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Florentin Smarandache

Mathematics and Science Department, University of New Mexico, New Mexico 87301, USA

email: fsmarandache@gmail.com

Bijan Davvaz

Department of Mathematics, Yazd University, Yazd, Iran

email: davvaz@yazd.ac.ir

Pabitra Kumar Maji

Department of Mathematics, Bidhan Chandra College, Asansol 713301, Burdwan (W), West Bengal, India.

email: pabitra_maji@yahoo.com

Harish Garg

School of Mathematics, Thapar Institute of Engineering & Technology, Deemed University, Patiala-147004, Punjab, India

email: harish.garg@thapar.edu

Jianming Zhan

Department of Mathematics, Hubei University for Nationalities, Hubei Province, 445000, P. R. C.

email: zhanjianming@hotmail.com

Surapati Pramanik

Department of Mathematics, Nandalal Ghosh B.T. College, Narayanpur, Dist- North 24 Parganas, West Bengal 743126, India

email: sura_pati@yaho.co.in

Muhammad Irfan Ali

Department of Mathematics, COMSATS Institute of Information Technology Attock, Attock 43600, Pakistan

email: mirfanali13@yahoo.com

Said Broumi

Department of Mathematics, Hassan II Mohammedia-Casablanca University, Kasablanka 20000, Morocco

email: broumisaid78@gmail.com

Mumtaz Ali

University of Southern Queensland, Darling Heights QLD 4350, Australia

email: Mumtaz.Ali@usq.edu.au

Oktay Muhtaroğlu

Department of Mathematics, Tokat Gaziosmanpaşa University, 60250 Tokat, Turkey

email: oktay.muhtaroglu@gop.edu.tr

Ahmed A. Ramadan

Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

email: aramadan58@gmail.com

Sunil Jacob John

Department of Mathematics, National Institute of Technology Calicut, Calicut 673 601 Kerala, India

email: sunil@nitc.ac.in

Aslıhan Sezgin

Department of Statistics, Amasya University, Amasya, Turkey

email: aslihan.sezgin@amasya.edu.tr

Alaa Mohamed Abd El-latif

Department of Mathematics, Faculty of Arts and Science, Northern Border University, Rafha, Saudi Arabia

email: alaa_8560@yahoo.com

Kalyan Mondal

Department of Mathematics, Jadavpur University, Kolkata, West Bengal 700032, India

email: kalyanmathematic@gmail.com

Jun Ye

Department of Electrical and Information Engineering, Shaoxing University, Shaoxing, Zhejiang, P.R. China

email: yehjun@aliyun.com

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, 71516-Assiut, Egypt

email: drshehata2009@gmail.com

İdris Zorlutuna

Department of Mathematics, Cumhuriyet University, Sivas, Turkey

email: izorlu@cumhuriyet.edu.tr

Murat Sarı

Department of Mathematics, Yıldız Technical University, İstanbul, Turkey

email: sarim@yildiz.edu.tr

Contact

Editor-in-Chief

Name: Prof. Dr. Naim Çağman

Email: journalofnewtheory@gmail.com
Phone: +905354092136
Address: Departments of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, Tokat, Turkey

Editors

Name: Assoc. Prof. Dr. Faruk Karaaslan

Email: karaaslan.faruk@gmail.com
Phone: +905058314380
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çankırı Karatekin University, 18200, Çankırı, Turkey

Name: Assoc. Prof. Dr. İrfan Deli

Email: irfandeli@kilis.edu.tr
Phone: +905426732708
Address: M.R. Faculty of Education, Kilis 7 Aralık University, Kilis, Turkey

Name: Asst. Prof. Dr. Serdar Enginoğlu

Email: serdarenginoglu@gmail.com
Phone: +905052241254
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, 17100, Çanakkale, Turkey

CONTENT

1. A Generalized Mathematical Model of Hard-to-treat Infections with Culturing and Antibiotic Susceptibility Testing / Pages 1 - 14
Reuben GWERYİNA, Genesis Shiandiem TAYEN, Francis Shienbee KADUNA

2. Some Structures on Pythagorean Fuzzy Topological Spaces / Pages 15 - 25
Taha ÖZTÜRK, Adem YOLCU

3. On Some Neutrosophic Algebraic Equations / Pages 26 - 32
Mohammad ABOBALA

4. On Some Special Elements İn Neutrosophic Rings and Refined Neutrosophic Rings / Pages 33 - 39
Mohammad ABOBALA

5. Spectral Properties of the Anti-Periodic Boundary-Value-Transition Problems / Pages 40 - 49
Serdar PAŞ, Kadriye AYDEMİR, Fahreddin MUHTAROV

6. Characterization of the Evolute Offset of Ruled Surfaces with B-Darboux Frame / Pages 50 - 55
Gül UGUR KAYMANLİ

7. A Classical and Bayesian Approach for Parameter Estimation in Structural Equation Models / Pages 56 - 75
Naci MURAT, Mehmet Ali CENGİZ

8. Model Selection in Beta Regression Analysis Using Several Information Criteria and Heuristic Optimization / Pages 76 - 84
Emre DÜNDER, Mehmet Ali CENGİZ

9. Similarity Measures of Pythagorean Neutrosophic Sets with Dependent Neutrosophic Components Between T and F / Pages 85 - 94
Jansi RAJAN, Mohana KRISHNASWAMY

10. Brief Review of Soft Sets and Its Application in Coding Theory / Pages 95 - 106
Samy Mohammed MOSTAFA, Fatema Faisal KAREEM, Hussein Ali JAD

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A Generalised Mathematical Model of Hard-to-treat Infections with Culturing and Antibiotic Susceptibility Testing

Reuben Iortyer Gweryina[1], Genesis Shiandiem Tayen[2], Francis Shienbee Kaduna[3]

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Abstract − A mathematical model for hard-to-treat infections with culturing and antibiotic susceptibility testing (CAST) as an intervention strategy in a population is formulated and analysed. The analysis of the model has been done qualitatively to investigate the existence and stability of equilibria. Using the Lyapunov function, the disease-free equilibrium of the model proved to be globally asymptotically stable with respect to the threshold quantity $R_c < 1$. Of course, this entails local stability. A similar approach is employed in proving the global stability of the endemic equilibrium state in the case $R_c > 1$. However, the local stability of the endemic equilibrium is investigated using the method of row elimination. The model was validated using the Tuberculosis case in South Africa, and the result reveals that patients without adopting CAST strategy are prone to drug resistance and delay in quick response to the treatment regimen. On the contrary, individuals who have adopted the strategy have shown greater recovery potential from the infection. Based on that, self - medication, blind prescription should be avoided to curtail the consequences of drug resistance.

Keywords *– Hard- to- treat infections, Culturing, Antibiotic susceptibility testing, Global stability, Lyapunov function*

1. Introduction

In the world of medicine, the consequences of an improper diagnosis of most hard-to-treat infections such as Mycobacterium Tuberculosis, Typhoid fever, Gonorrhoea, staphylococcus etc. are responsible for high human mortality and morbidity [1]. To this paper, hard-to-treat infections refer to diseases/infections that prove incurable without culturing and antibiotic susceptibility testing. It is primarily culturing to ascertain the main cause of an infection and determine drug resistance strains under laboratory-controlled conditions to give a correct diagnosis and treatment [2]. It is observed that those who carry out culture testing tends not to have antibiotic-resistant strains and do heal quickly [3]. To achieve this, it is necessary to carry out antibiotic susceptibility testing, which involves culturing the disease in the presence of antibiotics. If the bacteria grow, they are resistant to the drugs, but if it fails to multiply, it implies that the drugs are effective and the bacteria are not resistant [4, 5].

[6] explains the need for rapid diagnostic testing to determine the causative organism of infection and maintained that diagnosis through culture before treatment plays a valuable and critical role in the cure of patients and those at risk of developing the infection. [7] identify disk diffusion and broth dilution techniques

[1] gweryina.reuben@uam.edu.ng (Corresponding Author); [2] tayengenesis@gmail.com; [3] kadunafrancis@gmail.com
[1,2,3] Department of Mathematics/Statistics/Computer Science, College of Science, Federal University of Agriculture, Makurdi, Nigeria

for bacterial culture and antibiotic susceptibility testing used in veterinary medicine and [8] made a case for integrating culture-based and molecular methods in agro-ecosystems to understand better ways of bacterial inhibition. [9] gave an overview of the current methods available to identify antimicrobial susceptibility testing of anaerobes (aerobic bacterial).

The number of mortality and morbidity cases recorded against drug resistance to diseases due to non-culturing and antibiotic susceptibility testing is alarming, and mathematical models under this area are few and not widely explored. People have made attempts to model this kind of infections on specific diseases as far back, as seen in [10, 11]. [12] projected that rapid expansion of Tuberculosis (TB) culture and drug sensitivity testing (DST) among South African adults could save >47,000 lives and prevent >7,000 multidrug-resistant (MDR)-TB cases during the 10 years from 2008 to 2017. This corresponds to a reduction of 17% in total TB mortality, 14% in MDR-TB incidence, and 47% in MDR-TB mortality. Their model projected that culture and DST impact depends most strongly on the speed and sensitivity of culture, treatment rates in diagnosed TB patients, and TB case detection rates in the absence of culture. In the paper, Detection of antibiotic resistance is essential for gonorrhoea point-of-care (POC) testing: a mathematical modelling study, [13] addressed clinically relevant situations to evaluate the potential impact of gonorrhoea POC tests on antibiotic-resistant Gonorrhoea and can guide the introduction of POC tests. [14] used mathematical modelling to provide a framework that integrates information regarding the transmission and control of foodborne pathogens and antimicrobial resistance. [15] provided a highlight on critical questions in the management of Gonorrhoea that can be addressed by mathematical models and identify key data needs. Their overarching aim is to articulate a shared agenda across gonococcus-related fields from microbiology to epidemiology that will catalyse a comprehensive evidence-based clinical and public health strategy to manage gonococcal infections antimicrobial resistance.

Because of the above, the present study uses this opportunity to consider the general dynamics of hard – to - treat infections with antibiotic susceptibility testing as a robust way of enhancing proper medical treatment. This paper's organisation begins with an introduction in Section 1 and follows model formulation in Section 2. The analysis of the model is presented in Section 3 with numerical simulations and discussion in Section 4. Finally, the conclusion is given in Section 5.

2. Model formulation and the Feasible Region

The model classifies the total population at time t, denoted by N, into susceptible individuals S, infected individuals without CAST strategy I_w, infected individuals with CAST strategy I_c and individuals who recovered from the infection R. It is assumed that individuals are recruited at a constant rate ϕ to the susceptible population S and recovered individual also become susceptible at γ rate. Susceptible individuals can be infected with disease following the contact with infected individuals at an average rate $\lambda = \beta\left(\frac{I_w+\theta I_c}{N}\right)$, where β is effective contact rate and θ is the modification parameter which takes the values $0 \leq \theta \leq 1$. When $\theta = 1$ implies that CAST strategy is ineffective in disease control while when $\theta = 0$ signifies that the strategy can effectively control the spread of the infection. Individuals with culture and antibiotic susceptibility testing can acquire the infection at a reduced rate of $(1-\pi)\lambda$ and a higher recovery rate of ρ_c compared to those without. It is also assumed that some of the infectives I_w move to join I_c class with a rate α depending on CAST strategy. It is also assumed that the natural death rate occurs in all populations at a per-capita rate of μ. It is noted that the recovery rate of infective due to CAST strategy is greater than those without the strategy ($\rho_c > \rho_w$). The mortality rate due to I_c is lower than that of I_w ($\delta_c < \delta_w$). The variables and parameters of the model (1) are hereby presented in Table 1.

Table 1: Parameters of the Model.

Parameter	Interpretation
S	Number of susceptible persons
I_w	Number of infected persons without CAST strategy
I_c	Number of infected persons with CAST strategy
R	Number of recovered persons due to treatment
Φ	Recruitment number of susceptible persons
π	The fraction of susceptible persons who become infected and do not adopt CAST strategy
α	The rate of adopting CAST strategy
β	Effective contact rate
λ	The force of infection
μ	Natural death rate
θ	Modification parameter
δ_w	Disease induced death rate for infectives without CAST strategy
δ_c	Disease induced death rate for infectives with CAST strategy
ρ_c	Recovery rate based on CAST intervention strategy
ρ_w	Recovery rate based on ordinary medical test prescription (without CAST strategy)
γ	The rate at which recovered persons regain susceptibility

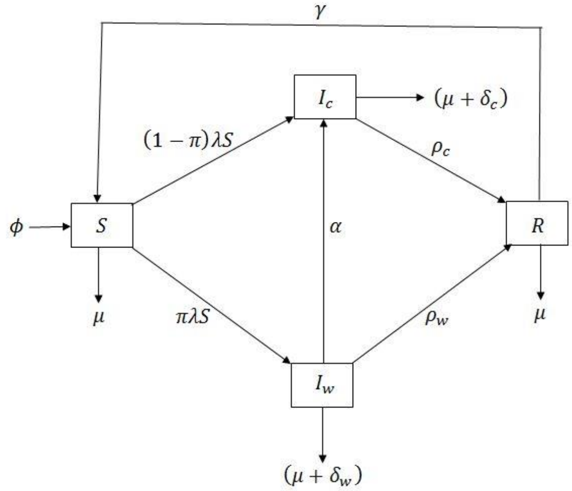

Fig. 1. Flow diagram of the generalised model of hard-to-treat infections

2.1. Model Equations

Using the description of model and Fig. 1, we derive the differential equations below.

$$\left.\begin{aligned}\frac{dS}{dt} &= \phi - \lambda S + \gamma R - \mu S\\ \frac{dI_w}{dt} &= \pi\lambda S - (\mu + \alpha + \rho_w + \delta_w)I_w\\ \frac{dI_c}{dt} &= (1-\pi)\lambda S + \alpha I_w - (\mu + \rho_c + \delta_c)I_c\\ \frac{dR}{dt} &= \rho_w I_w + \rho_c I_c - (\mu + \gamma)R\end{aligned}\right\}, \tag{1}$$

where

$$\lambda = \beta\left(\frac{I_w + \theta I_c}{N}\right). \tag{2}$$

Adding the whole equations of (1) yields

$$\frac{dN}{dt} = \phi - \mu N - \delta_w I_w - \delta_c I_c. \tag{3}$$

2.2. The feasible region

This sub-section examines the model's invariant region (1) whereby the system is mathematically and epidemiologically well-posed.

Theorem 2. 1. The model (1) is a dynamical system on the biologically feasible region:

$$D = \{(S, I_w, I_c, R) \in \mathbb{R}_+^4 : N_+ \leq N(t) \leq S^0\},$$

where $S^0 = \frac{\phi}{\mu}$ and $N_+ = \frac{\phi}{\mu + \delta_w + \delta_c}$.

Proof. The proof follows a two-step approach [16]

Step 1. We prove that the solution $S(t),\ I_w(t),\ I_c(t)$ and $R(t)$ of the model (1) based on the initial conditions such that $S(0),\ I_w(0),\ I_c(0)$ and $R(0)$ are non-negative. Let $\tilde{t} = sup\{t > 0 : S > 0, I_w \geq 0, I_c \geq 0, R \geq 0\}$. Then, $\tilde{t} > 0$ and it shows from the first equation of the model (1) that

$$\frac{dS}{dt} = \phi - (\mu + \lambda(t))S + \gamma R \geq \phi - (\mu + \lambda(t))S,$$

The above inequality equation has the form

$$\frac{d}{dt}\left[S(t)exp\left\{\mu t + \int_0^t \lambda(s)ds\right\}\right] \geq \phi\exp\left\{\mu t + \int_0^t \lambda(s)ds\right\}.$$

Thus,

$$S(\tilde{t})exp\left\{\mu\tilde{t} + \int_0^{\tilde{t}} \lambda(s)ds\right\} - S(0) \geq \int_0^{\tilde{t}} \phi\exp\left\{\mu p + \int_0^p \lambda(v)dv\right\}dp,$$

So that

$$S(\tilde{t}) \geq S(0)exp\left\{-\left(\mu\tilde{t} + \int_0^{\tilde{t}} \lambda(s)ds\right)\right\} + (0)exp\left\{-\left(\mu\tilde{t} + \int_0^{\tilde{t}} \lambda(s)ds\right)\right\} \times \int_0^{\tilde{t}} \phi\exp\left\{\mu p + \int_0^p \lambda(v)dv\right\}dp > 0.$$

Similarly, it can be proven that $I_w \geq 0, I_c \geq 0, and\ R \geq 0$ for all $t > 0$.

Step 2. We now show that the total population at time $t, N(t)$ satisfies the boundedness property

$N_+ \leq N(t) \leq S^0$ whenever $N_+ \leq N(t_0) \leq S^0$.

From equation (3), one has that

$$\phi - (\mu + \delta_w + \delta_c)N(t) \leq \frac{dN}{dt} \leq \phi - \mu N(t). \tag{4}$$

Applying the Gronwall inequality to the equation (4) yields

$$\frac{\phi}{\mu + \delta_w + \delta_c}\left[1 - e^{-(\mu+\delta_w+\delta_c)t}\right] + S(0)e^{-(\mu+\delta_w+\delta_c)t} \leq S(0)e^{-\mu t} + \frac{\phi}{\mu}(1 - e^{-\mu t})$$

which implies that

$$N_+ \leq N(t) \leq S^0.$$

Bringing step 1 and step 2 together, Theorem 2.1 follows from the classical theory of dynamical systems.

3. Existence of model equilibria and stability

3.1. Local stability of disease-free equilibrium (DFE)

The disease-free equilibrium point occurs at the state in which there is no infection. Hence, the DFE point of the model (1) is given by $\mathrm{E}_0 = (S^0, 0,0,0)$, where the infected compartment tends to zero and S^0 is the same as in Theorem 2.1.

The local stability of the DFE (E_0) depends on the control reproduction number, R_c, which is computed by a next-generation operator [17]. Using their notations F and V which denote the matrix of the new infections and transition matrix respectively, we have

$$F = \beta\begin{pmatrix} \pi & \pi\theta \\ 1-\pi & (1-\pi)\theta \end{pmatrix} \text{ and } V = \begin{pmatrix} \mu+\alpha+\rho_w+\delta_w & 0 \\ -\alpha & \mu+\rho_c+\delta_c \end{pmatrix},$$

with

$$V^{-1} = \begin{pmatrix} \dfrac{1}{\mu+\alpha+\rho_w+\delta_w} & 0 \\ \dfrac{\alpha}{(\mu+\alpha+\rho_w+\delta_w)(\mu+\rho_c+\delta_c)} & \dfrac{1}{\mu+\rho_c+\delta_c} \end{pmatrix}.$$

Therefore, the control reproduction number is

$$R_c = \rho(FV^{-1}) = \beta\left(\frac{\pi(d_2+\alpha\theta)+(1-\pi)d_1\theta}{d_1 d_2}\right), \tag{5}$$

where

$$d_1 = \mu+\alpha+\rho_w+\delta_w \text{ and } d_2 = \mu+\rho_c+\delta_c.$$

Therefore, by Theorem 2 in [17], we can claim the following result.

Lemma 3.1. The DFE of the model (1) is locally asymptotically stable if $R_c < 1$ and unstable otherwise.

Biologically, lemma 3.1 implies that an adequate pool of few infected individuals into the susceptible population will not generate an outbreak of infection except $R_c > 1$. Therefore, to ensure better control of infection, the global asymptotic stability of DFE is needed as addressed in the next subsection.

3.2. Global Stability of the DFE

The global investigation of stability at the disease-free state using Lyapunov function's construction depends on the infected compartments only.

Lemma 3.2. The DFE of the model (1) is globally asymptotically stable in D provided that $R_c < 1$ and unstable if $R_c > 1$.

Proof. Following the work of [18], we consider the Lyapunov function

$$L = AI_w + BI_c, \tag{6}$$

where $A > 0$ and $B > 0$, with the derivatives of L defined by

$$\frac{dL}{dt} = A\frac{dI_w}{dt} + B\frac{dI_c}{dt}. \tag{7}$$

Thus, substituting the corresponding right-hand side of (1) into (7) gives

$$\frac{dL}{dt} = (\pi A + (1-\pi)B)\lambda S - (d_1 A - \alpha B)I_w - d_2 B I_c\,. \tag{8}$$

Therefore, setting the coefficients of λS to the numerator of R_c (excluding β) and that of I_w to the denominator of R_c, we have

$$\begin{aligned}\pi A + (1-\pi)B &= \pi(d_2 + \alpha\theta) + (1-\pi)d_1\theta,\\ d_1 A - \alpha B &= d_1 d_2,\end{aligned}$$

from which we obtain

$$A = d_2 + \alpha\theta > 0 \text{ and } B = d_1\theta > 0.$$

Now replacing the expressions for A and B in (8) above yields

$$\begin{aligned}\frac{dL}{dt} &= \beta\left(\frac{I_w + \theta I_c}{N}\right)S(\pi(d_2 + \alpha\theta) + (1-\pi)d_1\theta) - d_1 d_2(I_w + \theta I_c),\\ &= d_1 d_2(I_w + \theta I_c)\left(\beta\left(\frac{\pi(d_2 + \alpha\theta) + (1-\pi)d_1\theta}{d_1 d_2}\right)\frac{S}{N} - 1\right).\end{aligned}$$

Since $S \leq N$ is in the region of the invariant set, it then follows that

$$\begin{aligned}\frac{dL}{dt} &\leq d_1 d_2(I_w + \theta I_c)\left(\beta\left(\frac{\pi(d_2 + \alpha\theta) + (1-\pi)d_1\theta}{d_1 d_2}\right) - 1\right),\\ &= d_1 d_2(I_w + \theta I_c)(R_c - 1).\end{aligned}$$

This shows that

$$\frac{dL}{dt} < 0 \text{ if } R_c < 1$$

Equality holds at $R_c = 1$ and $I_w = I_c = 0$. Therefore, we can conclude from the LaSalle's invariance principle stated in Theorem 3.1 below that the DFE is globally asymptotically stable since $S \to \frac{\phi}{\mu}$ as $t \to \infty$ at $I_w = I_c = 0$.

Theorem 3.1. [19] (La Salle Invariance Principle). Let $H(x)$ be a locally Lipschitz function defined over a domain $G \subset R^n$ and $\Omega \subset G$ be a compact set that is positively invariant concerning $\dot{x} = H(x)$. Let $V(x)$ be a continuously differentiable positive definite function on G such that $\dot{V}(x) \leq 0$ in Ω for all $x \in G$. Let $E = \{x \in \Omega \lceil \dot{V}(x) = 0\}$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \longrightarrow \infty$.

3.3. Existence of Endemic Equilibrium State (EES)

The endemic equilibrium state defines the persistence of an infection in the population. Suppose $E^{**} = (S^{**}, {I_w}^{**}, {I_c}^{**}, R^{**}) > 0$ is the endemic equilibrium of the model (1). Then,

$$\begin{aligned}0 &= \phi - \lambda^{**}S^{**} + \gamma R^{**} - \mu S^{**},\\ 0 &= \pi\lambda^{**}S^{**} - d_1 {I_w}^{**},\\ 0 &= (1-\pi)\lambda^{**}S^{**} + \alpha {I_w}^{**} - d_2 {I_c}^{**},\\ 0 &= \rho_w {I_w}^{**} + \rho_c {I_c}^{**} - d_3 R^{**}.\end{aligned}$$

Therefore,

$$\begin{cases} S^{**} = \dfrac{\phi + \gamma R^{**}}{\lambda^{**} + \mu},\\ {I_w}^{**} = \dfrac{\pi}{d_1}\lambda^{**}S^{**},\\ {I_c}^{**} = \left(\dfrac{\alpha\pi}{d_1 d_2} + \dfrac{1-\pi}{d_2}\right)\lambda^{**}S^{**},\\ R^{**} = \left(\dfrac{\pi}{d_1}\left(\dfrac{\rho_w}{d_3} + \dfrac{\rho_c\alpha}{d_2 d_3}\right) + \dfrac{(1-\pi)\rho_c}{d_2 d_3}\right)\lambda^{**}S^{**}. \end{cases} \tag{9}$$

To be specific, the value of S^{**} is

$$S^{**} = \frac{\phi}{\mu + \lambda^{**}\left(1 - \frac{\gamma}{d_3}\left(\frac{\pi}{d_1}\left(\rho_w + \frac{\rho_c \alpha}{d_2}\right) + (1-\pi)\frac{\rho_c}{d_2}\right)\right)}. \tag{10}$$

Recall from (2) that the force of infection at the equilibrium state is

$$\lambda^{**} = \beta\left(\frac{I_w^{**} + \theta I_c^{**}}{N^{**}}\right) = \frac{R_c - 1}{K}, \tag{11}$$

with

$$K = \frac{\pi}{\mu}\left(1 + \frac{\rho_w}{d_3} + \frac{\alpha}{d_2}\left(1 + \frac{\rho_c}{d_3}\right)\right) + \frac{1-\pi}{d_2}\left(1 + \frac{\rho_c}{d_3}\right).$$

Note that $\lambda^{**} \neq 0$ defines the endemic equilibrium which exists at the point, $R_c > 1$. From the above, the following result can be inferred.

Lemma 3.3. If $R_c > 1$, then the model (1) admits a unique positive endemic equilibrium state.

3.4. Local Stability of EES

The linearised form of system (1) at E** gives the Jacobian, J,

$$J = \begin{pmatrix} -\left(\mu + \beta\frac{(I_w{}^{**} + \theta I_c{}^{**})}{N^{**}}\right) & -\beta\frac{S^{**}}{N^{**}} & -\beta\theta\frac{S^{**}}{N^{**}} & \gamma \\ \pi\beta\frac{(I_w{}^{**} + \theta I_c{}^{**})}{N^{**}} & -\left(d_1 - \pi\beta\frac{S^{**}}{N^{**}}\right) & \pi\beta\theta\frac{S^{**}}{N^{**}} & 0 \\ (1-\pi)\beta\frac{(I_w{}^{**} + \theta I_c{}^{**})}{N^{**}} & \alpha + (1-\pi)\beta\frac{S^{**}}{N^{**}} & -\left(d_2 - (1-\pi)\beta\theta\frac{S^{**}}{N^{**}}\right) & 0 \\ 0 & \rho_w & \rho_c & -(\mu+\gamma) \end{pmatrix}. \tag{12}$$

The transition by row reduction into an upper triangular matrix of the Jacobian is given by

$$T_U = \begin{pmatrix} -g_1 & -g_2 & -g_3 & \gamma \\ 0 & -A_1 & -A_2 & A_3 \\ 0 & 0 & -(A_1B_2 + A_2B_1) & (A_3B_1 + A_1B_3) \\ 0 & 0 & 0 & Q \end{pmatrix}, \tag{13}$$

where

$A_1 = g_1g_5 + g_2g_4, \quad A_2 = g_3g_4 - g_1g_6, A_3 = \gamma g_4,$

$B_1 = g_1g_8 - g_2g_7, \quad B_2 = g_1g_9 + g_3g_7, \quad B_3 = \gamma g_7,$

and $Q = (A_1B_2 + A_2B_1)(\rho_w A_3 - g_{10}A_1) + (\rho_c A_1 - \rho_w A_2)(A_3B_1 + A_1B_3),$

with

$$g_1 = \mu + \beta\frac{(I_w{}^{**} + \theta I_c{}^{**})}{N^{**}}, g_2 = \beta\frac{S^{**}}{N^{**}}, \quad g_3 = \beta\theta\frac{S^{**}}{N^{**}}, g_4 = \pi\beta\frac{(I_w{}^{**} + \theta I_c{}^{**})}{N^{**}},$$

$$g_5 = \left(d_1 - \pi\beta\frac{S^{**}}{N^{**}}\right), \quad g_6 = \pi\beta\theta\frac{S^{**}}{N^{**}}, \quad g_7 = (1-\pi)\beta\frac{(I_w{}^{**} + \theta I_c{}^{**})}{N^{**}},$$

$$g_8 = \alpha + (1-\pi)\beta\frac{S^{**}}{N^{**}}, \quad g_9 = \left(d_2 - (1-\pi)\beta\theta\frac{S^{**}}{N^{**}}\right), \quad g_{10} = (\mu + \gamma).$$

For the system to be Locally Asymptotically Stable at the endemic state, we now show that all the diagonal elements of the upper triangular matrix, which are the eigenvalues of (12) are negative.

Then, from (13)

$$\lambda_1 = -g_1 = -\left(\mu + \frac{(R_c - 1)}{K}\right) < 0 \text{ iff } R_c > 1. \tag{14}$$

Similarly,

$$\lambda_2 = -A_1 = -\left(d_1\left(\mu + \frac{(R_c - 1)}{K}\right) - \mu\beta\pi\frac{S^{**}}{N^{**}}\right) < 0 \text{ iff } R_c > 1. \tag{15}$$

For $\lambda_3 = -(A_1B_2 + A_2B_1) < 0,$

it implies that $(A_1B_2 + A_2B_1) > 0$, and detail simplification gives

$$\begin{aligned}\lambda_3 = -\mu\beta\frac{S^{**}}{N^{**}}\left(\theta\left(\beta\frac{S^{**}}{N^{**}}(\mu\pi - \alpha) - d_1(1-\pi)(1+\mu)\right) - \left(\mu + \left(\frac{R_c - 1}{K}\right)\right)(d_2\pi + \alpha\theta)\right)\\ -d_1d_2\left(\left(\frac{R_c - 1}{K}\right)^2 + 2\mu\left(\frac{R_c - 1}{K}\right) + \mu^2\right) < 0 \text{ iff } R_c > 1.\end{aligned} \tag{16}$$

Lastly,

$$\lambda_4 = Q = (A_1B_2 + A_2B_1)(\rho_w A_3 - g_{10}A_1) + (\rho_c A_1 - \rho_w A_2)(A_3B_1 + A_1B_3).$$

Since $(A_1B_2 + A_2B_1)$ and $(A_3B_1 + A_1B_3)$ are positive, we are left to show that $(\rho_w A_3 - g_{10}A_1)$ and $(\rho_c A_1 - \rho_w A_2)$ are negative. This implies that

$\rho_w A_3 - g_{10}A_1 < 0 \Leftrightarrow \rho_w A_3 < g_{10}A_1$ yields

$$\rho_w < g_{10}\frac{A_1}{A_3}, \tag{17}$$

and

$\rho_c A_1 - \rho_w A_2 < 0 \Leftrightarrow \rho_c A_1 < \rho_w A_2.$

This gives

$$\rho_c\frac{A_1}{A_2} < \rho_w. \tag{18}$$

Combining equations (17) and (18), we get the inequality

$$\rho_c\frac{A_1}{A_2} < \rho_w < g_{10}\frac{A_1}{A_3}$$

from which we arrived at

$\lambda_4 < 0$ iff $\rho_c > \frac{-\mu(\mu+\gamma)\theta S^{**}}{\gamma\pi N^{**}({I_w}^{**}+\theta {I_c}^{**})} = -(\mu+\gamma)\mu\beta\theta\frac{S^{**}}{N^{**}}\frac{K}{\gamma\pi(R_c-1)}$, provided $R_c > 1$.

Lemma 3.4. The endemic equilibrium is locally asymptotically stable iff $R_c > 1$.

3.5. Global Stability of the Endemic Equilibrium E^{**}

Lemma 3.5. The endemic equilibrium point of the model (1) is globally asymptotically stable if and only if $R_c|_{\alpha=\gamma=0} > 1$.

PROOF. We consider a nonlinear Lyapunov function of Volterra type as applied in [20]

$$L_1 = S - S^{**} - S^{**}\ln\left(\frac{S}{S^{**}}\right) + \frac{1}{\pi}\left[I_w - {I_W}^{**}\ln\left(\frac{I_w}{{I_W}^{**}}\right)\right] + \frac{1}{1-\pi}\left[I_c - {I_c}^{**}\ln\left(\frac{I_c}{{I_c}^{**}}\right)\right].$$

The derivative of L_1 with respect to time is given by

$$L_1' = \left(1-\frac{S^{**}}{S}\right)\frac{dS}{dt}+\frac{1}{\pi}\left(1-\frac{{I_w}^{**}}{I_w}\right)\frac{dI_w}{dt}+\frac{1}{1-\pi}\left(1-\frac{{I_c}^{**}}{I_c}\right)\frac{dI_c}{dt}. \quad (19)$$

Putting the equations of the model (1) at $\gamma = 0$ in (19), we have

$$L_1' = \left(1-\frac{S^{**}}{S}\right)\left[\phi-\beta\left(\frac{I_w+\theta I_C}{N}\right)S-\mu S\right]+\frac{1}{\pi}\left(1-\frac{{I_w}^{**}}{I_w}\right)\left[\beta\pi\left(\frac{I_w+\theta I_C}{N}\right)S-(\mu+\alpha+\rho_w+\delta_w)I_w\right]$$
$$+\frac{1}{1-\pi}\left(1-\frac{{I_c}^{**}}{I_c}\right)\left[\beta(1-\pi)\left(\frac{I_w+\theta I_C}{N}\right)S+\alpha I_w-(\mu+\rho_c+\delta_c)I_c\right].$$

For convenience, let $H(I)=\frac{I_w+\theta I_C}{N}$, then simplification gives

$$L_1' = \left(1-\frac{S^{**}}{S}\right)[\phi-\beta H(I)S-\mu S]+\left(\frac{1}{\pi}\right)\times\left(1-\frac{{I_w}^{**}}{I_w}\right)[\beta\pi H(I)S-(\mu+\alpha+\rho_w+\delta_w)I_w]$$
$$+\left(\frac{1}{1-\pi}\right)\times\left(1-\frac{{I_c}^{**}}{I_c}\right)[\beta(1-\pi)H(I)S+\alpha I_w-(\mu+\rho_c+\delta_c)I_c]. \quad (20)$$

Using the following equilibrium relations of the model (1) obtained at $\gamma = 0$,

$$\phi = \beta H(I^{**})S^{**}+\mu S^{**},$$
$$\mu+\alpha+\rho_w+\delta_w = \frac{\beta\pi H(I^{**})S^{**}}{{I_w}^{**}},$$
$$\mu+\rho_c+\delta_c = \frac{\beta(1-\pi)H(I^{**})S^{**}+\alpha {I_w}^{**}}{{I_w}^{**}},$$

Then, equation (20) becomes

$$L_1' = \mu S^{**}\left(1-\frac{S^{**}}{S}-\frac{S}{S^{**}}\right)+\beta H(I^{**})S^{**}\left[1-\frac{H(I)S}{H(I^{**})S^{**}}-\frac{S^{**}}{S}+\frac{H(I)}{H(I^{**})}\right]$$
$$+\beta H(I^{**})S^{**}\left[\frac{H(I)S}{H(I^{**})S^{**}}-\frac{I_w}{{I_w}^{**}}+\frac{{I_w}^{**}H(I)S}{I_w H(I^{**})S^{**}}+1\right]$$
$$+\beta H(I^{**})S^{**}\left[\frac{H(I)S}{H(I^{**})S^{**}}-\frac{I_c}{{I_c}^{**}}+\frac{{I_c}^{**}H(I)S}{I_c H(I^{**})S^{**}}+1\right]$$
$$+\frac{\alpha}{1-\pi}{I_w}^{**}\left[\frac{I_w}{{I_w}^{**}}-\frac{I_c}{{I_c}^{**}}-\frac{{I_c}^{**}I_w}{I_c {I_w}^{**}}+1\right] \quad (21)$$

Adding the second, third and fourth terms of (21) and using the fact that $\frac{H(I)}{H(I^{**})}\leq 1$, since $H(I)$ is a decreasing function, we get

$$L_1' = \mu S^{**}\left(2-\frac{S^{**}}{S}-\frac{S}{S^{**}}\right)+\beta H(I^{**})S^{**}\left[4-\frac{I_w}{{I_w}^{**}}-\frac{S{I_w}^{**}}{S^{**}I_w}-\frac{I_c}{{I_c}^{**}}-\frac{{I_c}^{**}S}{I_c S^{**}}\right]$$
$$+\beta H(I^{**})S^{**}\left[-\frac{S^{**}}{S}+\frac{S}{S^{**}}\right]+\frac{\alpha}{1-\pi}{I_w}^{**}\left[\frac{I_w}{{I_w}^{**}}-\frac{I_c}{{I_c}^{**}}-\frac{{I_c}^{**}I_w}{I_C {I_w}^{**}}+1\right]. \quad (22)$$

But $\beta H(I^{**})=\lambda^{**}=\frac{R_c-1}{K}$ from (11) and setting $\alpha = 0$ in (22), we have

$$L_1' = \mu S^{**}\left(2-\frac{S^{**}}{S}-\frac{S}{S^{**}}\right)+\left(\frac{R_c-1}{K}\right)S^{**}\left[4-\frac{I_w}{{I_w}^{**}}-\frac{S{I_w}^{**}}{S^{**}I_w}-\frac{I_c}{{I_c}^{**}}-\frac{{I_c}^{**}S}{I_c S^{**}}\right]-\left(\frac{R_c-1}{K}\right)S^{**}\left[\frac{(S^{**})^2-S^2}{SS^{**}}\right],$$

from which we arrived at

$$L_1' \leq \mu S^{**}\left(2-\frac{S^{**}}{S}-\frac{S}{S^{**}}\right)+\left(\frac{R_c-1}{K}\right)S^{**}\left[4-\frac{I_w}{{I_w}^{**}}-\frac{S{I_w}^{**}}{S^{**}I_w}-\frac{I_c}{{I_c}^{**}}-\frac{{I_c}^{**}S}{I_cS^{**}}\right].$$

Thus,

$L_1' \leq 0$ if and only if $R_c > 1,\ 2 \leq \frac{S^{**}}{S}+\frac{S}{S^{**}}$ and $4 \leq \frac{I_w}{{I_w}^{**}}+\frac{S{I_w}^{**}}{S^{**}I_w}+\frac{I_c}{{I_c}^{**}}+\frac{{I_c}^{**}S}{I_cS^{**}}$. However, $L_1' = 0$ if $R_c = 1, S = S^{**},\ I_w = {I_w}^{**}$ and $I_c = {I_c}^{**}$. Hence, $\{(S,\ I_w,\ I_C) = (S^{**},\ {I_w}^{**},\ {I_c}^{**})\}$ is the only singleton set in D, which is the largest compact subset where $L_1' = 0$. At this point, we can conclude by invariance principle in Theorem 3.1 that the endemic equilibrium is globally asymptotically stable.

3.6. Threshold Analysis

The control reproduction number, R_c, of a model system with CAST strategy defined by (5) is a threshold quantity that determines whether the disease will invade the host population. If R_c is less than unity, the disease will be under control, and if it is not, then there will be an outbreak of disease.

In the absence of CAST strategy, we have

$$\lim_{(\alpha,\theta,\rho_c,\delta_c)\to(0,1,0,0)} R_c = \frac{\beta\pi}{d_1}+\frac{\beta(1-\pi)}{d_2} = R_0, \tag{23}$$

where R_0 is the basic reproduction number.

Thus, the difference between equations R_c of (5) and R_0 of (23) is

$$R_0 - R_c = \frac{\beta(1-\pi)}{d_2}(1-\theta)-\frac{\beta\pi\alpha\theta}{d_1d_2}. \tag{24}$$

Clearly from equation (24), $R_0 - R_c$ is positive if $\theta = 0$. This epidemiologically implies that CAST strategy could be essential for effective treatment of hard-to-treat infections. On the other hand, $\theta = 1$ biologically shows that $R_0 - R_c$ is negative and thus ineffective in curtailing this kind of infections.

