Number 34 Year 2021

Journal of New Theory

ISSN: 2149-1402

Editor-in-Chief
Naim Çağman

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated by J. New Theory or JNT) is a mathematical journal focusing on new mathematical theories or a mathematical theory's applications to science.

J. New Theory is an open-access journal.

JNT was founded on 18 November 2014, and its first issue was published on 27 January 2015.

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

E-mail: journalofnewtheory@gmail.com

Language: English only.

Article Processing Charges: It has no processing charges.

Publication Frequency: Quarterly

Review Process: Blind Peer Review

The policy of Screening for Plagiarism: For a manuscript to be accepted from 2020 for pre-review, its reference-excluded similarity rate obtained via iThenticate, Turnitin, etc. should be a maximum of **30%**.

Publication Ethics: The governance structure of J. New Theory and its acceptance procedures are transparent and designed to ensure the highest quality of published material. Journal of New Theory adheres to the international standards developed by the Committee on Publication Ethics (COPE).

Aim: Journal of New Theory aims to share new ideas in pure or applied mathematics with the world of science.

Scope: Journal of New Theory is an international, online, open access, and peer-reviewed journal. Journal of New Theory publishes original research articles, reports, reviews, editorial, letters to the editor, technical notes etc. from all science branches that use mathematics theories. Journal of New Theory concerns the studies in the areas of, but not limited to:

· Fuzzy Sets,
· Soft Sets,
· Neutrosophic Sets,
· Decision-Making
· Algebra
· Number Theory
· Analysis
· Theory of Functions
· Geometry
· Applied Mathematics
· Topology
· Fundamental of Mathematics
· Mathematical Logic
· Mathematical Statistics

You can submit your manuscript in the JNT style as pdf, docx, or tex. The manuscript preparation rules, article templates (LaTeX and Microsoft Word) can be accessed from the following links.

- 20210330_JNT_Template
- 20210330_JNT_Template_Word
- 20210330_JNT_Template_LaTeX
- 20210330_JNT_Manuscript_Preparation_Rules
- 2021030_JNT_Copyright_Form

Kindly ensure that your manuscript's similarity rate by iThenticate or Turnitin is 30% or less, excluding references. Moreover, use the equation editor for all the mathematical expressions in your study. For example, use $A = cos3x + x$ instead of A = cos3x + x. Add your current ORCID ID. Add an Introduction and a Conclusion. Do not add an acknowledgment to editors or referees. Do not include your title in your personal information. Do not abbreviate your name and surname (first name, middle name (if any), and last name).

Contact

Editor-in-Chief

Name: Prof. Dr. Naim Çağman

Email: journalofnewtheory@gmail.com
Phone: +905354092136
Address: Departments of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, Tokat, Turkey

Editors

Name: Assoc. Prof. Dr. Faruk Karaaslan

Email: karaaslan.faruk@gmail.com
Phone: +905058314380
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çankırı Karatekin University, 18200, Çankırı, Turkey

Name: Assoc. Prof. Dr. İrfan Deli

Email: irfandeli@kilis.edu.tr
Phone: +905426732708
Address: M.R. Faculty of Education, Kilis 7 Aralık University, Kilis, Turkey

Name: Assoc. Prof. Dr. Serdar Enginoğlu

Email: serdarenginoglu@gmail.com
Phone: +905052241254
Address: Departments of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, 17100, Çanakkale, Turkey

CONTENT

Journal of New Theory
ISSN: 2149-1402

34 (2021) 01-11
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

On Ostrowski-Type Inequalities via Strong s-Godunova-Levin Functions

Badreddine Meftah[1], **Assia Azaizia**[2]

Article History
Received: 23 Mar 2019
Accepted: 19 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — In this paper, we first introduce a new class of convex functions called strong s-Godunova-Levin functions, which encompass the strong Godunova-Levin, s-Godunova-Levin, and Godunova-Levin function classes. By relying on the identity given by Cerone *et al.* [Demonstratio Math., 37 (2004))] and by some simple technical methods, we derive some new Ostrowski-type inequalities for functions whose derivatives in absolute value at a certain power $q \geq 1$ lies in the above cited new class of functions. Some special cases are discussed. The results obtained can be considered a generalization of certain known results.

Keywords — *Ostrowski inequality, Hölder inequality, power mean inequality, strong s-Godunova-Levin functions*

Mathematics Subject Classification (2020) — 26D10, 26D15

1. Introduction

Let H be an interval in $\mathbb{R}$. The following concepts are known in the literature.

Definition 1.1. [1] A function $\eta : H \to \mathbb{R}$ is said to be convex if

$$\eta\left(tx + (1-t)y\right) \leq t\eta\left(x\right) + (1-t)\,\eta\left(y\right)$$

holds for all $x, y \in H$ and $t \in [0, 1]$.

Definition 1.2. [2] A function $\eta : H \to \mathbb{R}$ is called strongly convex with modulus c if

$$\eta\left(tx + (1-t)y\right) \leq t\eta\left(x\right) + (1-t)\,\eta\left(y\right) - ct(1-t)\left|x-y\right|^2$$

holds for all $x, y \in H$ and $t \in (0, 1)$.

Definition 1.3. [3] A nonnegative function $\eta : H \to \mathbb{R}$ is said to be p-convex if

$$\eta\left(tx + (1-t)y\right) \leq \eta\left(x\right) + \eta(y)$$

holds for all $x, y \in H$ and all $t \in [0, 1]$.

[1]badrimeftah@yahoo.fr (Corresponding Author); [2]azaiziaassia@gmail.com
[1]Laboratory of Telecommunication, Faculty of Sciences and Technology, University of 8 Mai 1945, Guelma, Algeria
[2]Department of Mathematics, Faculty of Mathematics, Computer Science, and Science of Matter, University of 8 Mai 1945, Guelma, Algeria

Definition 1.4. [4] A nonnegative function $\eta : H \to \mathbb{R}$ is said to be strongly p-convex if

$$\eta\left(tx+(1-t)\,y\right) \leq \eta\left(x\right)+\eta(y)-ct(1-t)\left|x-y\right|^{2}$$

holds for all $x, y \in H$ and all $t \in [0,1]$.

Definition 1.5. [5] A function $\eta : H \to [0,+\infty)$ is said to be Godunova-Levin function if

$$\eta\left(tx+(1-t)\,y\right) \leq \tfrac{\eta(x)}{t}+\tfrac{\eta(y)}{1-t}$$

holds for all $x, y \in H$ and $t \in (0,1)$.

Definition 1.6. [4] A function $\eta : H \to [0,+\infty)$ is said to be strong Godunova-Levin function if

$$\eta\left(tx+(1-t)\,y\right) \leq \tfrac{\eta(x)}{t}+\tfrac{\eta(y)}{1-t}-ct(1-t)\left|x-y\right|^{2}$$

holds for all $x, y \in H$ and all $t \in (0,1)$.

Definition 1.7. [6] A function $\eta : H \to [0,+\infty)$ is said to be s-Godunova-Levin function, where $s \in [0,1]$, if

$$\eta\left(tx+(1-t)\,y\right) \leq \tfrac{\eta(x)}{t^{s}}+\tfrac{\eta(y)}{(1-t)^{s}}$$

holds for all $x, y \in H$ and all $t \in (0,1)$.

The most important inequality to study the error estimation for different numerical quadrature rules is undoubtedly that known as the Ostrowski inequality which can be stated as follows:

Theorem 1.8. [7] Let $\eta : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$ is an interval, be a mapping, and $e, g \in U^{\circ}$, with $e < g$. If $|\eta'| \leq \mathbb{M}$ for all $x \in [e,g]$, then

$$\left|\eta(x)-\tfrac{1}{g-e}\int_{e}^{g}\eta\left(t\right)dt\right| \leq \mathbb{M}\left(g-a\right)\left[\frac{1}{4}+\tfrac{\left(x-\frac{e+g}{2}\right)^{2}}{(g-e)^{2}}\right] \tag{1}$$

In recent decades, the inequality (1) has generated much interest from researchers, several papers dealing with its generalizations and extensions has appeared, see [8–19], and references have been cited therein.

In [20], Cerone and Dragomir have shown the following identity:

Lemma 1.9. [20] Let $\eta : H \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on H°, where $e, g \in H$ with $e < g$. If $\eta' \in L[e,g]$, then

$$\eta\left(\upsilon\right)-\tfrac{1}{g-e}\int_{e}^{g}\eta\left(u\right)du=\tfrac{(x-e)^{2}}{g-e}\int_{0}^{1}t\eta'\left(t\upsilon+(1-t)\,a\right)dt-\tfrac{(g-x)^{2}}{g-e}\int_{0}^{1}t\eta'\left(t\upsilon+(1-t)\,g\right)dt$$

for each $\upsilon \in [e,g]$.

Based on the above lemma, they have established some Ostrowski-type inequalities via different types of convexity. We cited the results therein.

Theorem 1.10. Let $\eta : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. If $|\eta'|$ is convex on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}\left|\eta\left(x\right)-\tfrac{1}{b-a}\int_{a}^{b}\eta\left(u\right)du\right| \leq & \tfrac{b-a}{6}\left[\left(\tfrac{x-a}{b-a}\right)^{2}\left|\eta'\left(a\right)\right|+\left(\tfrac{b-x}{b-a}\right)^{2}\left|\eta'\left(b\right)\right|\right.\\ & \left.+\left(1+2\left(\tfrac{x-\frac{a+b}{2}}{b-a}\right)^{2}\right)\left|\eta'\left(x\right)\right|\right]\end{aligned}$$

In [21], Noor et al. have established the following Ostrowski-type inequalities for differential s-Godunova-Levin functions.

Theorem 1.11. Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $\eta' \in L([a,b])$ for all $x \in [a,b]$. If $|\eta'|$ is s-Godunova-Levin function of the second kind and $|\eta'| \leq M$, then the following inequality holds,

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{M\left((b-x)^2+(x-a)^2\right)}{(b-a)(1-s)}$$

Theorem 1.12. Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $\eta' \in L([a,b])$ for all $x \in [a,b]$. If $|\eta'|^q$ is s-Godunova-Levin function of the second kind where $q \geq 1$ and $|\eta'| \leq M$, then the following inequality holds,

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{M\left((b-x)^2+(x-a)^2\right)}{(b-a)(1-s)^{\frac{1}{q}}2^{1-q}}$$

Theorem 1.13. Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $\eta' \in L([a,b])$ for all $x \in [a,b]$. If $|\eta'|^q$ is s-Godunova-Levin function of the second kind where $q, p > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $|\eta'| \leq M$, then the following inequality holds,

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{M\left((b-x)^2+(x-a)^2\right)}{(b-a)(1-s)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}$$

The last result is based on the Hölder inequality, which can be stated as follows:

Theorem 1.14. [22, Hölder Inequality for Integrals] Let $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[e,g]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[e,g]$, then

$$\int_e^g |f(u)\,g(u)|\,du \leq \left(\int_e^g |f(u)|^p\,du\right)^{\frac{1}{p}}\left(\int_e^g |g(u)|^q\,du\right)^{\frac{1}{q}}$$

with equality if and only if $\alpha\,|f(u)|^p = \beta\,|g(u)|^q$ almost everywhere for some constants α and β.

Motivated by the above and some other existing results, we establish some new Ostrowski-type inequalities for functions whose derivatives in absolute values lie in a new class of convex functions called strong s-Godunova-Levin functions.

2. Main Results

Definition 2.1. A function $\eta : H \to [0,+\infty)$ is said to be strong s-Godunova-Levin functions with modulus $c > 0$, where $s \in [0,1]$, if

$$\eta(tx+(1-t)y) \leq \frac{\eta(x)}{t^s} + \frac{\eta(y)}{(1-t)^s} - ct(1-t)|x-y|^2$$

holds for all $x, y \in H$ and all $t \in (0,1)$.

Remark 2.2. Clearly all strong s-Godunova-Levin functions is s-Godunova-Levin functions. Moreover, we note that Definition 2.1 recaptures all definitions cited above by fixing the value of s or by tending c towards 0, with exception of Definitions 1 and 2.

Theorem 2.3. Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), where $a < b$, and $\eta' \in L[a,b]$. If $|\eta'|$ is strong s-Godunova-Levin functions with modulus $c > 0$, where $s \in [0,1)$, then the following inequality

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(a)|}{(1-s)(2-s)}\right) + \frac{(b-x)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(b)|}{(1-s)(2-s)}\right) - \frac{c}{12}\left(\frac{(b-x)^4+(x-a)^4}{b-a}\right)$$

holds for all $x \in [a,b]$.

Proof. From Lemma 1.9, modulus, and strong s-Godunova-Levin convexity of $|\eta'|$, we obtain

$$\begin{aligned}\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| &\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)b)\right|dt \\ &\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left(\frac{|\eta'(x)|}{t^s} + \frac{|\eta'(a)|}{(1-t)^s} - ct(1-t)(x-a)^2\right)dt \\ &\quad + \frac{(b-x)^2}{b-a}\int_0^1 t\left(\frac{|\eta'(x)|}{t^s} + \frac{|\eta'(b)|}{(1-t)^s} - ct(1-t)(b-x)^2\right)dt \\ &= \frac{(x-a)^2}{b-a}\left(\left|\eta'(x)\right|\int_0^1 t^{1-s}dt + \left|\eta'(a)\right|\int_0^1 t(1-t)^{-s}dt \right. \\ &\quad \left. - c(x-a)^2\int_0^1 t^2(1-t)\,dt\right) \\ &\quad + \frac{(b-x)^2}{b-a}\left(\left|f'(x)\right|\int_0^1 t^{1-s}dt + \left|f'(b)\right|\int_0^1 t(1-t)^{-s}dt \right. \\ &\quad \left. - c(b-x)^2\int_0^1 t^2(1-t)\,dt\right) \\ &= \frac{(x-a)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(a)|}{(1-s)(2-s)}\right) + \frac{(b-x)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(b)|}{(1-s)(2-s)}\right) \\ &\quad - \frac{c}{12}\left(\frac{(b-x)^4+(x-a)^4}{b-a}\right)\end{aligned}$$

The proof is completed. □

Corollary 2.4. In Theorem 2.3, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{4(1-s)(2-s)}\left(\left|\eta'(a)\right| + 2(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right| + \left|\eta'(b)\right|\right) - \frac{c(b-a)^3}{96}$$

Corollary 2.5. In Theorem 2.3, if we tend c to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(a)|}{(1-s)(2-s)}\right) + \frac{(b-x)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(b)|}{(1-s)(2-s)}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le \tfrac{b-a}{4(1-s)(2-s)}\left(\left|f'(a)\right| + 2(1-s)\left|f'\left(\tfrac{a+b}{2}\right)\right| + \left|f'(b)\right|\right)$$

Remark 2.6. In Corollary 2.5, if we assume that $|\eta'(u)| \le M$, then the first inequality recaptures Corollary 3.1 from [21].

Corollary 2.7. In Theorem 2.3, if we take $s = 0$, we obtain

$$\left|\eta(x) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le \tfrac{(x-a)^2}{b-a}\left(\tfrac{|\eta'(x)|+|\eta'(a)|}{2}\right) + \tfrac{(b-x)^2}{b-a}\left(\tfrac{|\eta'(x)|+|\eta'(b)|}{2}\right) - \tfrac{c}{12}\left(\tfrac{(b-x)^4+(x-a)^4}{b-a}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le \tfrac{b-a}{8}\left(\left|\eta'(a)\right| + 2\left|\eta'\left(\tfrac{a+b}{2}\right)\right| + \left|\eta'(b)\right| - \tfrac{c(b-a)^2}{12}\right)$$

Corollary 2.8. In Corollary 2.7, if we tend c to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le \tfrac{(x-a)^2}{b-a}\left(\tfrac{|\eta'(x)|+|\eta'(a)|}{2}\right) + \tfrac{(b-x)^2}{b-a}\left(\tfrac{|\eta'(x)|+|\eta'(b)|}{2}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le \tfrac{b-a}{8}\left(\left|\eta'(a)\right| + 2\left|\eta'\left(\tfrac{a+b}{2}\right)\right| + \left|\eta'(b)\right|\right)$$

Theorem 2.9. Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$, and $\eta' \in L[a,b]$. If $|\eta'|^q$ is strong s-Godunova-Levin functions with modulus $c > 0$, where $s \in [0,1)$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality

$$\begin{aligned}\left|\eta(x) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le & \tfrac{b-a}{(p+1)^{\frac{1}{p}}}\left(\left(\tfrac{x-a}{b-a}\right)^2\left(\tfrac{|\eta'(x)|^q+|\eta'(a)|^q}{1-s} - \tfrac{c}{6}(x-a)^2\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\tfrac{b-x}{b-a}\right)^2\left(\tfrac{|\eta'(x)|^q+|\eta'(b)|^q}{1-s} - \tfrac{c}{6}(b-x)^2\right)^{\frac{1}{q}}\right)\end{aligned}$$

holds for all $x \in [a,b]$.

Proof. From Lemma 1.9, properties of modulus, and Hölder inequality, we have

$$\begin{aligned}\left|\eta(x) - \tfrac{1}{b-a}\int_a^b \eta(u)\,du\right| \le & \tfrac{(x-a)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)a)\right|dt + \tfrac{(b-x)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)b)\right|dt \\ \le & \tfrac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\eta'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ & + \tfrac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\eta'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{1}{(p+1)^{\frac{1}{p}}} \left(\frac{(x-a)^2}{b-a} \left(\int_0^1 \left|\eta'(tx+(1-t)a)\right|^q dt \right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} \left(\int_0^1 \left|\eta'(tx+(1-t)b)\right|^q dt \right)^{\frac{1}{q}} \right)$$

$$\leq \frac{1}{(p+1)^{\frac{1}{p}}} \left(\frac{(x-a)^2}{b-a} \left(\int_0^1 \left(\frac{|\eta'(x)|^q}{t^s} + \frac{|\eta'(a)|^q}{(1-t)^s} - ct(1-t)(x-a)^2 \right) dt \right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} \left(\int_0^1 \left(\frac{|\eta'(x)|^q}{t^s} + \frac{|\eta'(b)|^q}{(1-t)^s} - ct(1-t)(b-x)^2 \right) dt \right)^{\frac{1}{q}} \right)$$

$$= \frac{1}{(p+1)^{\frac{1}{p}}} \left(\frac{(x-a)^2}{b-a} \left(\left|\eta'(x)\right|^q \int_0^1 t^{-s} dt + \left|\eta'(a)\right|^q \int_0^1 (1-t)^{-s} dt - c(x-a)^2 \int_0^1 t(1-t)\, dt \right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} \left(\left|\eta'(x)\right|^q \int_0^1 t^{-s} dt + \left|\eta'(b)\right|^q \int_0^1 (1-t)^{-s} dt - c(b-x)^2 \int_0^1 t(1-t)\, dt \right)^{\frac{1}{q}} \right)$$

$$= \frac{b-a}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{b-a} \right)^2 \left(\frac{|\eta'(x)|^q + |\eta'(a)|^q}{1-s} - \frac{c}{6}(x-a)^2 \right)^{\frac{1}{q}} + \left(\frac{b-x}{b-a} \right)^2 \left(\frac{|\eta'(x)|^q + |\eta'(b)|^q}{1-s} - \frac{c}{6}(b-x)^2 \right)^{\frac{1}{q}} \right)$$

The proof is completed. □

Corollary 2.10. In Theorem 2.9, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| \eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left(\left(\frac{|\eta'(a)|^q + \left|\eta'\left(\frac{a+b}{2}\right)\right|^q}{1-s} - \frac{c(b-a)^2}{24} \right)^{\frac{1}{q}} + \left(\frac{\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q}{1-s} - \frac{c(b-a)^2}{24} \right)^{\frac{1}{q}} \right)$$

Corollary 2.11. In Theorem 2.9, if we tend c to 0, i.e. $c \to 0^+$, we obtain

$$\left| \eta(x) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{b-a} \right)^2 \left(\frac{|\eta'(x)|^q + |\eta'(a)|^q}{1-s} \right)^{\frac{1}{q}} + \left(\frac{b-x}{b-a} \right)^2 \left(\frac{|\eta'(x)|^q + |\eta'(b)|^q}{1-s} \right)^{\frac{1}{q}} \right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| \eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left(\left(\frac{\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(a)|^q}{1-s} \right)^{\frac{1}{q}} + \left(\frac{\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q}{1-s} \right)^{\frac{1}{q}} \right)$$

Remark 2.12. In Corollary 2.11, if we assume that $|\eta'(u)| \leq M$, then the first inequality gives

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}}\left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right)M,$$

which is the correct result of Corollary 3.2 from [21].

Corollary 2.13. In Theorem 2.9, if we take $s = 0$, we obtain

$$\begin{aligned}\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq &\frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{b-a}\right)^2\left(|\eta'(x)|^q + |\eta'(a)|^q - \frac{c}{6}(x-a)^2\right)^{\frac{1}{q}}\right.\\ &\left. + \left(\frac{b-x}{b-a}\right)^2\left(|\eta'(x)|^q + |\eta'(b)|^q - \frac{c}{6}(b-x)^2\right)^{\frac{1}{q}}\right)\end{aligned}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\begin{aligned}\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq &\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(a)|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right.\\ &\left. + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right)\end{aligned}$$

Corollary 2.14. In Corollary 2.13, if we tend c to 0, i.e. $c \to 0^+$, we obtain

$$\begin{aligned}\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq &\frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{b-a}\right)^2\left(|\eta'(x)|^q + |\eta'(a)|^q\right)^{\frac{1}{q}}\right.\\ &\left. + \left(\frac{b-x}{b-a}\right)^2\left(|\eta'(x)|^q + |\eta'(b)|^q\right)^{\frac{1}{q}}\right)\end{aligned}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(a)|^q\right)^{\frac{1}{q}} + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q\right)^{\frac{1}{q}}\right)$$

Theorem 2.15. Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) where $a < b$, and $\eta' \in L[a,b]$. If $|\eta'|^q$ strong s-Godunova-Levin functions with modulus $c > 0$, where $s \in [0,1)$ and $q > 1$, then we have

$$\begin{aligned}\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq &\frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(a)|^q}{(1-s)(2-s)} - \frac{c(x-a)^2}{12}\right)^{\frac{1}{q}}\\ &+ \frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(b)|^q}{(1-s)(2-s)} - \frac{c(b-x)^2}{12}\right)^{\frac{1}{q}}\end{aligned}$$

for all $x \in [a,b]$.

Proof. From Lemma 1.9, properties of modulus, and power mean inequality, we get

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\int_0^1 t\,|\eta'(tx + (1-t)a)|\,dt + \frac{(b-x)^2}{b-a}\int_0^1 t\,|\eta'(tx + (1-t)b)|\,dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\eta'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\eta'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^2}{(b-a)2^{1-\frac{1}{q}}}\left(\int_0^1 t\left|\eta'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{(b-a)2^{1-\frac{1}{q}}}\left(\int_0^1 t\left|\eta'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\int_0^1 t\left(\frac{|\eta'(x)|^q}{t^s}+\frac{|\eta'(a)|^q}{(1-t)^s}-ct(1-t)(x-a)^2\right)dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\int_0^1 t\left(\frac{|\eta'(x)|^q}{t^s}+\frac{|\eta'(b)|^q}{(1-t)^s}-ct(1-t)(b-x)^2\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^2}{(b-a)2^{1-\frac{1}{q}}}\left(\left|\eta'(x)\right|^q\int_0^1 t^{1-s}dt+\left|\eta'(a)\right|^q\int_0^1 t(1-t)^{-s}dt\right.$$
$$\left.-c(x-a)^2\int_0^1 t^2(1-t)dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\left|\eta'(x)\right|^q\int_0^1 t^{1-s}dt+\left|\eta'(b)\right|^q\int_0^1 t(1-t)^{-s}dt\right.$$
$$\left.-c(b-x)^2\int_0^1 t^2(1-t)dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q+|\eta'(a)|^q}{(1-s)(2-s)}-\frac{c(x-a)^2}{12}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q+|\eta'(b)|^q}{(1-s)(2-s)}-\frac{c(b-x)^2}{12}\right)^{\frac{1}{q}}$$

The proof is completed. □

Corollary 2.16. In Theorem 2.15, if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b \eta(u)du\right| \leq \frac{b-a}{2^{3-\frac{1}{q}}}\left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q+|\eta'(a)|^q}{(1-s)(2-s)}-\frac{c(b-a)^2}{48}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2^{3-\frac{1}{q}}}\left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q+|\eta'(b)|^q}{(1-s)(2-s)}-\frac{c(b-a)^2}{48}\right)^{\frac{1}{q}}$$

Corollary 2.17. In Theorem 2.15, if we tend c to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(a)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(b)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{2^{3-\frac{1}{q}}}\left(\left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}} + \left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(a)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}}\right)$$

Remark 2.18. In Corollary 2.17, if we assume that $|\eta'(u)| \leq M$, then the first inequality recaptures Corollary 3.3 from [21].

Corollary 2.19. In Theorem 2.15, if we take $s = 0$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(a)\right|^q - \frac{c(x-a)^2}{6}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(b)\right|^q - \frac{c(b-x)^2}{6}\right)^{\frac{1}{q}}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{8}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(a)\right|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(b)\right|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right)$$

Corollary 2.20. In Corollary 2.19, if we tend c to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(a)\right|^q\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(b)\right|^q\right)^{\frac{1}{q}}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{8}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(a)\right|^q\right)^{\frac{1}{q}} + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(b)\right|^q\right)^{\frac{1}{q}}\right)$$

3. Conclusion

Ostrowski-type inequalities are of great importance when studying the error estimation for different numerical quadrature rules. It suffices to take for example $x = (a + b)/2$, and we obtain the rule of midpoint or fix some values of x and use the triangular inequality to estimate the error of the Simpson rule and the Trapezoidal rule. In this study, we introduce the concept of strong s-Godunova-Levin functions and established new Ostrowski-type inequalities for this new class of functions and their associated corollaries. The results obtained generalize those of [22].

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Mathematics in Science and Engineering 187, Academic Press, Inc., Boston, MA, 1992.

[2] B. T. Polyak, *Existence Theorems and Convergence of Minimizing Sequences in Extremum Problems with Restictions*, Soviet Mathematics Doklady 7 (1966) 72–75.

[3] S. S. Dragomir, J. E. Pečarić, L. E. Persson, *Some Inequalities of Hadamard Type*, Soochow Journal of Mathematics 21(3) (1995) 335–341.

[4] H. Angulo, J. Gimenez, A. M. Moros, K. Nikodem, *On Strongly h-convex Functions*, Annals of Functional Analysis 2(2) (2011) 85–91.

[5] E. K. Godunova, V. I. Levin, *Inequalities for Functions of A Broad Class That Contains Convex, Monotone and Some Other Forms of Functions*, (Russian) Numerical Mathematics and Mathematical Physics (Russian) 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985.

[6] S. S. Dragomir, *Nequalities of Hermite-Hadamard Type for h-convex Functions on Linear Spaces*, Proyecciones 34(4) (2015) 323–341.

[7] A. M. Ostrowski, *Über Die Absolutabweichung Einer Differentiierbaren Funktion Von Ihrem Integralmittelwert*, (German) Commentarii Mathematici Helvetici 10(1) (1937) 226–227.

[8] G. Farid, U. N. Katugampola, M. Usman, *Ostrowski-type Fractional Integral Inequalities for s-Godunova-Levin Functions via Katugampola Fractional Integrals*, Open Journal of Mathematical Science 1 (2017) 97–110.

[9] G. Farid, A. U. Rehman, M. Usman, *Ostrowski-type Fractional Integral Inequalities for s-Godunova-Levin Functions via k-fractional Integrals*, Proyecciones 36(4) (2017) 753–767.

[10] S. Kermausuor, *Simpson's Type Inequalities for Strongly (s, m)-convex Functions in The Second Sense and Applications*, Open Journal of Mathematical Science 3(1) (2019) 74–83.

[11] B. Meftah, *Some New Ostrowski's Inequalities for Functions Whose n^{th} Derivatives are r-convex*, International Journal of Analysis Article ID 6749213 (2016) 7 pages.

[12] B. Meftah, *Some New Ostrowski's Inequalities for n-times Differentiable Mappings Which are Quasi-convex*, Facta Universitatis, Series: Mathematics and Informatics 32(3) (2017) 319–327.

[13] B. Meftah, *Some New Ostrowski's Inequalities for Functions Whose n^{th} Derivatives are Logarithmically Convex*, Annales Mathematicae Silesianae 32(1) (2018) 275–284.

[14] B. Meftah, *Some Ostrowski's Inequalities for Functions Whose n^{th} Derivatives are s-convex*, Analele Universitatii din Oradea Fascicola Matematica 25(2) (2018) 185–212.B.

[15] Meftah, A. Azaizia, *Fractional Ostrowski-type Inequalities for Functions Whose First Derivatives are MT-preinvex*, Revista De Matemáticas De la Universidad del Atlántico Páginas 6(1) (2019) 33–43.

[16] E. Set, M. E. Özdemir, M. Z. Sarıkaya, A. O. Akdemir, *Ostrowski-type Inequalities for Strongly Convex Functions*, Georgian Mathematical Journal 25(1) (2018) 109–115.

[17] M. Tunç, *Ostrowski-type Inequalities via h-convex Functions with Applications to Special Means*, Journal of Inequalities and Applications 2013(326) (2013) 1–10.

[18] M. Z. Sarikaya, N. Alp, *On Hermite-Hadamard-Fejér Type Integral Inequalities for Generalized Convex Functions via Local Fractional Integrals*, Open Journal of Mathematical Science 3(1) (2019) 273–284.

[19] E. Set, M. E. Özdemir, M. Z. Sarikaya, *New Inequalities of Ostrowski's Type for s-convex functions in the Second Sense with Applications*, Facta Universitatis, Series: Mathematics and Informatics 27(1) (2012) 67–82.

[20] P. Cerone, S. S. Dragomir, *Ostrowski-type Inequalities for Functions Whose Derivatives Satisfy Certain Convexity Assumptions*, Demonstratio Mathematica 37(2) (2004) 299–308.

[21] M. A. Noor, K. I. Noor, M. U. Awan, *Fractional Ostrowski Inequalities for s-Godunova-Levin Functions*, International Journal of Analysis and Applications 5(2) (2014) 167–173.

[22] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and Its Applications (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht 1993.

Journal of
New Theory
ISSN: 2149-1402

34 (2021) 12-19
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

The nth Power of Generalized (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Matrix Sequences and Some Combinatorial Properties

Şükran Uygun[1]

Article History
Received: 15 Apr 2019
Accepted: 15 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract − In this study, new formulas for the n^{th} power of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas special matrix sequences are established by using determinant and trace of the matrices. By these formulas, some identities for (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences are obtained. The formulas for finding the n^{th} power for classic Jacobsthal and Jacobsthal Lucas matrix sequences are also derivable if we choose $s=t=1$.

Keywords − *Jacobsthal numbers, recurrence relations, special matrices*

Mathematics Subject Classification (2020) – 11B37, 11C20

1. Introduction

In the literature, the researchers investigated the n^{th} power of the matrices by different methods. In [1], Williams studied the n^{th} power of a 2×2 matrix. Laughlin found identities deriving from the n^{th} power of some matrices in [2,3]. Belbachir investigated linear recurrent sequences and powers of a square matrix in [4]. There are certainly new developments on special integer and matrix sequences by constructing recurrence relation. In [5], the authors studied sums and products for recurring sequences. Halıcı and Akyuz derived combinatorial identities by using the trace, the determinant and the n^{th} power of a special matrix whose entries are Horadam numbers [6,7]. Among these integer sequences, the Jacobsthal and Jacobsthal Lucas numbers have been studied extensively in the last decade years in [8-12]. The Jacobsthal numbers j_n are terms of the sequence $\{0,1,1,3,5,11,\cdots\}$, defined by the recurrence relation, $j_n=j_{n-1}+2j_{n-2}$, for $n\geq 2$, beginning with the values $j_0=0$, $j_1=1$. The Jacobsthal Lucas numbers c_n are the terms of the sequence $\{2,1,5,7,17,\cdots\}$, defined by the recurrence relation, $c_n=c_{n-1}+2c_{n-2}$, for $n\geq 2$, beginning with the values $c_0=2$ and $c_1=1$ in [13]. (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences are defined by using the following recurrence relation,

$$j_n(s,t)=sj_{n-1}(s,t)+2t\,j_{n-2}(s,t) \quad (j_0(s,t)=0 \text{ and } j_1(s,t)=1) \tag{1}$$

and

$$c_n(s,t)=c_{n-1}(s,t)+2c_{n-2}(s,t) \quad (c_0(s,t)=2 \text{ and } c_1(s,t)=1)$$

where $s>0, t\neq 0$ and $s^2+8t>0$ [8].

