan's second curvature tensor P^i_{jkh} satisfies the condition (2.2), where λ_m and μ_m are non-null covariant vectors field, it's called *a generalized $\mathcal{B}P$-recurrent space*. We shall denote it briefly by $G\mathcal{B}P\text{-}RF_n$.

Transvecting the condition (2.2) by y^j, using (1.6a), (1.9a), (1.1a) and (1.4c), we get

$$\mathcal{B}_m P^i_{kh} = \lambda_m P^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) \quad (2.3)$$

Contracting the indices i and h in (2.2) and (2.3), using (1.9d), (1.9e) and in view of (1.2), we get

$$\mathcal{B}_m P_{jk} = \lambda_m P_{jk} + \mu_m (n-1)\, g_{jk} \qquad (2.4)$$

$$\mathcal{B}_m P_k = \lambda_m P_k + \mu_m (n-1)\, y_k \qquad (2.5)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 2.1. In G$\mathcal{B}$P-RF$_n$, v(hv)-torsion tensor P^i_{kh} , P-Ricci tensor P_{jk} and the curvature vector P_k (for Cartan's second curvature tensor P^i_{jkh}) are given by (2.3),(2.4) and (2.5), respectively.

Trausvecting (2.2) and (2.3) by g_{ir} , using (1.9b), (1.9c), (1.7) and in view of (1.2), we get

$$\mathcal{B}_m P_{jkrh} = \lambda_m P_{jkrh} + \mu_m \left(g_{rh}\, g_{jk} - g_{rk}\, g_{jh} \right) - 2\, y^n \mathcal{B}_n C_{irm}\; P^i_{jkh} \qquad (2.6)$$

$$\mathcal{B}_m P_{krh} = \lambda_m P_{krh} + \mu_m (g_{hr}\, y_k - g_{kr}\, y_h) - 2\, y^n \mathcal{B}_n C_{irm}\; P^i_{kh} \qquad (2.7)$$

Therefore, we have

Theorem 2.2. In G$\mathcal{B}$P-RF$_n$, the associate curvature tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} and the associate tensor P_{jkh} of v(hv)-torsion tensor P^i_{kh} (for Cartan's second curvature tensor P^i_{jkh}) is given by the equations (2.6) and (2.7), respectively.

Taking covariant derivative (Berwald's covariant differential operator) of the equation (1.10a) with respect to x^m and using condition (2.2), yields

$$\lambda_m P^i_{jkh} + \mu_m (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) = \mathcal{B}_m \left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \right)$$

By using the equation (1.10a), the above equation can be written as

$$\mathcal{B}_m \left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k}\right) = \lambda_m \left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k}\right) + \mu_m (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \qquad (2.8)$$

Equation (2.8) shows that the tensor $\left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \right)$ can't vanish, because the vanishing of it would implies the vanishing of the covariant vector field μ_m , i.e. $\mu_m = 0$, a contradiction.

Thus, it is concluded the following.

Theorem 2.3. In G$\mathcal{B}$P-RF$_n$, the tensor $\left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \right)$ is non-vanishing and this tensor is generalized recurrent.

Transvecting equation (2.8) by y^j, using equations (1.6a), (1.9f), (1.1a) and (1.4c), we get the same equation (2.3).

Further, transvecting (2.8) by y_i , using equations (1.6c), (1.9g), (1.1a) and (1.4c), we get the same equation (2.6).

Now, transvecting equation (2.8) by y^k, using equations (1.6a), (1.11b), (1.9f), (1.1a), (1.4c) and in view of (1.2), we get

$$\mathcal{B}_m \left(C^i_{jh|k}\, y^k \right) = \lambda_m \left(C^i_{jh|k}\, y^k \right) + \mu_m (\delta^i_h\, y_j - g_{jh}\, y^i) \qquad (2.9)$$

Transvecting (2.9) by g_{ir} , using (1.4d), (1.1a), (1.7) and in view of (1.2), we get

$$\mathcal{B}_m \left(C_{jrh|k}\, y^k\right) = \lambda_m \left(C_{jrh|k}\, y^k\right) + \mu_m (g_{rh}\, y_j - g_{jh}\, y_r) - 2\, y^n \mathcal{B}_n C_{irm} \left(C^i_{jh|k} y^k\right) \qquad (2.10)$$

Therefore, it is concluded the following.

Theorem 2.4. In G$\mathcal{B}$P-RF$_n$, we have the identities (2.9) and (2.10).

Trausvecting (2.9) and (2.10) by g^{jh}, using (1.4e), (1.4f), (1.1a) and in view of (1.2), we get

$$\mathcal{B}_m \left(C^i_{|k}\, y^k\right) = \lambda_m \left(C^i_{|k}\, y^k\right) \#(2.11)$$

$$\mathcal{B}_m \left(C_{r|k}\, y^k\right) = \lambda_m \left(C_{r|k}\, y^k\right) - 2\, y^n \mathcal{B}_n C_{irm} \left(C^i_{|k}\, y^k\right), \text{where } \mathcal{B}_m\, g^{jh} = 0 \qquad (2.12)$$

Therefore, we have

Theorem 2.5. In G$\mathcal{B}$P-RF$_n$, the tensor $\left(C^i_{|k}\, y^k\right)$ is recurrent and the tensor $\left(C_{r|k}\, y^k\right)$ is given by the equation (2.11).

3. The Certain Identities for Curvature Tensor P^i_{jkh}

In this section we shall obtain certain identities for some tensors to be generalized recurrent in our space of $G\mathcal{B}R\text{-}TRF_n$.
For a Riemannian space V$_4$, the projective curvature tensor P^i_{jkh} (Cartan's second curvature tensor) and the divergence of W-tensor in terms of the divergence of projective curvature tensor can be expressed as [9]

$$W^i_{jkh} = P^i_{jkh} + \frac{1}{3}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) \tag{3.1}$$

Taking covariant derivative of first order (Berwald's covariant differential operator) of (3.1) with respect to x^m, we get

$$\mathcal{B}_m W^i_{jkh} = \mathcal{B}_m P^i_{jkh} + \frac{1}{3}\mathcal{B}_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) \tag{3.2}$$

Using the condition (2.2) in (3.2), we get

$$\mathcal{B}_m W^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{3}\mathcal{B}_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right)$$

In view of equation (3.1) and by using (1.7), the above equation can be written as

$$\begin{aligned}\mathcal{B}_m W^i_{jkh} = {} & \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - \frac{1}{3}\lambda_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) \\ & + \frac{1}{3}\delta^i_k \mathcal{B}_m R_{jh} - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) g_{jk} + \frac{2}{3} R^i_h\, y^n \mathcal{B}_n C_{jkm}\end{aligned} \tag{3.3}$$

This, shows that

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$

if and only if

$$\delta^i_k \mathcal{B}_m R_{jh} - \lambda_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \left(\mathcal{B}_m R^i_h\right) g_{jk} + 2 R^i_h\, y^n \mathcal{B}_n C_{jkm} = 0 \tag{3.4}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.1. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective curvature tensor W^i_{jkh} is generalized recurrent if and only if (3.4) holds.

Transvecting (3.3) by y^j , using (1.6a), (1.15a), (1.1a), (1.16c) and (1.4c), yields

$$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) - \frac{1}{3}\lambda_m\left(\delta^i_k H_h - R^i_h H_k\right) + \frac{1}{3}\delta^i_k \mathcal{B}_m H_h - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) y_k \tag{3.5}$$

This, shows that

$$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) \tag{3.6}$$

if and only if

$$\delta^i_k \mathcal{B}_m H_h - \lambda_m\left(\delta^i_k H_h - R^i_h H_k\right) - \left(\mathcal{B}_m R^i_h\right) y_k = 0 \tag{3.7}$$

Therefore, it is concluded the following theorem

Theorem 3.2. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective torsion tensor W^i_{kh} is given by the equation (3.6) if and only if (3.7) holds.

Transvecting (3.5) by y^k, using (1.6a), (1.15b), (1.1b), (1.2) and (1.20c), we get

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) - \frac{1}{3}\lambda_m(H_h y^i - (n-1) R^i_h H) + \frac{1}{3} y^i \mathcal{B}_m H_h - \frac{1}{3}(\mathcal{B}_m R^i_h) F^2$$

This, shows that

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) \tag{3.8}$$

if and only if

$$y^i \mathcal{B}_m H_h - \lambda_m(H_h y^i - (n-1) R^i_h H) - (\mathcal{B}_m R^i_h) F^2 = 0 \tag{3.9}$$

Thus, the following is derived.

Theorem 3.3. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective deviation tensor W^i_h is given by the equation (3.8) if and only if (3.9) holds.

Also, the projective curvature tensor P^i_{jkh} (for a Riemannian space V_4) is defined by [9]

$$P^i_{jkh} = R^i_{jkh} - \frac{1}{3}(\delta^i_h R_{jk} - \delta^i_k R_{jh}) \tag{3.10}$$

Taking covariant derivative of third order (Berwald's covariant differential operator) of (3.10) with respect to x^m, we get

$$\mathcal{B}_m P^i_{jkh} = \mathcal{B}_m R^i_{jkh} - \frac{1}{3}(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}) \tag{3.11}$$

Using the condition (2.2) in (3.11), we get

$$\mathcal{B}_m R^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{3}(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh})$$

By using (3.10), the above equation can be written as

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{3}(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}) - \frac{1}{3}\lambda_m(\delta^i_h R_{jk} - \delta^i_k R_{jh}) \tag{3.12}$$

This, shows that

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

if and only if

$$(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}) - \lambda_m(\delta^i_h R_{jk} - \delta^i_k R_{jh}) = 0 \tag{3.13}$$

Thus, it is concluded the following theorem

Theorem 3.4. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Cartan's third curvature tensor R^i_{jkh} is generalized recurrent if and only if (3.13) holds.

Transvecting (3.12) by y^j, using (1.6a), (1.16b), (1.1a) and (1.16c), yields

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{3}(\delta^i_h \mathcal{B}_m y_k - \delta^i_k \mathcal{B}_m y_h) - \frac{1}{3}\lambda_m(\delta^i_h y_k - \delta^i_k y_h)$$

This, shows that

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) \tag{3.14}$$

if and only if

$$(\delta^i_h \mathcal{B}_m y_k - \delta^i_k \mathcal{B}_m y_h) - \lambda_m(\delta^i_h y_k - \delta^i_k y_h) = 0 \tag{3.15}$$

Further, transvecting (3.13) by y^k, using (1.6a), (1.20a), (1.1a), (1.1b) and (1.2), we get

$$\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) + \frac{1}{3}(\delta^i_h \mathcal{B}_m F^2 - y^i \mathcal{B}_m y_h) - \frac{1}{3}\lambda_m(\delta^i_h F^2 - y_h y^i) \tag{3.16}$$

This, shows that

$$\mathcal{B}_m H_h^i = \lambda_m H_h^i + \mu_m(\, \delta_h^i \, F^2 - \, y_h \, y^i \,) \tag{3.17}$$

if and only if

$$(\, \delta_h^i \, \mathcal{B}_m \, F^2 - y^i \, \mathcal{B}_m \, y_h \,) - \lambda_m(\, \delta_h^i \, F^2 - \, y_h \, y^i \,) = 0 \tag{3.18}$$

Transvecting (3.13) by g_{ip} , using (1.7), (1.6a), (1.20b) and (1.2), yields

$$\begin{aligned} \mathcal{B}_m H_{kp.h} \; = \; & \lambda_m \, H_{kp.h} \, + \mu_m(g_{hp} \, y_k - g_{kp} \, y_h) - 2 \, H_{kh}^i \; y^n \, \mathcal{B}_n \, C_{ipm} \\ & + \frac{1}{3}(\, g_{hp} \, \mathcal{B}_m y_k - g_{kp} \, \mathcal{B}_m \, y_h \,) - \frac{1}{3} \, \lambda_m(\, g_{hp} \, y_k - g_{kp} \, y_h \,) \end{aligned} \tag{3.19}$$

This, shows that

$$\mathcal{B}_m H_{kp.h} \; = \lambda_m \, H_{kp.h} \; + \mu_m(g_{hp} \, y_k - g_{kp} \, y_h) \tag{3.20}$$

if and only if

$$(\, g_{hp} \, \mathcal{B}_m y_k - g_{kp} \, \mathcal{B}_m \, y_h \,) - \lambda_m(\, g_{hp} \, y_k - g_{kp} \, y_h \,) - 6 \, H_{kh}^i \; y^n \, \mathcal{B}_n \, C_{ipm} = 0 \tag{3.21}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.5. In G$\mathcal{B}$P-RF$_{\text{n}}$ (for $n = 4$), Berwald's covariant derivative of the first order for the h(v)-torsion tensor H_{kh}^i , the deviation tensor H_h^i and the tensor $H_{kp.h}$ are given by the equations (3.14), (3.17) and (3.20), respectively, if and only if (3.15) (3.18) and (3.21)holds.

Contracting the indices i and h in (3.12), using (1.16e) and in view of (1.2), we get

$$\mathcal{B}_m \, R_{jk} = \lambda_m R_{jk} + (n \, - 1) \, \mu_m \, g_{jk} + \frac{1}{3}(\, n \, - 1 \,) \, \mathcal{B}_m R_{jk} - \frac{1}{3}(\, n - 1) \, \lambda_m \, R_{jk} \tag{3.22}$$

This, shows that

$$\mathcal{B}_m \, R_{jk} = \lambda_m R_{jk} + (n \, - 1) \, \mu_m \, g_{jk}$$

if and only if

$$\mathcal{B}_m R_{jk} = \; \lambda_m \, R_{jk} \tag{3.23}$$

Therefore, it is concluded the following.

Theorem 3.6. In G$\mathcal{B}$P-RF$_{\text{n}}$ (for $n = 4$), Berwald's covariant derivative of the first order for the R-Ricci tensor R_{jk} is non- vanishing if and only if R-Ricci tensor R_{jk} is recurrent.

Transvecting (3.22) by y^j, using (1.6a), (1.16c) and (1.1a), yields

$$\mathcal{B}_m \, H_k = \lambda_m H_k + (n \, - 1) \, \mu_m \, y_k + \frac{1}{3}(\, n \, - 1 \,) \, \mathcal{B}_m H_k - \frac{1}{3}(\, n - 1) \, \lambda_m \, H_k \tag{3.24}$$

This, shows that

$$\mathcal{B}_m \, H_k = \lambda_m H_k + (n \, - 1) \, \mu_m \, y_k \tag{3.25}$$

if and only if

$$\mathcal{B}_m H_k = \; H_k \tag{3.26}$$

Further, transvecting (3.22) by y^k , using (1.6a), (1.16d) and (1.1a), we get

$$\mathcal{B}_m \, R_j = \lambda_m R_j + (n \, - 1) \, \mu_m \, y_j + \frac{1}{3}(\, n \, - 1 \,) \, \mathcal{B}_m R_j - \frac{1}{3}(\, n - 1) \, \lambda_m \, R_j \tag{3.27}$$

This, shows that

$$\mathcal{B}_m \, R_j = \lambda_m R_j + (n \, - 1) \, \mu_m \, y_j \tag{3.28}$$

if and only if

$$\mathcal{B}_m R_j = \; \lambda_m \, R_j$$

Therefore, it is concluded the following.

Theorem 3.7. In G$\mathcal{B}$P-$\mathrm{RF_n}$ (for $n = 4$), Berwald's covariant derivative of the first order for the curvature vector H_k and the curvature vector R_j are non- vanishing if and only if the curvature vector H_k and the curvature vector R_j are recurrent.

4. Conclusion

A Finsler space is called generalized $\mathcal{B}$P-recurrent if it satisfies the condition (2.2).

In G$\mathcal{B}$P-$\mathrm{RF_n}$, Berwald's covariant derivative of the first order for v(hv)-torsion tensor P^i_{kh}, P-Ricci tensor P_{jk} and the curvature vector P_k (for Cartan's second curvature tensor P^i_{jkh}) are given by (2.3),(2.4) and (2.5), respectively.

In G$\mathcal{B}$P-$\mathrm{RF_n}$, the associate curvature tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} and the associate tensor P_{jkh} of v(hv)-torsion tensor P^i_{kh} (for Cartan's second curvature tensor P^i_{jkh}) is given by the equations (2.6) and (2.7), respectively also we have the identities (2.9) and (2.10).
In G$\mathcal{B}$P-$\mathrm{RF_n}$ (for $n = 4$), the necessary and sufficient condition of Weyl's projective curvature tensor W^i_{jkh} to be generalized recurrent are given by the equation (3.4).

In G$\mathcal{B}$R-$\mathrm{TRF_n}$ (for $n = 4$), the necessary and sufficient conditions of Berwald's covariant derivative of the first order for the torsion tensor W^i_{kh}, the deviation tensor W^i_h, the h(v)-torsion tensor H^i_{kh}, the deviation tensor H^i_h and the tensor $H_{kp.h}$ are given by equations (3.6), (3.8), (3.14), (3.17) and (3.20), respectively.
In G$\mathcal{B}$R-$\mathrm{TRF_n}$ (for $n = 4$), the necessary and sufficient conditions of Cartan's third curvature tensor R^i_{jkh} is generalized recurrent and given by equation (3.11).
Author recommend the need for continuing research and development in generalized $\mathcal{B}$P-recurrent spaces and interlard it with the properties of special spaces for Finsler space.

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References

[1] P. N. Pandey, S. Saxena, A. Goswani, *On A Generalized H-recurrent Space*, Journal of International Academy of Physical Sciences 15 (2011) 201-211.

[2] Z. Ahsan, M. Ali, *On Some Properties of W-curvature Tensor*, Palestine Journal of Mathematics, Palestine 3(1) (2014) 61-69.

[3] F. Y. A. Qasem, *On Transformation in Finsler Spaces*, D. Phil Thesis, University of Allahabad, (Allahabad) (India) (2000).

[4] M. Matsumoto, *On h-isotropic and C^h-recurrent Finsler*, Journal of Mathematics of Kyoto University 11 (1971), 1- 9.

[5] F. Y. A. Qasem, A. A. M. Saleem, *On W^i_{jkh} Generalized Birecurrent Finsler Space*, Journal of the Faculties of Education, University of Aden (Aden) (Yemen) (11) (2010) 21-32.

[6] W. H. A. Hadi, *Study of Certain Generalized Birecurrent in Finsler Space*, PhD Dissertation, University of Aden (2016) Aden, Yemen.
[7] F. Y. A. Qasem, A. A. A. Abdallah, *On Study Generalized $\mathcal{B}R$-recurrent Finsler Space*, International Journal of Mathematics and its Applications 4(2-B) (2016) 113-121.
[8] F. Y. A. Qasem, A. A. M. Saleem, *On Generalized $\mathcal{B}N$-recurrent Finsler Space*, Electronic Aden University Journal 7 (2017) 9-18.
[9] A. A. A. Abdallah, *On Generalized $\mathcal{B}R$-recurrent Finsler Space*, Master Thesis, University of Aden, (2017) Aden, Yemen.
[10] F. Y. A. Qasem, A. A. A. Abdallah, *On Certain Generalized $\mathcal{B}R$-recurrent Finsler Space*, International Journal of Applied Science and Mathematics, 3(3) (2016) 111-114.
[11] F. Y. A. Qasem, A. A. A. Abdallah, *On Generalized $\mathcal{B}R$-recurrent Finsler Space*, Electronic Aden University Journal (6) (2017) 27-33.
[12] F. Y. A. Qasem, S. M. S. Baleedi, *On A Generalized $\mathcal{B}K$-recurrent Finsler Space*, International Journal of Science: Basic and Applied Research 28(3) (2016) 195-203.
[13] S. M. S. Baleedi, *On Certain Generalized $\mathcal{B}K$-recurrent Finsler Space,* Master Thesis, University of Aden (2017) Aden, Yemen.
[14] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin Göttingen- Heidelberg, (1959), 2nd Edit. (in Russian), Nauka, Moscow (1981).
[15] F. Y. A. Qasem, *On Generalized H-birecurrent Finsler Space*, International Journal of Mathematics and its Applications 4(2-B) (2016) 51-57.
[16] F. Y. A. Qasem, A. M. A. Al-qashbari, M. M. Q. Husien, *On Study Generalized R^h-trirecurrent Affinely Connected Space*, Journal of Scientific and Engineering Research 6(11) (2019) 179-186.

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Generalized Fuzzy TOPSIS to Solve Multi-Criteria Decision-Making Problems

Rana Muhammad Zulqarnain[1], Muhammad Saeed[2], Bagh Ali[3], Sohaib Abdal[4], Muhammad Saqlain[5], Muhammad Irfan Ahamad[6], Zeeshan Zafar[7]

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Abstract − Multiple Criteria Decision Making (MCDM) is a process in which we choose the best alternative from all feasible alternatives. In this paper, we study fuzzy sets with some basic concepts and fuzzy TOPSIS (technique for order preference by similarity to ideal solution) method. We proposed the TOPSIS method under a fuzzy environment and expressed the rating of each alternative and weight of each criterion in the form of a triangular fuzzy number. Finally, we used the proposed method for decision making in the garments industry for the selection of supplier.

Keywords − *Fuzzy set, triangular fuzzy number (TFN), fuzzy TOPSIS, MCDM*

1. Introduction

Nowadays TOPSIS is most familiar with MCDM in different fields. Hwang and Yoon [1] proposed the TOPSIS method to solve MCDM problems and choose the best alternative with the shortest distance from a positive ideal solution and farthest distance from the negative ideal solution. Many researchers used the TOPSIS method for decision making, medical diagnoses, and other different areas of life reported in the literature [2]–[10].

Later, Chen [11] introduced the concept of the vertex method to measure the distance among two TFN and extended the TOPSIS method under a fuzzy environment. For calculating fuzzy positive ideal solution (FPIS) and fuzzy negative ideal solution (FNIS) and ranking of all alternatives, he presented the closeness coefficient, according to the concept of TOPSIS. But in [12], the authors challenged the Chen fuzzy TOPSIS method and claimed that Chen's method is not appropriate, he claimed that the weighted normalized fuzzy ratings are not TFNs. To overcome these limitations, they proposed a new improved fuzzy TOPSIS method in which the membership functions for the weighted normalized fuzzy ratings were presented. They also proposed a simple

[1]ranazulqarnain7777@gmail.com (Corresponding author); [2]muhammad.saeed@umt.edu.pk; [3]baghalisewag@mail.nwpu.edu.cn; [4]sohaib@stumail.nwu.edu.cn; [5]msaqlain@lgu.edu.pk; [6]mirfan230@hotmail.com; [7]zams706@live.com
[1,4]School of Mathematics, Northwest University, Xi'an 710127, China
[2]School of Science, Department of Mathematics, University of Management and Technology, Lahore
[3]Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an 710129, China
[5]Department of Mathematics, Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan
[6,7]Shaanxi Key Laboratory of Earth Surface System and Environmental Carrying Capacity, College of Urban and Environmental Sciences, Northwest University, Xi'an, 710127, China

method with a mean of relative areas for ranking of fuzzy numbers and established an improved fuzzy TOPSIS method by using presented ranking fuzzy numbers.

On the base of alpha level sets, a fuzzy TOPSIS method was proposed [12] by Wang and Elhag and discussed the relation among fuzzy TOPSIS and fuzzy weighted average. In [13], the authors gave the concept of a direct approach to the fuzzy extension of the TOPSIS method, they claimed that the proposed method more efficient than the previously proposed method and free of limitation. For group decision-making problems the extended TOPSIS method is based on fuzzy numbers presented in [14]. Parveen and Kamble [15] proposed the fuzzy TOPSIS method with hexagonal fuzzy numbers and compare with other MCDM problems, they also presented the difficulties faced by the women in society by using the newly proposed method. The authors proposed a decision-making method on an interval-valued fuzzy soft matrix [16] known as "interval-valued fuzzy soft max-min decision-making method". Zulqarnain et al. [17, 18] used the "interval-valued fuzzy soft max-min decision-making method" for decision making and medical diagnoses. In [19], the authors extended the TOPSIS method to Pythagorean fuzzy data for the solution of MCDM in which experts provided the feasible alternatives for assessment information. Mahmut used the fuzzy TOPSIS method for the selection of equipment in the mining industry and concluded that this method is very helpful for decision-makers to solve decision-making problems in the mining industry [20]. To improve the efficiency of the TOPSIS method in decision-making Ding constructed the integrated fuzzy TOPSIS method in [21].

Fuzzy TOPSIS use in different industries for hiring workers and also used for decision making, medical diagnosis for MCDM problems reported in the literature [22]–[26], the authors compare and decided that the fuzzy TOPSIS method is more efficient than classical TOPSIS. Ahmad and Mohamad [27] presented an evaluation among fuzzy TOPSIS and simplified fuzzy TOPSIS and detected that the fuzzy TOPSIS method is more suitable comparative to simplified fuzzy TOPSIS. In [28], the author used the fuzzy TOPSIS method for the evaluation of the power plant. The intuitionistic fuzzy TOPSIS method is used for the selection of the best choice for an auto company in [29]. Chu [30] used the fuzzy TOPSIS method for the selection of plant location. Fuzzy TOPSIS method is used for the selection of the best candidate for personnel selection according to the following criteria experience, education, technical skills, and relocation in [31] and Dual Hesitant Fuzzy Geometric Bonferroni Mean Operators and Diminishing Choquet hesitant 2-tuple linguistic aggregation operator are developed in [34, 35].

1.1 Motivation and Contribution

For the linguistic assessments, the technique of classical TOPSIS is used, but due to the uncertainty and imprecise nature of the linguistic assessments, we proposed fuzzy TOPSIS. In this paper, we discuss the fuzzy set with some operations and fuzzy TOPSIS. We presented the generalization of TOPSIS under a fuzzy environment and use the proposed method in the garments industry for supplier selection.