4. Numerical results and discussion

As an application of our model developed on hard– to – treat infections, we focus on the case study of 2014 Mycobacterium Tuberculosis (TB) outbreak in South Africa.

4.1. Parameter estimation

According to the Population Reference Bureau in 2018, the total population of South Africa denoted by N was estimated as of 2014 to be 53,700,000. The Global TB Report 2015 estimated that South Africa had the second highest TB incidence rate in 593 cases per 100, 000 population. Thus, we have the total number of infected individuals with TB as in 2014 to be

$$I = \frac{593}{100{,}000} \times 53,700,000 = 318,441.$$

Meanwhile those individuals I_C infected with TB who adopted the CAST strategy was 101, 423 [21], and the total number of infected individuals I_w without considering the strategy becomes

$$I_w = I - I_c = 318{,}441 - 101{,}423 = 217{,}018.$$

On the other hand, the total number of people R who have recovered from TB during the year under review by [4] was 251, 344. To this effect, the number of individuals susceptible to TB in 2014 evidently satisfies the relation

$$S = N - (I_w + I_c + R) = 53{,}130{,}215.$$

The death rate is defined as the inverse of the life expectancy at birth. As in the year 2014, the life expectancy of South Africans was 60.99 years. Therefore, the natural death rate, μ, is estimated to be $\mu = \frac{1}{60.99} = 0.0163961$ per year. Also, the recruitment number ϕ can be estimated from the relation in the feasible region as

$$\phi \cong N \times \mu = 880{,}472.21$$

The rest of the parameters can be similarly estimated and appropriately assumed as presented in Table 2

Table 2: Values for population-independent parameters of the model $\left(yr^{-1}\right)$

Variable/Parameter	Value	Source
N	53,700,000	[4]
S	53,130,215	Estimated
I_w	217,018	Estimated
I_c	101,423	[21]
R	251,344	[4]
Φ	880,472.21	Estimated
π	0.5	Assumed
α	$0 \le \alpha \le 1$	Variable
β	0.6983	[22]
μ	0.0163961	Estimated
θ	$0 \le \theta \le 1$	Variable
δ_w	0.06908	Assumed
δ_c	0.03384	Assumed
ρ_c	0.06667	[4]
ρ_w	0.075	[4]
γ	0.007893	Assumed

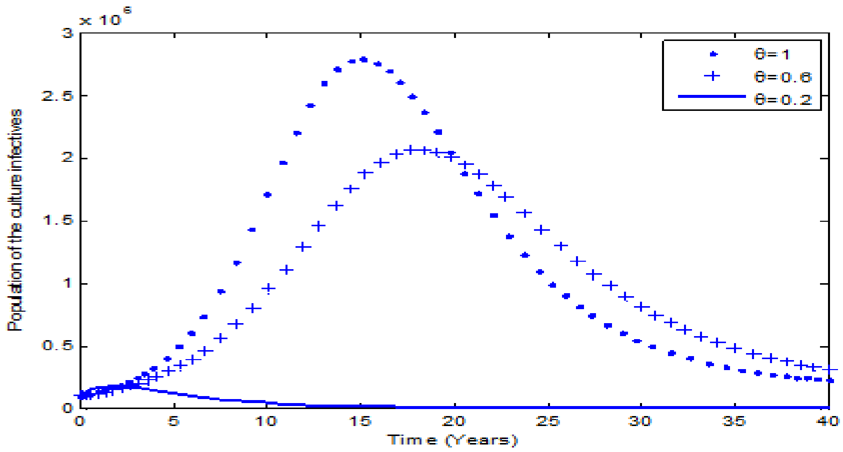

Fig. 2. Effect of θ on TB infectives, I_C with CAST strategy

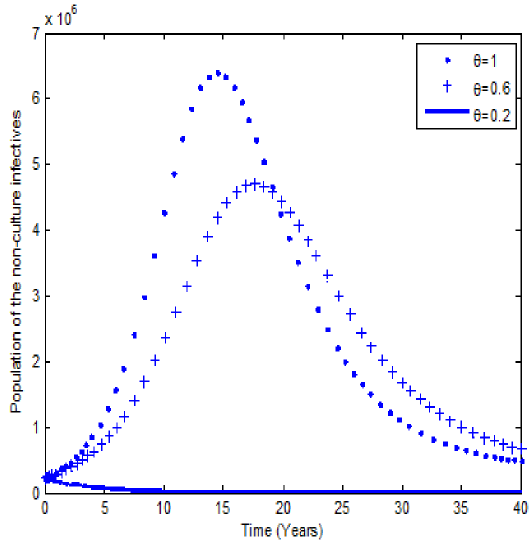

Fig. 3. Effect of θ on TB infectives, I_w without CAST strategy

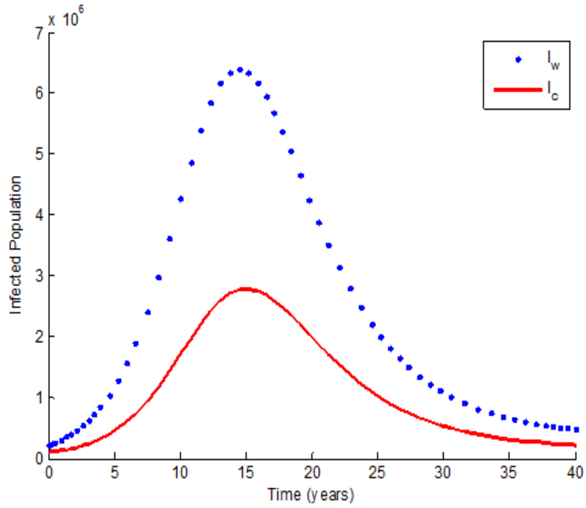

Fig. 4. The effect of the modification parameter $(\theta = 1)$ on the dynamics of TB infectives

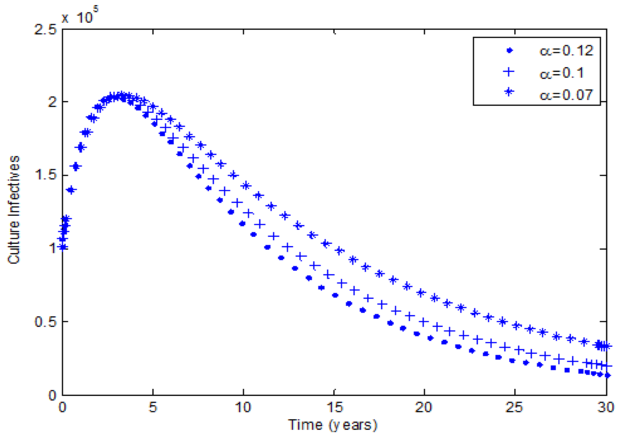

Fig. 5. Effect of α on TB infectives, I_C with CAST strategy.

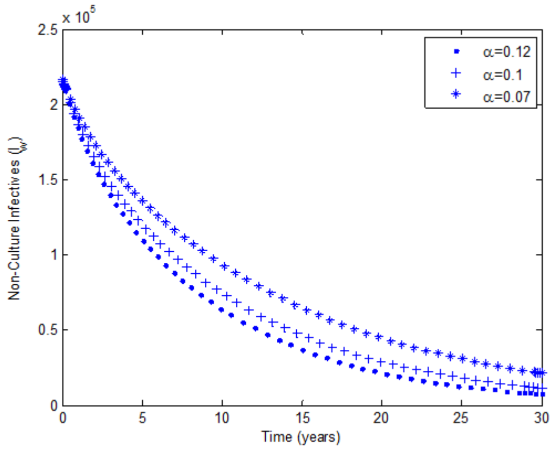

Fig. 6. Effect of α on TB infectives, I_C without CAST strategy

Fig. 2 illustrated the dynamical behaviour of infectives I_C who have gone for culture and conducted antibiotic susceptibility testing concerning the modification parameter, θ. The number of infected individuals remains high at $\theta = 1$, implying that, CAST strategy fails at that point and begins to decline as the value of θ decreases demonstrating the effectiveness of culture and antibiotic susceptibility testing. A similar consideration was carried out in Fig. 3 on those infectives I_W that have gone only for ordinary prescription treatment and indicates the same scenario only that the number of people with CAST strategy has a comparative advantage in quick response to treatment than those without the intervention strategy. A clear comparison of the above two experiments is given in Fig. 4 at $\theta = 1$ in which $I_C < I_W$. This inequality shows the significance of CAST strategy as a prerequisite to the proper treatment of infectious diseases and further discourages the ordinary diagnosis/blind prescription treatment of patients suffering from hard-to-treat infections. This result agrees with the works of [3, 6] that mandated the use of culture before medication as it prevents drug resistance and promotes timely cure from infections. The impact of α on the infectives is also given in Fig. 5 and 6. In both Figures, it is important to say that infection is easily treated in individuals who embrace CAST strategy but prove difficult for those who have gone for blind prescription or ordinary diagnosis at $\alpha = 0$. This outcome is consistent with [12] that says culture and drug sensitivity test can save more lives and prevent multi-drug resistance in patients.

5. Conclusion

This paper aims to model the role of culture and antibiotic susceptibility testing on the treatment of hard-to-treat infections. To this end, incidence function that accounts for individuals' behaviour with (out) culture has been introduced. Stability analysis concerning $\boldsymbol{R_c}$ being the key objective of any epidemiological study has been done, and the investigation reveals that the basic equilibria of the model are stable, both local and global using appropriate standard stability methods. Threshold analysis of the effective reproduction number $\boldsymbol{R_c}$ has proven that CAST strategy is very critical in mitigating and controlling the hard-to-treat diseases. Numerically, we simulate the proposed model using tuberculosis data from South Africa as a case study. The result confirms that individuals who present themselves for treatment of infection without culture and antibiotic susceptibility testing have a slow recovery pace and thus increases their mortality. Based on these findings; priority should be on culture and drug sensitivity testing by health practitioners before prescribing drugs to patients, since this will reduce fatality and boast recovery rates of individuals from hard-to-use infections. Additionally, since the study focuses on a generalised model for non-specific infection, we expect future research to target specific diseases as each disease may have its peculiar transmission dynamics.

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Some Structures on Pythagorean Fuzzy Topological Spaces

Taha Yasin Öztürk[1], **Adem Yolcu**[2]

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Abstract — In this paper, we introduced some operations such as Pythagorean fuzzy interior, Pythagorean fuzzy closure, Pythagorean fuzzy boundary, Pythagorean fuzzy basic on Pythagorean fuzzy topological spaces. Also, the notions of Pythagorean fuzzy open (closed) functions and Pythagorean fuzzy homeomorphism are introduced and their basic properties are investigated.

Keywords — *Pythagorean fuzzy topological spaces, Pythagorean fuzzy interior, Pythagorean fuzzy closure, Pythagorean fuzzy boundary, Pythagorean fuzzy open (closed) functions, Pythagorean fuzzy homeomorphism.*

1. Introduction

In 1965, to dispose uncertain or vaguee information in decision making, fuzzy set theory was first introduced by Zadeh [1]. Fuzzy set theory was characterized by a membership function which assigns to each target a membership value ranging between 0 and 1. Further, Chang [2] introduced the fuzzy topological spaces and studied some basic notions of topology such as open set, closed set and continuity. Later, Lowen [3, 4] also made different studies on fuzzy topological spaces. Intuitionistic fuzzy set (IFS), initially proposed by Atanassov [5], is incorporated the degree of non-membership γ into the analysis along with the membership degree μ in such a way that $\mu+\gamma \leq 1$. Coker [6] introduced the concept of intuitionistic fuzzy topological spaces and studied some notions such as continuity and compactness. Then, different studies were carried out on intuitionistic fuzzy topological spaces [7–9]. Although intuitionistic fuzzy set theory is popular, in some pactical decision-making processes the sum of degree of membership and the degree of non-membership, in which an alternative that meets the criteria of expertise is given, can be larger than 1; but their square sum is 1 or less.
Yager developed pythagorean fuzzy set (PFS) [10] characterized by a membership degree and non-membership degree which satisfies the condition that the square sum of its membership and non-membership degree is less than or equal to 1. Obviously, PFS is more effective than IFS. Yager [11] showed this situation with an example. An expert gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his support against membership is $\frac{1}{2}$. Since the sum of the two values is bigger than 1. They are not avaliable for IFS. But they are avaliable for PFS since $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \leq 1$. The PFS also has been studied from different perspectives such as in decision-making technologies [11–14], aggregation operator [15–18], information measure [19, 20], the extensions of PFS [21–24] and basic properties [10, 25, 26]. Besides, In 2019, Olgun et al. [27] introduce pythagorean fuzzy topological spaces and studied some properties. After that, In 2020, Naeem et. al. studied Pythagorean m-polar Fuzzy Topology with TOPSIS Approach in Exploring Most Effectual Method for Curing from COVID-19 [28].

[1]taha36100@hotmail.com ; [2]yolcu.adem@gmail.com (Corresponding Author)
[1,2]Department of Mathematic, Kafkas University, Kars, Turkey

In this study, we investigated some basic notions of pythagorean fuzzy topological spaces such as pythagorean fuzzy interior, pythagorean fuzzy closure, pythagorean fuzzy boundary and pythagorean fuzzy basic. Finally, we also defined pythagorean fuzzy open (closed) function and pythagorean fuzzy homeomorphism.

2. Preliminaries

In this section, we will give some preliminary information for the present study.

Definition 2.1. [1] Let X be an universe. A fuzzy set (FS for short) A in X, $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$ is the membership function of the fuzzy set A; $\mu_A(x) \in [0,1]$ is the membership of $x \in X$ in A.

Definition 2.2. [5] Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS for short) A in X is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

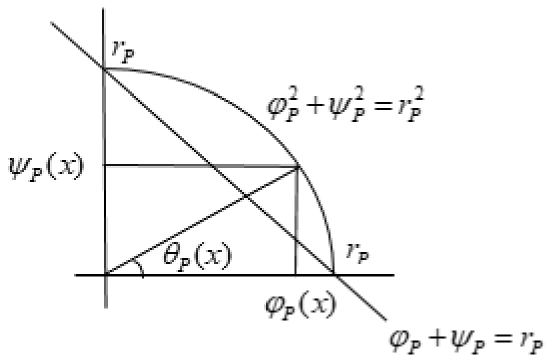
The degree of indeterminacy $I_A = 1 - \mu_A(x) - \gamma_A(x)$.

Definition 2.3. [10] Let X be a universe of discourse. A pythagorean fuzzy set P in X is given by $P = \{\langle x, \varphi_P(x), \psi_P(x)\rangle : x \in X\}$ where the functions $\varphi_P(x) : X \to [0,1]$ denotes the degree of membership and $\psi_P(x) : X \to [0,1]$ denotes the degree of non-membership of the element $x \in X$ to the set P, respectively, with the condition that $0 \leq (\varphi_P(x))^2 + (\psi_P(x))^2 \leq 1$.

The degree of indeterminacy $I_P = \sqrt{1 - (\varphi_P(x))^2 - (\psi_P(x))^2}$.

Remark 2.4. It is easy to check that PFSs generalize IFSs. That is, all intuitionistic fuzzy degrees are part of the Pythagorean fuzzy degrees. In actual decision-making problems, the PFS characterizes a larger membership space than the IFS. Namely, the PFS a higher capability than the IFS to model vagueness in real decision-making problems. Yager [10] proposed a novel concept of PFS to model the condition that the sum of the degree to which an alternative x_i satisfies and dissatisfies with respect to the attribute C_j is bigger than 1, while the IFS cannot deal with it.

Fig. 1. Comparison of intuitionistic fuzzy subsets and Pythagorean fuzzy subsets



Definition 2.5. [10] Let $P_1 = \{\langle x, \varphi_{P_1}(x), \psi_{P_1}(x)\rangle : x \in X\}$ and $P_2 = \{\langle x, \varphi_{P_2}(x), \psi_{P_2}(x)\rangle : x \in X\}$ be two pythagorean fuzzy sets over X. Then,

a the pythagorean fuzzy complement of P_1 is defined by

$$P_1^c = \{\langle x, \psi_{P_1}(x), \varphi_{P_1}(x)\rangle : x \in X\},$$

b the pythagorean fuzzy intersection of P_1 and P_2 is defined by

$$P_1 \cap P_2 = \{\langle x, \min\{\varphi_{P_1}(x), \varphi_{P_2}(x)\}, \max\{\psi_{P_1}(x), \psi_{P_2}(x)\}\rangle : x \in X\},$$

c the pythagorean fuzzy union of P_1 and P_2 is defined by

$$P_1 \cup P_2 = \{\langle x, \max\{\varphi_{P_1}(x), \varphi_{P_2}(x)\}, \min\{\psi_{P_1}(x), \psi_{P_2}(x)\}\rangle : x \in X\},$$

d we say P_1 is a pythagorean fuzzy subset of P_2 and we write $P_1 \subseteq P_2$ if $\varphi_{P_1}(x) \leq \varphi_{P_2}(x)$ and $\psi_{P_1}(x) \geq \psi_{P_2}(x)$ for each $x \in X$,

e $0_X = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_X = \{\langle x, 1, 0\rangle : x \in X\}$.

Definition 2.6. [27] Let $X \neq \emptyset$ be a set and τ be a family of pythagorean fuzzy subsets of X. If

T1 0_X, $1_X \in \tau$,

T2 for any P_1, $P_2 \in \tau$, we have $P_1 \cap P_2 \in \tau$,

T3 for any $\{P_i\}_{i\in I} \subseteq \tau$, we have $\underset{i\in I}{\cup} P_i \in \tau$

then τ is called a pythagorean fuzzy topology on X and the pair $(X, \tau)_p$ is said to be a pythagorean fuzzy topological space ($PFTS$ for short).Each member of τ is called a pythagorean fuzzy open set ($PFOS$ for short). The complement of a pythagorean fuzzy open set is called a pythagorean fuzzy closed set ($PFCS$ for short).

Remark 2.7. As any fuzzy set or intuitionistic fuzzy set can be considered as a pythagorean fuzzy set, we observe that any fuzzy topological space or intuitionistic fuzzy topological space is a pythagorean fuzzy topological space as well. Conversely, it is obvious that pythagorean fuzzy topological space needs not to be a fuzzy topological space or intuitionistic fuzzy topological space. Even a pythagorean fuzzy open set may be neither a fuzzy set nor an intuitionistic fuzzy set (see following example).

Example 2.8. [27] Let $X = \{x_1, x_2\}$. Consider the following family of pythagorean fuzzy subsets $\tau = \{0_X, 1_X, P_1, ..., P_5\}$ where

$$\begin{aligned}
P_1 &= \{\langle x_1, 0.5, 0.7\rangle, \langle x_2, 0.2, 0.4\rangle\},\\
P_2 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.3, 0.9\rangle\},\\
P_3 &= \{\langle x_1, 0.4, 0.8\rangle, \langle x_2, 0.1, 0.95\rangle\},\\
P_4 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.3, 0.4\rangle\},\\
P_5 &= \{\langle x_1, 0.5, 0.7\rangle, \langle x_2, 0.2, 0.9\rangle\}.
\end{aligned}$$

Observe that $(X, \tau)_p$ is a pythagorean fuzzy topological space.

Definition 2.9. [27] Let X and Y be two non-empty sets, let $f : X \to Y$ be a function and let A and B be Pythagorean fuzzy subsets of X and Y, respectively. Then, the membership and non-membership functions of image of A with respect to f that is denoted by $f[A]$ are defined by

$$\mu_{f[A]}(y) = \begin{cases} \sup\limits_{z\in f^{-1}(y)} \mu_A(z), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$v_{f[A]}(y) = \begin{cases} \inf\limits_{z\in f^{-1}(y)} v_A(z), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

respectively. The membership and non-membership functions of pre-image of B with respect to f that is denoted by $f^{-1}[B]$ are defined by

$\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$ and $v_{f^{-1}[B]}(x) = v_B(f(x))$ respectively.

In the study [27], they showed that $\mu^2_{f[A]} + v^2_{f[A]} \leq 1$ pythagorean fuzzy membership condition is provide for pythagorean fuzzy image and pre-image.

Proposition 2.10. [27] Let X and Y be two non-empty sets and $f : X \to Y$ be a pythagorean fuzzy function. Then, we have

1. $f^{-1}[B^c] = (f^{-1}[B])^c$ for any pythagorean fuzzy subset B of Y.
2. $(f[A])^c \subseteq f[A^c]$ for any pythagorean fuzzy subset A of X.
3. If $B_1 \subseteq B_2$ then $f^{-1}[B_1] \subseteq f^{-1}[B_2]$ where B_1 and B_2 are pythagorean fuzzy subset of Y.
4. If $A_1 \subseteq A_2$ then $f[A_1] \subseteq f[A_2]$ where A_1 and A_2 are pythagorean fuzzy subset of X.
5. $f\left[f^{-1}[B]\right] \subseteq B$ for any pythagorean fuzzy subset B of Y.
6. $A \subseteq f^{-1}[f[A]]$ for any pythagorean fuzzy subset A of X.

Definition 2.11. [27] Let $(X,\tau_1)_p$ and $(Y,\tau_2)_p$ be two pythagorean fuzzy topological spaces and $f: X \to Y$ be a pythagorean fuzzy function. Then, f is said to be pythagorean fuzzy continuous if for any pythagorean fuzzy subset A of X and for any neighbourhood V of $f[A]$ there exists a neighbourhood U of A such that $f[U] \subseteq V$.

Theorem 2.12. [27] Let $(X,\tau_1)_p$ and $(Y,\tau_2)_p$ be two pythagorean fuzzy topological spaces. A function $f: X \to Y$ is pythagorean fuzzy continuous iff for each open (closed) pythagorean fuzzy subset B of Y we have $f^{-1}[B]$ is an open (closed) pythagorean fuzzy subset of X.

3. Basic Results

Definition 3.1. Let $\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i\in I}$ be a family of pythagorean fuzzy sets over X. Then,

a $\underset{i\in I}{\cap} P_i = \{\langle x, \inf\{\varphi_{P_i}(x)\}, \sup\{\psi_{P_i}(x)\}\rangle : x \in X\}$,

b $\underset{i\in I}{\cup} P_i = \{\langle x, \sup\{\varphi_{P_i}(x)\}, \inf\{\psi_{P_i}(x)\}\rangle : x \in X\}$.

Note that $\underset{i\in I}{\cap} P_i$ and $\underset{i\in I}{\cup} P_i$ are pythagorean fuzzy sets over X. We shall $\underset{i\in I}{\cap} P_i$ define $\underset{i\in I}{\cap} P_i = \left\{\left\langle x, \alpha_{\underset{i\in I}{\cap} P_i}, \beta_{\underset{i\in I}{\cap} P_i}\right\rangle : x \in X\right\}$ such that $\alpha_{\underset{i\in I}{\cap} P_i} = \inf\{\varphi_{P_i}(x)\}$ and $\beta_{\underset{i\in I}{\cap} P_i} = \sup\{\psi_{P_i}(x)\}$. In order to for $\underset{i\in I}{\cap} P_i$ to be pythagorean fuzzy set we must have that $\alpha^2_{\underset{i\in I}{\cap} P_i}(x) + \beta^2_{\underset{i\in I}{\cap} P_i}(x) \leq 1$. We see since $\beta^2_{\underset{i\in I}{\cap} P_i}(x) = \sup\{\psi^2_{P_i}(x)\}$, then

$$\begin{aligned}\beta^2_{\underset{i\in I}{\cap} P_i}(x) &= \sup\{\psi^2_{P_i}(x)\} = \sup\{r_i^2 - \varphi^2_{P_i}, r_i^2 - \psi^2_{P_i}\}\\ &\leq \sup\{r_i^2 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}, r_i^2 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\}\end{aligned}$$

$$\begin{aligned}\beta^2_{\underset{i\in I}{\cap} P_i}(x) &\leq \sup\{1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\}, 1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\\ &\leq 1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\end{aligned}$$

where $\varphi^2_{P_i} + \psi^2_{P_i} = r_i^2$ for every $i \in I$. From this we see that $\alpha^2_{\underset{i\in I}{\cap} P_i}(x) + \beta^2_{\underset{i\in I}{\cap} P_i}(x) \leq \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\} + 1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\} \leq 1$. Thus, $\underset{i\in I}{\cap} P_i$ is a pythagorean fuzzy set.

The prof is trivial for $\underset{i\in I}{\cup} P_i$.

Theorem 3.2. Let $\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i\in I}$ be a family of pythagorean fuzzy sets over X. Then,

i $\overline{\underset{i\in I}{\cap} P_i} = \underset{i\in I}{\cup} \overline{P_i}$,

ii $\overline{\underset{i\in I}{\cup} P_i} = \underset{i\in I}{\cap} \overline{P_i}$.

Proof. i) We have $\underset{i\in I}{\cap} P_i = \{\langle x, \inf\{\varphi_{P_i}(x)\}, \sup\{\psi_{P_i}(x)\}\rangle : x \in X\}$. Then

$$\overline{\underset{i\in I}{\cap} P_i} = \{\langle x, \sup\{\psi_{P_i}(x), \inf\{\varphi_{P_i}(x)\}\}\rangle : x \in X\}$$

and $\overline{P_i} = \{\langle x, \psi_{P_i}(x), \varphi_{P_i}(x)\rangle : x \in X\}$ and so $\underset{i\in I}{\cup}\overline{P_i} = \{\langle x, \sup\{\psi_{P_i}(x), \inf\{\varphi_{P_i}(x)\}\}\rangle : x \in X\}$. That is, $\overline{\underset{i\in I}{\cap} P_i} = \underset{i\in I}{\cup}\overline{P_i}$.

ii) It is proved similar to (i) $\square$

Definition 3.3. Let $(X,\tau)_p$ be a $PFTS$ and $P = \{\langle x, \varphi_P(x), \psi_P(x)\rangle : x \in X\}$ be a PFS over X. Then the pythagorean fuzzy interior, pythagorean fuzzy closure and pythagorean fuzzy boundary of P are defined by;

a $int(P) = \cup\{G : G \text{ is a } PFOS \text{ in } X \text{ and } G \subseteq P\}$,

b $cl(P) = \cap\{K : K \text{ is a } PFCS \text{ in } X \text{ and } P \subseteq K\}$,

c $Fr(P) = cl(P) \cap cl(P^c)$.

It is clear that,

a $int(P)$ is the biggest pythagorean fuzzy open set contained P,

b $cl(P)$ is the smallest pythagorean fuzzy closed set containing P.

Remark 3.4. From the definition pythagorean fuzzy union and intersection, it is obvious that pythagorean fuzzy interior, closure and boundary is a pythagorean fuzzy set.

Example 3.5. Let $X = \{x_1, x_2, x_3\}$. Consider the following family of pythagorean fuzzy sets $\tau = \{1_X, 0_X, P_1, P_2, P_3, P_4,\}$ where

$$\begin{aligned} P_1 &= \{\langle x_1, 0.6, 0.8\rangle, \langle x_2, 0.7, 0.6\rangle, \langle x_3, 0.3, 0.2\rangle\}, \\ P_2 &= \{\langle x_1, 0.7, 0.9\rangle, \langle x_2, 0.2, 0.5\rangle, \langle x_3, 0.1, 0.9\rangle\}, \\ P_3 &= \{\langle x_1, 0.7, 0.8\rangle, \langle x_2, 0.7, 0.5\rangle, \langle x_3, 0.3, 0.2\rangle\}, \\ P_4 &= \{\langle x_1, 0.6, 0.9\rangle, \langle x_2, 0.2, 0.6\rangle, \langle x_3, 0.1, 0.9\rangle\}. \end{aligned}$$

It is clear that $(X,\tau)_p$ is a pythagorean fuzzy topological space. Now, assume that,

$$P = \{\langle x_1, 0.8, 0.5\rangle, \langle x_2, 0.9, 0.3\rangle, \langle x_3, 0.4, 0.1\rangle\}$$

is a pythagorean fuzzy subset over X. Then

$$\begin{aligned} int(P) &= 0_X \cup P_1 \cup P_2 \cup P_3 \cup P_4 \\ &= P_3 = \{\langle x_1, 0.7, 0.8\rangle, \langle x_2, 0.7, 0.5\rangle, \langle x_3, 0.3, 0.2\rangle\}. \end{aligned}$$

On the other hand, in order to find the pythagorean fuzzy closure of P, it necessary to determine the pythagorean fuzzy closed sets over X. Then

$$\begin{aligned} P_1^c &= \{\langle x_1, 0.8, 0.6\rangle, \langle x_2, 0.6, 0.7\rangle, \langle x_3, 0.2, 0.3\rangle\}, \\ P_2^c &= \{\langle x_1, 0.9, 0.7\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.9, 0.1\rangle\}, \\ P_3^c &= \{\langle x_1, 0.8, 0.7\rangle, \langle x_2, 0.5, 0.7\rangle, \langle x_3, 0.2, 0.3\rangle\}, \\ P_4^c &= \{\langle x_1, 0.9, 0.6\rangle, \langle x_2, 0.6, 0.2\rangle, \langle x_3, 0.9, 0.1\rangle\}. \end{aligned}$$

Hence,

$$cl(P) = 1_X$$

Similarly to find the pythagorean fuzzy boundary of P,

$$P^c = \{\langle x_1, 0.5, 0.8\rangle, \langle x_2, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.4\rangle\}$$

$$\begin{aligned} cl\left(P^c\right) &= 1_X \cap P_1^c \cap P_2^c \cap P_3^c \cap P_4^c \\ &= P_3^c = \{\langle x_1, 0.8, 0.7\rangle, \langle x_2, 0.5, 0.7\rangle, \langle x_3, 0.2, 0.3\rangle\} \end{aligned}$$

$$\begin{aligned} Fr(P) &= cl(P) \cap cl\left(P^c\right) \\ &= 1_X \cap P_3^c \\ &= \{\langle x_1, 0.8, 0.7\rangle, \langle x_2, 0.5, 0.7\rangle, \langle x_3, 0.2, 0.3\rangle\}. \end{aligned}$$

Proposition 3.6. Let $(X,\tau)_p$ be a $PFTS$ and P, P_1, P_2 be PFSs over X. Then the following properties hold;

i $int\,(P) \subseteq P$,

ii $int\,(int\,(P)) = int\,(P)\,,$

iii $P_1 \subseteq P_2 \Rightarrow int\,(P_1) \subseteq int\,(P_2)\,,$

iv $int\,(P_1 \cap P_2) = int\,(P_1) \cap int\,(P_2)\,,$

v $int\,(1_X) = 1_X$, $int\,(0_X) = 0_X$.

Proof. (i), (ii), (iii) and (v) can be easily obtained from the definition of the pythagorean fuzzy interior.

(iv) From $int\,(P_1 \cap P_2) \subseteq int\,(P_1)$ and $int\,(P_1 \cap P_2) \subseteq int\,(P_2)$ we obtain $int\,(P_1 \cap P_2) \subseteq int\,(P_1) \cap int\,(P_2)$. On the other hand, from the facts $int\,(P_1) \subseteq P_1$ and $int\,(P_2) \subseteq P_2 \Rightarrow int\,(P_1) \cap int\,(P_2) \subseteq P_1 \cap P_2$ and $int\,(P_1) \cap int\,(P_2) \in \tau$ we have $int\,(P_1) \cap int\,(P_2) \subseteq int\,(P_1 \cap P_2)$. So, proof of the axioms (iv) is obtained from these two inequalities. □

Theorem 3.7. Let $J : PFS(X) \to PFS(X)$ be a mapping. The family $\tau = \{P \in PFS(X) : J(P) = P\}$ is a pythagorean fuzzy topology over X, if the mapping J satisfies the following conditions:

i $J(P) \subseteq P$,

ii $J(1_X) = 1_X$,

iii $J\,(J(P)) = J(P)$,

iv $J\,(P_1 \cap P_2) = J\,(P_1) \cap J\,(P_2)\,.$

Also, $J(P) = int\,(P)$ for each pythagorean fuzzy set P in this pythagorean fuzzy topological space.

Proof. Straightforward. □

Proposition 3.8. Let $(X,\tau)_p$ be a $PFTS$ and P, P_1, P_2 be PFSs over X. Then the following properties hold;

i $P \subseteq cl\,(P)\,,$

ii $cl\,(cl\,(P)) = cl\,(P)\,,$

iii $P_1 \subseteq P_2 \Rightarrow cl\,(P_1) \subseteq cl\,(P_2)\,,$

iv $cl\,(P_1 \cup P_2) = cl\,(P_1) \cup cl\,(P_2)\,,$

v $cl\,(1_X) = 1_X$, $cl\,(0_X) = 0_X$.

Proof. (i), (ii), (iii) and (v) can be easily obtained from the definition of the pythagorean fuzzy closure.

(iv) From $cl\,(P_1) \subseteq cl\,(P_1 \cup P_2)$ and $cl\,(P_2) \subseteq cl\,(P_1 \cup P_2)$ we obtain $cl\,(P_1) \cup cl\,(P_2) \subseteq cl\,(P_1 \cup P_2)$. On the other hand, from the facts $P_1 \subseteq cl\,(P_1)$ and $P_2 \subseteq cl\,(P_2) \Rightarrow P_1 \cup P_2 \subseteq cl\,(P_1) \cup cl\,(P_2)$ and $cl\,(P_1) \cup cl\,(P_2) \in PFCS$ we have $cl\,(P_1 \cup P_2) \subseteq cl\,(P_1) \cup cl\,(P_2)$. Thus, proof of the axioms (iv) is obtained from these two inequalities. □

Theorem 3.9. Let $C : PFS(X) \to PFS(X)$ be a mapping. The family $\tau = \{P \in PFS(X) : C(P^c) = P^c\}$ is a pythagorean fuzzy topology over X, if the mapping C satisfies the following conditions:

i $P \subseteq C(P)$,

ii $C(0_X) = 0_X$,

iii $C(C(P)) = C(P)$,

iv $C(P_1 \cup P_2) = C(P_1) \cup C(P_2)$.

Also, $C(P) = cl(P)$ for each pythagorean fuzzy set P in this pythagorean fuzzy topological space.

Proof. Straightforward. □

Theorem 3.10. Let $(X, \tau)_p$ be a $PFTS$ and P be a PFS over X. Then,

a $cl(P^c) = (int(P))^c$,

b $int(P^c) = (cl(P))^c$.

Proof. (a) Let $P = \{\langle x, \varphi_P(x), \psi_P(x)\rangle : x \in X\}$ and assume that the family of PFSs contained in P are indexed by the family $\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i \in I}$. Then we see that $int(P) = \{\langle x, \sup\{\varphi_{P_i}(x)\}, \inf\{\psi_{P_i}(x)\}\rangle : x \in X\}$ and hence $(int(P))^c = \{\langle x, \inf\{\psi_{P_i}(x)\}, \sup\{\varphi_{P_i}(x)\}\rangle : x \in X\}$. Since $P^c = \{\langle x, \psi_P(x), \varphi_P(x)\rangle : x \in X\}$ and $\varphi_{P_i}(x) \leq \varphi_P(x)$, $\psi_{P_i}(x) \geq \psi_P(x)$ for each $i \in I$, we obtain that

$\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i \in I}$ is the family of PFSs containing P^c, i.e. $cl(P^c) = \{\langle x, \inf\{\psi_{P_i}(x)\}, \sup\{\varphi_{P_i}(x)\}\rangle : x \in X\}$. Therefore, $cl(P^c) = (int(P))^c$ immediately.

(b) This analagous to (a). □

Proposition 3.11. Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then, the following are equivalent to each other;

a f is a pythagorean fuzzy continuous function,

b $f[cl(P)] \subseteq cl(f[P])$ for each PFS P in X,

c $cl\left(f^{-1}[K]\right) \subseteq f^{-1}[cl(K)]$ for each PFS K in Y,

d $f^{-1}[int(K)] \subseteq int\left(f^{-1}[K]\right)$ for each PFS K in Y.

Proof. a)⇒b) Let $f : X \to Y$ be a pythagorean fuzzy continuous function and P be a PFS over X. Then, $f[P] \subseteq cl(f[P])$ and $P \subseteq f^{-1}[cl(f[P])]$. Since $cl(f[P])$ is a pythagorean fuzzy closed set in Y and f is a pythagorean fuzzy continuous function, $f^{-1}[cl(f[P])]$ is a pythagorean fuzzy closed set in X. On the other hand, if $cl(P)$ is the smallest pythagorean fuzzy closed set containing P, then $cl(P) \subseteq f^{-1}[cl(f[P])]$ and so, $f[cl(P)] \subseteq cl(f[P])$.

b)⇒c) Suppose that $P = f^{-1}[K]$. From (b), $f[cl(P)] = f\left[cl\left(f^{-1}[K]\right)\right] \subseteq cl(f[P]) = cl\left(f\left[f^{-1}[K]\right]\right) \subseteq cl(K)$. Then, $cl\left(f^{-1}[K]\right) = cl(P) \subseteq f^{-1}[f[cl(P)]] \subseteq f^{-1}[cl(K)]$.

c)⇒d) Since $int(K) = (cl(K^c))^c$, then $cl\left(f^{-1}[K]\right) = cl(P) \subseteq f^{-1}[f[cl(P)]] \subseteq f^{-1}[cl(K)]$.

Assume that, G is a pythagorean fuzzy open set in Y. Then, $int(G) = G$. From (d), $f^{-1}[G] = f^{-1}[int(G)] \subseteq int\left(f^{-1}[G]\right) \subseteq f^{-1}[G]$. Therefore, f is a pythagorean fuzzy continuous function. □

Definition 3.12. Let $(X, \tau)_p$ be a $PFTS$.

a A subfamily Γ of τ is called a pythagorean fuzzy basic (PFB for short) for τ, if for each $P \in \tau$, $P = 0_X$ or there exists $\Gamma' \subseteq \Gamma$ such that $P = \cup\Gamma'$.

b A subfamily Φ of τ is called a pythagorean fuzzy subbase ($PFSB$ for short) for τ, if the family $\Gamma = \left\{\cap\Phi' : \Phi' \text{ is a finite subset of } \Phi\right\}$ is a pythagorean fuzzy basic for τ.

Example 3.13. Considering the pythagorean fuzzy topology in Example 1, the family

$$\Phi = \{P_1, P_2\}$$

is a pythagorean fuzzy subbase for τ and

$$\Gamma = \{P_1, P_2, P_4\}$$

is a pythagorean fuzzy basic for τ.

Theorem 3.14. Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then,

i f is a pythagorean fuzzy continuous function iff for each $B \in \Gamma$ we have $f^{-1}[B]$ is a pythagorean fuzzy open subset of X such that Γ is a pythagorean fuzzy basic for τ_2.

ii f is a pythagorean fuzzy continuous function iff for each $K \in \Phi$ we have $f^{-1}[K]$ is a pythagorean fuzzy open subset of X such that Φ is a pythagorean fuzzy subbase for τ_2.

Proof. i) Let f be a pythagorean fuzzy continuous function. Since each $B \in \Gamma \subseteq \tau_2$ and f is a pythagorean fuzzy continuous function, then $f^{-1}[B] \in \tau_1$.

Concersely, suppouse that Γ is a pythagorean fuzzy basic for τ_2 and $f^{-1}[B] \in \tau_1$ for each $B \in \Gamma$. Then for arbitrary a pythagorean fuzzy open set $P \in \tau_2$,

$$f^{-1}[P] = f^{-1}\left[\underset{B\in\Gamma}{\cup} B\right] = \underset{B\in\Gamma}{\cup} f^{-1}[B] \in \tau_1.$$

That is, f is a pythagorean fuzzy continuous function.

ii) Let f be a pythagorean fuzzy continuous function. Since each $K \in \Phi \subseteq \tau_2$ and f is a pythagorean fuzzy continuous function, then $f^{-1}[K] \in \tau_1$.