[1]suygun@gantep.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Arts and Sciences, Gaziantep University, Gaziantep, Turkey

The Binet formula enables us to state (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas number easily. It can be clearly obtained from the roots r_1 and r_2 of the characteristic equation as the form $x^2 = sx + 2t$, where

$$r_1 = \frac{s+\sqrt{s^2+8t}}{2} \text{ and } r_2 = \frac{s-\sqrt{s^2+8t}}{2}$$

The Binet formula for (s,t)-Jacobsthal numbers and (s,t)-Jacobsthal Lucas numbers are given, respectively, by

$$j_n(s,t) = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \text{ and } c_n(s,t) = {r_1}^n + {r_2}^n$$

In [9], for any integer $n \geq 1$, (s,t)-Jacobsthal matrix sequence is defined as

$$J_n(s,t) = s\,J_{n-1}(s,t) + 2t\,J_{n-2}\,(s,t) \tag{2}$$

with initial conditions $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J_1 = \begin{pmatrix} s & 2 \\ t & 0 \end{pmatrix}$ and (s,t)-Jacobsthal Lucas matrix sequence is defined as

$$C_n(s,t) = s\,C_{n-1}(s,t) + 2t\,C_{n-2}\,(s,t) \tag{3}$$

with initial conditions $C_0 = \begin{pmatrix} s & 4 \\ 2t & -s \end{pmatrix}$ and $C_1 = \begin{pmatrix} s^2+4t & 2s \\ st & 4t \end{pmatrix}$. Some important properties for (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas matrix sequences are given as in [9]

a) $J_n = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ t\,j_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}$

b) $C_n = \begin{pmatrix} c_{n+1}(s,t) & 2c_n(s,t) \\ t\,c_n(s,t) & 2tc_{n-1}(s,t) \end{pmatrix}$

c) $J_{m+n} = J_m\,J_n$

d) $J_n = {J_1}^n$

e) $C_{n+1} = C_1\,J_n$

2. The n^{th} Power of Generalized (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Matrix Sequences and Some Combinatorial Properties

In [1], Williams gave a well-known formula for any integer $n \geq 1$, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^n = \begin{cases} \dfrac{{r_1}^n - {r_2}^n}{r_1 - r_2} A - \dfrac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2, & r_1 \neq r_2 \\ nr^{n-1}A - (n-1)\det(A)\,r^{n-2} I_2, & r_1 = r_2 \end{cases} \tag{4}$$

where r_1, r_2 being the roots of the associated characteristic equation $r^2 - (a+d)r + \det(A) = 0$ of the matrix A and I_2 is the identity matrix 2×2.

Corollary 1. For any integer $n \geq 1$, the n^{th} power of $J_1(s,t)$ and $C_1(s,t)$ are

$${J_1}^n(s,t) = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \begin{pmatrix} s & 2 \\ t & 0 \end{pmatrix} - \frac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2 \tag{5}$$

where $r_1 = \frac{s+\sqrt{s^2+8t}}{2}$ and $r_2 = \frac{s-\sqrt{s^2+8t}}{2}$

$$C_1{}^n(s,t) = \frac{s_1{}^n - s_2{}^n}{s_1 - s_2}\begin{pmatrix} s^2+4t & 2s \\ st & 4t \end{pmatrix} - \frac{s_1{}^{n-1} - s_2{}^{n-1}}{s_1 - s_2} I_2 \tag{6}$$

where $s_1 = \frac{s^2+8t+s\sqrt{s^2+8t}}{2}$ and $s_2 = \frac{s^2+8t-s\sqrt{s^2+8t}}{2}$.

If we choose $s = t = 1$ in (5) and (6), we get the n^{th} power of classic Jacobsthal and Jacobsthal Lucas matrix sequences:

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n = \frac{2^n - (-1)^n}{3}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - \frac{2^{n-1} - (-1)^{n-1}}{3} I_2$$

and

$$\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}^n = \frac{2^n - (-1)^n}{3}\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix} - \frac{2^{n-1} - (-1)^{n-1}}{3} I_2$$

Proof. The proof is obtained by using the eigenvalues of $J_1(s,t)$ and $C_1(s,t)$ and (2), (3), (4). □

Corollary 2. For any integer $n \geq 1$, the determinants of $J_1{}^n(s,t)$ and $C_1{}^n(s,t)$ are

$$\det\left(J_1{}^n(s,t)\right) = (-2t)^n \text{ and } \det\left(C_1{}^n(s,t)\right) = (2t)^n(s^2+8t)^n$$

Proof. By using the property of the determinant of a matrix is the product of eigenvalues of this matrix, we get the determinant of $J_1(s,t)$ and $C_1(s,t)$ is $-2t$ and $(2t)(s^2+8t)$, respectively. The determinant of the n^{th} power of a matrix is the n^{th} power of the product of the eigenvalues. So, the results are easily seen.
□

If we choose $s = t = 1$, we get classic Jacobsthal and Jacobsthal Lucas matrix sequences, and the determinant of them are obtained as

$$\det(J_1{}^n) = (-2)^n,\ \det(C_1{}^n) = (18)^n$$

Laughlin, in [2,3] gave if A is a 2×2 matrix as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the nth power of A is given by

$$A^n = \begin{pmatrix} x_n - dx_{n-1} & bx_{n-1} \\ cx_{n-1} & x_n - ax_{n-1} \end{pmatrix} \tag{7}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i$, T is the trace of A, and D is the determinant of A.

Corollary 3. The n^{th} power of $J_1(s,t)$ and $C_1(s,t)$ are

$$J_1{}^n(s,t) = \begin{pmatrix} x_n & 2x_{n-1} \\ tx_{n-1} & x_n - sx_{n-1} \end{pmatrix} \tag{8}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i}(2t)^i$, and

$$C_1{}^n(s,t) = \begin{pmatrix} y_n - 4t\, y_{n-1} & 2s\, y_{n-1} \\ st\, y_{n-1} & y_n - (s^2+4t)\, y_{n-1} \end{pmatrix} \tag{9}$$

such that $y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (s^2+4t)^{n-i}(2t)^i$.

If we choose $s = t = 1$ in (8) and (9), we get the n^{th} power of classic Jacobsthal and Jacobsthal Lucas matrix sequences,

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} x_n & 2x_{n-1} \\ x_{n-1} & x_n - x_{n-1} \end{pmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (2)^i$ and

$$\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}^n = \begin{pmatrix} y_n - 4\, y_{n-1} & 2\, y_{n-1} \\ y_{n-1} & y_n - 5\, y_{n-1} \end{pmatrix}$$

such that $y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 9^{n-i} 2^i$.

PROOF. The proof is obtained by (2), (3), and (7). □

Corollary 4. The n^{th} element of (s,t)-Jacobsthal Lucas matrix sequence is given as

$$C_n(s,t) = C_1(s,t) J_{n-1}(s,t) = \begin{pmatrix} (s^2+4t)x_{n-1} + 2st\, x_{n-2} & 2(s\, x_{n-1} + 4t\, x_{n-2}) \\ t(s\, x_{n-1} + 4t\ x_{n-2}) & 2t(x_{n-1} - s\, x_{n-2}) \end{pmatrix}$$

or

$$C_n(s,t) = s\, J_n(s,t) + 4t\, J_{n-1}(s,t) = \begin{pmatrix} s\, x_n + 4t\ x_{n-1} & 2(s\, x_{n-1} + 4t\, x_{n-2}) \\ t(s\, x_{n-1} + 4t\ x_{n-2}) & x_n - s\, x_{n-1} + 4t(x_{n-1} - s\, x_{n-2}) \end{pmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i} (2t)^i$, for any integer $n \geq 1$.

PROOF. By (d, e), (2-3), (7), the proofs are easily obtained. □

Theorem 5. For any integer $n \geq 1$, the following property is satisfied,

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n-i}{i} s^{n-2i} (2t)^i = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i} (s^2+8t)^i \tag{10}$$

PROOF. The eigenvalues of $J_1(s,t)$ are $r_1 = \frac{s+\sqrt{s^2+8t}}{2}$ and $r_2 = \frac{s-\sqrt{s^2+8t}}{2}$. The eigenvalues of ${J_1}^n(s,t)$ are ${r_1}^n$ and ${r_2}^n$. By using (8), it is obtained that ${J_1}^n(s,t) = \begin{pmatrix} x_n & 2x_{n-1} \\ tx_{n-1} & x_n - sx_{n-1} \end{pmatrix}$. The trace of ${J_1}^n(s,t)$ is tr$({J_1}^n(s,t))$ $=2x_n$- sx_{n-1}. Because the sum of the eigenvalues is equal to the trace of the matrix, ${r_1}^n + {r_2}^n = 2x_n - sx_{n-1}$

$$2x_n - s\, x_{n-1} = 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i} (-D)^i - s \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} T^{n-1-2i} (-D)^i$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i} (2t)^i \left(\frac{i}{n-i} \right)$$

By binomial expansion

$${r_1}^n + {r_2}^n = \left(\frac{s+\sqrt{s^2+8t}}{2} \right)^n + \left(\frac{s-\sqrt{s^2+8t}}{2} \right)^n$$

$$= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i} (s^2+8t)^i$$

The equality of the results completes the proof. □

If we choose $s = t = 1$ in (10), we get the same result for classic Jacobsthal and Jacobsthal Lucas matrix sequences as

$$\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \frac{n}{n-i} 2^i = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2i} 9^i$$

By the Binet formula of (s,t)-Jacobsthal Lucas sequence, the following is obtained,

$$c_n(s,t) = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2i} s^{n-2i} (s^2+8t)^i$$

and

$$c_n(s,t) = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \frac{n}{n-i} s^{n-2i} (2t)^i$$

Corollary 6. The n^{th} element of Jacobsthal matrix sequence is also demonstrated by using the elements of (s,t)-Jacobsthal sequences,

$$J_n(s,t) = j_n(s,t) J_1 - j_{n-1}(s,t) I_2 \tag{11}$$

Proof. By (a, d) and Binet formulas, we get

$$\begin{aligned} J_n(s,t) &= \begin{pmatrix} j_{n-1}(\mathrm{s,t}) & 2\, j_n(\mathrm{s,t}) \\ t\, j_n(\mathrm{s,t}) & 2t\ j_{n-1}(\mathrm{s,t}) \end{pmatrix} = J_1^n(s,t) \\ &= \frac{r_1^{\ n} - r_2^{\ n}}{r_1 - r_2} J_1(s,t) - \frac{r_1^{\ n-1} - r_2^{\ n-1}}{r_1 - r_2} I_2 \\ &= j_n(\mathrm{s,t}) J_1(s,t) - j_{n-1}(\mathrm{s,t}) I_2 \end{aligned}$$ □

If we choose $s = t = 1$ in (11), we get the same result for classic Jacobsthal and Jacobsthal Lucas matrix sequences as

$$J_n = j_{n-1} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - j_{n-1} I_2$$

By binomial expansion, the following is derived,

$$\frac{r_1^{\ n} - r_2^{\ n}}{r_1 - r_2} = \frac{1}{\sqrt{s^2+8t}} \left[\left(\frac{s+\sqrt{s^2+8t}}{2} \right)^n - \left(\frac{s-\sqrt{s^2+8t}}{2} \right)^n \right] = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} s^{n-2i-1} (s^2+8t)^i$$

and

$$\begin{aligned} J_n(s,t) &= \frac{r_1^{\ n} - r_2^{\ n}}{r_1 - r_2} J_1(s,t) - \frac{r_1^{\ n-1} - r_2^{\ n-1}}{r_1 - r_2} I_2 \\ &= \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} s^{n-1-2i} (s^2+8t)^i J_1(s,t) \\ &\quad - \frac{1}{2^{n-2}} \sum_{i=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \binom{n-1}{2i+1} s^{n-2-2i} (s^2+8t)^i I_2 \end{aligned}$$

Lemma 7. In [2], for all $g \epsilon \mathbb{R}$ or $\mathbb{Z}$, for any integer $n \geq 1$, if

$$A = \frac{1}{g^2 + Tg + D}(A + gI)(gA + DI) \tag{12}$$

then

$$A^n = \left(\frac{gD}{g^2 + Tg + D}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{D}{g^2}\right)^i \left(\frac{g}{D}\right)^r A^r \tag{13}$$

Corollary 8. For all $g \epsilon \mathbb{R}$ or $\mathbb{Z}$, for any integer $n \geq 1$,

$$J_1{}^n(s,t) = \left(\frac{-2tg}{g^2 + sg - 2t}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{(-2t\,)^{i-r}}{g^{2i-r}}\right) J_1{}^r\ (s,t)$$

and

$$C_1{}^n(s,t) = \left(\frac{2t(s^2+8t)g}{g^2 + (s^2+8t)g + 2t(s^2+8t)}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{(2t(s^2+8t)\,)^{i-r}}{g^{2i-r}}\right) C_1{}^r(s,t)$$

Example 9. If $s = t = 1$, we get classic the Jacobsthal and Jacobsthal Lucas matrix sequences. For $n = 4$, the following is obtained,

$$J_1{}^4 = \begin{pmatrix} J_5 & 2J_4 \\ J_4 & 2J_3 \end{pmatrix} = \left(\frac{-2tg}{g^2 + g - 2}\right)^4 \sum_{r=0}^{s} \sum_{i=0}^{r} \binom{4}{i} \binom{4}{r-i} (-2)^{i-r} g^{r-2i} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

and

$$C_1{}^4 = \begin{pmatrix} c_5 & 2c_4 \\ c_4 & 2c_3 \end{pmatrix} = \left(\frac{18g}{g^2 + 9g + 18}\right)^4 \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{18^{i-r}}{g^{r-2i}}\right) \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}^r$$

Theorem 10. For any integer $n \geq 1$,

$$j_{nk}(\mathrm{s,t}) = j_n(\mathrm{s,t}) \sum_{i=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor} \binom{k-1-i}{i} c_n{}^{k-1-2i}(\mathrm{s,t})\,(2t)^{in} \tag{14}$$

Proof. By using the property $j_{n+1}(s,t) + 2t\, j_{n-1}(s,t) = c_n(s,t)$ and the Binet formula of $(\mathrm{s,t})$-Jacobsthal sequence, the following is obtained,

$$(J_1{}^n)^k = J_1{}^{nk} = J_{nk} = \begin{pmatrix} j_{nk+1}(s,t) & 2j_{nk}(s,t) \\ tj_{nk}(s,t) & 2tj_{nk-1}(s,t) \end{pmatrix}$$

$$(J_1{}^n)^k = (J_n)^k = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ tj_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}^k$$

$$= \begin{pmatrix} x_k - 2t\, j_{n-1}(s,t)x_{k-1} & 2j_n(s,t)x_{k-1} \\ tj_{nk}(s,t)x_{k-1} & x_k - j_{n+1}(s,t)x_{k-1} \end{pmatrix}$$

where

$$x_k = 2 \sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \binom{k-i}{i} T^{k-2i} (-D)^i$$

$$= 2\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} \begin{bmatrix} \left(j_{n+1}(s,t) + 2t\, j_{n-1}(s,t)\right)^{k-2i} \\ (\, 2t\, (j_{n+1}(s,t)\, j_{n-1}(s,t) - \, {j_n}^2(s,t))^i \end{bmatrix}$$

$$= 2\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} (c_n(s,t))^{k-2i}(-2t)^{in}$$

By the equality of the matrices, the proof is completed. □

Theorem 11.

$$j_{nk+r}(s,t) = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} \left(c_n(s,t)\right)^{k-2i}(-2t)^{in}\left[j_r(s,t) + \frac{k-2i}{k-i}\frac{(-2t)^r\, j_{n-1}(s,t)}{c_n(s,t)}\right] \tag{15}$$

PROOF. By using (a), (10) and [9], we get

$$J_1{}^n(s,t) = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ tj_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}$$

Then,

$$J_1{}^{nk+r}(s,t) = \begin{pmatrix} j_{nk+r+1}(s,t) & 2j_{nk+r}(s,t) \\ tj_{nk+r}(s,t) & 2tj_{nk+r-1}(s,t) \end{pmatrix}$$

and

$$J_1{}^{nk+r}(s,t) = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ tj_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}^k \begin{pmatrix} j_{r+1}(s,t) & 2j_r(s,t) \\ tj_r(s,t) & 2tj_{r-1}(s,t) \end{pmatrix}$$

$$= \begin{pmatrix} x_k - 2t\, j_{n-1}(s,t)x_{k-1} & 2j_n(s,t)x_{k-1} \\ tj_n(s,t)x_{k-1} & x_k - j_{n+1}(s,t)x_{k-1} \end{pmatrix} \begin{pmatrix} j_{r+1}(s,t) & 2j_r(s,t) \\ tj_r(s,t) & 2tj_{r-1}(s,t) \end{pmatrix}$$

where

$$x_k = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i}\ T^{k-2i}\,(-D)^i$$

$$= 2\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i}\ (c_n(s,t))^{k-2i}(-2t)^{in}$$

By the equality of the matrices,

$$j_{nk+r}(s,t) = (\, x_k - 2t\, j_{n-1}(s,t)x_{k-1})j_r(s,t) + 2tj_n(s,t)x_{k-1}j_{r-1}(s,t)$$

$$= j_r(s,t)x_k - 2t(\, j_{n-1}(s,t)j_r(s,t) - \, j_n(s,t)j_{r-1}(s,t))\, x_{k-1}$$

$$= j_r(s,t)x_k - (-2t)^r\, j_{n-r-1}(s,t)\, x_{k-1}$$

$$= j_r(s,t)\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i}\ (c_n(s,t))^{k-2i}(-1)^{in-i}(2t)^{in}$$

$$+(-2t)^r\, j_{n-r-1}(s,t) \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-i}{i}\ (c_n(s,t))^{k-1-2i}(-1)^{in-i}(2t)^{in}$$

$$= \sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \binom{k-i}{i} \left[\begin{array}{c} (c_n(s,t))^{k-2i}(-1)^{in-i}(2t)^{in} \\ \left[j_r(s,t) + \frac{k-2i}{k-i} \frac{(-2t)^r\, j_{n-r-1}(s,t)}{c_n(s,t)} \right] \end{array} \right] \qquad \square$$

If we choose $s = t = 1$ in (15), we get the property of the classic Jacobsthal sequences,

$$j_{nk+r} = \sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \binom{k-i}{i} (c_n)^{k-2i}(2)^{in}(-1)^{in-i} \left[j_r + \frac{k-2i}{k-i} \frac{(-2)^r\, j_{n-r-1}}{c_n} \right]$$

3. Conclusion

The paper aims to find the nth power of 2×2 special matrices whose entries are the elements of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas number sequences. From the results, some properties of the (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences are established. We develop new methods for finding the nth element of (s,t)-Jacobsthal sequences.

Conflict of Interest

The authors declare no conflict of interest.

References

[1] K. S. Williams, *The n^{th} Power of a 2×2 Matrix*, Mathematics Magazine 65(5) (1992) 336-336.

[2] J. Mc Laughlin, *Combinatorial Identities Deriving from the n^{th} Power of a 2×2 Matrix*, Integers: Electronic Journal of Combinatorial Number Theory 4 (2004) 1-15.

[3] J. Mc Laughlin, B. Sury, *Powers of Matrix and Combinatorial Identities*, Integers: Electronic Journal of Combinatorial Number Theory 5 (2005) 1-9.

[4] H. Belbachir, *Linear Recurrent Sequences and Powers of a Square Matrix*, Integers: Electronic Journal of Combinatorial Number Theory 6 (2006) 1-17.

[5] G. E. Bergum, V. E. Hoggatt Jr., Sums and products for recurring sequences, The Fibonacci Quarterly, 13(2) (1975) 115-120.

[6] Z. Akyüz, S. Halıcı, *On Some Combinatorial Identities Involving the Terms of Generalized Fibonacci and Lucas Sequences*, Hacettepe Journal of Mathematics and Statistics 42(4) (2013) 431-435.

[7] Z. Akyüz, S. Halıcı, *Some Identities Deriving from the n^{th} Power of a Special Matrix*. Advances in Difference Equations 1 (2012) 1-6.

[8] S. Uygun, *The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Sequences*, Applied Mathematical Sciences 70(9) (2015) 3467-3476.

[9] K. Uslu, S. Uygun, *The (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas Matrix Sequences*, ARS Combinatoria 108 (2013) 13-22.

[10] S. Uygun, *Some Sum Formulas of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Matrix Sequences*, Applied Mathematics 7 (2016) 61-69.

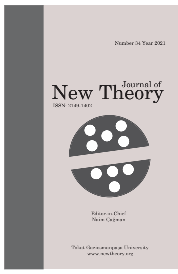
[11] S. Halıcı, M. Uysal, *A Study on Some identities involving (s_k,t)-Jacobsthal Numbers*, Notes on Number Theory and Discrete Mathematics 26(4) (2020) 4 74-79. DOI: 10.7546/nntdm.2020.26.4.74-79

[12] A. A. Wani, P. Catarino, S. Halıcı, *On a Study of (s,t)-generalized Pell Sequence and Its Matrix Sequence*, Journal of Mathematics 51(9) (2019) 17-32.

[13] A. F. Horadam, *Jacobsthal Representation Numbers*, The Fibonacci Quarterly 34(1) (1996) 40-54.

Journal of New Theory
ISSN: 2149-1402

34 (2021) 20-27
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Convergence of Multiset Sequences

Suma Pachilangode[1], **Sunil Jacob John**[2]

Article History
Received: 17 Feb 2020
Accepted: 18 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — In this paper, we introduce the concept of the multiset sequence and its convergence. A few special examples of multiset sequences, e.g. a prime identifier, are also given. A metric is defined in multisets for statistical convergences of multiset sequences. Wijsman and Hausdorff convergence of multiset sequences are discussed.

Keywords — *Multiset, multiset sequence, prime identifier sequence, set theoretic limit, Wijsman convergence, Hausdorff convergence*

Mathematics Subject Classification (2020) — 03E99, 40A05

1. Introduction

During the process of information retrieval, duplicates may occur at various stages of the process. In such situations, the need for multisets and multiset operations arises. For example, in a cyber investigation, hitting on a particular website and phone number in a tower on some time interval are some of such situations where multisets are more suitable than ordinary sets.

Multiset (in short mset) or Bag is a collection of objects in which repetition is allowed [1]. Multiset theory can be used in situations, where the classical set theory proves inadequate. Research in the multiset theory is still at the infant stage. The need for multisets was pointed out by Knuth in 1981 [1]. The papers [2–9] are on the multiset theory and its applications in Mathematics and Computer Science. The relations and operations with multisets [10], relations, and functions in multiset context [11], are some of the developments in this field.

The concept of convergence of sequences of real numbers has been extended by several authors to the convergence of sequences of sets [12–17]. Statistical convergence for sequences of sets and some basic theorems are established by Nuray and Rhoades [18]. These papers include topics, such as statistical convergence and ideal convergence of set sequences.

In this paper, we define mset sequences and investigate their various properties. There is also a comparison of mset sequences with set sequences. A few special examples of mset sequences, e.g. a prime identifier, are also given. Here we are attempting to extend the concept of convergence on classical set sequences to mset sequences. A metric is introduced on mset for statistical convergence, and making use of this metric, Wijsman and Hausdorff convergences are defined.

Sequences have been used in various fields, such as computer science, for a variety of purposes, and convergence of these sequences could be found as well. These wide range of applications of sequences and their convergences in different real-life situations is the motivation of our work.

[1]sumamuraleemohan@gmail.com (Corresponding Author); [2]sunil@nitc.ac.in
[1,2]Department of Mathematics, National Institute of Technology Calicut, Calicut 673601, Kerala, India

In Section 2, some of the preliminaries that are necessary for further sections of the paper are given. In Section 3, an extension of the sequence of sets into the multiset context is presented. The following section has some examples of the same process. The final section covers the convergence of these multiset sequences.

2. Preliminary

In this section, we recall some basic definitions and properties of multisets that are necessary for this paper.

Definition 2.1. [1] A collection of elements containing duplicates is called multiset. The word multiset is often shortened to mset. If the elements of a multiset are taken from a set X, then it is said to be drawn from X. A multiset M drawn from X can be considered a function $C_M : X \to W$, where W is the set of non-negative integers. For each $x \in X$, $C_M(x)$ is the characteristic value or count value of x in M and indicates the number of occurrence of x in M. Since characteristic value actually characterizes a multiset, most of our assertions are based on this characteristic value.

Note 2.2. Let M be an mset drawn from X with x_1 appearing k_1 times, x_2 appearing k_2 times, and x_n appearing $k_n$times. Then, M is written as $M = \{k_1|x_1, k_2|x_2, \cdots, k_n|x_n\}$. $C_M(x) = k$ is sometimes denoted as $x \in^k M$.

Definition 2.3. [8] Let M_1 and M_2 be two msets drawn from a set X. M_1 is a submultiset (shortly submset) of M_2 if $C_{M_1}(x) \leq C_{M_2}(x)$ for all x in X and is written as $M_1 \subseteq M_2$.

Definition 2.4. [8] Two msets M_1 and M_2 are *equal* if $M_1 \subseteq M_2$ and $M_2 \subseteq M_1$. In other words, $C_{M_1}(x) = C_{M_2}(x),\ \forall x \in X$.

Definition 2.5. [1] Addition of two multisets M_1 and M_2 drawn from a set X results in a new multiset $M = M_1 \oplus M_2$ such that $\forall x \in X,\ C_M(x) = C_{M_1}(x) + C_{M_2}(x)$.

Definition 2.6. [8] Subtraction of two multisets M_1 and M_2 drawn from a set X results in a new multiset $M = M_1 \ominus M_2$ such that $\forall x \in X,\ C_M(x) = \max\{C_{M_1}(x) - C_{M_2}(x), 0\}$.

Definition 2.7. [11] For an mset $M = \{k_1|x_1,\ k_2|x_2, \cdots, k_n|x_n\}$, the set $S = \{x_1, x_2, \cdots x_n\}$ is known as the *root set* of M.

Definition 2.8. [1] The union of M_1 and M_2 is a multiset, denoted by $M = M_1 \cup M_2$, with the count value $C_M(x) = \max\{C_{M_1}(x), C_{M_2}(x)\}$, for every $x \in X$.

Definition 2.9. [1] The intersection of M_1 and M_2 is a multiset, denoted by $M = M_1 \cap M_2$, with the count value $C_M(x) = \min\{C_{M_1}(x), C_{M_2}(x)\}$, for every $x \in X$.

Definition 2.10. [11] For an mset M, the power mset $\tilde{P}(M)$ is the set of all the submsets of M.

Definition 2.11. [11]$[X]^m$ is the collection of all the msets derived from X with multiplicity at most m for every element of $x \in X$.

Definition 2.12. [11] The complement of a multiset $M \in [X]^m$ is the multiset M^c with count value $C_{M^c}(x) = m - C_M(x)$.

Definition 2.13. Partition of a positive integer n is a non-increasing sequence $(n_1, n_2, \cdots n_k)$ such that $n_1 + n_2 + \ldots n_k = n$, where the elements n_i are positive integers $\forall i \in N$.

Note 2.14. Partition of a positive integer is a multiset and conversely, every multiset represents a partition of an integer. The mset $M = \{k_1|x_1, k_2|x_2, \cdots k_n|x_n\}$ is the partition of the integer $n = k_1x_1 + k_2x_2 + \cdots, k_nx_n$. In such case, we write $M = P(n)$.

3. Multiset Sequence

Definition 3.1. A sequence in which all the terms are sets is known as a set sequence. A set sequence is a function from $N \to P(X)$, where N is the set of positive integers and $P(X)$ is the power set of a nonempty set X.

Example 3.2. If $A_n = \{1, 2, \cdots, n\}$, then $\{A_n\}$ is a set sequence.

Definition 3.3. A sequence in which all the terms are multiset is known as a multiset sequence or shortly mset sequence. An mset sequence can be considered a function from $N \to \tilde{P}(X)$, where $\tilde{P}(X)$ is the power mset of a nonempty set X. Many of the properties of set sequences are also satisfied by mset sequences with appropriate modification. Some of them are discussed here.

Definition 3.4. An mset sequence $\{M_n\}$ drawn from X is said to be bounded from below if there exists an mset A drawn from X such that $A \subseteq M_n$ for each $n \in N$.

Definition 3.5. An mset sequence $\{M_n\}$ drawn from X is said to be bounded from above if there exists an mset B drawn from X such that $M_n \subseteq B$ for each $n \in N$.

Definition 3.6. An mset sequence which is bounded from below and above is known as a bounded sequence.

Definition 3.7. Let $\{M_n\}$ be an mset sequence. This sequence is an expanding sequence or non-decreasing sequence, if $M_n \subseteq M_{n+1}$ for each n.

Definition 3.8. Let$\{M_n\}$ be an mset sequence. This sequence is a contracting sequence or non-increasing sequence, if $M_{n+1} \subseteq M_n$ for each n.

Definition 3.9. An mset sequence which is either expanding or contracting is a monotone mset sequence.

Note 3.10. We can construct a monotone mset sequence from the given mset sequences.

Definition 3.11. Let $\{M_n\}$ be an mset sequence drawn from a set X. Consider the mset sequences $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, and $\{D_n\}$, defined as

$$A_n = \cap_{i=1}^{n} M_n, \quad B_n = \cup_{i=n}^{\infty} M_n, \quad C_n = \cup_{i=1}^{n} M_n, \quad \text{and} \quad D_n = \cap_{i=n}^{\infty} M_n$$

Then, $\{A_n\}$ and $\{B_n\}$ are the contracting mset sequences, while $\{C_n\}$ and $\{D_n\}$ are the expanding mset sequences.

Theorem 3.12. Distributive Laws: Let $\{M_n\}$ be an mset sequence and M be any mset, such that M_n, for all $n \in N$ and M are elements of $[X]^m$ for a nonempty set X and a positive integer m. Then,

(i) $M \cap (\cup_{n=1}^{\infty} M_n) = \cup_{n=1}^{\infty} (M \cap M_n)$

(ii) $M \cup (\cap_{n=1}^{\infty} M_n) = \cap_{n=1}^{\infty} (M \cup M_n)$

Proof.

(i) Let $M \cap (\cup_{n=1}^{\infty} M_n) = P$ and $\cup_{n=1}^{\infty} (M \cap M_n) = Q$. Then, P and Q are msets drawn from X. For an arbitrary $x \in X$, let $C_P(x) = k$. Then, $C_M(x) \geq k$, since P is a subset of M.

<u>Case 1</u>: If $C_M(x) = k$, then $C_{M \cap M_j}(x) = k$ for those j with $C_{M_j}(x) \geq k$ and $C_{M \cap M_j}(x) < k$ for those j with $C_{M_j}(x) < k$. So, $C_Q(x) = k$.

<u>Case 2</u>: If $C_M(x) > k$, then $C_{M_n}(x) \leq k$ for each n and there exists at least one r with $C_{M_r}(x) = k$. Therefore, for each $n \in N$, $C_{M \cap M_n}(x) \leq k$ and in particular $C_M \cap M_r(x) = k$. Hence, $C_Q(x) = k$. Thus, in both cases, $C_P(x) = C_Q(x)$. Since x is an arbitrary element, this is true for every element of X and this proves (i).