1.2 Structure of Article

The following paper is organized as follows: in section 2, first, we discuss some basic definitions of fuzzy sets. In section 3, we study about fuzzy TOPSIS method and construct a graphical model for fuzzy TOPSIS. In section 4, we use the proposed method for the selection of suppliers in the garments industry. lastly, the conclusion is made in section 5.

2. Preliminaries

In this section, we recall some definitions of the fuzzy set with some operations.

Definition 2.1. [32] A fuzzy set A in M is characterized by a membership function $f_A(y_i)$ which associates with each object of M in the interval [0, 1], with the value of $f_A(y_i)$ where y_i representing the grade of membership of y in A.

Definition 2.2. [33] A fuzzy subset μ is convex, on the universal set ℝ iff for all $c, d \in \mathbb{R}, \mu(\alpha c + \beta d) \geq \mu(c) \wedge \mu(d)$, where $\alpha + \beta = 1$.

Definition 2.3. [33] On the universal set V, a fuzzy subset μ is entitled as a normal fuzzy subset if here a subset c_i such that $\mu(c_i) = 1$.

Definition 2.4. [33] Stated upon the universal set S, a fuzzy number is a fuzzy subset that exists as together convex and normal.

Definition 2.5. [26] If C = (x_1, y_1, z_1) for all $x_1, y_1, z_1 \in \mathbb{R}$ is a fuzzy number with piecewise linear membership function defined as follows

$$\delta_C\,(t) = \begin{cases} \frac{t-x_1}{y_1-x_1} & if\ x_1 \leq t \leq y_1 \\ 1 & if\ t = y_1 \\ \frac{z_1-t}{z_1-y_1} & if\ y_1 \leq t \leq z_1 \\ 0 & Otherwise \end{cases}$$

Then C = (x_1, y_1, z_1) is called a triangular fuzzy number (TFN).

Definition 2.6 [11] If C = (x_1, y_1, z_1) and D = (x_2, y_2, z_2) are two TFN, then distance between them can be defined as

$$d\,(C, D) = \sqrt{\frac{1}{3}\ \left((x_1 - x_2)^2,\ (y_1 - y_2)^2, (z_1 - z_2)^2\right)}$$

3. Fuzzy TOPSIS Algorithm [11]

In this section, we present the fuzzy TOPSIS method with an algorithm and construct a model for the fuzzy TOPSIS method.

Let M = $\{M_1, M_2, M_3, \ldots, M_m\}$ be a set of *m* alternatives and N = $\{N_1, N_2, N_3, \ldots, N_n\}$ be a set of evaluation criteria and D = $\{D_1, D_2, D_3, \ldots, D_l\}$ be a set of *l* decision-makers.

Step 1: Fuzzy Rating Scale selection for Linguistic Variables

The criteria for linguistic variables and alternatives are given in table 1.

Step 2: Fuzzy linguistic ratings for alternatives and criteria of weights for decision-makers

"$\widetilde{x_{ij}^k}$ be a fuzzy rating for k^{th} decision-maker for the i^{th} alternatives and j^{th} criterion, represented as follows

$$\widetilde{x_{ij}^k} = (a_{ij}^k, b_{ij}^k, c_{ij}^k)$$

The weight for k^{th} decision-maker and j^{th} criteria are given as follows

$$\widetilde{w_j^k} = (w_{j1}^k, w_{j2}^k, w_{j3}^k)$$

Step 3: Aggregated fuzzy ratings for the alternatives

$\widetilde{x_{ij}}$ be an aggregated fuzzy rating for the i^{th} alternative w.r.t the j^{th} criteria are given as follows

$$\widetilde{x_{ij}} = (a_{ij}, b_{ij}, c_{ij})$$

$$a_{ij} = \min_k \{a_{ij}^k\} \quad b_{ij} = \frac{1}{l}\sum_{k=1}^{l}\{b_{ij}^k\} \quad c_{ij} = \min_k \{c_{ij}^k\}$$

and

$\widetilde{w_j} = (\widetilde{w_{j1}}, \widetilde{w_{j2}}, \widetilde{w_{j3}})$ be an aggregated fuzzy weight for the j^{th} criteria represents in the following equation.

$$w_{j1} = \min_k \{w_{j1}^k\} \qquad w_{j2} = \frac{1}{l}\sum_{k=1}^{l}\{w_{j2}^k\} \qquad w_{j3} = \min_k \{w_{j3}^k\}$$

Step 4: Construction of Aggregated Fuzzy Decision Matrix (AFDM) and Aggregated Fuzzy Weight Matrix (AFWM)

Fuzzy MCDM problem can be converted to an AFDM as follows

$$\mathrm{D} = \begin{matrix} M_1 \\ M_2 \\ \vdots \\ M_m \end{matrix} \begin{bmatrix} \widetilde{x_{11}} & \widetilde{x_{12}} & \cdots & \widetilde{x_{1n}} \\ \widetilde{x_{21}} & \widetilde{x_{22}} & \cdots & \widetilde{x_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{x_{m1}} & \widetilde{x_{m2}} & \cdots & \widetilde{x_{mn}} \end{bmatrix}$$

Moreover, the AFWM is defined as follows

$$\breve{W} = [\widetilde{w_1}, \widetilde{w_2}, \widetilde{w_3}, \dots, \widetilde{w_n}]^T$$

Where $\widetilde{w_j}$ be an aggregated fuzzy weight for the j^{th} criterion.

Step 5: Normalization of the FDM (NFDM)

The NFDM is given as

$$\breve{R} = [\widetilde{r_{ij}}]_{m\times n} = \begin{bmatrix} \widetilde{r_{11}} & \widetilde{r_{12}} & \cdots & \widetilde{r_{1n}} \\ \widetilde{r_{21}} & \widetilde{r_{22}} & \cdots & \widetilde{r_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{r_{m1}} & \widetilde{r_{m2}} & \cdots & \widetilde{r_{mn}} \end{bmatrix} \text{ where } \widetilde{r_{ij}} = \left(\frac{a_{ij}}{c_j^*}, \frac{b_{ij}}{c_j^*}, \frac{c_{ij}}{c_j^*}\right) \text{ and } c_j^* = \max_k c_{ij} \text{ (benefit criteria)}$$

$$\widetilde{r_{ij}} = \left(\frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_j}\right),\ a_j^- = \min_i a_{ij}, \text{ where } a_j^- \text{ represents the cost criteria}$$

To normalize the decision matrix.

Step 6: Weighted Normalized Fuzzy Decision Matrix (WNFDM)

WNFDM gave as follows

$$\breve{V} = [\widetilde{v_{ij}}]_{m\times n} = \left[\widetilde{w_j(\cdot)\,\widetilde{r_{ij}}}\right] = \begin{bmatrix} \widetilde{w_1}(\cdot)\,\widetilde{r_{11}} & \widetilde{w_2}(\cdot)\,\widetilde{r_{12}} & \cdots & \widetilde{w_n}(\cdot)\,\widetilde{r_{1n}} \\ \widetilde{w_1}(\cdot)\,\widetilde{r_{21}} & \widetilde{w_2(\cdot)\,\widetilde{r_{22}}} & \cdots & \widetilde{w_2}(\cdot)\,\widetilde{r_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{w_1}(\cdot)\,\widetilde{r_{m1}} & \widetilde{w_2}(\cdot)\,\widetilde{r_{m2}} & \cdots & \widetilde{w_n(\cdot)\,\widetilde{r_{mn}}} \end{bmatrix}$$

Step 7: Determination of FPIS and FNIS

To find the FPIS and the FNIS we used the following equations

$$M^* = (\widetilde{v_1^*}, \widetilde{v_2^*}, \widetilde{v_3^*}, \dots, \widetilde{v_n^*}) \qquad \text{where } \widetilde{v_j^*} = (c_j^*, c_j^*, c_j^*) \text{ and } c_j^* = \max_i \left\{ v_j^{(3^{rd}\ \widetilde{component})} \right\}$$

$$M^- = (\widetilde{v_1^-}, \widetilde{v_2^-}, \widetilde{v_3^-}, \dots, \widetilde{v_n^-}) \qquad \text{where } \widetilde{v_j^-} = (c_j^-, c_j^-, c_j^-) \text{ and } c_j^- = \min_i \left\{ v_j^{(1^{st}\ \widetilde{component})} \right\}$$

Where $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

Step 8: Calculation of d_i^* and d_i^-

The distances from FPIS and FNIS of all weighted alternative $\widetilde{v_{ij}}$ where $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

d_i^* and d_i^- can be calculated as follows

$$d_i^* = \sum_{j=1}^{n} d(\widetilde{v_{ij}}, \widetilde{v_i^*}), \quad i = 1, 2, 3, \ldots, m$$

$$d_i^- = \sum_{j=1}^{n} d(\widetilde{v_{ij}}, \widetilde{v_i^-}), \quad i = 1, 2, 3, \ldots, m$$

Step 9: Determination of Closeness Coefficient CC_i

CC_i of alternatives can be calculated as follows

$$CC_i = \frac{d_i^-}{d_i^* + d_i^-} \text{ for all } i = 1, 2, 3, \ldots, m\text{''}$$

Step 10: Ranking the alternatives

An alternative closeness coefficient's value is near to 1 represents that it is near to FPIS and away from FNIS.

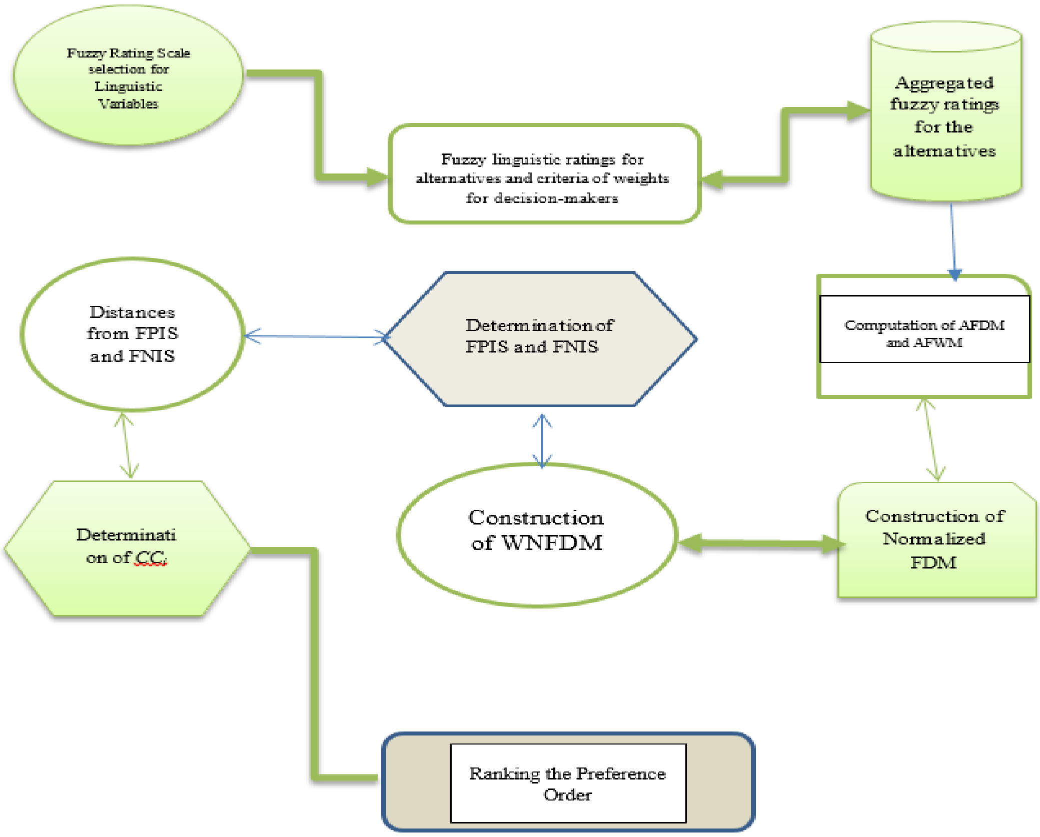

Figure 1. Algorithm of proposed Fuzzy TOPSIS

4. Application of fuzzy TOPSIS method

A garments industry wants to hire a supplier from out of two supplier's M = {M_1, M_2}. For the selection of the best supplier, the managing director of the industry hires a team of three decision-makers as follows D = {D_1, D_2, D_3}. The managing director of garments industry decided the evaluation criteria for the selection of the best supplier for the industry given as follows N = {N_1, N_2, N_3, N_4}

$$N = \begin{cases} Benifit\ Criteria \\ Cost\ Criteria \end{cases} j_1 = \begin{cases} N_1 \\ N_2 \\ N_3 \end{cases} \qquad j_2 = \{N_4$$

Solution by Fuzzy TOPSIS

Step 1: Fuzzy Rating Scale selection for Linguistic Variables

The rating scale for linguistic variables given in the following

Table 1. Ratings for Linguistic Variables

Criteria Weights	Alternatives	TFN
L	VP	(1,1,3)
L	P	(1,3,5)
M	F	(3,5,7)
H	G	(5,7,9)
VH	VG	(7,9,9)

Where weights of criteria represent "very low (VL), low (L), medium (M), high (H), and very high (VH). Similarly, rating for alternatives VP, P, F, G, VG represents very poor, poor, fair, good, and very good" respectively.

Step 2: Fuzzy linguistic ratings for alternatives and criteria of weights for decision-makers

Every decision-maker allocate some weight for each criterion given in the following table

Table 2. Criteria Weightage by the DMs $\widetilde{w_j^k} = (w_{j1}^k, w_{j2}^k, w_{j3}^k)$

	D₁	**D₂**	**D₃**
N_1	H (5, 7, 9) $\widetilde{w_1^1} = (\widetilde{w_{11}^1}, \widetilde{w_{12}^1}, \widetilde{w_{13}^1})$	M (3, 5, 7) $\widetilde{w_1^2} = (\widetilde{w_{11}^2}, \widetilde{w_{12}^2}, \widetilde{w_{13}^2})$	M (3, 5, 7) $\widetilde{w_1^3} = (\widetilde{w_{11}^3}, \widetilde{w_{12}^3}, \widetilde{w_{13}^3})$
N_2	VH (7, 9, 9) $\widetilde{w_2^1} = (\widetilde{w_{21}^1}, \widetilde{w_{22}^1}, \widetilde{w_{23}^1})$	H (5, 7, 9) $\widetilde{w_2^2} = (\widetilde{w_{21}^2}, \widetilde{w_{22}^2}, \widetilde{w_{23}^2})$	H (5, 7, 9) $\widetilde{w_2^1} = (\widetilde{w_{21}^3}, \widetilde{w_{22}^3}, \widetilde{w_{23}^3})$
N_3	VH (7, 9, 9) $\widetilde{w_3^1} = (\widetilde{w_{31}^1}, \widetilde{w_{32}^1}, \widetilde{w_{33}^1})$	H (5, 7, 9) $\widetilde{w_3^2} = (\widetilde{w_{31}^2}, \widetilde{w_{32}^2}, \widetilde{w_{33}^2})$	H (5, 7, 9) $\widetilde{w_3^3} = (\widetilde{w_{31}^3}, \widetilde{w_{32}^3}, \widetilde{w_{33}^3})$
N_4	M (3, 5, 7) $\widetilde{w_4^1} = (\widetilde{w_{41}^1}, \widetilde{w_{42}^1}, \widetilde{w_{43}^1})$	L (1, 3, 5) $\widetilde{w_4^2} = (\widetilde{w_{41}^2}, \widetilde{w_{42}^2}, \widetilde{w_{43}^2})$	L (1, 3, 5) $\widetilde{w_4^3} = (\widetilde{w_{41}^3}, \widetilde{w_{42}^3}, \widetilde{w_{43}^3})$

The aggregated fuzzy weights $\widetilde{w_j} = (w_{j1}, w_{j2}, w_{j3})$ for each criterion *"j = 1, 2, 3, 4"* are calculated as follows.

$$w_{11} = \min_k \{w_{11}^k\} = \min\{5, 3, 3\} = 3$$

$$w_{12} = \frac{1}{3}\sum_{k=1}^{3}\{w_{12}^k\} = \frac{1}{3}[7+5+5] = 5.667$$

$$w_{13} = \min_k \{w_{13}^k\} = \max\{9, 7, 7\} = 9$$

So,

$$\widetilde{w_1} = (w_{11}, w_{12}, w_{13}) = (3, 5.667, 9)$$

Similarly, we can get

$$\widetilde{w_2} = (w_{21}, w_{22}, w_{23}) = (5, 7.667, 9)$$

$$\widetilde{w_3} = (w_{31}, w_{32}, w_{33}) = (5, 7.667, 9)$$

$$\widetilde{w_4} = (w_{41}, w_{42}, w_{43}) = (1, 3.667, 7)$$

Therefore, the aggregated weight vector is

$$\widetilde{W} = [\widetilde{w_1}, \widetilde{w_2}, \widetilde{w_3}, \widetilde{w_4}]^T$$

$$\widetilde{W} = [(3, 5.667, 9), (5, 7.667, 9), (5, 7.667, 9), (1, 3.667, 7)]^T$$

Alternative rating for decision-makers given as follows

Table 3. Rating of alternatives for DM

		D₁	**D₂**	**D₃**
N_1	**M_1**	F $\widetilde{x_{11}^{(1)}}=(a_{11}^{(1)}, b_{11}^{(1)}, c_{11}^{(1)})$= (3, 5, 7)	F $\widetilde{x_{11}^{(2)}}=(a_{11}^{(2)}, b_{11}^{(2)}, c_{11}^{(2)})$= (3, 5, 7)	F $\widetilde{x_{11}^{(3)}}=(a_{11}^{(3)}, b_{11}^{(3)}, c_{11}^{(3)})$= (3, 5, 7)
	M_2	F $\widetilde{x_{21}^{(1)}}=(a_{21}^{(1)}, b_{21}^{(1)}, c_{21}^{(1)})$= (5, 7, 9)	F $\widetilde{x_{21}^{(2)}}=(a_{21}^{(2)}, b_{21}^{(2)}, c_{21}^{(2)})$= (3, 5, 7)	F $\widetilde{x_{21}^{(3)}}=(a_{21}^{(3)}, b_{21}^{(3)}, c_{21}^{(3)})$= (3, 5, 7)
N_2	**M_1**	VG $\widetilde{x_{12}^{(1)}}=(a_{12}^{(1)}, b_{11}^{(1)}, c_{12}^{(1)})$= (7, 9, 9)	VG $\widetilde{x_{12}^{(2)}}=(a_{12}^{(2)}, b_{11}^{(2)}, c_{12}^{(2)})$= (7, 9, 9)	VG $\widetilde{x_{12}^{(3)}}=(a_{12}^{(3)}, b_{11}^{(3)}, c_{12}^{(3)})$= (7, 9, 9)
	M_2	G $\widetilde{x_{22}^{(1)}}=(a_{22}^{(1)}, b_{22}^{(1)}, c_{22}^{(1)})$= (5, 7, 9)	VG $\widetilde{x_{22}^{(2)}}=(a_{22}^{(2)}, b_{22}^{(2)}, c_{22}^{(2)})$= (3, 5, 7)	G $\widetilde{x_{22}^{(3)}}=(a_{22}^{(3)}, b_{22}^{(3)}, c_{22}^{(3)})$= (5, 7, 9)
N_3	**M_1**	P $\widetilde{x_{13}^{(1)}}=(a_{13}^{(1)}, b_{13}^{(1)}, c_{13}^{(1)})$= (1, 3, 5)	F $\widetilde{x_{13}^{(2)}}=(a_{13}^{(2)}, b_{13}^{(2)}, c_{13}^{(2)})$= (3, 5, 7)	P $\widetilde{x_{13}^{(3)}}=(a_{13}^{(3)}, b_{13}^{(3)}, c_{13}^{(3)})$= (1, 3, 5)
	M_2	P $\widetilde{x_{23}^{(1)}}=(a_{23}^{(1)}, b_{23}^{(1)}, c_{23}^{(1)})$= (1, 3, 5)	P $\widetilde{x_{23}^{(2)}}=(a_{23}^{(2)}, b_{23}^{(2)}, c_{23}^{(2)})$= (1, 3, 5)	P $\widetilde{x_{23}^{(3)}}=(a_{23}^{(3)}, b_{23}^{(3)}, c_{23}^{(3)})$= (1, 3, 5)
N_4	**M_1**	F $\widetilde{x_{14}^{(1)}}=(a_{14}^{(1)}, b_{14}^{(1)}, c_{14}^{(1)})$= (3, 5, 7)	F $\widetilde{x_{14}^{(2)}}=(a_{14}^{(2)}, b_{14}^{(2)}, c_{14}^{(2)})$= (3, 5, 7)	P $\widetilde{x_{14}^{(3)}}=(a_{14}^{(3)}, b_{14}^{(3)}, c_{14}^{(3)})$= (1, 3, 5)
	M_2	P $\widetilde{x_{24}^{(1)}}=(a_{24}^{(1)}, b_{24}^{(1)}, c_{24}^{(1)})$= (1, 3, 5)	P $\widetilde{x_{24}^{(2)}}=(a_{24}^{(2)}, b_{24}^{(2)}, c_{24}^{(2)})$= (1, 3, 5)	F $\widetilde{x_{24}^{(3)}}=(a_{24}^{(3)}, b_{24}^{(3)}, c_{24}^{(3)})$= (3, 5, 7)

Step 3: Aggregated fuzzy ratings for the alternatives

$\widetilde{x_{ij}}$ be an aggregated fuzzy rating for the i^{th} alternative w.r.t the j^{th} criteria can be calculated as follows

$$\widetilde{x_{11}} = (a_{11}, b_{11}, c_{11}), \text{ where}$$

$$a_{11} = \min_k \{a_{11}^k\} = \min\{3, 3, 3\} = 3$$

$$b_{11} = \frac{1}{3}\sum_{k=1}^{l}\{b_{11}^k\} = \frac{1}{3}[5+5+5] = 5$$

$$c_{11} = \min_k \{c_{11}^k\} = \max\{7, 7, 7\} = 7$$

Therefore

$$\widetilde{x_{11}} = (3.000, 5.000, 7.000)$$

Similarly, we can find other values given in Table 4

Step 4: Construction of AFDM

Table 4. AFDM $\widetilde{D} = \widetilde{x_{ij}}$

	N_1	**N_2**	**N_3**	**N_4**
M_1	$\widetilde{x_{11}}$ = (3.000,5.000,7.000)	$\widetilde{x_{12}}$ = (7.000,9.000,9.000)	$\widetilde{x_{13}}$ = (1.000,3.667,7.000)	$\widetilde{x_{14}}$ = (1.000,4.333,7.000)
M_2	$\widetilde{x_{21}}$ = (3.000,6.333,9.000)	$\widetilde{x_{22}}$ = (5.000,7.667,9.000)	$\widetilde{x_{23}}$ = (1.000,3.000,5.000)	$\widetilde{x_{24}}$ = (1.000,3.667,7.000)

Step 5: NFDM

We can calculate the NFDM as follows

$$\widetilde{r_{11}} = \left(\frac{a_1^-}{c_{11}}, \frac{a_1^-}{b_{11}}, \frac{a_1^-}{a_{11}}\right), \text{ where } N_1 \text{ is the cost criteria}$$

$$\widetilde{r_{11}} = \left(\frac{3}{7}, \frac{3}{5}, \frac{3}{3}\right) \text{ where } a_1^- = \min_i a_{i1} = \min\{3.000, 3.000\} = 3$$

Similarly, we can get other values

Therefore, NFDM is given in the following table.