Concersely, assume that Φ is a pythagorean fuzzy subbase for τ_2 and $f^{-1}[K] \in \tau_1$ for each $K \in \Phi$. Then for arbitrary a pythagorean fuzzy open set $P \in \tau_2$,

$$\begin{aligned} f^{-1}[P] &= f^{-1}\left[\underset{i_j\in I}{\cup} (K_{i_1} \cap K_{i_2} \cap ... \cap K_{i_n})\right] \\ &= \underset{i_j\in I}{\cup} \left(f^{-1}[K_{i_1}] \cap f^{-1}[K_{i_2}] \cap ... \cap f^{-1}[K_{i_n}]\right) \in \tau_1 \end{aligned}$$

That is, f is a pythagorean fuzzy continuous function. □

Definition 3.15. Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then,

a f is called a pythagorean fuzzy open function if $f[P]$ is a pythagorean fuzzy open set over Y for every pythagorean fuzzy open set P over X.

b f is called a pythagorean fuzzy closed function if $f[K]$ is a pythagorean fuzzy closed set over Y for every pythagorean fuzzy closed set K over X.

Example 3.16. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Consider the following families of pythagorean fuzzy sets $\tau_1 = \{0_X, 1_X, P_1, P_2, P_3, P_4\}$ and $\tau_2 = \{0_Y, 1_Y, S_1, S_2, S_3, S_4\}$ where

$$\begin{aligned} P_1 &= \{\langle x_1, 0.3, 0.5\rangle, \langle x_2, 0.6, 0.2\rangle, \langle x_3, 0.6, 0.5\rangle\}, \\ P_2 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.8, 0.3\rangle, \langle x_3, 0.7, 0.6\rangle\}, \\ P_3 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.8, 0.2\rangle, \langle x_3, 0.7, 0.5\rangle\}, \\ P_4 &= \{\langle x_1, 0.3, 0.5\rangle, \langle x_2, 0.6, 0.3\rangle, \langle x_3, 0.6, 0.6\rangle\}, \end{aligned}$$

$$\begin{aligned} S_1 &= \{\langle y_1, 0.6, 0.2\rangle, \langle y_2, 0.3, 0.5\rangle, \langle y_3, 0.6, 0.5\rangle\}, \\ S_2 &= \{\langle y_1, 0.8, 0.3\rangle, \langle y_2, 0.6, 0.5\rangle, \langle y_3, 0.7, 0.6\rangle\}, \\ S_3 &= \{\langle y_1, 0.8, 0.2\rangle, \langle y_2, 0.6, 0.5\rangle, \langle y_3, 0.7, 0.5\rangle\}, \\ S_4 &= \{\langle y_1, 0.6, 0.3\rangle, \langle y_2, 0.3, 0.5\rangle, \langle y_3, 0.6, 0.6\rangle\}, \end{aligned}$$

It is clear that $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ are pythagorean fuzzy topological spaces. If pythagorean fuzzy function $f : X \to Y$ is defined as;

$$\begin{aligned} f(x_1) &= y_2 \\ f(x_2) &= y_1 \\ f(x_3) &= y_3 \end{aligned}$$

Then f is a pythagorean fuzzy open function. However f is not pythagorean fuzzy closed function on pythagorean fuzzy topological spaces $(X, \tau_1)_p$.

Theorem 3.17. Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then,

i f is a pythagorean fuzzy open function if $f[int(P)] \subseteq int(f[P])$ for each pythagorean fuzzy set P over X.

ii f is a pythagorean fuzzy closed function if $cl(f[P]) \subseteq f[cl(P)]$ for each pythagorean fuzzy set P over X.

Proof. i) Let f be a pythagorean fuzzy open function and P be a PFS over X. Then, $int(P)$ is a pythagorean fuzzy open set and $int(P) \subseteq P$. Since f is a pythagorean fuzzy open function, $f[int(P)]$ is a pythagorean fuzzy open set over Y and $f[int(P)] \subseteq f[P]$. Thus, $f[int(P)] \subseteq int(f[P])$ is obtained.

Conversely, suppouse that P is any pythagorean fuzzy open set over X. Then $P = int(P)$. From the condition of theorem, we have $f[int(P)] \subseteq int(f[P])$. Then $f[P] = f[int(P)] \subseteq int(f[P]) \subseteq f[P]$. This implies that $f[P] = int(f[P])$. That is, f is a pythagorean fuzzy open function.

ii) Let f be a pythagorean fuzzy closed function and P be a PFS over X. Since f is a pythagorean fuzzy closed function then $f[cl(P)]$ is a pythagorean fuzzy closed set over Y and $f[P] \subseteq f[cl(P)]$. Thus, $cl(f[P]) \subseteq f[cl(P)]$ is obtained.

Conversely, assume that P is any pythagorean fuzzy closed set over X. Then $P = cl(P)$. From the condition of theorem, we have $cl(f[P]) \subseteq f[cl(P)] = f[P] \subseteq cl(f[P])$. This means that, $cl(f[P]) = f[P]$. That is, f is a pythagorean fuzzy closed function. □

Definition 3.18. Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then f is a called a pythagorean fuzzy homeomerphism, if

i f is a bijection,

ii f is a pythagorean fuzzy continuous function,

iii f^{-1} is a pythagorean fuzzy continuous function.

Theorem 3.19. Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then the following conditions are equivalent;

i f is a pythagorean fuzzy homeomerphism,

ii f is a pythagorean fuzzy continuous function and pythagorean fuzzy open function,

iii f is a pythagorean fuzzy continuous function and pythagorean fuzzy closed function.

Proof. The proof can be easily obtained by using the previous theorems on continuity, opennes and closedness are omitted. □

Conclusion

In this paper, we introduced the concept of pythagorean fuzzy interior, pythagorean fuzzy closure, pythagorean fuzzy boundary, pythagorean fuzzy basic on pythagorean fuzzy topological spaces. We also study pythagorean fuzzy open (closed) function and pythagorean fuzzy homeomorphism. Some basic properties of these concepts are explored. We hope that, the results of this study may help in the investigation of pythagorean fuzzy topological spaces in many researches.

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On Some Neutrosophic Algebraic Equations

Mohammad Abobala[1]

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Abstract − This paper is devoted to studying linear equations, and quadratic equations over a neutrosophic field $F(I)$ and refined neutrosophic field $F(I_1, I_2)$. This work introduces a full description of the solution's algorithm in $F(I)$ and $F(I_1, I_2)$, and discusses the solution's algorithm for a linear system of neutrosophic equations over $F(I)$ and $F(I_1, I_2)$ for the first time.

Keywords – *Neutrosophic field, refined neutrosophic field, neutrosophic linear system, neutrosophic quadratic equation*

1. Introduction

Neutrosophy is a new kind of logic founded by F. Smarandache to deal with indeterminacy in nature and reality. According to Smarandache's work, every idea can be represented by three corresponding values (degree of truth, falsity, and indeterminacy). Recently, neutrosophy has found its way into algebraic studies. Many neutrosophic algebraic structures were defined and handled, such as neutrosophic group, neutrosophic ring, and neutrosophic field. See [1-5].

Refined neutrosophic structures such as refined neutrosophic groups and refined neutrosophic rings were firstly presented in the works of Agboola et al. [6,7] by using Smarandache's idea in splitting the indeterminacy I into many different logical degrees. Laterally, refined neutrosophic algebraic structures were studied widely in [2,8-13].

Through this paper, we try to establish the basic theory of neutrosophic algebraic equations. We introduce a full description of basic algorithms which solve the linear neutrosophic equation, neutrosophic quadratic equation, and neutrosophic linear system in a neutrosophic field $F(I)$ and refined neutrosophic field $F(I_1, I_2)$. Also, we construct some examples to clarify the validity of this work.

Our work's main idea is to transform the neutrosophic equation into an easy equivalent system of classical equations, and then we can build the desired algorithms.

2. Preliminaries

Definition 2.1. [3] Let $(R, +, \times)$ be a ring. Then, $R(I) = \{a + bI : a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with the condition $I^2 = I$.

If R is a field, then $R(I)$ is called a neutrosophic field.

A neutrosophic field is not a field by classical meaning, since I is not invertible.

[1]mohammadabobala777@gmail.com (Corresponding Author)
[1]Faculty of Science, Tishreen University, Lattakia, Syria

Definition 2.2. [1] Let R be a ring and $R(I)$ be the related neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I \ : \ a_0 \in P_0 \, , a_1 \in P_1\}$; P_0, P_1 are two subsets of R.

(a) We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Remark 2.3. [6] The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$$I_1{}^2 = I_1 \, , I_2{}^2 = I_2 \, , I_1 I_2 = I_2 I_1 = I_1$$

Definition 2.4. [6] If X is a set, then $X(I_1, I_2) = \{(a, bI_1, cI_2) : a \, , b \, , c \in X \}$ is called the refined neutrosophic set generated by X, I_1, I_2.

Definition 2.5. [6] Let $(R, +, \times)$ be a ring, $(R(I_1, I_2) \, , + \, , \times)$ is called a refined neutrosophic ring generated by R, I_1, I_2.

Theorem 2.6. [6] Let $(R(I_1, I_2) \, , + \, , \times)$ be a refined neutrosophic ring, then it is a ring. It is called a neutrosophic field if R is a classical field.

3. Main Discussion

Definition 3.1. Let $F(I)$ be a neutrosophic field. Then, a neutrosophic algebraic linear equation is defined as follows:

$$AX + B = 0; A = a_0 + a_1 I, B = b_0 + b_1 I, X = x_0 + x_1 I; x_i, a_i, b_i \in F$$

A neutrosophic quadratic equation is defined as follows:

$$AX^2 + BX + C = 0; A = a_0 + a_1 I, B = b_0 + b_1 I, C = c_0 + c_1 I, X = x_0 + x_1 I; x_i, a_i, b_i, c_i \in F$$

Theorem 3.2. Let $F(I)$ be a neutrosophic field, $AX + B = 0$ be a linear neutrosophic equation. Then, it is equivalent to the following two classical linear equations:

(a) $a_0 x_0 + b_0 = 0$.

(b) $(a_0 + a_1)(x_0 + x_1) + (b_0 + b_1) = 0$.

Proof. By computing $AX + B = 0$, we find $a_0 x_0 + b_0 + (a_0 x_1 + a_1 x_0 + a_1 x_1 + b_1)I = 0$. Thus,

$$a_0 x_0 + b_0 = 0 \text{ (equation (a))}$$

$$a_0 x_1 + a_1 x_0 + a_1 x_1 + b_1 = 0 \text{ (*)}$$

By adding (a) to (*), we get $(a_0 + a_1)(x_0 + x_1) + (b_0 + b_1) = 0$ (equation (b)).

Remark 3.3. It is easy to get an algorithm to solve neutrosophic linear equation $AX + B = 0$ in a neutrosophic field $F(I)$. We should solve the equivalent system, and then we get the desired solution.

Example 3.4. Consider the following neutrosophic linear equation $(1 + 2I)X + (2 - 3I) = 0$ (*) over the neutrosophic field of real numbers $R(I)$. The equivalent system is:

(a) $x_0 + 2 = 0$. (Its solution is $x_0 = -2$.)

(b) $3(x_0 + x_1) + (-1) = 0$. Its solution is $x_0 + x_1 = \frac{1}{3}$, thus $x_1 = \frac{7}{3}$.

The solution of (*) is $X = -2 + \frac{7}{3} I$.

Example 3.5. Consider the neutrosophic linear equation $(1 + 2I)X + (2 - 3I) = 0$ (*) over $Z_3(I)$ the neutrosophic field of integers modulo 3. The equivalent system is:

(a) $x_0 + 2 = 0$. (Its solution is $x_0 = -2 \equiv 1 (\text{mod } 3)$)

(b) $3(x_0 + x_1) + (-1) = 0$. It is a non solvable in Z_3. Hence (*) is not solvable in $Z_3(I)$.

Theorem 3.6. Let $F(I)$ be a neutrosophic field, $AX^2 + BX + C = 0$ be a quadratic neutrosophic equation. Then, it is equivalent to the following two classical linear equations:

(a) $a_0{x_0}^2 + b_0x_0 + c_0 = 0$.

(b) $(a_0 + a_1)(x_0 + x_1)^2 + (b_0 + b_1)(x_0 + x_1) + c_0 + c_1 = 0$.

Proof. By computing $AX^2 + BX + C = 0$, we get

$$(a_0{x_0}^2 + b_0x_0 + c_0) + (2a_0x_0x_1 + a_0{x_1}^2 + a_1{x_0}^2 + 2a_1x_0x_1 + a_1{x_1}^2 + b_0x_1 + b_1x_0 + b_1x_1 + c_1)I = 0$$

Thus, $a_0{x_0}^2 + b_0x_0 + c_0 = 0$ (equation (a)) and $2a_0x_0x_1 + a_0{x_1}^2 + a_1{x_0}^2 + 2a_1x_0x_1 + a_1{x_1}^2 + b_0x_1 + b_1x_0 + b_1x_1 + c_1$(*), by adding (a) to (*) we get $(a_0 + a_1)(x_0 + x_1)^2 + (b_0 + b_1)(x_0 + x_1) + c_0 + c_1 = 0$ (equation (b)).

Remark 3.7. To solve a quadratic neutrosophic equation $AX^2 + BX + C = 0$ in a neutrosophic field $F(I)$. It is sufficient to solve the equivalent system presented in Theorem 3.6.

Example 3.8. Consider the following quadratic neutrosophic equation $(1 + I)X^2 + (2 - I)X + 3I = 0$ (*) over the neutrosophic field of real numbers $R(I)$. The equivalent system is:

(a) ${x_0}^2 + 2x_0 = 0$. (It has two solutions $x_0 = 0$, or $x_0 = -2$.

(b) $(2)(x_0 + x_1)^2 + (1)(x_0 + x_1) + 3 = 0$. (It has no solutions in R, thus (*) is not solvable in $R(I)$.)

Example 3.9. Consider the following quadratic neutrosophic equation $(1 + I)X^2 + (2 - I)X + 3I = 0$ (*) over the neutrosophic field of complex numbers $C(I)$. The equivalent system is:

(a) ${x_0}^2 + 2x_0 = 0$. (It has two solutions $x_0 = 0$, or $x_0 = -2$.)

(b) $(2)(x_0 + x_1)^2 + (1)(x_0 + x_1) + 3 = 0$. It has two solutions $x_0 + x_1 = \frac{-1+i\sqrt{23}}{4}$ or $\frac{-1-i\sqrt{23}}{4}$, thus $x_1 \in \{\frac{-1+i\sqrt{23}}{4}, \frac{-1-i\sqrt{23}}{4}\}$ if $x_0 = 0$. If $x_0 = -2$, then $x_1 \in \{2 + \frac{-1+i\sqrt{23}}{4}, 2 + \frac{-1-i\sqrt{23}}{4}\}$. The solutions of equation (*) are $X = \frac{-1+i\sqrt{23}}{4}I$, or $X = \frac{-1-i\sqrt{23}}{4}I$, or $X = -2 + \left(2 + \frac{-1+i\sqrt{23}}{4}\right)I$, or $X = -2 + \left(2 + \frac{-1-i\sqrt{23}}{4}\right)I$.

Theorem 3.10. Let $A_1X_1 + A_2X_2 + \cdots + A_nX_n = C$ (*) be a neutrosophic linear equation with n-variables over a neutrosophic field $F(I)$. Suppose that $C = c_0 + c_1I$, $A_i = a_0^{(i)} + a_1^{(i)}I, X = x_0^{(i)} + x_1^{(i)}I$; $c_i, x_j^{(i)}, a_j^{(i)} \in F$. Then, (*) has the following equivalent system of classical linear equations:

(a) $a_0^{(1)}x_0^{(1)} + a_0^{(2)}x_0^{(2)} + \cdots + a_0^{(n)}x_0^{(n)} = c_0$.

(b) $(a_0^{(1)} + a_1^{(1)})\left(x_0^{(1)} + x_1^{(1)}\right) + \left(a_0^{(2)} + a_1^{(2)}\right)\left(x_0^{(2)} + x_1^{(2)}\right) + \cdots + \left(a_0^{(n)} + a_1^{(n)}\right)\left(x_0^{(n)} + x_1^{(n)}\right) = c_0 + c_1$.

Proof.

First of all, we should compute A_iX_i. We have $A_iX_i = a_0^{(i)}x_0^{(i)} + \left(a_0^{(i)}x_1^{(i)} + a_1^{(i)}x_0^{(i)} + a_1^{(i)}x_1^{(i)}\right)I$, we remark that $a_0^{(i)}x_0^{(i)} + \left(a_0^{(i)}x_1^{(i)} + a_1^{(i)}x_0^{(i)} + a_1^{(i)}x_1^{(i)}\right) = \left(a_0^{(i)} + a_1^{(i)}\right)(x_0^{(i)} + x_1^{(i)})$. Hence,

$$A_iX_i = a_0^{(i)}x_0^{(i)} + \left(a_0^{(i)}x_1^{(i)} + a_1^{(i)}x_0^{(i)} + a_1^{(i)}x_1^{(i)}\right)I = a_0^{(i)}x_0^{(i)} + [\left(a_0^{(i)} + a_1^{(i)}\right)\left(x_0^{(i)} + x_1^{(i)}\right) - a_0^{(i)}x_0^{(i)}]I$$

Now, we can write $\sum_{i=1}^{n} A_iX_i = \left(\sum_{i=1}^{n} a_0^{(i)}x_0^{(i)}\right) + I(\sum_{i=1}^{n}\left(a_0^{(i)} + a_1^{(i)}\right)\left(x_0^{(i)} + x_1^{(i)}\right) - \sum_{i=1}^{n} a_0^{(i)}x_0^{(i)}) = C = c_0 + c_1I$. Thus, $\left(\sum_{i=1}^{n} a_0^{(i)}x_0^{(i)}\right) = c_0$ (equation (a)). And $\sum_{i=1}^{n}\left(a_0^{(i)} + a_1^{(i)}\right)\left(x_0^{(i)} + x_1^{(i)}\right) - \sum_{i=1}^{n} a_0^{(i)}x_0^{(i)} = c_1$ (*), by adding (a) to (*) we get $\sum_{i=1}^{n}\left(a_0^{(i)} + a_1^{(i)}\right)\left(x_0^{(i)} + x_1^{(i)}\right) = c_0 + c_1$. (equation (b)).

Remark 3.11. We can solve a linear system of neutrosophic equations in a neutrosophic field $F(I)$ by solving its equivalent system in the classical field F.

Example 3.12. Consider the following neutrosophic linear system over the neutrosophic field of real numbers:

(1) $(1+I)X+(2-I)Y=1+3I$.

(2) $(2+I)X+5IY=-1+I$.

The equivalent system of (1) is

$$x_0+2y_0=1 \text{ (I)}$$
$$2(x_0+x_1)+(y_0+y_1)=4 \text{ (II)}$$

The equivalent system of (2) is

$$2x_0+0.y_0=-1 \text{ (III)}$$
$$3(x_0+x_1)+5(y_0+y_1)=0 \text{ (IV)}$$

From (I), (III), we get $x_0=-\frac{1}{2}, y_0=\frac{3}{4}$. From (II), (IV), we get $x_0+x_1=\frac{20}{7}, y_0+y_1=-\frac{12}{7}$.

Definition 3.13. Let $F(I_1,I_2)$ be a refined neutrosophic field. We define

(a) $AX+B=(0,0,0); A=(a_0,a_1I_1,a_2I_2), B=(b_0,b_1I_1,b_2I_2), X=(x_0,x_1I_1,x_2I_2); b_i,a_i,x_i \in F$. (Refined neutrosophic linear equation with one variable).

(b) $AX^2+BX+C=(0,0,0); C=(c_0,c_1I_1,c_2I_2)$. (Refined quadratic neutrosophic equation).

Theorem 3.14. Let $F(I_1,I_2)$ be a refined neutrosophic field, $AX+B=(0,0,0)$ be a refined linear neutrosophic equation. Then, it is equivalent to the following system of classical linear equations:

(a) $a_0x_0+b_0=0$.

(b) $(a_0+a_2)(x_0+x_2)+(b_0+b_2)=0$.

(c) $(a_0+a_1+a_2)(x_0+x_1+x_2)+(b_0+b_1+b_2)=0$.

Proof.

We compute

$$AX+B=(a_0x_0+b_0,[a_0x_1+a_1x_0+a_1x_1+a_1x_2+a_2x_1+b_1]I_1,[a_0x_2+a_2x_0+a_2x_2+b_2]I_2)$$

So, we get

$$a_0x_0+b_0=0 \text{ (equation (a))}$$

$$a_0x_2+a_2x_0+a_2x_2+b_2=0 \text{ (*)}, a_0x_1+a_1x_0+a_1x_1+a_1x_2+a_2x_1+b_1 \text{ (**)}$$

By adding (a) to (*), we find $(a_0+a_2)(x_0+x_2)+(b_0+b_2)=0$ (equation (b)).

By adding (b) to (**), we find $(a_0+a_1+a_2)(x_0+x_1+x_2)+(b_0+b_1+b_2)=0$ (equation (c)).

Example 3.15. Consider the following refined neutrosophic linear equation (*) $(2,I_1,3I_2)X+(4,7I_1,-5I_2)=(0,0,0)$ over the refined neutrosophic field $Q(I_1,I_2)$.

The equivalent system is:

(a) $2x_0+4=0$. It has a solution $x_0=-2$.

(b) $5(x_0+x_2)+(-1)=0$. It has a solution $x_0+x_2=\frac{1}{5}$, hence $x_2=\frac{11}{5}$.

(c) $6(x_0+x_1+x_2)+(6)=0$. It has a solution $x_0+x_1+x_2=-1$, hence $x_1=\frac{-6}{5}$.

The solution of equation (*) is $X=(-2,\frac{-6}{5}I_1,\frac{11}{5}I_2)$.

Theorem 3.16. Let $F(I_1, I_2)$ be a refined neutrosophic field, $AX^2 + BX + C = (0,0,0)$ be a refined quadratic neutrosophic equation over $F(I_1, I_2)$. Then, it is equivalent to the following system of classical quadratic equations:

(a) $a_0{x_0}^2 + b_0x_0 + c_0 = 0$.

(b)$(a_0 + a_2)(x_0 + x_2)^2 + (b_0 + b_2)(x_0 + x_2) + c_0 + c_2 = 0$.

(c) $(a_0 + a_1 + a_2)(x_0 + x_1 + x_2)^2 + (b_0 + b_1 + b_2)(x_0 + x_1 + x_2) + c_0 + c_1 + c_2 = 0$.

Proof. The proof is similar to the previous theorem.

Example 3.17. Consider the following refined neutrosophic quadratic equation over the refined neutrosophic field of complex numbers $C(I_1, I_2)$,

$(*)$ $(1,0,I_2)X^2 + (1,I_1,0)X + (-2,I_1,I_2) = (0,0,0)$, the equivalent system is

(a) ${x_0}^2 + x_0 - 2 = 0$. It has two solutions $x_0 = 1$ or $x_0 = -2$.

(b) $2(x_0 + x_2)^2 + (x_0 + x_2) - 1 = 0$. It has two possible solutions $x_0 + x_2 = -1$ or $x_0 + x_2 = \frac{1}{2}$. Thus if $x_0 = -2$, then $x_2 = 1$ or $x_2 = \frac{5}{2}$, and if $x_0 = 1$, then $x_2 = -2$ or $x_2 = \frac{-1}{2}$.

(c) $2(x_0 + x_1 + x_2)^2 + 2(x_0 + x_1 + x_2) = 0$. It has two possible solutions $x_0 + x_1 + x_2 = 0$ or $x_0 + x_1 + x_2 = -1$.

If $x_0 + x_2 = -1$, then $x_1 = 1$ or $x_1 = 0$, and if $x_0 + x_2 = \frac{1}{2}$, then $x_1 = \frac{-1}{2}$ or $x_1 = -\frac{3}{2}$.

The set of solutions of equation (*) are

$$\left\{(1,I_1,-2I_2),\left(1,\frac{-1}{2}I_1,-\frac{1}{2}I_2\right),(-2,I_1,I_2),\left(-2,-\frac{1}{2}I_1,\frac{5}{2}I_2\right),(1,0,-2I_2),\left(1,-\frac{3}{2}I_1,-\frac{1}{2}I_2\right),(-2,0,I_2),(-2,-\frac{3}{2}I_1,\frac{5}{2}I_2)\right\}$$

Theorem 3.18. Let $A_1X_1 + \cdots + A_nX_n = C$; $C = (c_0, c_1I_1, c_2I_2)$, $X_i = \left(x_0^{(i)}, x_1^{(i)}I_1, x_2^{(i)}I_2\right)$, $A_i = \left(a_0^{(i)}, a_1^{(i)}I_1, a_2^{(i)}I_2\right)$ be a linear equation with n-variables over a refined neutrosophic field $F(I_1, I_2)$. Then, it is equivalent to the following system of classical linear equations over the classical field F:

(a) $\sum_{i=1}^{n} a_0^{(i)}x_0^{(i)} = c_0$.

(b) $\sum_{i=1}^{n}\left(a_0^{(i)} + a_2^{(i)}\right)\left(x_0^{(i)} + x_2^{(i)}\right) = c_0 + c_2$.

(c) $\sum_{i=1}^{n}\left(a_0^{(i)} + a_1^{(i)} + a_2^{(i)}\right)\left(x_0^{(i)} + x_1^{(i)} + x_2^{(i)}\right) = c_0 + c_1 + c_2$.

Proof.

We shall prove that $\sum_{i=1}^{n} A_iX_i = (\sum_{i=1}^{n} a_0^{(i)}x_0^{(i)}, \left[\sum_{i=1}^{n}\left(a_0^{(i)} + a_1^{(i)} + a_2^{(i)}\right)\left(x_0^{(i)} + x_1^{(i)} + x_2^{(i)}\right) - \sum_{i=1}^{n}\left(a_0^{(i)} + a_2^{(i)}\right)\left(x_0^{(i)} + x_2^{(i)}\right)\right]I_1, \left[\sum_{i=1}^{n}\left(a_0^{(i)} + a_2^{(i)}\right)\left(x_0^{(i)} + x_2^{(i)}\right) - \sum_{i=1}^{n} a_0^{(i)}x_0^{(i)}\right]I_2)$.

We will use induction on n. For $n = 1$, the theorem is true easily. Suppose that it is true for k. We must prove it for $k + 1$.

$$\begin{aligned}\sum_{i=1}^{k+1} A_iX_i &= \sum_{i=1}^{k} A_iX_i + A_{k+1}X_{k+1}\\ &= \left(\sum_{i=1}^{k} a_0^{(i)}x_0^{(i)}, \left[\sum_{i=1}^{k}(a_0^{(i)} + a_1^{(i)} + a_2^{(i)})(x_0^{(i)} + x_1^{(i)} + x_2^{(i)}) - \sum_{i=1}^{k}(a_0^{(i)} + a_2^{(i)})(x_0^{(i)} + x_2^{(i)})\right]I_1,\right.\end{aligned}$$

$$\left[\sum_{i=1}^{k}(a_0^{(i)}+a_2^{(i)})(x_0^{(i)}+x_2^{(i)})-\sum_{i=1}^{k}a_0^{(i)}x_0^{(i)}\right]I_2\Bigg)+(a_0^{(k+1)},a_1^{(k+1)}I_1,a_2^{(k+1)}I_2)(x_0^{(k+1)},x_1^{(k+1)}I_1,x_2^{(k+1)}I_2)$$
$$=(m,nI_1,tI_2)$$

We have $m=(\sum_{i=1}^{k}a_0^{(i)}x_0^{(i)})+a_0^{(k+1)}x_0^{(k+1)}=\sum_{i=1}^{k+1}a_0^{(i)}x_0^{(i)}$.

$$t=\sum_{i=1}^{k}(a_0^{(i)}+a_2^{(i)})(x_0^{(i)}+x_2^{(i)})-\sum_{i=1}^{k}a_0^{(i)}x_0^{(i)}+[a_0^{(k+1)}x_2^{(k+1)}+a_2^{(k+1)}x_0^{(k+1)}+a_2^{(k+1)}x_2^{(k+1)}]$$
$$=\sum_{i=1}^{k}(a_0^{(i)}+a_2^{(i)})(x_0^{(i)}+x_2^{(i)})-\sum_{i=1}^{k}a_0^{(i)}x_0^{(i)}+[(a_0^{(k+1)}+a_2^{(k+1)})(x_0^{(k+1)}+x_2^{(k+1)})-a_0^{(k+1)}x_0^{(k+1)}]$$
$$=\sum_{i=1}^{k+1}\left(a_0^{(i)}+a_2^{(i)}\right)\left(x_0^{(i)}+x_2^{(i)}\right)-\sum_{i=1}^{k+1}a_0^{(i)}x_0^{(i)}$$

By following the same argument, we find that

$$n=\sum_{i=1}^{k+1}\left(a_0^{(i)}+a_1^{(i)}+a_2^{(i)}\right)\left(x_0^{(i)}+x_1^{(i)}+x_2^{(i)}\right)-\sum_{i=1}^{k+1}\left(a_0^{(i)}+a_2^{(i)}\right)\left(x_0^{(i)}+x_2^{(i)}\right)$$

By putting m,n,t in equation (*) we get:

(a) $\sum_{i=1}^{n}a_0^{(i)}x_0^{(i)}=c_0$.

(I) $\sum_{i=1}^{n}\left(a_0^{(i)}+a_2^{(i)}\right)\left(x_0^{(i)}+x_2^{(i)}\right)-\sum_{i=1}^{n}a_0^{(i)}x_0^{(i)}=c_2$.

(II) $\sum_{i=1}^{n}\left(a_0^{(i)}+a_1^{(i)}+a_2^{(i)}\right)\left(x_0^{(i)}+x_1^{(i)}+x_2^{(i)}\right)-\sum_{i=1}^{n}\left(a_0^{(i)}+a_2^{(i)}\right)\left(x_0^{(i)}+x_2^{(i)}\right)=c_1$.

We add (a) to (I) to get equation (b). Also, we add (b) to (II) to get equation (c). Hence our proof is complete.

Remark 3.19. According to the previous theorem, we can solve any linear system of refined neutrosophic linear equations by transforming it into a classical equivalent system.

Example 3.20. Consider the following system of refined linear neutrosophic equations over the refined neutrosophic field of real numbers $R(I_1,I_2)$.

(1) $(1,I_1,0)X+(0,I_1,I_2)Y=(1,0,I_2)$.

(2) $(2,0,I_2)X+(-1,I_1,-I_2)Y=(3,0,0)$. Where $X=(x_0,x_1I_1,x_2I_2),Y=(y_0,y_1I_1,y_2I_2)$.

The equivalent system of equation (1) is:

$1.x_0+0.y_0=1$ **(I)**,

$1.(x_0+x_2)+1.(y_0+y_2)=2$ **(II)**,

$2.(x_0+x_1+x_2)+2(y_0+y_1+y_2)=2$ (**III**).

The equivalent system of equation (2) is:

$2.x_0-1.y_0=3$ **(a)**,

$3.(x_0+x_2)-2(y_0+y_2)=3$ (**b**),

$3.(x_0+x_1+x_2)-1(y_0+y_1+y_2)=3$ **(c)**.

By solving (I) with (a), we find $x_0=1,y_0=-1$.

By solving (II) with (b), we find $x_0 + x_2 = \frac{7}{5}, y_0 + y_2 = \frac{3}{5}$. Thus, $x_2 = \frac{2}{5}, y_2 = \frac{8}{5}$.

By solving (III) with (c), we find $x_0 + x_1 + x_2 = 1, y_0 + y_1 + y_2 = 0$. Thus, $x_1 = -\frac{2}{5}, y_1 = -\frac{3}{5}$.

The solution of the system (1) and (2) is:

3. $X = (1, -\frac{2}{5}I_1, \frac{2}{5}I_2), Y = (-1, -\frac{3}{5}I_1, \frac{8}{5}I_2)$.

4. Conclusion

In this article, we have introduced an algorithm to solve linear and quadratic equations in a neutrosophic field $F(I)$ and refined neutrosophic field $F(I_1, I_2)$. Also, we have introduced an algorithm to solve a linear system of neutrosophic equations over $F(I)$ and $F(I_1, I_2)$ by turning it into an easy classical equivalent system of linear equations.

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On Some Special Elements in Neutrosophic Rings and Refined Neutrosophic Rings

Mohammad Abobala[1]

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Abstract — Idempotent elements in a ring R are the elements with the condition $\boldsymbol{a^2 = a}$. This paper introduces the criterion of any element in a refined neutrosophic ring to be idempotent. Also, the concept of symmetric and supersymmetric elements in a neutrosophic ring $R(I)$, and a refined neutrosophic ring $R(I_1, I_2)$ are defined. Also, the invertibility of these elements is discussed.

Keywords – *Neutrosophic ring, refined neutrosophic ring, idempotent element, symmetric element, supersymmetric element*

1. Introduction

Neutrosophic algebra is a new trend in pure mathematics; it is considered a combination between the neutrosophic set introduced by Smarandache and classical algebra.

Many neutrosophic algebraic structures were defined and studied in a wide range such as neutrosophic groups, neutrosophic rings, neutrosophic vector spaces, and neutrosophic modules. See [1-6].

Recently, many generalized concepts came to light, such as refined neutrosophic rings, Boolean rings, and n-refined neutrosophic rings [7-14]. These generalizations were built over the idea of splitting the indeterminacy I into many logical degrees. In the case of refined structures, I is splitting into two sub-indeterminacies I_1, I_2 with the following property $I_1 I_2 = I_1, {I_1}^2 = I_1, {I_2}^2 = I_2$ [9]. Also, I is splitting into n sub-indeterminacies $I_1, \ldots, I_n$ in the case of n-refined structures. See [12,13].

Idempotents in a ring R are the elements with the property $a^2 = a$. They were handled and classified in neutrosophic rings with semi idempotents in [15,16]. Through this paper, we introduce the condition of any element in a refined neutrosophic ring $R(I_1, I_2)$ to be idempotent. Two new kinds of special elements (symmetric and supersymmetric elements) in neutrosophic rings and refined neutrosophic rings are presented and classified. These elements have many interesting properties, especially in neutrosophic fields and refined neutrosophic fields. Also, their algebraic structure will be discussed in previous cases.

2. Preliminaries

Definition 2.1. [13] Let $(R, +, \times)$ be a ring. Then, $R(I) = \{a + bI : a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with the condition $I^2 = I$.

[1]mohammadabobala777@gmail.com (Corresponding Author)
[1]Faculty of Science, Tishreen University, Lattakia, Syria

If R is a field, then $R(I)$ is called a neutrosophic field.

A neutrosophic field is not a field by classical meaning, since I is not invertible.

Definition 2.2. [1] Let R be a ring and $R(I)$ be the related neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I \ : \ a_0 \in P_0 , a_1 \in P_1\}$; P_0, P_1 are two subsets of R.

(a) We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Remark 2.3. [11] The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$$\mathrm{I_1}^2 = \mathrm{I}_1 , {I_2}^2 = I_2 , I_1 I_2 = I_2 I_1 = I_1$$

Definition 2.4. [11] If X is a set, then $X(I_1, I_2) = \{(a, bI_1, cI_2) : a , b , c \in X \}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.5. [11] Let $(R, +,\times)$ be a ring, $(R(I_1, I_2) , + ,\times)$ is called a refined neutrosophic ring generated by R ,I_1, I_2.

Theorem 2.6. [11] Let $(R(I_1, I_2) , + ,\times)$ be a refined neutrosophic ring, then it is a ring. It is called a neutrosophic field if R is a classical field**.**

Definition 2.7. [17] Let R be a ring, a be any element in R. Then, it is called idempotent if and only if $a^2 = a$.

3. Idempotents in $R(I_1, I_2)$

Theorem 3.1. Let R be any ring (noncommutative ring in general), $R(I_1, I_2)$ be its corresponding refined neutrosophic ring. Assume that $x = (a, bI_1, cI_2)$ is an arbitrary element in $R(I_1, I_2)$. Then, x is idempotent in $R(I_1, I_2)$ if and only if $a, a + c, a + b + c$ are idempotents in R.

Proof. Let $x = (a, bI_1, cI_2)$ be an idempotent in $R(I_1, I_2)$, then

$$x^2 = (a^2, [a.b + b.a + b^2 + b.c + c.b]I_1, [a.c + c.a + c^2]I_2) = x$$

Thus, $[a^2 = a]$,(*)$[a.b + b.a + b^2 + b.c + c.b = b]$, (**)$[a.c + c.a + c^2 = c]$, hence a is an idempotent in R.

Now, we compute $(a + c)^2 = a^2 + a.c + c.a + c^2$, we can find from (**) that $(a + c)^2 = a^2 + c = a + c$, thus $a + c$ is idempotent in R.

Also, we have $(a + b + c)^2 = a^2 + b^2 + c^2 + a.b + b.a + a.c + c.a + b.c + c.b$, by (*) we get

$$(a + b + c)^2 = (a^2 + c^2 + a.c + c.a) + (a.b + b.a + b^2 + b.c + c.b) = (a + c) + b = a + b + c$$

Thus, $a + b + c$ is idempotent in R.

For the converse, we suppose that $a, a + c, a + b + c$ are idempotents in R, then we get

(1) $[a^2 = a]$.

(2) $(a + c)^2 = a^2 + a.c + c.a + c^2 = a + c$. By using equation (1), we get $a.c + c.a + c^2 = c$.

(3) $(a + b + c)^2 = a^2 + b^2 + c^2 + a.b + b.a + a.c + c.a + b.c + c.b = a + b + c$. By using (1) and (2), we get $(a + b + c)^2 = a + b^2 + a.b + b.a + (a.c + c.a + c^2) + b.c + c.b = a + b + c$. Thus,

$$a + b^2 + a.b + b.a + (c) + b.c + c.b = a + b + c$$

Hence, $b^2 + a.b + b.a + b.c + c.b = b$.

Now, we compute $x^2 = (a^2, [a.b + b.a + b^2 + b.c + c.b]I_1, [a.c + c.a + c^2]I_2) = (a, bI_1, cI_2) = x$. So, it is idempotent in the refined neutrosophic ring $R(I_1, I_2)$.

Example 3.2. Let $R = Z_3$ be the ring of integers modulo 3, $R(I_1, I_2) = \{(a, bI_1, cI_2) : a, b, c \in Z_3\}$ be its corresponding refined neutrosophic ring. The set of idempotents in R is

$M = \{0,1\}$, the set of idempotents in $R(I_1, I_2)$ according to Theorem 3.1 is:

$$N = \{(1, I_1, 2I_2), (1,0,2I_2), (1,2I_1, 0), (1,0,0), (0,0,0), (0, I_1, 0), (0,0, I_2), (0,2I_1, I_2)\}$$

The following theorem determines the number of idempotents in $R(I_1, I_2)$.

Theorem 3.3. If the ring R has m idempotents, then the corresponding refined neutrosophic ring $R(I_1, I_2)$ has m^3 idempotents.

Proof. According to Theorem 3.1, for each idempotent $a \in R$, we have $x = (a, bI_1, cI_2)$ is idempotent in $R(I_1, I_2)$, if and only if $a + c, a + b + c$ are idempotents in R, thus c can be taken by m ways, and b is the same. By this argument, we get the fact that $R(I_1, I_2)$ has $m \times m \times m = m^3$ idempotents.

4. Symmetric Elements

This section is devoted to studying a new kind of special elements in a neutrosophic ring and a refined neutrosophic ring with its algebraic structures.