The proof of (ii) is similar. □

Theorem 3.13. De Morgan's Laws : Let X be a nonempty set and m be a positive integer. For an mset sequence $\{M_n\}$, where each $M_n \in [X]^m$,

(i) $(\cup_{n=1}^{\infty} M_n)^c = \cap_{n=1}^{\infty}(M_n)^c$

(ii) $(\cap_{n=1}^{\infty} M_n)^c = \cup_{n=1}^{\infty}(M_n)^c$

PROOF. (i) Let$(\cup_{n=1}^{\infty} M_n)^c = P$ and $\cap_{n=1}^{\infty}(M_n)^c = Q$. For $x \in X$, let $C_P(x) = k$. Then, $C_{\cup M_n}(x) = m - k$. $C_{M_n}(x) \leq m - k$ for each $n \in N$ and there exists at least one M_r with $C_{M_r}(x) = m - k$, $C_{(M_n)^c}(x) \geq k$ for each $n \in N$, and in particular $C_{(M_r)^c}(x) = k$. So, $C_Q(x) = k$. This completes the proof of (i).

(ii) The proof is similar to that of (i). □

4. Examples of Multiset Sequence

In this section, we introduce some multiset sequences that are of practical importance.

1. $\{N_n\}$, where $N_n = \{1|1, 2|2, \cdots, n|n\}$ is an mset sequence in which the n^{th} term contains $\frac{n(n+1)}{2}$ elements.

2. The prime factorises n completely, and let F_n be the mset of these factors, including 1. Then, $\{F_n\}$ is an mset sequence. For example,

$$F_1 = \{1\}$$

$$F_2 = \{1, 2\}$$

$$F_3 = \{1, 3\}$$

$$F_4 = \{1, 2, 2\}$$

$$F_{36} = \{1, 2, 2, 3, 3\}$$

3. For every positive integer n, define an mset $M_n = \{a_n|n, a_{n-1}|(n-1) \cdots, a_1|1\}$, where $a_i = [\frac{n}{i}]$, integer part of $\frac{n}{i}$.Then, $\{M_n\}$ is an multiset sequence with many properties, which are listed below. A remarkable one is that one can determine by using this sequence whether an integer is prime or not.

$$M_1 = \{1|1\}$$

$$M_2 = \{1|2, 2|1\}$$

$$M_3 = \{1|3, 1|2, 3|1\}$$

$$M_4 = \{1|4, 1|3, 2|2, 4|1\}$$

$$M_5 = \{1|5, 1|4, 1|3, 2|2, 5|1\}$$

Properties of $\{M_n\}$

- The root set of the n^{th} term of M_n is $\{1, 2, \cdots, n\}$.
- $M_1 \subset M_2 \subset M_3 \subset \cdots$. So, $\{M_n\}$ is an expanding sequence.
- $M_n \in [X]^n$, for each n.
- The number of elements in M_n is $\sum_{k=1}^{n} n(D_k)$. Here, $D_k = \{m \in N : m \text{ divides } k\}$ and $n(D_k)$ denotes the number of elements in D_k.

  Illustration: For $M_6 = \{1|6, 1|5, 1|4, 2|3, 3|2, 6|1\}$, $D_1 = \{1\}$, $D_2 = \{1, 2\}$, $D_3 = \{1, 3\}$, $D_4 = \{1, 2, 4\}$, $D_5 = \{1, 5\}$, $D_6 = \{1, 2, 3, 6\}$. Then, $\sum_{k=1}^{6} n(D_k) = 14$.

- If $M_n \in P(k)$, then $M_{n+1} \in P(k + \sum j)$, where $P(n)$ is the partition set of n and $j \in D_{n+1}$

  Illustration : For $M_5 = \{1|5, 1|4, 1|3, 2|2, 5|1\}$, $M_5 \in P(21)$. For $M_6 = \{1|6, 1|5, 1|4, 2|3, 3|2, 6|1\}$, $M_6 \in P(33) = P(21 + 12)$. Here, $\sum_{j \in D_6} j = 12$.

- $P(n) \subset \tilde{P}(M_n)$, where $P(n)$ is the partition set of n and $\tilde{P}(M_n)$ is the power multiset of M_n.

  Illustration : For $M_3 = \{1|3, 1|2, 3|1\}$, $\tilde{P}(M_3) = \{\phi, \{1|1\}, \{1|2\}, \{1|3\}, \{1|2, 1|3\}, \{1|2, 1|1\}, \{1|3, 1|1\}, \{2|1\}, \{1|2, 1|3, 1|1\}, \{1|2, 2|1\}, \{1|3, 2|1\}, \{3|1\}, \{1|2, 1|3, 2|1\}, \{1|2, 3|1\}, \{1|3, 3|1\}, M_3\}$ and $P(3) = \{\{1|3\}, \{1|2, 1|1\}, \{3|1\}\}$. So, $P(3) \subset \tilde{P}(M_3)$.

- $M_k \ominus M_{k-1} = D_k$

  Illustration : For $M_5 = \{1|5, 1|4, 1|3, 2|2, 5|1\}$ and $M_6 = \{1|6, 1|5, 1|4, 2|3, 3|2, 6|1\}$, $M_6 \ominus M_5 = \{1|6, 1|3, 1|2, 1|1\} = \{6, 3, 2, 1\} = D_6$.

- If $M_k \ominus M_{k-1} = \{1,\ k\}$, then k is a prime number, otherwise, composite.

  Illustration : $M_{13} \ominus M_{12} = \{1,\ 13\}$. Thus, 13 is a prime number, but $M_{12} \ominus M_{11} = \{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$, so 12 is not a prime number.

5. Convergence of Multiset Sequences

In this section, the convergences of multiset sequences are discussed. The concepts of Wijsman convergence, Hausdorff convergence, and statistical convergence are extended to mset sequences.

Definition 5.1. For an mset sequence $\{M_n\}$, $\cup_{k=1}^{\infty} \cap_{j \geq k} M_j$ is the limit infimum of $\{M_n\}$ and $\cap_{k=1}^{\infty} \cup_{j \geq k} M_j$ is the limit supremum of $\{M_n\}$.

Definition 5.2. If the limit supremum and limit infimum of an mset sequence are equal, then the sequence is said to be convergent and the common mset is known as the set theoretic limit or simply the limit of the sequence $\{M_n\}$.

Proposition 5.3.

(i) For a non-decreasing mset sequence, the set theoretic limit is $\cup_{n=1}^{\infty} M_n$, and that for a non-increasing mset sequence is $\cap_{n=1}^{\infty} M_n$.

(ii) For an mset sequence $\{M_n\}$, lim inf $M_n \subseteq$ lim sup M_n.

Definition 5.4. Let M be an mset derived from a metric space (X, d). Then, (M, d_M) is an mset metric space, if d_M is a metric on M.

Note 5.5. The metric d is also a metric on M. Since this metric is calculated without considering the multiplicity of elements, it is not treated as a good one.

Considering the multiplicity of each element of M, we can define a d_M metric as follows:

Definition 5.6. Let (X, d) be a metric space and M be an mset drawn from X. Let $d_M : M \times M \to \mathbb{R}$ be a mapping defined by $d_M(x, y) = d(x, y) + |C_M(x) - C_M(y)|$ such that $\mathbb{R}$ is the set of real numbers.

Proposition 5.7. d_M is a metric on M.

Proof.

(i) For each $x, y \in M$, $d_M(x, y) \geq 0$. $d_M(x, y) = 0$, which means $d(x, y) = 0$ and $C_M(x) = C_m(y)$. That is, $x = y$ in M.

(ii) From the definition, $d_M(x, y) = d_M(y, x)$.

(iii) For x, y, z in M,

$$\begin{aligned} d_M(x,y) &= d(x,y) + |C_M(x) - C_M(y)| \\ &\leq d(x,z) + d(z,y) + |C_M(x) - C_M(z)| + |C_M(z) - C_M(y)| \\ &= d_M(x,z) + d_M(z,y). \end{aligned}$$

□

Definition 5.8. Let (M, d_M) be an mset metric space drawn from a metric space (X, d) and A be a submset of M. For any x in M, $d_M(x, A) = inf\{d_M(x, a) : a \in A\}$.

Definition 5.9. Let (M, d_M) be an mset metric space and $M_n, (n = 1, 2, 3, \cdots)$ are submsets of M. Then, the sequence $\{M_n\}$ is said to be Wijsman convergent to an mset $A \subseteq M$ if for each $x \in M$, $\lim_{n\to\infty} d_M(x, M_n) = d_M(x, A)$. In this case, it is written as Wlim $M_n = A$.

Definition 5.10. Let (M, d_M) be an mset metric space and $M_n \subseteq M$, for $n = 1, 2, \cdots$. Then the sequence $\{M_n\}$ is said to be a Wijsman Cauchy sequence, if for each $\varepsilon > 0$, there is a positive integer N such that $|d_M(x, M_n) - d_M(x, M_m)| < \varepsilon$ for each $m, n > N$ and for each $x \in M$.

Theorem 5.11. Let (M, d_M) be an mset metric space and $M_n, (n = 1, 2, 3, \cdots)$ are submsets of M. If the sequence $\{M_n\}$ is a Wijsman convergent sequence, then it is also a Wijsman Cauchy sequence.

Proof. Suppose $\{M_n\}$ is a Wijsman convergent sequence converging to A. Then, for each $x \in M$,

$$\lim_{n\to\infty} d_M(x, M_n) = d_M(x, A)$$

So, for given $\varepsilon > 0$, there is at least one positive integer N such that

$$|d_M(x, M_k) - d_M(x, A)| < \frac{\varepsilon}{2}, \ \forall k \geq N$$

Choose $m, n > N$. Then,

$$|d_M(x, M_m) - d_M(x, A)| < \frac{\varepsilon}{2}$$

and

$$|d_M(x, M_n) - d_M(x, A)| < \frac{\varepsilon}{2}$$

Thus,

$$|d_M(x, M_m) - d_M(x, M_n)| \leq |d_M(x, M_m) - d_M(x, A)| + |d_M(x, A) - d_M(x, M_n)| < \ \varepsilon$$

Therefore, $\{M_n\}$ is a Wijsman Cauchy sequence. □

Definition 5.12. Let (M, d_M) be an mset metric space and $M_n, (n = 1, 2, 3, \cdots)$ are submsets of M. Then, $\{M_n\}$ is said to be Hausdorff convergent to an mset $A \subseteq M$ if $\lim_{n\to\infty} sup_{x\in M}|d_M(x, M_n) - d_M(x, A)| = 0$

Theorem 5.13. Let (M, d_M) be an mset metric space and $M_n, (n = 1, 2, 3, \cdots)$ are submsets of M. If $\{M_n\}$ is a Hausdorff convergent sequence, then it is also a Wijsman convergent sequence.

Proof. The proof is obtained directly from the definitions of the Wijsman and Hausdorff convergences. □

Definition 5.14. Let (M, d_M) be an mset metric space and $M_n, (n = 1, 2, 3, \cdots)$ are submsets of M. Then, $\{M_n\}$ is said to be Wijsman statistically convergent to an mset $A \subset M$ if $\lim_{n\to\infty} \frac{1}{n}\{k \leq n : |d_M(x, M_k) - d_M(x, A)| \geq \varepsilon\} = 0, \forall x \in M$ and $\forall \varepsilon > 0$.

Definition 5.15. Let (M, d_M) be an mset metric space and $M_n, (n = 1, 2, 3, \cdots)$ are submsets of M. Then, $\{M_n\}$ is said to be Hausdorff statistically convergent to an mset $A \subset M$ if $\lim_{n\to\infty} \frac{1}{n}\{k \leq n : sup_{x\in M}|d_M(x, M_k) - d_M(x, A)| \geq \varepsilon\} = 0, \forall x \in M$ and $\forall \varepsilon > 0$.

6. Conclusion

This paper attempts to probe into the development of the multiset theory in novel scenarios such as mset sequences. It delves into the sequences and their convergences to obtain various results and properties analogous to the set sequences. The work here only scratches the surface of the possibilities of convergence of msets. An expanded scope of the research in this paper can dive much deeper into the same, and further research can be conducted on their various applications in different fields.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] D. E. Knuth, *The Art of Computer Programming*, Seminumerical Algorithms, Addison-Wesley 2 (1981).

[2] E. A. Bender, *Partitions of Multisets*, Discrete Mathematics (1974) 301-311.

[3] C. S. Calude, G. Paun, G. Rozenberg, A. Salomaa, *Multiset Processing* LNCS 2235, Springer Verlag (2001) 347-358.

[4] A. Syropoulos, *Mathematics of Multisets*, In: Calude C.S., Paun G., Rozenberg G., Salomaa A. (eds) Multiset Processing. WMC 2000. Lecture Notes in Computer Science, Springer, Berlin, Heidelberg 2235 (2001).

[5] N. J. Wildberger, *A New Look at Multisets*, School of Mathematics, UNSW Sydney 2053 (2003).

[6] D. Singh, A. M. Ibrahim, T. Yohanna, J. N. Singh, *An Overview of The Application of Multiset*, Novi Sad Journal of Mathematics 37 (2007) 73-92.

[7] K. P. Girish, S. J. John, *Rough Multisets and Information Multisystems*, Advances in Decision Sciences Article ID 495392 2011 (2011) 17 pages.

[8] R. R. Yager, *On the Theory of Bags*, International Journal of General Systems 13 (1986) 23-37.

[9] S. P. Jena, S. K. Ghosh, B. K. Tripathy, *On the Theory of Bags and Lists*, Information Sciences 132 (2001) 241-254.

[10] C. Brink, *Multisets and the Algebra of Relevance logic*, Non-Classical Logic 5 (1988) 75-95.

[11] K. P. Girish, S. J. John, *Relations and Functions in Multiset Context*, Information Sciences 179 (2009) 758-768.

[12] U. Ulusu, E. Dündar, *I-Lacunary Statistical Convergence of Sequences of Sets*, Filomat 28(8) (2014) 1567–1574.

[13] A. R. Benson, R. Kumar, A. Tomkins, *Sequences of Sets*, International Conference on Knowledge Discovery and Data Mining (2018) 19-23 London, United Kingdom.

[14] H. Gumus, *On Wijsman Ideal Convergent Set of Sequences Defined by an Orlicz Function*, Filomat 30(13) (2016) 3501-3509.

[15] O. Talo, Y. Sever, *On Kuratowski I-Convergence of Sequences of Closed Sets*, Filomat 31(4) (2017) 899-912.

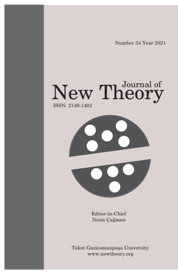
[16] M. Baronti, P. Papini, *Convergence of Sequences of Sets*, Methods of Functional Analysis in Approximation Theory 76 (1986) 133-155.

[17] R. A. Wijsman, *Convergence of Sequences of Convex Sets, Cones and Functions. II*, Transactions of the American Mathematical Society 123(1) (1966) 32-45.

[18] F. Nuray, B. E. Rhoades, *Statistical Convergence of Sequences of Sets*, Fasciculi Mathematici 49 (2012) 87-99.

Journal of New Theory
ISSN: 2149-1402

34 (2021) 28-36
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Best Proximity Point Results on Strong b-Metric Spaces

Mustafa Aslantaş[1]

Article History
Received: 20 Aug 2020
Accepted: 15 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — In the present paper, we prove the two best proximity point results on strong b-metric spaces with coefficient λ by introducing two new concepts which are named as BW b-contraction and proximal BW b-contraction. Thus, we generalise and improve many results available in the literature. To support our main results, some nontrivial and illustrative examples are given.

Keywords — *Best proximity point, BW-contraction, strong b-metric space*

Mathematics Subject Classification (2020) — 54E05, 54E35

1. Introduction and Preliminaries

In 1922, Banach [1] proved a fundamental theorem as named the Banach contraction principle, which is considered the beginning of fixed point theory on metric space. Due to its applicability, this principle has been extended and generalized by many authors in various ways [2–11]. In this sense, Boyd and Wong [12] obtained a fixed point theorem, a well-known generalization of the Banach contraction principle, as follows:

Theorem 1.1. Let $(\mho, \vartheta)$ be a complete metric space and $T : \mho \to \mho$ be a mapping such that

$$\vartheta(T\varsigma, T\varrho) \leq \psi\left(\vartheta(\varsigma, \varrho)\right)$$

for all $\varsigma, \varrho \in \mho$ where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous from the right function such that $\psi(\gamma) < \gamma$ for all $\gamma > 0$. Then, T has a unique fixed point $\xi \in \mho$.

The set of functions ψ is denoted Ψ.

On the other hand, Czerwik [13, 14] introduced the notion of a b-metric which is a generalization of a metric with a view of generalizing the Banach contraction principle.

Definition 1.2. Let $\mho$ be a non-empty set and $\vartheta : \mho \times \mho \to [0, +\infty)$ be a function such that for all $\varsigma, \varrho, \xi \in \mho$,

(1) $\vartheta(\varsigma, \varrho) = 0$ if and only if $\varsigma = \varrho$,
(2) $\vartheta(\varsigma, \varrho) = \vartheta(\varrho, \varsigma)$,
(3) $\vartheta(\varsigma, \xi) \leq \lambda[\vartheta(\varsigma, \varrho) + \vartheta(\varrho, \xi)]$.

Then ϑ is called a b-metric on ς and (ς, ϑ) is called a b-metric space.

[1]maslantas@karatekin.edu.tr (Corresponding Author)
[1]Department of Mathematics, Science Faculty, Çankırı Karatekin University, Çankırı, Turkey

It is clear that every metric space is a b-metric space, but not conversely. Indeed, let us consider the set $\mho = \mathbb{R}$ is endowed with the b-metric defined as $\vartheta(\varsigma, \varrho) = (\varsigma - \varrho)^2$ for all $\varsigma, \varrho \in \mho$. Then, $(\mho, \vartheta)$ is a b-metric space, but it is not a standard metric space. Note that b-metric may not be continuous. To remedy this deficiency, Kirk and Shahzad [15] introduced a strong b-metric space as follows:

Definition 1.3. [15] Let $\mho$ be a nonempty set. A map $\vartheta : \mho \times \mho \to [0, \infty)$ is a strong b-metric on $\mho$ if for all $\varsigma, \varrho, \xi \in \mho$ and $\lambda \geq 1$ the following conditions hold:
(i) $\varsigma = \varrho$ if and only if $\vartheta(\varsigma, \varrho) = 0$;
(ii) $\vartheta(\varsigma, \varrho) = \vartheta(\varrho, \varsigma)$;
(iii) $\vartheta(\varsigma, \xi) \leq \vartheta(\varsigma, \varrho) + \lambda\vartheta(\varrho, \xi)$.
Moreover, the triple $(\mho, \lambda, \vartheta)$ is called a strong b-metric space.

Lemma 1.4. [15] Every strong b-metric is continuous.

Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ. Each strong b-metric ϑ on $\mho$ generates T_0 topology τ_ϑ, which has, as a base, the family open p-balls

$$B(\varsigma, \varepsilon) = \{\varrho \in \mho : \vartheta(\varsigma, \varrho) < \varepsilon\}$$

for all $\varsigma \in \mho$ and $\varepsilon > 0$ A sequence $\{\varsigma_n\}$ in ς is said to be a Cauchy sequence if

$$\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0.$$

A sequence $\{\varsigma_n\}$ converges to a point ς in $\mho$ if and only if

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma) = 0.$$

$(\mho, \vartheta)$ is said to be complete if every Cauchy sequence $\{\varsigma_n\}$ in $\mho$ converges with respect to τ_ϑ to a point $\varsigma \in \mho$.

Recently, the fixed point theory has been extended and generalized in different ways for nonself mappings $T : \Gamma \to \Lambda$, where Γ and Λ are the subsets of a metric space $(\mho, \vartheta)$. Indeed, if $\Gamma \cap \Lambda = \emptyset$, it cannot have a solution of equation $T\varsigma = \varsigma$. Hence, it is sensible to investigate if there is a point ς such that $\vartheta(\varsigma, T\varsigma)$ is minimum. The concept of best proximity point has been emerged with this idea. A point ς is called a best proximity point if $\vartheta(\varsigma, T\varsigma) = \vartheta(\Gamma, \Lambda)$. Since every best proximity point is a natural generalization of fixed point, many authors have studied this topic [16–19].

Now, we recall some fundamental definitions and results on strong b-metric spaces which are useful for our main results.

Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ and Γ and Λ be nonempty subsets of $\mho$. We denote the following subsets of Γ and Λ, respectively,

$$\Gamma_0 = \{\varsigma \in \Gamma : \vartheta(\varsigma, \varrho) = \vartheta(\Gamma, \Lambda) \text{ for some } \varrho \in \Lambda\}$$

and

$$\Lambda_0 = \{\varrho \in \Lambda : \vartheta(\varsigma, \varrho) = \vartheta(\Gamma, \Lambda) \text{ for some } \varrho \in \Gamma\}$$

where $\vartheta(\Gamma, \Lambda) = \inf\{\vartheta(\varsigma, \varrho) : \varsigma \in \Gamma \text{ and } \varrho \in \Lambda\}$.

Definition 1.5. [20] Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ and Γ and Λ be nonempty subsets of $\mho$ with $\Gamma_0 \neq \emptyset$. Then, the pair (Γ, Λ) is said to exhibit the weak p-property if

$$\left.\begin{array}{l} \vartheta(\varsigma_1, \varrho_1) = \vartheta(\Gamma, \Lambda) \\ \vartheta(\varsigma_2, \varrho_2) = \vartheta(\Gamma, \Lambda) \end{array}\right\} \Longrightarrow \vartheta(\varsigma_1, \varsigma_2) \leq \vartheta(\varrho_1, \varrho_2)$$

for all $\varsigma_1, \varsigma_2 \in \Gamma_0$ and $\varrho_1, \varrho_2 \in \Lambda_0$.

Definition 1.6. [21] Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ, a mapping $T : \Gamma \to \Lambda$ is said to be a proximal contraction if there exists a nonnegative number $\alpha < 1$ such that, for all $u_1, u_2, \varsigma_1, \varsigma_2$ in Γ,

$$\left.\begin{array}{l}\vartheta(u_1, T\varsigma_1) = \vartheta(\Gamma, \Lambda)\\ \vartheta(u_2, T\varsigma_2) = \vartheta(\Gamma, \Lambda)\end{array}\right\} \Longrightarrow \vartheta(u_1, u_2) \leq \alpha\vartheta(\varsigma_1, \varsigma_2)$$

Definition 1.7. Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ and Γ and Λ be nonempty subsets of $\mho$. If every sequence $\{\varrho_n\}$ in Λ satisfying the condition $\vartheta(\varsigma, \varrho_n) \to \vartheta(\varsigma, \Lambda)$ for some ς in Γ has a subsequence $\{\varrho_{n_k}\}$ such that $\varrho_{n_k} \to \varrho \in \Lambda$, then Λ is called an approximately compact with respect to Γ.

In the present paper, we prove two best proximity point results on strong b-metric spaces with coefficient λ by introducing two new concepts which are named as *BW* b-contraction and proximal *BW* b-contraction. Thus, we generalize and improve many results avaliable in the literature. Besides, to support our main results, some nontrivial and illustrative examples are given.

2. Best Proximity Point Results with Weak p-Property

We begin the following new concept of *BW* b-contraction mapping in this section.

Definition 2.1. Let $(\mho, \vartheta)$ be strong b-metric space with coefficient λ, Γ and Λ be nonempty subsets of $\mho$ and $T \to \Gamma \to \Lambda$ be a mapping. T is called *BW* b-contraction mapping if there exists $\psi \in \Psi$ such that

$$\vartheta(T\varsigma, T\varrho) \leq \psi\left(\vartheta(\varsigma, \varrho)\right)$$

for all $\varsigma, \varrho \in \Gamma$

Theorem 2.2. Let Γ and Λ be closed subsets of complete strong b-metric space $(\mho, \vartheta)$ with $\Gamma_0 \neq \emptyset$ and $T : \Gamma \to \Lambda$ be a *BW* b-contraction mapping. Assume that the pair (Γ, Λ) has the weak p-property and $T(\Gamma_0) \subseteq \Lambda_0$. Then, T has a best proximity point in Γ.

Proof. Let $\varsigma_0 \in \Gamma_0$ be an arbitrary point. Since $T\varsigma_0 \in T(\Gamma_0) \subseteq \Lambda_0$, there exists $\varsigma_1 \in \Gamma_0$ such that

$$\vartheta(\varsigma_1, T\varsigma_0) = \vartheta(\Gamma, \Lambda)$$

Similarly, there exists $\varsigma_2 \in \Gamma_0$ such that

$$\vartheta(\varsigma_2, T\varsigma_1) = \vartheta(\Gamma, \Lambda)$$

Since (Γ, Λ) has the weak p-Property, we have

$$\vartheta(\varsigma_1, \varsigma_2) \leq \vartheta(T_0, T\varsigma_1)$$

Continuing this process, we can construct a sequence $\{\varsigma_n\}$ such that

$$\vartheta(\varsigma_{n+1}, T\varsigma_n) = \vartheta(\Gamma, \Lambda) \tag{1}$$

and

$$\vartheta(\varsigma_n, \varsigma_{n+1}) \leq \vartheta(T\varsigma_{n-1}, T\varsigma_n) \tag{2}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\varsigma_{n_0} = \varsigma_{n_0+1}$, then the proof is done. Assume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \in \mathbb{N}$. Using contractivity of T, we get

$$\begin{aligned}\vartheta(\varsigma_n, \varsigma_{n+1}) &\leq \vartheta(T\varsigma_{n-1}, T\varsigma_n) \\ &\leq \psi\left(\vartheta(\varsigma_{n-1}, \varsigma_n)\right) \\ &< \vartheta(\varsigma_{n-1}, \varsigma_n)\end{aligned} \tag{3}$$

for all $n \geq 1$. Thus, $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is a nonincreasing sequence in $\mathbb{R}$. Therefore, the sequence $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is convergent. Hence, there is $u \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = u.$$

We want to show that $u = 0$. Suppose that $u > 0$.

$$\begin{aligned} 0 &< u = \lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) \\ &\leq \lim_{n\to\infty} \vartheta\left(T\varsigma_{n-1}, T\varsigma_n\right) \\ &\leq \lim_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n)) \\ &\leq \limsup_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n)) \\ &\leq \psi(u) \\ &< u. \end{aligned}$$

This is a contradiction. Therefore, we have $\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = 0$. After that, we want to show that $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$. Assume the contrary. Hence, there is two subsequences $\{\varsigma_{n_k}\}$ and $\{\varsigma_{m_k}\}$ with $m_k > n_k \geq k$ and $\varepsilon > 0$ such that

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \geq \varepsilon \tag{4}$$

for all $k \geq 1$, where m_k is the smallest natural number satisfying (10) corresponding n_k. Therefore, we get

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) < \varepsilon.$$

Hence, we have

$$\begin{aligned} \varepsilon &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \\ &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \\ &< \varepsilon + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \end{aligned} \tag{5}$$

Letting $k \to \infty$ in (5), $\lim_{k\to\infty} \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) = \varepsilon$. Also, we have

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \leq \lambda\vartheta(\varsigma_{n_k}, \varsigma_{n_k+1}) + \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) + \lambda\vartheta(\varsigma_{m_k+1}, \varsigma_{m_k}) \tag{6}$$

and

$$\vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \leq \lambda\vartheta(\varsigma_{n_k+1}, \varsigma_{n_k}) + \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) + \lambda\vartheta(\varsigma_{m_k}, \varsigma_{m_k+1}). \tag{7}$$

Taking limit as $k \to \infty$ in (6) and (7), we get

$$\lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) = \varepsilon.$$

Then, we have

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \\ &\leq \limsup_{n\to\infty} \psi(\vartheta(\varsigma_{n_k}, \varsigma_{m_k})) \\ &\leq \psi(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

This is a contradiction. Hence, $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$ and so $\{\varsigma_n\}$ is a Cauchy sequence in Γ. Similarly, it can be seen that $\{T\varsigma_n\}$ is a Cauchy sequence in Λ. Since $(\mho, \vartheta)$ is a complete strong b-metric space with coefficient λ and Γ and Λ are closed subsets of $\mho$, there exist $\varsigma^* \in \Gamma$ and $\varrho^* \in \Lambda$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma^*) = 0$$

and

$$\lim_{n\to\infty} \vartheta(T\varsigma_n, \varrho^*) = 0$$

From Lemma 1.4, letting $n \to \infty$ in (1), we have

$$\vartheta(\varsigma^*, \varrho^*) = \vartheta(\Gamma, \Lambda). \tag{8}$$

Now, assume that $\varsigma^* \neq \varsigma_n$ for all $n \in \dot{\mathbb{N}}$. Then, we have

$$\begin{aligned} \vartheta(\varrho^*, T\varsigma^*) &= \lim_{n\to\infty} \vartheta(T\varsigma_n, T\varsigma^*) \\ &\leq \limsup_{n\to\infty} \psi(\vartheta(\varsigma_n, \varsigma^*)) \\ &< \vartheta(\varsigma_n, \varsigma^*) \end{aligned}$$

Assume that $\varsigma^* = \varsigma_n$ for some $n \in \mathbb{N}$. Then, we can find a subsequence $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ such that $\varsigma^* \neq \varsigma_{n_k}$ for all $k \in \mathbb{N}$ and so we can consider this subsequence in the above steps. Hence, we have $\vartheta(\varrho^*, T\varsigma^*) = 0$ and so $\varrho^* = T\varsigma^*$. Thus, ς^* is a best proximity point of T. □

Example 2.3. Let $\mho = \mathbb{N}$ and $\vartheta : \mho \times \mho \to \mathbb{R}$ be a function defined by

$$\vartheta(\varsigma, \varrho) = \begin{cases} 0 \ , & \varsigma = \varrho \\ 3 \ , & \varsigma, \varrho \in \{2n-1 : n \geq 1\} \text{ and } \varsigma \neq \varrho \\ 1 \ , & \text{otherwise} \end{cases}$$

It is clear that $(\mho, \vartheta)$ is a strong b-metric space with coefficient $\lambda \geq 2$. Now, we will show that $(\mho, \vartheta)$ is complete. Indeed, let $\{\varsigma_n\}$ be a Cauchy sequence. Then, for every $\varepsilon > 0$, we have

$$\vartheta(\varsigma_n, \varsigma_m) < \varepsilon$$

for all $m, n \geq n_0$. Hence, we get $\varsigma_n = \varsigma_m = \varsigma$ for all $m, n \geq n_0$. Define the sets $\Gamma = \{2n-1 : n \geq 1\}$ and $\Lambda = \{2n : n \geq 1\}$. Then, we have $\Gamma = \Gamma_0$, $\Lambda = \Lambda_0$, and $\vartheta(\Gamma, \Lambda) = 1$. Moreover, (Γ, Λ) has the weak p-Property. Let $T : \Gamma \to \Lambda$ and $\psi : [0, \infty) \to [0, \infty)$ be mappings defined as

$$T(2n-1) = 2n$$

for all $n \geq 1$ and

$$\psi(t) = \frac{t}{3}$$

for all $t \in [0, \infty)$. Then, it can be seen that $\psi \in \Psi$ and T is a BW b-contraction mapping. Furthermore, we have $T(\Gamma_0) \subseteq \Lambda_0$. Hence, all the hypotheses of Theorem 2.2 are satisfied. Therefore, T has a best proximity point in Γ.

Taking $\Gamma = \Lambda = \mho$ in Theorem 2.2, we give the following fixed point result.

Corollary 2.4. Let $(\mho, \vartheta)$ be a complete strong b-metric space with coefficient λ and $T : \mho \to \mho$ be a BW b-contraction mapping. Then, T has a fixed point in $\mho$.

If we take $\lambda = 1$ in Theorem 2.2 and Corollary 2.4 , we obtain the following results, respectively.

Corollary 2.5. Let Γ and Λ be closed subsets of complete metric space $(\mho, \vartheta)$ with $\Gamma_0 \neq \emptyset$ and $T : \Gamma \to \Lambda$ be a BW b-contraction mapping. Assume that the pair (Γ, Λ) has the weak p-Property and $T(\Gamma_0) \subseteq \Lambda_0$. Then, T has a best proximity point in Γ.

Corollary 2.6. Let $(\mho, \vartheta)$ be a complete metric space and $T : \mho \to \mho$ be a BW b-contraction mapping. Then, T has a fixed point in $\mho$.