Table 5. NFDM $\tilde{R} = \widetilde{r_{ij}}$

	N₁	N₂	N₃	N₄
M₁	$\widetilde{r_{11}}$ = (0.429,0.600,1.000)	$\widetilde{r_{12}}$ = (0.778, 1.000,1.000)	$\widetilde{r_{13}}$ = (0.143,0.524,1.000)	$\widetilde{r_{14}}$ = (0.143,0.619,1.000)
M₂	$\widetilde{r_{21}}$ = (0.333, 0.474, 1.000)	$\widetilde{r_{22}}$ = (0.556,0.852,1.000)	$\widetilde{r_{23}}$ = (0.143,0.429,0.714)	$\widetilde{r_{24}}$ = (0.143,0.524,1.000)

Step 6: WNFDM

Now we get the WNFDM

Table 6. WNFDM $\tilde{V}$ = $[\widetilde{v_{ij}}]$

	N₁	N₂	N₃	N₄
M₁	$\widetilde{v_{11}}$ = (1,286, 3.400, 9.000)	$\widetilde{v_{12}}$ = (3.889, 7.667, 9.000)	$\widetilde{v_{13}}$ = (0.714, 4.016, 9.000)	$\widetilde{v_{14}}$ = (0.143, 2.270, 7.000)
M₂	$\widetilde{v_{21}}$ = (1.000, 2.684, 9.000)	$\widetilde{v_{22}}$ = (2.778, 6.531, 9.000)	$\widetilde{v_{23}}$ = (0.714, 3.286, 6.429)	$\widetilde{v_{24}}$ = (0.143, 1.921, 7.000)

Step 7: Determination of FPIS and FNIS

To calculate FPIS and FNIS given in the following table

Table 7. The calculated values of FPIS and FNIS

		FPIS		
M^*	$\widetilde{v_1^*}$ = (9, 9, 9)	$\widetilde{v_2^*}$ = (9, 9, 9)	$\widetilde{v_3^*}$ = (9, 9, 9)	$\widetilde{v_4^*}$ = (7, 7, 7)
		FNIS		
M^-	$\widetilde{v_1^-}$ = (1, 1, 1)	$\widetilde{v_2^-}$ = (2.778, 2.778, 2,778)	$\widetilde{v_3^-}$ = (0.714, 0.714, 0.714)	$\widetilde{v_3^-}$ = (0.143, 0.143, 0.143)

Step 8: Calculation of d_i^* and d_i^-

The distances from FPIS and FNIS of all weighted alternative $\widetilde{v_{ij}}$, where "*i = 1, 2, 3, …, m*" and "*j = 1, 2, 3, …, n*". d_i^* and d_i^- can be calculated as follows $d(v_{1j}, \widetilde{v_j^*})$ where *j = 1, 2, 3, 4*

for *j = 1*

$$d(v_{11}, \widetilde{v_1^*}) = d((1{,}286, 3.400, 9.000),\ (9, 9, 9))$$

$$d(v_{11}, \widetilde{v_1^*}) = \sqrt{\frac{1}{3}\ ((1{,}286-9)^2,\ (3.400-9)^2, (9.000-9)^2)} = 5.503$$

Similarly, for *j = 2*

$$d(v_{12}, \widetilde{v_2^*}) = d((3.889, 7.667, 9.000),\ (9, 9, 9))$$

$$d(v_{12}, \widetilde{v_2^*}) = \sqrt{\frac{1}{3}\ ((3.889-9)^2,\ (7.667-9)^2, (9.000-9)^2)} = 3.049$$

for *j = 3*

$$d(v_{13}, \widetilde{v_3^*}) = d((0.714, 4.016, 9.000),\ (9, 9, 9))$$

$$d(v_{13}, \widetilde{v_3^*}) = \sqrt{\frac{1}{3}\ ((0.714-9)^2,\ (4.016-9)^2, (9.000-9)^2)} = 5.582$$

for *j = 4*

$$d(v_{14}, \widetilde{v_4^*}) = d((0.143, 2.270, 7.000),\ (7, 7, 7))$$

$$d(v_{14}, \widetilde{v_4^*}) = \sqrt{\frac{1}{3}\ ((0.143-7)^2,\ (2.270-7)^2, (7.000-7)^2)} = 4.809$$

The remaining values $d(v_{2j}, \widetilde{v_j^*})$, $d(v_{1j}, \widetilde{v_1^-})$, $d(v_{2j}, \widetilde{v_j^-})$ for "*j = 1, 2, 3, 4*" are left for the sake of brevity and are given in the following Table

Table 8. Distances $d\ (M_i, M^*)$ and $d\ (M_i, M^-)$ from FPIS and FNIS for the alternatives M_i

	N₁	**N₂**	**N₃**	**N₄**
FPIS M₁	$d(v_{11}, \widetilde{v_1^*}) = 5.503$	$d(v_{12}, \widetilde{v_2^*}) = 3.049$	$d(v_{13}, \widetilde{v_3^*}) = 5.582$	$d(v_{14}, \widetilde{v_4^*}) = 4.809$
FPIS M₂	$d(v_{21}, \widetilde{v_1^*}) = 5.884$	$d(v_{22}, \widetilde{v_2^*}) = 3.864$	$d(v_{23}, \widetilde{v_3^*}) = 5.997$	$d(v_{24}, \widetilde{v_4^*}) = 4.926$
FNIS M₁	$d(v_{11}, \widetilde{v_1^-}) = 4.824$	$d(v_{12}, \widetilde{v_2^-}) = 4.613$	$d(v_{13}, \widetilde{v_3^-}) = 5.149$	$d(v_{14}, \widetilde{v_4^-}) = 4.145$
FNIS M₂	$d(v_{21}, \widetilde{v_1^-}) = 4.72$	$d(v_{22}, \widetilde{v_2^-}) = 4.195$	$d(v_{23}, \widetilde{v_3^-}) = 3.617$	$d(v_{24}, \widetilde{v_4^-}) = 4.089$

d_i^* be each weighted alternative from FPIS is computed as

$$d_i^* = \sum_{j=1}^{n} d(v_{ij}, \widetilde{v_j^*});\ i = 1, 2$$

Now d_1^* for the alternative M_1 form FPIS M^* is calculated as follows

$d_1^* = \sum_{j=1}^{n} d(v_{1j}, \widetilde{v_j^*}) = d(v_{11}, \widetilde{v_1^*}) + d(v_{12}, \widetilde{v_2^*}) + d(v_{13}, \widetilde{v_3^*}) + d(v_{14}, \widetilde{v_4^*})$ = 5.503 3.049 5.582 4.809 = 18.943

Similarly, we can find d_2^*, d_1^-, d_2^- and their respective values are given in the following Table

Table 9. The distance of each weighted alternative

$\mathbf{d_1^*}$	$\mathbf{d_2^*}$	$\mathbf{d_1^-}$	$\mathbf{d_2^-}$
18.943	20.671	18.731	16.621

Step 9: Determination of Closeness Coefficient CC_i

Finally, the closeness coefficient CC_i of alternatives "i =1, 2" calculated as follows

$$CC_i = \frac{d_i^-}{d_i^* + d_i^-}$$

$$CC_1 = \frac{18.731}{18.731+18.943} = 0.497$$

$$CC_2 = \frac{16.621}{16.621+20.671} = 0.445$$

Step 10: Ranking the alternatives

The ranking order for the alternatives is $M_1 > M_2$, i.e., M_1 is the best supplier according to the given criteria.

5. Conclusion

In this paper, we proposed a fuzzy TOPSIS method. By using crisp data it is more difficult to solve decision-making problems under an uncertain environment, to overcome such uncertainties fuzzy TOPSIS is more appropriate. Finally, to show the applicability and validity of the proposed technique with an illustrated example of the best supplier in the garments industry is presented. We consider this technique will be helpful in problem-solving and will expand the area of investigations for more accuracy in real-life issues.

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References

[1] K. Yoon, C. L. Hwang, *Multi Attribute Decision Making Methods and Applications,* Springer-Verlag, Berlin Heidelberg, 1981.

[2] M. Zulqarnain, F. Dayan, and M. Saeed, *TOPSIS Analysis for The Prediction of Diabetes Based on General Characteristics of Humans,* International Journal of Pharmaceutical Sciences and Research 9(7) (2018) 2932-2939.

[3] M. Zulqarnain, S. Abdal, M. I. Ahamad, Z. Zafar, B. E. Ali, S. Younas, A. Maalik, A. Mariam, F. Dayan, *Application of TOPSIS Method in Decision Making via Soft Set*, Biomedical Journal of Scientific and Technical Research 24(3) (2017) 18208-18215.

[4] S. M. S. Tonekaboni, K. Shahroudi, *Application of Topsis Method To Supplier Selection in Iran Auto Supply Chain*, Journal of Global Strategic Management 6(2) (2012) 123-123.

[5] A. Sarkar, *A TOPSIS Method to Evaluate The Technologies*, International Journal of Quality & Reliability Management 31(1) (2013) 2-13.

[6] Z. Pavić, V. Novoselac, *Notes on TOPSIS Method*, International Journal of Engineering Research and General Science 1(2) (2013) 5-12.

[7] J. Wielki, M. Jurczyk-Bunkowska, D. Madera, *Application of TOPSIS Method for Evaluation of IT Application in The Hospital*, 20th European Conference on Knowledge Management ECKM 2 (2019) 1096-1104.

[8] R. M. Zulqarnain, M. Saeed, N. Ahmad, F. Dayan, B. Ahmad, *Application of TOPSIS Method for Decision Making,* International Journal of Scientific Research in Mathematical and Statistical Sciences 7(2) (2020) 76-81.

[9] R. M. Zulqarnain, S. Abdal, B. Ali, L. Ali, F. Dayan, M. I. Ahamad, Z. Zafar, *Selection of Medical Clinic for Disease Diagnosis by Using TOPSIS Method,* International Journal of Pharmaceutical Sciences Review and Research 61(1) (2020) 22-27.

[10] R. M. Zulqarnain, X. L. Xin, M. Saeed, N. Ahmad, F. Dayan, B. Ahmad, *Recruitment of Medical Staff in Health Department by Using TOPSIS Method,* International Journal of Pharmaceutical Sciences Review and Research 62(1) (2020) 1-7.

[11] C. T. Chen, *Extensions of The TOPSIS for Group Decision-Making under Fuzzy Environment*, Fuzzy Sets and Systems 114(1) (2020) 1-9.

[12] T. C. Chu, Y. C. Lin, *Improved Extensions of The TOPSIS for Group Decision Making under Fuzzy Environment*, Journal of Information and Optimization Sciences 23(2) (2002) 273-286.

[13] Y. M. Wang, T. M. S. Elhag, *Fuzzy TOPSIS Method Based on Alpha Level Sets with An Application to Bridge Risk Assessment*, Expert Systems with Applications 31(2) (2006) 309-319.

[14] L. Dymova, P. Sevastjanov, A. Tikhonenko, *An Approach to Generalization of Fuzzy TOPSIS Method,* Information Sciences 238 (2013) 149-162.

[15] N. Parveen, P. N. Kamble, *Decision-Making Problem Using Fuzzy TOPSIS Method with Hexagonal Fuzzy Number,* Computing in Engineering and Technology (2020) 421-430.

[16] M. Zulqarnain, M. Saeed, *A New Decision Making Method on Interval Valued Fuzzy Soft Matrix (IVFSM),* British Journal of Mathematics & Computer Science 20(5) (2017) 1-17.

[17] R. M. Zulqarnain, M. Saeed, B. Ali, N. Ahmad, L. Ali, S. Abdal, *Application of Interval valued Fuzzy Soft Max-Min Decision Making Method,* International Journal of Mathematical Research. 9(1) (2020) 11-19.

[18] R. M. Zulqarnain, X. L. Xin, M. Saeed, N. Ahmad, F. Dayan, B. Ahmad, *Application of Interval Valued Fuzzy Soft Max-Min Decision Making Method in Medical Diagnosis*, International Journal of Pharmaceutical Sciences Review and Research 62(1) (2020) 56-60.

[19] M. Akram, W. A. Dudek, F. Ilyas, *Group Decision-Making Based on Pythagorean Fuzzy TOPSIS Method,* International Journal of Intelligent Systems 34(7) (2019) 1455-1475.

[20] M. Yavuz, *Equipment Selection by Using Fuzzy TOPSIS Method*, IOP Conf. Series: Earth and Environmental Science 44(4) (2016) 1-5.
[21] J. F. Ding, *An Integrated Fuzzy Topsis Method for Ranking Alternatives and Its Application,* Journal of Marine Science and Technology 19(4) (2011) 341-352.
[22] M. Sevkli, S. Zaim, A. Turkyilmaz, M. Satir, *An Application of Fuzzy TOPSIS Method for Supplier Selection,* IEEE International Conference on Fuzzy Systems (2010) 18-23 Barcelona, Spain.
[23] S. Eslamian Shiraz, Ü. Şengül, M. Eren, *Determination of Extended Fuzzy TOPSIS Method of Criteria Leading to Supplier Selection for Industries*, Asian Social Science 10(4) (2014) 183-194.
[24] A. Yeşim Yayla, A. Yildiz, A. Özbek, *Fuzzy TOPSIS Method in Supplier Selection and Application in The Garment Industry*, Fibres and Textiles in Eastern Europe 93(4) (2012) 20-23.
[25] M. Zulqarnain, *Choose Best Criteria for Decision Making Via Fuzzy Topsis Method*, Mathematics and Computer Science 2(6) (2017) 113-119.
[26] G. R. Jahanshahloo, F. H. Lotfi, M. Izadikhah, *Extension of The TOPSIS Method for Decision-Making Problems with Fuzzy Data*, Applied Mathematics and Computation 181(2) (2006) 1544-1551.
[27] S. A. S. Ahmad, D. Mohamad, *A Comparative Analysis between Fuzzy Topsis and Simplified Fuzzy Topsis*, AIP Conference Proceedings 1870(1) (2017).
[28] K. Boran, *An Evaluation of Power Plants in Turkey: Fuzzy TOPSIS Method*, Energy Sources, Part B: Economics, Planning, and Policy 12(2) (2017) 119-125.
[29] M. Zulqarnain, F. Dayan, *Selection of Best Alternative For An Automotive Company by Intuitionistic Fuzzy TOPSIS Method*, International Journal of Scientific and Technology research 6(10) (2017) 126-132.
[30] T. C. Chu, *Selecting Plant Location via A Fuzzy TOPSIS Approach*, The International Journal of Advanced Manufacturing Technology 20(11) (2002) 859-864.
[31] K. Aliyeva, *Multifactor Personnel Selection by the Fuzzy TOPSIS Method*, International Conference on Theory and Applications of Fuzzy Systems and Soft Computing, Springer 1 (2019) 478-483.
[32] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.
[33] A. R. R. Supriya Kumar De, Ranjit Biswas, *An Application of Intuitionistic Fuzzy Sets in Medical Diagnosis*, Fuzzy Sets and Systems 117 (2001) 209-213.
[34] R. N. Jamil, T. Rashid, *Application of Dual Hesitant Fuzzy Geometric Bonferroni Mean Operators in Deciding an Energy Policy for the Society,* Mathematical Problems in Engineering Article ID 4541982 (2018)14 pages.
[35] I. Beg, R. N Jamil, T. Rashid, *Diminishing Choquet Hesitant 2-tuple Linguistic Aggregation Operator for Multiple Attributes Group Decision Making*, International Journal of Analysis and Applications 17(1) (2019) 76-104.

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Several Types of $\mathcal{B}^{\#}$-closed Sets in Ideal Nanotopological Spaces

Raghavan Asokan[1], **Ochanan Nethaji**[2], **Ilangovan Rajasekaran**[3]

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Abstract — In this paper, we made an attempt to unveil to notions of nano $\mathcal{B}_g^{\#}$-closed sets and $I_{\mathcal{B}_g^{\#}}$-closed sets are introduce and their properties are discussed with suitable examples. They are characterizations in the context of an ideal nanotopological spaces.

Keywords — *Nano $\mathcal{B}^{\#}$-set, nano $t^{\#}$-set, nano $\mathcal{B}_{gs}^{\#}$-set and nano $I_{\mathcal{B}_g^{\#}}$-closed set.*

1. Introduction

An ideal I [1] on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

1. $A \in I$ and $B \subset A$ imply $B \in I$ and
2. $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a space (X, τ) with an ideal I on X if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [2]. The closure operator defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [3] is a Kuratowski closure operator which generates a topology $\tau^\star(I, \tau)$ called the $\star$-topology which is finer then τ. We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or an ideal space.

Some new notions in the concept of ideal nano topological spaces were introduced by Parimala et al. [4,5] and Rajasekaran et.al [6] were introduced nano $\mathcal{B}^{\#}$-set and nano $t^{\#}$-set.

In this paper, we made an attempt to unveil to notions of nano $\mathcal{B}_g^{\#}$-closed sets and $I_{\mathcal{B}_g^{\#}}$-closed sets are introduce and their properties are discussed with suitable examples. They are characterizations in the context of an ideal nanotopological spaces.

[1]rasoka_mku@yahoo.co.in; [2]jionetha@yahoo.com; [3]sekarmelakkal@gmail.com(Corresponding Author)
[1]Dept. of Mathematics, School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India.
[2]Department of Mathematics, Kamaraj College, Tuticorin - 628 003, Tamilnadu, India.
[3]Dept. of Mathematics, Tirunelveli Dakshina Mara Nadar Sangam College, T. Kallikulam, Tirunelveli Dt, Tamil Nadu, India.

2. Preliminaries

Definition 2.1. [7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.

3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [8] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$.

2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by $n\text{-}int(A)$ and $n\text{-}cl(A)$, respectively.

Definition 2.3. A subset A of a space $(U, \mathcal{N})$ is called a

1. nano semi-open [8] if $H \subseteq n\text{-}cl(n\text{-}int(H))$.

2. nano pre-open [8] if $H \subseteq n\text{-}int(n\text{-}cl(H))$.

The complements of the above mentioned sets are called their respective closed sets.

Definition 2.4. A subset H of a space $(U, \mathcal{N})$ is called a

1. nano g-closed (briefly, ng-closed) [9] if $n\text{-}cl(H) \subseteq G$, whenever $H \subset G$ and G is n-open.

2. nano gp-closed (briefly, ngp-closed) [10] if $n\text{-}pcl(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.

3. nano gs-closed (briefly, ngs-closed) [11] if $n\text{-}scl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano semi-open.

The complements of the above mentioned sets are called their respective closed sets.

Definition 2.5. [6] A subset H of a space $(U, \tau_R(X))$ is called a

1. nano $t^{\#}$-set (briefly, $nt^{\#}$-set) if $n\text{-}int(H) = n\text{-}cl(n\text{-}int(H))$.

2. nano $\mathcal{B}^{\#}$-set (briefly, $n\mathcal{B}^{\#}$-set) if $H = P \cap Q$, where P is n-open and Q is $nt^{\#}$-set.

Remark 2.6. [6] In a space $(U, \mathcal{N})$, each n-open set is $n\mathcal{B}^{\#}$-set.

Theorem 2.7. In a space $(U, \mathcal{N})$,

1. each n-closed set is ng-closed set. [9]

2. each ng-closed set is ngp-closed set. [10]
3. each ng-closed set is ngs-closed set. [11]

A nano topological space $(U,\mathcal{N})$ with an ideal I on U is called [4] an ideal nano topological space and is denoted by $(U,\mathcal{N},I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes [4] the family of nano open sets containing x.

In future an ideal nano topological spaces $(U,\mathcal{N},I)$ is referred as a space.

Definition 2.8. [4] Let $(U,\mathcal{N},I)$ be a space with an ideal I on U. Let $(.)^\star_n$ be a set operator from $\wp(U)$ to $\wp(U)$ ($\wp(U)$ is the set of all subsets of U). For a subset $A \subseteq U$, $A^\star_n(I,\mathcal{N}) = \{x \in U : G_n \cap A \notin I,$ for every $G_n \in G_n(x)\}$ is called the nano local function (briefly, n-local function) of A with respect to I and $\mathcal{N}$. We will simply write $A^\star_n$ for $A^\star_n(I,\mathcal{N})$.

Theorem 2.9. [4] Let $(U,\mathcal{N},I)$ be a space and A and B be subsets of U. Then

1. $A \subseteq B \Rightarrow A^\star_n \subseteq B^\star_n$.
2. $A^\star_n = n\text{-}cl(A^\star_n) \subseteq n\text{-}cl(A)$ ($A^\star_n$ is a n-closed subset of $n\text{-}cl(A)$).
3. $(A^\star_n)^\star_n \subseteq A^\star_n$.
4. $(A \cup B)^\star_n = A^\star_n \cup B^\star_n$.
5. $V \in \mathcal{N} \Rightarrow V \cap A^\star_n = V \cap (V \cap A)^\star_n \subseteq (V \cap A)^\star_n$.
6. $J \in I \Rightarrow (A \cup J)^\star_n = A^\star_n = (A - J)^\star_n$.

Theorem 2.10. [4] Let $(U,\mathcal{N},I)$ be a space with an ideal I and $A \subseteq A^\star_n$, then $A^\star_n = n\text{-}cl(A^\star_n) = n\text{-}cl(A)$.

Definition 2.11. [4] Let $(U,\mathcal{N},I)$ be a space. The set operator $n\text{-}cl^\star$ called a nano $\star$-closure is defined by $n\text{-}cl^\star(A) = A \cup A^\star_n$ for $A \subseteq X$.

It can be easily observed that $n\text{-}cl^\star(A) \subseteq n\text{-}cl(A)$.

Theorem 2.12. [5] In a space $(U,\mathcal{N},I)$, if A and B are subsets of U, then the following results are true for the set operator $n\text{-}cl^\star$.

1. $A \subseteq n\text{-}cl^\star(A)$.
2. $n\text{-}cl^\star(\phi) = \phi$ and $n\text{-}cl^\star(U) = U$.
3. $If A \subset B$, then $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(B)$.
4. $n\text{-}cl^\star(A) \cup n\text{-}cl^\star(B) = n\text{-}cl^\star(A \cup B)$.
5. $n\text{-}cl^\star(n\text{-}cl^\star(A)) = n\text{-}cl^\star(A)$.

Definition 2.13. [12] A subset A of a space $(U,\mathcal{N},I)$ is $n\star$-dense in itself (resp. $n\star$-perfect and $n\star$-closed) if $A \subseteq A^\star_n$ (resp. $A = A^\star_n$, $A^\star_n \subseteq A$).

Definition 2.14. [13] A subset A of a space $(U,\mathcal{N},I)$ is called a weakly nano I-locally closed set (briefly, $\mathcal{W}$-nI-LC) if $A = P \cap Q$ where P is n-open and Q is $n\star$-closed.

Definition 2.15. [12] A subset A of a space $(U,\mathcal{N},I)$ is called a nano I_g-closed (briefly nI_g-closed) if $A^\star_n \subseteq B$ whenever $A \subseteq B$ and B is n-open.

Theorem 2.16. [13] For a subset A of a space $(U,\mathcal{N},I)$, the following are equivalent,

1. A is $\mathcal{W}$-nI-LC,
2. $A = G \cap n\text{-}cl^\star(A)$ for few n-open set G,
3. $n\text{-}cl^\star(A) - A = A^\star_n - A$ is n-closed,
4. $(U - A^\star_n) \cup A = A \cup (U - n\text{-}cl^\star(A))$ is n-open,
5. $A \subset n\text{-}int(A \cup (U - A^\star_n))$.

3. On nano $I_{\mathcal{B}_g^{\#}}$-closed sets

Definition 3.1. A subset H of a space $(U, \mathcal{N})$ is called a

1. nano $\mathcal{B}_g^{\#}$-closed set if (briefly, $n\mathcal{B}_g^{\#}$-closed set) $n\text{-}cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $n\mathcal{B}^{\#}$-set. The complement of $n\mathcal{B}_g^{\#}$-open if $H^c = U - H$ is $n\mathcal{B}_g^{\#}$-closed.
2. nano $\mathcal{B}_g^{\#}$-set if (briefly, $n\mathcal{B}_g^{\#}$-set) $H = P \cap Q$ where P is ng-open and $nt^{\#}$-set.
3. nano $\mathcal{B}_{gs}^{\#}$-set if (briefly, $n\mathcal{B}_{gs}^{\#}$-set) $H = P \cap Q$ where P is ngs-open and $nt^{\#}$-set.
4. nano $\mathcal{B}_{gp}^{\#}$-set if (briefly, $n\mathcal{B}_{gp}^{\#}$-set) $H = P \cap Q$ where P is ngp-open and $nt^{\#}$-set.

Remark 3.2. The diagram holds for any subset of a space $(U, \mathcal{N})$:

$$\begin{array}{ccccc} & & n\mathcal{B}^{\#}\textbf{-set} & & \\ & & \downarrow & & \\ n\mathcal{B}_{gp}^{\#}\textbf{-set} & \longleftarrow & n\mathcal{B}_g^{\#}\textbf{-set} & \longrightarrow & n\mathcal{B}_{gs}^{\#}\textbf{-set} \end{array}$$

In this diagram, none of the implications are reversible.

Example 3.3. Let $U = \{a, b, c, d, e\}$ with $U/R = \{\{e\}, \{a, b\}, \{c, d\}\}$ and $X = \{b, e\}$. Then $\mathcal{N} = \{\phi, U, \{e\}, \{a, b\}, \{a, b, e\}\}$. Let the ideal be $I = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Then

1. $\{c\}$ is $n\mathcal{B}_g^{\#}$-set but not $n\mathcal{B}^{\#}$-set.
2. $\{b, c, d, e\}$ is $n\mathcal{B}_{gp}^{\#}$-set and $n\mathcal{B}_{gs}^{\#}$-set but not $n\mathcal{B}_g^{\#}$-set.

Definition 3.4. A subset H of a space $(U, \mathcal{N}, I)$ is called a

1. $nI_{\mathcal{B}_g^{\#}}$-closed set if $H_n^{\star} \subseteq P$ whenever $H \subseteq P$ and P is $n\mathcal{B}^{\#}$-set.
2. $nI_{\mathcal{B}_{gs}^{\#}}$-closed set if $H_n^{\star} \subseteq P$ whenever $H \subseteq P$ and P is $n\mathcal{B}_{gs}^{\#}$-set.
3. $nI_{\mathcal{B}_{gp}^{\#}}$-closed set if $H_n^{\star} \subseteq P$ whenever $H \subseteq P$ and P is $n\mathcal{B}_{gp}^{\#}$-set.
4. nI_{gp}-closed set if $H_n^{\star} \subseteq P$ whenever $H \subseteq P$ and P is ngp-open.
5. nI_{gs}-closed set if $H_n^{\star} \subseteq P$ whenever $H \subseteq P$ and P is ngs-open.

The complements of the above mentioned closed sets are called their respective open sets.

Theorem 3.5. Let $(U, \mathcal{N}, I)$ be a space and $H \subseteq U$, then

1. H is $n\star$-closed $\Rightarrow$ H is $nI_{\mathcal{B}_g^{\#}}$-closed.
2. H is $nI_{\mathcal{B}_g^{\#}}$-closed $\Rightarrow$ H is nI_g-closed.
3. H is $nI_{\mathcal{B}_{gp}^{\#}}$-closed $\Rightarrow$ H is $nI_{\mathcal{B}_g^{\#}}$-closed.
4. H is $nI_{\mathcal{B}_{gs}^{\#}}$-closed $\Rightarrow$ H is $nI_{\mathcal{B}_g^{\#}}$-closed.
5. H is $nI_{\mathcal{B}_{gs}^{\#}}$-closed $\Rightarrow$ H is nI_{gs}-closed.
6. H is $nI_{\mathcal{B}_{gp}^{\#}}$-closed $\Rightarrow$ H is nI_{gp}-closed.
7. H is nI_{gp}-closed $\Rightarrow$ H is nI_g-closed.
8. H is nI_{gs}-closed $\Rightarrow$ H is nI_g-closed.

PROOF. 1. Let H be $n\star$-closed set and P be $n\mathcal{B}^{\#}$-set in U such that $H \subseteq P$. Since H is $n\star$-closed, $H_n^\star \subseteq H$, so $H_n^\star \subseteq P$. Hence H is $nI_{\mathcal{B}_g^{\#}}$-closed set.