Definition 4.1. Let R be a ring, $R(I)$ be the corresponding neutrosophic ring. An arbitrary element $x = a + bI \in R(I)$ is called symmetric if and only if $a = b$. The set of all symmetric elements in a neutrosophic ring is denoted by $S(I)$.

Definition 4.2. Let R be a ring, $R(I_1, I_2)$ be the corresponding refined neutrosophic ring. An arbitrary element $x = (a, bI_1, cI_2) \in R(I_1, I_2)$ is called symmetric if and only if $a = b = c$. The set of all symmetric elements in a refined neutrosophic ring is denoted by $S(I_1, I_2)$.

Theorem 4.3. Let $R(I)$ be a neutrosophic ring, $S(I)$ be the set of all symmetric elements. Then, $(S(I), +)$ is a subgroup of $(R(I), +)$ and $(S(I), +) \cong (R, +)$.

Proof. Let $x = a + aI, y = b + bI$ be two arbitrary elements in $S(I)$, $x - y = (a - b) + (a - b)I \in S(I)$, thus $S(I)$ is a subgroup of $(R(I), +)$. (It is known that $(R(I), +)$ is an abelian group by the definition of the ring).

We define $f: R \to S(I); f(a) = a + aI$, suppose that $a, b \in R$, then $f(a + b) = (a + b) + (a + b)I = f(a) + f(b)$.

f is a well-defined map since if $a = b$, then $a + aI = b + bI$, i.e. $f(a) = f(b)$. Clearly, f is bijective; thus, it is an isomorphism.

Theorem 4.4: Let $R(I_1, I_2)$ be a refined neutrosophic ring, $S(I_1, I_2)$ be the set of all symmetric elements. Then, $(S(I_1, I_2), +)$ is a subgroup of $(R(I_1, I_2), +)$ and $(S(I_1, I_2), +) \cong (R, +)$.

Proof. The proof is similar to that of Theorem 4.3.

Theorem 4.5. Let $K(I)$ be a neutrosophic field, $S(I)$ be the set of all symmetric elements. If $Char(K) = 2$, then $S(I)$ is a field and $S(I) \cong K$.

Proof. We must prove that $(S(I)/\{0\}, .)$ is a group. Let $x = a + aI, y = b + bI$ be two arbitrary elements in $S(I)/\{0\}$, we have $x.y = (a.b) + (a.b + a.b + a.b)I = (a.b) + (a.b)I \in S(I)$, since $a.b + a.b = 2a.b = 0$ (under the assumption $Char(K) = 2$). The inverse of x is $x^{-1} = a^{-1} + a^{-1}I$ because $x.x^{-1} = (aa^{-1}) + (aa^{-1} + aa^{-1} + aa^{-1})I = 1 + I$. $1 + I$ is an identity concerning multiplication, that is because $(a + aI).(1 + I) = a + (a + a + a)I = a + aI$.

We define $f: S(I) \to K; f(a + aI) = a$, f is a well-defined bijective map.

Let $x = a + aI, y = b + bI$ be two arbitrary elements in $S(I)$, $f(x + y) = (a + b) = f(x) + f(y), f(x.y) = f(a.b + a.bI) = a.b = f(x).f(y)$. Hence f is an isomorphism.

Example 4.6. Let $K = Z_2$ be a field with $Char(K) = 2$, $K(I) = \{0,1, I, 1 + I\}$, $S(I) = \{0,1 + I\}$.

We can see that $S(I)$ is a field, the identity concerning multiplication is $1 + I$, and $S(I) \cong Z_2 = K$.

Theorem 4.7. Let $K(I_1, I_2)$ be a refined neutrosophic field, $S(I_1, I_2)$ be the set of all symmetric elements.

If $Char(K) = 2$, then $S(I_1, I_2)$ is a field and $S(I_1, I_2) \cong K$.

Proof. We must prove that $(S(I_1, I_2)/\{0\}, .)$ is a group. Let $x = (a, aI_1, aI_2), y = (b, bI_1, bI_2)$ be two arbitrary elements in $S(I_1, I_2)/\{0\}$, we have $x.y = (a.b, [5a.b]I_1, [3a.b]I_2) = (a.b, a.bI_1, a.bI_2) \in S(I_1, I_2)$, since $5a.b = 3a.b = a.b$ (under the assumption $Char(K) = 2$).

The inverse of x is $x^{-1} = (a^{-1}, a^{-1}I_1, a^{-1}I_2)$ because $x.x^{-1} = (a.a^{-1}, [5a.a^{-1}]I_1, [3a.a^{-1}]I_2) = (1,1.I_1, 1.I_2)$. ($5aa^{-1} = 5 = 1 + 4 = 0, 3aa^{-1} = 3 = 1 + 2 = 1$, under the assumption $Char(K) = 2$).

$(1,1.I_1, 1.I_2)$ is an identity concerning multiplication, that is because $(a, aI_1, aI_2).(1,1.I_1, 1.I_2) = (a, aI_1, aI_2)$.

We define $f: S(I_1, I_2) \to K; f(a, aI_1, aI_2) = a$, f is an isomorphism (It can be proved by a similar way to the previous theorem.).

Example 4.8. Let $K = Z_2$ be a field with $Char(K) = 2$,

$$K(I_1, I_2) = \{(0,0,0), (1,0,0), (0,1.I_1, 0), (0,0,1.I_2), (1,1.I_1, 1I_2), (1,1I_1, 0), (0,1.I_1, 1.I_2), (1,0,1.I_2\}$$

$S(I_1, I_2) = \{(0,0,0), (1,1.I_1, 1I_2)\}$, which is a field isomorphic to $K = Z_2$.

The following theorem determines which elements in a neutrosophic field $K(I)$ are invertible.

Theorem 4.9. Let K be a field, $K(I)$ be the corresponding neutrosophic field. An arbitrary element $z = a + bI \in K(I)$ is invertible if and only if $a \neq 0$ and $a \neq -b$.

Proof. Let $z = a + bI \in K(I)$ be an invertible element in $K(I)$. There is $m = x + yI \in K(I); z.m = 1$. Thus, $(a.x) + (a.y + b.x + b.y)I = 1$, this means $x = a^{-1}, a.y + b(a^{-1}) + by = 0$. Hence,

$$y = \frac{-b.a^{-1}}{a+b}, \text{ this implies } a \neq 0 \text{ and } a \neq -b$$

Conversely, suppose that $a \neq 0$ and $a \neq -b$, then there is $m = x + yI \in K(I)$, where $x = a^{-1}$, $y = \frac{-b.a^{-1}}{a+b}$ with $z.m = 1$.

Result 4.10. If $K(I)$ is a neutrosophic field with $Char(K) \neq 2$, then all symmetric elements different from zero are invertible.

Proof. Let K be a field with $Char(K) \neq 2$, $x = a + aI$ be a symmetric element different from zero. It is clear that $a \neq -a$, thus x is invertible according to Theorem 4.10.

Example 4.11. Let $K = Z_5$ be the field of integers modulo 5. We have $x = 3 + 3I$ a symmetric element. The inverse of x is $x^{-1} = 2 + 4I$.

The inverse of a symmetric element is not supposed to be symmetric in general.

The following theorem determines which elements are invertible in a refined neutrosophic field $K(I_1, I_2)$.

Theorem 4.12. Let $K(I_1, I_2)$ be a refined neutrosophic field. An arbitrary element $t = (a, bI_1, cI_2)$ is invertible if and only if $a \neq 0, a + c \neq 0, a + b + c \neq 0$.

Proof. Suppose that $t = (a, bI_1, cI_2)$ is invertible. Then, there is $m = (x, yI_1, zI_2); m.t = 1_{K(I_1, I_2)}$.

$m.t = (a.x, [a.y + b.x + b.z + b.y + c.y]I_1, [a.z + c.x + c.z]I_2) = (1,0,0)$, this means $x = a^{-1}$, $z = \frac{-ca^{-1}}{a+c}$, $y = (a + b + c)^{-1}.(-b.a^{-1} + \frac{bc}{a(a+c)})$, which implies that $a \neq 0, a + c \neq 0, a + b + c \neq 0$.

Conversely, if $a \neq 0, a + c \neq 0, a + b + c \neq 0$, then there is $m = (x, yI_1, zI_2); m.t = 1_{K(I_1,I_2)}$, where $x = a^{-1}, z = \frac{-ca^{-1}}{a+c}, y = (a + b + c)^{-1}.(-b.a^{-1} + \frac{bc}{a(a+c)})$.

Result 4.13. Let K be a field with $Char(K) \neq 2$ and $Char(K) \neq 3$, then all symmetric elements different from zero in the corresponding refined neutrosophic field $K(I_1, I_2)$ are invertible since the conditions of Theorem 4.12 are true in this case.

Example 4.14. Let $K = Z_5$ be the field of integers modulo 5 and $Char(K) = 5$, let $x = (3, 3I_1, 3I_2)$ is an element in the refined neutrosophic field $K(I_1, I_2)$. According to Theorem 12.3, the inverse of x is $x^{-1} = (2,3I_1, 4I_2)$.

The inverse of a symmetric element in a refined neutrosophic field is not supposed to be a symmetric element in general.

5. Super Symmetric Elements

The following section is discussing a generalized kind of symmetric elements.

Definition 5.1. Let R be a ring, $R(I)$ be the corresponding neutrosophic ring. An arbitrary element $x = a + bI \in R(I)$ is called supersymmetric if and only if $a = m.c, b = n.c\,; c \in R$ and $m, n \in Z$. The set of all supersymmetric elements in a neutrosophic ring is denoted by $SS(I)$.

Definition 5.2. Let R be a ring, $R(I_1, I_2)$ be the corresponding refined neutrosophic ring. An arbitrary element $x = (a, bI_1, cI_2) \in R(I_1, I_2)$ is called supersymmetric if and only if $a = m.d, b = n.d, c = s.d; d \in R$ and $m, n, s \in Z$. The set of all supersymmetric elements in a refined neutrosophic ring is denoted by $SS(I_1, I_2)$.

Theorem 5.3. Let $R(I)$ be a neutrosophic ring, $SS(I)$ be the set of all supersymmetric elements. Then, $SS(I)$ is closed under the multiplication of $R(I)$.

Proof. Let $x = m.a + n.aI, y = s.b + t.bI; a, b \in R$ and $m, n, s, t \in Z$, we have $x.y = (mn)(a.b) + [(mt + ns + nt)(a.b)]I \in SS(I)$. Since $mn, mt + ns + nt \in Z$.

Remark 5.4. $SS(I)$ is not an additive subgroup of $(R(I), +)$ in general. We clarify it by the following example:

Let R be the ring of real numbers, $x = \sqrt{2} + 3\sqrt{2I}, y = \sqrt{3} - 4\sqrt{3}I$ be two elements in $SS(I)$, $x + y = (\sqrt{2} + \sqrt{3}) + (3\sqrt{2} - 4\sqrt{3})I$, we can see that $x + y$ is not in $SS(I)$, since $3\sqrt{2} - 4\sqrt{3}$ cannot be written as $m.(\sqrt{2} + \sqrt{3})$, where m is an integer.

Theorem 5.5. Let $R(I_1, I_2)$ be a refined neutrosophic ring, $SS(I_1, I_2)$ be the set of all supersymmetric elements. Then, $SS(I_1, I_2)$ is closed under the multiplication of $R(I_1, I_2)$.

Proof. The proof is similar to that of Theorem 5.3.

Remark 5.6. $SS(I_1, I_2)$ is not an additive subgroup of $(R(I_1, I_2), +)$ in general. We illustrate an example.

Let R be the ring of real numbers, $x = (\sqrt{2}, \sqrt{2}\,I_1, 3\sqrt{2}\,I_2), y = (\sqrt{3}, 2\sqrt{3}I_1, 5\sqrt{3}I_2)$ be two elements in $SS(I_1, I_2)$, $x + y = (\sqrt{2} + \sqrt{3}, [\sqrt{2} + 2\sqrt{3}]I_1, [3\sqrt{2} + 5\sqrt{3}]I_2)$, we can see that $\sqrt{2} + 2\sqrt{3}$ can not be written as $m.(\sqrt{2} + \sqrt{3})$ where m is an integer.

The following theorem introduces a special case, which $SS(I)$ and $SS(I_1, I_2)$, will be two additive subgroups of $(R(I), +)$ and $(R(I_1, I_2), +)$, respectively.

Theorem 5.7. Let $R = Z$ be the ring of integers. Then,

(a) $(SS(I), +)$ is a subgroup of $(R(I), +)$.

(b) $(SS(I), +.)$ is a subring of $(R(I), +, .)$.

(c) $(SS(I_1, I_2), +)$ is a subgroup of $(R(I_1, I_2), +)$.

(d) $(SS(I_1, I_2), +, .)$ is a subring of $(R(I_1, I_2), +, .)$.

Proof.

(a) Let $x = m.a + n.aI, y = s.b + t.bI; a, b \in Z$ and $m, n, s, t \in Z,$ we have $x - y = (m.a - s.b)(1) + [(n.a - t.b)(1)]I \in SS(I)$, thus $(SS(I), +)$ is an additive subgroup of $(R(I), +)$.

(b) It holds directly from (a) and Theorem 5.3.

(c) It can be proved by a similar argument of (a).

(d) It holds directly from (c) and Theorem 5.5.

We discuss the invertibility properties of supersymmetric elements in a neutrosophic field and a refined neutrosophic field by the following theorem.

Theorem 5.8. Let K be a field, $K(I)$ be the corresponding neutrosophic field, $K(I_1, I_2)$ be the corresponding refined neutrosophic field. Then,

(a) An arbitrary supersymmetric element $x = m.a + n.aI \in K(I); m, n \in Z$ and $a \in K$ is invertible if and only if $a \neq 0$ and $(m + n).a \neq 0$.

(b) An arbitrary supersymmetric element $x = (m.a, n.aI_1, s.aI_2) \in K(I_1, I_2); m, n, s \in Z$ and $a \in K$ is invertible if and only if $a \neq 0, (m + s).a \neq 0, (m + n + s).a \neq 0$.

Proof.

(a) According to Theorem 4.9, $x = a + bI$ is invertible if and only if $a \neq 0$ and $a + b \neq 0$, thus $x = m.a + n.aI$ is invertible if and only if $a \neq 0$ and $m.a + n.a = (m + n).a \neq 0$.

(b) It can be proved by a similar argument of section (a), and by using Theorem 4.12.

6. Conclusion

In this article, we have determined the criterion of idempotency in a refined neutrosophic ring. Also, we have introduced two new kinds of special elements in neutrosophic rings and refined neutrosophic rings. We have studied the invertibility conditions of these elements and their algebraic structure. This work should be extended to the case of n-refined neutrosophic rings defined in [12], where the necessary and sufficient condition for nilpotency is still unknown.

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Spectral Properties of the Antiperiodic Boundary-Value-Transition Problems

Serdar Paş[1], Kadriye Aydemir[2], Fahreddin Muhtarov[3]

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Abstract − This work is concerned with the boundary-value-transition problem consisting of a two-interval Sturm-Liouville equation

$$Lu := -u''(x) + q(x)u(x) = \lambda u(x)\,, x \in [-1,0) \cup (0,1]$$

together with antiperiodic boundary conditions, given by

$$u(-1) = -u(1)$$

$$u'(-1) = -u'(1)$$

and transition conditions at the interior point $x = 0$, given by

$$u(+0) = Ku(-0)$$

$$u'(+0) = \frac{1}{K}u'(-0)$$

where $q(x)$ is a continuous function in the intervals $[-1,0)$ and $(0,1]$ with finite limit values $q(\pm 0)$, $K \neq 0$ is the real number, and λ is the complex eigenvalue parameter. In this study, we shall investigate some properties of the eigenvalues and eigenfunctions of the considered problem.

Keywords *–Antiperiodic Sturm-Liouville problem, eigenvalue, eigenfunction, transition condition*

1. Introduction

A simple model for the movement of electrons in a crystal lattice, consisting of the ions in the crystal lattice and crystal with a periodic potential time-independent Schrödinger equation that describes the effects of forces from other electrons. The wave function of the electron meets the one-dimensional Schrödinger equation with the periodic potential. $T(x)$. Let t be a period that is $T(x+t) = T(x)$. By changing the variable

$$u(x) = \varphi\left(\frac{x}{t}\right),\ q(x) = \frac{2mt^2}{\hbar}T\left(\frac{x}{ta}\right), \text{and}\ \ \lambda = \frac{2mE}{\hbar^2}$$

we have

$$-u'' + q(x)u = \lambda u \tag{1.1}$$

where u is the normalized wavefunction, λ is energy parameter, $q(x+1) = q(x)$. The spectrum of (1.1) is absolutely continuous and occurs a combination of closed intervals or 'bands' separated by 'gaps'. The presence of these bands and gaps has important implications for the conductivity properties of crystals.

[1] serdarpas@gmail.com; [2] kadriyeaydemr@gmail.com (Corresponding Author); [3] fahreddinmuhtarov@gmail.com
[1,2] Department of Mathematics, Faculty of Arts and Sciences, Amasya University, Amasya, Turkey
[3] Department of Mathematics, Mechanics-Mathematics Faculty, Baku State University, Baku, Azerbaijan

The study published by Birkhoff [1] investigated the asymptotic behaviour of the solutions of linear differential equations depending on the eigenvalue parameter given by

$$\frac{d^n y}{dx^n}+\lambda a_{n-1}(x,\lambda)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+\lambda^n a_0(x,\lambda)y=0$$

In this work, asymptotic formulas of solutions of the considered linear differential equations related to the eigenvalue parameter have been studied, and it is defined as the concept of regular boundary conditions. In the literature, such conditions are called regular boundary conditions in the sense of Birkhoff. He proved the theorem associate with the completeness of systems consisting of eigenfunctions and associated functions (i.e., root functions) of the differential operator corresponding to the problem.

In the study of Tamarkin [2], it is found the asymptotic of basic solutions for linear differential equations dependent on parameters. He defined the concept of strong regular conditions, and he studied the properties of eigenfunctions under these boundary conditions and the expansion in the series of eigenfunctions. In later years, the investigation of new concrete problems posed by physics led to the rapid development of Sturm-Liouville theory. Today, the Sturm-Liouville problems remain one of the most current issues needed by spectral theory.

In the study of Lee [3] showed the periodic analogues of spectral and oscillation theory concerned with the standard Sturm-Liouville problem.

Berghe et al. [4] investigated the eigenvalues of boundary value problems under periodic and quasi-periodic boundary conditions and explained that a simple linearly dependent multistep method could reduce the error of approximate eigenvalues.

Liu [5] prove existence for the solutions of the periodic Sturm-Liouville problem consisting of the n-th order functional differential equation with impulses effects, given by

$$\begin{cases} x^n(s)=f\Big(s,x(s),x\big(\alpha_1(s)\big)\dots,x\big(\alpha_m(s)\big)\Big), & s\in[0,S] \\ \Delta x^i(s_k)=I_{i,k}\big(x(s_k),\dots x^{n-1}(s_k)\big), & k=1,2,\dots,r \end{cases}$$

and the periodic boundary conditions, given by

$$x^i(0)=x^i(S), \qquad i=0,1,\dots,n-1$$

This method is based on Mawhin's theory and some technical inequalities.

In the study of Wang [6], by using a fixed-point theorem for operators on a cone, some results of first-order periodic Sturm-Liouville problem of impulsive dynamic equations with time scales are established. Examples are provided to show the results in this paper.

The article by Malathi et al. [7] discusses the shooting algorithm and the Floquet theory. In the shooting algorithm for an eigenvalue of a Sturm-Liouville problem, the equation is solved as an initial value problem on the interval $[a,b]$. Floquet theory is used to show a non-trivial solution of boundary value problems, and the application of shooting techniques approximates the eigenvalues. The numerical results of Sturm-Liouville eigenvalue problems with periodic boundary conditions are given.

In the studies [8-12], boundary value transmission problems are discussed for the two-linked regular Sturm-Liouville equations.

This study investigates some properties of eigenvalues and characteristic function of the antiperiodic Sturm-Liouville value transition problem together with boundary-transition conditions on $[-1,0)\cup(0,1]$.

2. Eigenvalues and Corresponding Eigenfunctions of The Problem

In this study, in the Hilbert space $L_2(-1,0)\oplus L_2(0,1)$ we shall examine some spectral properties of a boundary-value-transition problem consisting of a two-interval Sturm-Liouville equation

$$Lu:=-u''(x)+q(x)u(x)=\lambda u(x), \qquad x\in[-1,0)\cup(0,1] \tag{2.1}$$

together with antiperiodic boundary conditions, given by

$$u(-1) = -u(1), \qquad u'(-1) = -u'(1) \tag{2.2}$$

and transition conditions at the interior point $x = 0$, given by

$$u(+0) = Ku(-0), \quad u'(+0) = \frac{1}{K}u'(-0) \tag{2.3}$$

where$q(x)$ is a continuous function in the intervals $[-1,0)$ and $(0,1]$ with finite limit values $q(\pm 0)$, $K \neq 0$ is the real number, and λ is the complex eigenvalue parameter.

Theorem 2.1. All eigenvalues of the boundary-value-transition problem $(2.1) - (2.3)$ are real.

Proof. Let (λ, u) be an eigenvalue-eigenfunction pair, $\bar{u}$ be the complex conjugate of u, $\bar{\lambda}$ be the complex conjugate of λ. Since K is a real number and $q(x)$ is a real-valued function, we get

$$-\bar{u}''(x) + q(x)\bar{u}(x) = \bar{\lambda}\bar{u}(x) \tag{2.4}$$

$$\bar{u}(-1) = -\bar{u}(1), \qquad \bar{u}'(-1) = -\bar{u}'(1)$$

$$\bar{u}(+0) = K\bar{u}(-0), \qquad \bar{u}'(+0) = \frac{1}{K}\bar{u}'(-0)$$

Now, multiplying the equation (2.1) by $\bar{u}$ and the equation (2.4) by u we have

$$-u''\bar{u} + q(x)u\bar{u} = \lambda u\bar{u}$$

and

$$-u\bar{u}'' + q(x)u\bar{u} = \bar{\lambda}u\bar{u}$$

respectively. Subtracting these two equalities gives

$$u\bar{u}'' - u''\bar{u} = (\lambda - \bar{\lambda})u\bar{u}$$

Taking in view the identity $u\bar{u}'' - u''\bar{u} = (u\bar{u}' - u'\bar{u})'$ we have

$$(u\bar{u}' - u'\bar{u})' = (\lambda - \bar{\lambda})u\bar{u}$$

Now integrating over $[-1, 0)$ we obtain

$$\int_{-1}^{-0} (u\bar{u}' - u'\bar{u})'\, dx = \int_{-1}^{-0} (\lambda - \bar{\lambda})u\bar{u}dx$$

Hence,

$$u(-0)\bar{u}'(-0) - u'(-0)\bar{u}(-0) - u(-1)\bar{u}'(-1) + u'(-1)\bar{u}(-1) = \int_{-1}^{-0} (\lambda - \bar{\lambda})u\bar{u}dx$$

Similarly, we can show that

$$u(1)\bar{u}'(1) - u'(1)\bar{u}(1) - u(+0)\bar{u}'(+0) + u'(+0)\bar{u}(+0) = \int_{+0}^{1} (\lambda - \bar{\lambda})u\bar{u}dx$$

Since $u(x)$ satisfies the transition conditions (2.3), we have

$$u(+0) = Ku(-0)$$

$$u'(+0) = \frac{1}{K}u'(-0)$$

Similarly, we get

$$\bar{u}(+0) = K\bar{u}(-0)$$

$$\bar{u}'(+0) = \frac{1}{K}\bar{u}'(-0)$$

and

$$u(1)\bar{u}'(1) - u'(1)\bar{u}(1) - Ku(-0)\frac{1}{K}\bar{u}'(-0) + \frac{1}{K}u'(-0)K\bar{u}(-0) = \int_{+0}^{1}(\lambda - \bar{\lambda})u\bar{u}dx$$

Thus, we get that

$$0 = (\lambda - \bar{\lambda})\left[\int_{-1}^{-0} u\bar{u}dx + \int_{+0}^{1} u\bar{u}dx\right] = (\lambda - \bar{\lambda})\|u\|_H^2$$

Since the eigenfunction u is nonzero, the last equality gives $\lambda = \bar{\lambda}$. Consequently, λ is real, which completes the proof.

Theorem 2.2. Let (λ_m, u_m) and (λ_n, u_n) be two eigenpairs of the boundary-value-transition problem (2.1) − (2.3). If $\lambda_m \neq \lambda_n$ then the eigenfunctions u_m and u_n are orthogonal in the Hilbert space $H \coloneqq L_2(-1,0) \oplus L_2(0,1)$. That is,

$$\int_{-1}^{-0} u_m(x)u_n(x)dx + \int_{+0}^{1} u_m(x)u_n(x)dx = 0$$

Proof. Since u_m and u_n are eigenfunctions corresponding to the eigenvalues λ_m and λ_n, respectively, we get the following equalities,

$$-u_m'' + q(x)u_m = \lambda_m u_m$$

$$-u_n'' + q(x)u_n = \lambda_n u_n$$

Multiplying the first equality by u_n and the second equality by u_m and taking the difference yields

$$u_m u_n'' - u_m'' u_n = (\lambda_m - \lambda_n)u_m u_n$$

Applying the well-known Lagrange's formulae commonly known as Green's identity we get

$$u_m(-0)u_n'(-0) - u_m'(-0)u_n(-0) - u_m(-1)u_n'(-1) + u_m'(-1)u_n = \int_{-1}^{-0} (\lambda_m - \lambda_n)u_m u_n dx \quad (2.5)$$

By the boundary conditions (2.2) we have

$$u_m(-1) = -u_m(1),\ u_m'(-1) = -u_m'(1)$$

and

$$u_n(-1) = -u_n(1),\ u_n'(-1) = -u_n'(1)$$

Substituting these into the equation (2.5), we get

$$u_m(-0)u_n'(-0) - u_m'(-0)u_n(-0) - u_m(1)u_n'(1) + u_m'(1)u_n(1) = \int_{-1}^{-0} (\lambda_m - \lambda_n)u_m u_n dx$$

Similarly, we can show that

$$u_m(1)u_n'(1) - u_m'(1)u_n(1) - u_m(+0)u_n'(+0) + u_m'(+0)u_n(+0) = \int_{+0}^{1} (\lambda_m - \lambda_n) u_m u_n dx \quad (2.6)$$

Since u_m and u_n satisfy the transition conditions (2.3), we get

$$u_m(+0) = K u_m(-0), u_m'(+0) = \tfrac{1}{K} u_m'(-0)$$

and

$$u_n(+0) = K u_n(-0),\ u_n'(+0) = \tfrac{1}{K} u_n'(-0)$$

Substituting these into the equation (2.6), we obtain

$$u_m(1)u_n'(1) - u_m'(1)u_n(1) - K u_m(-0)\frac{1}{K}u_n'(-0) + \frac{1}{K}u_m'(-0)K u_n(-0) = \int_{+0}^{1} (\lambda_m - \lambda_n) u_m u_n dx$$

from which it follows that

$$0 = (\lambda_m - \lambda_n)\left[\int_{-1}^{-0} u_m u_n dx + \int_{+0}^{1} u_m u_n dx\right]$$

Since $\lambda_m \neq \lambda_n$, we get that

$$\int_{-1}^{-0} u_m u_n dx + \int_{+0}^{1} u_m u_n dx = 0,$$

That is $\langle u_m , u_n \rangle = 0$, which completes the proof.

2.1. Construction of the Hilbert Space and Differential Operator for Given Boundary-Value-Transition Problem

Let us define the inner product of $\varphi(x),\ \omega(x) \in H$ by the equality

$$\langle \varphi, \omega \rangle = \int_{-1}^{-0} \varphi(x)\overline{\omega(x)} dx + \int_{+0}^{1} \varphi(x)\overline{\omega(x)} dx$$

where

$$H = \{\varphi(x) \mid \varphi(x) \in L_2(-1,0) \oplus L_2(0,1)\}$$

We can show that the inner-product axioms are obviously satisfied.

Lemma 2.1.1. The inner product space $(H, \langle \cdot , \cdot \rangle)$ is a Hilbert space.

Proof. It is sufficient to show that every Cauchy sequence in the space H is convergent to some limit point in H. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in H. Then for any $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $\|\varphi_n - \varphi_m\| < \varepsilon^2$ whenever $n, m \geq n_0(\varepsilon)$. Since

$$\begin{aligned}\|\varphi_n - \varphi_m\|_H^2 &= \langle \varphi_n - \varphi_m , \varphi_n - \varphi_m \rangle_H \\ &= \|\varphi_n - \varphi_m\|_{L_2(-1,0)}^2 + \|\varphi_n - \varphi_m\|_{L_2(0,1)}^2 < \varepsilon^2\end{aligned}$$

we have

$$\|\varphi_n - \varphi_m\|_{L_2(-1,0)}^2 < \varepsilon^2,\ \|\varphi_n - \varphi_m\|_{L_2(0,1)}^2 < \varepsilon^2$$

Consequently, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in both Hilbert spaces $L_2(-1,0)$ and $L_2(0,1)$. Since the spaces $L_2(-1,0)$ and $L_2(0,1)$ are complete, any Cauchy sequence taken from these spaces are convergent sequences. So, there are $\varphi_l \in L_2(-1,0)$ and $\varphi_r \in L_2(0,1)$ such that

$$\|\varphi_n - \varphi_l\|^2_{L_2(-1,0)} \longrightarrow 0 \ (n \to \infty), \qquad \|\varphi_n - \varphi_r\|^2_{L_2(0,1)} \longrightarrow 0 \ (n \to \infty)$$

Consequently

$$\|\varphi_n - \tilde{\varphi}\|^2_H = \|\varphi_n - \varphi_l\|^2_{L_2(-1,0)} + \|\varphi_n - \varphi_r\|^2_{L_2(0,1)} \longrightarrow 0 \ (n \to \infty)$$

where $\tilde{\varphi} := \begin{cases} \varphi_l, & x \in [-1,0) \\ \varphi_r, & x \in (0,1] \end{cases} \in H$. Therefore, the completeness of the inner-product space H is proved.

Now we will define a linear operator $A : H \to H$ associated with the boundary value transition problem $(2.1) - (2.3)$ as follows:

Let the domain $D(A)$ be define as follows:

$$D(A) = \{ \varphi \in H \mid \text{The functions } \varphi_1(x), \varphi_2(x), \ \varphi_1'(x) \text{ and } \varphi_2'(x) \text{ are absolute continuous in the intervals } [-1,0] \text{ and } [0,1], \text{ there are finite limit values } \varphi(\pm 0) \text{ and } \varphi'(\pm 0), \text{ and } -\varphi_1'' + q(x)\varphi_1 \in L_1(-1,0), \ -\varphi_2'' + q(x)\varphi_2 \in L_2(0,1), \ \varphi_1(-1) = -\varphi_2(1), \ \varphi_1{}'(-1) = -\varphi_2{}'(1), \ \varphi_1(0) = K\varphi_2(0), \ \varphi_1{}'(0) = \frac{1}{K}\varphi_2{}'(0) \} \quad (2.7)$$

and the operator $A: D(A) \to H$ be defined by

$$A\varphi := -\varphi'' + q(x)\varphi \quad (2.8)$$

where

$$\varphi_1(x) = \begin{cases} \varphi(x), & x \in [-1,0) \\ \varphi(-0), & x = 0 \end{cases} \quad \text{and} \quad \varphi_2(x) = \begin{cases} \varphi(x), & x \in (0,1] \\ \varphi(+0), & x = 0 \end{cases}$$

The eigenvalues and the eigenfunctions of the boundary value transition problem are defined as the eigenvalues and eigenfunctions of the operator A, respectively.

The following lemma is easy to prove.

Lemma 2.1.2. The operator A is the linear operator.

Theorem 2.1.2. The linear operator A defined by $(2.7) - (2.8)$ is symmetric in the Hilbert space

$H = L_2(-1,0) \oplus L_2(0,1)$.

Proof. Let $\varphi, \omega \in D(A) \subset H$. By the definition of A we have

$$\langle A\varphi, \omega \rangle_H = -\int_{-1}^{-0} \varphi''(x)\overline{\omega(x)}\,dx + \int_{-1}^{-0} q(x)\varphi(x)\overline{\omega(x)}\,dx - \int_{+0}^{1} \varphi''(x)\overline{\omega(x)}\,dx + \int_{+0}^{1} q(x)\varphi(x)\overline{\omega(x)}\,dx \quad (2.9)$$

Integrating by parts twice, we obtain

$$\int_{-1}^{-0} \varphi''(x)\overline{\omega(x)}\,dx = \varphi'(x)\overline{\omega(x)}\Big|_{-1}^{-0} - \varphi(x)\overline{\omega'(x)}\Big|_{-1}^{-0} + \int_{-1}^{-0} \varphi(x)\,\overline{\omega''(x)}dx,$$

and therefore,

$$\int_{-1}^{-0} \left(-\varphi''(x) + q(x)\varphi(x)\right)\overline{\omega(x)}\,dx = \int_{-1}^{-0} \varphi(x)\left(\overline{-\omega''(x) + q(x)\omega(x)}\right)dx + W(\varphi,\overline{\omega};-0) - W(\varphi,\overline{\omega};-1) \quad (2.10)$$

By similar technique as above, one can show that

$$\int_{+0}^{1}\left(-\varphi''(x)+q(x)\varphi(x)\right)\overline{\omega(x)}\,dx=\int_{+0}^{1}\varphi(x)\left(\overline{-\omega''(x)+q(x)\omega(x)}\right)dx+W(\varphi,\overline{\omega};1)-W(\varphi,\overline{\omega};+0) \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we obtain

$$\begin{aligned}\langle A\varphi,\omega\rangle_H=&\int_{-1}^{-0}\varphi(x)\left(\overline{-\omega''(x)+q(x)\omega(x)}\right)dx+\int_{+0}^{1}\varphi(x)\left(\overline{-\omega''(x)+q(x)\omega(x)}\right)dx\\&+W(\varphi,\overline{\omega};-0)-W(\varphi,\overline{\omega};-1)+W(\varphi,\overline{\omega};1)-W(\varphi,\overline{\omega};+0)\end{aligned} \tag{2.12}$$

Hence, (2.12) takes the form

$$\langle A\varphi,\omega\rangle_H-\langle \varphi,A\omega\rangle_H=W(\varphi,\overline{\omega};-0)-W(\varphi,\overline{\omega};-1)+W(\varphi,\overline{\omega};1)-W(\varphi,\overline{\omega};+0)$$

Since $\varphi,\omega\in D(A)$, this yield

$$\begin{aligned}\varphi(-0)\overline{\omega'(-0)}-\varphi'(-0)\overline{\omega(-0)}-\varphi(-1)\overline{\omega'(-1)}+\varphi'(-1)\overline{\omega(-1)}+\varphi(1)\overline{\omega'(1)}-\varphi'(1)\overline{\omega(1)}\\-\varphi(+0)\overline{\omega'(+0)}+\varphi'(+0)\overline{\omega(+0)}=0\end{aligned}$$

Then the equality

$$\langle A\varphi,\omega\rangle_H=\langle \varphi,A\omega\rangle_H$$

is valid for all $\varphi,\omega\in D(A)$. This completes the proof of Theorem.

2.2. Some Auxiliary Initial Value Problems and Solutions

In this section, we will use solutions of some auxiliary initial value problems, given only on the sub-intervals $[-1,0]$ and $[0,1]$, which are closely related to the boundary value transition problem $(2.1)-(2.3)$. The initial value problem

$$-u''(x)+q(x)u(x)=\lambda u(x),\qquad x\in[-1,0]$$
$$u(-1)=1$$
$$u'(-1)=0$$

has a unique solution $u=\phi_1(x,\lambda)$ for each $\lambda\in\mathbb{C}$ for the theory of ordinary differential equations and this solution is analytical in the whole complex plane concerning the variable λ for each $x\in[-1,0]$. (See, [13])

Let $\phi_2(x,\lambda)$ be the solution of the initial-value problem given by

$$-u''(x)+q(x)u(x)=\lambda u(x),\qquad x\in[0,1]$$
$$u(1)=1$$
$$u'(1)=0$$

This solution is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[0,1]$. (See, [13])

Similarly, for each $\lambda\in\mathbb{C}$, the initial-value problem

$$-u''(x)+q(x)u(x)=\lambda u(x),\qquad x\in[0,1]$$
$$u(1)=0$$
$$u'(1)=1$$

has a unique solution $u=\chi_2(x,\lambda)$ and the initial-value problem

$$-u''(x)+q(x)u(x)=\lambda u(x),\qquad x\in[0,1]$$

$$u(-1) = 0$$
$$u'(-1) = 1$$

has a unique solution $u = \chi_1(x,\lambda)$. These solutions are also analytical in the whole complex plane concerning the variable λ for each fixed x, that is, $\chi_1(x,\lambda)$ and $\chi_2(x,\lambda)$ are entire functions of $\lambda \in \mathbb{C}$ for each fixed x(see, [13]).

2.3. The Characteristic Function

Theorem 2.3.1. The eigenvalues of the boundary-value-transition problem $(2.1) - (2.3)$ are coincide with the zeros of the characteristic function

$$\Delta(\lambda) = \left[\phi_2(+0,\lambda) - K\phi_1(-0,\lambda)\right]\left[\chi_2'(+0,\lambda) - \frac{1}{K}\chi_1'(-0,\lambda)\right] - \left[\chi_2(+0,\lambda) - K\chi_1(-0,\lambda)\right]\left[\phi_2'(+0,\lambda) - \frac{1}{K}\phi_1'(-0,\lambda)\right] \quad (2.13)$$

Proof. Since for each $\lambda \in \mathbb{C}$ the Wronskian $W(\phi_1,\chi_1;x)$ is independent on $x \in [-1,0]$ and $W(\phi_1,\chi_1;-1) = 1 \neq 0$, the functions $\phi_1(x,\lambda)$, $\chi_1(x,\lambda)$ are linearly independent solutions of the equation (2.1) in the interval $[-1,0]$. Therefore, the general solution of the equation (2.1) on the left interval $[-1,0]$ can be expressed in the form

$$y = c_1\phi_1(x,\lambda) + c_2\chi_1(x,\lambda)$$

Similarly, the general solution of the same differential equation on the right interval $[0,1]$ can be expressed in the form

$$y = c_3\phi_2(x,\lambda) + c_4\chi_2(x,\lambda)$$

Thus, the general solution of the differential equation (2.1) on the interval $[-1,0) \cup (0,1]$ can be written in the form

$$y = \begin{cases} c_1\phi_1(x,\lambda) + c_2\chi_1(x,\lambda), & x \in [-1,0) \\ c_3\phi_2(x,\lambda) + c_4\chi_2(x,\lambda), & x \in (0,1] \end{cases}$$

Applying the antiperiodic boundary conditions (2.2) we obtain

$$c_1\phi_1(-1,\lambda) + c_2\chi_1(-1,\lambda) = c_3\phi_2(1,\lambda) + c_4\chi_2(1,\lambda)$$
$$c_1\phi_1'(-1,\lambda) + c_2\chi_1'(-1,\lambda) = c_3\phi_2'(1,\lambda) + c_4\chi_2'(1,\lambda)(2.3.2) \quad (2.14)$$

By the definition of the solutions ϕ_1, χ_1, ϕ_2 and χ_2 we get

$$\phi_1(-1,\lambda) = 1,\ \phi_1'(-1,\lambda) = 0,\ \phi_2(1,\lambda) = 1,\ \phi_2'(1,\lambda) = 0$$
$$\chi_1(-1,\lambda) = 0,\ \chi_1'(-1,\lambda) = 1,\ \ \chi_2(1,\lambda) = 0,\ \chi_2'(1,\lambda) = 1$$

Substituting these equalities into (2.14), we obtain that $c_1 = c_3 = A$ and $c_2 = c_4 = B$. Then, the general solution can be written in the form

$$y = A\phi(x,\lambda) + B\chi(x,\lambda)$$

Substituting this into transition conditions (2.3), we obtain the following linear system of equations concerning the variables A and B, given by

$$\big(\phi_2(+0,\lambda) - K\phi_1(-0,\lambda)\big)A + \big(\chi_2(+0,\lambda) - K\chi_1(-0,\lambda)\big)B = 0$$
$$\left(\phi_2'(+0,\lambda) - \frac{1}{K}\phi_1'(-0,\lambda)\right)A + \left(\chi_2'(+0,\lambda) - \frac{1}{K}\chi_1'(-0,\lambda)\right)B = 0$$

This homogeneous system of linear equations has a nontrivial solution $(A,B) \neq (0,0)$ if the determinant of this system is equal to zero, i.e.

$$\begin{vmatrix} \phi_2(+0,\lambda) - K\phi_1(-0,\lambda) & \chi_2(+0,\lambda) - K\chi_1(-0,\lambda) \\ \phi_2'(+0,\lambda) - \frac{1}{K}\phi_1'(-0,\lambda) & \chi_2'(+0,\lambda) - \frac{1}{K}\chi_1'(-0,\lambda) \end{vmatrix} = 0$$

Hence, $\Delta(\lambda)=0$. This completes the proof.