3. Best Proximity Point Results with Proximal Contraction

Definition 3.1. Let Γ and Λ be subsets of strong b-metric space $(\mho, p)$ and $T : \Gamma \to \Lambda$ be a mapping. T is called a proximal BWb-contraction if the following condition satisfies

$$\left.\begin{array}{l}\vartheta(u_1, T\varsigma_1) = \vartheta(\Gamma, \Lambda)\\ \vartheta(u_2, T\varsigma_2) = \vartheta(\Gamma, \Lambda)\end{array}\right\} \Longrightarrow \vartheta(u_1, u_2) \leq \psi(\vartheta(\varsigma, \varrho))$$

for all $u_1, u_2, \varsigma_1, \varsigma_2 \in \Gamma$.

Theorem 3.2. Let $(\mho, \vartheta)$ be a complete strong b-metric space with coefficient λ, and Γ and Λ be closed subsets of $\mho$ with $\Gamma_0 \neq \emptyset$. Assume that $T : \Gamma \to \Lambda$ be a mapping satisfying $T(\Gamma_0) \subseteq \Lambda_0$ and Λ is an approximately compact w.r.t Γ. If T is a proximal BW b-contraction, then T has a best proximity poit in Γ.

PROOF. Let $\varsigma_0 \in \Gamma_0$ be an arbitrary point. Since $T\varsigma_0 \in T(\Gamma_0) \subseteq \Lambda_0$, there exists $\varsigma_1 \in \Gamma_0$ such that

$$\vartheta(\varsigma_1, T\varsigma_0) = \vartheta(\Gamma, \Lambda)$$

Similarly, there exists $\varsigma_2 \in \Gamma_0$ such that

$$\vartheta(\varsigma_2, T\varsigma_1) = \vartheta(\Gamma, \Lambda)$$

Since T is a proximal BW b-contraction, we have

$$\vartheta(\varsigma_1, \varsigma_2) \leq \psi(\vartheta(\varsigma_0, \varsigma_1))$$

Continuing this process, we can construct a sequence $\{\varsigma_n\}$ in Γ such that

$$\vartheta(\varsigma_{n+1}, T\varsigma_n) = \vartheta(\Gamma, \Lambda) \tag{9}$$

and

$$\vartheta(\varsigma_n, \varsigma_{n+1}) \leq \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\vartheta(\varsigma_{n_0}, \varsigma_{n_0+1}) = 0$, then the proof is done. Assume that $\vartheta(\varsigma_n, \varsigma_{n+1}) > 0$ for all $n \in \mathbb{N}$. Hence, we have

$$\begin{array}{rcl}\vartheta(\varsigma_n, \varsigma_{n+1}) & \leq & \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))\\ & < & \vartheta(\varsigma_{n-1}, \varsigma_n)\end{array}$$

for all $n \geq 1$. Thus, $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is a nonincreasing sequences in $\mathbb{R}$. Therefore, the sequence $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is convergent. Hence, there is $u \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = u.$$

We want to show that $u = 0$. Suppose that $u > 0$.

$$\begin{array}{rcl}0 & < & u = \lim\limits_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1})\\ & \leq & \lim\limits_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))\\ & \leq & \lim\sup\limits_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))\\ & \leq & \psi(u)\\ & < & u.\end{array}$$

This is a contradiction. Therefore, we have $\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = 0$. After that, we want to show that $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$. Assume the contrary. Hence, there are two subsequences $\{\varsigma_{n_k}\}$ and $\{\varsigma_{m_k}\}$ with $m_k > n_k \geq k$ and $\varepsilon > 0$ such that

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \geq \varepsilon \tag{10}$$

for all $k \geq 1$, where m_k is the smallest natural number satisfying (10) corresponding n_k. Therefore, we get

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) < \varepsilon.$$

Thus, we have

$$\begin{aligned} \varepsilon &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \\ &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \\ &< \varepsilon + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \end{aligned} \tag{11}$$

Letting $k \to \infty$ in (11), $\lim_{k\to\infty} p(\varsigma_{n_k}, \varsigma_{m_k}) = \varepsilon$. Also, we have

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \leq \lambda\vartheta(\varsigma_{n_k}, \varsigma_{n_k+1}) + \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) + \lambda\vartheta(\varsigma_{m_k+1}, \varsigma_{m_k}) \tag{12}$$

and

$$\vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \leq \lambda\vartheta(\varsigma_{n_k+1}, \varsigma_{n_k}) + \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) + \lambda\vartheta(\varsigma_{m_k}, \varsigma_{m_k+1}). \tag{13}$$

Taking limit as $k \to \infty$ in (12) and (13), we get

$$\lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) = \varepsilon.$$

Then, we have

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \\ &\leq \lim \sup_{n\to\infty} \psi(\vartheta(\varsigma_{n_k}, \varsigma_{m_k})) \\ &\leq \psi(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

This is a contradiction. Hence, $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$ and so $\{\varsigma_n\}$ is a Cauchy sequence in Γ. Since (ς, ϑ) is a complete strong b-metric space, and Γ is a closed subset of $\mho$, there exists $\varsigma^* \in \Gamma$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma^*) = 0 \tag{14}$$

Also, we have

$$\begin{aligned} \vartheta(\varsigma^*, \Lambda) &\leq \vartheta(\varsigma^*, T\varsigma_n) \\ &\leq \lambda p(\varsigma^*, \varsigma_{n+1}) + \vartheta(\varsigma_{n+1}, T\varsigma_n) \\ &= \lambda\vartheta(\varsigma^*, \varsigma_{n+1}) + \vartheta(\Gamma, \Lambda) \\ &\leq \lambda\vartheta(\varsigma^*, \varsigma_{n+1}) + \vartheta(\varsigma^*, \Lambda) \end{aligned}$$

From (14), we get $\vartheta(\varsigma^*, T\varsigma_n) \to \vartheta(\varsigma^*, \Lambda)$ as $n \to \infty$. Since Λ is an approximately compact concerning Γ, there exists a subsequence $\{T\varsigma_{n_k}\}$ of $\{T\varsigma_n\}$ such that

$$T\varsigma_{n_k} \to \varrho^*$$

for some $\varrho^* \in \Lambda$. Letting $n \to \infty$ in (9), we have

$$\vartheta(\varsigma^*, \varrho^*) = \vartheta(\Gamma, \Lambda).$$

Besides, since $T\varsigma^* \in \Lambda_0$, there exists $\xi \in \Gamma_0$ such that

$$\vartheta(\xi, T\varsigma^*) = \vartheta(\Gamma, \Lambda)$$

Now, assume that $\varsigma^* \neq \varsigma_n$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\vartheta(\varsigma^*, \xi) &= \lim_{n\to\infty} \vartheta(\varsigma_{n+1}, \xi)\\ &\leq \lim_{n\to\infty} \psi(\vartheta(\varsigma_n, \varsigma^*))\\ &< \lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma^*)\\ &= 0\end{aligned}$$

Assume that $\varsigma^* = \varsigma_n$ for some $n \in \mathbb{N}$. Then, we can find a subsequence $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ such that $\varsigma^* \neq \varsigma_{n_k}$ for all $k \in \mathbb{N}$ and so we can consider this subsequence in the above steps. Therefore, $\varsigma^* = \xi$ and so T has a best proximity point in Γ. □

Example 3.3. Let $\mho = [0,1] \cup [2,\infty)$ and $\vartheta : \mho \times \mho \to \mathbb{R}$ be a function defined as

$$\vartheta(\varsigma, \varrho) = \begin{cases} 0 & , \quad \varsigma = \varrho \\ 3 & , \quad \varsigma, \varrho \in [0,1] \text{ and } \varsigma \neq \varrho \\ |\varsigma - \varrho| & , \quad \text{otherwise} \end{cases}$$

Then, $(\mho, \vartheta)$ is a complete strong b-metric space with coefficient $\lambda \geq 2$. Define the sets $\Gamma = [0,1]$ and $\Lambda = [2,\infty)$, then we have $\vartheta(\Gamma, \Lambda) = 1$, $\Gamma_0 = \{1\}$ and $\Lambda_0 = \{2\}$. Let $T : \Gamma \to \Lambda$ be a mapping defined as

$$T\varsigma = \begin{cases} 2 & , \quad \varsigma = 1 \\ \varsigma + 2 & , \quad \varsigma \in [0,1) \end{cases}$$

for all $\varsigma \in \Gamma$. Then, we have $T(\Gamma_0) \subseteq \Lambda_0$. Further, we define a function $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. Then, it can be seen that $\psi \in \Psi$ and T is a proximal BW b-contraction mapping. Moreover, we have $T(\Gamma_0) \subseteq \Lambda_0$. Hence, all the hypotheses of Theorem 3.2 are satisfied. Therefore, T has a best proximity point in Γ.

If we take $\lambda = 1$ in Theorem 3.2, we obtain the following results, respectively.

Corollary 3.4. Let Γ and Λ be closed subsets of complete metric space $(\mho, \vartheta)$ with $\Gamma_0 \neq \emptyset$ and $T : \Gamma \to \Lambda$ be a proximal *BW* b-contraction mapping satisfying $T(\Gamma_0) \subseteq \Lambda_0$. Then, T has a best proximity point in Γ.

Note that if we take $\Gamma = \Lambda = \mho$ in Definition 3.1, then the proximal *BW* b-contraction mapping becomes *BW* b-contraction mapping. Therefore, we can obtain Corollary 2.5 and Corollary 2.6 from Theorem 3.2.

4. Conclusion

The applications of the fixed point theorems comprise diverse disciplines of mathematics, statistics, and engineering dealing with various problems such as the theory of differential equations, approximation theory, potential theory, functional analysis, and topology. In this paper, we obtain some best proximity point results on strong b-metric spaces and present some generalizations of the fixed point results.

References

[1] S. Banach, *Sur Les Opérations Dans Les Ensembles Abstraits et Leur Applications Aux Équations Intégrales*, Fundamenta Mathematicae 3(1) (1922) 133-181.

[2] M. Abbas, T. Nazir, *Fixed Point of Generalized Weakly Contractive Mappings in Ordered Partial Metric Spaces*, Fixed Point Theory and Application 2012(1) (2012) 1-19.

[3] M. Aslantaş, H. Şahin, D. Türkoğlu, *Some Caristi Type Fixed Point Theorems*, The Journal of Analysis (2020) 1-15. https://doi.org/10.1007/s41478-020-00248-8

[4] L. Ćirić, B. Samet, H. Aydi, C. Vetro, *Common Fixed Points of Generalized Contractions on Partial Metric Spaces and An Application*, Applied Mathematics and Computation 218(6) (2011) 2398-2406.

[5] M. Riaz, S. T. Tehrim, *A Robust Extension of VIKOR Method for Bipolar Fuzzy Sets Using Connection Numbers of SPA Theory Based Metric Spaces*, Artificial Intelligence Review 54 (2020) 561-591.

[6] M. Riaz, S. T. Tehrim, *Bipolar Fuzzy Soft Mappings with Application to Bipolar Disorders*, International Journal of Biomathematics 12(07) (2019) 1-31.

[7] M. Riaz, F. Smarandache, A. Firdous, A. Fakhar, *On Soft Rough Topology with Multi-attribute Group Decision Making*, Mathematics 7(1) (2019) 1-18.

[8] S. T. Tehrim, M. Riaz, *A Novel Extension of TOPSIS to MCGDM with Bipolar Neutrosophic Soft Topology*, Journal of Intelligent and Fuzzy Systems 37(4) (2019) 5531-5549.

[9] M. Riaz, M. R. Hashmi, *Fixed Points of Fuzzy Neutrosophic Soft Mapping with Decision-making*, Fixed Point Theory and Applications 2018(7) (2018) 1-10.

[10] S. Romaguera, *Fixed Point Theorems for Generalized Contractions on Partial Metric Spaces*, Topology and Its Applications 159(1) (2012) 194-199.

[11] S. Romaguera, *A Kirk Type Characterization of Completeness for Partial Metric Spaces*, Fixed Point Theory and Applications (2010) Article ID 493298, 6 pages.

[12] D. W. Boyd, J. S. Wong, *On Nonlinear Contraction*, Proceedings of the American Mathematical Society 20(2) (1969) 458-464.

[13] S. Czerwik, *Contraction Mappings in b-metric Spaces*, Acta Mathematica et Informatica Universitatis Ostraviensis 1(1) (1993) 5-11.

[14] S. Czerwik, *Nonlinear Set-valued Contraction Mappings in b-metric Spaces*, Atti del Seminario Matematico e Fisico dell'Universitá di Modena 46 (1998) 263-276.

[15] W. Kirk, N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, Cham, 2014.

[16] İ. Altun, M. Aslantaş, H. Sahin, *Best Proximity Point Results for p-proximal Contractions*, Acta Mathematica Hungarica 162 (2020) 393-402.

[17] M. Aslantaş, H. Şahin, İ. Altun, *Best Proximity Point Theorems for Cyclic p-contractions with Some Consequences and Applications*, Nonlinear Analysis: Modelling and Control 26(1) (2021) 113-129.

[18] H. Şahin, *Best Proximity Point Theory on Vector Metric Spaces*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 70(1) 2021 130-142.

[19] H. Şahin, M. Aslantaş, I. Altun, *Feng–Liu Type Approach to Best Proximity Point Results for Multivalued Mappings*, Journal of Fixed Point Theory and Applications 22(11) (2020) 1-13.

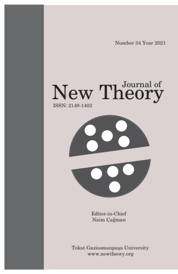
[20] M. Gabeleh, *Global Optimal Solutions of Non-self Mappings*, Scientific Bulletin, Series A: Applied Mathematics and Physics 75(3) (2013) 67-74.

[21] S. S. Basha, *Best Proximity Points: Optimal Solutions*, Journal of Optimization Theory and Applications 151 (2011) 210-216.

Journal of New Theory

ISSN: 2149-1402

34 (2021) 37-44
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



On Evolutes of Null Cartan Curves in Minkowski 4-Space

Arfah Arfah[1]

Article History
Received: 07 Dec 2020
Accepted: 28 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — This paper aims to discuss the theory of evolutes of null Cartan curves in Minkowski 4-space. In the second part, we present the basic concepts of curves in Minkowski 4-space with its Frenet equations. In the next section, the definition of evolutes of null Cartan curves in Minkowski 4-space is given, and we derive some theorems related to casual characters of those evolute curves. The last part provides an example for the theorems in the preceding section.

Keywords — *Evolutes, involutes, null Cartan curves, Minkowski space*

Mathematics Subject Classification (2020) — 53A04, 51B20

1. Introduction

Applications of geometry in many aspects of human life have motivated many mathematicians to find and develop many new theories of local and general properties of curves and surfaces in geometry. Many theories in classical differential geometry are extended to non-classical differential geometry, such as Lorentzian manifold. It started at the beginning of the twentieth century, when Einstein's theory opened a door for the use of new geometries. One of the theories in classical differential geometry which can be extended to Lorentzian space is the theory of involute-evolute of curves. The concept of involute and evolute of curves in Riemannian manifold was firstly introduced by Huygens in 1973 when he tried to create an accurate clock called isochronous pendulum clock [1]. There are many books and research articles providing explanations about the involute and evolute of curves both in Riemannian space and semi-Riemannian space [2–10].

In Lorentz-Minkowski space, a curve can locally be time-like, space-like or null depending on the casual character of the tangent vector of the curves. For non-null curves (time-like, space-like) it can easily analogue with the curve in Euclidean space. However, geometry of null curves is different from that of non-null curves since the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. The theory of null curves in Minkowski space has been studied by many mathematicians such as Ferrandez, Gimenez and Lucas [11], Inoguchi and Lee [12], and Qian and Kim [13]. Application of null curves has been studied by Duggal [14] and Mohajan [15].

In this study, we will discuss the theory of evolute curves of null Cartan curves in Minkowski 4-space. In the second part, we focus on the basic concepts of curves in Minkowski 4-space with its Frenet equations. In the next section, we introduce and give the general formula of evolute curves of null Cartan curves in Minkowski 4-space. We also provide some theorems and corollaries related to the casual characteristics of the evolute curves which are derived from the null Cartan curves. In the last part, an example is given as an application of the theorems in the previous section.

[1]arfahn70@gmail.com (Corresponding Author)
[1]Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey

2. Preliminary

Minkowski space $\mathbb{E}_1^4$ is the real vector space $\mathbb{R}^4$ equipped with the standard indefinite metric $\langle , \rangle$ defined as

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \tag{1}$$

for any vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. In Minkowski space, any vector $v \neq 0$ is said to be time-like if $\langle v, v\rangle < 0$, space-like if $\langle v, v\rangle > 0$ or $v = 0$ and null if $\langle v, v\rangle = 0$ and $v \neq 0$. The norm of a vector in $\mathbb{E}_1^3$ is defined by $\|v\| = \sqrt{|\langle v, v\rangle|}$.

Let $\alpha : I \to \mathbb{E}_1^3$ be a curve in Minkowski space. Locally, α can be time-like, space-like or null if its tangent vector is time-like, space-like or null, respectively. For non-null curves, the arc length s is defined by $s = \int_0^t \sqrt{|\langle \alpha', \alpha'\rangle|}dt$. If $\langle \alpha', \alpha'\rangle = 1$ the non-null curve is called the curve parametrized by the arc length. For null curves, since $\langle \alpha', \alpha'\rangle = 0$, the pseudo-arc length is defined by $s = \int_0^t \langle \alpha'', \alpha''\rangle^{\frac{1}{4}} dt$, and if $\langle \alpha'', \alpha''\rangle = 1$, then the null curve is parametrized by pseudo-arc length.

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the Frenet frame along the curve $\alpha(s)$ in $\mathbb{E}_1^4$. T, N, B_1 and B_2 are the tangent, principal normal, first binormal and second binormal vector fields, respectively. If α is a pseudo-null unit speed curve i.e., a space-like curve with light-like principal normal vector field parametrized by arc length s in $\mathbb{E}_1^4$, the Frenet equations of α are given by

$$T' = \kappa N, \quad N' = \tau B_1, \quad B_1' = \sigma N - \tau B_2, \quad B_2' = -\kappa T - \sigma B_1 \tag{2}$$

where κ and σ denote the curvature and bitorsion of α, respectively. T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$\begin{aligned} \langle T, T\rangle = \langle B_1, B_1\rangle = 1, \quad \langle N, N\rangle = \langle B_2, B_2\rangle = 0, \quad \langle N, B_2\rangle = 1, \\ \langle T, N\rangle = \langle T, B_1\rangle = \langle T, B_2\rangle = \langle N, B_1\rangle = \langle B_1, B_2\rangle = 0 \end{aligned} \tag{3}$$

The curvature κ in (2) has value 0 when α is a straight line and 1 in all other cases [16].

Let $\gamma : I \to (s)$ be an arbitrary null Cartan curve in $\mathbb{E}_1^4$. Then, there exists a unique Cartan frame $\{T, N, B_1, B_2\}$ given by

$$T = \frac{\gamma'}{\varphi}, \quad N = \left(\frac{1}{\varphi}\right)' \gamma' + \frac{1}{\varphi}\gamma'', \quad B_1 = -\frac{1}{\varphi}\gamma''' - \frac{\langle \gamma''', \gamma'''\rangle}{2\varphi^3}\gamma', \quad B_2 = \frac{1}{\varphi^3}(\gamma' \times \gamma'' \times \gamma''') \tag{4}$$

for any given $\varphi = \sqrt{\langle \varphi'', \varphi''\rangle} > 0$. The Frenet equations of the null curve γ is given by

$$T' = N, \quad N' = -k_1 T - B_1, \quad B' = -k_1 N + k_2 B_2, \quad B_2' = -k_2 T \tag{5}$$

where

$$k_1 = \frac{1}{2\varphi^2}(\langle \gamma''', \gamma'''\rangle + 2\varphi\varphi'' - 4(\varphi')^2), \quad k_2 = -\frac{1}{\varphi^4} \det\left(\gamma', \gamma'', \gamma''', \gamma^4\right) \tag{6}$$

Here, k_1 and k_2 are called the first and the second null curvatures of γ. The Cartan Frame $\{T, N, B_1, B_2\}$ satisfies the equations

$$\begin{aligned} \langle T, T\rangle = \langle B_1, B_2\rangle = 0, \quad \langle T, N\rangle = \langle N, N\rangle = \langle B_2, B_2\rangle = 1, \\ \langle T, N\rangle = \langle T, B_2\rangle = \langle B_1, N\rangle = \langle B_1, B_2\rangle = \langle B_2, N\rangle = 0, \end{aligned} \tag{7}$$

and

$$N \times T \times B_1 = B_2, \quad N \times B_2 \times T = T, \quad N \times B_1 \times B_2 = B_1, \quad T \times B_2 \times B_1 = N \tag{8}$$

(see [17]).

A null curve lies on pseudo-sphere in $\mathbb{E}_1^4$ with radius r if and only if $k_2 = \pm\frac{1}{r}$ [18]. In addition, a null curve which has non-zero constant k_1 and k_2 in $\mathbb{E}_1^4$ are called null helices [14]. Furthermore, a null Cartan curve in $\mathbb{E}_1^4$ is a Bertrand null curve if and only if k_1 is non-zero constant and k_2 is zero [19]. In Euclidean case, if β is an evolute of α, then for a given point P on β and the corresponding point P' on α the principal normal line of β at P is parallel to the tangent line of α at P' [8].

3. Evolutes of Null Cartan Curves

Definition 3.1. The curve $\gamma^*(s)$ is the evolute of null Cartan curve $\gamma(s)$ if and only if for all $s \in I \subseteq \mathbb{R}$, the the tangent line of $\gamma^*(s)$ intersects $\gamma(s)$ orthogonally.

Let $\gamma(s)$ be a null Cartan curve parametrized by pseudo-arc length s and γ^* be its evolute curves. If x^* be the point of contact on the evolute to the tangent line which intersects γ at $x(s)$, then $x^* - x$ lies on the tangent line of γ^* and perpendicular to the tangent vector of γ. Since γ is a null Cartan curve, $x^* - x$ can be represented as linear combination of the principal normal vector $N(s)$ and the second binormal vector $B_2(s)$ of curve $\gamma(s)$. Therefore, it can be written as

$$x^* = x(s) + p(s)(s)N(s) + q(s)B_2(s) \tag{9}$$

Next, we will find the function $p(s)$ and $q(s)$ by considering the causal characters of the curves.

Theorem 3.2. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then,

$$\gamma^*(s) = \gamma(s) + \frac{1}{k_2}B_2(s) \tag{10}$$

PROOF. If we take the derivative of equation (9), we have

$$\begin{aligned}(\gamma^*)' =& T + p'N + p(-k_1T - B_1) + q'B_2 + q(-k_2T)\\ =&(1 - pk_1 - qk_2)T + p'N - pB_1 + q'B_2\end{aligned} \tag{11}$$

Since $(\gamma^*)'(s)$ is the tangent of $\gamma^*(s)$, which is perpendicular to the tangent vector T of γ, $\gamma^*(s)$ is proportional to $\gamma^*(s) - \gamma(s) = pN + qB_2$. Therefore, we get

$$1 - pk_1 - qk_2 = 0 \quad p' = \lambda p, \quad p = 0, \quad q' = \lambda q \tag{12}$$

for some smooth real function λ in $\mathbb{E}^4_1$. Consequently, From Equation (12) we have

$$p = 0, \qquad q = \frac{1}{k_2} \tag{13}$$

Substituting these value into Equation (9) obtains Equation (10). □

Theorem 3.3. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then, the distance between $\gamma^*(s)$ and $\gamma(s)$ is $\frac{1}{k_2}$.

PROOF. From equation (10), we have

$$\gamma^*(s) - \gamma(s) = \frac{1}{k_2}B_2(s) \tag{14}$$

Therefore,

$$\|\gamma^*(s) - \gamma(s)\| = \sqrt{\left\langle \frac{1}{k_2}B_2(s), \frac{1}{k_2}B_2(s) \right\rangle} = \frac{1}{k_2}$$

□

Theorem 3.4. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then, $\gamma^*(s)$ is a space-like curve.

PROOF. From Equations (11) and (13) we have

$$(\gamma^*(s))' = -\frac{k_2'}{k_2^2}B_2 \tag{15}$$

Therefore,

$$\langle (\gamma^*(s))', (\gamma^*(s))' \rangle = \left\langle -\frac{k_2'}{k_2^2}B_2, -\frac{k_2'}{k_2^2}B_2 \right\rangle = \left(\frac{k_2'}{k_2^2}\right)^2 > 0$$

Thus, the proof is completed □

Theorem 3.5. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s and $\{T^*, N^*, B_1^*, B_2^*\}$ be the Frenet frame of $\gamma^*(s)$. If $\{T, N, B_1, B_2\}$ and k_2 are the Frenet frame and the non-constant second null curvature of $\gamma(s)$, then

$$T^* = -B_2, \quad N^* = \frac{k_2^3}{k_2'}T, \quad B_1^* = \frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}T + N, \quad B_2^* = \frac{k_2'}{k_2^3}B_1 \tag{16}$$

Proof. Let s^* be the arc length parameter of γ^*. Therefore, by Equation (15) we have

$$\frac{d\gamma^*}{ds^*} \cdot \frac{ds^*}{ds} = -\frac{k_2'}{k_2^2}B_2 \Longrightarrow T^* \frac{ds^*}{ds} = -\frac{k_2'}{k_2^2}B_2$$

Taking the norm of the Equation above yields $\frac{ds^*}{ds} = \pm\frac{k_2'}{k_2^2}$. As a result we get

$$T^* = -B_2 \tag{17}$$

Differentiating (17) towards parameter s yields

$$\frac{dT^*}{ds^*}\frac{ds^*}{ds} = k_2 T \Longrightarrow \kappa N^* = \frac{k_2^3}{k_2'}T \tag{18}$$

From (18) we find that N^* is a null principal normal vector field since T is a null vector. Therefore, γ^* is a pseudo-null curve in $\mathbb{E}_1^4$. Take $\kappa = 1$ by assuming that γ^* is a non-straight line.

Taking the derivative of N^* towards s^* yields,

$$\frac{dN}{ds^*} = \frac{dN^*}{ds}\frac{ds}{ds^*} = \left(\frac{3k_2^2(k_2')^2 - k_2^3 k_2''}{(k_2')^2}T + \frac{k_2^3}{k_2'}N\right)\frac{k_2^2}{k_2'} = \frac{3k_2^4(k_2')^2 - k_2^5 k_2''}{(k_2')^3}T + \frac{k_2^5}{(k_2')^2}N$$

As a consequence, we have

$$\|\frac{dN}{ds^*}\| = \frac{k_2^5}{(k_2')^2} \tag{19}$$

Using Equation (2) we find

$$B_1^* = \frac{\frac{dN}{ds^*}}{\|\frac{dN}{ds^*}\|} = \frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}T + N \tag{20}$$

and

$$B_2^* = \frac{k_2'}{k_2^3}B_1 \tag{21}$$

satisfying Equation (3). □

Theorem 3.5 results in the following corollary.

Corollary 3.6. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then, $\gamma^*(s)$ is a pseudo-null curve i.e., a space-like curve with light-like principal normal vector field.

Theorem 3.7. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. If γ^* is a non-straight line, then the curvature, torsion and bitorsion of γ^* are given by

$$\kappa = 1, \quad \tau = \frac{k_2^5}{(k_2')^2}, \quad \sigma = -\frac{1}{k_2}\left(\frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}\right)^2 - k_1 \tag{22}$$

Proof. Since $\gamma^*(s)$ is a pseudo-null in $\mathbb{E}_1^4$ and a non-straight line, from Equations (2) and (19), we have

$$\kappa = 1, \quad \tau = \|\frac{dN^*}{ds^*}\| = \frac{k_2^5}{(k_2')^2}$$

Differentiating Equation (21) towards parameter s^*, we have

$$\frac{dB_2^*}{ds^*} = \frac{dB_2^*}{ds}\frac{ds}{ds^*} = \left(\frac{k_2^3 k_2'' - 3k_2^2(k_2')^2}{k_2^6}B_1 + \frac{k_2'}{k_2^2}(-k_1 N + k_2 B_2)\right)\frac{k_2^2}{k_2'} = -k_1 N + \frac{k_2 k_2'' - 3(k_2')^2}{k_2^2 k_2'}B_1 + k_2 B_2$$

Therefore, using (2), we find

$$\begin{aligned}\sigma &= -\left\langle B_1^*, \frac{B_2^*}{ds^*}\right\rangle \\ &= \left\langle \frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}T + N, -k_1 N + \frac{k_2 k_2'' - 3(k_2')^2}{k_2^2 k_2'}B_1 + k_2 B_2\right\rangle \\ &= -\frac{1}{k_2}\left(\frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}\right)^2 - k_1\end{aligned}$$

□

Theorem 3.7 results in some corollaries as follow:

Corollary 3.8. If $\gamma(s)$ lies on pseudo-sphere in $\mathbb{E}_1^4$ with radius r, $\gamma(s)$ has no evolute curve.

Corollary 3.9. Let $\gamma(s)$ be a planar null Cartan curve. Then, there is no evolute of $\gamma(s)$.

Corollary 3.10. If $\gamma(s)$ is a Bertrand null curve, $\gamma(s)$ has no evolute curve.

The proof of corollaries 3.8, 3.9, and 3.10 is clear since k_2 is a constant, which implies the tangent vectors of γ^* vanish everywhere.

4. Example

In this section, an example of the evolute of the null Cartan curves is provided as an application of the theorems in the previous section.

Example 4.1. Let $\gamma : I \to E_1^3$ be a null Cartan curve parametrized by pseudo-arc length s and given as

$$\begin{aligned}\gamma(s) = \Bigg(&\frac{1}{\sqrt{56}}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} + \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right), \frac{1}{\sqrt{56}}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} - \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right), \\ &\frac{2s^2}{9\sqrt{14}}\left(2\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \\ &\frac{2s^2}{9\sqrt{14}}\left(2\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg)\end{aligned}$$

By direct calculation using (4), we find

$$T = \left(\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{s} + \frac{s^{2-\frac{3\sqrt{6}}{2}}}{s}\right), \frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{s} - \frac{s^{2-\frac{3\sqrt{6}}{2}}}{s}\right), \frac{s\sqrt{14}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right), \frac{s\sqrt{14}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$

$$\begin{aligned}N = \Bigg(&\frac{\sqrt{14}}{56}\left(\frac{(2+3\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}}{2s^2} + \frac{(2-3\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}}{2s^2}\right), \frac{\sqrt{14}}{56}\left(\frac{(2+3\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}}{2s^2} - \frac{(2-3\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}}{2s^2}\right), \\ &-\frac{\sqrt{14}}{28}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{14}}{28}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),\end{aligned}$$

$$B_1=\left(-\frac{3\sqrt{14}}{56s^3}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}+(5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{3\sqrt{14}}{56s^3}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}-(5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right)\right.$$
$$\left.\frac{\sqrt{7}}{28s}\left(13\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),\frac{\sqrt{7}}{28s}\left(13\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

$$B_2=\left(-\frac{\sqrt{7}}{28s^2}\left(-s^{2+\frac{3\sqrt{6}}{2}}-s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{\sqrt{7}}{28s^2}\left(-s^{2+\frac{3\sqrt{6}}{2}}+s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{3\sqrt{21}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right),\right.$$
$$\left.\frac{3\sqrt{21}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)$$

By using (6), we find

$$k_1=-\frac{6}{s^2},\qquad k_2=-\frac{3\sqrt{3}}{2s^2}$$

Substituting k_2 and B_2 into (9), we find the evolute curve of γ as

$$\gamma^*(s)=\left(\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}}+\frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right)+\frac{\sqrt{21}}{126}\left(s^{2+\frac{3\sqrt{6}}{2}}-s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}}-\frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right)\right.$$
$$+\frac{\sqrt{21}}{126}\left(s^{2+\frac{3\sqrt{6}}{2}}+s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{s^2\sqrt{14}}{63}\left(2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right)$$
$$+\frac{s^2\sqrt{7}}{7}\sin\left(\frac{\sqrt{2}}{2}\ln s\right),\frac{s^2\sqrt{14}}{63}\left(2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)$$
$$\left.-\frac{s^2\sqrt{7}}{7}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)$$

By using Equation (16), we get the Frenet frame of $\gamma^*(s)$ as follows:

$$T^*=\left(\frac{\sqrt{7}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}-s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right),\frac{\sqrt{7}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}+s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right),\frac{3\sqrt{21}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)}{14},-\frac{3\sqrt{21}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)}{14}\right),$$

$$N^*=\left(\frac{s\sqrt{14}}{112}\left(s^{2+\frac{3\sqrt{6}}{2}}+s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{s\sqrt{14}}{112}\left(s^{2+\frac{3\sqrt{6}}{2}}-s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{s^3\sqrt{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)}{56},\frac{s^3\sqrt{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)}{56}\right),$$

$$B_1^*=\left(-\frac{\sqrt{14}}{168}\left(\frac{(3\sqrt{6}-4)s^{2+\frac{3\sqrt{6}}{2}}}{s^2}-\frac{(3\sqrt{6}+4)s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right),\frac{\sqrt{14}}{56}\left(\frac{(3\sqrt{6}-4)s^{2+\frac{3\sqrt{6}}{2}}}{s^2}+\frac{(3\sqrt{6}+4)s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right),\right.$$
$$\left.-\frac{\sqrt{7}}{14}\left(2\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),\frac{\sqrt{7}}{14}\left(-2\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)+\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

$$B_2^*=\left(\frac{\sqrt{14}}{63}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}+(5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{\sqrt{14}}{63}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}-(5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right)\right.$$
$$-\frac{2s^2\sqrt{7}}{189}\left(13\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$
$$\left.-\frac{2s^2\sqrt{7}}{189}\left(13\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

Finally, by using Equation (20), we get

$$\kappa = 1, \quad \tau = -\frac{81\sqrt{3}}{32s^4}, \quad \sigma = 2\sqrt{3} + \frac{3}{s^2}$$

5. Conclusion

Based on the definitions, theorems, and an example in the previous section, we find that the evolute of the null Cartan curve in Minkowski 4-space is a pseudo-null curve - i.e., a space-like curve with light-like principal normal vector field. Furthermore, there is no evolute of null Cartan helices, null Bertrand curves, and null curves lying on the pseudo-sphere in $\mathbb{E}_1^4$.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] U. C. Merzbach, C. B. Boyer, *A History of Mathematics Third Edition*, John Wiley and Sons Inc., 2010.