2. Let H be $nI_{\mathcal{B}_g^{\#}}$-closed set and $H \subseteq P$ where $P \in \mathcal{N}$. Since each n-open set is $n\mathcal{B}^{\#}$-set, so P is $n\mathcal{B}^{\#}$-set. Since H is $nI_{\mathcal{B}_g^{\#}}$-closed set, we obtain that $H_n^\star \subseteq P$ and hence H is nI_g-closed set.
3. It follows from Remark 3.2 and Definition 3.4.
4. It follows from Remark 3.2 and Definition 3.4.
5. Let $H \subseteq P$ where P is ngs-open set in U. Since each ngs-open set is $n\mathcal{B}_{gs}^{\#}$-set, so P is $n\mathcal{B}_{gs}^{\#}$-set. Since H is $nI_{\mathcal{B}_{gs}^{\#}}$-closed set, we have $H_n^\star \subseteq P$. Hence H is nI_{gs}-closed set.
6. Let $H \subseteq P$ where P is ngp-open set in U. Since each ngp-open set is $n\mathcal{B}_{gp}^{\#}$-set, so P is $n\mathcal{B}_{gp}^{\#}$-set. Since H is $nI_{\mathcal{B}_{gp}^{\#}}$-closed set, we have $H_n^\star \subseteq P$. Hence H is nI_{gp}-closed set.
7. It follows from Theorem 2.7 and Definition 3.4(4).
8. It follows from Theorem 2.7 and Definition 3.4(5).

Remark 3.6. These relations are shown in the diagram.

$$\begin{array}{ccccc}
 & & n\star\text{-closed} & & \\
 & & \downarrow & & \\
nI_{\mathcal{B}_{gp}^{\#}}\text{-closed} & \longrightarrow & nI_{\mathcal{B}_{g}^{\#}}\text{-closed} & \longleftarrow & nI_{\mathcal{B}_{gs}^{\#}}\text{-closed} \\
\downarrow & & \downarrow & & \downarrow \\
nI_{gp}\text{-closed} & \longrightarrow & nI_{g}\text{-closed} & \longleftarrow & nI_{gs}\text{-closed}
\end{array}$$

The converses of each statement in Theorem 3.5 are not true as shown in the following Examples.

Example 3.7. In Example 3.3, Then

1. $\{c\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed but not $n\star$-closed.
2. $\{d\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed but not $nI_{\mathcal{B}_{gp}^{\#}}$-closed.
3. $\{a, c\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed but not $nI_{\mathcal{B}_{gs}^{\#}}$-closed.
4. $\{b, c, d\}$ is nI_{gp}-closed but not $nI_{\mathcal{B}_{gp}^{\#}}$-closed.
5. $\{a, e\}$ is nI_{gs}-closed but not $nI_{\mathcal{B}_{gs}^{\#}}$-closed.
6. $\{c\}$ is nI_g-closed but not nI_{gp}-closed.
7. $\{d\}$ is nI_g-closed but not nI_{gs}-closed.

Example 3.8. Let $U = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{d\}, \{a, c\}\}$ and $X = \{c, d\}$. Then $\mathcal{N} = \{\phi, U, \{d\}, \{a, c\}, \{a, c, d\}\}$. Let the ideal be $I = \{\phi, \{d\}\}$. Then
$\{b, c\}$ is nI_g-closed but not $nI_{\mathcal{B}_g^{\#}}$-closed.

Remark 3.9. The following Example shows that the family of $n\mathcal{B}^{\#}$-sets and the family of $nI_{\mathcal{B}_g^{\#}}$-closed sets are independent of a space $(U, \mathcal{N}, I)$.

Example 3.10. In Example 3.3, then

1. $\{e\}$ is $n\mathcal{B}^{\#}$-set but not $nI_{\mathcal{B}_g^{\#}}$-closed.
2. $\{a\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed but not $n\mathcal{B}^{\#}$-set.

Theorem 3.11. If H is both $n\mathcal{B}^{\#}$-set and $nI_{\mathcal{B}_g^{\#}}$-closed set, then H is $n\star$-closed.

Proof. Let H be a both $n\mathcal{B}^{\#}$-set and $nI_{\mathcal{B}_g^{\#}}$-closed set. Then $H_n^{\star} \subseteq H$, whenever H is $n\mathcal{B}^{\#}$-set and $H \subseteq H$. Hence H is $n\star$-closed.

Theorem 3.12. If A and B are $nI_{\mathcal{B}_g^{\#}}$-closed sets, then $A \cup B$ is $nI_{\mathcal{B}_g^{\#}}$-closed set.

Proof. Let $A \cup B \subseteq U$, where U is $n\mathcal{B}^{\#}$-set. Since A and B are $nI_{\mathcal{B}_g^{\#}}$-closed, $A_n^{\star} \subseteq U$ and $B_n^{\star} \subseteq U$, whenever $A \subseteq U, B \subseteq U$ and U is $n\mathcal{B}^{\#}$-set. Therefore, $(A \cup B)_n^{\star} = A_n^{\star} \cup B_n^{\star} \subseteq U$. Hence $A \cup B$ is $nI_{\mathcal{B}_g^{\#}}$-closed set.

Remark 3.13. The intersection of two $nI_{\mathcal{B}_g^{\#}}$-closed sets but not $nI_{\mathcal{B}_g^{\#}}$-closed set.

Example 3.14. In Example 3.3, $H = \{c, e\}$ and $K = \{d, e\}$ are $nI_{\mathcal{B}_g^{\#}}$-closed. But $H \cap K = \{e\}$ is not $nI_{\mathcal{B}_g^{\#}}$-closed.

Theorem 3.15. If A is $nI_{\mathcal{B}_g^{\#}}$-closed set such that $A \subseteq B \subseteq A_n^{\star}$, then B is also $nI_{\mathcal{B}_g^{\#}}$-closed set.

Proof. Let G be $n\mathcal{B}^{\#}$-set in U such that $B \subseteq G$. Then $A \subseteq G$. Since A is $nI_{\mathcal{B}_g^{\#}}$-closed set, $A_n^{\star} \subseteq G$. Now $B_n^{\star} \subseteq (A_n^{\star})_n^{\star} \subseteq A_n^{\star} \subseteq G$. Therefore B is also $nI_{\mathcal{B}_g^{\#}}$-closed set.

Proposition 3.16. For any space $(U, \mathcal{N}, I)$, each singleton $\{x\}$ of U is $n\mathcal{B}^{\#}$-set.

Proof. Let $x \in U$. If $\{x\} \in \mathcal{N}$, then $\{x\}$ is $n\mathcal{B}^{\#}$-set. If $\{x\} \notin \mathcal{N}$, then $n\text{-}int(\{x\}) = \phi = n\text{-}cl(n\text{-}int(\{x\}))$, so $\{x\}$ is $n\mathcal{B}^{\#}$-set.

Corollary 3.17. For each $x \in U, \{x\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed set if and only if {x} is $n\star$-closed set.

Proof. Necessity: Let $\{x\}$ be $nI_{\mathcal{B}_g^{\#}}$-closed set. Since $\{x\}$ is both $n\mathcal{B}^{\#}$-set and $nI_{\mathcal{B}_g^{\#}}$-closed set, then $\{x\}$ is $n\star$-closed.

Sufficiency: Let $\{x\}$ be $n\star$-closed set. We know that each $n\star$-closed set is $nI_{\mathcal{B}_g^{\#}}$-closed set. Therefore $\{x\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed set.

Theorem 3.18. Let A be $nI_{\mathcal{B}_g^{\#}}$-closed set. Then $A_n^{\star} - A$ does not contain any non-empty complement of $n\mathcal{B}^{\#}$-set.

Proof. Let A be $nI_{\mathcal{B}_g^{\#}}$-closed set. Suppose that F is the complement of $n\mathcal{B}^{\#}$-set and $F \subseteq A_n^{\star} - A$. Since $F \subseteq A_n^{\star} - A \subseteq U - A, A \subseteq U - F$ and $U - F$ is $n\mathcal{B}^{\#}$-set. Therefore, $A_n^{\star} \subseteq U - F$ and $F \subseteq U - A_n^{\star}$. However, since $F \subseteq A_n^{\star} - A$, we have $F = \phi$.

Theorem 3.19. For a subset A of a space $(U, \mathcal{N}, I)$, the following are equivalent.

1. A is $n\star$-closed.
2. A is $\mathcal{W}$-nI-LC and $nI_{\mathcal{B}_g^{\#}}$-closed.

Proof. (1)⇒(2). Obvious.

(2)⇒(1). Since A is $\mathcal{W}$-nI-LC, by Theorem 2.16, $A = G \cap n\text{-}cl^{\star}(A)$, where G is n-open in U. So, $A \subseteq G$ and G is $n\mathcal{B}^{\#}$-set in U. Since A is $nI_{\mathcal{B}_g^{\#}}$-closed, $A_n^{\star} \subseteq G$ and $A_n^{\star} \cup A \subseteq G$. Therefore $n\text{-}cl^{\star}(A) \subseteq G \cap n\text{-}cl^{\star}(A) = A$. Hence A is $n\star$-closed set in U.

Remark 3.20. The following Example shows that the family of $\mathcal{W}$-nI-LC and the family of $nI_{\mathcal{B}_g^{\#}}$-closed sets are independent of a space $(U, \mathcal{N}, I)$.

Example 3.21. In Example 3.3, then

1. $\{e\}$ is $\mathcal{W}$-nI-LC but not $nI_{\mathcal{B}_g^{\#}}$-closed.
2. $\{c\}$ is $nI_{\mathcal{B}_g^{\#}}$-closed but not $\mathcal{W}$-nI-LC.

Theorem 3.22. Let $(U, \mathcal{N}, I)$ be a space and $A \subseteq U$. Then A is $nI_{\mathcal{B}_g^{\#}}$-open if and only if $F \subseteq n\text{-}int^{\star}(A)$ whenever F is the complement of $n\mathcal{B}^{\#}$-set and $F \subseteq A$.

Proof. Suppose A is $nI_{\mathcal{B}_g^{\#}}$-open. If F is the complement of $n\mathcal{B}^{\#}$-set and $F \subseteq A$, then $U - A \subseteq U - F$ and so $(U - A)_n^{\star} \subseteq U - F$ and $[(U - A) \cup (U - A)_n^{\star}] \subseteq [U - F] \cup [U - A]$. Hence $n\text{-}cl^{\star}(U - A) \subseteq U - F$. Therefore $F \subseteq n\text{-}int^{\star}(A)$.

Conversely, suppose the condition holds. Let G be $n\mathcal{B}^{\#}$-set such that $U - A \subseteq G$. Then $U - G \subseteq A$ and $U - G$ is the complement of $n\mathcal{B}^{\#}$-set. By assumption, $U - G \subseteq n\text{-}int^{\star}(A)$ which implies that $n\text{-}cl^{\star}(U - A) \subseteq G$ and $(U - A)_n^{\star} \subseteq G$. Therefore $U - A$ is $nI_{\mathcal{B}_g^{\#}}$-closed and so A is $nI_{\mathcal{B}_g^{\#}}$-open.

Theorem 3.23. Let $(U, \mathcal{N}, I)$ be a space and $A \subseteq U$. If A is $nI_{\mathcal{B}_g^{\#}}$-open and $n\text{-}int^{\star}(A) \subseteq B \subseteq A$, then B is $nI_{\mathcal{B}_g^{\#}}$-open.

Proof. It follows from Theorem 3.15.

Theorem 3.24. Let $(U, \mathcal{N}, I)$ be a space. Then each subset of U is $nI_{\mathcal{B}_g^{\#}}$-closed if and only if each $n\mathcal{B}^{\#}$-set is $n\star$-closed.

Proof. ($\Rightarrow$) Let $G \subseteq U$ be any $n\mathcal{B}^{\#}$-set. Since G is both $n\mathcal{B}^{\#}$-set and $nI_{\mathcal{B}_g^{\#}}$-closed, by Theorem 3.11, G is $n\star$-closed.

($\Leftarrow$) Let $A \subseteq U$ and G is $n\mathcal{B}^{\#}$-set such that $A \subseteq G$, then $A_n^{\star} \subseteq G_n^{\star} \subseteq G$. Therefore A is $nI_{\mathcal{B}_g^{\#}}$-closed set.

References

[1] R. Vaidyanathaswamy, *Set Topology*, Chelsea Publishing Company, New York, 1946.

[2] K. Kuratowski, *Topology*, Vol I. Academic Press (New York) 1966.

[3] R. Vaidyanathaswamy, *The Localization Theory in Set Topology*, Proc. Indian Acad. Sci. 20 (1945) 51-61.

[4] M. Parimala, T. Noiri, S. Jafari, *New types of nano topological spaces via nano ideals*, https://www.researchgate.net/publication/315892279.

[5] M. Parimala, S. Jafari, *On some new notions in nano ideal topological spaces*, Eurasian Bulletin of Mathematics 1(3) (2018) 85-93.

[6] I. Rajasekaran, O. Nethaji, *A new form of some nano sets*, Bulletin of the international mathematical virtual institute 9(2) (2019) 355-360.

[7] Z. Pawlak, *Rough sets*, International journal of computer and Information Sciences 11(5) (1982) 341-356.

[8] M. L. Thivagar, C. Richard, *On nano forms of weakly open sets*, International Journal of Mathematics and Statistics Invention 1(1) (2013) 31-37.

[9] K. Bhuvaneshwari, K. Mythili Gnanapriya, *Nano Generalized closed sets*, International Journal of Scientific and Research Publications 4(5) (2014) 1-3.

[10] K. Bhuvaneswari, K. Mythili Gnanapriya, *On nano generalised pre closed sets and nano pre generalised closed sets in nano topological spaces*, International Journal of Innovative Research in Science, Engineering and Technology 3(10) (2014) 16825-16829.

[11] K. Bhuvaneswari, A. Ezhilarasi, *On nano semi-generalised and nano generalised-semi closed sets in nano topological spaces*, International Journal of Mathematics and Computer Applications Research (IJMCAR) 4(3) (2014) 117-124.

[12] M. Parimala, S. Jafari, S. Murali, *Nano ideal generalized closed sets in nano ideal topological spaces*, Annales Univ. Sci. Budapest 60 (2017) 3-11.

[13] O. Nethaji, R. Asokan, I. Rajasekaran, *Novel concept of ideal nanotopological spaces*, appear.

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Spherical Bipolar Fuzzy Sets and Its Application in Multi Criteria Decision Making Problem

Rayappan Princy [1], Krishnaswamy Mohana[2]

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Abstract − This paper introduces the concept of Spherical Bipolar Fuzzy sets (SBFS) as a combination of Spherical Fuzzy sets and Bipolar Valued Fuzzy Sets along with their properties. Arithmetic operations involving addition, multiplication and subtraction are presented together with their proofs. A multi criteria decision making method is established in which the evaluation values of alternatives respective to criteria are represented in SBFS. Finally, a numerical example shows the application of the proposed method.

Keywords − *Intuitionistic fuzzy set (IFS), IFS2, neutrosophic fuzzy set (NFS), bipolar valued fuzzy set, spherical fuzzy set*

1.Introduction

Researchers have introduced many extensions and generalizations of fuzzy sets in the literature. It starts from ordinary fuzzy sets and extends to recently developed types of fuzzy sets [1-11].

Atanassov's intuitionistic fuzzy sets of second type (IFS2) are characterized by a membership degree and a non- membership degree satisfying the condition that the square sum of its membership degree and non-membership degree is equal to or less than one, which is a generalization of Intuitionistic Fuzzy Sets (IFS). The motivation of introducing IFS2 is that in the real-life decision process, the sum of the support (membership) degree and the against (non- membership) degree to which an alternative satisfying a criterion provided by the decision maker may be larger than 1 but their square sum is equal to or less than 1 [12,13]. Third dimension of IFS2 is hesitancy degree, which can be calculated by $\pi_{\tilde{p}} = \left(1 - \mu_{\tilde{p}}{}^2(u) - \vartheta_{\tilde{p}}{}^2(u)\right)^{1/2}$. In the recent years IFS2 have been employed in the solutions of multi-criteria decision making problems [14,15].

Similar to IFS2, Smarandache's neutrosophic sets (NS) are represented by the following three dimensions: a truthness degree, an indeterminacy degree, and a falsity degree. NS do not deal with the hesitancy of a system but also decrease indecisiveness of inconsistent information [1,3]. Also, various authors have given their contributions towards Neutrosophic using various new methods and ideas [16 -20].

Based on the analogy with the spherical coordinates of two points, Antonov [21] introduced the new stereo metrical interpretation of the IFS elements and gave the geometrical interpretation of the distance

[1]princy.pjs@gmail.com (Corresponding Author); [2]riyaraju1116@gmail.com
[1]Nirmala College for Women, Coimbatore, India
[2]Nirmala College for Women, Coimbatore, India

between two intuitionistic fuzzy points. Yang and Chiclana [9] proposed a spherical representation, which allowed us to define a distance function between intuitionistic fuzzy sets. They have showed that the spherical distance is different from the existing distances in that it is non- linear with respect to the change of the corresponding fuzzy membership degrees, and thus it seems more appropriate than usual linear distances for non- linear contexts in 3D spaces. Their work is just on the usage of IFS on a sphere. On the surface of a sphere, the following condition is proposed:

Let $\tilde{A} = \{\langle u, \mu_{\tilde{A}}(u), \vartheta_{\tilde{A}}(u) \rangle : u \in U\}$ be an intuitionistic fuzzy set. They have,

$$\mu_{\tilde{A}} + \vartheta_{\tilde{A}} + \pi_{\tilde{A}} = 1\#(1)$$

Which can be equivalently transformed to

$$x^2 + y^2 + z^2 = 1\#(2)$$

where, $x^2 = \mu_{\tilde{A}}(u)$, $y^2 = \vartheta_{\tilde{A}}(u)$, $z^2 = \pi_{\tilde{A}}(u)$.

In the spherical representation, hesitancy can be calculated based on the given membership and non-membership values since they only consider the surface of the sphere. Besides, they measure the spherical arc distance between two IFSs. Furthermore, Gong et al. [22] introduced an approach generalizing Yang and Chiclana's work. They applied the spherical distance measure to obtain the difference between two IFSs. They first introduced an ideal opinion and each individual opinion in group decision, they briefly constructed a non- linear optimization model.

The spherical fuzzy sets introduced in this paper are based on the fact that the hesitancy of a decision maker can be defined independently from membership and non-membership degrees, satisfying the following condition:

$$0 \le {\mu_{\tilde{A}}}^2(u) + {\vartheta_{\tilde{A}}}^2(u) + {\pi_{\tilde{A}}}^2(u) \le 1\#(3)$$

On the surface of the sphere, Equation (3) becomes

$${\mu_{\tilde{A}}}^2(u) + {\vartheta_{\tilde{A}}}^2(u) + {\pi_{\tilde{A}}}^2(u) = 1, \forall\, u \in U\#(4)$$

Since Yang and Chiclana [9] and Gong et al. [22] measure the arc distance on the surface of the sphere, Euclidean distance is not measured in these works. In our spherical fuzzy sets approach, the sphere is not solid but a spherical volume. Based on this fact, Euclidean distance measurement is meaningful. This also means that any two points within the spherical volume are also on the surface of another sphere. Euclidean distance gives the shortest distance between two points in the sphere.

Bipolar-valued fuzzy sets, which was introduced by Lee [24,25] is an extension of fuzzy sets whose membership degree range is extended from the interval $[0,1]$ to $[-1,1]$. The membership degrees of the Bipolar valued fuzzy sets signify the degree of satisfaction to the property analogous to a fuzzy set and its counter-property in a bipolar valued fuzzy set, if the membership degree is 0 it means that the elements are unrelated to the corresponding property. Furthermore, if the membership degree is on $(0,1]$ it indicates that the elements somewhat fulfil the property, and if the membership degree is on $[-1,0)$ it indicates that elements somewhat satisfy the entire counter property.

In this paper, we introduce the SFS with its basic operations such as addition, subtraction, multiplication and aggregation. The idea behind SFS is to let decision makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assign the parameters of that membership function with a larger domain.

2. Preliminaries

Definition 2.1. [12,13] Let a set U be a universe of discourse. An IFS $\tilde{A}$ is an object having the form, $\tilde{A} = \{\langle u, \mu_{\tilde{A}}(u), \vartheta_{\tilde{A}}(u) \rangle \mid u \in U\}$ where the function $\mu_{\tilde{A}}: U \to [0,1], \vartheta_{\tilde{A}}: U \to [0,1]$ and $0 \le \mu_{\tilde{A}}(u) + \vartheta_{\tilde{A}}(u) \le 1$ are degree of membership, non-membership of u to $\tilde{A}$, respectively. For any IFS $\tilde{A}$ and $u \in U$, $\pi_{\tilde{A}} = 1 - \mu_{\tilde{A}}(u) - \vartheta_{\tilde{A}}(u)$ is called degree of hesitancy of u to $\tilde{A}$.

However, in the real life, the decision makers might express their preferences about membership degrees and non-membership degrees of an alternative with respect to a criterion dissatisfies the condition that the sum of the membership and non-membership degrees should be less than or equal to1.Instead of requiring the decision makers to alter their preference information, Atanassov [12] proposed a novel concept of IFS2 (intuitionistic fuzzy sets of second type) to model this situation. This concept provides a larger preference area for decision makers.

Definition 2.2. [12,13] Let a set U be a universe of discourse. An IFS2 $\tilde{P}$ is an object having the form, $\tilde{P} = \{\langle u, \mu_{\tilde{P}}(u), \vartheta_{\tilde{P}}(u) \rangle \mid u \in U\}$ where the function $\mu_{\tilde{P}}: U \to [0,1], \vartheta_{\tilde{P}}: U \to [0,1]$ and $0 \le \mu_{\tilde{P}}(u) + \vartheta_{\tilde{P}}(u) \le 1$ are degree of membership, non-membership of u to $\tilde{P}$, respectively.

For any IFS2 $\tilde{A}$ and $u \in U, \pi_{\tilde{P}} = 1 - \mu_{\tilde{P}}(u) - \vartheta_{\tilde{P}}(u)$ is called degree of hesitancy of u to $\tilde{P}$.

The novel concept of SFS (Spherical Fuzzy Sets) provides a larger preference domain for decision makers and DM can define their hesitancy information about of an alternative with respect to a criterion.

Intuitionistic and IFS2 fuzzy membership functions are composed of membership, non-membership and hesitancy parameters, which can be calculated by $\pi_{\tilde{I}} = 1 - \mu - \vartheta$ or $\pi_{\tilde{P}} = \left(1 - \mu_{\tilde{P}}{}^2(u) - \vartheta_{\tilde{P}}{}^2(u)\right)^{1/2}$, respectively. Neutrosophic membership functions are also defined by three parameters truthiness, falsity and indeterminacy, whose sum can be between 0 and 3, and the value of each is between 0 and 1 independently. In spherical fuzzy sets, while the squared sum of membership, non-membership and hesitancy parameters can be between 0 and, each of them can be defined between 0 and 1 independently.

Definition 2.3. [26] A Spherical Fuzzy Set (SFS) $\tilde{A}_s$ of the universe of discourse U is given by,

$$\tilde{A}_s = \{\langle \mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u), \pi_{\tilde{A}_s}(u) | u \in U \rangle\} \#(5)$$

where, $\mu_{\tilde{A}_s}: U \to [0,1], \vartheta_{\tilde{A}_s}: U \to [0,1], \pi_{\tilde{A}_s}: U \to [0,1]$ and

$$0 \le \mu_{\tilde{A}_s}{}^2(u) + \vartheta_{\tilde{A}_s}{}^2(u) + \pi_{\tilde{A}_s}{}^2(u) \le 1, \forall\, u \in U \#(6)$$

For each u, the numbers $\mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u)\; and\; \pi_{\tilde{A}_s}(u)$ are the degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$, respectively.

Definition 2.4. [26] Basic operators of Spherical Fuzzy Sets:

Union:

$$\tilde{A}_s \cup \tilde{B}_s = \left\{ \begin{matrix} \max\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, \min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, \\ \min\left\{\left(1 - \left(\left(\max\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}\right)^2 + \left(\min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}\right)^2\right)\right)^{1/2}, \max\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\right\} \end{matrix} \right\}$$

Intersection:

$$\tilde{A}_s \cap \tilde{B}_s = \left\{ \begin{matrix} \min\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, \min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, \\ \max\left\{\left(1 - \left(\left(min\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}\right)^2 + \left(\max\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}\right)^2\right)\right)^{1/2}, \min\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\right\} \end{matrix} \right\}$$

Addition:

$$\tilde{A}_s \oplus \tilde{B}_s = \left\{ \begin{matrix} \left(\mu_{\tilde{A}_s}{}^2 + \mu_{\tilde{B}_s}{}^2 - \mu_{\tilde{A}_s}{}^2 \mu_{\tilde{B}_s}{}^2\right)^{1/2}, \vartheta_{\tilde{A}_s} \vartheta_{\tilde{B}_s} \\ \left(\left(1 - \mu_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s} + \left(1 - \mu_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s} - \pi_{\tilde{A}_s}\, \pi_{\tilde{B}_s}\right)^{1/2} \end{matrix} \right\}$$

Multiplication:

$$\tilde{A}_s \otimes \tilde{B}_s = \left\{ \begin{matrix} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s},\ \left(\vartheta_{\tilde{A}_s}{}^2 + \vartheta_{\tilde{B}_s}{}^2 - \vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)^{1/2}, \\ \left(\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s} + \left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s} - \pi_{\tilde{A}_s}\pi_{\tilde{B}_s}\right)^{1/2} \end{matrix} \right\}$$

Multiplication by a scalar; $\boldsymbol{\lambda > 0}$:

$$\lambda.\tilde{A}_s = \left\{ \begin{matrix} \left(1-\left(1-\mu_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2},\ \vartheta_{\tilde{A}_s}{}^{\lambda}, \\ \left(\left(1-\mu_{\tilde{A}_s}{}^2\right)^{\lambda} - \left(1-\mu_{\tilde{A}_s}{}^2 - \pi_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2} \end{matrix} \right\}$$

Power of $\boldsymbol{\tilde{A}_s}$; $\boldsymbol{\lambda > 0}$:

$$\tilde{A}_s{}^{\lambda} = \left\{ \begin{matrix} \mu_{\tilde{A}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2} \\ \left(\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda} - \left(1-\vartheta_{\tilde{A}_s}{}^2 - \pi_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2} \end{matrix} \right\}$$

Definition 2.5. Let X be the universe. Then a bipolar valued fuzzy sets A on X is defined by positive membership function $\mu_A^+ : X \to [0,1]$ and a negative membership function $\mu_A^- : X \to [-1,0]$. For sake of easiness, we shall practice the symbol $A = \{< x,\ \mu_A^+(x),\ \mu_A^-(x) > : x \in X\}$.