Theorem 2.3.2. If $K^2\neq 1$, then for the characteristic function $\Delta(\lambda)$ the following asymptotic formulas hold

$$\Delta(\lambda)=(1-K)\left(1-\frac{1}{K}\right)\cos^2\sqrt{\lambda}+(1+K)\left(1+\frac{1}{K}\right)\sin^2\sqrt{\lambda}+O\left(\frac{1}{\sqrt{\lambda}}e^{2t}\right)$$

as $|\lambda|\to\infty$, where $t=Im\sqrt{\lambda}$.

Proof. By applying well-known properties of Volterra integral equations, we can derive the following asymptotic formulas

$$\phi_1(x,\lambda)=\cos\left(\sqrt{\lambda}(x+1)\right)+O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x+1|}\right)$$

$$\phi_1{}'(x,\lambda)=-\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x+1)\right)+O\left(e^{|t||x+1|}\right)$$

$$\phi_2(x,\lambda)=\cos\left(\sqrt{\lambda}(x-1)\right)+O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x-1|}\right)$$

$$\phi_2{}'(x,\lambda)=-\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x-1)\right)+O\left(e^{|t||x-1|}\right)$$

$$\chi_1(x,\lambda)=\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(x+1)\right)+O\left(\frac{1}{\lambda}e^{|t||x+1|}\right)$$

$$\chi_1'(x,\lambda)=\cos\left(\sqrt{\lambda}(x+1)\right)+O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x+1|}\right)$$

$$\chi_2(x,\lambda)=\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(x-1)\right)+O\left(\frac{1}{\lambda}e^{|t||x-1|}\right)$$

$$\chi_2'(x,\lambda)=\cos\left(\sqrt{\lambda}(x-1)\right)+O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x-1|}\right)$$

as $|\lambda|\longrightarrow\infty$, where O denote the Landau symbol. Substituting these asymptotic formulas into (2.13) we arrive at

$$\begin{aligned}\Delta(\lambda)&=\left[(1-K)\cos\sqrt{\lambda}+O\left(\frac{1}{\sqrt{\lambda}}e^{|t|}\right)\right]\left[\left(1-\frac{1}{K}\right)\cos\sqrt{\lambda}+O\left(\frac{1}{\sqrt{\lambda}}e^{|t|}\right)\right]\\&\quad-\left[(-1-K)\tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}+O\left(\tfrac{1}{\lambda}e^{|t|}\right)\right]\left[\left(1+\tfrac{1}{K}\right)\sqrt{\lambda}\sin\sqrt{\lambda}+O\left(e^{|t|}\right)\right]\\&=(1-K)\left(1-\tfrac{1}{K}\right)\cos^2\sqrt{\lambda}+(1+K)\left(1+\tfrac{1}{K}\right)\sin^2\sqrt{\lambda}+O\left(\tfrac{1}{\sqrt{\lambda}}e^{2t}\right)\end{aligned}$$

This completes the proof.

Theorem 2.3.3. If $K^2\neq 1$, then the boundary-value-transition problem $(2.1)-(2.3)$ has a countable set of eigenvalues without finite accumulation point.

Proof. Denote by $\Delta_1(\lambda)$ the leading term of $\Delta(\lambda)$, that is

$$\Delta_1(\lambda)=(1-K)\left(1-\frac{1}{K}\right)\cos^2\sqrt{\lambda}+(1+K)\left(1+\frac{1}{K}\right)\sin^2\sqrt{\lambda}$$

This function has a countable set of zeros $\lambda_n',\ n=1,2,\cdots$ without a finite accumulation point. Applying now the well-known Rouche's theorem (see, for example, [13]) to the appropriate circles we conclude that the characteristic function $\Delta(\lambda)$ has a countable set zeros $\lambda_n,\ n=1,2,\cdots$ which satisfies the asymptotic equality $\lambda_n=\lambda_n'+O\left(\frac{1}{n}\right)$. The proof is complete.

3. Conclusions

In this study, antiperiodic Sturm-Liouville problems, including transition conditions, were investigated for the first time in the literature. A Hilbert space suitable for the problem is established. Then, an operator is defined on this Hilbert space that is the same as the problem's eigenvalues. It has been proved that the eigenvalues are real and the eigenfunctions are orthogonal. The problem's characteristic function is defined, and the asymptotic formula is obtained for the characteristic function. Finally, the asymptotic formula for eigenvalues was found using the asymptotic formula of the characteristic function.

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Characterization of the Evolute Offset of Ruled Surfaces with B-Darboux Frame

Gul Ugur Kaymanli[1]

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Abstract — In this paper, we first introduce the evolute offset of non-cylindirical ruled surfaces with B-Darboux frame in Euclidean 3-space. Then some geometric properties of this evolute offset of non-cylindirical ruled surface are studied. That is, we examine the striction curve, distribution parameter, orthogonal trajectory of evolute offset of ruled surfaces in terms of B-Darboux frame of given ruled surfaces, and we study the developability of ruled surface generated by B-Darboux vectors in three dimensional Euclidean space.

Keywords — *B-Darboux Frame, Curvatures, Evolute Offset, Ruled Surface*

1. Introduction

In geometry, a ruled surface in three dimensional space is a surface which is stated as the set of points swept by a moving straight line. By using the directions, many offset of this surface such as parallel, Bertrand, Mannheim, involute-evolute, and Smarandache have been defined. In kinematics, mechanism, geophysics, Computer-Aided Geometric Design, and geometric modelling, both offsets and ruled surfaces are extensively worked on. In general, offsets of surfaces are usually more complicated than their origin surfaces. Because of this, analysing offsets surfaces or curves by the help of the properties of the base surface or curve is important. Therefore, many researchers have been working on this subject.

After [1] Pottmann et al. in 1996 studied rational ruled surfaces and their offsets, Kasap et al. in [2], Akyigit et. al. in [3] had a research on the involute-evolute offsets of ruled surface in 2009 and involute-evolute curves in Galilean Space in 2010, respectively. The involute evolute partner of both d-curves in Euclidean 3-space and pseudo null curves in Minkowski 3-space found in [4]- [5]. While Yoon worked on, in 2016, the evolute offsets of ruled surfaces in three dimensional Lorentzian space [6], recently, Senturk and Yuce in [7]- [8] studied evolute offsets of ruled surfaces using Darboux frame in 3 dimensional Euclidean space.

In this paper, after giving necessary definitions and theorems in preliminary section, the evolute offset of ruled surfaces with B-Darboux frame is defined in the following chapter. Some geometric properties of this evolute offset of non-cylindirical ruled surface are studied. That is, we examine the striction curve, distribution parameter and the developability of evolute offset of given ruled surfaces in terms of B-Darboux frame in Euclidean 3- space.

[1]gulugurk@karatekin.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Science, Cankiri Karatekin University, Cankiri, Turkey

2. Preliminary

Let $M(u(s),v(s))$ be an oriented surface and $\alpha(s)$ be a unit speed curve on M in E^3. If $\mathbf{t}$ is the unit tangent vector of α, $\mathbf{U}$ is the unit normal vector of M and $\mathbf{V}=\mathbf{U}\wedge\mathbf{t}$, then $\{\mathbf{t},\mathbf{V},\mathbf{U}\}$ is called the Darboux frame of $\alpha(s)$. Therefore, the Darboux formulas are written as

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{V}\\ \mathbf{U}\end{bmatrix}=\begin{bmatrix}0 & \kappa_g & \kappa_n\\ -\kappa_g & 0 & \tau_g\\ -\kappa_n & -\tau_g & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{V}\\ \mathbf{U}\end{bmatrix} \tag{1}$$

where

$$\kappa_g=\left\langle \mathbf{t}',\mathbf{V}\right\rangle,\kappa_n=\left\langle \mathbf{t}',\mathbf{U}\right\rangle,\tau_g=\left\langle \mathbf{V}',\mathbf{U}\right\rangle \tag{2}$$

are called the normal curvature, geodesic curvature and the geodesic torsion of α, respectively [9]. As an alternative to the Darboux frame, B-Darboux frame is defined as a new adapted frame on the surface [10], [11]. Its mathematical properties derive from the observation that, while the tangent vector $\mathbf{t}$ for a curve on a surface is unique, we can pick any practical arbitrary basis vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ for the remainder of the proposed frame in the normal plane of the surface.

Theorem 2.1. [12] Assume that $r(s)=M(u(s),v(s))$ is a unit speed curve on a surface M in E^3. Let us consider the Darboux frame $\{\mathbf{t},\mathbf{V},\mathbf{U}\}$ along this curve on the surface. Then, the variation equation of the B-Darboux frame $\{\mathbf{t},\mathbf{B}_1,\mathbf{B}_2\}$ on the surface given as

$$\frac{d}{dt}\begin{bmatrix}\mathbf{t}\\ \mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}=\begin{bmatrix}0 & n_1 & n_2\\ -n_1 & 0 & 0\\ -n_2 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}, \tag{3}$$

where the B-Darboux curvatures are obtained as

$$\begin{aligned} n_1 &= \kappa_g\sin\phi+\kappa_n\cos\phi\\ n_2 &= \kappa_n\sin\phi-\kappa_g\cos\phi, \end{aligned} \tag{4}$$

where the angle ϕ between the vectors $\mathbf{U}$ and $\mathbf{B}_1$ are obtained by

$$\phi-\phi_o=\int\tau_g dt \tag{5}$$

here ϕ_o is an arbitrary integration constant.

A straightforward computation shows that the following relations among the B-Darboux curvatures, the normal curvature and the geodesic curvature holds:

$$\kappa_g^2+\kappa_n^2=n_1^2+n_2^2. \tag{6}$$

If $\alpha(s)$ be a curve and $X(s)$ be a generator vector, then the parametrization of ruled surface $\varphi(s,v)$ is

$$\varphi(s,v)=\alpha(s)+vX(s). \tag{7}$$

The striction curve on the ruled surface consists of the foot of the common perpendicular line of the consecutive rulings on the main ruling. It is written as

$$c(s)=\alpha(s)-\frac{\langle\alpha_s,X_s\rangle}{\langle X_s,X_s\rangle}X(s). \tag{8}$$

Theorem 2.2. The ruled surface is developable if consecutive rulings intersect [13].

The distribution parameter of the ruled surface is determined as (see [14], [15])

$$P_X=\frac{\det(\alpha_s,X,X_s)}{\langle X_s,X_s\rangle}. \tag{9}$$

Theorem 2.3. The ruled surface is named as developable if and only if $P_X=0$ [13].

The ruled surface is called as a non-cylindirical ruled surface if $\langle X_s, X_s \rangle \neq 0$.

A unit direction vector of straight line X is span by the vectors $\{\mathbf{t}, \mathbf{B_1}\}$. Therefore, it is given as

$$X = \cos\phi \mathbf{t} + \sin\phi \mathbf{B_1}. \tag{10}$$

where ϕ is the angle between the vectors $\mathbf{t}$ and X [15]. In [12], the distrubition parameter and the striction curve of ruled surface with B-Darboux frame are determined as

$$P_X = \frac{n_2 \sin\phi \cos\phi}{(\phi' + n_1)^2 + (n_2)^2 (\cos\phi)^2}, \tag{11}$$

and

$$c(s) = \alpha(s) - \frac{(\phi' + n_1) \sin\phi}{(\phi' + n_1)^2 + (n_2)^2 (\cos\phi)^2} X(s). \tag{12}$$

respectively.

The ruled surface with B-Darboux frame is a developable provided that

$$n_2 \cos\phi \sin\phi = 0. \tag{13}$$

In that case, we have the followings:

i) If $n_2 = 0$ then $\frac{\kappa_g}{\kappa_n} = \tan\phi$ which is trivial. Specially, if $\phi = 0$, then α is geodesic curve and never asymptotic line.

ii) If $\cos\phi = 0$ then base curve is orthogonal trajectory.

iii) If $\sin\phi = 0$ then main ruled surface is the tangent developable.

An orthogonal trajectory of a family of curves is a curve which intersect each curve of the family orthogonally. For the ruled surface $\varphi(s, v)$, the orthogonal trajectory is

$$\cos\phi ds = -dv. \tag{14}$$

3. Evolute Offsets of Ruled Surface with B-Darboux Frame

Definition 3.1. Two ruled surfaces $\varphi(s, v)$ with B-Darboux frame $\{\mathbf{t}, \mathbf{B}_1, \mathbf{B}_2\}$ and $\varphi^*(s, v)$ with B-Darboux frame $\{\mathbf{t}^*, \mathbf{B}_1^*, \mathbf{B}_2^*\}$ are given

$$\varphi(s, v) = \alpha(s) + vX(s) \tag{15}$$

$$\varphi^*(s, v) = \alpha^*(s) + vX^*(s). \tag{16}$$

$\varphi(s, v)$ is said to be involute offsets of $\varphi^*(s, v)$ or $\varphi^*(s, v)$ is said to be evolute offset of $\varphi(s, v)$ if frame vectors $\mathbf{t}$ of $\varphi(s, v)$ and $\mathbf{B}_1^*$ of $\varphi^*(s, v)$ are linearly dependent at the points of their corresponding rullings.

A unit direction vector of straight line X^* of φ^* is spanned by the vectors $\{\mathbf{t}^*, \mathbf{B}_1^*\}$. Therefore, it is given

$$X^* = \cos\phi^* \mathbf{t}^* + \sin\phi^* \mathbf{B}_1^*, \tag{17}$$

where ϕ^* is the angle between the vectors $\mathbf{t}^*$ and X^*.

If $\mathbf{t}, \mathbf{B}_1$ and $\mathbf{B}_2$ are the B-Darboux vectors of φ, then the B-Darboux vectors of evolute offset φ^* of φ, as in Figure 1, are written as

$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{B}_1^* \\ \mathbf{B}_2^* \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & -\sin\psi \\ 1 & 0 & 0 \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \tag{18}$$

where the ψ is the angle between $\mathbf{B}_2$ and $\mathbf{B}_2^*$.

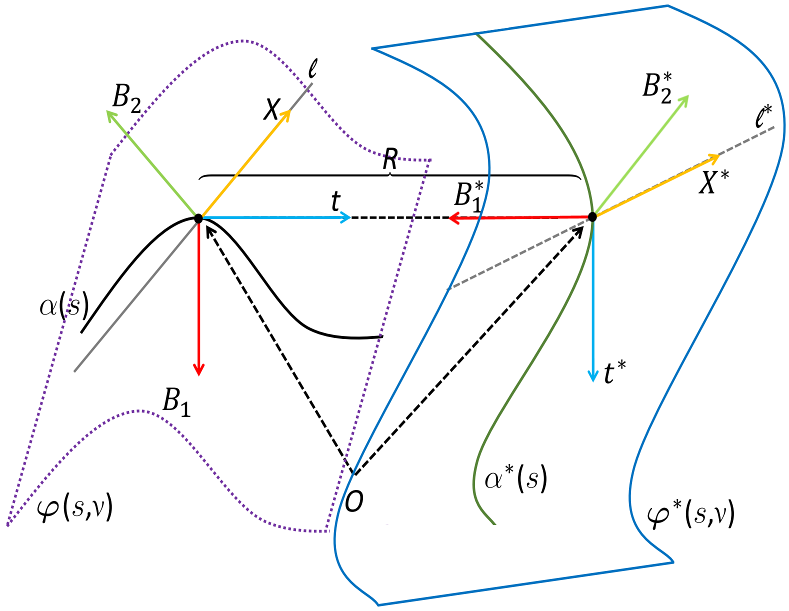

Fig. 1. Relation between φ and φ^*.

It is easy to see that

$$\alpha^*(s) = \alpha(s) + R\mathbf{t} \tag{19}$$

where R is the distance function between the corresponding points of the base curves $\alpha(s)$ and $\alpha^*(s)$, and it is given as $R(s) = c - s$ [4].

$$X^*(s) = \sin\phi^*\mathbf{t} + \cos\phi^*\cos\psi\mathbf{B}_1 - \cos\phi^*\sin\psi\mathbf{B}_2 \tag{20}$$

in terms of B-Darboux frame of φ. The parametrization of the offset ruled surface φ^*, using the equations (19) and (20), can be stated

$$\begin{aligned} \varphi^*(s,v) &= \alpha^*(s) + vX^*(s) \\ &= \alpha(s) + R\mathbf{t} + v[\sin\phi^*\mathbf{t} + \cos\phi^*\cos\psi\mathbf{B}_1 - \cos\phi^*\sin\psi\mathbf{B}_2]. \end{aligned} \tag{21}$$

Taking derivative of the equation (20) with respect to s, we get

$$\begin{aligned} X_s^* &= \cos\phi^*((\phi^*)' - n_1\cos\psi + n_2\sin\psi)\mathbf{t} \\ &\quad +[\sin\phi^*(n_1 - (\phi^*)'\cos\psi) - \psi'\cos\phi^*\sin\psi]\mathbf{B_1} \\ &\quad +[\sin\phi^*(n_2 + (\phi^*)'\sin\psi) - \psi'\cos\phi^*\cos\psi]\mathbf{B_2}. \end{aligned} \tag{22}$$

Striction curve of the ruled surface φ^* in terms of B-Draboux frame of φ is calculated by

$$\begin{aligned} c^*(s) = \alpha(s) + (c-s)\mathbf{t} - \tfrac{c-s}{<X_s^*,X_s^*>} \quad & (\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) \\ & -\psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi))X^*(s). \end{aligned} \tag{23}$$

Distribution parameter of the evolute offset φ^* in terms of B-Draboux frame of ruled surface φ is given as

$$\begin{aligned} P_{X^*} &= \tfrac{c-s}{||X_s^*||^2}(\psi'\cos\phi^*\sin\phi^*(n_1\cos\psi - n_2\sin\psi) - (\phi^*)'(n_1\sin\psi + n_2\cos\psi) \\ &\quad - \cos^2(\phi^*)(n_2\sin\psi - n_1\cos\psi)(n_1\sin\psi + n_2\cos\psi)). \end{aligned} \tag{24}$$

Corollary 3.2. If the tangent vector $\mathbf{t}^*$ and X^* are linearly dependent, then φ^* is developable.

For the distribution parameters of ruled surfaces $\varphi_{\mathbf{t}}$, $\varphi_{\mathbf{B}_1}$, and $\varphi_{\mathbf{B}_2}$, respectively, one can get

$$\begin{aligned} P_{\mathbf{t}} &= \frac{\det(\alpha_s,\mathbf{t},\mathbf{t}_s)}{\langle\mathbf{t}_s,\mathbf{t}_s\rangle} &&= \frac{\det(\mathbf{t},\mathbf{t},\mathbf{t}_s)}{\langle\mathbf{t}_s,\mathbf{t}_s\rangle} &&= 0, \\ P_{\mathbf{B}_1} &= \frac{\det(\alpha_s,\mathbf{B}_1,\mathbf{B}_{1s})}{\langle\mathbf{B}_{1s},\mathbf{B}_{1s}\rangle} &&= \frac{\det(\mathbf{t},\mathbf{B}_1,-n_1\mathbf{t})}{\langle -n_1\mathbf{t},-n_1\mathbf{t}\rangle} &&= 0, \\ P_{\mathbf{B}_2} &= \frac{\det(\alpha_s,\mathbf{B}_2,\mathbf{B}_{2s})}{\langle\mathbf{B}_{2s},\mathbf{B}_{2s}\rangle} &&= \frac{\det(\mathbf{t},\mathbf{B}_2,-n_2\mathbf{t})}{\langle -n_2\mathbf{t},-n_2\mathbf{t}\rangle} &&= 0. \end{aligned} \tag{25}$$

Similarly, the distrubition parameters of evolute offsets of the ruled surface spanned by B-Darboux frame vectors are calculated

$$\begin{aligned} P_{\mathbf{t}^*} &= \frac{c-s}{||\mathbf{t}^*||^2}(n_1\cos\psi - n_2\sin\psi)(n_1\sin\psi + n_2\cos\psi), \\ P_{\mathbf{B}_1^*} &= 0, \\ P_{\mathbf{B}_2^*} &= \frac{c-s}{||\mathbf{B}_2^*||^2}(n_2\sin\psi - n_1\cos\psi)(n_1\sin\psi + n_2\cos\psi). \end{aligned} \tag{26}$$

Corollary 3.3. The ruled surface spanning $\mathbf{t}^*$ is a developable only if $\frac{n_1}{n_2} = \pm\cot\psi$ satisfies.

Corollary 3.4. The ruled surface spanning $\mathbf{B}_1^*$ is always developable.

Corollary 3.5. The ruled surface spanning $\mathbf{B}_2^*$ is a developable, either $\frac{n_1}{n_2} = \tan\psi$ or $\frac{n_1}{n_2} = -\cot\psi$ satisfies.

When we take v is constant, we obtain the curve $\beta^*(s) = \alpha^*(s) + vX^*(s)$ on the evolute offsets of ruled surface whose tangent vector field is calculated

$$\begin{aligned} T^* &= v\cos\phi^*((\phi^*)' - n_1\cos\psi + n_2\sin\psi)\mathbf{t} \\ &\quad +[Rn_1 + v(\sin\phi^*(n_1 - (\phi^*)'\cos\psi) - \psi'\cos\phi^*\sin\psi)]\mathbf{B_1} \\ &\quad +[Rn_2 + v(\sin\phi^*(n_2 + (\phi^*)'\sin\psi) - \psi'\cos\phi^*\cos\psi)]\mathbf{B_2}. \end{aligned} \tag{27}$$

So, it is easy to get $< T^*, X^* >= R\cos\phi^*(n_1\cos\psi - n_2\sin\psi)$.
The orthogonal trajectory of the evolute offsets φ^* is written as

$$R\cos\phi^*(n_1\cos\psi - n_2\sin\psi)ds = -dv. \tag{28}$$

Theorem 3.6. The shortest distance between the rullings of the evolute offset $\varphi^*(s,v) = \alpha^*(s) + vX^*(s)$ along the orthogonal trajectories is given

$$v = \frac{R(\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) - \psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi))}{< X_s^*, X_s^* >}.$$

Proof. Suppose $\alpha^*(s_1)$ and $\alpha^*(s_2)$ are two points in succesive rullings on evolute offset along the orthogonal trajectories. The distance between these two points is given

$$l(v) = \int_{s_1}^{s_2} ||T^*||ds.$$

Calculating $||T^*||$, we get

$$l(v) = \int_{s_1}^{s_2} \begin{aligned} &(R^2(n_1^2 + n_2^2) + 2Rv(\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) \\ &-\psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi)) - v^2 < X_s^*, X_s^* >)^{\frac{1}{2}}ds. \end{aligned} \tag{29}$$

In order to find the distance, we need to minimize the integrant. Therefore, using $l'(v) = 0$, we have

$$R(\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) - \psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi)) - v < X_s^*, X_s^* >= 0$$

which proves the theorem. □

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A Classical and Bayesian Approach for Parameter Estimation in Structural Equation Models

Naci Murat[1], **Mehmet Ali Cengiz**[2]

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Abstract − Structural Equation Models (SEMs) with latent variables provide a general framework for modelling relationships in multivariate data. Although SEMs are most commonly used in studies involving intrinsically latent variables, such as happiness, quality of life, or stress, they also provide a parsimonious framework for covariance structure modelling. For this reason, they have become increasingly used outside of traditional social science applications. Frequentist inferences are based on point estimates and hypothesis tests for the measurement and latent variable parameters. Although most of the literature on SEMs is frequentist, Bayesian approaches have been proposed in the last years. This study aims to provide an easily accessible overview of a Classic and a Bayesian approach to SEMs. Due to the flexibility of the Bayesian approach, it is straightforward to apply the method in a comprehensive class of SEM-type modelling frameworks, allowing nonlinearity, interactions, missing data, mixed categorical, count, and continuous observed variables. The WinBUGS software package, which is freely available, can be used to implement Bayesian SEM analysis. Bayesian model fitting typically relies on MCMC, which involves simulating draws from the joint posterior distribution of the model unknowns (parameters and latent variables) through a computationally intensive procedure. The advantage of MCMC is that there is no need to rely on broad sample assumptions because exact posterior distributions can be estimated for any function of the model unknowns. In small to moderate samples, these exact posteriors can provide a more realistic measure of model uncertainty. Therefore, we use the MCMC method for the Bayesian approach in this study. All approaches given above are applied to the data obtained from Samsun Chamber of Commerce and Industry.

Keywords *− Structural equation models, Bayesian approach, MCMC, Bayesian structural equation models*

1. Introduction

The Structural Equation Model (SEM) is a multivariate statistical modelling technique that reveals the cause-effect relationship between measurable variables and non-measurable (implicit) variables. SEM consists of Observed/Measured Variables and unobservable or unmeasurable variables (Latent Variables) that can function as endogenous and exogenous. Since implicit variables cannot be measured directly, it is essential to define the measurable variables that the researcher wants to examine, and that is thought to represent the implicit variable. Measurable variables that describe implicit variables can be one or more. Therefore, the fact that makes the implicit variable measurable is assessing variables or variables that define the implicit variable.

In SEM, which is based on the causality relationship between implicit variables, each of the implicit variables is a linear function of the set of variables that were observed or measured. The parameters of these

[1] nacimurat@omu.edu.tr (Corresponding Author); [2] macengiz@omu.edu.tr
[1] Department of Industrial Engineering, Faculty of Engineering, Ondokuz Mayıs University, Samsun, Turkey
[2] Department of Statistics, Faculty of Art and Sciences, Ondokuz Mayıs University, Samsun, Turkey

linear functions are obtained using an analysis of covariance. It is tested that using the goodness of fit tests whether the researcher's model is compatible with the data's variance-covariance structure. If the model predictions are accepted at the end of the test, the linear relationship between the implicit variables is assumed to be reasonable. SEM is a hybrid method that combines factor analysis and path analysis. Perhaps the main reason why SEM is so widely used today is that direct or indirect relationships between observable and unobservable variables can be analysed in a single model. SEM can also be considered as multiple regression analysis, and factor analysis performed simultaneously. Therefore, YEM; is also named with definitions such as causal analysis, causal modelling, concurrent structural modelling, covariance structure analysis, path analysis, or confirmatory factor analysis.

SEM is based on three basic analytical developments [1]. These are, respectively, path analysis, latent variable model, and general covariance estimation methods. Wright [2] started his first studies on road analysis and along with other studies, road analysis was developed and basic rules were established [2,3]. Today, SEM is widely used in many fields such as behavioral sciences, educational sciences, economics, marketing, health sciences and social sciences. In examining the structural relations of production practices, delivery time and productivity in Japan and Korea, the effect of time-based production on customer-specific production and adding value to the customer, the development of customer satisfaction index in the Turkish mobile phone industry, evaluation of customer satisfaction in the telephone industry with multi-level structural equation models, brand In the measurement of the value of the supply chain management, the effect of e-supply chain competence on competitive advantage and organizational performance, the effect of supplier development on purchasing performance, in modelling student success, In performing risk analysis in the coal mine construction project, The effects of total quality management practices applied in enterprises on employee performance, the examination of factors affecting individuals' adoption of internet banking, in fraudulent financial reporting determination of auditor responsibility, investigation of the effect of critical control (success) factors in enterprise resource planning (ERP) applications, determinants of capital structure selection. Effects of depression and disease severity on quality of life, Use of SEM in decision tree models, Operation management, symptoms related to ecstatic; the role of fear of blood, injection and injury (KEY), estimation of post-stress traumas of child welfare institutions, processing speed, relationship between intelligence, creativity and school performance, use of incremental goodness of fit indexes in market research studies, structural equation for river water quality data model, role ambiguity, role conflict, relationships between job satisfaction and performance, structural equation technique and interactional stress and coping model.

Why not take advantage of our abilities, which we regularly use and call intuition, common sense, and sixth sense, for scientific purposes? Bayes Theory emerges as an alternative inference in the scientific use of such abilities. The classic inference is to conclude the population we do not have information about with sample data. Statistical operations such as confidence intervals and hypothesis tests are the basis of classical inference. However, "Life; It is the art of drawing sufficient conclusions from insufficient a priori." Thomas Bayes, who has a similar opinion with Samuel Butler, has a different perspective on the audience's inference based on the observed sample data. In general, he formed the chain of logic from causes to effects, from results to causes. In the last 30 years, using an approach different from other common basic approaches in statistical analysis has increased. This approach is Bayesian inference and is based on the well-known theorem put forward by Thomas Bayes in 1763. Thomas Bayes did not even predict that his simple probabilistic theorem would be a statistical method of inference. However, in the last 30 years, this theorem has influenced many statisticians and mathematicians, and Bayes statistics has been accepted as the primary method of statistical inference. Many researchers such as Jeffreys, de Finetti, Savage, and Lindley contributed to the development of Bayesian analysis. In recent years, the technical applicability of Bayes analysis has also rapidly developed using computers and has opened up new application areas. As a result of these developments, Bayesian analysis was expanded with researchers such as Berger [4] and Bernardo and Smith [5].

Today, Bayesian analysis is successfully applied in every discipline. Many applied studies have been revealed. A broad overview of these studies will be given below. It is often difficult to calculate the Bayesian factor. Different calculation methods have been proposed by Kass and Raftery [6]. A simple approach is used in the Bayesian Knowledge Criteria (BIC) YEM. For example, [7] on the LISREL model, Lee and Song [8] on the two-level SEM, Jedidi, Jagpal [9] applied to finite mixed SEM. Also, posterior simulation based approaches are used in Bayes Factor calculations. DiCiccio, Kass [10] detailed many methods from the Laplace approach to importance sampling. Gelman and Meng [11] developed a road sampling approach. Lee [12] gave the Bayesian approach with WinBUGS applications on linear, nonlinear and loss observation structural equation models. Wang and Fan [13] examined the factors affecting myopia disease with Bayesian structural equation models. Bayesian approach for semi-parametric structural equation models is given by Guo, Zhu [14]. In the first part of this study, firstly, literature information about structural equation models, secondly Bayesian approach and finally Bayesian structural equation models are given. In the second section under the title of basic information, basic concepts related to structural equation models and Bayesian approach are presented. Linear and nonlinear structural equation models are presented with a Bayesian approach in the material method section. In the fourth chapter, 2011 data of Samsun Chamber of Commerce and Industry member satisfaction are used. There are 4 factors that are thought to affect the overall satisfaction of the room. These 4 factors are guidance, solution, personnel and representation factors, respectively. Factors affecting the general satisfaction of the chamber were determined by both classical and Bayesian structural equation models.

2. Materials and Methods

2.1. Linear Structural Equation Modelling

Due to the nature of the problems and the design of the questionnaires in behaviour, education, medical and social sciences, data are usually obtained as sequential categorical variables. Examples of these variables; Scales such as attitude scales, likert scales, rating scales can be given. When questioned about some attitudes, the scale was “I do not approve”, “I have no idea”, “I approve”, while questioning about the effect of a drug was “worsened”, “did not change”, “got better” and when questioned about a political event, definitely “I don’t agree”, “I don’t agree”, “I have no idea”, “I agree”, “I absolutely agree”. Consider a five-point scale associated with responses to a political event. A common approach is to treat these integer values as continuous values drawn from the normal distribution. This approach does not cause serious problems if the histograms of the observation value are symmetrical and the frequencies of the central values are high. This will emerge in many cases when the “I have no idea” option is chosen. To claim that the observed variables are multivariate normally distributed, in most cases we have to choose the middle category. For example, “I have no idea” or “no change”. In many cases likert scales may have clutter at both ends. For example, such as “strongly agree (strongly disagree)” or “agree (disagree)”. Therefore, histograms are either skewed or bimodal as opposed to the variables involved. Treating such ordered categorical variables as normal may lead to erroneous results [15,16]. A better approach for evaluating discrete data is to consider these data as latent continuous variables from a specified threshold normal distribution. For a given data set, the ratios of 1, 2, 3, and 4 values are 0.05, 0.05, 0.40, and 0.5, respectively. From the histogram given in Figure 1, it is seen that the dashed data are skewed to the right.

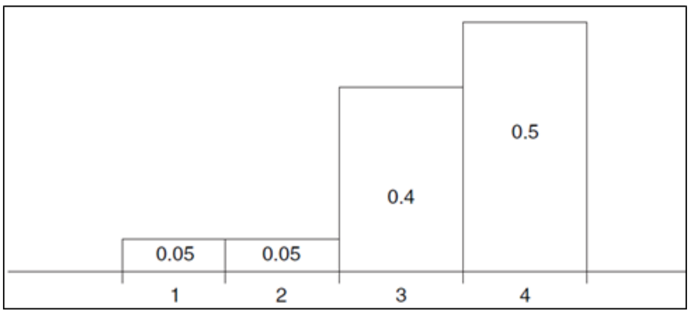

Fig. 1. Historical development of YEM [17]

The threshold approach for the analysis of this discrete data is to consider the discrete categorical data as normal variable y. There are no precise continuous measurements of y but they are related to the observed ordered categorical variable z. This relationship is expressed as follows:

$$z = k \qquad if \quad \propto_{k-1} < y < \propto_k \quad k = 1,2,3,4$$

Here, $-\infty < \propto_1 < \propto_2 < \propto_3 < \infty$ where $\propto_1, \propto_2$ and $\propto_3$ are threshold values. Then, the histogram of sequential categorical observations given in Figure 1 can be in a view with $N[0.1]$ distribution with appropriate threshold values as in Figure 2.

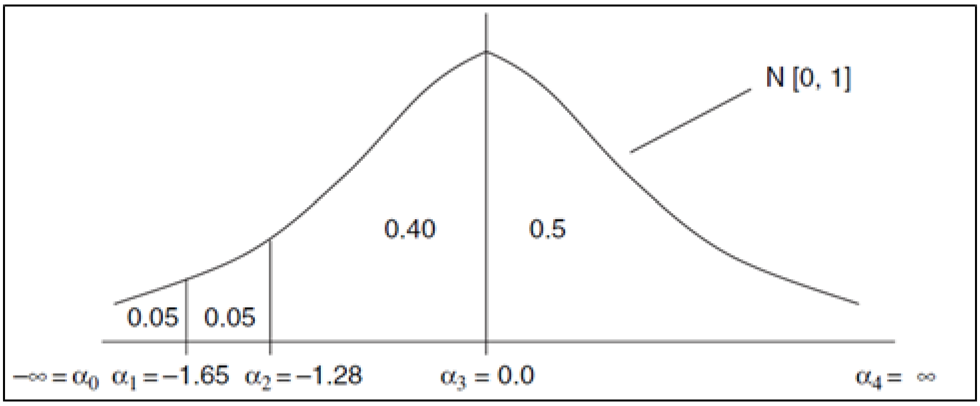

Fig. 2. Histogram chart of the scale

While the difference $\propto_2 - \propto_1$ may differ from the difference $\propto_3 - \propto_2$, unequal scales are allowed. Therefore, this threshold approach allows flexible modelling. As associated with a common normal distribution, it also allows parameters to be easily interpreted. It should be noted that temporary integer values ($k = 1,2,3,4$) are used only to represent the category; Only the frequencies of these values are important in statistical analysis. Structural equation modelling consisting of continuous and discrete data does not have a simple structure. Because it is necessary to calculate multiple integrals associated with cell probabilities determined by ordered categorical results [12].

Some multi-step methods have been introduced to reduce the computational difficulties of these integrals. The basic procedure of these multi-stage methods is polychromic and polyserial correlations, estimating the threshold value in the first stage, deriving the asymptotic distribution of the predictions in the second stage, and analysing the structural equation model with the generalized least-squares approach and covariance structural equation model in the last stage. There are different methods at the first stage to manage the different procedures given in PRELIS and LISREL. Different methods initially applied PRELIS and LISREL [18], LISCOMP and MPLUS [19], and Lee et al. [20] causes different operations. Multistep estimators, however, are not statistically optimal and need to invert a large matrix of size that increases very rapidly with the number of variables that can be observed at each stage of generalized least squares minimization. Besides multi-step operations, Reboussin and Liang [21] proposed an equality estimation approach and Shi and Lee [22] developed a Monte Carlo EM algorithm for the maximum likelihood analysis of a factor analysis model.

When dealing with sequential categorical variables in Bayesian analysis, the basic idea is to treat the expressed latent continuous measurements as hypothetically lost data and amplify them with observed data in posterior analysis. Using this data magnification strategy, the model based on the full data set becomes continuously variable. Sequences of observations of structural parameters, latent variables, and thresholds in infinitive analysis are simulated from the composite posterior distribution using a hybrid algorithm that is the combination of Gibbs sampling [23] and MH algorithm [24,25]. Combined Bayesian estimates of unknown thresholds, structural parameters, and latent variables are produced together with the standard error estimates of these estimates by using simulated observations. In addition to these point estimates, Bayesian model selection can be reviewed using the Bayes factor [12].

2.2. Application Material

The questions regarding the Samsun Chamber of Commerce and Industry member satisfaction survey used in this study. Table 1 is also given.

Table 1. Survey questions and related factors

Factors	Question	Factor Name	
a1	H1	General	General When you think about it in general, how satisfied you with are being a member of our chamber?
a1	H2		Generally speaking, how satisfied are you with the services of our room?
a2	H3	Guidance	Guidance Room responds to our requests in a timely manner
a2	H4		The efficiency of the chamber in strengthening the dialogue between the public authority and the industrialist is sufficient.
a2	H5		The efficiency of chamber services in the development of the sectors is sufficient.
a2	H6		Chamber efficiency is sufficient in terms of national and international expansion of the members.
a3	H7	Solution	Solution I find the room management successful in understanding our problems / needs related to the sector.
a3	H8		I can reach the management / concerned people when we need it
a3	H9		Room management has the ability to produce solutions to your sectoral problems
a3	H10		Our individual problems are taken into consideration by the room management.
a3	H11		I find room management successful in providing an environment and coordination that helps to solve individual problems.
a4	H12	Personal	Staff Attitudes and behaviours of personnel in business relations
a4	H13		Personnel being innovative and productive
a4	H14		Personnel bringing suggestions and guidance
a5	H15	Representation	Representation How satisfied are you with the representation level of the Chamber management?
a5	H16		How satisfied are you with the chamber management in terms of member relations?