[2] M. Bilici, M. Calışkan, *On the Involutes of the space-like Curve with a time-like Binormal in Minkowski 3-space*, International Mathematical Forum 4 (2009) 1497–1509.

[3] B. Bukcu, M. K. Karacan, *On the Involute and Evolute Curves of the time-like Curve in Minkowski 3-Space*, Demonstratio Mathematica 40 (2007) 721–732.

[4] B. Bukcu, M. K. Karacan, *On Involute and Evolute Curves of space-like Curve with a space-like Principal Normal in Minkowski 3-space*, International Journal of Mathematical Combinatorics 1 (2009) 27–37.

[5] D. Fuchs, *Evolutes and Involutes of Spatial Curves*, The American Mathematical Monthly 120(3) (2013) 217–231.

[6] M. M. Lipschutz, *Schaum's Outline of Differential Geometry*, McGraw-Hill, 1969.

[7] U. Öztürk, E. B. K. Öztürk, K. İlarslan, *On The Involute-Evolute of the Pseudonull Curve in Minkowski 3-Space*, Journal of Applied Mathematics 6(10) (2013) 1–6.

[8] T. Sochi, *Introduction to Differential Geometry of Space Curves and Surfaces*, CreateSpace Independent Publishing, 2017.

[9] G. U. Kaymanlı, *Characterization of the Evolute Offset of Ruled Surfaces with B-Darboux Frame*, Journal of New Theory 33 (2020) 50–55.

[10] M. Hanif, H. Z. Hua, E. Nesovic, *On Involutes of Order k of a Null Cartan Curve in Minkowski Spaces*, Filomat 33(8) 2295–2305.

[11] A. Ferrandez, A. Gimenez, P. Lucas, *Null Generalized Helices in Lorentz-Minkowski Spaces*, Journal of Physics A General Physics 35(39) (2002) 8243–8251.

[12] J. Inoguchi, S. Lee. *Null Curves in Minkowski 3-Space*, International Electronic Journal of Geometry 1(2) (2008) 40–83.

[13] J. Qian, Y. H. Kim, *Directional Associated Curves of a Null Curve in* $\mathbb{E}_1^3$, Bulletin of The Korean Mathematical Society 52(1) (2015) 183–200.

[14] K. L. Duggal, D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[15] H. K. Mohajan, *Minkowski Geometry and Spacetime Manifold in Relativity*, Journal of Environmental Techniques 1 (2013) 101–109.

[16] S. Yılmaz, M. Turgut, *Determination of Frenet Apparatus of Partially Null and Pseudo Null Curves in Minkowski Space-time*, International Journal of Contemporary Mathematical Sciences 3(27) (2008) 1337–1341.

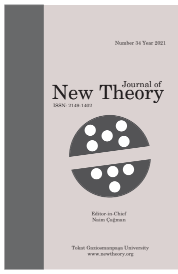
[17] X. Liu, Z. Wang, *On Lightlike Hypersurfaces and Lightlike Focal Sets of Null Cartan Curves in Lorentz-Minkowski Spacetime*, Journal of Nonlinear Science and Application 8 (2015) 628–639.

[18] M. Sasaki, *Notes on Null Curve in Minkowski Spaces*, Turkish Journal of Mathematics 34 (2010) 417–424.

[19] A. C. Çöken, Ü. Çiftçi, *On the Cartan Curvatures of a Null Curve in Minkowski Spacetime*, Geometry Dedicata 114 (2005) 71–78.

Journal of New Theory
ISSN: 2149-1402

34 (2021) 45-63
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Rough Approximations of Complex Quadripartitioned Single Valued Neutrosophic Sets

Hüseyin Kamacı[1]

Article History
Received: 27 Jan 2021
Accepted: 04 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract − The quadripartitioned single valued neutrosophic set consisting of the real-valued amplitude terms: truth-membership grade, contradiction-membership grade, ignorance-membership grade and falsity-membership grade, cannot handle complex-valued information. In this paper, the ranges of grades of truth-membership, contradiction-membership, ignorance-membership and falsity-membership are extended from the interval [0,1] to unit circle in the complex plane, and thus the notion of complex quadripartitioned single valued neutrosophic set is proposed. Further, some fundamental operations and relations on the complex quadripartitioned single valued neutrosophic sets are studied. Secondly, the rough approximations of complex quadripartitioned single valued neutrosophic sets are derived, and then their related remarkable properties are discussed. Finally, a formulation is proposed to measure rough degree of complex quadripartitioned single valued neutrosophic sets in the approximate space.

Keywords − *Single valued neutrosophic set, quadripartitioned single valued neutrosophic set, complex quadripartitioned single valued neutrosophic set, rough approximations, rough degree*

Mathematics Subject Classification (2020) − 03E72, 94D05

1. Introduction

To tackle real world issues, the techniques generally employed in classical mathematics are not always beneficial due to uncertainties and ambiguities. In 1965, Zadeh [1] proposed the fuzzy set (FS) as an effective mathematical tool to deal with such issues. In the following years, Atanassov [2] created an intuitionistic fuzzy set (IFS) that offers both the truth-membership degree and the falsity-membership degree of an object into the set. Many authors established several fuzzy models in the different aspects, i.e., relations, aggregation operators, matrix representations [3–11]. Smarandache [12] developed the neutrosophic logic sprouted from branch of philosophy *neutrosophy* which means the study of neutralities, and then initiated the theory of neutrosophic sets (NSs), a generalization of the IFSs, in which each element is characterized by a truth-membership function, indeterminate-membership function and the falsity-membership function, each of which belongs to the the non-standard unit interval $]0^-, 1^+[$. In 2010, Wang et al. [13] said that the NS is difficult to truly apply to practical problems in real world scenarios, and therefore enlivened the idea of single valued neutrosophic set (SVNS), in which each element is characterized by a truth-membership function, indeterminate-membership function and the falsity-membership function, each of which belongs to the the unit interval $[0, 1]$. For more details, refer to [14–16]. Many authors studied the generalized types of NSs and SVNSs such as interval-valued [17–19], bipolar-valued [20–22], cubic [23–26], and their practical applications [27–29].

[1]huseyin.kamaci@hotmail.com (Corresponding Author)
[1]Department of Mathematics, Faculty of Science and Arts, Yozgat Bozok University, Yozgat, Turkey

In 2017, Ali and Smarandache [30] introduced the framework of complex neutrosophic set (CNS) characterized by complex-valued truth-membership function, complex-valued indeterminate-membership function and complex-valued falsity-membership function. They put forward that the CNS is the mainstream over all because it is not only the extension of all the current frameworks but also represents the information in a complete and comprehensive way. Al-Quran and Alkhazaleh [31] studied the relations between the (single valued) CNSs with their applications in decision making.

In 1982, Pawlak [32] developed the notion of rough set which expresses vagueness in the concepts of the lower and upper approximations of a set and it employs the boundary region of a set. In [33,34], the authors established the models of rough FSs and rough IFSs. In 2014, Broumi et al. [35] introduced a hybrid structure of rough NSs and discussed its basic operations in the approximation space. In [36], the multi-attribute decision making method based on the rough accuracy score function with rough neutrosophic attribute values was constructed. Samuel and Narmadhagnanam [37] studied the tangent logarithmic distance measure and cosecant similarity measure between rough NSs. In 2018, Abdel-Basset and Mohamed [38] proposed the combination of SVNS and rough set will deal with all aspects of vagueness, incompleteness and inconsistency of data and information. Nowadays, many authors have concerned with the rough approximations of NSs and SVNSs in crisp and neutrosophic spaces, in which both constructive and axiomatic approaches are employed.

By splitting the indeterminacy in the structure of SVNSs into two parts as Unknown (or ignorance) and Contradiction, Chatterjee et al. [39] proposed the notion of quadripartitioned single valued neutrosophic set (QSVNS) based on Belnap's [40] four-valued logic. Mohan and Krishnaswamy [41] presented the axiomatic characterizations of the combined structure of QSVNS and rough set. In [42, 43], the researchers discussed the bipolarity hybridizations of QSVNSs, and some basic set-theoretic terminologies of the emerging QSVNSs. Currently, QSVNS theory has become a very successful and flourishing area of research in different aspects of both theory and practice.

In this study, we introduce the complex quadripartitioned single valued neutrosophic sets (CQSVNSs) by extending the QSVNSs, whose complex-valued truth-membership function, complex-valued contradiction-membership function, complex-valued ignorance-membership function and complex-valued falsity-membership function are the combination of real-valued truth amplitude term in association phase term, real-valued contradiction amplitude term in association phase term, real-valued ignorance amplitude term in association phase term and real-valued falsity amplitude term in association phase term, respectively. Moreover, their set-theoretic operations such as intersection, union, complement, cartesian product, algebraic products are derived. We develop the rough approximations of CQSVNSs and discuss their axiomatic characterizations. Further, we investigate the approximate precision degree and the rough degree in the novel model.

The structure of the paper is organized as follows. In Section 2, some concepts required in our work are briefly recalled. Section 3 is devoted to the construction, operations and relations of CQSVNSs. Section 4 introduces the model of rough CQSVNS in the approximation space. In Section 5, the level cut sets of lower and upper approximations and the rough degree of CQSVNS in the approximation space are studied. Section 6 gives brief conclusion and future research directions.

2. Preliminaries

In this section, we give some preliminary information that will be useful in the following sections.

Definition 2.1. [13] Let $\mathcal{A}$ be a universe of discourse. A single valued nuetrosophic set (SVNS) N in $\mathcal{A}$ is characterized in the following form

$$N = \{(a, \langle t_N(a), \iota_N(a), f_N(a)\rangle) : a \in \mathcal{A}\} \tag{1}$$

where $t_N, \iota_N, f_N : \mathcal{A} \to [0,1]$ are termed the functions of truth-membership, indeterminacy-membership and falsity-membership, respectively. Also, $t_N(a)$, $\iota_N(a)$ and $f_N(a)$ denote the grades of truth-membership, indeterminacy-membership and falsity-membership of $a \in \mathcal{A}$ to the set N respectively with the condition $0 \leq t_N(a) + \iota_N(a) + f_N(a) \leq 3$ for each $a \in \mathcal{A}$.

Definition 2.2. [30] Let $\mathcal{A}$ be a universe of discourse. A complex single valued nuetrosophic set (CSVNS) C in $\mathcal{A}$ is characterized in the following form

$$C = \{(a, \langle t_C(a), \iota_C(a), f_C(a)\rangle) : a \in \mathcal{A}\} \tag{2}$$

where $t_C(a) = \Gamma_C(a).e^{i\gamma_C(a)}$, $\iota_C(a) = \Delta_C(a).e^{i\delta_C(a)}$ and $f_C(a) = \Omega_C(a).e^{i\omega_C(a)}$ (for $i = \sqrt{-1}$) denote the complex truth-membership grade, complex indeterminacy-membership grade and complex falsity-membership grade of $a \in \mathcal{A}$ to the set C, respectively. In addition, the amplitude terms $\Gamma_C(a)$, $\Delta_C(a)$, $\Omega_C(a)$ and the phase terms $\gamma_C(a)$, $\delta_C(a)$, $\omega_C(a)$ satisfy the following conditions: $0 \leq \Gamma_C(a) + \Delta_C(a) + \Omega_C(a) \leq 3$ for $\Gamma_C(a), \Delta_C(a), \Omega_C(a) \in [0, 1]$ and $0 \leq \gamma_C(a) + \delta_C(a) + \omega_C(a) \leq 6\pi$ for $\gamma_C(a), \delta_C(a), \omega_C(a) \in [0, 2\pi]$.

Chatterjee et al. [39] split the indeterminacy in structure of SVNS into two parts signifying contradiction and unknown (ignorance), and thereby initiated the theory of quadripartitioned single valued neutrosophic set. The term quadripartitioned means something that is divided into the four characteristic features.

Definition 2.3. [39] Let $\mathcal{A}$ be a universe of discourse. A quadripartitioned single valued nuetrosophic set (QSVNS) Q in $\mathcal{A}$ is an object having the following form

$$Q = \{(a, \langle t_Q(a), c_Q(a), u_Q(a), f_Q(a)\rangle) : a \in \mathcal{A}\} \tag{3}$$

where $t_Q, c_Q, u_Q, f_Q : \mathcal{A} \to [0, 1]$ are termed the functions of truth-membership, contradiction-membership, ignorance-membership and falsity-membership, respectively. Also, $t_Q(a)$, $c_Q(a)$, $u_Q(a)$ and $f_Q(a)$ denote the grades of truth-membership, contradiction-membership, ignorance-membership and falsity-membership of $a \in \mathcal{A}$ to the set Q respectively with the condition $0 \leq t_Q(a) + c_Q(a) + u_Q(a) + f_Q(a) \leq 4$ for each $a \in \mathcal{A}$.

Remark 2.4. A QSVNS Q can be decomposed to yield two SVNS, say Q_T and Q_F, where the respective membership functions of both these are described as $t_{Q_T}(a) = t_Q(a) = t_{Q_F}(a)$; $\iota_{Q_T}(a) = c_Q(a)$; $\iota_{Q_F}(a) = u_Q(a)$; $f_{Q_T}(a) = f_Q(a) = f_{Q_F}(a)$ for all $a \in \mathcal{A}$.

In this respect, it needs to be specified that while performing set-theoretic operations on these SVNSs, the behavior of ι_{Q_T} is treated similar to that of t_{Q_T} while the behavior of ι_{Q_F} is modelled in a way similar to that of f_{Q_F}.

Assume $\mathcal{A}$ and $\mathcal{B}$ is any non-empty crisp sets. The subset of cartesian product of $\mathcal{A}$ and $\mathcal{B}$ is called a relation from $\mathcal{A}$ to $\mathcal{B}$. Especially, the subset of cartesian product of $\mathcal{A} \times \mathcal{A}$ is a relation on $\mathcal{A}$. The relation $\Re$ on $\mathcal{A}$ is said to be

1. reflexive when $(a_j, a_j) \in \Re$ for all $a_j \in \mathcal{A}$.
2. symmetric when $(a_j, a_k) \in \Re \Rightarrow (a_k, a_j) \in \Re$ for all $a_j, a_k \in \mathcal{A}$.
3. transitive when $(a_j, a_k) \in \Re$ and $(a_k, a_l) \in \Re \Rightarrow (a_j, a_l) \in \Re$ for all $a_j, a_k, a_l \in \mathcal{A}$.

If $\Re$ is reflexive, symmetric and transitive then it is called an equivalence relation on $\mathcal{A}$.

Definition 2.5. [32] Let $\mathcal{A}$ be any non-empty crisp set and $\Re$ an equivalence relation on $\mathcal{A}$. Then, $(\mathcal{A}, \Re)$ is said to be (Pawlak) approximation space. If B is a subset of $\mathcal{A}$, then the sets

$$\underline{appr}_{\Re}(B) = \{b : [b]_{\Re} \subseteq B\} \tag{4}$$

and

$$\overline{appr}_{\Re}(B) = \{b : [b]_{\Re} \cap B \neq \emptyset\} \tag{5}$$

are called the lower and upper approximations of B , respectively, where $[b]_{\Re}$ stands for the equivalence class of $\Re$ containing the object $b \in B \subseteq \mathcal{A}$. The pair $appr_{\Re}(B) = (\underline{appr}_{\Re}(B), \overline{appr}_{\Re}(B))$ is said to be rough set of B in the (Pawlak) approximation space $(\mathcal{A}, \Re)$. Especially, if $\underline{appr}_{\Re}(B) = \overline{appr}_{\Re}(B)$ then B is called a definable. The positive region, negative region and boundary region of B are defined as $\mathfrak{P}_{\Re}(B) = \underline{appr}_{\Re}(B)$, $\mathfrak{N}_{\Re}(B) = \mathcal{A} - \overline{appr}_{\Re}(B)$ and $\mathfrak{B}_{\Re}(B) = \overline{appr}_{\Re}(B) - \underline{appr}_{\Re}(B)$, respectively.

3. Complex Quadripartitioned Single Valued Neutrosophic Set Theory

In this section, we initiate the theory of complex quadripartitioned single valued neutrosophic set and discuss some basic complex quadripartitioned single valued neutrosophic operations and relations.

3.1. Construction of Complex Quadripartitioned Single Valued Neutrosophic Set

It can be observed that QSVNSs are insufficient to describe the complex information based on the four-valued logic. To eliminate this drawback, the framework of complex quadripartitioned single valued neutrosophic set is constructed as follows.

Definition 3.1. Let $\mathcal{A}$ be a universe of discourse. A complex quadripartitioned single valued neutrosophic set (CQSVNS) $\mathfrak{C}$ in $\mathcal{A}$ is characterized by a truth-membership function $t_\mathfrak{C}$, a contradiction-membership function $c_\mathfrak{C}$, an ignorance-membership function $u_\mathfrak{C}$ and a falsity-membership function $f_\mathfrak{C}$ that assign an element $a \in \mathcal{A}$ a complex-valued degree of $t_\mathfrak{C}(a)$, $c_\mathfrak{C}(a)$, $u_\mathfrak{C}(a)$ and $f_\mathfrak{C}(a)$ in $\mathfrak{C}$. The values $t_\mathfrak{C}(a)$, $c_\mathfrak{C}(a)$, $u_\mathfrak{C}(a)$, $f_\mathfrak{C}(a)$ and their sum may all within the unit circle in the complex plane, and are of the form: $t_\mathfrak{C}(a) = \Gamma_\mathfrak{C}(a).e^{i\gamma_\mathfrak{C}}(a)$, $c_\mathfrak{C}(a) = \Lambda_\mathfrak{C}(a).e^{i\lambda_\mathfrak{C}}(a)$, $u_\mathfrak{C}(a) = \Psi_\mathfrak{C}(a).e^{i\psi_\mathfrak{C}}(a)$ and $f_\mathfrak{C}(a) = \Omega_\mathfrak{C}(a).e^{i\omega_\mathfrak{C}}(a)$ (where $i = \sqrt{-1}$). In addition, the amplitude terms $\Gamma_\mathfrak{C}(a)$, $\Lambda_\mathfrak{C}(a)$, $\Psi_\mathfrak{C}(a)$, $\Omega_\mathfrak{C}(a)$ and the phase terms $\gamma_\mathfrak{C}(a)$, $\lambda_\mathfrak{C}(a)$, $\psi_\mathfrak{C}(a)$, $\omega_\mathfrak{C}(a)$ satisfy the following conditions:

$$0 \leq \Gamma_\mathfrak{C}(a) + \Lambda_\mathfrak{C}(a) + \Psi_\mathfrak{C}(a) + \Omega_\mathfrak{C}(a) \leq 4 \quad \text{for} \quad \Gamma_\mathfrak{C}(a), \Lambda_\mathfrak{C}(a), \Psi_\mathfrak{C}(a), \Omega_\mathfrak{C}(a) \in [0,1] \tag{6}$$

and

$$0 \leq \gamma_\mathfrak{C}(a) + \lambda_\mathfrak{C}(a) + \psi_\mathfrak{C}(a) + \omega_\mathfrak{C}(a) \leq 8\pi \quad \text{for} \quad \gamma_\mathfrak{C}(a), \lambda_\mathfrak{C}(a), \psi_\mathfrak{C}(a), \omega_\mathfrak{C}(a) \in [0, 2\pi] \tag{7}$$

Simply, a CQSVNS can be given in the following form:

$$\begin{aligned} \mathfrak{C} &= \{(a, \langle t_\mathfrak{C}(a), c_\mathfrak{C}(a), u_\mathfrak{C}(a), f_\mathfrak{C}(a)\rangle) : a \in \mathcal{A}\} \\ &= \{(a, \langle \Gamma_\mathfrak{C}(a).e^{i\gamma_\mathfrak{C}}(a), \Lambda_\mathfrak{C}(a).e^{i\lambda_\mathfrak{C}}(a), \Psi_\mathfrak{C}(a).e^{i\psi_\mathfrak{C}}(a), \Omega_\mathfrak{C}(a).e^{i\omega_\mathfrak{C}}(a)\rangle) : a \in \mathcal{A}\} \end{aligned} \tag{8}$$

The complex membership value $\langle \Gamma_\mathfrak{C}(a).e^{i\gamma_\mathfrak{C}}(a), \Lambda_\mathfrak{C}(a).e^{i\lambda_\mathfrak{C}}(a), \Psi_\mathfrak{C}(a).e^{i\psi_\mathfrak{C}}(a), \Omega_\mathfrak{C}(a).e^{i\omega_\mathfrak{C}}(a)\rangle$ for a of $\mathfrak{C}$ is simply denoted $((\Gamma_\mathfrak{C}, \gamma_\mathfrak{C}), (\Lambda_\mathfrak{C}, \lambda_\mathfrak{C}), (\Psi_\mathfrak{C}, \psi_\mathfrak{C}), (\Omega_\mathfrak{C}, \omega_\mathfrak{C}))$ and named as complex quadripartitioned single valued neutrosophic number (CQSVNN).

Example 3.2. Bronchitis is an inflammation of the lining of the bronchial tubes that carry air to the lungs. The bronchitis can be acute or chronic. The symptoms of acute bronchitis are usually a mild headache, cough and production of mucus. The sets of symptoms of acute bronchitis is $\mathcal{A} = \{a_1 \ (\textit{a mild headache}),\ a_2 \ (\textit{cough}),\ a_3 \ (\textit{production of mucus})\}$. While these symptoms usually improve in about a week, they may take a few weeks. By using the data of many patients who survived the disease, a doctor (expert) can create the following CQSVNS in $\mathcal{A}$ depends on the membership "recovery time of symptoms".

$$\mathfrak{C} = \left\{ \begin{array}{l} (a_1, \langle 0.4e^{i2\pi(1)}, 0.7e^{i2\pi(\frac{3}{4})}, 0.6e^{i2\pi(\frac{2}{3})}, 0.6e^{i2\pi(\frac{3}{5})}\rangle) \\ (a_2, \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.7e^{i2\pi(\frac{2}{3})}, 0e^{i2\pi(0)}, 0.6e^{i2\pi(\frac{1}{3})}\rangle), \\ (a_3, \langle 0.4e^{i2\pi(\frac{1}{3})}, 0.6e^{i2\pi(\frac{1}{4})}, 0.2e^{i2\pi(\frac{2}{5})}, 1e^{i2\pi(0)}\rangle) \end{array} \right\}$$

Definition 3.3. Let $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$. For $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$ and $\beta_1, \beta_2, \beta_3, \beta_4 \in [0, 2\pi]$, the $((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4))$-level cut set of $\mathfrak{C}$, denoted by $\mathfrak{C}^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$, is defined as follows:

$$\mathfrak{C}^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \left\{ a \in \mathcal{A} : \left(\begin{array}{l} \Gamma_\mathfrak{C}(a) \geq \alpha_1,\ \Lambda_\mathfrak{C}(a) \geq \alpha_2,\ \Psi_\mathfrak{C}(a) \leq \alpha_3,\ \Omega_\mathfrak{C}(a) \leq \alpha_4, \\ \gamma_\mathfrak{C}(a) \geq \beta_1,\ \lambda_\mathfrak{C}(a) \geq \beta_2,\ \psi_\mathfrak{C}(a) \leq \beta_3,\ \omega_\mathfrak{C}(a) \leq \beta_4 \end{array} \right) \right\} \tag{9}$$

Example 3.4. Consider the CQSVNS $\mathfrak{C}$ in Example 3.2. Then, $((0.3, \frac{\pi}{5}), (0.5, \frac{\pi}{2}), (0.7, \frac{4\pi}{3}), (1, \pi))$-level cut set of $\mathfrak{C}$ is $\mathfrak{C}^{(\frac{\pi}{5},\frac{\pi}{2},\frac{4\pi}{3},\pi)}_{(0.3,0.5,0.7,1)} = \{a_3\}$.

Proposition 3.5. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$. If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ then $(\mathfrak{C}_1)^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} \subseteq (\mathfrak{C}_2)^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$.

PROOF. It can be proved easily according to the Definition 3.3, therefore omitted. □

Definition 3.6. A CQSVNS $\mathfrak{C}$ in $\mathcal{A}$ is said to be a null CQSVNS, denoted by Φ, if its complex membership degrees are respectively $t_\Phi(a) = \Gamma_\Phi(a).e^{i\gamma_\Phi}(a) = 0$, $c_\Phi(a) = \Lambda_\Phi(a).e^{i\lambda_\Phi}(a) = 0$, $u_\Phi(a) = \Psi_\Phi(a).e^{i\psi_\Phi}(a) = e^{i2\pi}$ and $f_\Phi(a) = \Omega_\Phi(a).e^{i\omega_\Phi}(a) = e^{i2\pi}$ for all $a \in \mathcal{A}$.

Definition 3.7. A CQSVNS $\mathfrak{C}$ in $\mathcal{A}$ is said to be a absolute CQSVNS, denoted by $\widehat{\mathcal{A}}$, if its complex membership degrees are respectively $t_{\widehat{\mathcal{A}}}(a) = \Gamma_{\widehat{\mathcal{A}}}(a).e^{i\gamma_{\widehat{\mathcal{A}}}}(a) = e^{i2\pi}$, $c_{\widehat{\mathcal{A}}}(a) = \Lambda_{\widehat{\mathcal{A}}}(a).e^{i\lambda_{\widehat{\mathcal{A}}}}(a) = e^{i2\pi}$, $u_{\widehat{\mathcal{A}}}(a) = \Psi_{\widehat{\mathcal{A}}}(a).e^{i\psi_{\widehat{\mathcal{A}}}}(a) = 0$ and $f_{\widehat{\mathcal{A}}}(a) = \Omega_{\widehat{\mathcal{A}}}(a).e^{i\omega_{\widehat{\mathcal{A}}}}(a) = 0$ for all $a \in \mathcal{A}$.

3.2. Operations of Complex Quadripartitioned Single Valued Neutrosophic Set

In this part, we study the set-theoretic operations on the CQSVNSs and the properties related to them.

Definition 3.8. Let $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be three CQSVNSs in $\mathcal{A}$. Then,

(a) $\mathfrak{C}_1$ is said to be a CQSVN subset of $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$, if the following conditions are satisfied:

$$\begin{pmatrix} t_{\mathfrak{C}_1}(a) \le t_{\mathfrak{C}_2}(a), \text{ i.e., } \Gamma_{\mathfrak{C}_1}(a) \le \Gamma_{\mathfrak{C}_2}(a) \text{ and } \gamma_{\mathfrak{C}_1}(a) \le \gamma_{\mathfrak{C}_2}(a) \\ c_{\mathfrak{C}_1}(a) \le c_{\mathfrak{C}_2}(a), \text{ i.e., } \Lambda_{\mathfrak{C}_1}(a) \le \Lambda_{\mathfrak{C}_2}(a) \text{ and } \lambda_{\mathfrak{C}_1}(a) \le \lambda_{\mathfrak{C}_2}(a) \\ u_{\mathfrak{C}_1}(a) \ge u_{\mathfrak{C}_2}(a), \text{ i.e., } \Psi_{\mathfrak{C}_1}(a) \ge \Psi_{\mathfrak{C}_2}(a) \text{ and } \psi_{\mathfrak{C}_1}(a) \ge \psi_{\mathfrak{C}_2}(a) \\ f_{\mathfrak{C}_1}(a) \ge f_{\mathfrak{C}_2}(a), \text{ i.e., } \Omega_{\mathfrak{C}_1}(a) \ge \Omega_{\mathfrak{C}_2}(a) \text{ and } \omega_{\mathfrak{C}_1}(a) \ge \omega_{\mathfrak{C}_2}(a) \end{pmatrix} \tag{10}$$

(b) $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are said to be a CQSVN equal, denoted by $\mathfrak{C}_1 = \mathfrak{C}_2$, if the following conditions are satisfied:

$$\begin{pmatrix} t_{\mathfrak{C}_1}(a) = t_{\mathfrak{C}_2}(a), \text{ i.e., } \Gamma_{\mathfrak{C}_1}(a) = \Gamma_{\mathfrak{C}_2}(a) \text{ and } \gamma_{\mathfrak{C}_1}(a) = \gamma_{\mathfrak{C}_2}(a) \\ c_{\mathfrak{C}_1}(a) = c_{\mathfrak{C}_2}(a), \text{ i.e., } \Lambda_{\mathfrak{C}_1}(a) = \Lambda_{\mathfrak{C}_2}(a) \text{ and } \lambda_{\mathfrak{C}_1}(a) = \lambda_{\mathfrak{C}_2}(a) \\ u_{\mathfrak{C}_1}(a) = u_{\mathfrak{C}_2}(a), \text{ i.e., } \Psi_{\mathfrak{C}_1}(a) = \Psi_{\mathfrak{C}_2}(a) \text{ and } \psi_{\mathfrak{C}_1}(a) = \psi_{\mathfrak{C}_2}(a) \\ f_{\mathfrak{C}_1}(a) = f_{\mathfrak{C}_2}(a), \text{ i.e., } \Omega_{\mathfrak{C}_1}(a) = \Omega_{\mathfrak{C}_2}(a) \text{ and } \omega_{\mathfrak{C}_1}(a) = \omega_{\mathfrak{C}_2}(a) \end{pmatrix} \tag{11}$$

(c) the complement of $\mathfrak{C}$, denoted by $\sim \mathfrak{C}$, is defined as

$$\sim \mathfrak{C} = \{(a, \langle t_{\sim\mathfrak{C}}(a), c_{\sim\mathfrak{C}}(a), u_{\sim\mathfrak{C}}(a), f_{\sim\mathfrak{C}}(a)\rangle) : a \in \mathcal{A}\}, \tag{12}$$

where $t_{\sim\mathfrak{C}}(a) = f_{\mathfrak{C}}(a)$, $c_{\sim\mathfrak{C}}(a) = u_{\mathfrak{C}}(a)$, $u_{\sim\mathfrak{C}}(a) = c_{\mathfrak{C}}(a)$, and $f_{\sim\mathfrak{C}}(a) = t_{\mathfrak{C}}(a)$ for all $a \in \mathcal{A}$.