Definition 2.6. [24] Let A and B be two bipolar valued fuzzy sets then their union, intersection, and complement are well-defined as follows:

i. $\mu_{A\cup B}^+(x) = \max\{\mu_A^+(x), \mu_B^+(x)\}$

ii. $\mu_{A\cup B}^-(x) = \min\{\mu_A^-(x), \mu_B^-(x)\}$

iii. $\mu_{A\cap B}^+(x) = \min\{\mu_A^+(x), \mu_B^+(x)\}$

iv. $\mu_{A\cap B}^-(x) = \max\{\mu_A^-(x), \mu_B^-(x)\}$

v. $\mu_{\bar{A}}^+(x)$= 1-μ_A^+(x) and $\mu_{\bar{A}}^-(x) = -1 - \mu_A^+(x)$ for all $x \in X$.

3. Spherical Bipolar Fuzzy Sets

Definition 3.1. A spherical bipolar fuzzy set $\tilde{A}_s$ of the universe of discourse U is given by,

$$\tilde{A}_s = \left\{< u, \mu_{\tilde{A}_s}{}^+(\mathrm{u}), \vartheta_{\tilde{A}_s}{}^+(\mathrm{u}), \pi_{\tilde{A}_s}{}^+(\mathrm{u}), \mu_{\tilde{A}_s}{}^-(\mathrm{u}), \vartheta_{\tilde{A}_s}{}^-(\mathrm{u}), \pi_{\tilde{A}_s}{}^-(\mathrm{u}) | \mathrm{u} \in \mathrm{U}\right\}$$

where,

$\mu_{\tilde{A}_s}{}^+(u): U \to [0,1], \vartheta_{\tilde{A}_s}{}^+(u): U \to [0,1], \pi_{\tilde{A}_s}{}^+(u): U \to [0,1]$,

$\mu_{\tilde{A}_s}{}^-(u): U \to [-1,0], \vartheta_{\tilde{A}_s}{}^-(u): U \to [-1,0], \pi_{\tilde{A}_s}{}^-(u): U \to [-1,0]$ and

$0 \le \mu_{\tilde{A}_s}{}^{2^+}(u) + \vartheta_{\tilde{A}_s}{}^{2^+}(u) + \pi_{\tilde{A}_s}{}^{2^+}(u) \le 1,$

$-1 \le -\left(\mu_{\tilde{A}_s}{}^{2^-}(u) + \vartheta_{\tilde{A}_s}{}^{2^-}(u) + \pi_{\tilde{A}_s}{}^{2^-}(u)\right) \le 0\ \forall\ u \in U.$

For each u, the numbers $\mu_{\tilde{A}_s}{}^+(u), \vartheta_{\tilde{A}_s}{}^+(u), \pi_{\tilde{A}_s}{}^+(u)$ are the positive membership, non-membership and the hesitancy of u to $\tilde{A}_s$ and the numbers $\mu_{\tilde{A}_s}{}^-(u), \vartheta_{\tilde{A}_s}{}^-(u), \pi_{\tilde{A}_s}{}^-(u)$ are the negative degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$ respectively.

Definition 3.2. The basic operators of Spherical Bipolar Fuzzy sets are:

(i) Union

$$\tilde{A}_s \cup \tilde{B}_s = \left\{ \begin{array}{c} < \max\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right), \min\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right), \\ min\left\{\left(1-\left(\left(\max\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right)\right)^2 + \left(\min\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right)\right)^2\right)\right)^{1/2}, \max\left\{\pi_{\tilde{A}_s}{}^{+}, \pi_{\tilde{B}_s}{}^{+}\right\}\right\}, \\ \min\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right), \max\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right), \\ \max\left\{\left(1-\left(\left(\min\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right)\right)^2 + \left(\max\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right)\right)^2\right)\right)^{1/2}, \min\left\{\pi_{\tilde{A}_s}{}^{-}, \pi_{\tilde{B}_s}{}^{-}\right\}\right\} > \end{array} \right\}$$

(ii) Intersection

$$\tilde{A}_s \cap \tilde{B}_s = \left\{ \begin{array}{c} < \min\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right), \max\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right), \\ \max\left\{\left(1-\left(\left(\min\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right)\right)^2 + \left(\max\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right)\right)^2\right)\right)^{1/2}, \min\left\{\pi_{\tilde{A}_s}{}^{+}, \pi_{\tilde{B}_s}{}^{+}\right\}\right\}, \\ \max\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right), \min\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right), \\ \min\left\{\left(1-\left(\left(\max\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right)\right)^2 + \left(\min\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right)\right)^2\right)\right)^{1/2}, \max\left\{\pi_{\tilde{A}_s}{}^{-}, \pi_{\tilde{B}_s}{}^{-}\right\}\right\} > \end{array} \right\}$$

(iii) Complement

$$\tilde{A}_s{}^{c} = \left\{< u, 1-\mu_{\tilde{A}_s}{}^{+}(u), 1-\vartheta_{\tilde{A}_s}{}^{+}(u), 1-\pi_{\tilde{A}_s}{}^{+}(u), -1-\mu_{\tilde{A}_s}{}^{-}(u), -1-\vartheta_{\tilde{A}_s}{}^{-}(u), -1-\pi_{\tilde{A}_s}{}^{-}(u)\right\}, u \in U .$$

(iv) Addition

$$\tilde{A}_s \oplus \tilde{B}_s = \left\{ \begin{array}{c} \left(\mu_{\tilde{A}_s}{}^{2} + \mu_{\tilde{B}_s}{}^{2} - \mu_{\tilde{A}_s}{}^{2}\mu_{\tilde{B}_s}{}^{2}\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2} + \left(1-\mu_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2} - \pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \quad \mu_{\tilde{A}_s}{}^{-}\mu_{\tilde{B}_s}{}^{-}, \\ \left(\vartheta_{\tilde{A}_s}{}^{2^-} + \vartheta_{\tilde{B}_s}{}^{2^-} - \vartheta_{\tilde{A}_s}{}^{2^-}\vartheta_{\tilde{B}_s}{}^{2^-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2^-}\right)\pi_{\tilde{A}_s}{}^{2^-} + \left(1-\vartheta_{\tilde{A}_s}{}^{2^-}\right)\pi_{\tilde{B}_s}{}^{2^-} - \pi_{\tilde{A}_s}{}^{2^-}\pi_{\tilde{B}_s}{}^{2^-}\right]^{1/2} \end{array} \right\}$$

(v) Multiplication

$$\tilde{A}_s \otimes \tilde{B}_s = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \left(\vartheta_{\tilde{A}_s}{}^{2} + \vartheta_{\tilde{B}_s}{}^{2} - \vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2} + \left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2} - \pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \\ \left(\mu_{\tilde{A}_s}{}^{2^-} + \mu_{\tilde{B}_s}{}^{2^-} - \mu_{\tilde{A}_s}{}^{2^-}\mu_{\tilde{B}_s}{}^{2^-}\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^{-}\vartheta_{\tilde{B}_s}{}^{-}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2^-}\right)\pi_{\tilde{A}_s}{}^{2^-} + \left(1-\mu_{\tilde{A}_s}{}^{2^-}\right)\pi_{\tilde{B}_s}{}^{2^-} - \pi_{\tilde{A}_s}{}^{2^-}\pi_{\tilde{B}_s}{}^{2^-}\right]^{1/2} \end{array} \right\}$$

(vi) Multiplication by a scalar; $\lambda > 0$

$$\lambda . \tilde{A}_s = \left\{ \begin{array}{c} \left(1-\left(1-\mu_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \ \vartheta_{\tilde{A}_s}{}^{\lambda}, \\ \left(\left(1-\mu_{\tilde{A}_s}{}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{A}_s}{}^{2} - \pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, -\mu_{\tilde{A}_s}{}^{-\lambda}, \\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2^-}\right)^{\lambda}\right)^{1/2}, -\left(\left(1-\vartheta_{\tilde{A}_s}{}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{A}_s}{}^{2^-} - \pi_{\tilde{A}_s}{}^{2^-}\right)^{\lambda}\right)^{1/2} \end{array} \right\}$$

(vii) Power of $\tilde{A}_s$; $\lambda > 0$

$$\tilde{A}_s{}^{\lambda} = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \left(\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -\vartheta_{\tilde{A}_s}{}^{-\lambda}, \\ -\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)^{1/2} \end{array} \right\}$$

Definition 3.3. For these SBFS, $\tilde{A}_s = \{< \mu_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{A}_s}{}^{+}, \pi_{\tilde{A}_s}{}^{+}, \mu_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{A}_s}{}^{-}, \pi_{\tilde{A}_s}{}^{-}\}$ and $\tilde{B}_s = \{< \mu_{\tilde{B}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}, \pi_{\tilde{B}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}, \pi_{\tilde{B}_s}{}^{-}\}$, the following are valid under the condition λ, λ_1, $\lambda_2 > 0$.

i. $\tilde{A}_s \oplus \tilde{B}_s = \tilde{B}_s \oplus \tilde{A}_s$

ii. $\tilde{A}_s \otimes \tilde{B}_s = \tilde{B}_s \otimes \tilde{A}_s$

iii. $\lambda(\tilde{A}_s \oplus \tilde{B}_s) = \lambda\tilde{A}_s \oplus \lambda\tilde{B}_s$

iv. $\lambda_1\tilde{A}_s \oplus \lambda_2\tilde{A}_s = (\lambda_1+\lambda_2)\tilde{A}_s$

v. $(\tilde{A}_s \otimes \tilde{B}_s)^{\lambda} = \tilde{A}_s{}^{\lambda} \otimes \tilde{B}_s{}^{\lambda}$

vi. $\tilde{A}_s{}^{\lambda_1} \otimes \tilde{A}_s{}^{\lambda_2} = \tilde{A}_s{}^{\lambda_1+\lambda_2}$

Proof. According to Definition 3.2, we will prove (*i*) - (*iii*) and (*v*) since (*iv*) and (*vi*) are similar to the proofs of (*iii*) and (*v*) respectively.

i.

$$\tilde{A}_s \oplus \tilde{B}_s = \left\{ \begin{array}{c} \left(\mu_{\tilde{A}_s}{}^{2}+\mu_{\tilde{B}_s}{}^{2}-\mu_{\tilde{A}_s}{}^{2}\mu_{\tilde{B}_s}{}^{2}\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}+\left(1-\mu_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \mu_{\tilde{A}_s}{}^{-}\mu_{\tilde{B}_s}{}^{-}, \\ \left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}+\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

$$\tilde{B}_s \oplus \tilde{A}_s = \left\{ \begin{array}{c} \left(\mu_{\tilde{B}_s}{}^{2}+\mu_{\tilde{A}_s}{}^{2}-\mu_{\tilde{B}_s}{}^{2}\mu_{\tilde{A}_s}{}^{2}\right)^{1/2}, \vartheta_{\tilde{B}_s}\vartheta_{\tilde{A}_s}, \\ \left[\left(1-\mu_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}+\left(1-\mu_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\pi_{\tilde{A}_s}{}^{2}\right]^{1/2}, \mu_{\tilde{B}_s}{}^{-}\mu_{\tilde{A}_s}{}^{-}, \\ \left(\vartheta_{\tilde{B}_s}{}^{2-}+\vartheta_{\tilde{A}_s}{}^{2-}-\vartheta_{\tilde{B}_s}{}^{2-}\vartheta_{\tilde{A}_s}{}^{2-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\pi_{\tilde{A}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

And so, $\tilde{A}_s \oplus \tilde{B}_s = \tilde{B}_s \oplus \tilde{A}_s$.

ii.

$$\tilde{A}_s \otimes \tilde{B}_s = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \left(\vartheta_{\tilde{A}_s}{}^{2}+\vartheta_{\tilde{B}_s}{}^{2}-\vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}+\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \\ \left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^{-}\vartheta_{\tilde{B}_s}{}^{-}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}+\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

$$\tilde{B}_s \otimes \tilde{A}_s = \left\{ \begin{array}{c} \mu_{\tilde{B}_s}\mu_{\tilde{A}_s}, \left(\vartheta_{\tilde{B}_s}{}^{2}+\vartheta_{\tilde{A}_s}{}^{2}-\vartheta_{\tilde{B}_s}{}^{2}\vartheta_{\tilde{A}_s}{}^{2}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}+\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\pi_{\tilde{A}_s}{}^{2}\right]^{1/2}, \\ \left(\mu_{\tilde{B}_s}{}^{2-}+\mu_{\tilde{A}_s}{}^{2-}-\mu_{\tilde{B}_s}{}^{2-}\mu_{\tilde{A}_s}{}^{2-}\right)^{1/2}, \vartheta_{\tilde{B}_s}{}^{-}\vartheta_{\tilde{A}_s}{}^{-}, \\ \left[\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\pi_{\tilde{A}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

And so $\tilde{A}_s \otimes \tilde{B}_s = \tilde{B}_s \otimes \tilde{A}_s$.

iii.
$$\lambda(\tilde{A}_s\oplus\tilde{B}_s)=\lambda\left\{\begin{array}{c}\left(\mu_{\tilde{A}_s}^2+\mu_{\tilde{B}_s}^2-\mu_{\tilde{A}_s}^2\mu_{\tilde{B}_s}^2\right)^{1/2},\vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s},\\ \left[\left(1-\mu_{\tilde{B}_s}^2\right)\pi_{\tilde{A}_s}^2+\left(1-\mu_{\tilde{A}_s}^2\right)\pi_{\tilde{B}_s}^2-\pi_{\tilde{A}_s}^2\pi_{\tilde{B}_s}^2\right]^{1/2},\ \mu_{\tilde{A}_s}^-\mu_{\tilde{B}_s}^-,\\ \left(\vartheta_{\tilde{A}_s}^{2-}+\vartheta_{\tilde{B}_s}^{2-}-\vartheta_{\tilde{A}_s}^{2-}\vartheta_{\tilde{B}_s}^{2-}\right)^{1/2},\\ \left[\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)\pi_{\tilde{A}_s}^{2-}+\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)\pi_{\tilde{B}_s}^{2-}-\pi_{\tilde{A}_s}^{2-}\pi_{\tilde{B}_s}^{2-}\right]^{1/2}\end{array}\right\}.$$

$$=\left\{\begin{array}{c}\left(1-\left(1-\left(\mu_{\tilde{A}_s}^2+\mu_{\tilde{B}_s}^2-\mu_{\tilde{A}_s}^2\mu_{\tilde{B}_s}^2\right)\right)^\lambda\right)^{1/2},\vartheta_{\tilde{A}_s}^\lambda\vartheta_{\tilde{B}_s}^\lambda,\\ \left[\begin{array}{c}\left(1-\left(\mu_{\tilde{A}_s}^2+\mu_{\tilde{B}_s}^2-\mu_{\tilde{A}_s}^2\mu_{\tilde{B}_s}^2\right)\right)^\lambda-\\ \left(\left(1-\left(\mu_{\tilde{A}_s}^2+\mu_{\tilde{B}_s}^2-\mu_{\tilde{A}_s}^2\mu_{\tilde{B}_s}^2\right)-\left(1-\mu_{\tilde{B}_s}^2\right)\pi_{\tilde{A}_s}^2-\left(1-\mu_{\tilde{A}_s}^2\right)\pi_{\tilde{B}_s}^2+\pi_{\tilde{A}_s}^2\pi_{\tilde{B}_s}^2\right)\right)^\lambda\end{array}\right]^{1/2},\\ -\left(\mu_{\tilde{A}_s}^-\right)^\lambda\left(\mu_{\tilde{B}_s}^-\right)^\lambda,-\left(1-\left(1-\left(\vartheta_{\tilde{A}_s}^{2-}+\vartheta_{\tilde{B}_s}^{2-}-\vartheta_{\tilde{A}_s}^{2-}\vartheta_{\tilde{B}_s}^{2-}\right)\right)^\lambda\right)^{1/2},\\ -\left[\begin{array}{c}\left(1-\left(\vartheta_{\tilde{A}_s}^{2-}+\vartheta_{\tilde{B}_s}^{2-}-\vartheta_{\tilde{A}_s}^{2-}\vartheta_{\tilde{B}_s}^{2-}\right)\right)^\lambda-\\ \left(\left(\begin{array}{c}1-\left(\vartheta_{\tilde{A}_s}^{2-}+\vartheta_{\tilde{B}_s}^{2-}-\vartheta_{\tilde{A}_s}^{2-}\vartheta_{\tilde{B}_s}^{2-}\right)-\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)\pi_{\tilde{A}_s}^{2-}\\ -\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)\pi_{\tilde{B}_s}^{2-}+\pi_{\tilde{A}_s}^{2-}\pi_{\tilde{B}_s}^{2-}\end{array}\right)\right)^\lambda\end{array}\right]^{1/2}\end{array}\right\}$$

Now,

$$\lambda\tilde{A}_s\oplus\lambda\tilde{B}_s=\left\{\begin{array}{c}\left(1-\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda\right)^{1/2},\ \vartheta_{\tilde{A}_s}^\lambda,\\ \left(\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda-\left(1-\mu_{\tilde{A}_s}^2-\pi_{\tilde{A}_s}^2\right)^\lambda\right)^{1/2},-\mu_{\tilde{A}_s}^{-\lambda},\\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda\right)^{1/2},-\left(\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{A}_s}^{2-}-\pi_{\tilde{A}_s}^{2-}\right)^\lambda\right)^{1/2}\end{array}\right\}$$

$$\oplus$$

$$\left\{\begin{array}{c}\left(1-\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda\right)^{1/2},\ \vartheta_{\tilde{B}_s}^\lambda,\\ \left(\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda-\left(1-\mu_{\tilde{B}_s}^2-\pi_{\tilde{B}_s}^2\right)^\lambda\right)^{1/2},-\mu_{\tilde{B}_s}^{-\lambda},\\ -\left(1-\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda\right)^{1/2},-\left(\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{B}_s}^{2-}-\pi_{\tilde{B}_s}^{2-}\right)^\lambda\right)^{1/2}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(1-\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda+1-\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda-\left(1-\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda\right)\left(1-\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda\right)\right)^{1/2},\vartheta_{\tilde{A}_s}^\lambda\vartheta_{\tilde{B}_s}^\lambda,\\ \left[\begin{array}{c}1-\left(1-\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda\right)\left(\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda-\left(1-\mu_{\tilde{A}_s}^2-\pi_{\tilde{A}_s}^2\right)^\lambda\right)+\\ \left(1-\left(1-\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda\right)\left(\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda-\left(1-\mu_{\tilde{B}_s}^2-\pi_{\tilde{B}_s}^2\right)^\lambda\right)\right)-\\ \left(\left(1-\mu_{\tilde{A}_s}^2\right)^\lambda-\left(1-\mu_{\tilde{A}_s}^2-\pi_{\tilde{A}_s}^2\right)^\lambda\right)\left(\left(1-\mu_{\tilde{B}_s}^2\right)^\lambda-\left(1-\mu_{\tilde{B}_s}^2-\pi_{\tilde{B}_s}^2\right)^\lambda\right)\end{array}\right]^{1/2},-\left(\mu_{\tilde{A}_s}^-\right)^\lambda\left(\mu_{\tilde{B}_s}^-\right)^\lambda,\\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda+1-\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda-\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda\right)\left(1-\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda\right)\right)^{1/2},\\ -\left[\begin{array}{c}1-\left(1-\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda\right)\left(\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{A}_s}^{2-}-\pi_{\tilde{A}_s}^{2-}\right)^\lambda\right)+\\ \left(1-\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda\right)\left(\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{B}_s}^{2-}-\pi_{\tilde{B}_s}^{2-}\right)^\lambda\right)\right)-\\ \left(\left(1-\vartheta_{\tilde{A}_s}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{A}_s}^{2-}-\pi_{\tilde{A}_s}^{2-}\right)^\lambda\right)\left(\left(1-\vartheta_{\tilde{B}_s}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{B}_s}^{2-}-\pi_{\tilde{B}_s}^{2-}\right)^\lambda\right)\end{array}\right]^{1/2}\end{array}\right\}$$

$$= \left\{ \begin{array}{c} \left(1-\left(1-\mu_{\tilde{A}_s}{}^2\right)^\lambda\left(1-\mu_{\tilde{B}_s}{}^2\right)^\lambda\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^\lambda \vartheta_{\tilde{B}_s}{}^\lambda, \\ \left(\left(1-\mu_{\tilde{A}_s}{}^2\right)^\lambda\left(1-\mu_{\tilde{B}_s}{}^2\right)^\lambda-\left(1-\mu_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^\lambda\left(1-\mu_{\tilde{B}_s}{}^2-\pi_{\tilde{B}_s}{}^2\right)^\lambda\right)^{1/2}, -\left(\mu_{\tilde{A}_s}{}^-\right)^\lambda\left(\mu_{\tilde{B}_s}{}^-\right)^\lambda, \\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)^\lambda\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)^\lambda\right)^{1/2}, \\ -\left(\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)^\lambda\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)^\lambda-\left(1-\vartheta_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^\lambda\left(1-\vartheta_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^\lambda\right)^{1/2} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(1-\left(1-\left(\mu_{\tilde{A}_s}{}^2+\mu_{\tilde{B}_s}{}^2-\mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)\right)^\lambda\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^\lambda\vartheta_{\tilde{B}_s}{}^\lambda, \\ \left[\begin{array}{c} \left(1-\left(\mu_{\tilde{A}_s}{}^2+\mu_{\tilde{B}_s}{}^2-\mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)\right)^\lambda - \\ \left(\left(1-\left(\mu_{\tilde{A}_s}{}^2+\mu_{\tilde{B}_s}{}^2-\mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)-\left(1-\mu_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2-\left(1-\mu_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2+\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right)\right)^\lambda \end{array} \right]^{1/2}, \\ -\left(\mu_{\tilde{A}_s}{}^-\right)^\lambda\left(\mu_{\tilde{B}_s}{}^-\right)^\lambda, -\left(1-\left(1-\left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)\right)^\lambda\right)^{1/2}, \\ -\left[\begin{array}{c} \left(1-\left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)\right)^\lambda - \\ \left(\left(\begin{array}{c} 1-\left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)-\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-} \\ -\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-} \end{array} \right)\right)^\lambda \end{array} \right]^{1/2} \end{array} \right\}$$

And so, $\lambda(\tilde{A}_s \oplus \tilde{B}_s) = \lambda\tilde{A}_s \oplus \lambda\tilde{B}_s$.

iv. $$(\tilde{A}_s \otimes \tilde{B}_s)^\lambda = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2+\left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2-\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right]^{1/2}, \\ \left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^-\vartheta_{\tilde{B}_s}{}^-, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}+\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\right]^{1/2} \end{array} \right\}^\lambda$$

$$= \left\{ \begin{array}{c} \left\{\left\{\mu_{\tilde{A}_s}{}^\lambda\mu_{\tilde{B}_s}{}^\lambda\right\}\right\}, \left(1-\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)\right)^\lambda\right)^{\frac{1}{2}}, \\ \left[\begin{array}{c} \left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)\right)^\lambda - \\ \left(\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2+\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right)\right)^\lambda \end{array} \right]^{1/2} \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^\lambda\right)^{1/2}, -\left(\vartheta_{\tilde{A}_s}{}^-\right)^\lambda\left(\vartheta_{\tilde{B}_s}{}^-\right)^\lambda, \\ \left[\begin{array}{c} \left(1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)\right)^\lambda - \\ \left(\left(\begin{array}{c} 1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-} - \\ \left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-} \end{array} \right)\right)^\lambda \end{array} \right]^{1/2} \end{array} \right\}$$

Now,

$$\tilde{A}_s{}^\lambda \otimes \tilde{B}_s{}^\lambda = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}{}^\lambda, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^\lambda\right)^{1/2}, \left(\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^\lambda-\left(1-\vartheta_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^\lambda\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^\lambda\right)^{1/2}, -\vartheta_{\tilde{A}_s}{}^{-\lambda}, \\ -\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^\lambda-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^\lambda\right)^{1/2} \end{array} \right\}$$

$$\otimes$$

$$\left\{\begin{matrix}\mu_{\tilde{B}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}\right)^{1/2}, \left(\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}-\left(1-\vartheta_{\tilde{B}_s}{}^2-\pi_{\tilde{B}_s}{}^2\right)^{\lambda}\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -(\vartheta_{\tilde{B}_s}{}^{-})^{\lambda}, \\ -\left(\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}\end{matrix}\right\}$$

$$=\left\{\begin{matrix}\mu_{\tilde{A}_s}{}^{\lambda}\mu_{\tilde{B}_s}{}^{\lambda}, \begin{pmatrix}1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}+1-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}- \\ \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\right)\left(1-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}\right)\end{pmatrix}^{1/2}, \\ \begin{bmatrix}1-\left(1-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^{\lambda}\right)+ \\ 1-\left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}-\left(1-\vartheta_{\tilde{B}_s}{}^2-\pi_{\tilde{B}_s}{}^2\right)^{\lambda}\right)- \\ \left(\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}-\left(1-\vartheta_{\tilde{B}_s}{}^2-\pi_{\tilde{B}_s}{}^2\right)^{\lambda}\right)\end{bmatrix}^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}+1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)\left(1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)\right)^{1/2}, \\ -\vartheta_{\tilde{A}_s}{}^{-\lambda}\vartheta_{\tilde{B}_s}{}^{-\lambda}, \\ \begin{bmatrix}1-\left(1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)+ \\ 1-\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)- \\ \left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)\end{bmatrix}^{1/2}\end{matrix}\right\}$$

$$=\left\{\begin{matrix}\mu_{\tilde{A}_s}{}^{\lambda}\mu_{\tilde{B}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}\right)^{1/2}, \\ \left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\left(1-\vartheta_{\tilde{B}_s}{}^2\right)^{\lambda}-\left(\left(1-\vartheta_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^{\lambda}\left(1-\vartheta_{\tilde{B}_s}{}^2-\pi_{\tilde{B}_s}{}^2\right)^{\lambda}\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -\vartheta_{\tilde{A}_s}{}^{-\lambda}\vartheta_{\tilde{B}_s}{}^{-\lambda}, \\ -\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\left(1-\mu_{B_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}\end{matrix}\right\}$$

$$=\left\{\begin{matrix}\left\{\left\{\mu_{\tilde{A}_s}{}^{\lambda}\mu_{\tilde{B}_s}{}^{\lambda}\right\}\right\}, \left(1-\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)\right)^{\lambda}\right)^{\frac{1}{2}}, \\ \begin{bmatrix}\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)\right)^{\lambda}- \\ \left(\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2+\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right)\right)^{\lambda}\end{bmatrix}^{1/2} \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -(\vartheta_{\tilde{A}_s}{}^{-})^{\lambda}\left(\vartheta_{\tilde{B}_s}{}^{-}\right)^{\lambda}, \\ \begin{bmatrix}\left(1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)\right)^{\lambda}- \\ \left(\begin{pmatrix}1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}- \\ \left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\end{pmatrix}\right)^{\lambda}\end{bmatrix}^{1/2}\end{matrix}\right\}$$

Thus, $(\tilde{A}_s \otimes \tilde{B}_s)^{\lambda} = \tilde{A}_s{}^{\lambda} \otimes \tilde{B}_s{}^{\lambda}$.