3. Results

In this study, member satisfaction survey data with 616 companies randomly selected among Samsun Chamber of Commerce and Industry members in 2011 were used. The aim is to reveal the structural relationship between general satisfaction from Samsun Chamber of Commerce and Industry and guidance, solution, personnel and representation. The application consists of 2 steps. In the first step, the structural equation model was examined with confirmatory factor analysis using the LISREL package program. In the second step, the Bayesian structural equation model was analysed using the WinBUGS package program.

CLASSIC SOLUTION WITH LISREL

All relevant observed variables were associated with latent variables using a one-way road vehicle. The main reason why we draw the path diagram is that it allows easy visualization of the relationship between variables. After completing the figural representation of the relationship between variables, the solution phase was started. In the solution phase, the fit indices specified in the material and method section were examined and the final solution was obtained. Goodness of fit criteria are extremely important in obtaining the final solution. Although there is no comparison in the literature regarding the superiority of goodness of fit criteria, the LISREL program has highlighted the RMSEA value under the path diagram. Since the RMSEA value was not within the required fit criteria in the initial solution, the correction indices suggested by the program were examined and the solution process was repeated. The correction indices suggested as a result of the LISREL solution were used and associated as shown in Figure 3. The decrease in chi-square value is taken into account when using correction indices. The new solution is obtained by performing the correction process that provides the highest decrease in the chi-square value. Below, all the situations under the Estimates option in the LISREL program are shown on the path diagram.

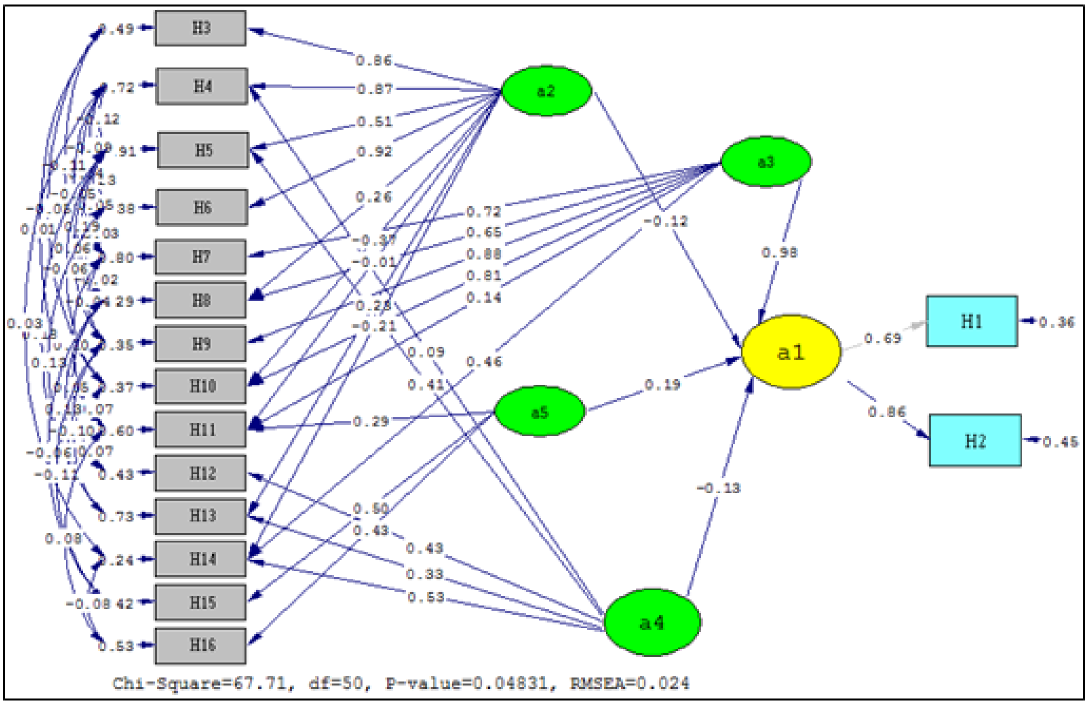

Fig. 3. Threshold values for the scale

The non-standardized coefficients obtained as a result of the LISREL solution are shown in Figure 3. The chi-square value was obtained as 67.71 at the program output. The ratio of the chi-square value to the degrees of freedom 50 is obtained as 1.35. This ratio shows us that the model established is a very powerful model. Another supporting indicator is the approximate root mean square error (RMSEA) value. The fact that this value is close to (0.024) 0 shows that the fit of the model is good. After checking the general fit of the model, the significance of the model parameters was examined.

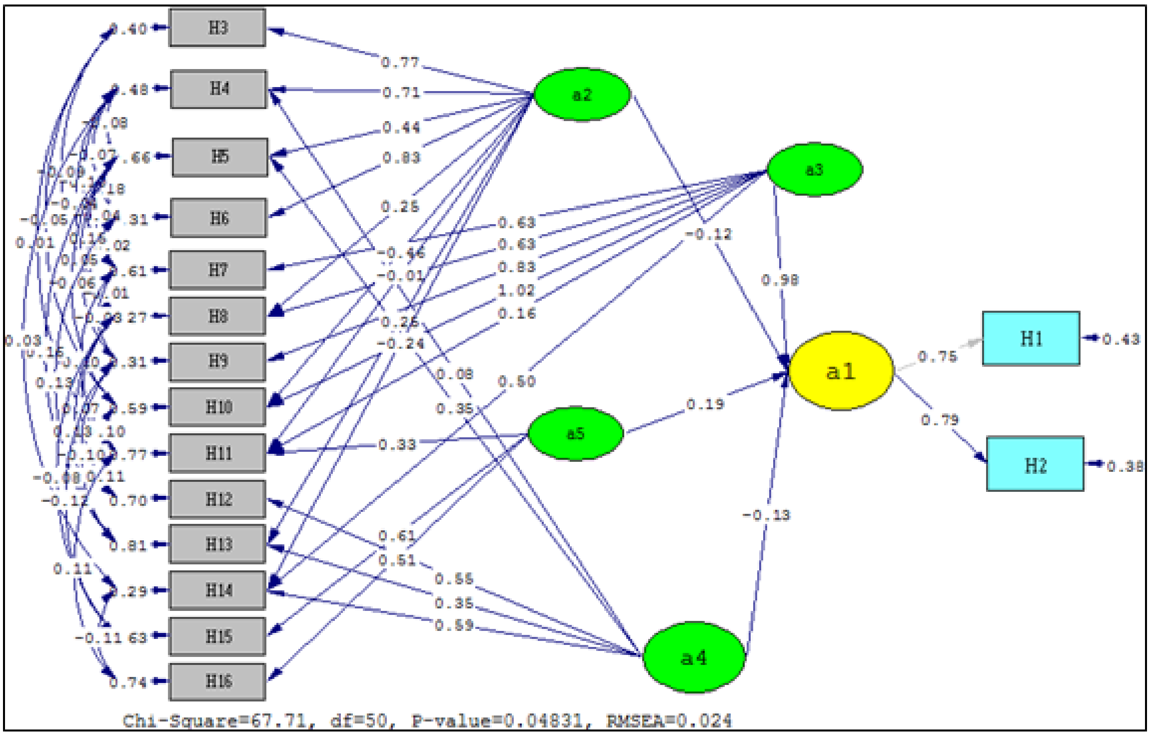

Fig 4. Unstandardized results

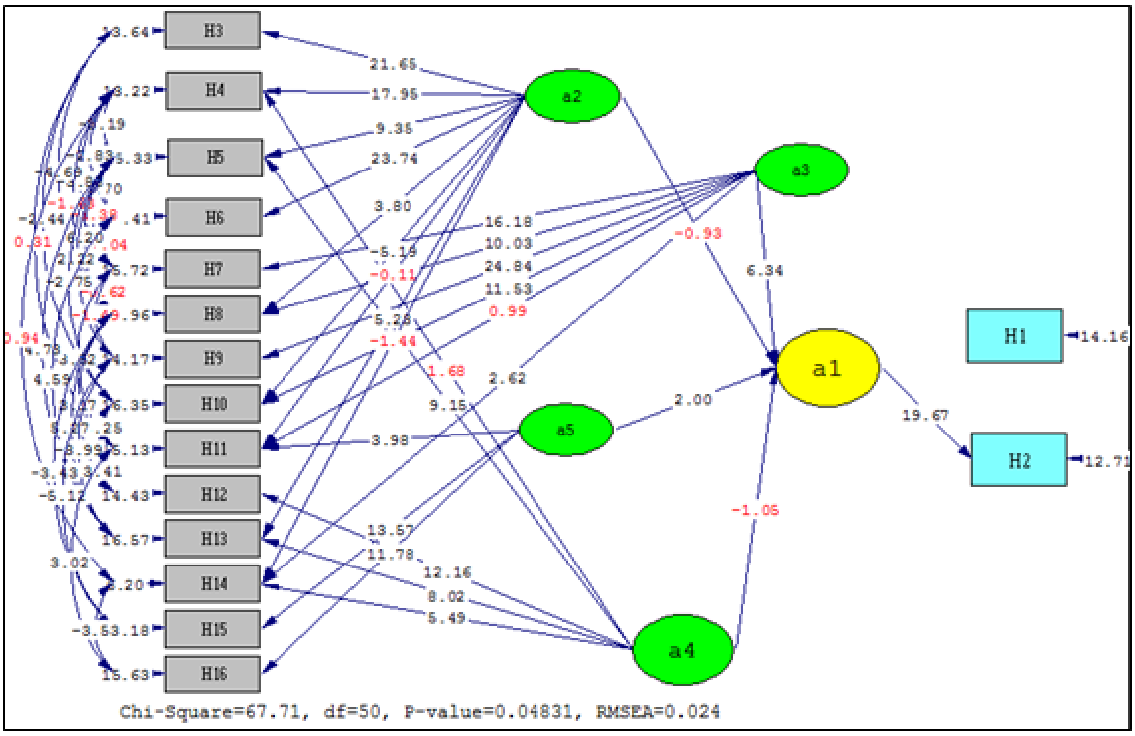

Fig. 5. Unstandardized results

Figure 4 contains the standardized path coefficients for the parameters, while the t-values of the non-statistically significant path coefficients on the path diagram are given in Figure 5. The t-values of the non-significant path coefficients are shown in red in the path diagram. When the t-values of the variables of guidance, solution, personnel and representation, which are thought to have an effect on general satisfaction, were examined, it was seen that the guidance ($t = -0.93$) and personnel ($t = -1.06$) variables were meaningless.

Interpreting Measurement Model Results

After the path diagram is obtained, the process of interpreting the analysis results is started. The results obtained are given below.

Table 2. Measurement model results

Factor / Expression	Standardized Loads	*t*-values	R^2
Factor a1			
H1	0.75		0.57
H2	0.79	19.67	0.62
Factor a2			
H3	0.77	21.65	0.60
H4	0.71	17.95	0.52
H5	0.44	9.35	0.34
H6	0.83	23.74	0.69
H8	0.25	3.80	0.73
H10	-0.46	$\mathbf{-5.19}$	0.41
H11	-0.01	$\mathbf{-0.11}$	0.23
H13	0.25	5.23	0.19
H14	-0.24	$\mathbf{-1.44}$	0.71
Factor a3			
H7	0.63	16.18	0.39
H8	0.63	10.03	0.73
H9	0.83	24,84	0.69
H10	1.02	11.53	0.41
H11	0.16	$\mathbf{0.99}$	0.23
H14	0.50	2.62	0.71
Factor a4			
H12	0.55	12.16	0.30
H13	0.35	8.02	0.19
H14	0.59	5.49	0.71
H4	0.08	$\mathbf{1.68}$	0.52
H5	0.35	9.15	0.34
Factor a5			
H15	0.61	13.57	0.37
H16	0.51	11.78	0.26
H11	0.33	3.98	0.23

Measurement model results are given in Table 2. The standardized loads included in the measurement model results show the correlation between each observed variable and the implicit variable it is related to. Considering the first indicator of the implicit variable a1, H1, the correlation coefficient is 0.75. When the correlation coefficient is squared, R2 of H1 is 0.56. It is seen that the variability related to the implicit variable a1 is mostly explained by H2 (0.62). Fit criteria for the measurement model are given in Table 3.

Table 3. Fit criteria for the model

Fit Measurement	**Good Fit**	**Acceptable Fit**	**Results**
χ^2	$0 \le \chi^2 \le 2df$		67.71 $(P = 0.04831)$
P of Close Fit	≥ 0.05		1.00
$\frac{\chi^2}{df}$	$0 \le \frac{\chi^2}{df} \le 2$	$2 \le \frac{\chi^2}{df} \le 3$	1.35
RMSEA	$0 \le$RMSEA≤ 0.05	$0.05 \le$ RMSEA ≤ 0.08	0.024
NFI	$0.95 \le$ NFI ≤ 1	$0.90 \le$ NFI ≤ 0.95	0.99
NNFI	$0.97 \le$ NNFI ≤ 1	$0.95 \le$ NNFI ≤ 0.97	1.00
CFI	$0.97 \le$ CFI ≤ 1	$0.95 \le$ CFI ≤ 0.97	1.00
GFI	$0.95 \le$ GFI ≤ 1	$0.90 \le$ GFI ≤ 0.95	0.99
AGFI	$0.90 \le$ AGFI ≤ 1	$0.85 \le$ AGFI ≤ 0.90	0.96
PGFI	≥ 0.95		0.36
AIC			239.71
ECVI			0.39
IFI	≥ 0.95	$0.90 \le$ IFI ≤ 0.94	1.00
RFI	≥ 0.90		0.99
Critical N			688.06

The adaptation criteria obtained for the model are presented in Table 3. When the results are examined, it is seen that the goodness of fit criteria are within the ranges recommended by the literature. It was seen that the value (1.35) obtained by dividing the Chi-square value by the degrees of freedom was within the acceptable range.

Table 4. Structural relationship coefficient values

Factor / Expression	**Standardized Loads**	***t*-values**	**R^2**
Factor a1			
a2	0.23	$\mathbf{1.61}$	
a3	0.59	3.20	$\mathbf{0.93}$
a4	0.32	4.28	
a5	-0.01	$\mathbf{-0.14}$	

In Table 4, standardized loads and t-value are given regarding the structural relationship between the general satisfaction implicit variable and the implicit variables of guidance, solution, personnel and representation. According to the results, the path coefficients between the general satisfaction.

Table 5. Frequency disturbution from response to the questions

	Likert Scale					
	1	**2**	**3**	**4**	**5**	**Sum**
H1	2	67	119	294	134	616
H2	36	210	139	184	47	616
H3	16	127	177	168	128	616
H4	28	115	121	161	191	616
H5	27	97	197	130	165	616
H6	13	89	129	195	190	616
H7	45	106	184	183	98	616
H8	9	143	86	292	86	616
H9	30	159	157	217	53	616
H10	8	44	182	332	50	616
H11	8	51	220	249	88	616
H12	4	44	259	255	54	616
H13	3	42	206	190	175	616
H14	29	49	205	292	41	616
H15	4	57	122	361	72	616
H16	5	16	114	264	217	616

Implicit variable and the counselling and representation implicit variable were not found to be significant. Only the structural relationship between the general satisfaction implicit variable and the solution and personnel implicit variable was found to be significant. Bayesian solution has been implemented with WinBUGS package program. Before starting the Bayesian solution, frequency tables for 16 questions were prepared. The main purpose of extracting the frequency tables is to determine the percentage rates for each question of the 5-point Likert scale used in the questionnaire form and the threshold values required for analysis based on these rates. The frequency distribution of each question is given in the table below. The threshold values were started to be calculated by obtaining the percentages of the scale categories from the frequency table for each question. Threshold value calculation is made as one minus of the number of categories used in Likert scale. Threshold values in Table 6 were obtained from the reverse of the normal cumulative distribution by using the relevant frequency tables.

Table 6. Distribution percentage of response to the questions

	Likert Scale				
	1	**2**	**3**	**4**	**5**
H1	0.003247	0.108766	0.193182	0.477273	0.217532
H2	0.058442	0.340909	0.225649	0.298701	0.076299
H3	0.025974	0.206169	0.287338	0.272727	0.207792
H4	0.045455	0.186688	0.196429	0.261364	0.310065
H5	0.043831	0.157468	0.319805	0.211039	0.267857
H6	0.021104	0.144481	0.209416	0.316558	0.308442
H7	0.073052	0.172078	0.298701	0.297078	0.159091
H8	0.01461	0.232143	0.13961	0.474026	0.13961
H9	0.048701	0.258117	0.25487	0.352273	0.086039
H10	0.012987	0.071429	0.295455	0.538961	0.081169
H11	0.012987	0.082792	0.357143	0.404221	0.142857
H12	0.006494	0.071429	0.420455	0.413961	0.087662
H13	0.00487	0.068182	0.334416	0.308442	0.284091
H14	0.047078	0.079545	0.332792	0.474026	0.066558
H15	0.006494	0.092532	0.198052	0.586039	0.116883
H16	0.008117	0.025974	0.185065	0.428571	0.352273

The expressions to be used in the analysis in WinBUGS are given in the table below for both measurement models and structural equation model.

Structure of model in WinBUGS

Table 7. Symbolic representation of implicit and measurable variables

Factor	Questions	Node	Implicit	Variable	Node
a1	H1	1	a1	a2	gam[1]
a1	H2	lam[1]	a1	a3	gam[2]
a2	H3	1	a1	a4	gam[3]
a2	H4	lam[2]	a1	a5	gam[4]
a2	H5	lam[3]			
a2	H6	lam[4]			
a3	H7	1			
a3	H8	lam[5]			
a3	H9	lam[6]			
a3	H10	lam[7]			
a3	H11	lam[8]			
a4	H12	1			
a4	H13	lam[9]			
a4	H14	lam[10]			
a5	H15	1			
a5	H16	lam[11]			

Measurement of The Equality

```
for(j in 1:P){

y[i,j]~dnorm(mu[i,j],psi[j])I(thd[j,z[i,j]],thd[j,z[i,j]+1])
        ephat[i,j]<-y[i,j]-mu[i,j]
    }
    mu[i,1]<-eta[i]
    mu[i,2]<-lam[1]*eta[i]
    mu[i,3]<-xi[i,1]
    mu[i,4]<-lam[2]*xi[i,1]
    mu[i,5]<-lam[3]*xi[i,1]
    mu[i,6]<-lam[4]*xi[i,1]
    mu[i,7]<-xi[i,2]
    mu[i,8]<-lam[5]*xi[i,2]
    mu[i,9]<-lam[6]*xi[i,2]
    mu[i,10]<-lam[7]*xi[i,2]
    mu[i,11]<-lam[8]*xi[i,2]
    mu[i,12]<-xi[i,3]
    mu[i,13]<-lam[9]*xi[i,3]
    mu[i,14]<-lam[10]*xi[i,3]
    mu[i,15]<-xi[i,4]
    mu[i,16]<-lam[11]*xi[i,4]
```

Structural Equation

```
xi[i,1:4]~dmnorm(u[1:4],phi[1:4,1:4])
        eta[i]~dnorm(nu[i],psd)
        nu[i]<-gam[1]*xi[i,1]+gam[2]*xi[i,2]+gam[3]*xi[i,3]+gam[4]*xi[i,4]
        dthat[i]<-eta[i]-nu[i]
```

Threshold Values

```
thd=structure(
        .Data=c(-200.000.-2.722,-1.216,-0.510.
0.781,200.000.
-200.000.-1.568,-0.255, 0.319, 1.430.200.000.
-200.000.-1.944,-0.732, 0.049, 0.814,200.000.
-200.000.-1.691,-0.732,-0.180. 0.496,200.000.
-200.000.-1.708,-0.837, 0.053, 0.619,200.000.
-200.000.-2.031,-0.972,-0.319, 0.500.200.000.
-200.000.-1.453,-0.690. 0.110. 0.998,200.000.
-200.000.-2.180.-0.685,-0.289, 1.082,200.000.
-200.000.-1.658,-0.505, 0.155, 1.366,200.000.
-200.000.-2.227,-1.376,-0.306, 1.397,200.000.
-200.000.-2.227,-1.306,-0.118, 1.068,200.000.
-200.000.-2.484,-1.419, -0.004, 1.355,200.000.
-200.000.-2.585,-1.453,-0.234, 0.571,200.000.
-200.000.-1.674,-1.142,-0.102, 1.502,200.000.
-200.000.-2.484,-1.287,-0.533, 1.191,200.000.
-200.000.-2.404,-1.824,-0.775, 0.379,200.000),
        .Dim=c(16,6)),
```

Primarily, the point at which convergence was achieved was determined and this point was used as the burning period. Two methods were used to check whether convergence was achieved. The first of these is to examine the trace graphs for the related parameters. The trace graphs of the path coefficients for each measurement equations are given below. When the trace charts are examined, it is seen that the predicted values in the parameters gradually become stagnant and not take extreme values.

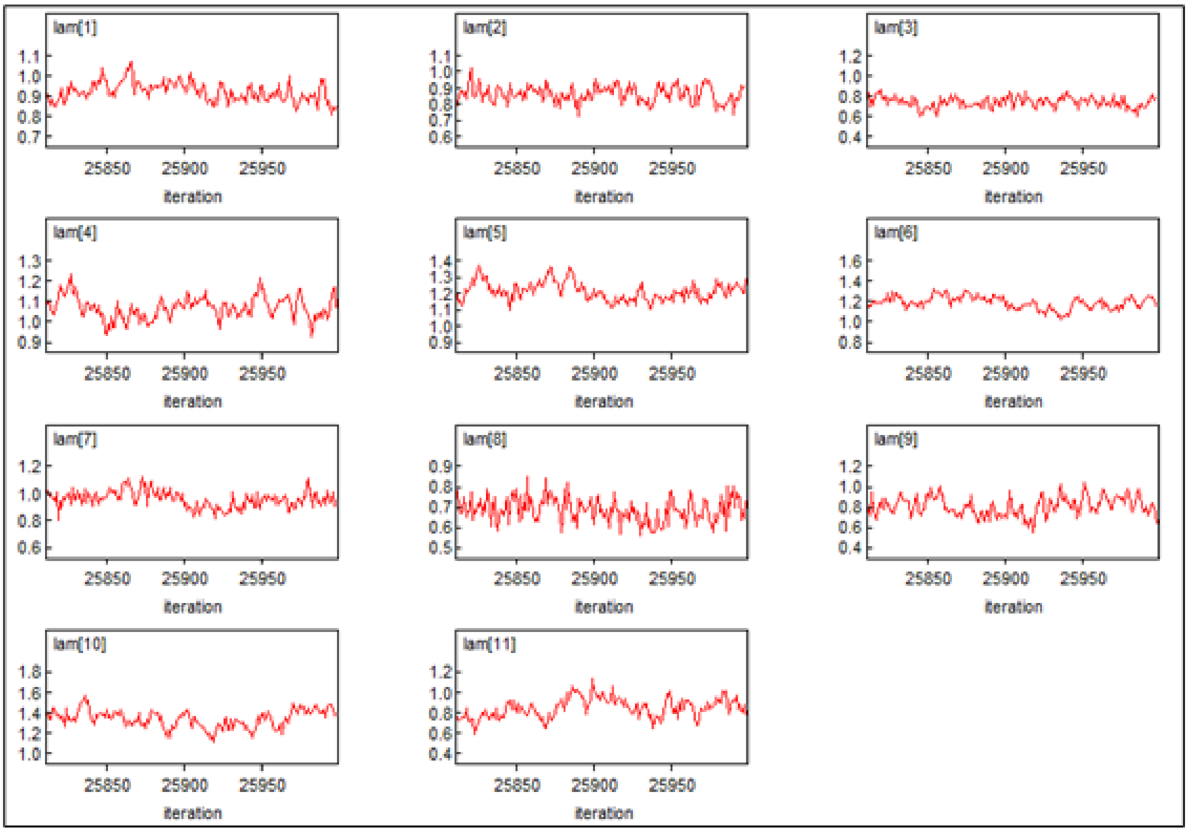

Fig. 6. Trace graphics related to measurement model parameters

When the trace graphs of the measurement model parameters in Figure 6 are examined, it is seen that there is no extreme value. After 11000 samples, it can be seen from the trace graphs that convergence is achieved.

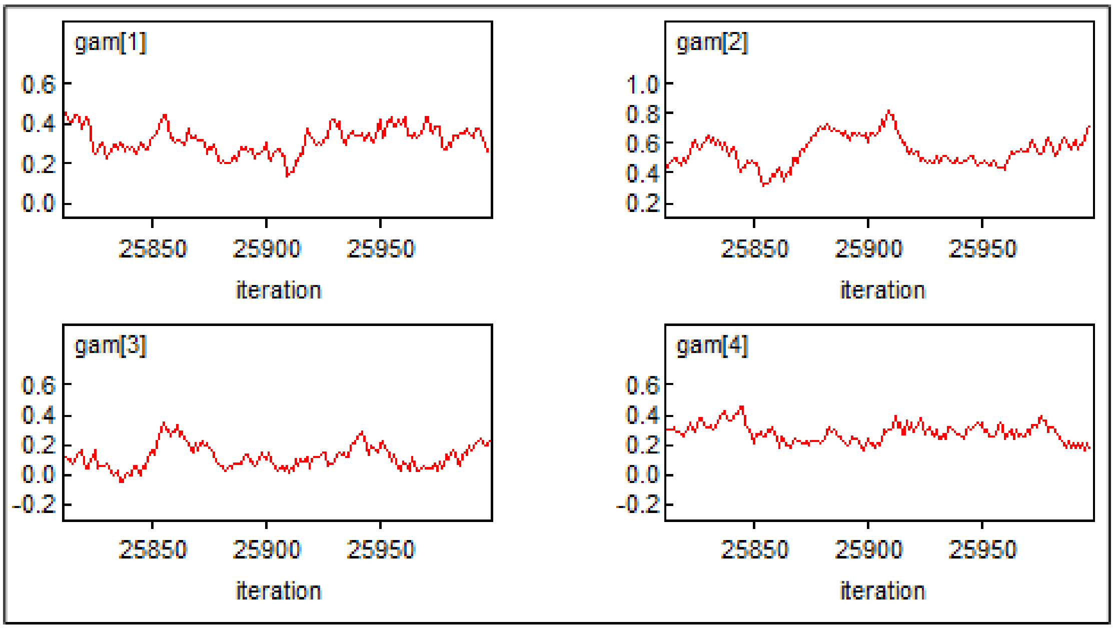

Fig. 7. Trace graphics related to measurement model parameters

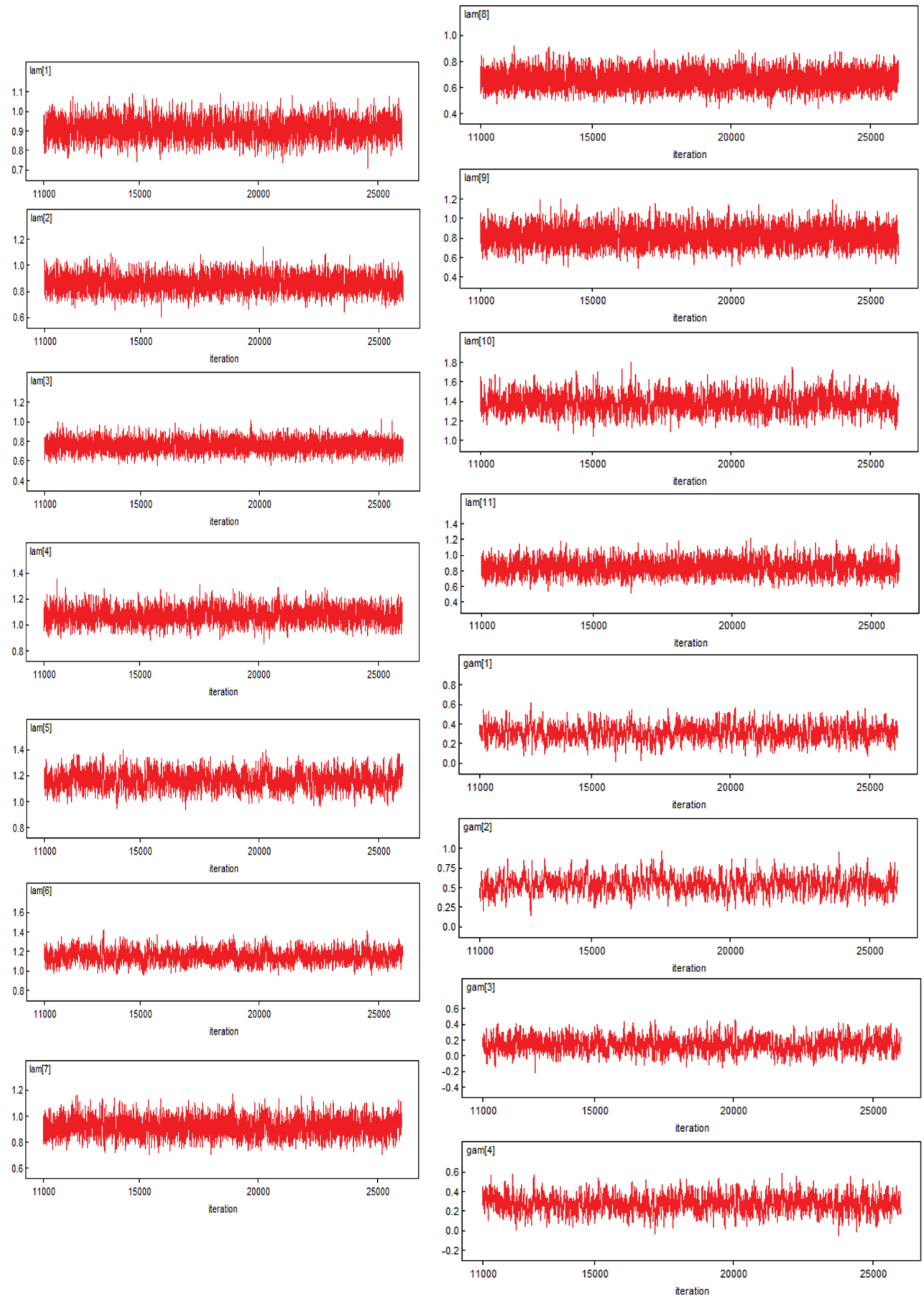

Fig. 8. Trace graphics related to structural model parameters

When the trace graphs of the structural equation model parameters in Figure 8 are examined, it is seen that there is no extreme value as in the measurement model parameters. After 11000 samples, it can be seen from the trace graphs that convergence is achieved. The interpretation of trace charts alone does not provide us with precise information on whether convergence is achieved. Secondly, the Thumb rule, which is a stronger method, is used to determine whether convergence is achieved. As a rule, MC errors for each parameter must be less than 5% of the standard deviation values. In the table below, it can be checked whether there is convergence for both the measurement and structural equation parameters.

Table 8. Convergence results of measurement model and structural equation model parameters

Node	Mean	Standard Deviation	Sd. (5%)	MC error	2.50%	Median	97.50%
gam[1]	0.3074	0.08184	0.004092	0.00328	0.1571	0.304	0.4626
gam[2]	0.5539	0.1152	0.00576	0.00549	0.348	0.5526	0.7694
gam[3]	0.1405	0.09059	0.00453	0.003564	−0.04055	0.1425	0.312
gam[4]	0.2714	0.08856	0.004428	0.003877	0.1012	0.2702	0.4458
lam[1]	0.9102	0.0487	0.002435	0.001313	0.8182	0.9095	1.005
lam[2]	0.8605	0.06729	0.003365	0.001976	0.7517	0.8582	0.9763
lam[3]	0.7595	0.06857	0.003429	0.002095	0.6467	0.757	0.8805
lam[4]	1.075	0.06825	0.003413	0.002101	0.9633	1.073	1.193
lam[5]	1.167	0.07395	0.003698	0.003261	1.045	1.166	1.296
lam[6]	1.154	0.07424	0.003712	0.003361	1.034	1.152	1.284
lam[7]	0.9194	0.07099	0.00355	0.00276	0.8021	0.9167	1.045
lam[8]	0.6654	0.06959	0.00348	0.002214	0.5485	0.6637	0.7891
lam[9]	0.8138	0.08949	0.004475	0.002305	0.65	0.811	0.9912
lam[10]	1.381	0.09121	0.004561	0.003211	1.217	1.378	1.565
lam[11]	0.8534	0.09682	0.004841	0.003385	0.6805	0.8497	1.04

As can be seen in Table 8, MC error values of all parameters related to measurement models and structural equation models are less than 5% of the standard deviation values. Buddha reveals the point at which convergence is achieved in the Bayesian solution more clearly than the trace graphs. The main purpose in finding the point at which convergence is achieved is to determine the burning period and to ensure that the estimates up to this period are not taken into account. After the burning period, 15000 updates were made, and parameter estimations were made for the model. The historical graphics for each parameter are represented in Figure 9.

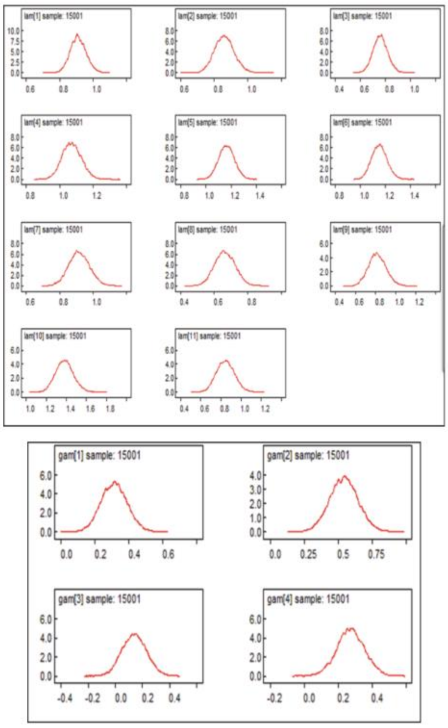

Fig. 9. History plots for parameters

Past graphs of the path coefficients related to both measurement models and structural equation model are given. From these graphs, it can be seen that there are no excessive fluctuations, and that each parameter converges. The Bayesian structural equation results obtained over 26000 samples, 11000 of which were taken using the burning period, are given in the table below.

Table 9. Bayesian prediction results

Node	Mean	Standard Deviation	Sd. (5%)	MC error	2.50%	Median	97.50%
gam[1]	0.3132	0.07689	0.003845	0.003078	0.1659	0.3129	0.4667
gam[2]	0.5436	0.1055	0.005275	0.004844	0.3391	0.5432	0.7545
gam[3]	**0.1401**	**0.08975**	**0.004488**	**0.003419**	**−0.03154**	**0.1398**	**0.3171**
gam[4]	0.2749	0.08222	0.004111	0.003308	0.1139	0.2741	0.4383
lam[1]	0.9111	0.04736	0.002368	0.001148	0.8211	0.9095	1.007
lam[2]	0.8605	0.05756	0.002878	0.001349	0.751	0.8595	0.9768
lam[3]	0.7586	0.05795	0.002898	0.001222	0.6478	0.7578	0.8758
lam[4]	1.073	0.05736	0.002868	0.001587	0.9642	1.071	1.188
lam[5]	1.166	0.0619	0.003095	0.002353	1.047	1.165	1.289
lam[6]	1.151	0.0609	0.003045	0.002406	1.036	1.15	1.275
lam[7]	0.9171	0.0623	0.003115	0.00192	0.7966	0.9156	1.042
lam[8]	0.6643	0.06102	0.003051	0.001395	0.5479	0.6627	0.7858
lam[9]	0.8176	0.0875	0.004375	0.002139	0.6521	0.815	0.9956
lam[10]	1.382	0.08879	0.00444	0.00332	1.216	1.381	1.563
lam[11]	0.8461	0.08819	0.00441	0.00268	0.6808	0.8449	1.023

Bayesian parameter estimation results are given in Table 9. Only the gamut [3] structural equation parameter is meaningless. Additionally, parameter estimates of classical and Bayesian measurement models were represented in the Table 10 and 11.

Table 10. Parameter estimates of classical and Bayesian measurement models

Factor/ Expression	LISREL	BAYES
Factor a1		
H1	0.75	1
H2	0.79	0.9111
Factor a2		
H3	0.77	1
H4	0.71	0.8605
H5	0.44	0.7586
H6	0.83	1.073
Factor a3		
H7	0.63	1
H8	0.63	1.166
H9	0.83	1.151
H10	1.02	0.9171
H11	0.16	0.6643
Factor a4		
H12	0.55	1
H13	0.35	0.8176
H14	0.59	1.382
Factor a5		
H15	0.61	1
H16	0.51	0.8461

Table 11. Classical and Bayesian structural model parameter estimates

Factor/ Expression	LISREL	BAYES
Factor a1		
a2	**0.23**	0.31
a3	0.59	0.54
a4	0.32	**0.14**
a5	**−0.01**	0.27

With LISREL, it was seen that the implicit variables of counselling and representation did not have an effect on general satisfaction in the classical solution, only the solution and staff implicit variables were effective. In the Bayesian approach, contrary to the classical analysis, it was found that the counselling implicit variable and the representation implicit variable were significant, while the staff implicit variable was not significant.

4. Conclusion

In this study, classical structural equation models and Bayesian structural equation models are emphasized. Both approaches were applied to the survey data obtained from Samsun Chamber of Commerce and Industry. The data consists of 16 observed variables measuring 5 latent variables and 616 observations. Structural relationship and measurement models designed in classical analysis were created. First, the model fit indices were examined, and the analysis process was started. When the measurement models created in LISREL and the structural relationship were examined, it was seen that the model fit was not good and correction indices were needed. Correction indices were discussed in two parts. The correction indices in the first part show the relationships between observed variables and latent variables. The correction indices in this section show the decrease in the chi-square value calculated to evaluate model fit as a result of associating the observed variables under one latent variable with another latent variable. The correction indices in the second part are based on the independence of the errors for the observed variables. Since the model fit was not achieved in the initial solution, the analysis was performed using correction indices. In the initial solution, both the RMSEA value was higher than 0.08 and the division of the chi-square to the degrees of freedom was over 3. These values made it necessary to use correction indices in classical analysis. After the recommended corrections were made in the measurement model, it was determined that the model fit well according to all the criteria used in the assessment of goodness of fit. When the obtained fit indices were examined, the RMSEA value was found to be 0.024 and the division of the chi-square to the degrees of freedom was found to be 1.35. After examining the model fit indices, parameter estimation was started. Otherwise, it would not be reasonable to examine the parameter estimates unless the model fit is achieved. The significance of the parameter estimates related to the measurement models and the structural model was examined using t-values. Path coefficient with a t-value below 1.96 was considered to be insignificant. One of the most important advantages of the path diagram in classical analysis is that the meaningless relations are shown in shape and in different colours. This advantage allows for interpretation and viewing all relationships in a single photo. It was determined that the structural relationship between the counselling and representation implicit variables symbolized by a2 and a5 of the general satisfaction implicit variable was insignificant. While there was no significant relationship between the two latent variables and the general satisfaction implicit variable, a significant relationship was found between the other two latent variables (solution and staff). With this structural relationship obtained, 93% of the general satisfaction implicit variable is explained.