(d) the intersection of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \cap \mathfrak{C}_2$, is defined as

$$\begin{aligned} \mathfrak{C}_1 \cap \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a), c_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a), u_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a), f_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \tag{13}$$

where

$$\Gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a) \wedge \Gamma_{\mathfrak{C}_2}(a), \quad \Lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a) \wedge \Lambda_{\mathfrak{C}_2}(a),$$

$$\Psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a) \vee \Psi_{\mathfrak{C}_2}(a), \quad \Omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a) \vee \Omega_{\mathfrak{C}_2}(a),$$

$$\gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a) \wedge \Gamma_{\mathfrak{C}_2}(a), \quad \lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a) \wedge \lambda_{\mathfrak{C}_2}(a),$$

$$\psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a) \vee \psi_{\mathfrak{C}_2}(a), \quad \omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a) \vee \omega_{\mathfrak{C}_2}(a).$$

(e) the union of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \cup \mathfrak{C}_2$, is defined as

$$\begin{aligned} \mathfrak{C}_1 \cup \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a), c_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a), u_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a), f_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \tag{14}$$

where

$$\Gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a) \vee \Gamma_{\mathfrak{C}_2}(a), \quad \Lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a) \vee \Lambda_{\mathfrak{C}_2}(a),$$

$$\Psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a) \wedge \Psi_{\mathfrak{C}_2}(a), \quad \Omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a) \wedge \Omega_{\mathfrak{C}_2}(a),$$

$$\gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a) \vee \Gamma_{\mathfrak{C}_2}(a), \quad \lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a) \vee \lambda_{\mathfrak{C}_2}(a),$$

$$\psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a) \wedge \psi_{\mathfrak{C}_2}(a), \quad \omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a) \wedge \omega_{\mathfrak{C}_2}(a).$$

Example 3.9. Let $\mathcal{A} = \{a_1, a_2\}$ be a universal set. Assume that two CQSVNS are

$$\mathfrak{C}_1 = \{(a_1, \langle 0.5e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(0)}, 0.1e^{i2\pi(1)}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.5e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{1}{10})}\rangle)\}$$

and

$$\mathfrak{C}_2 = \{(a_1, \langle 0.6e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(\frac{5}{7})}, 0.9e^{i2\pi(\frac{9}{10})}, 0.1e^{i2\pi(\frac{5}{6})}\rangle), (a_2, \langle 0.7e^{i2\pi(\frac{2}{3})}, 0.4e^{i2\pi(1)}, 0.2e^{i2\pi(\frac{1}{5})}, 0.8e^{i2\pi(\frac{1}{2})}\rangle)\}$$

The complement of $\mathfrak{C}_1$ is

$$\sim \mathfrak{C}_1 = \{(a_1, \langle 0.1e^{i2\pi(1)}, 1e^{i2\pi(0)}, 0.7e^{i2\pi(\frac{7}{10})}, 0.5e^{i2\pi(\frac{1}{2})}\rangle), (a_2, \langle 0.7e^{i2\pi(\frac{1}{10})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.5e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{1}{2})}\rangle)\}$$

The intersection of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is

$$\mathfrak{C}_1 \cap \mathfrak{C}_2 = \{(a_1, \langle 0.5e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{9}{10})}, 0.1e^{i2\pi(1)}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.4e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.8e^{i2\pi(\frac{1}{2})}\rangle)\}$$

The union of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is

$$\mathfrak{C}_1 \cup \mathfrak{C}_2 = \{(a_1, \langle 0.6e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{5}{7})}, 0.9e^{i2\pi(0)}, 0.1e^{i2\pi(\frac{5}{6})}\rangle), (a_2, \langle 0.7e^{i2\pi(\frac{2}{3})}, 0.5e^{i2\pi(1)}, 0.2e^{i2\pi(\frac{1}{5})}, 0.7e^{i2\pi(\frac{1}{10})}\rangle)\}$$

Proposition 3.10. For three CQSVNSs $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathcal{A}$, $\sim \mathfrak{C}$, $\mathfrak{C}_1 \cap \mathfrak{C}_2$ and $\mathfrak{C}_1 \cup \mathfrak{C}_2$ are also CQSVNSs in $\mathcal{A}$.

PROOF. By considering the concepts in Definition 3.8, these results can be proved easily. □

Proposition 3.11. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be three CQSVNSs in $\mathcal{A}$. Then, the following are hold.

(i) $\mathfrak{C}_1 * \mathfrak{C}_2$ and $\mathfrak{C}_2 * \mathfrak{C}_3 \Rightarrow \mathfrak{C}_1 * \mathfrak{C}_3$ for each $* \in \{\subseteq, =\}$

(ii) $\mathfrak{C}_1 \Diamond \mathfrak{C}_2 = \mathfrak{C}_2 \Diamond \mathfrak{C}_1$ for each $\Diamond \in \{\cap, \cup\}$

(iii) $\mathfrak{C}_1 \Diamond (\mathfrak{C}_2 \Diamond \mathfrak{C}_3) = (\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \Diamond \mathfrak{C}_3$ for each $\Diamond \in \{\cap, \cup\}$

(iv) $\mathfrak{C}_1 \Diamond (\mathfrak{C}_2 \Box \mathfrak{C}_3) = (\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \Box (\mathfrak{C}_1 \Diamond \mathfrak{C}_3)$ for each $\Diamond, \Box \in \{\cap, \cup\}$

(v) $(\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \Box \mathfrak{C}_3 = (\mathfrak{C}_1 \Box \mathfrak{C}_3) \Diamond (\mathfrak{C}_2 \Box \mathfrak{C}_3)$ for each $\Diamond, \Box \in \{\cap, \cup\}$

(vi) $\sim (\mathfrak{C}_1 \Diamond \mathfrak{C}_2) = \sim \mathfrak{C}_1 \Box \sim \mathfrak{C}_2$ for each $\Diamond, \Box \in \{\cap, \cup\}$ and $\Diamond \neq \Box$

PROOF. We will prove (vi), others can be demonstrated by similar techniques.
(vi): Assume that $\Diamond = \cap$ and $\Box = \cup$. According to the operations of complement and intersection in Definition 3.8, we can write

$$\sim (\mathfrak{C}_1 \cap \mathfrak{C}_2) = \{(a, \langle f_{\mathfrak{C}_1}(a) \vee f_{\mathfrak{C}_2}(a), u_{\mathfrak{C}_1}(a) \vee u_{\mathfrak{C}_2}(a), c_{\mathfrak{C}_1}(a) \wedge c_{\mathfrak{C}_2}(a), t_{\mathfrak{C}_1}(a) \wedge t_{\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\} \tag{15}$$

Likewise, we obtain for $d = 1, 2$,

$$\sim \mathfrak{C}_d = \{(a, \langle t_{\sim\mathfrak{C}_d}(a), c_{\sim\mathfrak{C}_d}(a), u_{\sim\mathfrak{C}_d}(a), f_{\sim\mathfrak{C}_d}(a)\rangle) : a \in \mathcal{A}\} = \{(a, \langle f_{\mathfrak{C}_d}(a), u_{\mathfrak{C}_d}(a), c_{\mathfrak{C}_d}(a), t_{\mathfrak{C}_d}(a)\rangle) : a \in \mathcal{A}\}$$

and so

$$\sim \mathfrak{C}_1 \cup \sim \mathfrak{C}_2 = \{(a, \langle f_{\mathfrak{C}_1}(a) \vee f_{\mathfrak{C}_2}(a), u_{\mathfrak{C}_1}(a) \vee u_{\mathfrak{C}_2}(a), c_{\mathfrak{C}_1}(a) \wedge c_{\mathfrak{C}_2}(a), t_{\mathfrak{C}_1}(a) \wedge t_{\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\} \quad (16)$$

From Eqs. (15) and (16), we have $\sim (\mathfrak{C}_1 \cap \mathfrak{C}_2) = \sim \mathfrak{C}_1 \cup \sim \mathfrak{C}_2$. It is shown in a similar way that $\sim (\mathfrak{C}_1 \cup \mathfrak{C}_2) = \sim \mathfrak{C}_1 \cap \sim \mathfrak{C}_2$. □

Definition 3.12. Let $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be three CQSVNSs in $\mathcal{A}$ and $n > 0$ be a real number. Then, the following operational laws are hold.

(a)

$$\begin{aligned} \mathfrak{C}_1 \oplus \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a), c_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a), u_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a), f_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\} \end{aligned} \quad (17)$$

where $\Gamma_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a) + \Gamma_{\mathfrak{C}_2}(a) - \Gamma_{\mathfrak{C}_1}(a)\Gamma_{\mathfrak{C}_2}(a)$, $\Lambda_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a) + \Lambda_{\mathfrak{C}_2}(a) - \Lambda_{\mathfrak{C}_1}(a)\Lambda_{\mathfrak{C}_2}(a)$, $\Psi_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a)\Psi_{\mathfrak{C}_2}(a)$, $\Omega_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a)\Omega_{\mathfrak{C}_2}(a)$, $\gamma_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a) + \gamma_{\mathfrak{C}_2}(a) - \gamma_{\mathfrak{C}_1}(a)\gamma_{\mathfrak{C}_2}(a)$, $\lambda_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a) + \lambda_{\mathfrak{C}_2}(a) - \lambda_{\mathfrak{C}_1}(a)\lambda_{\mathfrak{C}_2}(a)$, $\psi_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a)\psi_{\mathfrak{C}_2}(a)$, and $\omega_{\mathfrak{C}_1 \oplus \mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a)\omega_{\mathfrak{C}_2}(a)$.

(b)

$$\begin{aligned} \mathfrak{C}_1 \otimes \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a), c_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a), u_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a), f_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (18)$$

where $\Gamma_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a)\Gamma_{\mathfrak{C}_2}(a)$, $\Lambda_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a)\Lambda_{\mathfrak{C}_2}(a)$, $\Psi_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a) + \Psi_{\mathfrak{C}_2}(a) - \Psi_{\mathfrak{C}_1}(a)\Psi_{\mathfrak{C}_2}(a)$, $\Omega_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a) + \Omega_{\mathfrak{C}_2}(a) - \Omega_{\mathfrak{C}_1}(a)\Omega_{\mathfrak{C}_2}(a)$, $\gamma_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a)\gamma_{\mathfrak{C}_2}(a)$, $\lambda_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a)\lambda_{\mathfrak{C}_2}(a)$, $\psi_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a) + \psi_{\mathfrak{C}_2}(a) - \psi_{\mathfrak{C}_1}(a)\psi_{\mathfrak{C}_2}(a)$, and $\omega_{\mathfrak{C}_1 \otimes \mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a) + \omega_{\mathfrak{C}_2}(a) - \omega_{\mathfrak{C}_1}(a)\omega_{\mathfrak{C}_2}(a)$.

(c)

$$\begin{aligned} n\mathfrak{C} &= \{(a, \langle t_{n\mathfrak{C}}(a), c_{n\mathfrak{C}}(a), u_{n\mathfrak{C}}(a), f_{n\mathfrak{C}}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{n\mathfrak{C}}(a).e^{i\gamma_{n\mathfrak{C}}(a)}, \Lambda_{n\mathfrak{C}}(a).e^{i\lambda_{n\mathfrak{C}}(a)}, \\ \Psi_{n\mathfrak{C}}(a).e^{i\psi_{n\mathfrak{C}}(a)}, \Omega_{n\mathfrak{C}}(a).e^{i\omega_{n\mathfrak{C}}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (19)$$

where $\Gamma_{n\mathfrak{C}}(a) = 1 - (1 - \Gamma_{\mathfrak{C}}(a))^n$, $\Lambda_{n\mathfrak{C}}(a) = 1 - (1 - \Lambda_{\mathfrak{C}}(a))^n)$, $\Psi_{n\mathfrak{C}}(a) = (\Psi_{\mathfrak{C}}(a))^n$, $\Omega_{n\mathfrak{C}}(a) = (\Omega_{\mathfrak{C}}(a))^n$, $\gamma_{n\mathfrak{C}}(a) = 1 - (1 - \gamma_{\mathfrak{C}}(a))^n$, $\lambda_{n\mathfrak{C}}(a) = 1 - (1 - \lambda_{\mathfrak{C}}(a))^n$, $\psi_{n\mathfrak{C}}(a) = (\psi_{\mathfrak{C}}(a))^n$, and $\omega_{n\mathfrak{C}}(a) = (\omega_{\mathfrak{C}}(a))^n$.

(d)

$$\begin{aligned} \mathfrak{C}^n &= \{(a, \langle t_{\mathfrak{C}^n}(a), c_{\mathfrak{C}^n}(a), u_{\mathfrak{C}^n}(a), f_{\mathfrak{C}^n}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}^n}(a).e^{i\gamma_{\mathfrak{C}^n}(a)}, \Lambda_{\mathfrak{C}^n}(a).e^{i\lambda_{\mathfrak{C}^n}(a)}, \\ \Psi_{\mathfrak{C}^n}(a).e^{i\psi_{\mathfrak{C}^n}(a)}, \Omega_{\mathfrak{C}^n}(a).e^{i\omega_{\mathfrak{C}^n}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (20)$$

where $\Gamma_{\mathfrak{C}^n}(a) = (\Gamma_{\mathfrak{C}}(a))^n$, $\Lambda_{\mathfrak{C}^n}(a) = (\Lambda_{\mathfrak{C}}(a))^n$, $\Psi_{\mathfrak{C}^n}(a) = 1 - (1 - \Psi_{\mathfrak{C}}(a))^n$, $\Omega_{\mathfrak{C}^n}(a) = 1 - (1 - \Omega_{\mathfrak{C}}(a))^n$, $\gamma_{\mathfrak{C}^n}(a) = (\gamma_{\mathfrak{C}}(a))^n$, $\lambda_{\mathfrak{C}^n}(a) = (\lambda_{\mathfrak{C}}(a))^n$, $\psi_{\mathfrak{C}^n}(a) = 1 - (1 - \psi_{\mathfrak{C}}(a))^n$, and $\omega_{\mathfrak{C}^n}(a) = 1 - (1 - \omega_{\mathfrak{C}}(a))^n$.

Example 3.13. Consider the CQSVNSs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in Example 3.9 and $n = 2$. Then we have

$$\mathfrak{C}_1 \oplus \mathfrak{C}_1 = \left\{ \begin{array}{l} (a_1, \langle 0.8e^{i2\pi(\frac{2}{3})}, 0.76e^{i2\pi(\frac{32}{35})}, 0.9e^{i2\pi(0)}, 0.01e^{i2\pi(\frac{5}{6})}\rangle), \\ (a_2, \langle 0.82e^{i2\pi(\frac{5}{6})}, 0.7e^{i2\pi(1)}, 0.08e^{i2\pi(\frac{3}{25})}, 0.56e^{i2\pi(\frac{1}{20})}\rangle) \end{array} \right\},$$

$$\mathfrak{C}_1 \otimes \mathfrak{C}_1 = \left\{ \begin{array}{l} (a_1, \langle 0.3e^{i2\pi(\frac{1}{6})}, 0.14e^{i2\pi(\frac{1}{2})}, 1e^{i2\pi(\frac{9}{10})}, 0.19e^{i2\pi(1)}\rangle), \\ (a_2, \langle 0.28e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(1)}, 0.52e^{i2\pi(\frac{17}{25})}, 0.94e^{i2\pi(\frac{11}{20})}\rangle) \end{array} \right\},$$

$$2\mathfrak{C}_1 = \left\{ \begin{array}{l} (a_1, \langle 0.75e^{i2\pi(\frac{3}{4})}, 0.91e^{i2\pi(\frac{91}{100})}, 1e^{i2\pi(0)}, 0.01e^{i2\pi(1)}\rangle), \\ (a_2, \langle 0.64e^{i2\pi(\frac{3}{4})}, 0.75e^{i2\pi(1)}, 0.16e^{i2\pi(\frac{9}{25})}, 0.49e^{i2\pi(\frac{1}{100})}\rangle) \end{array} \right\},$$

and

$$\mathfrak{C}_1^2 = \left\{ \begin{array}{l} (a_1, \langle 0.25e^{i2\pi(\frac{1}{4})}, 0.49e^{i2\pi(\frac{49}{100})}, 1e^{i2\pi(0)}, 0.19e^{i2\pi(1)}\rangle), \\ (a_2, \langle 0.16e^{i2\pi(\frac{1}{4})}, 0.25e^{i2\pi(1)}, 0.64e^{i2\pi(\frac{21}{25})}, 0.91e^{i2\pi(\frac{19}{100})}\rangle) \end{array} \right\}$$

Proposition 3.14. If $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are three CQSVNSs in $\mathcal{A}$ and $n > 0$ is a real number then $\mathfrak{C}_1 \oplus \mathfrak{C}_2$, $\mathfrak{C}_1 \otimes \mathfrak{C}_2$, $n\mathfrak{C}$ and $\mathfrak{C}^n$ are also CQSVNSs in $\mathcal{A}$.

PROOF. By considering Definition 3.12, these results can be proved easily. □

Proposition 3.15. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$ and $n, m > 0$ be two real numbers. Then,

(i) $\mathfrak{C}_1 \blacklozenge \mathfrak{C}_2 = \mathfrak{C}_2 \blacklozenge \mathfrak{C}_1$ for each $\blacklozenge \in \{\oplus, \otimes\}$

(ii) $n(\mathfrak{C}_1 \oplus \mathfrak{C}_2) = n\mathfrak{C}_1 \oplus n\mathfrak{C}_2$

(iii) $n\mathfrak{C}_1 \oplus m\mathfrak{C}_1 = (n + m)\mathfrak{C}_1$

(iv) $(\mathfrak{C}_1 \oplus \mathfrak{C}_2)^n = \mathfrak{C}_1^n \otimes \mathfrak{C}_2^n$

(v) $\mathfrak{C}_1^n \otimes \mathfrak{C}_1^m = \mathfrak{C}_1^{n+m}$

(vi) $\sim (\mathfrak{C}_1 \blacklozenge \mathfrak{C}_2) = \sim \mathfrak{C}_1 \blacksquare \sim \mathfrak{C}_1$ for each $\blacklozenge, \blacksquare \in \{\oplus, \otimes\}$ and $\blacklozenge \neq \blacksquare$

PROOF. It can be proved similar to calculations in the proof of Proposition 3.11. □

Proposition 3.16. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be three CQSVNSs in $\mathcal{A}$. Then,

(i) $(\mathfrak{C}_1 \lozenge \mathfrak{C}_2) \blacklozenge \mathfrak{C}_3 = (\mathfrak{C}_1 \blacklozenge \mathfrak{C}_3) \lozenge (\mathfrak{C}_2 \blacklozenge \mathfrak{C}_3)$ for each $\lozenge \in \{\cap, \cup\}$ and $\blacklozenge \in \{\oplus, \otimes\}$

(ii) $(\mathfrak{C}_1 \lozenge \mathfrak{C}_2) \blacklozenge (\mathfrak{C}_1 \square \mathfrak{C}_2) = (\mathfrak{C}_1 \blacklozenge \mathfrak{C}_2)$ for each $\blacklozenge \in \{\oplus, \otimes\}$, $\lozenge, \square \in \{\cap, \cup\}$ and $\lozenge \neq \square$

PROOF. From Definitions 3.8 and 3.12, they can be proved easily. □

Definition 3.17. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$. Then, the cartesian product of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \times \mathfrak{C}_2$, is defined as

$$\begin{aligned} \mathfrak{C}_1 \times \mathfrak{C}_2 &= \left\{ \left((a_j, a_k), \left\langle \begin{array}{l} t_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k), c_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k), \\ u_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k), f_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) \end{array} \right\rangle \right) : (a_j, a_k) \in \mathcal{A} \times \mathcal{A} \right\}, \\ &= \left\{ \left((a_j, a_k), \left\langle \begin{array}{l} \Gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)}, \\ \Lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)}, \\ \Psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)}, \\ \Omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)} \end{array} \right\rangle \right) : (a_j, a_k) \in \mathcal{A} \times \mathcal{A} \right\} \end{aligned} \quad (21)$$

where $\Gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Gamma_{\mathfrak{C}_1}(a_j) \wedge \Gamma_{\mathfrak{C}_2}(a_k)$, $\Lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Lambda_{\mathfrak{C}_1}(a_j) \wedge \Lambda_{\mathfrak{C}_2}(a_k)$, $\Psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Psi_{\mathfrak{C}_1}(a_j) \vee \Psi_{\mathfrak{C}_2}(a_k)$, $\Omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Omega_{\mathfrak{C}_1}(a_j) \vee \Omega_{\mathfrak{C}_2}(a_k)$, $\gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \gamma_{\mathfrak{C}_1}(a_j) \wedge \Gamma_{\mathfrak{C}_2}(a_k)$, $\lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \lambda_{\mathfrak{C}_1}(a_j)) \wedge \lambda_{\mathfrak{C}_2}(a_k)$, $\psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)) = \psi_{\mathfrak{C}_1}(a_j) \vee \psi_{\mathfrak{C}_2}(a_k)$, and $\omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)) = \omega_{\mathfrak{C}_1}(a_j) \vee \omega_{\mathfrak{C}_2}(a_k)$.

Example 3.18. Consider the CQSVNSs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in Example 3.9. Then, the cartesian product of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is

$$\mathfrak{C}_1 \times \mathfrak{C}_2 = \left\{ \begin{array}{l} ((a_1,a_1), \langle 0.5e^{i2\pi(\frac{1}{4})}, 0.2e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{9}{10})}, 0.1e^{i2\pi(1)}\rangle), \\ ((a_1,a_2), \langle 0.5e^{i2\pi(\frac{1}{4})}, 0.4e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{1}{3})}, 0.8e^{i2\pi(1)}\rangle), \\ ((a_2,a_1), \langle 0.4e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(\frac{5}{7})}, 0.9e^{i2\pi(\frac{9}{10})}, 0.7e^{i2\pi(\frac{5}{6})}\rangle), \\ ((a_2,a_2), \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.4e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.8e^{i2\pi(\frac{1}{2})}\rangle) \end{array} \right\}$$

Also, the cartesian product of $\mathfrak{C}_1 \times \mathfrak{C}_1$ is

$$\mathfrak{C}_1 \times \mathfrak{C}_1 = \left\{ \begin{array}{l} ((a_1,a_1), \langle 0.5e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(0)}, 0.1e^{i2\pi(1)}\rangle), \\ ((a_1,a_2), \langle 0.4e^{i2\pi(\frac{1}{4})}, 0.5e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(1)}\rangle), \\ ((a_2,a_1), \langle 0.4e^{i2\pi(\frac{1}{4})}, 0.5e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(1)}\rangle), \\ ((a_2,a_2), \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.5e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{1}{10})}\rangle) \end{array} \right\}$$

Proposition 3.19. For two CQSVNSs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathcal{A}$, $\mathfrak{C}_1 \times \mathfrak{C}_2$ is a CQSVNS in $\mathcal{A} \times \mathcal{A}$.

PROOF. By considering Definition 3.17, this result can be demonstrated easily. □

Proposition 3.20. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be three CQSVNSs in $\mathcal{A}$. Then,

(i) $\mathfrak{C}_1 * \mathfrak{C}_2 \Rightarrow (\mathfrak{C}_1 \times \mathfrak{C}_3) * (\mathfrak{C}_2 \times \mathfrak{C}_3)$ for each $* \in \{\subseteq, =\}$

(ii) $\mathfrak{C}_1 \times (\mathfrak{C}_2 \times \mathfrak{C}_3) = (\mathfrak{C}_1 \times \mathfrak{C}_2) \times \mathfrak{C}_3$

(iii) $\mathfrak{C}_1 \times (\mathfrak{C}_2 \Diamond \mathfrak{C}_3) = (\mathfrak{C}_1 \times \mathfrak{C}_2) \Diamond (\mathfrak{C}_1 \times \mathfrak{C}_3)$ for each $\Diamond \in \{\cap, \cup\}$

(iv) $(\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \times \mathfrak{C}_3) = (\mathfrak{C}_1 \times \mathfrak{C}_3) \Diamond (\mathfrak{C}_2 \times \mathfrak{C}_3)$ for each $\Diamond \in \{\cap, \cup\}$

PROOF. We will prove the assertion (i), the other can be demonstrated in a similar way.

(i): Assume that $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$. By considering Eq. (10), for truth-membership grades, we have $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_2}(a_j)$, i.e. $\Gamma_{\mathfrak{C}_1}(a_j) \leq \Gamma_{\mathfrak{C}_2}(a_j)$ and $\gamma_{\mathfrak{C}_1}(a_j) \leq \gamma_{\mathfrak{C}_2}(a_j)$. There are three cases.

Case 1: If $t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_2}(a_j)$ then $t_{\mathfrak{C}_3}(a_k) \wedge t_{\mathfrak{C}_1}(a_j) = t_{\mathfrak{C}_3}(a_k)$ and $t_{\mathfrak{C}_3}(a_j) \wedge t_{\mathfrak{C}_2}(a_k) = t_{\mathfrak{C}_3}(a_k)$. It follows $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$.

Case 2: If $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j)$ then $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_1}(a_j)$ and $t_{\mathfrak{C}_3}(a_j) \wedge t_{\mathfrak{C}_2}(a_k) = t_{\mathfrak{C}_3}(a_k)$. Since $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_3}(a_k)$, it is obtained that $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$.

Case 3: If $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_2}(a_j) \leq t_{\mathfrak{C}_3}(a_k)$ then $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_1}(a_j)$ and $t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_2}(a_j)$. It follows $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$.

As a result of these three cases, $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$ for every $a_j, a_k \in \mathcal{A}$. By making similar calculations, it can be shown that $c_{\mathfrak{C}_1}(a_j) \wedge c_{\mathfrak{C}_3}(a_k) \leq c_{\mathfrak{C}_2}(a_j) \wedge c_{\mathfrak{C}_3}(a_k)$, $u_{\mathfrak{C}_1}(a_j) \vee u_{\mathfrak{C}_3}(a_k) \geq u_{\mathfrak{C}_2}(a_j) \vee u_{\mathfrak{C}_3}(a_k)$ and $f_{\mathfrak{C}_1}(a_j) \vee f_{\mathfrak{C}_3}(a_k) \geq f_{\mathfrak{C}_2}(a_j) \vee f_{\mathfrak{C}_3}(a_k)$ for every $a_j, a_k \in \mathcal{A}$. So we have $\mathfrak{C}_1 \times \mathfrak{C}_3 \subseteq \mathfrak{C}_2 \times \mathfrak{C}_3$ if $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$. This is obvious for situation of equality.

□

3.3. Relations on Complex Quadripartitioned Single Valued Neutrosophic Set

In this part, we discuss the complex quadripartitioned single valued neutrosophic relation and equivalence complex quadripartitioned single valued neutrosophic relation with desired properties.

Definition 3.21. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$. Then, a complex quadripartitioned single valued neutrosophic relation (CQSVN relation) from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ is a (non-null) CQSVN subset of $\mathfrak{C}_1 \times \mathfrak{C}_2$. Thus, a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ is denoted by $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$, where $\Im(\mathfrak{C}_1, \mathfrak{C}_2) \subseteq \mathfrak{C}_1 \times \mathfrak{C}_2$. $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$ can be represented as the set

$$\begin{aligned}\Im(\mathfrak{C}_1,\mathfrak{C}_2) &= \left\{ \left((a_j,a_k), \left\langle \begin{array}{c} t_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k), c_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k), \\ u_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k), f_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \end{array} \right\rangle \right) : (a_j,a_k) \in \mathcal{A}\times\mathcal{A} \right\}, \\ &= \left\{ \left((a_j,a_k), \left\langle \begin{array}{c} \Gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k).e^{i\gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)}, \\ \Lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k).e^{i\lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)}, \\ \Psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k).e^{i\psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)}, \\ \Omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k).e^{i\omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)} \end{array} \right\rangle \right) : (a_j,a_k) \in \mathcal{A}\times\mathcal{A} \right\} \quad (22)\end{aligned}$$

Especially, a CQSVN subset of $\mathfrak{C}_1 \times \mathfrak{C}_1$ is called a CQSVN relation on $\mathfrak{C}_1$ and denoted by $\Im(\mathfrak{C}_1)$.

Example 3.22. We consider $\mathfrak{C}_1 \times \mathfrak{C}_2$ given in Example 3.18. If

$$\Im(\mathfrak{C}_1,\mathfrak{C}_2) = \left\{ \begin{array}{l} ((a_1,a_1), \langle 0.3e^{i2\pi(\frac{2}{9})}, 0.2e^{i2\pi(\frac{1}{2})}, 1e^{i2\pi(1)}, 0.9e^{i2\pi(1)}\rangle), \\ ((a_1,a_2), \langle 0.2e^{i2\pi(\frac{1}{4})}, 0.1e^{i2\pi(\frac{1}{4})}, 1e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(1)}\rangle), \\ ((a_2,a_1), \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.2e^{i2\pi(\frac{5}{9})}, 0.9e^{i2\pi(1)}, 0.8e^{i2\pi(\frac{5}{6})}\rangle), \\ ((a_2,a_2), \langle 0.3e^{i2\pi(\frac{4}{9})}, 0.1e^{i2\pi(\frac{2}{5})}, 0.7e^{i2\pi(\frac{4}{5})}, 0.9e^{i2\pi(\frac{2}{3})}\rangle) \end{array} \right\},$$

then $\Im(\mathfrak{C}_1, \mathfrak{C}_2) \subseteq \mathfrak{C}_1 \times \mathfrak{C}_2$ and so $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$.

Definition 3.23. If $\Im$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ then the inverse $\Im^{-1}$ is a CQSVN relation from $\mathfrak{C}_2$ to $\mathfrak{C}_1$ and is defined as follows:

$$\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1) = \left\{ \left((a_k,a_j), \left\langle \begin{array}{c} t_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j), c_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j), \\ u_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j), f_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j) \end{array} \right\rangle \right) : (a_k,a_j) \in \mathcal{A}\times\mathcal{A} \right\}, \quad (23)$$

where $t_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j) = t_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)$, $c_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j) = c_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)$, $u_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j) = u_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)$ and $f_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j) = f_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)$.

Example 3.24. We consider the CQSVN relation $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$ from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ in Example 3.22. Then,

$$\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1) = \left\{ \begin{array}{l} ((a_1,a_1), \langle 0.3e^{i2\pi(\frac{2}{9})}, 0.2e^{i2\pi(\frac{1}{2})}, 1e^{i2\pi(1)}, 0.9e^{i2\pi(1)}\rangle), \\ ((a_1,a_2), \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.2e^{i2\pi(\frac{5}{9})}, 0.9e^{i2\pi(1)}, 0.8e^{i2\pi(\frac{5}{6})}\rangle), \\ ((a_2,a_1), \langle 0.2e^{i2\pi(\frac{1}{4})}, 0.1e^{i2\pi(\frac{1}{4})}, 1e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(1)}\rangle), \\ ((a_2,a_2), \langle 0.3e^{i2\pi(\frac{4}{9})}, 0.1e^{i2\pi(\frac{2}{5})}, 0.7e^{i2\pi(\frac{4}{5})}, 0.9e^{i2\pi(\frac{2}{3})}\rangle) \end{array} \right\},$$

is the inverse of $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$, and further is a CQSVN relation from $\mathfrak{C}_2$ to $\mathfrak{C}_1$.

Definition 3.25. If $\Im$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ and $\widetilde{\Im}$ is a CQSVN relation from $\mathfrak{C}_2$ to $\mathfrak{C}_3$ then the composition $\Im \circ \widetilde{\Im}$, is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_3$, is defined as follows:

$$(\Im\circ\widetilde{\Im})(\mathfrak{C}_1,\mathfrak{C}_3) = \left\{ \left((a_j,a_l), \left\langle \begin{array}{c} t_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l), c_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l), \\ u_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l), f_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l) \end{array} \right\rangle \right) : (a_j,a_l) \in \mathcal{A}\times\mathcal{A} \right\}, (24)$$

where $t_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l) = \big(\bigvee_{a_k}\{\Gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \wedge \Gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\}\big).e^{i(\bigvee_{a_k}\{\gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)\wedge\gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$,

$c_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l) = \big(\bigvee_{a_k}\{\Lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \wedge \Lambda_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\}\big).e^{i(\bigvee_{a_k}\{\lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)\wedge\lambda_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$,

$u_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l) = \big(\bigwedge_{a_k}\{\Psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \vee \Psi_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\}\big).e^{i(\bigwedge_{a_k}\{\psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)\vee\psi_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$,

$f_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j,a_l) = \big(\bigwedge_{a_k}\{\Omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \vee \Omega_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\}\big).e^{i(\bigwedge_{a_k}\{\omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)\vee\omega_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$.