Definition 3.4. Spherical Bipolar Weighted Arithmetic Mean (SBWAM) with respect to, $w = w_1, w_2, \ldots, w_n;\ w_i \in [0,1];\ \sum_{i=1}^{n} w_i = 1$, SBWAM is defined as,

$$SBWAM_w\left(A_{s_1}, A_{s_2}, \dots, A_{s_n}\right) = w_1 A_{s_1} + w_2 A_{s_2} + \cdots + w_n A_{s_n}$$

$$= \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^2\right)^{w_i}\right]^{1/2}, \prod_{i=1}^{n} \vartheta_{A_{s_i}}{}^{w_i}, \\ \left[\prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^2\right)^{w_i} - \prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^2 - \pi_{A_{s_i}}{}^2\right)^{w_i}\right]^{1/2}, \\ -\prod_{i=1}^{n}\left(-\mu_{A_{s_i}}{}^{2^-}\right)^{w_i}, -\left(1 - \prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2^-}\right)^{w_i}\right)^{1/2}, \\ -\left[\prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2^-}\right)^{w_i} - \left(\prod_{i=1}^{n} 1 - \vartheta_{A_{s_i}}{}^{2^-} - \pi_{A_{s_i}}{}^{2^-}\right)^{w_i}\right]^{1/2} \end{array} \right\} \quad (7)$$

Definition 3.5. Spherical Bipolar Weighted Geometric Mean (SBWGM) with respect to, $w = w_1, w_2, \dots, w_n$; $w_i \in [0,1]$; $\sum_{i=1}^{n} w_i = 1$, SBWGM is defined as,

$$SBWGM_w(A_1, A_2 \dots, A_n) = A_{s_1}{}^{w_1} + A_{s_2}{}^{w_2} + \cdots + A_{s_n}{}^{w_n}$$

$$= \left\{ \begin{array}{c} \prod_{i=1}^{n} \mu_{A_{s_i}}{}^{w_i}, \left[1 - \prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^2\right)^{w_i}\right]^{1/2}, \\ \left[\prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^2\right)^{w_i} - \prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^2 - \pi_{A_{s_i}}{}^2\right)^{w_i}\right]^{1/2}, \\ -\left(1 - \prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2^-}\right)^{w_i}\right)^{1/2}, \prod_{i=1}^{n}\left(-\vartheta_{A_{s_i}}{}^{2^-}\right)^{w_i}, \\ -\left[\prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2^-}\right)^{w_i} - \left(\prod_{i=1}^{n} 1 - \mu_{A_{s_i}}{}^{2^-} - \pi_{A_{s_i}}{}^{2^-}\right)^{w_i}\right]^{1/2} \end{array} \right\}$$
(8)

Definition 3.6. The score function and accuracy function of sorting Spherical Bipolar fuzzy set (SBFS) are defined by,

i. Score $\left(\tilde{A}_s\right) = \frac{1}{2}\left[\begin{array}{c}\left(\mu_{\tilde{A}_s} - \pi_{\tilde{A}_s}\right)^2 - \left(\vartheta_{\tilde{A}_s} - \pi_{\tilde{A}_s}\right)^2 + \left(\mu_{\tilde{A}_s}{}^- - \pi_{\tilde{A}_s}{}^-\right)^2 - \\ \left(\vartheta_{\tilde{A}_s}{}^- - \pi_{\tilde{A}_s}{}^-\right)^2\end{array}\right]$

ii. Accuracy $\left(\tilde{A}_s\right) = \frac{1}{2}\left[\mu_{\tilde{A}_s}{}^2 + \vartheta_{\tilde{A}_s}{}^2 + \pi_{\tilde{A}_s}{}^2 + \mu_{\tilde{A}_s}{}^{2^-} + \vartheta_{\tilde{A}_s}{}^{2^-} + \pi_{\tilde{A}_s}{}^{2^-}\right]$

4. Decision Making Method Based on The Spherical Fuzzy Weighted Aggregation Operator

In this section, we present a handling method for multi criteria decision making problem by means of two aggregation operators under the spherical bipolar environment.

Let $A = \{A_1, A_2, \dots, A_m\}$ be the set of all alternatives and let $c = \{C_1, C_2, \dots, C_n\}$be the set of criteria. Assume that the weight of the criteria $C_j(j = 1,2, \dots, n)$ entered by the decision maker is w_j, where $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. In the decision process, the evaluation information of the alternative A_i on the criteria is represented by the form of a Spherical Bipolar Fuzzy Set (SBFS):

$$\tilde{A}_i = \{\langle C_j, \mu_{\tilde{A}_i}{}^+(C_j), \vartheta_{\tilde{A}_i}{}^+(C_j), \pi_{\tilde{A}_i}{}^+(C_j), \mu_{\tilde{A}_i}{}^-(C_j), \vartheta_{\tilde{A}_i}{}^-(C_j), \pi_{\tilde{A}_i}{}^-(C_j) |\, C_j \in C\rangle\}$$

where, $0 \le \mu_{\tilde{A}_i}{}^{2^+}(C_j) + \vartheta_{\tilde{A}_i}{}^{2^+}(C_j) + \pi_{\tilde{A}_i}{}^{2^+}(C_j) \le 1$ and $-1 \le -(\mu_{\tilde{A}_i}{}^{2^-}(C_j) + \vartheta_{\tilde{A}_i}{}^{2^-}(C_j) + \pi_{\tilde{A}_i}{}^{2^-}(C_j)) \le 0$ for $j = 1,2, \dots, n$ and $i = 1,2, \dots, m$.

For convenience, the value of SBFS is denoted by,

$$\alpha_{ij} = \langle \mu_{ij}{}^+, \vartheta_{ij}{}^+, \pi_{ij}{}^+, \mu_{ij}{}^-, \vartheta_{ij}{}^-, \pi_{ij}{}^- \rangle \qquad (j = 1,2, \dots, m \; and \; i = 1,2, \dots, n)$$

Therefore, we get a SBFS decision matrix, $D = (\alpha_{ij})_{m\times n}$.

$$D = (\alpha_{ij})_{m\times n} = \begin{bmatrix} \langle {\mu_{11}}^+, {\vartheta_{11}}^+, {\pi_{11}}^+, {\mu_{11}}^-, {\vartheta_{11}}^-, {\pi_{11}}^- \rangle & \langle {\mu_{12}}^+, {\vartheta_{12}}^+, {\pi_{12}}^+, {\mu_{12}}^-, {\vartheta_{12}}^-, {\pi_{12}}^- \rangle & \cdots & \langle {\mu_{1n}}^+, {\vartheta_{1n}}^+, {\pi_{1n}}^+, {\mu_{1n}}^-, {\vartheta_{1n}}^-, {\pi_{1n}}^- \rangle \\ \langle {\mu_{21}}^+, {\vartheta_{21}}^+, {\pi_{21}}^+, {\mu_{21}}^-, {\vartheta_{21}}^-, {\pi_{21}}^- \rangle & \langle {\mu_{22}}^+, {\vartheta_{22}}^+, {\pi_{22}}^+, {\mu_{22}}^-, {\vartheta_{22}}^-, {\pi_{22}}^- \rangle & \cdots & \langle {\mu_{2n}}^+, {\vartheta_{2n}}^+, {\pi_{2n}}^+, {\mu_{2n}}^-, {\vartheta_{2n}}^-, {\pi_{2n}}^- \rangle \\ \vdots & \vdots & & \vdots \\ \langle {\mu_{m1}}^+, {\vartheta_{m1}}^+, {\pi_{m1}}^+, {\mu_{m1}}^-, {\vartheta_{m1}}^-, {\pi_{m1}}^- \rangle & \langle {\mu_{m2}}^+, {\vartheta_{m2}}^+, {\pi_{m2}}^+, {\mu_{m2}}^-, {\vartheta_{m2}}^-, {\pi_{m2}}^- \rangle & \cdots & \langle {\mu_{mn}}^+, {\vartheta_{mn}}^+, {\pi_{mn}}^+, {\mu_{mn}}^-, {\vartheta_{mn}}^-, {\pi_{mn}}^- \rangle \end{bmatrix}$$

Then, the aggregating spherical bipolar value α_i for $A_i (i = 1,2, \dots, m)$ is,

$\alpha_i = \langle {\mu_i}^+, {\vartheta_i}^+, {\pi_i}^+, {\mu_i}^-, {\vartheta_i}^-, {\pi_i}^- \rangle = SBWAM_w(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ or

$\alpha_i = \langle {\mu_i}^+, {\vartheta_i}^+, {\pi_i}^+, {\mu_i}^-, {\vartheta_i}^-, {\pi_i}^- \rangle = SBWGM_w(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ is obtained by equations (7) and (8).

In summary, the decision procedure for the proposed method can be summarized as follows:

Step 1. Construct the decision matrix provided by the decision maker as:

$$(\alpha_{ij})_{m\times n} = \langle {\mu_{ij}}^+, {\vartheta_{ij}}^+, {\pi_{ij}}^+, {\mu_{ij}}^-, {\vartheta_{ij}}^-, {\pi_{ij}}^- \rangle_{m\times n}.$$

Step 2. Compute α_i by calculating the weighted arithmetic average values by $SBWAM_w$ or $SBWGM_w$

Step 3. Calculate the score values of score (α_i) $i = 1,2, \dots, m$ for the collective overall SBF number of $\alpha_i (i = 1,2, \dots, m)$

Step 4. Give the ranking order of the alternative according to the score values, and then give the best choice.

5. Numerical Example

In this section, an example for a multi criteria decision making problem of engineering alternatives is used as a demonstration of the application of the proposed decision-making method in a realistic scenario, as well as the application and effectiveness of the proposed decision-making method.

Let us consider the decision-making problem.

There is an investment company which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money:

1) A_1 is a car company,
2) A_2 is a food company,
3) A_3 is a computer company,
4) A_4 is a arms company.

The investment company must take a decision according to the following three criteria:

1) C_1 is the risk,
2) C_2 is the growth,
3) C_3 is the environmental impact.

The weight vector of the criteria is given by,

$$w = (0.35, 0.25, 0.4)$$

Then,

STEP 1. Construct the decision matrix provided by the decision maker:

$$\begin{bmatrix} \begin{pmatrix} 0.2, 0.3, 0.4, \\ -0.1, -0.4, -0.6 \end{pmatrix} & \begin{pmatrix} 0.4, 0.5, 0.5, \\ -0.2, -0.3, -0.4 \end{pmatrix} & \begin{pmatrix} 0.2, 0.2, 0.5, \\ -0.5, -0.6, -0.2 \end{pmatrix} \\ \begin{pmatrix} 0.1, 0.2, 0.6, \\ -0.2, -0.5, -0.6 \end{pmatrix} & \begin{pmatrix} 0.6, 0.1, 0.2, \\ -0.3, -0.4, -0.5 \end{pmatrix} & \begin{pmatrix} 0.5, 0.1, 0.7, \\ -0.1, -0.3, -0.5 \end{pmatrix} \\ \begin{pmatrix} 0.2, 0.3, 0.3, \\ -0.5, -0.5, -0.6 \end{pmatrix} & \begin{pmatrix} 0.2, 0.3, 0.5, \\ -0.1, -0.2, -0.5 \end{pmatrix} & \begin{pmatrix} 0.2, 0.3, 0.8, \\ -0.1, -0.2, -0.3 \end{pmatrix} \\ \begin{pmatrix} 0.1, 0.4, 0.7, \\ -0.3, -0.5, -0.7 \end{pmatrix} & \begin{pmatrix} 0.1, 0.4, 0.7, \\ -0.3, -0.5, -0.7 \end{pmatrix} & \begin{pmatrix} 0.2, 0.5, 0.8 \\ -0.2, -0.4, -0.5 \end{pmatrix} \end{bmatrix}$$

STEP 2. Calculate the weighted arithmetic average value α_i for $i = 1,2,3,4$.

$$\alpha_1 = [0.2675, 0.2898, 0.4709, -0.0512, -0.4809, -0.4263]$$

$$\alpha_2 = [0.3509, 0.2898, 0.6633, -0.0281, -0.4080, -0.5475]$$

$$\alpha_3 = [0.2002, 0.2999, 0.6369, -0.0308, -0.4498, -0.4163]$$

$$\alpha_4 = [0.1487, 0.4374, 0.7481, -0.0651, -0.4640, -0.6509]$$

STEP 3. Calculate the score values s(α_i) $for\ i = 1,2,3,4$ for the collective overall SBF number $\alpha_i, i = 1,2,3,4$.

$$s(\alpha_1) = 0.0731$$

$$s(\alpha_2) = 0.1042$$

$$s(\alpha_3) = 0.0961$$

$$s(\alpha_4) = 0.2856$$

STEP 4. Rank of the investment company according to the score values is:

$$A_4 > A_2 > A_3 > A_1$$

Thus A_4 is the best alternative, (i.e.,) the best option to invest money is arms company.

6. Conclusion

In this paper, we studied the concept of Spherical Bipolar Fuzzy Sets (SBFS) by using Spherical Fuzzy Sets (SFS), Bipolar fuzzy set (BFS) and some of its basic operational relations. Two aggregation operators: Spherical bipolar weighted arithmetic average operator and Spherical bipolar weighted geometric average operator were proposed and they were applied to multi criteria decision making problems. Finally, a numerical example is provided to illustrate the application of the developed approach.

References

[1] F. Smarandache. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic.Rehoboth*: American Research Press (1999) 1-141.

[2] G. İntepe, E. Bozdağ, T. Koç, *The Selection of Technology Forecasting Method Using A Multi-Criteria Interval-Valued Intuitionistic Fuzzy Group Decision Making Approach*, Computers and Industrial Engineering 65(2) (2013) 277-285.

[3] F. Smarandache, *Neutrosophic Set- A Generalization of The Intuitionistic Fuzzy Set*, International Journal of Pure and Applied Mathematics 24(3) (2004) 1-15.

[4] I. Grattan-Guinness, *Fuzzy Membership Mapped onto Interval and Many-Valued Quantities*, Mathematical Logic Quarterly 22(1) (1976) 149-160.

[5] J. M. Garibaldi, T. Özen, *Uncertain Fuzzy Reasoning: A Case Study in Modelling Expert Decision Making*. IEEE Transactions on Fuzzy Systems 15(1) (2007) 16-30.

[6] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Fuzzy sets and Systems 20(1) (1986) 87-96.

[7] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.

[8] L. A. Zadeh, *The Concept of a Linguistic Variable and its application to Approximate Reasoning*, Information Sciences 8(3) (1975) 199-249.

[9] Y. Yang, F. Chiclana, *Intuitionistic Fuzzy Sets: Spherical Representation and Distances*, International Journal of Intelligent Systems 24(4) (2009) 399-420.

[10] R. R. Yager, *On The Theory of Bags*, International Journal of General Systems 13(1) (1986) 23–37.

[11] V. Torra, *Hesitant Fuzzy Sets*, International Journal of Intelligent Systems 25(6) (2010) 529-539.

[12] K. Atanassov, *Geometrical Interpretation of The Elements of The Intuitionistic Fuzzy Objects*, Preprint IM-MFAIS, 1-89, Sofia. Reprinted: International Journal Bioautomation 20 (S1) (2016) 27-42.

[13] K. Atanassov, *Intuitionistic Fuzzy Sets*, Springer, Heidelberg (1999).

[14] P. J. Ren, Z. Xu, X. J. Gou, *Pythagorean Fuzzy TODIM Approach to Multi-Criteria Decision Making*, Applied Soft Computing 31 (2016) 246–259.

[15] X. Gou, Z. Xu, P. Ren, *The Properties of Continuous Pythagorean Fuzzy Information*, International Journal of Intelligent Systems 31(5) (2016) 401-424.

[16] S. Broumi, M. Murugappan, M. Talea, A. Bakali, F. Smarandache, P. K. Singh, A. Dey, *Single Valued $(2n+1)$ Sided Polygonal Neutrosophic Numbers and Single Valued $(2n)$ Sided Polygonal Neutrosophic Numbers*, Neutrosophic Sets and Systems 25 (2019) 54-65.

[17] S. Broumi, M. Talea, A. Bakali, P. K. Singh, F. Smarandache, *Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB*, Neutrosophic Sets and Systems 24 (2019) 46-60.

[18] S. Broumi, M. Talea, A. Bakali, F. Smarandache, P. K. Singh, M. Murugappan, V. V. Rao, A *Neutrosophic Technique Based Efficient Routing Protocol for MANET Based on Its Energy and Distance*, Neutrosophic Sets and Systems 24 (2019) 61-69.

[19] S. Broumi, A. Dey, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari, R. Kumar, *Shortest Path Problem using Bellman Algorithm under Neutrosophic Environment*, Complex and Intelligent Systems 5 (2019) 409-416. https://doi.org/10.1007/s40747-019-0101-8

[20] S. Broumi, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari, M. Parimala, *Shortest Path Problem in Fuzzy, Intuitionistic Fuzzy and Neutrosophic Environment: An Overview*, Complex and Intelligent Systems 5 (2019) 371-378. https://doi.org/10.1007/s40747-019-0098-z

[21] I. Antonov, *On A New Geometrical Interpretation of The Intuitionistic Fuzzy Sets*, Notes on Intuitionistic Fuzzy Sets 1(1) (1995) 29–31.

[22] Z. Gong, X. Xu, Y. Yang, Y. Zhou, H. Zhang, *The Spherical Distance for Intuitionistic Fuzzy Sets and Its Application in Decision Analysis*, Technological and Economic Development of Economy 22(3) (2016) 393-415.

[23] R. Sambuc, *Function Φ-Flous, Application a l'aide au Diagnostic en Pathologie Thyroidienne*. University of Marseille (1975).

[24] K. M. Lee, 2000, *Bipolar Valued Fuzzy Sets and Their Operations*, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, (2000) 307-312.

[25] K. M. Lee, *Comparison of Interval Valued Fuzzy Sets, Intuitionistic Fuzzy Sets and Bipolar-Valued Fuzzy Sets*, Journal of Korean Institute of Intelligent Systems 14(2) (2004) 125-129.

[26] F. K. Gündoğu, C. Kahraman, *Spherical Fuzzy Sets and Spherical Fuzzy TOPSIS Method*, Journal of Intelligent & Fuzzy Systems 36(1) (2019) 337-352.

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Number Density, Dynamics, Concentration Places and Status of Species from the Genera Diving Ducks (Aythya Boil, 1822) and Mergansers (Merqus Linn., 1758) in the South Part of the Azerbaijan Sector of Caspian Sea

Babayev Ilyas Razzag[1], Mukhtarov Hafiz Shahhuseyn[2], Huseynov Rafig Azizagha[3]

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Abstract − The information about the number, density, the largest concentration places, and the status of species of genera Aythya and Merqus in the South part of the Azerbaijan sector of the Caspian Sea presented in the paper. It was revealed that 4 species of Aythya and 3 species of Merqus live in Azerbaijan.

Keywords − *Caspian Sea, coastal belt, concentration places, reproduction period, common, distribution*

1. Introduction

Four species of the genus diving ducks (common pochard - *Aythya ferina* Linn., 1758, white-eyed pochard - *Aytha nyroca* Ciuld, 1770, tufted duck - *Aythya fuligula* Linn., 1758, greater scaup - *Aythya marila* Linn., 1761) is spread in the southern coast of the Azerbaijan sector of the Caspian Sea. One of these species (the white-eyed pochard) is found annually, and the other is found both in winter and during migration [1]. The white-eyed pochard is included both in the Red Book of Azerbaijan and in the Red List of the International Union for Conservation of Nature [2, 3]. The common pochard is included only in the Red Book of the International Union for Conservation of Nature [3].

Three species of the genus Mergansers live in Azerbaijan, including the southern coast of the Azerbaijan sector of the Caspian Sea: smew (*Mergus albellus* Linn., 1758), red-breasted merganser (*Mergus serrator* Linn., 1758), goosander (*Mergus merganser* Linn., 1758). All 3 species are found in Azerbaijan during the periods of migration and wintering [1].

It is known that the negative impact of anthropogenic factors reduces the number of some bird species in nature, and some species are disappeared. The aim of the research is to study the number, density, habitats,

[1]ily4s.babayev@yandex.ru; [2]hafizmuxtarov@mail.ru (Corresponding Author); [3]rafiq.huseynov.59@mail.ru
[1,2] Azerbaijan National Academy of Science Institute of Zoology, Baku, Azerbaijan
[3]Sumgait State University, 43rd block, AZ5008, Sumgait, Azerbaijan

and status of the species in the genera of diving ducks and mergansers in the southern coast of the Azerbaijan sector of the Caspian Sea and on the basis of obtained data to develop scientifically-grounded measures to conserve these species.

2. Material and methods

The research was conducted in January-February 2014-2016. It took 115 days to collect the material. The study area was divided into separate areas differing from each other by environmental features. The total studied area was 1,766.5 km^2 (291 km^2 is the Shahdili-Alat strip, 293 km^2 is the southeast Shirvan coastline, 90 km^2 is the Salyan coastline, 891.5 km^2 is the Kyzylagach National Park).

The point counts method was used for counting birds in water areas. Peaks from 5 to 10 m high were selected for points. Birds were counted at a distance of 2-3 km. Birds counting was done after dividing the area into squares. The area of squares was 0.2 km^2 in the water reservoirs rich in cane and 18–20 km^2 in aquatorias [4, 5, 6]. The total number of birds in the Pirman harbor, Caspian and Aggush floodplains was calculated by the way of extrapolation.

The category of animals was determined by population density according to A.R.Kuzyakin [7] and G.T.Mustafayev [8]: The population with 0.1-0.9 individuals per 1 km^2 of the area is considered to be rare; the population of 1–10 individuals per 1 km^2 of the area is considered to be normal and the population with more than 10 individuals in the same area is considered to be large. During the study, telescopes, binoculars, cars, motor and non-motorboats were used.

3. Results and discussion

3.1. The common pochard.

The total number of the common pochard in the south part of the Azerbaijan sector of the Caspian Sea was 62015 in 2014, 54290 in 2015, 59795 in 2016 (Table 1). As the table shows in the period between 2014 and 2016 the 9,8 % of the total amount of the common pochard accounted for Khazar Shahdili-Alyat, 24,8 % for South-Eastern Shirvan, 5,1 % for Salyan, 0,6 % for Lankaran coastline, 0,7% for Big and Small Gyzylgaz lakes, 1,4% for Yenikend floodplain and 57,6% for Gyzylagach National Park.

Analyses of data show that the diving duck is numerous (10 or more specimens per 1 km^2) in the Shahdili-Alyat coastline, South - East Shirvan coastline, Salyan coastline, Big and Small Gyzylgaz lakes. And it is common (1 or more specimens per 1 km^2) in the Lankaran coastline, Gyzylagach National Park, Yenikend floodplain, and Shirvanovka lagoon.

Table 1. Number (specimen) and density of the common pochard in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Studied areas / Years	2014		2015		2016		Average	
	Number	Density 10 km^2	Number	Density 10 km^2	Number	Density 10 km^2	Number	Density 10 km^2
Shahdili-Alyat coastline (291 km^2)	6000	206,1	6300	216,4	5100	175,2	5800	199,3
South- East Shirvan coastline (111 km^2)	1800	162,1	19000	1711,7	23000	2072	14600	1315,3
Big and Small Gyzylgaz lakes (42 km^2)	300	71,4	450	107,1	530	126,1	426,6	101,5
Yenikend floodplain and Shirvanovka lagoon (141 km^2)	915	64,8	820	58,1	745	52,8	826,6	586
Salyan coastline (90 km^2)	8000	888,8	470	52,2	580	64,4	3016,6	335,1
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Gyzylagach National Park (891,5 km^2)	44500	499,1	27000	302,9	29500	333,9	33866,6	377,6
Lankaran coastline (180 km^2)	500	27,7	250	13,8	340	18,8	363,3	20,1
Number and density in the total studied areas (1746,5 km^2)	62015	355	54290	310	59795	342	58700	336,1

3.2. White-eyed pochard.

The total number of the white-eyed pochard was 998 specimen in winter of the 2014 year, 1231 specimen in the breeding period of the same year, 1470 specimen in winter of 2015, 1826 in the breeding period of the same year, 250 specimens in winter of 2016, 3029 specimens in the breeding period of the same year (Table 2).