In the Bayesian structural equation model solution, model structures were defined in two stages: measurement models and structural model. By creating the frequency tables for the questions prepared using a 5-point Likert, (4) threshold values equal to one less than the number of Likert categories were calculated. The inverse of the cumulative normal distribution was used in calculating the threshold values. With the calculation of the threshold values, the model was started to be analysed. The most important step of the analysis phase is convergence. Unless convergence was achieved, model parameters were not estimated. It was seen that all parameters related to the model converged in 11000 iterations. Convergence has been examined in two stages. In the first stage, the trace graphs of each parameter were examined, and a stationary structure was observed. The interpretation of trace charts alone is not sufficient as a precise information. In the second stage, according to the Thumb rule; The condition that MC error values for each parameter should be less than 5% of the standard deviation value of the same parameter was examined. All parameters were found to meet this requirement and 11000 was used as the burning period. The purpose of using 11000 as the burning period is to ignore the parameter values in these iterations where convergence is not achieved. After the burning period, the parameters related to the model were obtained at the end of 15000 iterations. When the structural relationship between general satisfaction and the other 4 latent variables was examined, it was seen that the personnel implicit variable represented by a4 was meaningless, unlike the classical analysis.

Guidance, solution, personnel and representation factors affecting general satisfaction were examined in the data obtained from Samsun Chamber of Commerce and Industry, and it was revealed that classical and

Bayesian approaches give different results in terms of parameter estimates. While only the solution and personnel implicit variables were significant in the classical approach, the implicit variables of guidance, solution and representation were found to be significant in the Bayesian approach. This study was applied to the standard Samsun Chamber of Commerce and Industry questionnaire, whose data were prepared previously. The results can be obtained differently by redesigning and obtaining the questionnaire forms. Model comparisons were not emphasized in this thesis. In the classical approach, Akaike information criterion was calculated, and AIC, BIC and DIC calculations in Bayesian approach will be our future studies.

Although there are many a priori selection methods in the Bayesian approach, a priori selection in this study is limited to only conjugate a priori. Theoretical information is given about the use of other a priori such as Jeffrey's a priori. Therefore, a priori comparison and comparison of the adaptation criteria in the Bayesian approach has prepared a theoretical background for the studies to be carried out in the following years. In this study, the scale type is taken as a fixed 5-point Likert. There are no studies on the use of Bayesian structural equation when the types of scales are different. Studies in this field will provide new gains to the literature.

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Model Selection in Beta Regression Analysis Using Several Information Criteria and Heuristic Optimization

Emre DÜNDER[1], Mehmet Ali CENGİZ[2]

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Abstract − In the context of generalized linear modeling (GLM), the beta regression analysis is used to estimate regression models when the dependent variable lies between (0,1). In this paper, we carried out a model selection process using several information criteria with heuristic optimization. We employed the differential evolution algorithm as a heuristic optimization method to select the best model for beta regression analysis. The results show that the alternative-type information criteria provide competitive results during the model selection process in beta regression analysis.

Keywords *− Beta regression, differential evolution algorithm, information criteria, model selection*

1. Introduction

In the regression analysis, the models are estimated according to the distribution of the dependent variable. Therefore, the distribution of the dependent variable should be appropriate, and the alternative choices can be used in the violation of normality. The generalized linear modelling (GLM) approach is employed to construct the regression models for several distribution-types such as Poisson, Gamma, Binomial, etc. Also, the selection of the optimal model including the best explanatory variable is very crucial. There are lots of attempts in the literature for the model selection in regression modeling. Sakate et al. [1] proposed a stepwise selection approach for Poisson regression analysis. Calcagno and Mazancourt [2] developed a software package to apply model selection for GLM. Örkcü [3] used and hybrid simulated annealing approach in regression analysis for model selection. Unler ve Murat [4] proposed a model selection approach based on the particle swarm optimization for binary dependent variables. The researchers also utilized the information criteria during the model selection [5-9].

Overall, we can see that the model selection process in GLM can be handled with three components:

1) GLM with an appropriate distribution,

2) The information criteria,

3) An appropriate optimization technique.

[1] emre.dunder@omu.edu.tr; [2] macengiz@omu.edu.tr
[1,2] Department of Statistics, Faculty of Arts and Science, Ondokuz Mayıs University, Samsun, Turkey

When the dependent variable is ranging between $(0,1)$ beta regression analysis can be used instead of the normal linear regression analysis [10]. By inspiring the current model approaches, we performed model selection in beta regression analysis using several information criteria.

To the best of our knowledge, this study is the first attempt to apply model selection under the alternative Bayesian and Information Complexity (ICOMP) type criteria for beta regression analysis. Also, we adopted the binary type of differential evolution algorithm to choose the optimal subset of the explanatory variables [11].

The article is organized as follows: In Section 2, we introduced the beta regression analysis and parameter estimates. In Section 3 we described the working mechanism of the binary differential evolution algorithm. In Section 4, we presented the information criteria which are used in the article. In Section 5, we introduced the numerical examples including the simulation studies and real data set applications. Finally, we presented the conclusion and discussion part in Section 6.

2. Beta regression analysis

In the regression analysis, beta regression analysis is used when the dependent variable includes ratio or percentage values in the interval $(0,1)$. The main reason for using beta regression analysis is that the ratio or percentage values of the data in the interval $(0,1)$ are suitable for the beta distribution due to the nature of the data [12]. The probability function of the beta distribution is expressed as follows:

$$f\,(y|\mu,\phi)=\frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma\big((1-\mu)\phi\big)}y^{\mu\phi-1}(1-y)^{(1-\mu)\phi-1},\qquad 0<y<1 \tag{1}$$

where μ shows the location parameter and ϕ shows the dispersion parameter. The parameter of μ is between $(0,1)$ and $\phi>0$. The expected value of the variable in beta distribution is $E(y)=\mu$ and the variance is $Var(y)=\mu\,(1-\mu)/(1+\phi)$ [13].

Based on the probability distribution, the log-likelihood function for the beta regression is written as:

$$L(\mu,\phi)=\log\Gamma(\phi)-\log\Gamma(\mu\phi)-\log\Gamma\big((1-\mu)\phi\big)+(\mu\phi-1)\log y+\big((1-\mu)\phi-1\big)\log(1-y) \tag{2}$$

By following the GLM approach, the beta regression can be written as the following form:

$$g(\mu)=\beta_0+\sum_{k=1}^{p}\beta_{ik}\,X_{ik},i=1,2,\dots,n \tag{3}$$

where $g(.)$ denotes the link function and $\beta=(\beta_1,\beta_2,\dots,\beta_p)$ vector represents the regression coefficients, and they are estimated with the numerical methods such as Newton-Rapson or Fisher scoring on the log-likelihood function. The detailed information can be found in [13].

Table 1. The link functions for beta regression

Link function	Formula
Logit	log(μ (1 − μ))
Log-log	-log(-log(μ))
Complementary log-log	log(-log((1 − μ)))
Probit	Φ^{-1}(μ)
Cauchy	tan(π(μ-0.5))
Log	log(μ)

The beta regression model can be estimated with various link functions. Table 1 shows the possible link functions that can be used within beta regression framework. In Table 1, $\log(.)$ shows the natural logarithm, $\Phi(.)$ denotes the cumulative link function of the standard normal distribution and $\tan(.)$ indicates the trigonometric tangent function [10,14].

3. Differential evolution algorithm for binary search

Heuristic techniques are very common to optimize a target function. Heuristic methods are implemented effectively inside the statistical modeling. Especially, the model selection is carried out with a proper heuristic optimization technique for determining the best variable subset. To carry out the model selection, each variable is coded as binary and the optimal set is chosen such as optimizing a target function.

The mentioned approach is employed in regression modeling and we can perform the model selection in beta regression analysis. Differential evolution is a very effective optimization method [15] and we used this method as a binary version. In our approach, the target functions are the information criteria that evaluates the several beta regressions models.

Differential evolution algorithm consists of four main steps:

1) Generating the initial population
2) Mutation
3) Crossover
4) Selection

As it occurs in the similar algorithms, the initial population is generated at the starting point [11]. The initial population is generated as the following way:

$$x_i^G = x_{i\,(L)} + rand_i\,[0,1]\,(x_{i\,(H)} - x_{i\,(L)}) \tag{4}$$

where $x = (x_1, x_2 \dots x_p)$ is the vector of the parameters, $x_{i\,(H)}$ and $x_{i\,(L)}$ are the bounds and $rand$ is the generated parameters as uniformly distributed. Since we employ a binary search, the bounds are limited between [0,1] and the parameters are rounded. Figure 1 represents the coding scheme for model selection.

Fig. 1: The coding of the explanatory variables

The mutation process is applied to make the search process more robust and durable. This process also allows new regions in the search space to be discovered. For this purpose, vectors called trial parameters are created. Trial parameters are formed by adding the weighted difference between two units onto a third unit [11]. The parameters re-derived in this way are evaluated within the objective function (i.e. information criteria) according to the values of the previous units. If the value of the objective function consisting of the re-derived vectors is better than the previous value, it is replaced with the more appropriate parameter vectors. This process for each size proceeds as follows:

$$v^{G+1} = x_{r3}{}^{G} + \mathrm{F}\,(x_{r1}{}^{G} - x_{r2}{}^{G}) \tag{5}$$

where $\mathrm{F} \in [0,1]$ is a scaling factor for $r_1 \neq r_2 \neq r_3$.

After the mutation stage, a crossover process is applied over the parameter vectors. Crossing is used to strengthen the success level of the parameter vectors obtained in the mutation stage and to define new vectors by acting from existing vectors. In the crossover stage, a crossover constant defined as $CR \in [0,1]$ is processed as follows:

$$x^{G+1} = \begin{cases} u^{G+1}, & f(u^{G+1}) \le f(x^G) \\ x^G, & f(u^{G+1}) > f(x^G) \end{cases} \tag{6}$$

where f(.) denotes the target function. This process is iteratively employed until reaching the optimum.

4. Information criteria and model selection

Information criterion is a measure that shows the performance of a statistical model. Mainly, information criteria attempt to penalize the bias and they are widely used within the scope of regression models, especially when choosing the most appropriate model.

The general structure of an information criterion is defined as follows:

$$\text{Information criterion} = -\,2\,LL + \text{Penalty} \tag{7}$$

where LL is the log-likelihood of the model.

The penalty of the criterion has a huge impact on the selected models in regression analysis. The formulations of the most common criteria such as AIC and BIC are given as follows:

$$AIC(k) = -2LL + 2k \tag{8}$$
$$BIC(k) = -2LL + k\log(n) \tag{9}$$

where k shows the number of free parameters and n shows the sample size. In regression models, k represents the number of explanatory variables.

Moreover, the different terms can be added as the penalty inside the information criteria. For example, ICOMP-type criteria include the covariance matrix of the statistical models with a complexity function [16,17]. There are also different criteria called as the information matrix-based information criterion (IBIC) and scaled unit information prior Bayesian information criterion (SPBIC) [18].

Table 2. Penalty terms of the seven information criteria

Information criteria	Penalty
ICOMPifim	2C(F^{-1})
ICOMPpeu	k+2C(F^{-1})
ICOMPpeuln	k+log(n)C(F^{-1})
ICOMPperf	n+2C(F^{-1})
CICOMP	k(1+log(n)) + 2C(F^{-1})
IBIC	klog(n/2π) + log $\lvert(\hat{\Sigma}_{model})^{-1}\rvert$
SPBIC	p (1 - log(k/ ($\beta^T(\hat{\Sigma}_{model})^{-1}\beta$)))

Table 2 demonstrates the penalty terms of five ICOMP-type and two Bayesian-type information criteria. In ICOMP-type criteria, $C(.)$ denotes the complexity function and F is the Fisher information matrix of the statistical model. We considered the C_{1F} complexity, shown as:

$$C_{1F}(.) = \frac{1}{4\lambda_m{}^2}\sum_{i=1}^{p}(\lambda_i - \lambda_m)^2 \tag{10}$$

where λ_m is the arithmetic mean of the eigenvalues which obtained from the inverse of Fisher information matrix [19]. In our study, we consider the Fisher information matrix of the beta regression model. In IBIC and SPBIC, $\hat{\Sigma}_{model}$ is the variance-covariance matrix of the beta regression models.

5. Numerical examples

In this part we performed a simulation study and real data applications on model selection for beta regression analysis. Mainly, we assessed the performance of the seven criteria in terms of the model selection capability.

We formulated the simulation settings from the following equation [20]:

$$\log\left(\frac{\mu}{1-\mu}\right) = \beta_o + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p \tag{11}$$

The expected value, μ is generated by considering the logit link function. After generating the expected values, we simulated the response variable from the beta distribution with a fixed dispersion parameter ϕ.

In simulation part, we obtained the ratio of correct selected variables (C) and incorrectly selected variables (I) for each correct and incorrect variable set. The number of runs is 100 for the simulation settings.

In real data analysis part, we evaluated the prediction errors and the number of the significant-insignificant variables. All the implementations were performed betareg and DEoptim packages, existing in R software [21, 22].

5.1 Simulation study-1

The simulation part includes a design which has the multicollinearity among the predictors. The explanatory variables are generated as the following way:

$$x_1 = 10 + \varepsilon_1 \tag{12}$$
$$x_2 = 10 + 0.3\varepsilon_1 + \alpha\varepsilon_2 \tag{13}$$
$$x_3 = 10 + 0.3\varepsilon_1 + 0.5604\alpha\varepsilon_2 + 0.8282\alpha\varepsilon_3 \tag{14}$$
$$x_4 = -8 + x_1 + 0.5x_2 + 0.3x_3 + 0.5\varepsilon_4 \tag{15}$$
$$x_5 = -5 + 0.5x_1 + x_2 + 0.5\varepsilon_5 \tag{16}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \sim N(0,1)$. We fixed $\alpha = 0.3$ for this setting and it causes the multicollinearity [23]. The true model includes x_1, x_2, x_3 and the generation process is conducted as follows:

$$L = 5 + 1.75x_1 + 1.25x_2 + 0.5x_3 + e \tag{17}$$
$$M = 1/(1 + exp(-L)) \tag{18}$$
$$Y \sim Beta(M\phi, (1-M)\phi) \tag{19}$$

As it is seen from above, the correct model includes $\{x_1, x_2, x_3\}$. We considered $\phi = 5, 50$ with an error term $e \sim N(0,1)$ and added five irrelevant variables as $d\ x\ R_d \sim U\,(0,1)\ d=1,2,3,4,5$. The sample size was chosen as $n = 50, 100, 300, 500$.

Table 3. The ratio of the correctly selected variables beta regression models for $\phi = 5$

Criteria	Sample size				Average
	$n = 50$	$n = 100$	$n = 300$	$n = 500$	
AIC	77.333	90.667	99.333	100.000	91.833
BIC	67.333	83.333	98.000	98.639	86.827
IBIC	60.000	74.000	90.667	96.599	80.316
SPBIC	68.667	86.667	98.667	99.320	88.330
CICOMP	50.000	66.000	88.667	96.599	75.316
ICOMPifim	68.667	86.667	99.333	100.000	88.667
ICOMPpeu	68.667	86.667	99.333	100.000	88.667
ICOMPpeuln	68.667	86.667	99.333	100.000	88.667
ICOMPperf	68.667	86.667	99.333	100.000	88.667

Table 4. The ratio of the incorrectly selected variables beta regression models for $\phi = 5$

Criteria	Sample size				Average
	$n = 50$	$n = 100$	$n = 300$	$n = 500$	
AIC	19.200	13.200	16.800	14.286	15.871
BIC	3.600	2.800	2.000	0.816	2.304
IBIC	0.400	0.000	0.000	0.000	0.100
SPBIC	8.800	5.600	4.000	3.265	5.416
CICOMP	1.200	0.000	0.000	0.000	0.300
ICOMPifim	8.400	5.600	4.400	4.898	5.824
ICOMPpeu	8.400	5.600	4.400	4.898	5.824
ICOMPpeuln	8.400	5.600	4.400	4.898	5.824
ICOMPperf	8.400	5.600	4.400	4.898	5.824

Table 5. The ratio of the correctly selected variables beta regression models for $\phi = 50$

Criteria	Sample size				Average
	$n = 50$	$n = 100$	$n = 300$	$n = 500$	
AIC	99.333	100.000	100.000	100.000	99.833
BIC	99.333	100.000	100.000	100.000	99.833
IBIC	95.333	100.000	100.000	100.000	98.833
SPBIC	98.667	100.000	100.000	100.000	99.667
CICOMP	94.000	100.000	100.000	100.000	98.500
ICOMPifim	97.333	100.000	100.000	100.000	99.333
ICOMPpeu	97.333	100.000	100.000	100.000	99.333
ICOMPpeuln	97.333	100.000	100.000	100.000	99.333
ICOMPperf	97.333	100.000	100.000	100.000	99.333

Table 6. The ratio of the incorrectly selected variables beta regression models for $\phi = 50$

Criteria	Sample size				Average
	$n = 50$	$n = 100$	$n = 300$	$n = 500$	
AIC	18.400	19.600	16.800	12.400	16.800
BIC	7.200	5.200	3.200	1.200	4.200
IBIC	0.000	0.000	0.000	0.000	0.000
SPBIC	4.000	4.000	2.000	0.800	2.700
CICOMP	0.000	0.000	0.000	0.000	0.000
ICOMPifim	1.600	3.200	2.400	1.200	2.100
ICOMPpeu	1.600	3.200	2.400	1.200	2.100
ICOMPpeuln	1.600	3.200	2.400	1.200	2.100
ICOMPperf	1.600	3.200	2.400	1.200	2.100

The simulation results are shown in Table 3-6. When checking the results, we see that AIC was able to select the correct variables in beta regression models. However, covariance-based information criteria give competitive results while determining the true models. ICOMP-type and BIC-type alternative criteria become superior to the classical ones while excluding the incorrect variables. The ratio of the incorrectly selected variables is lower than AIC and BIC. As the sample size increases, the model selection capability of ICOMP-type and BIC-type criteria become much obvious.

5.2 Real data analysis

In this part, we conducted real data analysis applications on two data sets. We considered two real benchmark data sets, Bodyfat and Vitamin. Bodyfat [24] consists $p = 17$ and Vitamin [25] consists $p = 11$ explanatory variables. In each data sets the range of the response variable is (0,1).

Table 7. Kolmogorov-Smirnov test for the beta distribution of the response variables

Data set	$M\phi$	$M(1-\phi)$	Statistic	p
Bodyfat	4.783	20.372	0.061	0.302
Vitamin	8.341	4.310	0.136	0.127

Table 7 shows the goodness of fit tests testing the fitness of the beta distribution for the response variables. The test results reveal that the response variables follow the beta distribution.

To test the predictive performance of the criteria in beta regression models, we used the standardized absolute errors [26] as follows:

$$SAE = \sum_{i=1}^{n} \frac{\varepsilon_i - E(\varepsilon)}{\sigma_\varepsilon} \tag{20}$$

where $\varepsilon = Y - \hat{Y}$ is the prediction error and $E(\varepsilon)$, σ_ε show the standart deviation of the expected values of the errors, respectively.

Table 8. The performance results for Bodyfat data set

Criteria	Significant (V)	Insignificant (V)	SAE
AIC	6	1	189.585
IBIC, SPBIC, CICOMP	3	0	183.348
BIC, ICOMPifim, ICOMPpeu, ICOMPpeuln, ICOMPperf	4	0	178.944

Table 9. The performance results for Vitamin data set

Criteria	Significant (V)	Insignificant (V)	SAE
AIC	7	2	2.08E-02
BIC, SPBIC	7	0	2.09E-02
ICOMPifim, ICOMPpeu, ICOMPpeuln, ICOMPperf	6	0	2.22E-02
IBIC, CICOMP	4	0	2.11E-02

Table 9 shows the real data analysis performance results of the information criteria for the selected beta regression models. The number of selected significant and insignificant variables are represented as Significant (V) and Insignificant (V), respectively.

The models selected by AIC are not satisfactory because of including insignificant variables and relatively higher errors. AIC seems to cause overfitting in the selected beta regression models. Most of the ICOMP-type criteria give promising results with low errors and a high number of significant variables. BIC-type alternative criteria also excluded the irrelevant variables, and the predictive errors are competitive. Although alternative ICOMP-type and BIC-type chose the different beta regression models, they tend to select only the significant variables and provide satisfactory prediction results.

6. Conclusion and Discussion

Model selection is one of the most important fields in regression analysis. The information criteria are much useful to carry out the model selection process. Within the scope of many regression models included in GLM, model selection results differ depending on the information criteria and selection mechanism. In this study, we focused on the model selection task beta regression analysis using several information criteria and a heuristic optimization approach. Firstly, we assessed the alternative information criteria on beta regression analysis for the model selection task.

We implemented the numerical examples with the simulation and real data set analysis. The simulation studies demonstrate the alternative information criteria provide better results since they can exclude the wrong variables in the selected beta regression models. Also, they tend to select correct variables as the sample size increases. Especially, we should emphasize that the alternative criteria provide satisfactory results in the presence of multicollinearity. The alternative ICOMP-type and BIC-type criteria are not affected by the multicollinearity and exclude the irrelevant variables. The increment of the sample size demonstrates this fact more obviously. The real data set examples also provide the model selection skills of the alternative criteria in both estimation and prediction results.

The different type of model selection strategies can be tried such as forward, backward and stepwise selection procedures for beta regression modeling. But it is well-known that the heuristic methods produce more efficient results. When using the information criteria inside the model selection process, the goal is to find the optimum value in the criteria so differential evolution algorithm is quite successful on the optimization process.

The information criteria that we used in the article are favorable on the model selection for beta regression. The classical criteria, especially for high multicollinearity, tend to select redundant variables. Overall, we can conclude that when the variance-covariance matrix exists as a penalty in the information criteria, it can improve the model selection results in beta regression analysis.

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Similarity Measures of Pythagorean Neutrosophic Sets with Dependent Neutrosophic Components Between T and F

Jansi Rajan[1], Mohana Krishnaswamy[2]

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Abstract — Clustering plays an important role in data mining, pattern recognition and machine learning. This paper proposes Pythagorean neutrosophic clustering methods based on similarity measures between Pythagorean neutrosophic sets with T and F are dependent neutrosophic components [PN-Set]. First, we define a generalized distance measure between PN-Sets and propose two distance-based similarity measures of PN-Sets. Then, we present a clustering algorithm based on the similarity measures of PN-Sets to cluster Pythagorean neutrosophic data. Finally, an illustrative example is given to demonstrate the application and effectiveness of the developed clustering methods.

Keywords — *Pythagorean neutrosophic Sets with T and F are dependent neutrosophic components, clustering algorithm, distance measure, similarity measure.*

3. Introduction

Fuzzy sets were firstly initiated by L.A. Zadeh [1] in 1965. Zadeh's idea of fuzzy set evolved as a new tool to deal with uncertainties in real-life problems and discussed only membership function. After the extensions of fuzzy set theory Atanassov [2] generalized this concept and introduced a new set called intuitionistic fuzzy set (IFS) in 1986, which can describe the non-membership grade of an imprecise event along with its membership grade under a restriction that the sum of both membership and non-membership grades does not exceed 1. IFS has its greatest use in practical multiple attribute decision-making problems. In some practical problems, the sum of membership and non-membership degree to which an alternative satisfying attribute provided by decision-maker (DM) may be bigger than 1.

Yager [3] was decided to introduce the new concept known as Pythagorean fuzzy sets. Pythagorean fuzzy sets have a limitation that their square sum is less than or equal to 1. IFS was failed to deal with indeterminate and inconsistent information which exist in beliefs system; therefore, Smarandache [4] in 1995 introduced a new concept known as neutrosophic set (NS) which generalizes fuzzy sets and intuitionistic fuzzy sets and so on. A neutrosophic set includes truth membership, falsity membership and indeterminacy membership.

In 2006, Smarandache introduced, for the first time, the degree of dependence (and consequently the degree of independence) between the components of the fuzzy set, and also between the components of the neutrosophic set. In 2016, the refined neutrosophic set was generalized to the degree of dependence or independence of subcomponents [5]. In neutrosophic set [5], if truth membership and falsity membership are

[1] mathematicsgasc@gmail.com (Corresponding Author); [2] riyaraju1116@gmail.com
[1,2] Department of Mathematics, Nirmala College for Women, Coimbatore, India

100% dependent and indeterminacy is 100% independent, that is $0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 2$. Sometimes in real life, we face many problems which cannot be handled by using neutrosophic for example when $u_A(x) + \zeta_A(x) + v_A(x) > 2$. So, Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PN-SET] of the condition is as their square sum does not exceed 2. Here, T and F are dependent neutrosophic components, and we make $u_A(x), v_A(x) as$ Pythagorean, then $\left(u_A(x)\right)^2 + \left(v_A(x)\right)^2 \leq 1$ with $u_A(x), v_A(x) \in [0,1]$. If $\zeta_A(x)$ is independent of them, then $0 \leq \zeta_A(x) \leq 1$. Then, $0 \leq \left(u_A(x)\right)^2 + \left(\zeta_A(x)\right)^2 + \left(v_A(x)\right)^2 \leq 2$, with $u_A(x), \zeta_A(x), v_A(x) \in [0,1]$.

Recently, Ye [6,7] presented the correlation coefficient of single-valued neutrosophic sets (SVNSsaa0 and the cross-entropy measure of SVNSs and applied them to single-valued neutrosophic decision-making problems. Then, Ye [8] proposed similarity measures between interval neutrosophic sets and their applications in multicriteria decision making. Xu [9] and Zhang [10] proposed a clustering algorithm. J. Ye [11] also introduced the clustering methods using Distance-based similarity measures of single-valued neutrosophic sets.

This paper proposes a Pythagorean neutrosophic clustering algorithm to deal with data represented by Pythagorean neutrosophic set with dependent neutrosophic components between T and F [PN-Set, in short]. We define a generalized distance measure between PN-Sets and propose two distance-based similarity measures of PN-Sets. Then, we present a clustering algorithm based on the similarity measures of PN-Sets to cluster Pythagorean neutrosophic data and gives an illustrative example.

4. Preliminaries

Definition 2.1 [2]

Let E be a universe. An intuitionistic fuzzy set A on E can be defined as follows:

$$A = \{< x, u_A(x), v_A(x) >: x \in E\}.$$

where $u_A: E \to [0,1]\ and\ v_A: E \to [0,1]$ such that $0 \leq u_A(x) + v_A(x) \leq 1$ for any $x \in E$.

Here, $u_A(x)$ and $v_A(x)$ is the degree of membership and degree of non-membership of the element x, respectively.

Definition 2.2 [12,13]

Let X be a nonempty set, and I the unit interval [0,1]. A Pythagorean fuzzy set S is an object having the form $A = \{(x, u_A(x), v_A(x)): x \in X\}$ where the functions $u_A: X \to [0,1]$ and $v_A: X \to [0,1]$ denote respectively the degree of membership and degree of non-membership of each element $x \in X$ to the set P, and $0 \leq \left(u_A(x)\right)^2 + (v_A(x))^2 \leq 1$ for each $x \in X$.

Definition 2.3[4]

Let X be a nonempty set (universe). A neutrosophic set A on X is an object of the form:

$$A = \left\{\left(x, u_A(x), \zeta_A(x), v_A(x)\right): x \in X\right\}.$$

Where $u_A(x), \zeta_A(x), v_A(x) \in [0,1], 0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 2, for\ all\ x\ in\ X$. $u_A(x)$ is the degree of membership, $\zeta_A(x)$ is the degree of indeterminacy and $v_A(x)$ is the degree of non-membership. Here $u_A(x)\ and\ v_A(x)$ are dependent components and $\zeta_A(x)$ is an independent component.

Definition 2.4 [4]

Let X be a nonempty set, and I the unit interval [0,1]. A neutrosophic set A and B of the form

$$A = \{(x, u_A(x), \zeta_A(x), v_A(x)) : x \in X\} \text{ and } B = \{(x, u_B(x), \zeta_B(x), v_B(x)) : x \in X\}$$

Then,

$$A^C = \{(x, v_A(x), 1 - \zeta_A(x), u_A(x)) : x \in X\} \text{ or } A^C = \{(x, v_A(x), \zeta_A(x), u_A(x)) : x \in X\}$$
$$A \cup B = \{(x, \max(u_A(x), u_B(x)), \min(\zeta_A(x), \zeta_B(x)), \min(v_A(x), v_B(x))) : x \in X\}$$
$$A \cap B = \{(x, \min(u_A(x), u_B(x)), \max(\zeta_A(x), \zeta_B(x)), \max(v_A(x), v_B(x)) : x \in X\}$$

3. Distance-Based Similarity Measures between PN-Sets

Definition 3.1

Let X be a nonempty set (universe). A PN-Set M on X is an object of the form:

$$M = \{(x, u_M(x), \zeta_M(x), v_M(x)) : x \in X\},$$

Where $u_M(x), \zeta_M(x), v_M(x) \in [0,1], 0 \leq (u_M(x))^2 + (\zeta_M(x))^2 + (v_M(x))^2 \leq 2$, for all $x \in X$. $u_M(x)$ is the degree of membership, $\zeta_M(x)$ is the degree of indeterminacy and $v_M(x)$ is the degree of non-membership. Here $u_M(x)\ and\ v_M(x)$ are dependent components and $\zeta_M(x)$ is an independent component.

Definition 3.2

Let X be a nonempty set and I the unit interval [0,1]. A PN-Sets M and N of the form

$$M = \{(x, u_M(x), \zeta_M(x), v_M(x)) : x \in X\} \text{ and } N = \{(x, u_N(x), \zeta_N(x), v_N(x)) : x \in X\}.$$

Then,

$$M^C = \{(x, v_M(x), 1 - \zeta_M(x), u_M(x)) : x \in X\} \text{or } M^C = \{(x, v_M(x), \zeta_M(x), u_M(x)) : x \in X\}$$
$$M \cup N = \{(x, \max(u_M(x), u_N(x)), \min(\zeta_M(x), \zeta_N(x)), \min(v_M(x), v_N(x))) : x \in X\}$$
$$M \cap N = \{(x, \min(u_M(x), u_N(x)), \max(\zeta_M(x), \zeta_N(x)), \max(v_M(x), v_N(x))) : x \in X\}$$

For two PN-Sets S and T in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, which are denoted by $M = \{(x_i, u_M(x_i), \zeta_M(x_i), v_M(x_i)) : x_i \in X\}$ and $N = \{(x_i, u_N(x_i), \zeta_N(x_i), v_N(x_i)) : x_i \in X\}$, where $u_M(x_i)$, $\zeta_M(x_i)$, $v_M(x_i)$, $u_N(x_i)$, $\zeta_N(x_i)$, $v_N(x_i) \in [0,1]$ for every $x_i \in X$. Let us consider the weight $w_i (i = 1,2, \dots, n)$ of an element $x_i (i = 1,2, \dots, n)$, with $w_i \geq 0\ (i = 1,2, \dots, n)$, and $\sum_{i=1}^{n} w_i = 1$. Then, we define the generalized PN weighted distance measure:

$$d_p(M, N) = \left\{\frac{1}{3} \sum_{i=1}^{n} w_i \left[\left|u_M^2(x_i) - u_N^2(x_i)\right|^p + \left|\zeta_M^2(x_i) - \zeta_N^2(x_i)\right|^p + \left|v_M^2(x_i) - v_N^2(x_i)\right|^p\right]\right\}^{\frac{1}{p}} \quad (1)$$

where $p > 0$. When $p = 1,2$, we can obtain the PN weighted Hamming distance, and the PN weighted Euclidean distance, respectively, as follows:

$$d_1(M,N) = \frac{1}{3}\sum_{i=1}^{n} w_i \left[\left|u_M^2(x_i) - u_N^2(x_i)\right| + \left|\zeta_M^2(x_i) - \zeta_N^2(x_i)\right| + \left|v_M^2(x_i) - v_N^2(x_i)\right|\right] \quad (2)$$

$$d_2(M,N) = \left\{\frac{1}{3}\sum_{i=1}^{n} w_i \left[\left|u_M^2(x_i) - u_N^2(x_i)\right|^2 + \left|\zeta_M^2(x_i) - \zeta_N^2(x_i)\right|^2 + \left|v_M^2(x_i) - v_N^2(x_i)\right|^2\right]\right\}^{\frac{1}{2}} \quad (3)$$

Therefore, Eqs. (2) and (3) are the special cases of (1). Then, for the distance measure, we have the following proposition.

Proposition 3.3. The above-defined distance $d_p(M,N)\ for\ p > 0$ satisfies the following properties:

(DP1) $0 \le d_p(M,N) \le 1$;

(DP2) $d_p(M,N) = 0$ if and only if M = N;

(DP3) $d_p(M,N) = d_p(N,M)$;

(DP4) If $M \subseteq N \subseteq O$, O is a PN-Set in X, then $d_p(M,O) \ge d_p(M,N)\ \ and\ \ d_p(M,O) \ge d_p(N,O)$.

Proof:

It is easy to see that $d_p(M,N)$ satisfies the properties (DP1)-DP43). Therefore, we only prove (DP4). Let M ⊆ $N \subseteq O$, then, $u_M(x_i) \le u_N(x_i) \le u_O(x_i), \zeta_M(x_i) \ge \zeta_N(x_i) \ge \zeta_O(x_i)\ and\ v_M(x_i) \ge v_N(x_i) \ge v_O(x_i)$ for every $x_i \in X$. Also, $u_M^2(x_i) \le u_N^2(x_i) \le u_O^2(x_i), \zeta_M^2(x_i) \ge \zeta_N^2(x_i) \ge \zeta_O^2(x_i)$, and $v_M^2(x_i) \le v_N^2(x_i) \le v_O^2(x_i)$, for every $x_i \in X$.

Then, we obtain the following relations:

$$\left|u_M^2(x_i) - u_N^2(x_i)\right|^p \le \left|u_M^2(x_i) - u_O^2(x_i)\right|^p, \left|u_N^2(x_i) - u_O^2(x_i)\right|^p \le \left|u_M^2(x_i) - u_O^2(x_i)\right|^p,$$

$$\left|\zeta_M^2(x_i) - \zeta_N^2(x_i)\right|^p \le \left|\zeta_M^2(x_i) - \zeta_O^2(x_i)\right|^p, \left|\zeta_N^2(x_i) - \zeta_O^2(x_i)\right|^p \le \left|\zeta_M^2(x_i) - \zeta_O^2(x_i)\right|^p,$$

$$\left|v_M^2(x_i) - v_N^2(x_i)\right|^p \le \left|v_M^2(x_i) - v_O^2(x_i)\right|^p, \left|v_N^2(x_i) - v_O^2(x_i)\right|^p \le \left|v_M^2(x_i) - v_O^2(x_i)\right|^p,$$

Hence,

$$\left|u_M^2(x_i) - u_N^2(x_i)\right|^p + \left|\zeta_M^2(x_i) - \zeta_N^2(x_i)\right|^p + \left|v_M^2(x_i) - v_N^2(x_i)\right|^p$$

$$\le \left|u_M^2(x_i) - u_O^2(x_i)\right|^p + \left|\zeta_M^2(x_i) - \zeta_O^2(x_i)\right|^p + \left|v_M^2(x_i) - v_O^2(x_i)\right|^p$$

$$\left|u_N^2(x_i) - u_O^2(x_i)\right|^p + \left|\zeta_N^2(x_i) - \zeta_O^2(x_i)\right|^p + \left|v_N^2(x_i) - v_O^2(x_i)\right|^p$$

$$\le \left|u_M^2(x_i) - u_O^2(x_i)\right|^p + \left|\zeta_M^2(x_i) - \zeta_O^2(x_i)\right|^p + \left|v_M^2(x_i) - v_O^2(x_i)\right|^p$$

Combining the above inequalities with the above-defined distance formula (1), we can obtain $d_p(M,O) \ge d_p(M,N)$ and $d_p(M,O) \ge d_p(N,O)$ for $p > 0$. Thus, the property (DP4) is satisfied.

This completes the proof.

Note that similarity and distance (dissimilarity) measures are complementary: when the first increases, the second decreases. Normalized distance measure and similarity measure are dual concepts.

Thus, $S(M,N) = 1 - d(M,N)$ and vice versa. The properties of distance measures below are complementary to those of similarity measure.

Proposition 3.4 Let A and B be two PN-Sets in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$; $S(A, B)$ is called a Pythagorean neutrosophic similarity measure, which should satisfy the following properties:

(SP1) $0 \leq S(M, N) \leq 1$;

(SP2) $S(M, N) = 0$ if and only if A = B;

(SP3) $S(M, N) = S(N, M)$;

(SP4) If $M \subseteq N \subseteq O$, C is a PN-Set in X, then $S(M, O) \geq S(M, N)$ and $S(M, O) \geq S(N, O)$.

Assume that there are two PN-sets $M = \{(x_i, u_M(x_i), \zeta_M(x_i), v_M(x_i)) : x_i \in X\}$ and $N = \{(x_i, u_N(x_i), \zeta_N(x_i), v_N(x_i)) : x_i \in X\}$ in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Thus, according to the relationship between the distance and the similarity measure, we can obtain the following PN similarity measure:

$$S_1(M, N) = 1 - d_p(M, N) = 1 - \left\{\frac{1}{3}\sum_{i=1}^{n} w_i[|u_M^2(x_i) - u_N^2(x_i)|^p + |\zeta_M^2(x_i) - \zeta_N^2(x_i)|^p + |v_M^2(x_i) - v_N^2(x_i)|^p]\right\}^{\frac{1}{p}} \quad (4)$$

Obviously, we can easily prove that $S_1(M, N)$ satisfies the properties (SP1) - (SP4) in Proposition 2 by the relationship between the distance and the similarity measure and the proof of Proposition 1, which is omitted here.

Furthermore, we can also propose another PN similarity measure:

$$S_2(M, N) = \frac{1 - d_p(M, N)}{1 + d_p(M, N)} = \frac{1 - \left\{\frac{1}{3}\sum_{i=1}^{n} w_i[|u_M^2(x_i) - u_N^2(x_i)|^p + |\zeta_M^2(x_i) - \zeta_N^2(x_i)|^p + |v_M^2(x_i) - v_N^2(x_i)|^p]\right\}^{\frac{1}{p}}}{1 + \left\{\frac{1}{3}\sum_{i=1}^{n} w_i[|u_M^2(x_i) - u_N^2(x_i)|^p + |\zeta_M^2(x_i) - \zeta_N^2(x_i)|^p + |v_M^2(x_i) - v_N^2(x_i)|^p]\right\}^{\frac{1}{p}}} \quad (5)$$

Then, the similarity measure $S_2(M, N)$ also satisfied the properties (SP1) - (SP4) in Proposition 2.

Proof:

It is easy to see that $S_2(M, N)$ satisfies the properties (SP1) - (SP3). Therefore, we only prove the property (SP4).

As we obtain $d_p(M, O) \geq d_p(M, N)$ and $d_p(M, O) \geq d_p(N, O)$ for p > 0 from the property (DP4) in Proposition 1, there are $1 - d_p(M, N) \geq 1 - d_p(M, O), 1 - d_p(N, O) \geq 1 - d_p(M, O), 1 + d_p(M, N) \leq 1 + d_p(M, O)$ and $1 + d_p(N, O) \leq 1 + d_p(M, O)$.Then, there are the following inequalities:

$$\frac{1 - d_p(M, N)}{1 + d_p(M, N)} \geq \frac{1 - d_p(M, O)}{1 + d_p(M, O)}$$

and

$$\frac{1 - d_p(N, O)}{1 + d_p(N, O)} \geq \frac{1 - d_p(M, O)}{1 + d_p(M, O)}$$

Then, there are $S(M, O) \leq S(M, N) and\ S(M, O) \leq S(N, O)$. Hence, the property (SP4) is satisfied.

This completes the proof.