Proposition 3.26. Let $\Im$ and $\widetilde{\Im}$ be two CQSVN relations from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ and from $\mathfrak{C}_2$ to $\mathfrak{C}_3$, respectively. Then, the following assertions are true.

(i) $(\Im^{-1})^{-1} = \Im$

(ii) $(\Im \circ \widetilde{\Im})^{-1} = \widetilde{\Im}^{-1} \circ \Im^{-1}$

PROOF. **(i):** The proof is straightforward.
(ii): If the composition $\Im \circ \widetilde{\Im}$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_3$ then the inverse $(\Im \circ \widetilde{\Im})^{-1}$ is a CQSVN relation from $\mathfrak{C}_3$ to $\mathfrak{C}_1$. By the definitions of inverse and composition of CQSVN relations, we can write

$$\begin{aligned}
&t_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = t_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l)\\
&= \Big(\bigvee_{a_k}\{\Gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \wedge \Gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k, a_l)\}\Big).e^{i(\bigvee_{a_k}\{\gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)\wedge\gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}\\
&= \Big(\bigvee_{a_k}\{\Gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) \wedge \Gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l, a_k)\}\Big).e^{i(\bigvee_{a_k}\{\gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j)\wedge\gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l,a_k)\})}\\
&= \Big(\bigvee_{a_k}\{\Gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l, a_k) \wedge \Gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j)\}\Big).e^{i(\bigvee_{a_k}\{\gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l,a_k)\wedge\gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j)\})}\\
&= t_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)
\end{aligned} \tag{25}$$

By using the similar techniques, we can demonstrate the equalities:
$c_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)=c_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)$, $u_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = u_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)$ and $f_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = f_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)$. So, we have $(\Im \circ \widetilde{\Im})^{-1} = \widetilde{\Im}^{-1} \circ \Im^{-1}$. □

Definition 3.27. A CQSVN relation $\Im$ on $\mathfrak{C}$ is said to be

(a) reflexive if $t_{\Im(\mathfrak{C})}(a_j, a_j) = e^{i2\pi}$, $c_{\Im(\mathfrak{C})}(a_j, a_j) = e^{i2\pi}$, $u_{\Im(\mathfrak{C})}(a_j, a_j) = 0$ and $f_{\Im(\mathfrak{C})}(a_j, a_j) = 0$ for all $a_j \in \mathcal{A}$.

(b) symmetric if $t_{\Im(\mathfrak{C})}(a_j, a_k) = t_{\Im(\mathfrak{C})}(a_k, a_j)$, $c_{\Im(\mathfrak{C})}(a_j, a_k) = c_{\Im(\mathfrak{C})}(a_k, a_j)$, $u_{\Im(\mathfrak{C})}(a_j, a_k) = u_{\Im(\mathfrak{C})}(a_k, a_j)$ and $f_{\Im(\mathfrak{C})}(a_j, a_k) = f_{\Im(\mathfrak{C})}(a_k, a_j)$ for all $a_j, a_k \in \mathcal{A}$.

(c) transitive if $\Im \circ \Im \subseteq \Im$.
(e.g., for amplitude term and phase term of truth-membership, it is characterized as follows: $\Gamma_{\Im(\mathfrak{C})}(a_j, a_l) \geq \bigvee_{a_k}\{\Gamma_{\Im(\mathfrak{C})}(a_j, a_k) \wedge \Gamma_{\Im(\mathfrak{C})}(a_k, a_l)\}$, $\gamma_{\Im(\mathfrak{C})}(a_j, a_l) \geq \bigvee_{a_k}\{\gamma_{\Im(\mathfrak{C})}(a_j, a_k) \wedge \gamma_{\Im(\mathfrak{C})}(a_k, a_l)\}$ for all $a_j, a_k, a_l \in \mathcal{A}$. Likewise, it can be interpreted in accordance with the concept of composition of CQSVN relations for contradiction-membership, ignorance-membership and falsity-membership.)

Example 3.28. For the CQSVNS $\mathfrak{C}_1\times\mathfrak{C}_1$ in Example 3.18, $\Im(\mathfrak{C}_1) = \mathfrak{C}_1\times\mathfrak{C}_1$ is a CQSVN relation on $\mathfrak{C}_1$. $\Im(\mathfrak{C}_1)$ is not reflexive (e.g., $t_{\Im(\mathfrak{C}_1)}(a_j, a_j) \neq e^{i2\pi}$). Since $t_{\Im(\mathfrak{C}_1)}(a_1, a_2) = t_{\Im(\mathfrak{C}_1)}(a_2, a_1)$, $c_{\Im(\mathfrak{C}_1)}(a_1, a_2) = c_{\Im(\mathfrak{C}_1)}(a_2, a_1)$, $u_{\Im(\mathfrak{C}_1)}(a_1, a_2) = u_{\Im(\mathfrak{C}_1)}(a_2, a_1)$ and $f_{\Im(\mathfrak{C}_1)}(a_1, a_2) = f_{\Im(\mathfrak{C}_1)}(a_2, a_1)$, $\Im(\mathfrak{C}_1)$ is symmetric. Since $\Im(\mathfrak{C}_1) \circ \Im(\mathfrak{C}_1) \subseteq \Im(\mathfrak{C}_1)$, $\Im(\mathfrak{C}_1)$ is transitive.

Proposition 3.29. Let $\Im$ be a CQSVN relations on $\mathfrak{C}$. Then,

(i) if $\Im$ is a reflexive CQSVN relation, then $\Im^{-1}$ is also reflexive.

(ii) if $\Im$ is a symmetric CQSVN relation, then $\Im^{-1}$ is also symmetric.

(iii) if $\Im$ is a transitive CQSVN relation, then $\Im^{-1}$ is also transitive.

PROOF. The proofs are straightforward. □

Definition 3.30. A CQSVN relation $\Im$ on $\mathfrak{C}$ is said to be equivalence CQSVN relation if $\Im$ is reflexive, symmetric and transitive.

Proposition 3.31. If $\Im$ is an equivalence CQSVN relation on $\mathfrak{C}$ then $\Im^{-1}$ is also an equivalence CQSVN relation on $\mathfrak{C}$.

PROOF. The proof is obvious from Definition 3.30 and Proposition 3.29. □

4. Rough Sets Combined Complex Quadripartitioned Single Valued Neutrosophic Sets

In this section, we introduce the concept of rough complex quadripartitioned single valued neutrosophic set by combining both rough set and CQSVNS. Also, we investigate the axiomatic characterizations.

Definition 4.1. Let $\mathcal{A}$ be a non-empty set and $\Re$ be a an equivalence relation on $\mathcal{A}$. Assume that $\mathfrak{C}$ is a CQSVNS in $\mathcal{A}$.
The lower approximation of $\mathfrak{C}$ in the approximation space $(\mathcal{A}, \Re)$, denoted by $\underline{appr}_{\Re}(\mathfrak{C})$, is defined as

$$\underline{appr}_{\Re}(\mathfrak{C}) = \left\{ \left(a_j, \left\langle \begin{array}{c} \Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)} \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\}, \tag{26}$$

where, for all $a_j \in \mathcal{A}$,

$$\Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k), \quad \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Lambda_{\mathfrak{C}}(a_k),$$

$$\Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Psi_{\mathfrak{C}}(a_k), \quad \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Omega_{\mathfrak{C}}(a_k),$$

$$\gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \gamma_{\mathfrak{C}}(a_k), \quad \lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \lambda_{\mathfrak{C}}(a_k),$$

$$\psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \psi_{\mathfrak{C}}(a_k), \quad \omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \omega_{\mathfrak{C}}(a_k)$$

The upper approximation of $\mathfrak{C}$ in the approximation space $(\mathcal{A}, \Re)$, denoted by $\overline{appr}_{\Re}(\mathfrak{C})$, is defined as

$$\overline{appr}_{\Re}(\mathfrak{C}) = \left\{ \left(a_j, \left\langle \begin{array}{c} \Gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)} \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\}, \tag{27}$$

where, for all $a_j \in \mathcal{A}$,

$$\Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k), \quad \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Lambda_{\mathfrak{C}}(a_k),$$

$$\Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Psi_{\mathfrak{C}}(a_k) \quad \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Omega_{\mathfrak{C}}(a_k),$$

$$\underline{appr}_{\Re}(\mathfrak{C})(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \gamma_{\mathfrak{C}}(a_k), \quad \lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \lambda_{\mathfrak{C}}(a_k),$$

$$\psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \psi_{\mathfrak{C}}(a_k), \quad \omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \omega_{\mathfrak{C}}(a_k)$$

It is easy to see that $\underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ are two CQSVNSs in $\mathcal{A}$. $appr_{\mathfrak{R}}(\mathfrak{C}) = (\underline{appr}_{\mathfrak{R}}(\mathfrak{C}), \overline{appr}_{\mathfrak{R}}(\mathfrak{C}))$ is called the rough complex quadripartitioned single valued neutrosophic set (rough CQSVNS) in the approximation space $(\mathcal{A}, \mathfrak{R})$. Furthermore, the positive region, negative region and boundary region of CQSVNS $\mathfrak{C}$ are defined as $\mathfrak{P}_{\mathfrak{R}}(\mathfrak{C}) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$, $\mathfrak{N}_{\mathfrak{R}}(\mathfrak{C}) =\sim \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\mathfrak{B}_{\mathfrak{R}}(\mathfrak{C}) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C}) \cap \sim \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$, respectively. If $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ then the CQSVNS $\mathfrak{C}$ is called a definable CQSVNS in $(\mathcal{A}, \mathfrak{R})$, otherwise $\mathfrak{C}$ is a rough. It can be easily demonstrated that null CQSVNS and absolute CQSVNS are definable.

Example 4.2. Suppose that $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is a universal set and

$$\mathfrak{C} = \left\{ \begin{array}{l} (a_1, \langle 0.5e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{2}{5})}, 1e^{i2\pi(1)}, 1e^{i2\pi(\frac{3}{10})}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{1}{2})}, 0e^{i2\pi(0)}\rangle), \\ (a_3, \langle 0.7e^{i2\pi(\frac{1}{9})}, 0.2e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{2}{7})}, 0.5e^{i2\pi(1)}\rangle), (a_4, \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.4e^{i2\pi(\frac{2}{3})}, 0.1e^{i2\pi(\frac{2}{5})}, 0.1e^{i2\pi(\frac{1}{10})}\rangle), \\ (a_5, \langle 0.2e^{i2\pi(\frac{1}{3})}, 0.5e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{2}{7})}, 0.7e^{i2\pi(\frac{3}{8})}\rangle), (a_6, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.2e^{i2\pi(\frac{1}{4})}\rangle) \end{array} \right\}$$

is a CQSVNS in $\mathcal{A}$. Also, let $\mathfrak{R}$ be an equivalence relation on $\mathcal{A}$ such that the equivalence classes are $[a_1]_{\mathfrak{R}} = \{a_1, a_3\}$, $[a_2]_{\mathfrak{R}} = \{a_2\}$, and $[a_4]_{\mathfrak{R}} = \{a_4, a_4, a_6\}$. Then, the lower and upper approximations of $\mathfrak{C}$, i.e. $\underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C})$, respectively, in the approximation space $(\mathcal{A}, \mathfrak{R})$ are as follows:

$$\left\{ \begin{array}{l} (a_1, \langle 0.5e^{i2\pi(\frac{1}{9})}, 0.2e^{i2\pi(\frac{1}{3})}, 1e^{i2\pi(1)}, 1e^{i2\pi(1)}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{1}{2})}, 0e^{i2\pi(0)}\rangle), \\ (a_3, \langle 0.5e^{i2\pi(\frac{1}{9})}, 0.2e^{i2\pi(\frac{1}{3})}, 1e^{i2\pi(1)}, 1e^{i2\pi(1)}\rangle), (a_4, \langle 0.1e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{3}{8})}\rangle), \\ (a_5, \langle 0.1e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{3}{8})}\rangle), (a_6, \langle 0.1e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{3}{8})}\rangle) \end{array} \right\},$$

and

$$\left\{ \begin{array}{l} (a_1, \langle 0.7e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{2}{5})}, 0.4e^{i2\pi(\frac{2}{7})}, 0.5e^{i2\pi(\frac{3}{10})}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{1}{2})}, 0e^{i2\pi(0)}\rangle), \\ (a_3, \langle 0.7e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{2}{5})}, 0.4e^{i2\pi(\frac{2}{7})}, 0.5e^{i2\pi(\frac{3}{10})}\rangle), (a_4, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.1e^{i2\pi(\frac{2}{7})}, 0.2e^{i2\pi(\frac{1}{10})}\rangle), \\ (a_5, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.1e^{i2\pi(\frac{2}{7})}, 0.2e^{i2\pi(\frac{1}{10})}\rangle), (a_6, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.1e^{i2\pi(\frac{2}{7})}, 0.2e^{i2\pi(\frac{1}{10})}\rangle) \end{array} \right\}$$

So, it is a rough CQSVNS.

Proposition 4.3. For the lower and upper approximations of CQSVNSs $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$, the following properties are hold.

- **(i)** $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}) \subseteq \mathfrak{C} \subseteq \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(ii)** $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \Rightarrow \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \subseteq \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \subseteq \overline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$
- **(iii)** $\underline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\overline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(iv)** $\underline{appr}_{\mathfrak{R}}(\overline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(v)** $\underline{appr}_{\mathfrak{R}}(\sim \mathfrak{C}) =\sim \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\sim \mathfrak{C}) =\sim \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(vi)** $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}_1 \cap \mathfrak{C}_2) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \cap \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C}_1 \cup \mathfrak{C}_2) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \cup \overline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$

PROOF.

(i): Let $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$, and $\underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ be lower and upper approximations of $\mathfrak{C}$, respectively. For every $a_j \in \mathcal{A}$, we calculate (by considering Definitions 3.8 (a) and 4.1), for the amplitude term of complex truth-membership,

$$\Gamma_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k) \leq \Gamma_{\mathfrak{C}}(a_j) \leq \bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k) = \Gamma_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j)$$

and for the phase term of complex falsity-membership,

$$\omega_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \omega_{\mathfrak{C}}(a_k) \geq \omega_{\mathfrak{C}}(a_j) \geq \bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \omega_{\mathfrak{C}}(a_k) = \omega_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j).$$

Proceeding with similar calculations, we obtain that

$$\Lambda_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) \leq \Lambda_{\mathfrak{C}}(a_j) \leq \Lambda_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j), \quad \Psi_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) \geq \Psi_{\mathfrak{C}}(a_j) \geq \Psi_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j),$$

$$\Omega_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) \geq \Omega_{\mathfrak{C}}(a_j) \geq \Omega_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j), \quad \gamma_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) \leq \gamma_{\mathfrak{C}}(a_j) \leq \gamma_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j),$$

$$\lambda_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) \leq \lambda_{\mathfrak{C}}(a_j) \leq \lambda_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j), \quad \psi_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) \geq \psi_{\mathfrak{C}}(a_j) \geq \psi_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j).$$

Therefore, we have $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}) \subseteq \mathfrak{C} \subseteq \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$

(ii): It is obvious from the definitions of lower and upper approximations of CQSVNS.

(iii): According to the definition of lower approximation of CQSVNS, we can write, for ever $a_j \in \mathcal{A}$,

$$\Gamma_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k) = \Gamma_{\mathfrak{C}}(a_{k_*})$$

where $a_{k_*} \in [a_j]_{\mathfrak{R}}$. It follows

$$\Gamma_{\underline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C}))}(a_j) = \bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Big(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k)\Big) = \Gamma_{\mathfrak{C}}(a_{k_*})$$

So, $\Gamma_{\underline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C}))}(a_j) = \Gamma_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j)$ for ever $a_j \in \mathcal{A}$. It can be shown similarly for other amplitude terms and phase terms. These demonstrate that $\underline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$. The property $\underline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ can be proved similarly.

(iv): The proof is similar to the proof of (iii).

(v): According to the Definitions 3.8 (c) and 4.1, we can obtain

$$\begin{aligned}
\underline{appr}_{\mathfrak{R}}(\sim \mathfrak{C}) &= \left\{ \left(a_j, \left\langle \begin{array}{c} \Big(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \gamma_{\sim\mathfrak{C}}(a_k))}, \\ \Big(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Lambda_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \lambda_{\sim\mathfrak{C}}(a_k))}, \\ \Big(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \Psi_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \psi_{\sim\mathfrak{C}}(a_k))}, \\ \Big(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \Omega_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \omega_{\sim\mathfrak{C}}(a_k))} \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\} \\
&= \left\{ \left(a_j, \left\langle \begin{array}{c} \Big(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \Omega_{\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \omega_{\mathfrak{C}}(a_k))}, \\ \Big(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \Psi_{\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \psi_{\mathfrak{C}}(a_k))}, \\ \Big(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Lambda_{\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \lambda_{\mathfrak{C}}(a_k))}, \\ \Big(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \gamma_{\mathfrak{C}}(a_k))}, \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\} \\
&= \sim \underline{appr}_{\mathfrak{R}}(\mathfrak{C}).
\end{aligned} \tag{28}$$

The property of $\overline{appr}_{\mathfrak{R}}(\sim \mathfrak{C}) = \sim \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ can be demonstrated similarly.

(vi): Based on the Definition 3.8 (c) and (d) and Definition 4.1, it can be proved similar to the proof of (v).

□

5. Level Cut Set-based Rough Degree of Complex Quadripartitioned Single Valued Neutrosophic Set

In this section, we introduce the approximate precision and rough degree of CQSVNS and give some theoretical results.

For the CQSVNS $\mathfrak{C}$ in $\mathcal{A}$, we know that $\underline{appr}_{\Re}(\mathfrak{C})$ and $\overline{appr}_{\Re}(\mathfrak{C})$ are two CQSVNSs. Thus, the $((\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3),(\alpha_4,\beta_4))$-level cut sets of $\underline{appr}_{\Re}(\mathfrak{C})$ and $\overline{appr}_{\Re}(\mathfrak{C})$ can be described as follows.

Definition 5.1. The $((\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3),(\alpha_4,\beta_4))$-level cut sets of $\underline{appr}_{\Re}(\mathfrak{C})$ and $\overline{appr}_{\Re}(\mathfrak{C})$, denoted by $(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ and $(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$, are defined as follows, respectively:

$$(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \left\{ a_j \in \mathcal{A} : \begin{pmatrix} \Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_1,\ \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_2, \\ \Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_3,\ \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_4, \\ \gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_1,\ \lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_2, \\ \psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_3,\ \omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_4 \end{pmatrix} \right\} \tag{29}$$

and

$$(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \left\{ a_j \in \mathcal{A} : \begin{pmatrix} \Gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_1,\ \Lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_2, \\ \Psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_3,\ \Omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_4, \\ \gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_1,\ \lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_2, \\ \psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_3,\ \omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_4 \end{pmatrix} \right\} \tag{30}$$

Definition 5.2. Let $(\mathcal{A},\Re)$ be an approximation space and $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$. Also, let the $((\alpha_1^2,\beta_1^2),(\alpha_2^2,\beta_2^2),(\alpha_3^2,\beta_3^2),(\alpha_4^2,\beta_4^2))$-level cut set of $\overline{appr}_{\Re}(\mathfrak{C})$ be not null. The level cut set-based approximate precision of CQSVNS $\mathfrak{C}$ can be defined as

$$\sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} = \frac{\left|(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}\right|}{\left|(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2)}_{(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)}\right|} \tag{31}$$

where the notation $|\cdot|$ denotes the cardinality of set and $0<\alpha_1^2 \leq \alpha_1^1 \leq 1$, $0<\alpha_2^2 \leq \alpha_2^1 \leq 1$, $0<\alpha_3^1 \leq \alpha_3^2 \leq 1$, $0<\alpha_4^1 \leq \alpha_4^2 \leq 1$, $0<\beta_1^2 \leq \beta_1^1 \leq 1$, $0<\beta_2^2 \leq \beta_2^1 \leq 1$, $0<\beta_3^1 \leq \beta_3^2 \leq 1$, $0<\beta_4^1 \leq \beta_4^2 \leq 1$.

The level cut set-based rough degree of CQSVNS $\mathfrak{C}$ is denoted and defined by

$$\rho(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} = 1-\sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \tag{32}$$

Note 5.3. From now on, the $((\alpha_1^2,\beta_1^2),(\alpha_2^2,\beta_2^2),(\alpha_3^2,\beta_3^2),(\alpha_4^2,\beta_4^2))$-level cut set of $\overline{appr}_{\Re}(\mathfrak{C})$ is not null.

Theorem 5.4. Let $(\mathcal{A},\Re)$ be an approximation space and $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$. Then, the approximate precision $\sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$ and the rough degree $\rho(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$ of CQSVNS $\mathfrak{C}$ provide the following properties.

(i) $0 \leq \sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq 1$

(ii) $0 \leq \rho(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq 1$

PROOF.

(i): By Proposition 4.3 (i), we know that $\underline{appr}_{\Re}(\mathfrak{C}) \subseteq \overline{appr}_{\Re}(\mathfrak{C})$. Since $0 < \alpha_p^2 \leq \alpha_p^1 \leq 1$, $0 < \beta_p^2 \leq \beta_p^1 \leq 1$ for $p = 1, 2$ and $0 < \alpha_q^1 \leq \alpha_q^2 \leq 1$, $0 < \beta_q^1 \leq \beta_q^2 \leq 1$ for $p = 3, 4$, we can say that

$$\left|(\underline{appr}_{\Re}(\mathfrak{C}))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}\right| \leq \left|(\overline{appr}_{\Re}(\mathfrak{C}))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}\right|$$

So, we have $0 \leq \sigma(\mathfrak{C})_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})} \leq 1$.

(ii): It is obvious from (i) and Eq. (32).

□

Example 5.5. Consider the lower approximation $\overline{appr}_{\Re}(\mathfrak{C})$ and upper approximation $\overline{appr}_{\Re}(\mathfrak{C})$ of $\mathfrak{C}$ in Example 4.2. We can find that the $((0.3, \frac{\pi}{3}), (0.7, \frac{\pi}{2}), (0.7, \frac{4\pi}{3}), (0.3, 0))$-level cut set of $\underline{appr}_{\Re}(\mathfrak{C})$ is

$$(\underline{appr}_{\Re}(\mathfrak{C}))_{(0.3,0.7,0.7,0.3)}^{(\frac{\pi}{3},\frac{\pi}{2},\frac{4\pi}{3},0)} = \{a_2\}$$

and $((0.2, \frac{\pi}{3}), (0.7, \frac{2\pi}{5}), (0.7, \frac{4\pi}{3}), (0.5, \frac{\pi}{5}))$-level cut set of $\overline{appr}_{\Re}(\mathfrak{C})$ is

$$(\overline{appr}_{\Re}(\mathfrak{C}))_{(0.2,0.7,0.7,0.5)}^{(\frac{\pi}{3},\frac{2\pi}{5},\frac{4\pi}{3},\frac{\pi}{5})} = \{a_2, a_4, a_5, a_6\}$$

Hence, we calculate the approximation precision and rough degree as

$$\sigma(\mathfrak{C})_{((0.3,0.2),(0.7,0.2),(0.7,0.7),(0.3,0.5))}^{((\frac{\pi}{3},\frac{\pi}{3}),(\frac{\pi}{2},\frac{2\pi}{5}),(\frac{4\pi}{3},\frac{4\pi}{3}),(0,\frac{\pi}{5}))} = \frac{1}{4}$$

and

$$\rho(\mathfrak{C})_{((0.3,0.2),(0.7,0.2),(0.7,0.7),(0.3,0.5))}^{((\frac{\pi}{3},\frac{\pi}{3}),(\frac{\pi}{2},\frac{2\pi}{5}),(\frac{4\pi}{3},\frac{4\pi}{3}),(0,\frac{\pi}{5}))} = \frac{3}{4}$$

Proposition 5.6. Let $(\mathcal{A}, \Re)$ be an approximation space, and $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$.

(i) If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ and $(\overline{appr}_{\Re}(\mathfrak{C}_1))_{(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)}^{(\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2)} = (\overline{appr}_{\Re}(\mathfrak{C}_2))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}$ then

$$\sigma(\mathfrak{C}_1)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})} \leq \sigma(\mathfrak{C}_2)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}$$

and

$$\rho(\mathfrak{C}_1)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})} \geq \rho(\mathfrak{C}_2)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}$$

(ii) If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ and $(\underline{appr}_{\Re}(\mathfrak{C}_1))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)} = (\underline{appr}_{\Re}(\mathfrak{C}_2))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}$ then

$$\sigma(\mathfrak{C}_1)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})} \geq \sigma(\mathfrak{C}_2)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}$$

and

$$\rho(\mathfrak{C}_1)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})} \leq \rho(\mathfrak{C}_2)_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}$$

PROOF.

(i): Since $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$, we have $(\underline{appr}_{\Re}(\mathfrak{C}_1))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)} \subseteq (\underline{appr}_{\Re}(\mathfrak{C}_2))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}$ by Propositions 3.5 and 4.3 (i). From the assumption, we have $(\overline{appr}_{\Re}(\mathfrak{C}_1))_{(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)}^{(\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2)} = (\overline{appr}_{\Re}(\mathfrak{C}_2))_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}$. Therefore, the proof is clear from Eqs. (31) and (32).

(ii): It can be proved similar to proof of (i).

□

6. Conclusion

The QSVNS based on the four-valued logic is an effective mathematical tool for managing ambiguity. In this study based on extension of these sets, we introduced the concept of CQSVNSs and carried out theoretical study of various set-theoretic operations on them. Then, we described the lower and upper approximations of CQSVNSs in the approximation space and discussed their properties. Meanwhile, we gave the definitions of rough CQSVN cut sets and then presented how to measure the rough degree of CQSVN in the approximation space. It is worth mentioning that the CQSVNs and rough CQSVNs can be used for dealing with many problems in real life. Future works may involve the different types of distance measures between two CQSVNs (or rough CQSVNSs) and their applications in the medical diagnosis, pattern recognition and clustering analysis.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8(3) (1965) 338–353.

[2] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems 20 (1986) 87–96.

[3] N. Ça?man, S. Engino?lu, *Fuzzy Soft Matrix Theory and Its Application in Decision Making*, Iranian Journal of Fuzzy Systems 9(1) (2012) 109–119.

[4] E. Aygün, H. Kamacı, *Some Generalized Operations in Soft Set Theory and Their Role in Similarity and Decision Making*, Journal of Intelligent and Fuzzy Systems 36(6) (2019) 6537–6547.

[5] H. Kamacı, *Interval-valued Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets and Their Applications*, Cumhuriyet Science Journal 40(2) (2019) 317–331.

[6] H. Kamacı, *Linear Diophantine Fuzzy Algebraic Structures*, Journal of Ambient Intelligence and Humanized Computing (2021) In Press. https://doi.org/10.1007/s12652-020-02826-x

[7] M. J. Khan, P. Kumam, P. Liu, W. Kumam, S. Ashraf, *A Novel Approach to Generalized Intuitionistic Fuzzy Soft Sets and Its Application in Decision Support System*, Mathematics 7(8) (2019) 1–21. https://doi.org/10.3390/math7080742

[8] M. Riaz, R. Hashmi, *Linear Diophantine Fuzzy Set and Its Applications towards Multi-attribute Decision Making Problems*, Journal of Intelligent and Fuzzy Systems 37 (2019) 5417–5439.

[9] M. Riaz, R. Hashmi, *MAGDM for Agribusiness in the Environment of Various Cubic m-polar Fuzzy Averaging Aggregation Operators*, Journal of Intelligent and Fuzzy Systems 37 (2019) 3671–3691.

[10] S. Petchimuthu, H. Garg, H. Kamacı, A.O. Atagün, *The Mean Operators and Generalized Products of Fuzzy Soft Matrices and Their Applications in MCGDM*, Computational and Applied Mathematics 39(2) (2020) 1–32 Article No: 68.

[11] S. Petchimuthu, H. Kamacı, *The Adjustable Approaches to Multi-criteria Group Decision Making Based on Inverse Fuzzy Soft Matrices*, Scientia Iranica (2020) In Press. doi: 10.24200/sci.2020.54294.3686

[12] F. Smarandache, *Neutrosophy: Neutrosophic Probability, Set, and Logic: Analytic Synthesis & Synthetic Analysis*, Rehoboth: American Research Press, USA, 1998.

[13] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, *Single Valued Neutrosophic Set*, Multispace and Multistructure 4 (2010) 410–413.

[14] I. Deli, Y. Şubaş, *A Ranking Method of Single Valued Neutrosophic Numbers and Its Applications to Multi-attribute Decision Making Problems*, International Journal of Machine Learning and Cybernetics 8 (2017) 1309–1322.

[15] F. Karaaslan, *Gaussian Single-valued Neutrosophic Numbers and Its Application in Multi-attribute Decision Making*, Neutrosophic Sets and Systems 22 (2018) 101–117.

[16] S. Naz, M. Akram, F. Smarandache, *Certain Notions of Energy in Single-valued Neutrosophic Graphs*, Axioms 7(3) (2018) 1–30.

[17] H. Garg, Nancy, *Multiple Attribute Decision Making Based on Immediate Probabilities Aggregation Operators for Single-valued and Interval Neutrosophic Sets*, Journal of Applied Mathematics and Computing 63 (2020) 619–653.

[18] N. L. A. M. Kamal, L. Abdullah, I. Abdullah, S. Alkhazaleh, F. Karaaslan, *Multi-valued Interval Neutrosophic Soft Set: Formulation and Theory*, Neutrosophic Sets and Systems 30 (2019) 149–170.

[19] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Hexis: Phoenox, AZ, USA, 2005.

[20] M. Abdel-Basset, M. Mohamed, M. Elhoseny, L. H. Son, F. Chiclana, A. E. N. H. Zaied, *Cosine Similarity Measures of Bipolar Neutrosophic Set for Diagnosis of Bipolar Disorder Diseases*, Artificial Intelligence in Medicine 101 (2019).

[21] A. Awang, M. Ali, L. Abdullah, *Hesitant Bipolar-valued Neutrosophic set: Formulation, Theory and Application*, IEEE Access 7 (2019) 176099–176114.

[22] S. T. Tehrim, M. Riaz, *A Novel Extension of TOPSIS to MCGDM with Bipolar Neutrosophic Soft Topology*, Journal of Intelligent and Fuzzy Systems 37 (2019) 5531–5549.

[23] M. Ali, I. Deli, F. Smarandache, *The Theory of Neutrosophic Cubic Sets and Their Applications in Pattern Recognition*, Journal of Intelligent and Fuzzy Systems 30 (2016) 1957–1963.

[24] M. Gulistan, M. Mohammad, F. Karaaslan, S. Kadry, S. Khan, H. A. Wahab, *Neutrosophic Cubic Heronian Mean Operators with Applications in Multiple Attribute Group Decision-making Using Cosine Similarity Functions*, International Journal of Distributed Sensor Networks 15(9) (2019) 1–21.

[25] Y. B. Jun, F. Smarandache, C. S. Kim, *Neutrosophic Cubic Sets*, New Mathematics and Natural Computation, 13 (2015) 41–54.

[26] H. Kamacı, *Neutrosophic Cubic Hamacher Aggregation Operators and Their Applications in Decision Making*, Neutrosophic Sets and Systems 33 (2020) 234–255.

[27] M. Akram, N. Ishfaq, S. Sayed, F. Smarandache, *Decision-making Approach Based on Neutrosophic Rough Information*, Algorithms 11(5) (2018) 1–20.