As the table shows in the period from 2014 to 2016 96,3 % of the total amount of the white-eyed pochard in the winter and 98,5% of it in the breeding period fell on the share of Gyzylagach National Park, 0,6 % in the winter and 0,8 % in the breeding period fell on the share of Big and Small Gyzylgaz lakes, 0,4 % in the winter and 0,6 % in the breeding period fell on the share of Yenikend floodplains, and only 0,7% in the winter fell on the share of Absheron-Gobustan coastline, 1,1 % on the share of South - East Shirvan coastline, 0,06 % on the share of Salyan coastline, 0,8 % on the share of Lankaran coastline. Table 2 shows that the main gathering place for white-eyed pochard in winter and during the breeding season is Gyzylagach National Park.

An analysis of the data shows that in the above stated specially protected areas, the white-eyed pochard is a common species (i.e., more than 1 or 1 bird occurs per 1 km2 both in the breeding season and in winter). In such coastal zones of the Caspian Sea as Shahdili-Alyat, South-East Shirvan, Salyan, and Lankaran coastlines specimens are found only in winter, in Big and Small Gyzylgaz lakes, Yenikend floodplain and in the Shirvanovka lagoon they are found both in winter and during the breeding season. The species is at risk of extinction (less than 0.1 birds occur per 1 km2).

Table 2. Number of the white-eyed pochard in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Studied areas \ Years	2014		2015		2016	
	In winter	In breeding	In winter	In breeding	In winter	In breeding
Shahdili-Alyat coastline (291 km^2)	6	-	28	-	-	-
South- East Shirvan coastline (111 km^2)	12	-	19	-	38	-
Big and Small Gyzylgaz lakes (42 km^2)	9	11	16	21	13	18
1	**2**	**3**	**4**	**5**	**6**	**7**
Yenikend floodplain and Shirvanovka lagoon (141km^2)	4	-	7	6	9	11
Salyan coastline (90 km^2)	3	-	-	-	-	-
Gyzylagach National Park (891,5 km^2)	960	1200	1400	1800	2450	1100
Lankaran coastline (180 km^2)	4	-	-	-	-	-
Number and density in the total studied areas (1746,5 km^2)	998	1231	1470	1826	2510	3029

3.3. Tufted duck.

The total number of tufted duck in the coastal strip of the southern coastline of the Azerbaijani sector of the Caspian Sea in winter was 75,480 in 2014, 65,861 in 2015, and 69,720 in 2016 (Table 3). As the table 3 within 2014-2016 years, averagely 26,8 % of the total number of tufted duck occurred in the Shahdili-Alyat coastline of the Caspian Sea, 29,8% in the South - East Shirvan coastline, 2,6% in the Salyan coastline, 0,3% in the Lankaran coastline, 0,3% in the Big and Small Gyzylgaz lakes, 0,6% in the Yenikend floodplain and Shirvanovka lagoon, 39,6% in the Gyzylagach National Park.

As it has shown from the above-mentioned facts the main places of congestion for tufted duck are Shahdili-Alyat coastline, South- East Shirvan coastline, and the Gyzylagach National Park.

An analysis of the data shows that the tufted duck is numerous (i.e., 10 or more birds per 1 km2) in the Shahdili-Alyat, South - East Shirvan, Salyan coastlines of the Caspian Sea and in the Kyzylagach National Park, and it is common (i.e. 1 or more birds per 1 km2) in the Big and Small Gyzylgaz lakes, Yenikend coastline, Shirvanovka lagoon and in the Lankaran coastal strip of the Caspian Sea.

Table 3. Number (specimen) of the tufted duck in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Studied areas \ Years	2014		2015		2016		Average	
	Number	Density, 10 km^2	Number	Density, 10 km^2	Number	Density, 10 km^2	Number	Density, 10 km^2
Shahdili-Alyat coastline (291 km^2)	16000	549,8	22500	773,1	18000	618,5	18833,3	641,1
South- East Shirvan coastline (111 km^2)	19000	1711,7	23000	2072	21000	1891,8	21000	1891,8
Big and Small Gyzylgaz lakes (42 km^2)	180	42,8	211	50,2	160	38	183,6	43,7
Yenikend floodplain and Shirvanovka lagoon (141 km^2)	400	38,3	350	24,9	560	39,7	436,6	30,9
Salyan coastline (90 km^2)	1100	122,2	2500	277,7	1900	211,1	1833,3	203,7
Gyzylagach National Park (891,5 km^2)	38600	432,9	17150	192,3	27800	311,8	27850	312,9
Lankaran coastline (180 km^2)	200	11,1	150	8,3	300	16,6	216,6	12
Number and density in the total studied areas (1746,5 km^2)	75480	432,1	65861	377,1	69720	399,1	70353	402,8

3.4. Tufted duck.

This species is not numerous. The number of the greater scaup in the southern coastline of the Azerbaijan sector of the Caspian Sea in the winter of 2014 amounted to 6994, 2015 - 2440, 2016 - 5468 (Table 4). The table 4 shows that in 2014-2016 years averagely 3,7% of the total amount of the greater scoup occurred in the Shahdili-Alyat coastline of the Caspian Sea, 7,5%- in the South - East Shirvan coastline, 0,1 %- in the Salyan coastline, 0,04 %- in the Lankaran coastline, 0,4% - in the Big and Small Gyzylgaz lakes, 0,1%- in the Yenikend floodplain and Shirvanovka lagoon, 88,9% - in the Gyzylagach National Park.

Thus, the main places of congestion for the greater scaup is the Gyzylagach National Park.

An analysis of the data shows that in the South - East Shirvan coastline, in the Kyzylagach National Park the greater scaup is common (i.e. 1 or more birds per 1 km^2), in the coastlines of Shahdili-Alat, Salyan, Lankaran, in the Big and Small Gyzylgaz lakes and the Yenikend floodplain the species is at risk of extinction (i.e., 0.1 or fewer birds per 1 km^2).

Table 4. Number and density (specimen) of the greater scaup in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Years / Studied areas	2014		2015		2016		Average	
	Number	Density 10 km^2	Number	Density 10 km^2	Number	Density 10 km^2	Number	Density 10 km^2
Shahdili-Alyat coastline (291 km^2)	28	0,9	250	8	270	9	182	6
South- East Shirvan coastline (111 km^2)	48	4,3	400	36	670	60	372	33
Big and Small Gyzylgaz lakes (42 km^2)	-	0	5	1,1	8	1,9	13	0,09
Yenikend floodplain and Shirvanovka lagoon (141 km^2)	4	0,2	3	0,2	12	0,8	6	0,4
Salyan coastline (90 km^2)	7	0,2	2	0,2	8	0,8	5	0,5
Gyzylagach National Park (891,5 km^2)	6900	77,3	1780	19,9	4500	50,4	4393	49,1
Lankaran coastline (180 km^2)	7	0,3	-	0	-	0	2,3	0,1
Number and density in the total studied areas (1746,5 km^2)	6994	40	2440	13,9	5468	31,3	4967,3	28,4

3.5. Tufted duck.

The total amount of smew in the southern coastline of the Azerbaijan sector of the Caspian Sea in the winter of 2014-was 676, 2015- 825, 2016 - 1120 (Table 5). As the table shows in 2014-2016 years 36,3% of the total amount of smew occurred in the Shahdili-Alyat coastline of the Caspian Sea, 41,7%- in the South - East Shirvan coastline, 0,7 %- in the Salyan coastline, 0,1 %- in the Lankaran coastline and 21,2 %- in the Gyzylagach National Park.

As it can be seen from the above, wintering places for smew are the Shahdili-Alyat and the South - East Shirvan coastlines of the Caspian Sea and the Gyzylagach National Park.

An analysis of the data shows that in the Shahdili-Alyat and South - East Shirvan coastlines the smew is common (i.e. 1 or more birds per 1 km2), in the Salyan and Lankaran coastlines and the Kyzylagach National Park it always was not numerous (i.e., 0.1 or fewer birds per 1 km2). The smew was not recorded in the Big and Small Gyzylgaz lakes, the Yenikend floodplain and Shirvanovka lagoon.

Table 5. Number and density (specimen) of the greater scaup in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Years	2014		2015		2016		Average	
	Number	Density 10 km²	Number	Density 10 km²	Number	Density 10 km²	Number	Density 10 km²
Shahdili-Alyat coastline (291 km²)	210	7,2	410	14	370	12,7	330	11,3
South- East Shirvan coastline (111 km²)	200	1,8	315	28,5	520	46,8	349	31
Big and Small Gyzylgaz lakes (42 km²)	0		0		0		0	0
Yenikend floodplain and Shirvanovka lagoon (141 km²)	0		0		0		0	0
Salyan coastline (90 km²)	6	0,6	7	0,7	5	0,5	6	0,6
Gyzylagach National Park (891,5 km²)	260	2,9	90	1	230	2	193,3	2,1
Lankaran coastline (180 km²)	0		4	0,2	0		1,3	0,07
Number and density in the total studied areas (1746,5 km²)	676	3,8	826	4,7	1120	6,4	907,3	5,1

3.6. Red-breasted merganser.

The total amount of red-breasted merganser in the southern coastline of the Azerbaijan sector of the Caspian Sea in the winter of 2014-was 632, 2015- 758, 2016 - 695 (Table 6). As it has shown from the table in 2014-2016 years 39,6% of the total amount of red-breasted merganser occurred in the Shahdili-Alyat coastline of the Caspian Sea, 48,3%- in the South - East Shirvan coastline, 1,5 %- in the Salyan coastline, 0,5 %- in the Lankaran coastline and 10,3 %- in the Gyzylagach National Park.

As it can be seen from the above, wintering places for red-breasted merganser are the Shahdili-Alyat and the South - East Shirvan coastlines of the Caspian Sea and the Gyzylagach National Park.

Table 6. Number and density (specimen) of the red – breasted merganser in winter in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Years / Studied areas	2014		2015		2016		Average	
	Number	Density 10 km²	Number	Density 10 km²	Number	Density 10 km²	Number	Density 10 km²
Shahdili-Alyat coastline (291 km²)	200	6,8	315	10,8	310	10,6	275	9,4
South- East Shirvan coastline (111 km²)	290	26,1	400	36	318	28,6	336	30,2
Big and Small Gyzylgaz lakes (42 km²)	0		0		0		0	0
Yenikend floodplain and Shirvanovka lagoon (141 km²)	0		0		0		0	0
Salyan coastline (90 km²)	12	1,3	8	0,8	11	1,2	10	1,1
Gyzylagach National Park (891,5 km²)	130	1,4	35	0,3	50	0,5	71,6	0,8
Lankaran coastline (180 km²)	4	0,2	0	0	6	0,3	3,3	0,1
Number and density in the total studied areas (1746,5 km²)	632	3,6	758	4,3	695	3,9	695	3,9

An analysis of the data shows that in the South - East Shirvan coastlines only the red – breasted merganser is common (i.e. 1 or more birds per 1 km²), in the Shahdili-Alyat, Salyan and Lankaran coastlines of the Caspian Sea, in the Kyzylagach National Park and in the Big and Small Gyzylgaz lakes, the Yenikend floodplain it is rare (i.e., 0.1 or fewer birds per 1 km²).

3.7. Goosander.

The total amount of goosander in the southern coastline of the Azerbaijan sector of the Caspian Sea in the winter of 2014-was 140, 2015- 138, 2016 - 177 (Table 7). As it has shown from the table 23% of the total amount of goosander occurred in the Shahdili-Alyat coastline of the Caspian Sea, 28% - in the South - East Shirvan coastline, 11,4 % - in the Salyan coastline, 1,5 % - in the Lankaran coastline and 35,2 % - in the Gyzylagach National Park.

An analysis of the data shows that Shahdili-Alyat, Salyan, and Lankaran coastlines of the Caspian Sea and in the Kyzylagach National Park the goosander is rare (i.e., 0.1 or fewer birds per 1 km^2) and in the Big and Small Gyzylgaz lakes, the Yenikend floodplain it as not recorded.

Table 7. Number and density (specimen) of the goosander in winter in the south coastline of the Azerbaijan sector of the Caspian Sea in 2014-2016

Studied areas / Years	2014		2015		2016		Average	
	Number	Density 10 km^2	Number	Density 10 km^2	Number	Density 10 km^2	Number	Density 10 km^2
Şahdili-Ələt coastline (291 km^2)	7	0,2	41	1,4	57	1,9	35	1,2
Shahdili-Alyat coastline (291 km^2)	36	3,2	28	2,5	67	6	43	3,8
South- East Shirvan coastline (111 km^2)	0		0		0		0	0
Big and Small Gyzylgaz lakes (42 km^2)	0		0		0		0	0
Yenikend floodplain and Shirvanovka lagoon (141 km^2)	13	1,4	21	2,3	18	2	17,3	1,9
Salyan coastline (90 km^2)	80	0,8	45	0,5	35	0,3	53	0,5
Gyzylagach National Park (891,5 km^2)	4	0,2	3	0,1	-	0	2,3	o,ı
Lankaran coastline (180 km^2)	140	0,8	138	0,7	177	1	151,6	0,8

4. Results and recommendations

1. The common pochard and tufted duck are numerous (respectively 33,6 specimens per 1 km^2 and 40,2 specimens per 1 km^2), greater scaup is common (28 specimens per 1 km^2), white-eyed pochard, smew, and red-breasted merganser are rare (respectively 0,5 specimens per 1 km^2, 0,3 specimens per 1 km^2) in the southern coastline of the Azerbaijan sector of the Caspian Sea. Only single specimens of the goosander were registered not depending on the effects of anthropogenic and abiotic factors.
2. The main concentration places of diving ducks and mergansers are Shahdili coastline, Kyzylagach National Park and South - East Shirvan coastline.

To maintain and increase the number of diving ducks and mergansers in the southern coastlines of the Azerbaijan sector of the Caspian Sea, the following recommendations should be implemented:

1) Increase in the area of Shahdili coastline;
2) Granting the status of the reserve to the Yenikend lagoon and the adjacent coastal waters of the Caspian;
3) Felling of old dense reed beds in the water bodies of the Kyzylagach National Park;

Removal of diving ducks and mergansers from the list of hunting birds.

References

[1] I. R. Babayev, Species composition, number, distribution of waterfowl and factors affecting on them in winter in the Caspian Sea coast bodering with South-Eastern Shirvan plain. Pr. of the Society of Azerbaycan Zoologists. Baku, “Elm”, 1 (2008) 601-606.

[2] Red Book of the Azerbaijan Republic, IV Section. Birds (Aves), Baku. (2013) 261-405 (In Azerbaijani).

[3] A. H. Gasymov, Ecology of the Caspian plankton. Baku, “Adiloghlu”. (2004) 550 pages.

[4] I. R. Babayev, The current status of habitats of wetland birds in southeast Azerbaijan, Abstracts of a scientific and practical conference dedicated to the 95th anniversary of academician H.Aliyev. El Alliance Company. (2002) 272-273.

[5] I. R. Babayev, F. Asgerov, F. T. Ahmadov, Biological diversity: Waterfowl of the Azerbaijan part of the Caspian Sea. “Nurlar”, Baku, (2006) 69 pages.

[6] I. R. Babayev, A. N. Abbasov, Number, distribution of waterfowl in the Kyzylagach National Park and factors affecting on them. Pr. of the Institute of Zoology. 28 (2) Baku, “Elm”, (2006) 123 -131.

[7] I. R. Babayev, The main places of accumulation of waterfowl and factorf affecting on them in the Absheron-Gobustan coasline of Caspian Pr. of the Society of Azerbaycan Zoologists. V 2. Baku, “Elm”, (2010) 816-824.

[8] I. R. Babayev, S. S. Rajabova, S. H. Samadova, Distribution and number of rare and disappearing birds and factors affecting on them in the Azerbaijan coastlines of the Caspian Sea. Pr. of the Institute of Zoology, 33 (2) Baku, “Muallim”, (2015) 5-16.

[9] National Atlas of the Azerbaijan Republic, General geographic map. State Cartography Committee, Baku. (2014) 22-23 (In Azerbaijani).

[10] I. R. Babayev, The wetland birds in the south part of the Azerbaijan coastlines of the Caspian Sea // Abstracts of a scientific and practical conference dedicated to the 80th anniversary of president of the Azerbaijan republic H.A.Aliyev, Baku, (2003) 144-146.

[11] D. G. Tuayev, Azerbaijan birds catalog, Baku, “Elm”, (2000) 240 pages.

[12] D. A. Scott, P. M. Rose, Atlas of Anatidae Populations in Africa and Western Eurasia // Wetlands International, Wagenigen, The Netherlands, (1996).

[13] S. Delany, D. A. Scott, Waterfowl Population Estimates, Third edition Wetlands International Global Series N 12, Wagenigen, The Netherlands. (2002).

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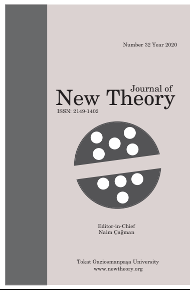

On Proper Class Coprojectively Generated by Modules With Projective Socle

Ayşe Çobankaya[1]

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Abstract — Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of modules and module homomorphism. $\mathcal{E}$ is called gd-closed sequence if $\operatorname{Im} f$ is gd-closed in B. In this paper, the proper class $\mathcal{GD} - \mathcal{C}losed$, which is coprojectively generated by modules with projective socle, be studied and also its relations among $\mathcal{N}eat, \mathcal{C}losed, \mathcal{D} - \overline{\mathcal{C}losed}, \mathcal{S} - \overline{\mathcal{C}losed}$ be investigated. Additionally, we examine coprojective modules of this class.

Keywords — *$\mathcal{GD} - \mathcal{C}losed$, g-semartinian modules, G-Dickson torsion theory, gdc-flat modules*

Introduction

Throughout the paper, we assume that all rings are associative with identity and all modules are unitary right modules. As usual, $Mod - R$ denotes the category of all right R-modules over a ring R.

A submodule N of a module M is *essential* (or *large*) in M, denoted $N \trianglelefteq M$, if for every $0 \neq K \leq M$, we have $N \cap K \neq 0$; and N is said to be *closed* in M if N has no proper essential extension in M. We also say in this case that N is a closed submodule of M. Closed submodules are important in rings and modules, and relative homological algebra. See, for example, [1] for their properties. More recently, many authors have studied their generalizations, some of which are neat submodules, $\mathcal{S}$-closed submodules, $\mathcal{D}$-closed submodules (see, for example, [2,3,6,11,12,15,18]).

The singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that, $mI = 0$ for some essential right ideal I of R. This takes the place of the torsion submodule in general setting. The module M is called nonsingular if $Z(M) = 0$, and singular if $M = Z(M)$, while the right singular ideal of R is $Z_r(R) = Z(R_R)$. The ring R is said to be right nonsingular if it is nonsingular as a right R-module. The Goldie torsion submodule $Z_2(M)$ of M is defined by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$. A module M is called Goldie torsion if $Z_2(M) = M$. A module M is called semiartinian if every non-zero homomorphic image of M contains a simple submodule, that is, $\operatorname{Soc}(M/N) \neq 0$ for every submodule $N \leq M$. A module M is called socle-free if $Soc(M) = 0$. It is well known that a simple module is either singular or projective. A module M is called *g-semiartinian* if every non-zero homomorphic image of M contains a singular simple submodule. The class of g-semiartinian modules is a torsion class of G-Dickson torsion theory, which is generated by singular simple modules [9]. The class of modules with projective socle is the torsion-free class of the same torsion theory. Note that the class of all g-semiartinian modules is closed under submodules, homomorphic images, direct sums and extensions, while the class of all socle-projective (nonsingular, socle-free) modules is closed under submodules, direct products, extensions.

[1](Corresponding Author)acaylak@cu.edu.tr;
[1]Department of Mathematics, Çukurova University, Adana, Turkey

The notion of s-closed submodules is a generalization of the notion of closed submodules and it was introduced in [16]. A submodule $N \leq M$ is called *s-closed* if M/N is nonsingular. Later, in [12], d-closed submodules were introduced. A submodule $N \leq M$ is called *d-closed* if M/N has zero socle. Inspired from these, a submodule N of a module M is called *gd-closed* if there is a submodule S in M such that $S \cap N = 0$ and $\mathrm{Soc}(M/(S \oplus N))$ is projective. Note that $\mathrm{Soc}(X)$ is projective for each nonsingular module X, and hence any s-closed submodule is gd-closed. Also, since zero module is projective, any d-closed submodule is gd-closed. A short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ is called gd-closed (respectively, s-closed, d-closed) sequence if $\mathrm{Im}(f)$ is gd-closed (respectively, s-closed, d-closed) in B. The class of gd-closed sequences is a proper class [19, Theorem 2.1], which is projectively generated by g-semiartinian modules ([8, Proposition 3.5]). In general, the class of s-closed (respectively d-closed) exact sequences is not a proper class, ([11], [12]). But the class of gd-closed exact sequence is a proper class by [19, Theorem 2.1]. We will call a module M is gdc-flat if every exact sequence ending with M is gd-closed sequence. First of all, it is obvious that projective modules are gdc-flat. Furthermore, nonsingular modules and modules with zero socle are less obvious examples of gdc-flat modules.

After this introductory section, this paper is divided into three sections. In Section 2, we recall some torsion theoretic concepts and then give some properties of proper classes. In Section 3, we prove some inclusion relations among $\mathcal{GD}-\mathcal{C}losed$ and the well-known proper classes, such as $\mathcal{S}-\overline{\mathcal{C}losed}$, $\mathcal{D}-\overline{\mathcal{C}losed}$, $\mathcal{N}eat$. We show that R is a C-ring if and only if $\mathcal{GD}-\mathcal{C}losed = \mathcal{S}-\overline{\mathcal{C}losed}$. (Proposition 3.7). For any ring R, we show that $\mathcal{GD}-\mathcal{C}losed = \mathcal{N}eat$ if and only if each g-semiartinian module T can be represented as $T = S \oplus P$, where S is semisimple goldie torsion module and P is a projective module (Proposition 3.8).

In Section 4, we investigate modules M such that each short exact sequence ending with M belongs to $\mathcal{GD}-\mathcal{C}losed$. Such modules are called gdc-flat modules. We prove that if the torsion submodule $\tau_{gd}(E)$ of an injective module E is projective, then E is gdc-flat; and the converse is true for C-rings (Theorem 4.8). Moreover, for C-rings, we prove that all injective modules are gdc-flat if and only if the injective hull of each g-semiartinian module is projective if and only if every semiartinian module embeds in a projective module (Theorem **??**). Finally, we prove that every $\mathcal{GD}-\mathcal{C}losed$ sequence is a $\mathcal{P}ure$ sequence if and only if every module with a projective socle is flat if and only if every gdc-flat module is flat if and only if every finitely generated module with a projective socle is flat (Proposition 4.12).

We use the notation $E(M)$, $Soc(M)$, $Sa(M)$, $Z(M)$, $Z_2(M)$ for the injective hull, socle, semi-artinian, singular, Goldie torsion submodule of a module M, respectively. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M will be denoted by M^+. For all other basic or background material, we refer the reader to [5, 14, 22].

Proper Classes and Torsion Theories

In this section, we denote by $\mathcal{P}$ a class of short exact sequences of modules and module homomorphisms. Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence belonging to $\mathcal{P}$. We call f and g a $\mathcal{P}$-monomorphism and a $\mathcal{P}$-epimorphism, respectively. Note that the monomorphism f and the epimorphism g uniquely determine the exact sequence $\mathcal{E}$ up to isomorphism. The following definition is given in the sense of Buchsbaum [4].

Definition 2.1. The class $\mathcal{P}$ is called proper if it satisfies the following conditions (see for example [5, 19, 23]).

(P-1) If a short exact sequence $\mathcal{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathcal{E}$.

(P-2) $\mathcal{P}$ contains all the splitting exact sequences.

(P-3) The composite of two $\mathcal{P}$-monomorphisms (respectively $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (respectively $\mathcal{P}$-epimorphism) when the composite is defined.

(P-4) If g and f are monomorphisms and gf is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms and gf is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

Let $\mathcal{P}$ be a proper class. A module M is called $\mathcal{P}$-projective if it is projective with respect to all short exact sequences in $\mathcal{P}$, that is, $\mathrm{Hom}(M;\mathcal{E})$ is exact for every $\mathcal{E}$ in $\mathcal{P}$. A module M is called $\mathcal{P}$-coprojective if every short exact sequence which ends with M belongs to $\mathcal{P}$. For a given class $\mathcal{M}$ of modules, we denote the smallest proper class for which each $M \in \mathcal{M}$ is $\overline{k}(\mathcal{M})$-coprojective by $\overline{k}(\mathcal{M})$. This proper class is called the proper class *coprojectively generated* by $\mathcal{M}$. The largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective is called the proper class *projectively generated* by $\mathcal{M}$.

We now give some examples of proper classes. Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of modules.

1. We denote the smallest proper class of short exact sequences by $\mathcal{S}plit$. This class consists of all splitting exact sequences of modules. Also, we denote the largest proper class of short exact sequence by $\mathcal{A}bs$. This class consists of all short exact sequences of modules.

2. A submodule N of a module M is called essential if every non-zero submodule of M has a non-zero intersection with N. A submodule N of a module M is called *closed* if it has no proper essential extension. $\mathcal{E}$ is called closed if $\mathrm{Im}\, f$ is closed in B. We denote the class of all closed short exact sequences by $\mathcal{C}losed$. It is known that $\mathcal{C}losed$ is a proper class (see [5, 10.5]) .