Example 3.5:

Assume that we have the following three PN-Sets in a universe of distance $X = \{x_1, x_2\}$:

$$A = \{(x_1, 0.1, 0.9, 0.6), (x_1, 0.1, 0.9, 0.6)\}$$

$$B = \{(x_1, 0.7, 0.8, 0.4), (x_1, 0.4, 0.6, 0.7)\}$$

$$C = \{(x_1, 0.8, 0.1, 0.3), (x_1, 0.4, 0.3, 0.1)\}$$

Then, there are $A \subseteq B \subseteq C$, with $u_A(x_i) \leq u_B(x_i) \leq u_C(x_i), \zeta_A(x_i) \leq \zeta_A(x_i) \leq \zeta_B(x_i)\zeta_C(x_i)$ and $v_A(x_i) \leq v_B(x_i)v_C(x_i)$ for each x_i in X=$\{x_1, x_2\}$, and the weight vector $w = (0.4,0.6)^T$.

By applying Eq. (4) (take p = 1), the similarity measures between the PN-Sets are as follows:

$$S_1(A,B) = 0.7427, S_1(B,C) = 0.7367, S_1(A,C) = 0.4793.$$

Hence, $S_1(A,C) \leq S_1(A,B)$ and $S_1(A,C) \leq S_1(B,C)$

By applying Eq. (5) for p = 1, the similarity measures between the PN-Sets are as follows:

$$S_2(A,B) = 0.5907, S_2(B,C) = 0.5832, S_2(A,C) = 0.3152.$$

Hence, $S_2(A,C) \leq S_2(A,B)$ and $S_2(A,C) \leq S_2(B,C)$.

4. Clustering Algorithm Based on the Similarity Measures of PN-Sets

In this section, we can apply the proposed similarity measures of PN-Sets to clustering analysis under a PN environment. Based on the intuitionistic fuzzy clustering algorithm proposed by Zhang [1] and Xu [9].

Definition 4.1 Assume that $A = (A_1, A_2, \dots, A_m)$ is a set of PN-Sets and $C = (S_{ij})_{m\times m}$ is a similarity matrix, where $S_{ij} = S_K(A_i, A_j)(k = 1,2)$ and $S_{ij} \in [0,1]$ for $i, j = 1,2, \dots, m$, with $S_{ii} = 1$ for $i = 1,2, \dots, m$, and $S_{ij} = S_{ji}$, for $i, j = 1,2, \dots, m$.

Definition 4.2 [9,10] Let $C = (S_{ij})_{m\times m}$ be a similarity matrix, if $C^2 = C_{\circ}\, C = (\bar{S}_{ij})_{m\times m}$, then C^2 is called a composition matrix of C, where $\bar{S}_{ij} = \max\limits_{k}\{min(S_{ik}, S_{kj})\}$, for $i, j = 1,2, \dots, m$.

Definition 4.3 [9,10] Let $C = (S_{ij})_{m\times m}$ be a similarity matrix, if $C^2 \subseteq C$, i.e., $\bar{S}_{ij} \leq S_{ij}$ for $i, j = 1,2, \dots, m$, then C is called an equivalent similarity matrix.

Definition 4.4 [9,10] Let $C = (S_{ij})_{m\times m}$ be a similarity matrix. Then, after finite time compositions of C:

$$C \to C^2 \to C^4 \to \cdots \to C^{2^k} \to \cdots, \qquad (6)$$

there must exist a positive integer k such that $C^{2^k} = C^{2^{(k+1)}}$, then C^{2^k} is also an equivalent similarity matrix.

Definition 4.5 [9,10] Let $C = (S_{ij})_{m\times m}$ be an equivalent similarity matrix. Then, $C_\lambda = (S_{ij}^\lambda)_{m\times m}$ is called the λ-cutting matrix of C, where

$$S_{ij}^\lambda = \begin{cases} 0, S_{ij} < \lambda; \\ 1, S_{ij} \geq \lambda \end{cases} \text{ for } i, j = 1,2, \dots, m, \qquad (7)$$

and λ is the confidence level with $\lambda \in [0,1]$.

Assume that $A = \{A_1, A_2, \dots, A_m\}$ is a set of PN-Set, where $A_j = \left\{\left(x_i, u_{A_j}, \zeta_{A_j}, v_{A_j}\right): x_i \in X\right\}$ $(j = 1,2, \dots, m)$ in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ is a PN. Let w_i be the weight for each element x_i $(i = 1,2, \dots, n)$, with $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$. Then, we can give the algorithm of clustering PN-Sets as follows:

Step 1. By use of Eqs. (4) or (5), one can calculate the similarity measure degrees of PN-Sets, and then construct a similarity matrix $C = (S_{ij})_{m\times m}$, where $S_{ij} = S_k(A_i, A_j)$ $(k = 1,2)$ for $i, j = 1,2, \dots, m$.

Step 2. The process of building the composition matrices is repeated until it holds that

$$C \to C^2 \to C^4 \to \cdots \to C^{2^k} = C^{2^{(k+1)}}$$

which implies that C^{2^k} is an equivalent similarity matrix, which is denoted by $\bar{C} = (\bar{S}_{ij})_{m\times m}$.

Step 3. For the equivalent similarity matrix $\bar{C} = (\bar{S}_{ij})_{m\times m}$, we can construct a λ-cutting matrix $\bar{C_\lambda} = (\bar{S}^{\lambda}_{ij})_{m\times m}$ of $\bar{C}$ by Eq(7); if all the elements of the ith row or column in $\bar{C_\lambda}$ are the same as the corresponding elements of the ith row or column, we conceive object sets A_i *and* A_j are the same class.

5. Illustrative Example

A car market is going to classify five different cars of A_j $(j = 1,2,\dots,5)$. Every car has six evaluation attributes: (i) x_1, fuel consumption; (ii) x_2, price; (iii) x_3, coefficient friction; (iv) x_4, comfortable degree; (v) x_5, safety. The characteristics of each car under the six attributes are represented by the form of PN-SETs, and then the Pythagorean neutrosophic data are as follows:

$$A_1 = \{x_1,(0.5,0.9,0.8), x_2,(0.6,0.7,0.7), x_3,(0.4,0.2,0.5), x_4,(0.7,0.8,0.5), x_5,(0.1,0.6,0.3)\}$$

$$A_2 = \{x_1,(0.1,0.7,0.8), x_2,(0.6,0.9,0.8), x_3,(0.5,0.2,0.4), x_4,(0.3,0.5,0.1), x_5,(0.7,0.3,0.5)\}$$

$$A_3 = \{x_1,(0.1,0.7,0.8), x_2,(0.5,0.6,0.7), x_3,(0.2,0.8,0.6), x_4,(0.4,0.3,0.9), x_5,(0.9,0.1,0.4)\}$$

$$A_4 = \{x_1,(0.3,0.6,0.7), x_2,(0.8,0.7,0.6), x_3,(0.1,0.9,0.5), x_4,(0.4,0.6,0.2), x_5,(0.5,0.2,0.7)\}$$

$$A_5 = \{x_1,(0.4,0.6,0.7), x_2,(0.9,0.6,0.1), x_3,(0.8,0.9,0.6), x_4,(0.5,0.2,0.3), x_5,(0.6,0.2,0.5)\}$$

If the weight vector of the attributes, x_i $(i = 1,2,3,4,5,6)$ $is\ w = \left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)^T$, then we utilize the two Pythagorean neutrosophic similarity measures to classify the five different cars of A_j $(j = 1,2,3,4,5)$ by the Pythagorean neutrosophic clustering algorithms.

5.1 Clustering Analysis using Eq. (4)

Step 1. Utilize the similarity measure formula (4) (take p = 2) to calculate the similarity measure between each pair of PN-SETs A_i *and* A_j $(i, j = 1,2,3,4,5)$ and construct the following similarity matrix:

$$C = \begin{bmatrix} 1 & 0.7358 & 0.6287 & 0.6824 & 0.6128 \\ 0.7358 & 1 & 0.7596 & 0.7339 & 0.6574 \\ 0.6287 & 0.7596 & 1 & 0.7095 & 0.6995 \\ 0.6824 & 0.7339 & 0.7095 & 1 & 0.7742 \\ 0.6168 & 0.6574 & 0.6995 & 0.7742 & 1 \end{bmatrix}$$

Step 2. Obtain equivalent similarity matrices by limited time composition of C:

$$C^2 = \begin{bmatrix} 1 & 0.7358 & 0.7358 & 0.7339 & 0.6574 \\ 0.7358 & 1 & 0.7596 & 0.7339 & 0.6574 \\ 0.7358 & 0.7596 & 1 & 0.7339 & 0.6574 \\ 0.7339 & 0.7339 & 0.7339 & 1 & 0.6574 \\ 0.6574 & 0.6574 & 0.6574 & 0.6574 & 1 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 1 & 0.7358 & 0.7358 & 0.7339 & 0.6574 \\ 0.7358 & 1 & 0.7596 & 0.7339 & 0.6574 \\ 0.7358 & 0.7596 & 1 & 0.7339 & 0.6574 \\ 0.7339 & 0.7339 & 0.7339 & 1 & 0.6574 \\ 0.6574 & 0.6574 & 0.6574 & 0.6574 & 1 \end{bmatrix}$$

Obviously, $C^4 = C^2$ implies that C^2 is an equivalent similarity matrix, denoted by $\bar{C}$.

Step 3. When λ has different values, we can construct a λ-cutting matrix $\bar{C_\lambda} = (\bar{S}^{\lambda}_{ij})_{m\times m}$ of $\bar{C}$ by Eq.(7) and obtain different categories, which give the following discussion:

(i) If $0 \leq \lambda \leq 0.6574$

$$\bar{C}_\lambda = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then the cars are the same category:$\{A_1, A_2, A_3, A_4, A_5\}$.

(ii) If $0.6574 < \lambda \leq 0.7339$

$$\bar{C}_\lambda = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into two categories: $\{A_1, A_2, A_3, A_4\}, \{A_5\}$.

(iii) If $0.7339 < \lambda \leq 0.7358$

$$\bar{C}_\lambda = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into three categories: $\{A_1, A_2, A_3\}, \{A_4\}, \{A_5\}$.

(iv) If $0.7358 < \lambda \leq 0.7596$

$$\bar{C}_\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into four categories: $\{A_1\}, \{A_2, A_3\}, \{A_4\}, \{A_5\}$.

(v) If $0.7596 < \lambda \leq 1$

$$\bar{C}_\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into five categories: $\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$.

5.2 Clustering Analysis Using Eq. (5)

Step 1. Utilize the similarity measure formula (5) (take p = 2) to calculate the similarity measure between each pair of PN-SETs $A_i \; and \; A_j \; (i, j = 1,2,3,4,5)$ and construct the following similarity matrix:

$$C = \begin{bmatrix} 1 & 0.5820 & 0.4585 & 0.5179 & 0.4459 \\ 0.5820 & 1 & 0.6124 & 0.5797 & 0.4896 \\ 0.4585 & 0.6124 & 1 & 0.5498 & 0.5379 \\ 0.5179 & 0.5797 & 0.5498 & 1 & 0.6316 \\ 0.4459 & 0.4896 & 0.5379 & 0.6316 & 1 \end{bmatrix}$$

Step 2. Obtain equivalent similarity matrices by limited time composition of C:

$$C^2 = \begin{bmatrix} 1 & 0.5820 & 0.5820 & 0.5797 & 0.4896 \\ 0.5820 & 1 & 0.6124 & 0.5797 & 0.4896 \\ 0.5820 & 0.6124 & 1 & 0.5797 & 0.4896 \\ 0.5797 & 0.5797 & 0.5797 & 1 & 0.4896 \\ 0.4896 & 0.4896 & 0.4896 & 0.4896 & 1 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 1 & 0.5820 & 0.5820 & 0.5797 & 0.4896 \\ 0.5820 & 1 & 0.6124 & 0.5797 & 0.4896 \\ 0.5820 & 0.6124 & 1 & 0.5797 & 0.4896 \\ 0.5797 & 0.5797 & 0.5797 & 1 & 0.4896 \\ 0.4896 & 0.4896 & 0.4896 & 0.4896 & 1 \end{bmatrix}$$

Obviously, $C^4 = C^2$ implies that C^2 is an equivalent similarity matrix, denoted by $\bar{C}$.

Step 3. When λ has different values, we can construct a λ-cutting matrix $\bar{C_\lambda} = \left(\bar{S}_{ij}^{\lambda}\right)_{m \times m}$ of $\bar{C}$ by Eq.(7) and obtain different categories, which give the following discussion:

(i) If $0 \leq \lambda \leq 0.4896$

$$\bar{C_\lambda} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then the cars are the same category:$\{A_1, A_2, A_3, A_4, A_5\}$.

(ii) If $0.4896 < \lambda \leq 0.5797$

$$\bar{C_\lambda} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into two categories: $\{A_1, A_2, A_3, A_4\}, \{A_5\}$.

(iii) If $0.5797 < \lambda \leq 0.5820$

$$\bar{C_\lambda} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into three categories: $\{A_1, A_2, A_3\}, \{A_4\}, \{A_5\}$.

(iv) If $0.5820 < \lambda \leq 0.6124$

$$\bar{C_\lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into four categories: $\{A_1\}, \{A_2, A_3\}, \{A_4\}, \{A_5\}$.

(v) If $0.6124 < \lambda \leq 1$

$$\bar{C_\lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then the cars can be divided into five categories: $\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$.

Conclusion

This paper introduced a generalized PN weighted distance measure and presented two distance-based similarity measures in a PN setting. Then, a PN clustering algorithm was established based on the two similarity measures. Finally, an illustrative example was given to demonstrate the application and effectiveness of the PN clustering methods.

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Brief Review of Soft Sets and Its Application in Coding Theory

Samy Mohammed Mostafa [1], Fatema Faisal Kareem [2], Hussein Ali Jad [3]

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Abstract In this paper, we will focus to one of the recent applications of PU-algebras in the coding theory, namely the construction of codes by soft sets PU-valued functions. First, we shall introduce the notion of soft sets PU-valued functions on PU-algebra and investigate some of its related properties. Moreover, the codes generated by a soft sets PU-valued function are constructed and several examples are given. Furthermore, example with graphs of binary block code constructed from a soft sets PU-valued function is constructed.

Keywords *- PU-algebra, Soft PU-algebra, code soft PU-algebra*

1. Introduction

Imai and Is´eki [1] in 1966 introduced the notion of a BCK-algebra. Is´eki [2] introduced BCI-algebras as a super class of the class of BCK-algebras. In [3], Hu and Li introduced a wide class of abstract algebras, BCH-algebras. They are shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Elkabany et al, in [4] introduced a new algebraic structure called PU-algebra, and they investigated severed basic properties. Moreover, they derived new view of several ideals on PU-algebra. Molodtsov [5] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free form the difficulties that have troubled the usual theoretical approaches. Maji et al [6,7] described the application of soft theory and studied several operations on the soft sets. Many Mathematicians have studied the concept of soft set of some algebraic structures. For example, see [8-13]. Coding theory is a mathematical domain with many applications in information theory, for more details see [14]. Various type of codes and their connections with other mathematical objects have been intensively studied. One of the recent applications was given in the Coding theory are BCK/ Hilbert/ R_0-algebras see [15-18]. In [12,18] provided an algorithm which allows to find a BCK/-algebra starting from a given binary block code. In [17] the authors presented some new connections between BCK- algebras and binary block codes. In [19,20] the authors established block-codes by using the notion of KU-valued functions.
In this paper, we will focus to one of the recent applications of PU-algebras in the coding theory, namely the Construction of codes by soft sets PU-valued functions. First, we shall introduce the notion of soft sets PU-valued functions on PU-algebra and investigate some of its related properties. Moreover, the codes generated by a soft sets PU-valued function are constructed and several Examples are given. Furthermore, Example with graphs of binary block code constructed from a soft sets PU-valued function is constructed.

[1] samymostafa@yahoo.com; [2] fa_sa20072000@yahoo.com; [3] hussein.aligad@yahoo.com
[1] Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt
[2] Department of Mathematics, Ibn-Al-Haitham college of Education, University of Baghdad, Iraq.
[3] Informatics Research Institute, City for Scientific Research and Technological Applications, Borg El Arab, Alexandria, Egypt

2. **Preliminaries**

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article.

Definition 2.1. [4] A PU-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:

(i) $0 * x = x$,

(ii) $(x * z) * (y * z) = y * x$, for any $x, y, z \in X$

On X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $y * x = 0$.

Example 2.2. [4] Let X = {0, 1, 2, 3, 4} be a set and $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then, $(X,*,0)$ is a PU-algebra.

Proposition 2.3. [4] In a PU-algebra $(X,*,0)$ the following hold, for all $x, y, z \in X$,

(a) $x * x = 0$

(b) $(x * z) * z = x$

(c) $x * (y * z) = y * (x * z)$

(d) $x * (y * x) = y * 0$

(e) $(x * y) * 0 = y * x$

(f) If $x \leq y$, then $x * 0 = y * 0$

(g) $(x * y) * 0 = (x * z) * (y * z)$

(h) $x * y \leq z$ if and only if $z * y \leq x$

(i) $x \leq y$ if and only if $y * z \leq x * z$

(j) In a PU-algebra $(X,*,0)$, the following are equivalent:

(1) $x = y$, (2) $x * z = y * z$, (3) $z * x = z * y$

(k) The right and the left cancellation laws hold in X .

(l) $(z * x) * (z * y) = x * y$

(m) $(x * y) * z = (z * y) * x$

(n) $(x * y) * (z * u) = (x * z) * (y * u)$ for all x, y, z and $u \in X$

Lemma 2.4. [4] If $(X,*,0)$ is a PU-algebra, then $(X, \leq)$ is a partially ordered set.

Definition 2.5. [4] A non-empty subset S of a PU-algebra $(X,*,0)$ is called a sub-algebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition 2.6. [4] A non-empty subset I of a PU-algebra $(X, *, 0)$ is called a ***new ideal*** of X if,

(i) $0 \in I$,

(ii) $(a * (b * x)) * x \in I$, for all $a, b \in I$ and $x \in X$.

Theorem 2.7 [4] Any sub-algebra S of a PU-algebra X is a ***new ideal*** of X.

Example 2.8 [4] Let $X = \{0, a, b, c\}$ be a set with $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then, $(X, *, 0)$ is a PU-algebra. It is easy to show that $I_1 = \{0, a\}$, $I_2 = \{0, b\}$, $I_3 = \{0, c\}$ are ***new ideals*** of X.

3. Brief review of soft set with examples in coding and fuzzy

Definition 3,1. [5] Molodtsov defined the notion of a soft sets as follows. Let U be an initial universe and E be the set of parameters. The parameters are usually "attributes, characteristics or properties of an object". Let $P(U)$ denote the power set of U and A is a subset of E.A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.In other words, a soft set over a universe is a U parameterized family of subsets of the universe U . For $e \in A$, $F(e)$ may be considered as the set of e-elements or e-approximate elements of the soft set (F, A) . Thus $(F, A) = \{F(e) \in P(U) : e \in A \subseteq E\}$. As an illustration, let us consider the following:

Example 3.2. (***soft***). Suppose a universe U is the set of eight Cars $U = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ be the set of Cars under consideration, E be a set of parameters.

$$E = \left\{ \begin{array}{l} e_1 = expensive, e_2 = beautiful, e_3 = manual\ gear, e_4 = cheap, \\ e_5 = automatic\ gear, e_6 = in\ good\ repair, e_7 = in\ bad\ \ repair \end{array} \right\}.$$

Then, the soft set (F, E) describes the attractiveness of the cars. which say Mr. X wants to buy. In this case, to define the soft set (F, E) means to point out the cars for each parameter, i.e. expensive, beautiful, manual gear, cheap, automatic gear, etc. Let $A = \{e_1, e_2, e_6\} \subseteq E$ and Now consider the mapping $F : A \to P(U)$ is given by $F(e_1) = \{C_2, C_3\}, F(e_2) = \{C_1, C_3, C_5\}, F(e_6) = \{C_1, C_3, C_6\}$.

Then, the soft set (F, A) is a parameterized family $\{F(e_i), i = 1,2,3\}$ of subsets of the universe given by

$$(F, A) = \{\{C_2, C_3\}, \{C_1, C_3, C_5\}, \{C_1, C_3, C_6\}\}$$

for example, $F(e_1)$ means car (expensive) whose functional value, called the e_1 - approximate value set, is the set $\{C_2, C_3\}$.Thus we can view the soft set (F, A) as consisting of a collection of approximations given by

$$(F, A) = \{F(e_1) = \{C_2, C_3\}, F(e_2) = \{C_1, C_3, C_5\}, F(e_6) = \{C_1, C_3, C_6\}\}$$

Each approximation has two parts:

(i) A predicate $F(e_1)\, or\, F(e_2)\ or\ F(e_6)$ and

(ii) The approximate set $\{C_2, C_3\}\ or\ \{C_1, C_3, C_5\}\ or \{C_1, C_3, C_6\}$, respectively.

The soft set (F, A) can also be represented by the set of ordered pairs given by

$$(F, A) = \{(e_1, F(e_1)), (e_2, F(e_2)), (e_6, F(e_6))\}$$

$$(F, A) = \{(e_1, \{C_2, C_3\}), (e_2, \{C_1, C_3, C_5\}),\ (e_6, \{C_1, C_3, C_6\})\}$$

It is worth noting that the sets $F(e), e \in A$ may be arbitrary, may be empty or may have non-empty intersection. Also, the soft set (F, A) can be divided by F_A.

Example 3.3 (coding soft) To store a soft set in a computer, a two-dimensional table is used to represent it. Table 1 (below) shows the tabular representation of the soft set (F, A), where if $h_i \in F(e_i)$, $then\, h_{i,j} = 1$, $otherwise\ h_{i,j} = 0$, where $h_{i,j}$ are the entries in the table.

U	e_1	e_2	e_6	Choice value
C_1	0	1	1	2
C_2	1	0	0	1
C_3	1	1	1	3
C_4	0	0	0	0
C_5	0	1	0	1
C_6	0	0	1	1
C_7	0	0	0	0
C_8	0	0	0	0

Let U be a universe and let A be a fuzzy set on the universe U, characterized by its membership function μ_A, such that $\mu_A : U \to [0,1], A \subseteq U$. Thus, the fuzzy set A can be completely defined as a set of ordered pairs given by $A = \{\langle x, \mu(x) : x \in U \rangle\}, \mu(x) \in [0,1]$.

Now let us consider the family of α -level sets for μ_A, given by $F(\alpha) = \{x \in U : \mu_A(x) \geq \alpha\}, \alpha \in [0,1]$, such that given F, we can find $\mu_A(x)$ by the formula: $\mu_A(x) = \sup\{\alpha \in [0,1] : x \in F(\alpha)\}$. Then, every Zadeh's fuzzy set A may be considered as the soft set $(F, [0,1])$. As an illustration, let us consider the following example.

Example 3.4. (fuzzy soft). Suppose that $U = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ and that we consider the single parameter quality of cars which are characterized by the value set whose terms are {expensive, beautiful, manual gear and cheap}. Let the terms beautiful and cheap for example be associated with its own fuzzy set as follows:

$$F_{beautiful} = \{(C_1,0.2),(C_2,0.7),(C_5,0.9),(C_6,0.1)\}$$

$$F_{cheap} = \{(C_1,0.9),(C_2,0.3),(C_3,0.1),(C_4,0.1),(C_5,0.2)\}$$

Then, the α – level set of $F_{beautiful}$ and F_{cheap} are given by;

$$F_{beautiful}(0.2) = \{C_1, C_2, C_5\},\ F_{cheap}(0.2) = \{C_1, C_2, C_4, C_5\}$$

$$F_{beautiful}(0.7) = \{C_2, C_5\}, F_{cheap}(0.7) = \{C_1\}$$

$$F_{beautiful}(0.9) = \{C_5\}, F_{cheap}(0.9) = \{C_1\}$$

$$F_{beautiful}(0.1) = \{C_1, C_2, C_5, C_6\}, F_{cheap}(0.1) = \{C_1, C_2, C_3, C_4, C_5\}$$

$$F_{beautiful}(0.3) = \{C_2, C_5\}, F_{cheap}(0.3) = \{C_1, C_2\}, \text{ where here } A = \{0.1,0.2,0.3,0.7,0.9\} \subset [0,1]$$

which can be regarded as the parameter set such that $F_{beautiful} : A \to P(U)$ gives the approximate value set $F_{beautiful}(\alpha)$, $for\ \alpha \in A$ .Thus, the soft set for the fuzzy set $F_{beautiful}$ can be written as:

$$(F_{beautiful}, A) = \{\ (0.1, \{C_1, C_2, C_5, C_6\}), (0.2, \{C_1, C_2, C_5\}), (0.3, \{C_2, C_5\}), (0.7, \{C_2, C_5\}), (0.9, \{C_5\})\ \}.$$

Similarly, the soft set for the fuzzy set F_{cheap} is given by;

$$(F_{cheap}, B) = \{\ (0.1, \{C_1, C_2, C_3, C_4, C_5\}), (0.2, \{C_1, C_2, C_4, C_5\}), (0.3, \{C_1, C_2\}), (0.7, \{C_1\}), (0.9, \{C_1\})\ \},$$

where $A, B = \{0.1,0.2,0.3,0.7,0.9\} \subset [0,1]$

Definition 3.5. The complement of a soft set (F, A) is denoted by (F^C, A) and is defined by $(F, A)^C$, where $F^C : A \to P(U)$ is a mapping given by $F^C(x) = U - F(x)\ \forall\ x \in X$.

Let the terms beautiful and cheap for example be associated with its own fuzzy set as follows:

$$F_{beautiful} = \{(C_1,0.2),(C_2,0.7),(C_5,0.9),(C_6,0.1)\}$$

$$F_{cheap} = \{(C_1,0.9),(C_2,0.3),(C_3,0.1),(C_4,0.1),(C_5,0.2)\}, \text{ then}$$

$$F^C{}_{beautiful} = \{(C_1,0.8),(C_2,0.3),(C_5,0.1),(C_6,0.9)\}, \text{ and}$$

$$F^C{}_{cheap} = \{(C_1,0.1),(C_2,0.7),(C_3,0.9),(C_4,0.9),(C_5,0.8)\}.$$

4. Soft PU-algebras

Mostafa et al [21] introduced the concept soft PU-algebras. Let X and A be a PU-algebra and a nonempty set, respectively. A pair (F, A) is called a soft set over X if and only if F is a mapping from a set of A into the power set of X . That is, $F : A \to P(X)$ such that $F(x) = \phi$ if $x \notin A$. A soft set over X can be represented by the set of ordered pairs $\{(x, F(x)) : x \in A, F(x) \in P(X)\}$.

It is clear to see that a soft set is a parameterized family of subsets of the set X .

Definition 4.1. Let (F,A) be a soft set over X. Then, (F,A) is called a soft PU-algebra over X, if $F(x)$ is a ***new ideal*** of X, for all $x \in A$.

Example 4.2. Let X = {0, a, b, c} be a set provided in Example 2.8. Define a mapping $F : X \to P(X)$ by: F (0) = {0}, F (a) = {0, a}, F (b) = {0, b} and F (c) = {0, c}. It is clear that (F,X) is a soft PU-algebra over X.

Definition 4.3. Let (F,A) and (G,B) be two soft PU-algebras over X. Then, (F,A) is called a soft PU-subalgebra of (G,B), denoted by $(F,A) \prec (G,B)$, if it satisfies:

(i) $A \subset B$,

(ii) $F(x)$ is sub-algebra of $G(x)$, for all $x \in A$.

Proposition 4.4. A soft set (F,A) over X is a soft PU-algebra, if and only if each $\Phi \neq F(\varepsilon)$ is a ***new ideal*** of X, for all $\varepsilon \in A$.

Theorem 4.5. Let (F,A) and (G,B) be two soft PU-algebras over X. If $A \bigcap B \neq \phi$, then the intersection $(F,A) \tilde{\bigcap} (G,B)$ is a soft PU-algebra over X.

5. Codes generate by a soft PU-algebra

Definition 5.1. Let $F : A \to P(X)$ be a mapping from a set $A \subseteq X$ into the power set of X, then F is called a soft PU-function on A.

Definition 5.2. A cut function of F, for $p \in P(X)$ is defined to be a mapping $F_p : A \to \{0,1\}$ such that $F_p(x) = 1 \Leftrightarrow p \subseteq F(x)$, for all x in A.

Obviously, F_p is the characteristic function of p-level subset (or, a p-cut) $f_p = \{x \in A : F_p(x) = 1\}$.

Example 5.3 Let $A = \{e_1, e_2\} \subseteq E$ and let X= { C_0, C_1, C_2, C_3 } be the set of Cars with the following Cayley table:

$*$	C_0	C_1	C_2	C_3
C_0	C_0	C_1	C_2	C_3
C_1	C_1	C_0	C_3	C_2
C_2	C_2	C_3	C_0	C_1
C_3	C_3	C_2	C_1	C_0

Then, (X, $*$, 0) is a PU-algebra. Let $A = \{e_1, e_2\} \subseteq E, where$

$$E = \{e_1 = expensive, e_2 = beautiful, e_3 = manual\ gear, e_4 = cheap\}$$

be a set of parameters and $F : A \to P$ (X) and given by

$$F = \left\langle \begin{matrix} e_1 & e_2 \\ F(e_1) & F(e_2) \end{matrix} \right\rangle = \left\langle \begin{matrix} e_1 & e_2 \\ \{C_2, C_3\} & \{C_1, C_3\} \end{matrix} \right\rangle$$

and $F_p : A \to \{0,1\}$. Table 1 (below) shows the tabular representation of the soft set (F, A), where if $C_i \in F(e_i)$, then $F_p(x)=1$, otherwise $F_p(x)=0$, where $F_p(x)$ are the entries in the table.

X	e_1	e_2
C_0	0	0
C_1	0	0
C_2	1	1
C_3	1	1

Definition 5.4. Let $F : A \to P(X)$ be a soft PU-function on A, and ~ a binary relation on $P(X)$, such that $\forall p,q \in P(X),[p \sim q \Leftrightarrow f_p = f_q]$. Then, ~ is an equivalence relation on $P(X)$.

Let $F(A) = \{p \in P(X): p = F(x), for some x \in A\}$, and for $p \in P(X)$, let $[p) = \{q \in P(X): p \subseteq q\}$.

Lemma 5.5. If $F : A \to P(X)$ is a soft PU-function on A, then for $p,q \in P(X)$

$$p \sim q \Leftrightarrow [p) \cap F(A) = [q) \cap F(A)$$

Proof. $\forall p,q \in P(X), p \sim q \Leftrightarrow f_p = f_q$

$$(\text{for } x \in A),[p \subseteq F(x) \Leftrightarrow q \subseteq F(x)] \Leftrightarrow \{x \in A: F(x) \in [p)\} = \{x \in A: F(x) \in [q)\} \Leftrightarrow$$
$$[p) \cap F(A) = [q) \cap F(A).$$

Example 5.6. Let X = {0, a, b} be a set with the operation $*$ defined by the following table.

$*$	0	a	b
0	0	a	b
a	b	0	a
b	a	b	0

and $P(X) = \{\phi,\{0\},\{a\},\{b\},\{0,a\},\{0,b\},\{a,b\},\{0,a,b\}\}$. Then, $(X,*,0)$ is a PU-algebra.

Let $F : X \to P(X)$ be a soft PU-function on X given by $F = \begin{pmatrix} 0 & a & b \\ \{0\} & \{0,a\} & \{0,b\} \end{pmatrix}$, then a cut function of F is given by the following table:

	0 {0}	a {0, a}	b {0, b}
F_ϕ	1	1	1
$F_{\{0\}}$	1	1	1
$F_{\{a\}}$	0	1	0
$F_{\{b\}}$	0	0	1
$F_{\{0,a\}}$	0	1	0
$F_{\{0,b\}}$	0	0	1
$F_{\{a,b\}}$	0	0	0
$F_{\{0,a,b\}}$	0	0	0

Hence, $f_\phi = f_{\{0\}} = \{0,a,b\}$, $f_{\{a\}} = f_{\{0,a\}} = \{a\}$, $f_{\{b\}} = f_{\{0,b\}} = \{b\}$ and $f_{\{a,b\}} = f_X = \phi$.

Lemma 5.7. Let $F : A \to P(X)$ be a soft PU-function on A, for every $x \in A$, if $F(x) = p$, then $p \in P(X)$ is an infimum of the class to which it belongs, i.e. $p = \inf [p]_\sim$.

Proof. If $q \in [p]_\sim$, then $p = F(x) \subseteq q$. Hence, $p = \inf [p]_\sim$.

Theorem 5.8. If $F : A \to P(X)$ is a soft PU-function on A, then for all $x \in A$,

$$F(x) = \sup\{p \in P(X) : F_p(x) = 1\}$$

Proof. Let $F(x) = q \in P(X)$. Then, $F_q(x) = 1$. Now, if any $p \in P(X), F_p(x) = 1$, then $p \subseteq F(x)$, i.e., $p \subseteq q$. Also, $q \in \{p \in P(X) : F_p(x) = 1\}$, thus q is the greatest element of that family. Thus,

$$F(x) = q = \sup\{p \in P(X) : F_p(x) = 1\}$$

Proposition 5.9. Let $F : A \to P(X)$ be a soft PU-function on A. If $q \subseteq p$, for all $p,q \in P(X)$, then $f_p \subseteq f_q$.

Proof. Let $q \subseteq p$, for all $p,q \in P(X)$ and $x \in f_p$, then $p \subseteq F(x)$. It follows that $q \subseteq F(x)$ and so $x \in f_q$. Hence, $f_p \subseteq f_q$.

Proposition 5.10. Let $F : A \to P(X)$ be a soft PU-function on A. Then, $[F(x) \neq F(y) \Leftrightarrow A_{F(x)} \neq A_{F(y)}]$, $\forall x, y \in A$.

Proof. The sufficiency is obvious. Assume that $A_{F(x)} \neq A_{F(y)}$, $\forall x, y \in A$. Then,

$$A_{F(x)} = \{z \in A : F(x) \subseteq F(z)\} \neq \{z \in A : F(y) \subseteq F(z)\} = A_{F(y)}$$

Corollary 5.11. Let $F : A \to P(X)$ be a soft PU-function on A. Then,

$$\forall x, y \in A,\ [F(y) \subseteq F(x) \Leftrightarrow A_{F(x)} \subseteq A_{F(y)}]$$

Proof. Clear.

Let $F : A \to P(X)$ be a soft PU-function on A, and $\sim$ a binary relation on $P(X)$, such that $\forall p,q \in P(X), [p \sim q \Leftrightarrow f_p = f_q]$. Then, $\sim$ is an equivalence relation on $P(X)$ and $[p]_\sim = \{q \in P(X) : p \sim q\}$ is an equivalence class containing p. Every soft PU-function on A determines a binary block code c of length n, in the following way:

To every class $[p]_\sim$, where $p \in P(X)$, there are corresponds a codeword $c_{[p]} = w_1 ... w_n$ such that $w_i = w_j \Leftrightarrow F_r(i) = j$, for $i \in A, j \in \{0,1\}$.

We use the following defined order on the set of codeword's belonging to a binary block code C, for any $x, y \in C, x = x_1 ... x_n$, $y = y_1 ... y_n$, $x \leq_c y \Leftrightarrow x_1, ..., x_n \leq_c y_1, ..., y_n$.

Example 5.12. Let X = {0, a, b} be a set with the operation $*$ defined by the following table.

$*$	0	a	b
0	0	a	b
a	b	0	a
b	a	b	0

Then, $(X, *, 0)$ is a PU-algebra. Let $F : X \to P(X)$ be a soft PU-function on X given by $F = \begin{pmatrix} 0 & a & b \\ \{0\} & \{0, a\} & \{0, b\} \end{pmatrix}$ and $F_p : X \to \{0,1\}$, then a cut function is given by the following table:

	0 {0}	a {0, a}	b {0, b}
$F_{\{0\}}$	1	1	1
$F_{\{0,a\}}$	0	1	0
$F_{\{0,b\}}$	0	0	1

Then, $C = \{111, 010, 001\}$, see Fig.1

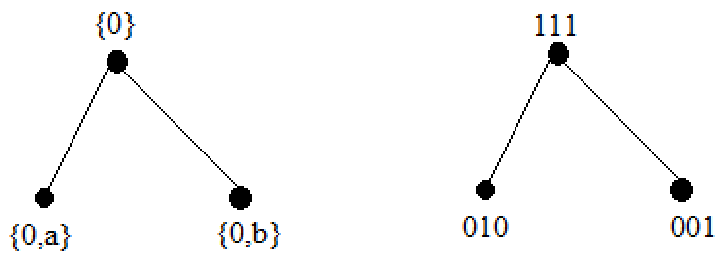

Fig 1. Graphs of the binary block code C

Theorem 5.13. Let X be a finite PU-algebra. Every $(P(X), \leq)$ determines a block-code C, such that $(P(X), \leq)$ is isomorphic with $(C, \leq_c)$.

Proof. Let $X = \{r_i : i = 1, ..., n\}$ be a finite PU-algebra in which r_1 is the least element and let a mapping $H : P(X) \to P(X)$ be identify PU-valued function on $P(X)$. The decomposition of H gives a family $\{H_r : r \in X\}$ which is the required code, under the above defined order $x \leq_c y \Leftrightarrow x_1, ..., x_n \leq_c y_1, ..., y_n$. Let $f : X \to \{H_r : r \in X\}$ be a function defined by $f(r) = H_r$, for all $r \in X$. By Lemma 4.6 every class contains exactly one element, and thus f is one to one. If $r, m \in X$ and $r \leq m$, then $H_m \subseteq H_r$, which means that $H_m \leq H_r$, and f is an isomorphism.

Example 5.14. Let $X = \{0, a, b\}$ be a set with the operation $*$ defined by the following table.

$*$	0	a	b
0	0	a	b
a	b	0	a
b	a	b	0

and $P(X) = \{\phi, \{0\}, \{a\}, \{b\}, \{0,a\}, \{0,b\}, \{a,b\}, \{0,a,b\}\}$. Then, $(X, *, 0)$ is a PU-algebra.

Let $F : X \to P(X)$ be a soft PU-function on X given by $F = \begin{pmatrix} 0 & a & b \\ \{0\} & \{0,a\} & \{0,b\} \end{pmatrix}$.

Now, let $H : P(X) \to P(X)$ be identify PU-valued function on $P(X)$, then a cut function is given by the following table:

	ϕ	{0}	{a}	{b}	{0, a}	{0, b}	{a, b}	{0, a, b}
F_{ϕ}	1	1	1	1	1	1	1	1
$F_{\{0\}}$	0	1	0	0	1	1	0	1
$F_{\{a\}}$	0	0	1	0	1	0	1	1
$F_{\{b\}}$	0	0	0	1	0	1	1	1
$F_{\{0,a\}}$	0	0	0	0	1	0	0	1
$F_{\{0,b\}}$	0	0	0	0	0	1	0	1
$F_{\{a,b\}}$	0	0	0	0	0	0	1	1
$F_{\{0,a,b\}}$	0	0	0	0	0	0	0	1

$$C = \{11111111, 01001101, 00101011, 00010111, 00001001, 00000101, 00000011, 00000001\}$$

6. Conclusion

Coding Theory is a mathematical domain with many applications in Information theory. Various type of codes and their connections with other mathematical objects have been intensively studied. One of these applications, namely connections between binary block codes and BCK-algebras, was recently studied in [16,17]. In this paper, we focused to one of the recent applications of PU-algebras in the coding theory, namely the construction of codes by soft sets PU-valued functions. First, we introduced the notion of soft sets PU-valued functions, on a set and investigated some of its related properties. Moreover, the codes generated by a soft sets PU-valued function were constructed and several examples are given. Furthermore, example with graphs of binary block code were constructed from a soft sets PU-valued function.

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