3. A submodule N of a module M is called pure if the sequence $0 \to N \otimes_R X \to M \otimes_R X$ is exact for every left R-module X. $\mathcal{E}$ is called a pure sequence if $\mathrm{Im}\, f$ is pure in B. The class of pure exact sequences is denoted by $\mathcal{P}ure$ and it is a proper class. (see, [10]).

4. A submodule N of a module M is called neat if the sequence $\mathrm{Hom}(S, M) \to \mathrm{Hom}(S, M/N) \to 0$ is exact for every simple module S. $\mathcal{E}$ is called a neat sequence if $\mathrm{Im}\, f$ is neat in B. The class of neat exact sequences is denoted by $\mathcal{N}eat$ and it is a proper class. It is projectively generated by simple modules (see [21]).

5. A submodule N of a module M is called extended s-closed if there is $S \subset M$ such that $S \cap N = 0$ and $M/(S \oplus N)$ is nonsingular. $\mathcal{E}$ is called extended s-closed sequence if $\mathrm{Im}\, f$ is extended s-closed in B. The class of extended s-closed exact sequences is denoted by $\mathcal{S} - \overline{\mathcal{C}losed}$ and it is a proper class. It is projectively generated by goldie torsion modules (see [11, Proposition 3.3, Proposition 3.4]).

6. A submodule N of a module M is called extended d-closed if there is $S \subset M$ such that $S \cap N = 0$ and $M/(S \oplus N)$ has zero socle. $\mathcal{E}$ is called extended d-closed sequence if $\mathrm{Im}\, f$ is extended d-closed in B. The class of extended d-closed exact sequences is denoted by $\mathcal{D} - \overline{\mathcal{C}losed}$ and it is a proper class. It is projectively generated by semiartinian modules (see [12, Proposition 5, Proposition 6]).

   In this paper, we study gd-closed submodules. A submodule N of a module M is called gd-closed if there is $S \subset M$ such that $S \cap N = 0$ and $M/(S \oplus N)$ has projective socle. $\mathcal{E}$ is called gd-closed sequence if $\mathrm{Im}\, f$ is gd-closed in B. G-semiartinian modules were introduced in [9]. A module M is g-semiartinian if every non-zero homomorphic image of M contains a simple singular module. The class of gd-closed exact sequences is denoted by $\mathcal{GD} - \mathcal{C}losed$ and it is a proper class. This class is projectively generated by g-semiartinian modules (see [19, Theorem 2.1]).

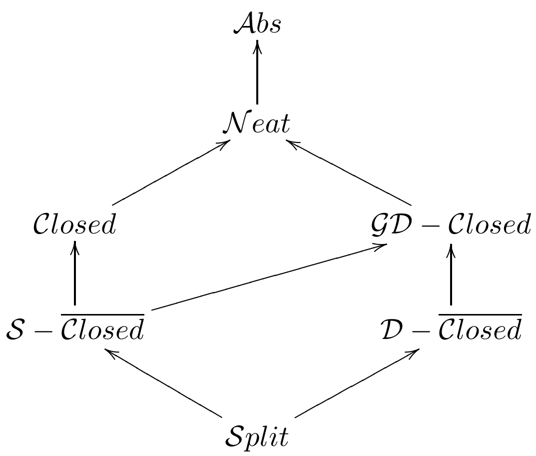
The following result can be obtained from [8, Proposition 3.5], so the proof is omitted.

Proposition 2.2. A submodule A of a module B is gd-closed in B if and only if $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, B/A) \to 0$ is exact for each g-semiartinian module T.

The class of all g-semiartinian modules is the torsion class of a torsion theory. This torsion theory is called G-dickson torsion theory and denoted by $\tau_{gd} = (\mathbb{T}_{GD}, \mathbb{F}_{GD})$ where $\mathbb{T}_{GD}$ and $\mathbb{F}_{GD}$ represent the torsion class and the torsion free class, respectively. Note that $\mathbb{F}_{GD}$ consists of the modules with projective socle. G-dickson torsion theory was introduced in [9]. Note that τ_{gd} is hereditary, that is, $\mathbb{T}_{GD}$ is closed under submodules. $\mathbb{T}_{GD}$ is closed under homomorphic images, direct sums and extensions as well. Furthermore, $\mathbb{F}_{GD}$ is closed under submodules, direct products, extensions and injective hulls. Any module M contains a unique maximal g-semiartinian submodule which is denoted by $\tau_{gd}(M)$. It is clear that $\tau_{gd}(M) = 0$ if and only if $\mathrm{Soc}(M)$ is projective. In addition $M/\tau_{gd}(M)$ has a projective socle. Note that $\tau_{gd}(M) = Z_2(M) \cap Sa(M)$.

Inclusion relations among the proper classes of modules

In this section, we prove some inclusion relations among the proper classes of modules that are considered in the present paper. These relations are given in the following diagram:



First of all, it is clear that $\mathcal{A}bs$ contains any proper class and $\mathcal{S}plit$ is contained in any proper class. It is clear that $\mathcal{S}-\overline{\mathcal{C}losed} \subset \mathcal{C}losed \subset \mathcal{N}eat$. Since every simple module is semiartinian, $\mathcal{D}-\overline{\mathcal{C}losed} \subset \mathcal{N}eat$. Also, $\mathcal{C}losed = \mathcal{N}eat$ exactly when R is a C-ring [17, Theorem 5]. In general, an extended s-closed submodule of a module is not extended d-closed submodule, and conversely, an extended d-closed submodule is not necessarily s-closed submodule. However, it was proven that $\mathcal{D}-\overline{\mathcal{C}losed} \subset \mathcal{S}-\overline{\mathcal{C}losed}$ if and only if R is C-ring [13, Proposition 3.5], and $\mathcal{D}-\overline{\mathcal{C}losed} \supset \mathcal{S}-\overline{\mathcal{C}losed}$ if and only if every simple module is singular [13, Proposition 3.6].

The following result can be obtained by [8, Proposition 3.7], so the proof is omitted.

Proposition 3.1. Let $\mathcal{K}$ be the class of all modules with projective socle. Then $\overline{k}(\mathcal{K}) = \mathcal{GD}-\mathcal{C}losed$.

Proposition 3.2. The following statements are equivalent for a ring R:

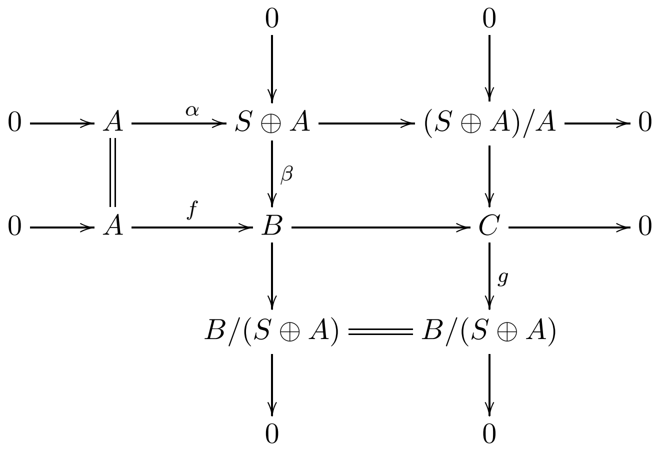
1. Every module has projective socle.
2. $\mathcal{A}bs = \mathcal{GD}-\mathcal{C}losed$.
3. Every g-semiartinian module is projective.
4. R is semisimple.

Proof. (1) $\Rightarrow$ (2) If every module has projective socle, then $\mathcal{A}bs = \mathcal{GD}-\mathcal{C}losed$. (2) $\Rightarrow$ (3) This is an obvious consequence of Proposition 2.2. (3) $\Rightarrow$ (4) Every simple module is projective by assumption, that is R is semisimple ring. (4) $\Rightarrow$ (1) By assumption, every module is semisimple, and so every simple module is projective. Thus, every module has projective socle. □

Proposition 3.3. $\mathcal{S}plit = \mathcal{GD}-\mathcal{C}losed$ if and only if every module with projective socle is projective.

Proof. Let C be a a module with projective socle. We consider the short exact sequence $0 \to A \to P \to C \to 0$ with P projective. Since $C \cong P/A$, the sequence is a $\mathcal{GD}-\mathcal{C}losed$ exact sequence, and thus it splits. Since C is a direct summand of P, C is a projective module. For the converse implication, let $0 \to A \xrightarrow{f} B \to C \to 0$ be a $\mathcal{GD}-\mathcal{C}losed$ exact sequence. Then there is a submodule

S of B such that $S \cap A = 0$ and $B/(S \oplus A)$ has projective socle. Now, consider the following diagram:



Since $B/(S \oplus A)$ is a projective module, β is $\mathcal{S}plit$-monomorphism and thus $f = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism. □

We know that $\mathcal{D} - \overline{\mathcal{C}losed}$ is projectively generated by semiartinian modules and $\mathcal{GD} - \mathcal{C}losed$ is projectively generated by g-semiartinian modules. Since each g-semiartinian module is semiartinian, it follows that $\mathcal{D} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$. See [9, Proposition 1] for the proof of the following proposition.

Proposition 3.4. Let R be a ring. Every semiartinian $R-$module is g-semiartinian if and only if every simple $R-$module is singular.

Proposition 3.5. Let R be a ring such that every semiartinian $R-$module is g-semiartinian. Then $\mathcal{GD} - \mathcal{C}losed = \mathcal{D} - \overline{\mathcal{C}losed}$.

Proof. $\mathcal{D} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$ because every g-semiartinian module is semiartinian. $\mathcal{GD} - \mathcal{C}losed \subset \mathcal{D} - \overline{\mathcal{C}losed}$ is clear by assumption.

□

It is known that $\mathcal{S} - \overline{\mathcal{C}losed}$ is projectively generated by goldie torsion modules. Since each g-semiartinian is goldie torsion module, it follows that $\mathcal{S} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$. For the proof of the following Proposition, see [9, Theorem 4].

Proposition 3.6. R is a C-ring if and only if the class of g-semiartinian modules and the class of goldie torsion modules are the same.

Proposition 3.7. R is a C-ring if and only if $\mathcal{GD} - \mathcal{C}losed = \mathcal{S} - \overline{\mathcal{C}losed}$.

Proof. If R is a C-ring, then the class of g-semiartinian modules and the class of Goldie torsion modules are the same. This implies that $\mathcal{S} - \overline{\mathcal{C}losed}$ is projectively generated by g-semiartinian modules, hence $\mathcal{GD} - \mathcal{C}losed = \mathcal{S} - \overline{\mathcal{C}losed}$. Conversely, let M be a singular module. Then $M \cong F/A$ with F is projective and A is essential in F. Assume that M has a zero socle. Now consider the exact sequence $\mathcal{E} : 0 \to A \to F \to M \to 0$. since zero module is projective, $\mathcal{E}$ is gd-closed exact sequence. By assumption, there exists a submodule S of F such that $A \cap S = 0$ and $F/(A \oplus S)$ is nonsingular. But A is essential in F, which gives a contradiction.

□

Since $\mathcal{N}eat$ is projectively generated by simple (singular) modules and simple (singular) modules are g-semiartinian, it follows that $\mathcal{GD} - \mathcal{C}losed \subset \mathcal{N}eat$.

Proposition 3.8. Let R be a ring. $\mathcal{GD} - \mathcal{C}losed = \mathcal{N}eat$ if and only if each g-semiartinian module T can be represented as $T = S \oplus P$, where S is semisimple goldie torsion module and P is a projective module.

Proof. Let $\mathcal{E} : 0 \to A \to B \to B/A \to 0$ be a neat exact sequence and T be a g-semiartinian module. By assumption, $\mathcal{E}$ is a gd-closed exact sequence. Therefore the sequence $\text{Hom}(T, B) \to \text{Hom}(T, B/A) \to 0$ is exact and this means that T is $\mathcal{N}eat$-projective. By [15, Theorem 2.6], T is a direct sum of a projective module and a semisimple module. Furthermore, since T is g-semiartinian, the semisimple component of T should be goldie torsion. Conversely, by assumption, we notice that every g-semiartinian module is $\mathcal{N}eat$-projective. It follows that every neat exact sequence is a gd-closed sequence.

□

Proposition 3.9. If R is Goldie torsion, then $\mathcal{GD} - \mathcal{C}losed = \mathcal{D} - \overline{\mathcal{C}losed}$.

Proof. If R is Goldie torsion, then every right module is Goldie torsion. Since each semiartinian modules are g-semiartinian, it follows that $\mathcal{GD} - \mathcal{C}losed \subset \mathcal{D} - \overline{\mathcal{C}losed}$. The implication $\mathcal{D} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$ is clear because every g-semiartinian module is semiartinian. □

gdc-flat Modules

An R-module M is called *flat* if the functor $M \otimes_R -$ is exact. Note that M is flat if and only if every short exact sequence $0 \to A \to B \to M \to 0$ of R-modules is pure-exact ([22, Proposition 3.67]). This connection between flatness and purity give rise to a lot of studies on some classes of modules defined throught closed, neat submodules. That is, the modules M such that any short exact sequence ending with M is included in $\mathcal{C}losed$ ($\mathcal{N}eat$), which are called *weakly-flat* (respectively, *neat-flat*), have been studied in [24], (respectively [2]). Gdc-flat modules were introduced in [7] throught this relation. A module M is called gdc-flat if every exact sequence ending with M is a gd-closed sequence. By Poposition 2.2, it follows that M is gdc-flat if and only if for any epimorphism $Y \to M$, $\text{Hom}(T, Y) \to \text{Hom}(T, M) \to 0$ is exact for any g-semiartinian module T.

Remark 4.1. Projective modules are gdc-flat since any exact sequence ending with a projective module splits and splitting sequences are gd-closed sequences.

The following Proposition gives a characterization of gd-closed sequences. See [7, Proposition 2.2].

Proposition 4.2. The following are equivalent for a module M.

1. M is gdc-flat.
2. There exists a gd-closed exact sequence with $0 \to K \to F \to M \to 0$ with F is projective.
3. There exists a gd-closed exact sequence with $0 \to K \to F \to M \to 0$ with F is gdc-flat.

Remark 4.3.
1. A module with projective socle is gdc-flat.
2. R is a semisimple ring if and only if every right R-module is gdc-flat.
3. The class of gdc-flat modules is closed under finite direct sums and direct summands.
4. Nonsingular modules are gdc-flat.
5. The class of gdc-flat modules is closed under extensions.
6. A g-semiartinian module is gdc-flat if and only it is projective by Proposition 2.2.

Definition 4.4. R is called right PS-ring if every simple right ideal of R is projective, that is $\text{Soc}(R_R)$ is projective ([20]).

Proposition 4.5. [7, Proposition 3.1] The following are equivalent.

1. R is a PS-ring.
2. Gdc-flat modules are exactly the modules with projective socle.
3. Every submodule of a gdc-flat module is gdc-flat.

4. Every right ideal of R is a gdc-flat module.

Proposition 4.6. Let $f : N \to M$ be an epimorphism of modules. If M is a gdc-flat module, then any g-semiartinian submodule of M is isomorphic to a g-semiartinian submodule of N. In particular, the torsion submodule $\tau_{gd}(M)$ of M embeds in a projective module.

Proof. Take a g-semiartinian submodule T of M and consider the inclusion map $\mathfrak{i} : T \to M$. Since M is gdc-flat, the sequence $\mathrm{Hom}(T, N) \to \mathrm{Hom}(T, M) \to 0$ is exact. It follows that there exists $g : T \to N$ such that $fg = \mathfrak{i}$. Clearly g is a monomorphism and $g(T)$ is a g-semiartinian submodule of N since g-semiartinian modules are closed under homomorphic images. For the particular case, take an epimorphism $f : P \to M$ with P is projective and let $T = \tau_{gd}(M)$. □

Definition 4.7. A ring R is said to be a C-ring if for every R-module of B and every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$.

The notion of a right C-ring has been introduced in [21]. It is known that R is a C-ring if and only if $\mathrm{Soc}(R/I) \neq 0$ for every essential ideal I of R .

Theorem 4.8. Let E be an injective module. If the torsion submodule $\tau_{gd}(E)$ of E is projective, then E is gdc-flat module. Furthermore, the converse statement holds for right C-rings.

Proof. Assume that $\tau_{gd}(E)$ is a projective module and $0 \to A \to F \xrightarrow{g} E \to 0$ be a short exact sequence with F is projective. Let T is a g-semiartinian module and $f : T \to E$ be a homomorphism. We seek a homomorphism $h : T \to F$ such that $gh = f$. As T is a g-semiartinian module, $f(T)$ is also g-semiartinian. It follows that there is an inclusion map $\mathfrak{i} : f(T) \to \tau_{gd}(E)$. Let $\mathfrak{i}' : f(T) \to E$ be the other inclusion map. The following commutative diagram is obtained by combining these maps:

$$\begin{array}{ccccccccc}
 & & & & T & & \\
 & & & & \downarrow{\scriptstyle f'} & & \\
 & & & & f(T) & \xrightarrow{\mathfrak{i}} & \tau_{gd}(E) \\
 & & & & \downarrow{\scriptstyle \mathfrak{i}'} & & \\
0 & \longrightarrow & A & \longrightarrow F \xrightarrow{g} & E & \longrightarrow & 0
\end{array}$$

Injectivity of E implies existence of a homomorphism $v : \tau_{gd}(E) \to E$ such that $v\mathfrak{i} = \mathfrak{i}'$. Projectivity of $\tau_{gd}(E)$ implies that there is a homomorphism $u : \tau_{gd}(E) \to F$ such that $gu = v$. Setting $h = u\mathfrak{i}f'$, we get that $gh = f$. Therefore E is gdc-flat by Proposition 4.2.

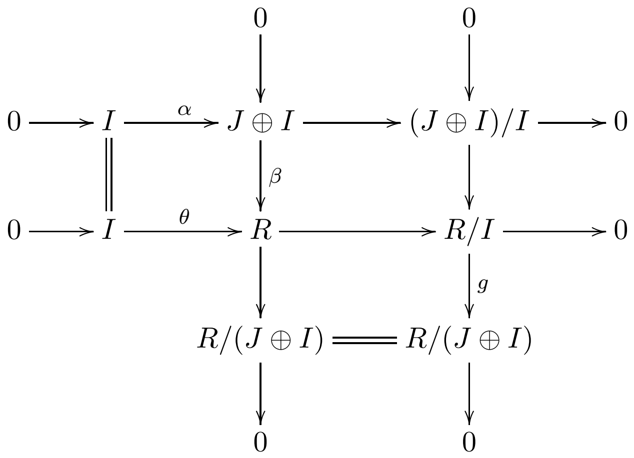
For the converse, suppose that E is gdc-flat. Proposition 4.6 implies that $\tau_{gd}(E)$ embeds in a projective module P. Since $\mathrm{Soc}(E/\tau_{gd}(E))$ is projective, $\tau_{gd}(E)$ is a gd-closed submodule of E and this means that it is a neat submodule of E. Since neat submodules and closed submodules coincide for a right C-ring, it follows that $\tau_{gd}(E)$ is closed in E. Hence $\tau_{gd}(E)$ is a direct summand of E because E is injective. It follows that $\tau_{gd}(E)$ is also injective. Therefore $\tau_{gd}(E)$ is a direct summand of P and it is projective. □

Definition 4.9. A ring R is called a right Kasch ring if each simple R-module embeds in R_R.

For a C-ring, it is true that the injective hull of a g-semiartinian module is also g-semiartinian. We use this fact in the following proof.

Proposition 4.10. Every cyclic module with projective socle is projective if and only if every cyclic gdc-flat module is projective.

Proof. ($\Rightarrow$) Let M be a cyclic gdc-flat module. Then there is a right ideal I of R such that $M \cong R/I$. Since I is gd-closed ideal of R, there is a right ideal J of R such that $J \cap I = 0$ and $\mathrm{Soc}(R/(J \oplus I))$ is projective. Consider the following commutative diagram:



$R/(J \oplus I)$ is projective since $\mathrm{Soc}(R/(J \oplus I))$ is projective. It follows that β is $\mathcal{S}plit$-monomorphism. Moreover, α is a $\mathcal{S}plit$-monomorphism. Therefore $\theta = \beta\alpha$ is a $\mathcal{S}plit$-monomorphism. Hence $M \cong R/I$ is projective since it is a direct summand of R.
($\Leftarrow$) Let M be a cyclic module with projective socle. Then it is gdc-flat by Remark 4.3 (1). Therefore M is projective. □

Proposition 4.11. The following statements are equivalent.

1. Every module with projective socle is projective.
2. Every gdc-flat module is projective.
3. Every gd-closed submodule of a projective module is a direct summand.

Proof. (1)$\Rightarrow$ (2) Let C be a gdc-flat module. Then there is a projective module B and gd-closed sequence $0 \to A \xrightarrow{f} B \to C \to 0$. It means that there is a submodule $S \subset B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/S \oplus A)$ is projective. Consider the diagram in the proof of Proposition 3.3. β is a $\mathcal{S}plit$-monomorphism since $B/(S \oplus A)$ is projective by assumption. It follows that $f = \beta\alpha$ is a $\mathcal{S}plit$-monomorphism as well. Since C is a direct summand of B, it is projective.
(2) $\Rightarrow$ (3) Let B be a projective module and A be a gd-closed submodule of B. Proposition 4.2 implies that C is gdc-flat since $\mathcal{E} : 0 \to A \to B \to C \to 0$ is gd-closed with B is projective. By assumption, C is projective, hence $\mathcal{E}$ splits. It follows that A is a direct summand of B.
(3)$\Rightarrow$ (1) Let C be a module and assume that it has a projective socle. Then there exists a projective module B and a gd-closed exact sequence $0 \to A \to B \to C \to 0$. Hence, A is a direct summand of B by assumption. Indeed, $B \cong A \oplus C$. Thus C is projective. □

Proposition 4.12. The following are equivalent.

1. Every gd-closed sequence is pure.
2. Every gdc-flat is flat.
3. Every module with projective socle is flat.
4. Every finitely generated module with projective socle is flat.

Proof. The implications (1)$\Rightarrow$(2), (2)$\Rightarrow$(3), (3)$\Rightarrow$(4) are obvious. (4)$\Rightarrow$(1) By [22, Corollary 3.49], every module having projective socle is flat. Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \to C \to 0$ be a gd-closed exact sequence, It means that there exists a submodule $S \subset B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A))$ is projective. Now consider the commutative diagram in the proof of Proposition 4.11. β is a $\mathcal{P}ure$-monomorphism since $B/(S \oplus A)$ is flat. Hence $f = \beta\alpha$ is a $\mathcal{P}ure$-monomorphism. □

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References

[1] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending modules*, Pitman Research Notes in Math. Ser. 313 Longman Scientific and Technical, Harlow, 1994.

[2] E. Büyükaşık, Y. Durğun, *Neat-flat Modules*, Comm. Algebra 44 (2016) 416-428.

[3] E. Büyükaşık, Y. Durğun, *Absolutely s-pure modules and neat-flat modules*, Comm. Algebra 43 (2)(2015) 384-399.

[4] A. David Buchsbaum, *A note on homology in categories*, Ann. of Math. (69) (2) (1959) 66-74.

[5] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules*, Birkhäuser Verlag, Basel, 2006.

[6] S. Crivei, S.Şahinkaya, *Modules whose closed submodules with essential socle are direct summands*, Taiwanese J. Math. 18(4)(2014) 989-1002.

[7] Y. Durğun, A. Çobankaya, *On subprojectivity domains of g-semiartinian modules,* J. Algebra Appl. (2021) https://doi.org/10.1142/S021949882150119X, (in press).

[8] Y. Durğun, A. Çobankaya, *Proper classes generated by t-closed submodules*, An. Şt. Univ. Ovidius Constanta 27 (2019) 83-95.

[9] Y. Durğun, A. Çobankaya, *G-Dickson Torsion Theory*, International Science, Mathematics and Engineering Sciences Congress (2019) 978-983.

[10] P. M. Cohn, *On the free product of associative rings* Math. Z. 71(1959) 380-398.

[11] Y. Durğun, S. Özdemir, *On S-Closed Submodules*, J. Korean Math. Soc. 54 (2017) 1281-1299.

[12] Y. Durğun, S. Özdemir, *On D-Closed Submodules*, Proc. Indian Acad. Sci (Math. Sci.) 130 (1)(2020) 14pp.

[13] Y. Durğun, *D-Extending Modules,* Hacet. J. Math. Stat. (49)(2020) 1-7, https://doi.org/10.15672/hujms. 460241.

[14] E. E. Enochs, O. M. G. Jenda, *Relative Homological Algebra* Walter de Gruyter, Berlin-New York, 2000.

[15] L. Fuchs, *Neat submodules over integral domains*, Period. Math. Hungar. 64(2)(2012) 131-143.

[16] K.R. Goodearl, *Singular torsion and the splitting properties*, Amer. Math. Soc. 124 , Providence, R. I. 1972.

[17] A. I. Generalov, *On weak and ω-high purities in the category of modules*, Mat. Sb. (N.S.) 105(147)(3) (1978), 389-402.

[18] Y. Kara, A. Tercan, *When some complement of a z-closed submodule is a summand*, Comm. Algebra, 46(7)(2018) 3071-3078.

[19] Kepka, T., On one class of purities, *Comment. Math. Univ. Carolinae*, 14 (1973), 139-154.

[20] W.K. Nicholson, J.F. Watters, *Rings with projective socle*, Proc. Amer. Math. Soc., 102 (1988) 443-450.

[21] G. Renault, *Étude de certains anneaux liés aux sous-modules compléments d'un A-module*, C. R. Acad. Sci. Paris 259 (1964) 4203-4205.

[22] J. Rotman, *An Introduction to Homological Algebra*, Universitext, Springer-Verlag, New York, 2009.

[23] E.G. Skljarenko, *Relative homological algebra in the category of modules*, Russian Math. Surveys, 333(201) (1978) 85-120.

[24] H. Zöschinger, *Schwach-Flache Moduln.* Comm. Algebra 41 (12) (2013) 4393-4407.