[28] I. Deli, S. Eraslan, N. Çağman, *ivnpiv-Neutrosophic Soft Sets and Their Decision Making Based on Similarity Measure*, Neural Computing and Applications 29(1) (2018) 187–2003.

[29] X. Peng, F. Smarandache, *Novel Neutrosophic Dombi Bonferroni Mean Operators with Mobile Cloud Computing Industry Evaluation*, Expert Sytems 36 (2019) 1–22. https://doi.org/10.1111/exsy.12411

[30] M. Ali, F. Smarandache, *Complex Neutrosophic Set*, Neural Computing and Applications 28 (2017) 1817–1834.

[31] A. Al-Quran, S. Alkhazaleh, *Relations between The Complex Neutrosophic Sets with Their Applications in Decision Making*, Axioms 7 (2018) 1–15.

[32] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences 11 (1982) 341–356.

[33] D. Dubios, H. Prade, *Rough Fuzzy Sets and Fuzzy Rough Sets*, International Journal of General Systems 17 (1990) 191–208.

[34] K. V. Thomas, L. S. Nair, *Rough Intuitionistic Fuzzy Sets in a Lattice*, International Mathematical Forum 6 (2011) 1327–1335.

[35] S. Broumi, F. Smarandache, M. Dhar, *Rough Neutrosophic Sets*, Italian Journal of Pure and Applied Mathematics 32 (2014) 493–502.

[36] K. Mondal, S. Pramanik, *Rough Neutrosophic Multi-attribute Decision Making Based on Rough Accuracy Score Functions*, Neutrosophic Sets and Systems 8 (2015) 14–21.

[37] A. E. Samuel, R. Narmadhagnanam, *Rough Neutrosophic Sets in Medical Diagnosis*, International Journal of Pure and Applied Mathematics 120 (2018) 79–87.

[38] M. Abdel-Basset, M. Mohamed, *The Role of Single Valued Neutrosophic Sets and Rough Sets in Smart City: Imperfect and Incomplete Information Systems*, Measurement 124 (2018) 47–55.

[39] R. Cahtterjee, P. Majumdar, S. K. Samanta, *On Some Similarity Measures and Entropy on Quadripartitioned Single Valued Neutrosophic Sets*, Journal of Intelligent and Fuzzy Systems 30 (2016) 2475–2485.

[40] N. D. Belnap, *A Useful Four-Valued Logic*, Modern Uses of Multiple-Valued Logic, Springer Netherlands (1977) 5–37.

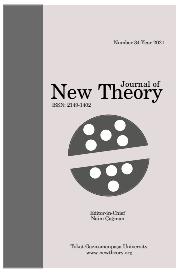
[41] M. Mohan, M. Krishnaswamy, *Axiomatic Characterizations of Quadripartitioned Single Valued Neutrosophic Rough Sets*, Journal of New Theory (30) (2020) 86–99.

[42] K. Sinha, P. Majumdar, *Bipolar Quadripartitioned Single Valued Neutrosophic Rough Set*, Neutrosophic Sets and Systems 38 (2020) 244–257.

[43] S. Roy, J.-G. Lee, A. Pal, S. K. Samanta, *Similarity Measures of Quadripartitioned Single Valued Bipolar Neutrosophic Sets and Its Application in Multi-criteria Decision Making Problems*, Symmetry 12(6) (2020) 1–16.

Journal of New Theory
ISSN: 2149-1402

34 (2021) 64-71
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



The Effects of Kernel Functions and Optimal Hyperparameter Selection on Support Vector Machines

Aslı Yaman[1], Mehmet Ali Cengiz[2]

Article History
Received: 08 Feb 2021
Accepted: 10 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract − Support Vector Machine (SVM) is a supervised machine learning method used for classification and regression. It is based on the Vapnik-Chervonenkis (VC) theory and Structural Risk Minimization (SRM) principle. Thanks to its strong theoretical background, SVM exhibits a high performance compared to many other machine learning methods. The selection of hyperparameters and the kernel functions is an important task in the presence of SVM problems. In this study, the effect of tuning hyperparameters and sample size for the kernel functions on SVM classification accuracy was investigated. For this, UCI datasets of different sizes and with different correlations were simulated. Grid search and 10-fold Cross-Validation methods were used to tune the hyperparameters. Then, SVM classification process was performed using three kernel functions, and classification accuracy values were examined.

Keywords − *Support vector machines, kernel function, tune parameter*

Mathematics Subject Classification (2020) − 62H30, 62P99

1. Introduction

The first mentions on Support Vector Machine (SVM) were made by Vapnik in 1979. However, after the presentation at the conference named COLT (Conference on Computational Learning Theory), held in America in 1992 [1], the use of SVM became widespread. Then, it was officially introduced by Vapnik in 1995 [2].

The SVM theory is based on the idea of Vapnik-Chervonenkis (VC) theory and Structural Risk Minimization (SRM). The VC theory is a subbranch of statistical learning theory. The main goal in learning problems is to reach the most accurate results with the minimum error. For this, the expected risk is desired to be minimum. The basic idea in SRM principle and VC theory is to select the model with the correct level of complexity to minimize the expected risk or generalization error among many models. The SRM principle aims to minimize the upper bound of the expected risk. For a function with distribution, the SRM principle converges to the optimal solution. SVM tries to keep both experimental risk and VC dimension to a minimum so that the expected risk reaches the minimum [3].

SVM aims to classify the observations most accurately by finding the optimal separating hyperplane between two or more classes. It is used in linear and non-linear classification and regression problems. Datasets in which training data cannot be separated linearly are transferred to a higher dimensional feature space using mapping functions. The dataset mapped to the feature space can be linearly separated using kernel functions

[1]asliyamann@gmail.com (Corresponding Author); [2]macengiz@omu.edu.tr
[1,2]Department of Statistics, Faculty of Arts and Sciences, Ondokuz Mayıs University, Samsun, Turkey

[4]. In feature space, SVM tries to solve the quadratic optimization problem to find the optimal separating hyperplane.

SVM is used in many different domains: pattern recognition (handwriting [5], face [6], speech [7], emotion [8], disease diagnosis [9], treatment success [10], time series [11], criminology [12], stock market prediction [13], etc.).

This study aims to introduce the two-class SVM classification theory and examine the effects of sample size and optimal hyperparameter selection on classification accuracy. Besides, determining the optimal values of the hyperparameters of kernel functions has a significant impact on SVM results. For tuning the hyperparameters, many algorithms have been proposed, such as grid search, random search, Bayesian optimization, simulated annealing, particle swarm optimization, genetic algorithm, etc. [14]. In this study, we used a grid search CV algorithm to tune hyperparameters. After tuning the hyperparameters, the SVM classification results were examined on the simulated dataset with different scenarios.

The rest of the paper is organized as follows. A brief review of the theory of SVM is described in Section 2. The experiments are presented in Section 3. Results are given in Section 4, and we conclude the paper with a summary of results by emphasizing the importance of this study and mentioning some viable future work.

2. Support Vector Machines

The theory of SVMs in classification problems is given in this section [15,16]. SVMs are used to optimally separate dataset belonging multiple classes by specifying a hyperplane. With linear SVM, the dataset can be separated completely (hard margin) or partially (soft margin), and the dataset cannot be separated linearly in any way with non-linear SVM.

2.1. Linear Support Vector Machines

Let $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), x \in R^n$ be the training dataset for SVM with separable two-class labels such as $y \in \{+1, -1\}$. The main purpose of SVM is to find the most suitable separating hyperplane that will enable to classify training observations correctly. Hyperplane represents a separating surface in a multidimensional space. There can be thousands of different hyperplanes between two classes, so the most suitable (optimal) hyperplane must be found for strong classification accuracy and better generalization performance.

To find the optimal separating hyperplane, it is necessary to determine the distance between training observations with different two-class labels called Margin. The maximum margin classifier will give the optimal separating hyperplane. Separating hyperplanes are formulated as in Equation 1,

$$D(x) = (w.z) + b = 0 \tag{1}$$

and should provide Equation 2 for both classes.

$$y_i[(w.z) + b] \geq 1, i = 1, \dots, n \tag{2}$$

The distance between hyperplane and origin is

$$d = \frac{|b|}{\|w\|} \tag{3}$$

in which b and w are the parameters of the optimal hyperplane. Here, $|\,.\,|$ is the absolute value and $\|\,.\,\|$ is the Euclidean norm of a vector. Assume two hyperplanes $(+d, -d)$ for two-class label $(+1, -1)$. Thus, the margin is computed as

$$\text{margin} = d_+ - d_- = \frac{|1-b|}{\|w\|} - \frac{|-1-b|}{\|w\|} = \frac{2}{\|w\|} \tag{4}$$

To maximize this margin (hard margin), the norm of w is minimized. Hence, the primal form of the optimization problem obtained for the maximum margin classifier or, in other words, the optimal separating hyperplane is as follows:

$$\text{minimize } \frac{1}{2}\|w\|^2$$

$$\text{subject to } y_i(w.z_i + b) \geq +1, i = 1,..,n \tag{5}$$

The optimal hyperplane problem is a classical optimization problem and can be solved by the Lagrangian multiplier method and Krush Kuhn Tucker (KKT) conditions, so the problem transforms into the dual form, and the dual form of the problem is solved as in Equation 6,

$$\text{maximize } W(x) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j y_i y_j z_i.z_j$$

$$\text{subject to } \sum_{i=1}^{n} \alpha_i y_i = 0, i = 1, \dots, n \tag{6}$$

The KKT theorem is important in the theory of SVM. According to KKT conditions, there are two different situations for $\alpha_i(y_i(w.z_i + b) - 1) = 0$, which are the correctly classified features outside of hyperplanes $(\alpha_i = 0)$ and the correctly classified features located on hyperplanes $(\alpha_i \geq 0)$, called *support vectors*. The structure of an SVM is shown in Figure 1.

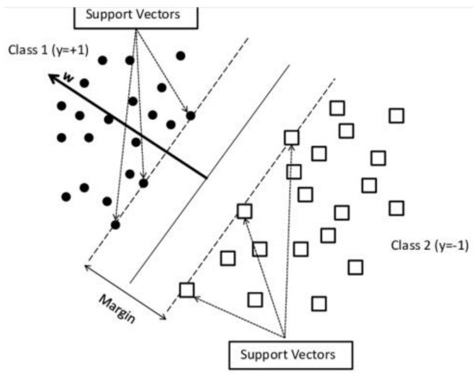


Fig 1: The structure of an SVM

2.2. Non-Linear Support Vector Machines

The previous sections mentioned that the dataset is completely and linearly separable (hard margin). Moreover, when the dataset is partially non-separable (soft margin), slack variables (ξ) are added to Equation 2, and the computations are performed as in the hard margin optimization. Then, separating hyperplane for the partially non-separable dataset is found as in Equation 7,

$$y_i[(w.z) + b] \geq 1 - \xi_i, \; i = 1, \dots, n \tag{7}$$

The optimal hyperplane for the partially non-separable case is obtained from Equation 8,

$$\text{minimize } \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i$$

$$\text{subject to } y_i(w.z_i + b) \geq 1 - \xi_i, \; i = 1,..,n, \; \xi_i \geq 0 \tag{8}$$

in which the regularization parameter C is constant.

Non-linear SVM classifier is used in cases where training observations cannot be separated by a linear decision surface. Generally, datasets cannot be separated linearly in real analysis. For such problems, mapping functions are used to transform the input space in which training observations cannot be separated linearly into a higher dimensional feature space where observations can be linearly separated [17]. To access this aim, kernel functions are used because the transition to a higher dimensional space with mapping functions and processing with dot products in this space is computationally difficult and time-consuming. Kernel function $K(.,.)$ is given in Equation 9,

$$K(x_i, x_j) = z_i.z_j = \varphi(x_i).\varphi(x_j) \tag{9}$$

The function satisfies Mercer's theorem. The most known kernel functions are linear, radial basis function, polynomial, sigmoid, dot product, and two-layer neural network kernel [18]. Some kernel function algorithms are given in Table 1.

Table 1. Kernel functions

Kernel Functions	Algorithms
Linear	$K(u',v) = u'v$
Polynomial	$K(u',v) = (u'v+1)^d$
Radial Basis Function	$K(u',v) = \exp(-\|u-v\|^2/\sigma^2)$

The non-linear separating hyperplane can be found as

$$\text{maximize } W(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j K(x_i, x_j)$$

$$\text{subject to } \sum_{i=1}^{n} y_i \alpha_i = 0, \quad 0 \le \alpha_i \le C, i = 1, \dots, n \tag{10}$$

and the decision function is as in Equation 11,

$$f(x) = \text{sign}(w.z + b) = \text{sign}(\sum_{i=1}^{n} \alpha_i y_i K(x_i, x) + b) \tag{11}$$

3. Experiments

It is aimed to determine the optimal hyperparameters of the kernel function to be used in SVM classification and to examine the effect of these optimal values on the classification accuracy.

For this purpose, firstly datasets with standard normal distribution according to different correlation levels and different sample sizes were created by using the "MASS" package in the R. Correlation levels for simulated datasets were determined as 0.25,0.50,0.75 and sample sizes were determined as 20,50,100 and 200, respectively. The number of features has been kept constant as 2.

For these datasets, optimal hyperparameter selection was performed according to kernel functions with Grid search and 10-fold Cross Validation (CV) methods with 30 iterations. The intervals for the hyperparameters determined as in Table 2.

Table 2. Hyperparameter searching intervals setting

	Hyperparameters			
Kernel Function	C	Sigma	Degree	Scale
Linear	{1, 2, … ,20}	-	-	-
Radial Basis Function	{1, 2, … ,20}	{0.1, 0.6, 1.1, … ,10}	-	-
Polynomial	{1, 2, … ,20}	-	{1,2,3}	$\{10^{-3}, 10^{-2}, \dots, 10^{1}\}$

2 class-SVM classification process was carried out by using "e1071" packages. Then, datasets for 3 different kernel functions (linear, polynomial and radial basis function) were analysed for optimal hyperparameter values and the kernel functions with the highest test accuracy were examined according to the obtained test prediction results.

4. Results

Firstly, optimal hyperparameters were selected according to 3 different correlation levels and SVM classifier performances were obtained for 3 different kernel functions when the number of observations was 20. Results were given in Table 3.

Table 3. Optimal values for kernel hyperparameters and classification accuracies when sample size was 20

Sample size	Kernel function	Correlation levels	Optimal hyperparameter values	Classification accuracy%
20	Linear	0.25	C=1	0.60
		0.50	C=1	0.60
		0.75	C=1	0.75
20	Polynomial	0.25	d=2, s=1, C=1	0.60
		0.50	d=2, s=1, C=1	0.60
		0.75	d=3, s=1, C=1	0.74
20	RBF	0.25	C=2, σ=2.5	0.60
		0.50	C=2, σ=2.6	0.60
		0.75	C= 2, σ=9.6	0.75

In Table 3, it was observed that when the sample size was 20, optimal hyperparameter values got almost the same values according to different correlation levels. Similar results were obtained for 3 kernel functions of SVM classification accuracies. While the sample size was 20, the most accurate classification percentage was obtained when the correlation level was 0.75 for all 3 kernel functions. While the sample size was small, it was concluded that the classification accuracy varied according to the correlation levels, not the kernel functions.

Secondly, for the number of observations 50, optimal hyperparameters were selected according to 3 different correlation levels and test accuracy percentages were obtained for 3 kernel functions according to the parameters. Results were given in Table 4.

Table 4. Optimal values for kernel hyperparameters and classification accuracies when sample size was 50

Sample size	Kernel function	Correlation levels	Optimal hyperparameter values	Classification accuracy%
50	Linear	0.25	C=5	0.89
		0.50	C=6	0.87
		0.75	C=3	0.88
50	Polynomial	0.25	d=2, s=1, C=18	0.98
		0.50	d=2, s=1, C=17	0.95
		0.75	d=3, s=1, C=18	0.99
50	RBF	0.25	C=2, ,σ=1.6	0.99
		0.50	C=19, σ=1.1	0.99
		0.75	C=1, σ=0.6	0.99

In Table 4, while the number of observations 50 with different correlation levels, it was seen that although the optimal hyperparameter values were different, more accurate classification percentages were obtained with the polynomial and RBF kernel functions.

The optimal hyperparameters were selected according to 3 different correlation levels and test accuracy percentages were obtained for 3 kernel functions according to the parameters for the number of observations 100. Results were given in Table 5.

Table 5. Optimal values for kernel hyperparameters and classification accuracies when sample size was 100

Sample size	Kernel function	Correlation levels	Optimal hyperparameter values	Classification accuracy%
100	Linear	0.25	C=3	0.76
		0.50	C=4	0.78
		0.75	C=1	0.76
100	Polynomial	0.25	d=2, s=1, C=8	0.96
		0.50	d=2, s=1, C=8	0.92
		0.75	d=2, s=1, C=8	0.96
100	RBF	0.25	C=12, σ =1.1	0.92
		0.50	C=20, σ =0.6	0.92
		0.75	C=20, σ =1.1	0.88

It is seen in the Table 5 that the highest classification accuracy values were obtained with the polynomial kernel. For the polynomial kernel, the result is that the optimal parameter values are the same despite different correlation levels. The same analyses were performed for the number of observations 200, and the results were obtained as in Table 6.

Table 6. Optimal values for kernel hyperparameters and classification accuracies when sample size was 200

Sample size	Kernel function	Correlation levels	Optimal hyperparameter values	Classification accuracy%
200	Linear	0.25	C=2	0.92
		0.50	C=5	0.90
		0.75	C=2	0.90
200	Polynomial	0.25	d=2, s=1, C=20	0.98
		0.50	d=2, s=1, C=12	0.99
		0.75	d=2, s=1, C=20	0.99
200	RBF	0.25	C=3, σ=0.6	0.98
		0.50	C=1, σ=1.6	0.98
		0.75	C=20, σ=1.1	0.99

In Table 6, the highest accuracy values are obtained with polynomial and radial kernel. Although the optimal hyperparameter values are different in the radial kernel according to different correlation levels, it is seen that similar results are obtained for the appropriate hyperparameter values in the polynomial kernel. Furthermore, Haberman's Survival dataset from the University of California Irvine (UCI) repository [19] was used in the experiments. It was described in Table 6 with the number of classes, instances, and features.

Table 6. Information about UCI dataset

Dataset	Number of Classes	Number of Instances	Number of Features
Haberman’s Survival	2	306	3

The analysis on the simulated datasets were also performed for the Haberman's Survival UCI dataset. The intervals specified in Table 2 were used to obtain the optimal hyperparameters. The optimal hyperparameter selection was performed according to kernel functions with Grid search and 10-fold CV methods with 30 iterations. After determining the optimal hyperparameter values, the SVM classification process was performed. The results were given in Table 7.

Table 7. SVM classification results of Haberman’s Survival dataset

Kernel function	Optimal hyperparameter values	Classification accuracy %
Linear	C=5	0.7368
Polynomial	C=12, d=3, s=0.1	0.7337
RBF	C=8, σ=0.1	0.7763

In Table 7, the highest SVM classification accuracy for the UCI dataset was achieved with the RBF kernel function with C=8 and σ=0.1 values. In addition, the graphical representation of obtaining optimal hyperparameter values for 3 kernel functions was given in Figure 2.

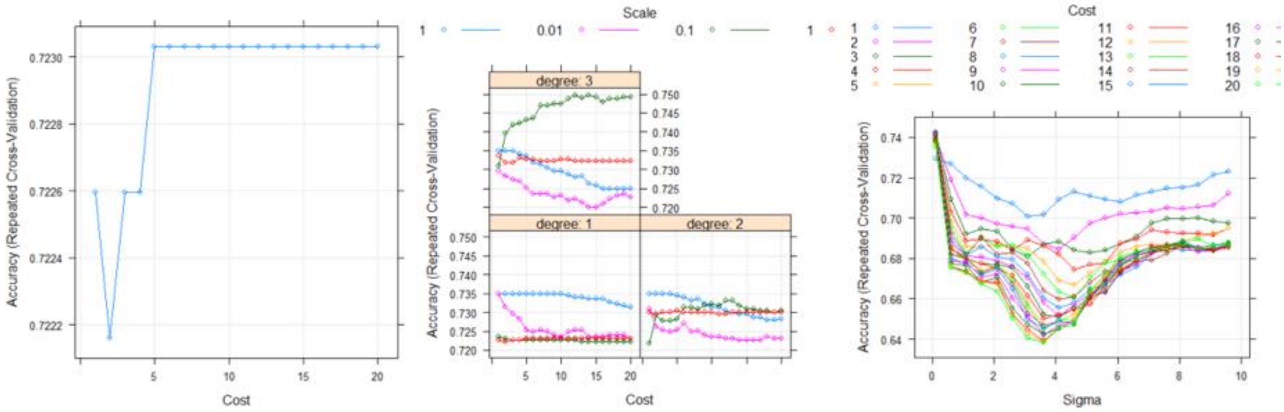


Fig 2: Optimal hyperparameters search for Haberman's Survival dataset

In Figure 2, the graphical representation of the values of the Cost parameter for the linear kernel according to SVM classification accuracy percentages was given on the left. It is seen that the highest accuracy values are obtained when the C parameter was greater than or equal to 5. In the middle, graphical representation of polynomial kernel parameters according to classification accuracy was given. It was observed that there is a relationship between C, scale and degree parameters and they have different effects on classification accuracy in different situations. On the right, a graphical representation of the values for the C and sigma parameters of the RBF kernel function was given.

5. Conclusion

The present study was focused on tuning the hyperparameters in SVM classification problems. Grid search and ten-fold CV methods were used to obtain optimal values of hyperparameters according to kernel functions with different sample sizes and correlation levels. Then, the classification accuracy values were examined by performing SVM classification.

SVM is a powerful method developed for classification and regression problems. Although grid search and five- or ten-fold CV yields successful results in finding the optimal value for hyperparameters, it may still pose a risk to determine the intervals for these parameters by the users. Therefore, in future studies, developing new approaches in addition to existing methods on the automatic selection of hyperparameter and kernel function will save time in analyses and produce more reliable results.

Conflict of Interest

The authors declare no conflict of interest.

References

[1] B. E. Boser, I. M. Guyon, V. N. Vapnik, *A Training Algorithm for Optimal Margin Classifiers*, In Proceedings of the Fifth Annual Workshop on Computational Learning Theory (1992) 144-152.

[2] V. N. Vapnik, *The Nature of Statistical Learning Theory*, New York: Springer-Verlag, 1995.

[3] V. Jakkula, *Tutorial on Support Vector Machine (SVM)*, School of EECS, Washington State University, (2006) 37.

[4] S. Han, C. Qubo, H. Meng, *Parameter Selection in SVM with RBF Kernel Function*, In World Automation Congress (2012) 1-4 Puerto Vallarta, Mexico.

[5] X. Chen, J. He, X. Wu, W. Yan, W. Wie, *Sleep Staging by Bidirectional Long Short-term Memory Convolution Neural Network*, Future Generation Computer Systems 109 (2020) 188-196.

[6] J. K. Appati, G. K. Gogovi, G. O. Fosu, *On the Selection of Appropriate Kernel Function for SVM in Face Recognition*, International Journal of Advanced Research in Computer Science and Software Engineering 4(3) (2014) 6-9.

[7] W. M. Campbell, J. P. Campbell, D. A. Reynolds, E. Singer, P. A. Torres-Carrasquillo, *Support Vector Machines for Speaker and Language Recognition*, Computer Speech and Language 20(2-3) (2006) 210-229.

[8] S. Bellamkonda, N. P. Gopalan, *A Facial Expression Recognition Model Using Support Vector Machines*, IJ Mathematical Sciences and Computing 4 (2018) 56-65.

[9] M. Rezaei, E. Zereshki, H. Sharini, M. Gharib Salehi, F. Naleini, *Detection of Alzheimer's Disease Based on Magnetic Resonance Imaging of The Brain Using Support Vector Machine Model*, Tehran University Medical Journal TUMS Publications 76(6) (2018) 410-416.

[10] Ö. Y. Akşehirli, H. Ankaralı, D. Aydın, Ö. Saraçlı, *An Alternative Approach in Medical Diagnosis: Support Vector Machines*, Türkiye Klinikleri Journal of Biostatistics 5(1) (2013) 19-28.

[11] F. E. Tay, L. Cao, *Application of Support Vector Machines in Financial Time Series Forecasting*, Omega 29(4) (2001) 309-317.

[12] P. Wang, R. Mathieu, J. Ke, H. J. Cai, *Predicting Criminal Recidivism with Support Vector Machine*, International Conference on Management and Service Science (2010) Wuhan, China.

[13] J. Karia, *Stock Market Prediction Using Machine Learning*, International Journal of Emerging Technology and Computer Science 3(2) (2018) 159-162.

[14] J. Wainer, P. Fonseca, *How to Tune the RBF SVM Hyperparameters? An Empirical Evaluation of 18 Search Algorithms." arXiv preprint arXiv:2008.11655* (2020).

[15] D. Fradkin, I. Muchnik, *Support Vector Machines for Classification,* DIMACS Series in Discrete Mathematics and Theoretical Computer Science 70 (2006) 13-20.

[16] S. R. Gunn, *Support Vector Machines for Classification and Regression*, ISIS Technical Report 14(1) (1998) 5-16.

[17] C. F. Lin, S. D. Wang, *Fuzzy Support Vector Machines*, IEEE Transactions on Neural Networks 13(2) (2002) 464-471.

[18] C. Savas, F. Dovis, *The Impact of Different Kernel Functions on The Performance of Scintillation Detection Based on Support Vector Machines*, Sensors 19(23) (2019) 1-16.

[19] D. Dua, C. Graff, *UCI Machine Learning Repository [http://archive.ics.uci.edu/ml]*, Irvine, CA: University of California, School of Information and Computer Science, (2019).

Journal of New Theory

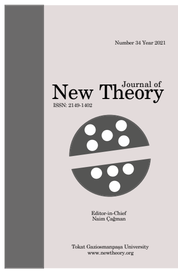
ISSN: 2149-1402

34 (2021) 72-81

Journal of New Theory

https://dergipark.org.tr/en/pub/jnt

Open Access



On Transmuted Power Function Distribution: Characterization, Risk Measures, and Estimation

Caner Tanış[1]

Article History
Received: 10 Feb 2021
Accepted: 26 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract − Transmuted power function distribution is generated using the quadratic rank transmutation method based on the mixture of the distributions of two order statistics. The distributions generating via Quadratic rank transmutation map are more flexible than the baseline ones since they have a potential to model various dataset. In this study, we provide some distributional properties and statistical inferences of transmuted power function distribution. We describe several previously unexamined properties, such as density shape, hazard shape, and the transmuted power function distribution measures. We also tackle the problem of point estimation for transmuted power function distribution. In this regard, maximum likelihood, least-squares, weighted least-squares, Anderson-Darling method, and Crámer–Von-Mises method are considered to estimate the two parameters of transmuted power function distribution. A comprehensive Monte Carlo simulation study is performed to compare these methods via bias and mean-squared errors.

Keywords *− Transmuted power function distribution, power function distribution, point estimation, risk measures, Monte Carlo simulation*

Mathematics Subject Classification (2020) − 62F10, 62P05

1. Introduction

Many authors very commonly discuss the point estimation and various characterization of the statistical distributions. Describing the distributions' statistical properties in detail is very significant to illustrate the usefulness of the distributions. Another critical point is the parameter estimation problem for the statistical distributions. It is well-known that the maximum likelihood method is very popular for point estimation. However, many researchers studied various alternative methods to the maximum likelihood method. In the last decade, there are many papers on the characterization and estimation of the distributions. Mahmoud and Mandouh [1] described some distributional properties of transmuted Fréchet distribution. Hamedani [2] examined some characteristics of transmuted complementary Weibull geometric distribution. Ahmad et al. [3] provided the characterization of transmuted Kumaraswamy distribution. Ahmad et al. [4] focused on a number of statistical properties and point estimation for transmuted Rayleigh distribution. Bhatti et al. [5] studied a couple of characterizations of transmuted Dagum distribution. Bhatti et al. [6] discussed several distributional properties of transmuted modified Burr II distribution. Bhatti et al. [7] examined some statistical properties of the transmuted geometric-quadratic hazard rate distribution. Tanış et al. [8] considered a comparison of the approximate Bayes and maximum likelihood estimation methods for log-Dagum distribution. Tanış and

[1]caner.tanis@gmail.com (Corresponding Author)
[1]Department of Statistics, Faculty of Arts and Sciences, Çankırı Karatekin University, Çankırı, Turkey

Saraçoğlu [9] compared the methods of estimation for log-Kumaraswamy distribution. Hanif et al. [10] discussed several estimation methods for Rician distribution. Anas et al. [11] performed partial characterisation of extreme value distribution. Hanif et al. [12] tackled the estimation of parameters' discrete inverse Weibull distribution using ranked set sampling. Hanif et al. [13] focused on the estimation of parameters' generalized exponential distribution. Karakaya and Tanış [14] compared the estimation methods for Akash distribution. Tanış and Saraçoğlu [15] provided a comparison of the methods of estimation for transmuted record type Weibull distribution. Karakaya and Tanış [16] discussed the estimation problem of Xgamma-Weibull distribution. Tanış et al. [17] described the estimation methods for transmuted lower record type Fréchet distribution.

The purpose of this paper is to examine some distributional properties and compare five estimation methods such as maximum likelihood, least-squares, weighted least-squares, Anderson-Darling, and Crámer–Von-Mises for transmuted power function distribution [18]. The paper is organized as follows: In Section 2, the transmuted power function distribution and distributional properties are described, such as density and hazard shapes with theorems. Then, some risk measures are defined for transmuted power function distribution in Section 3. Section 4 presents five methods of estimation for point estimation. Section 5 provides an extensive Monte Carlo simulation study to compare these estimation methods. Finally, the conclusions are presented in Section 6.

2. Transmuted Power Function Distribution

Transmuted power function distribution is proposed by Shahzad and Asghar [18] via a quadratic transmutation map (QRTM). The relationship between baseline distribution and transmuted distribution obtained by using QRTM are summarized by

$$F(x) = G(x)\big[1 + \lambda\big(1 - G(x)\big)\big] \tag{1}$$

where $|\lambda| \leq 1$, $G(x)$ denotes the cumulative distribution function (CDF) of baseline distribution, and $F(x)$ refers to the CDF of transmuted distribution newly generated by the QTRM. Consider the baseline distribution power function distribution with CDF $G(x;\beta) = x^{\beta}$ and the probability density function (PDF) $g(x;\beta) = \beta x^{\beta-1}$ then, the PDF and CDF of transmuted power function distribution are as follows:

$$F(x;\beta,\lambda) = x^{\beta}\{1 + \lambda(1 - x^{\beta})\} \tag{2}$$

and

$$f(x;\beta,\lambda) = \beta x^{\beta-1}\{1 + \lambda - 2\lambda x^{\beta}\} \tag{3}$$

respectively, where $\beta > 0$ is a shape parameter and $-1 \leq \lambda \leq 1$ [18]. Transmuted power function distribution can model the datasets in many fields, such as engineering, economics, hydrology, and social and behavioural sciences. Some statistical properties include mean, mode, median, variance, quantile function, reliability function, hazard function, order statistics, and generalized TL-moments with its special cases L-, TL-, LL LH-moments are described for transmuted power function distribution in [18]. In this paper, transmuted power function distribution is briefly denoted by $TPF(\beta,\lambda)$. Recently, many papers have produced about power function distribution in the literature. Some of these studies are listed as follows: Akhter [19] studied the estimation methods for power function distribution. Tahir et al. [20] proposed a new statistical distribution called Weibull-power function distribution. Okorie et al. [21] introduced the modified power function distribution. Bursa and Özel [22] provided a new extension of power function distribution, called exponentiated Kumaraswamy-power function distribution. Hassan and Salwa [23] proposed a new statistical distribution called exponentiated Weibull-power function distribution. Haq et al. [24] suggested the transmuted Weibull power function distribution. The cubic transmuted power function distribution was introduced by [25]. Arshad et al. [26] suggested the exponentiated power function distribution. Jabenn and Zaka [27] tackled the problem of percentile estimation for power function distribution.

2.1. Density and Hazard Shapes

In this subsection, we discuss the possible shapes of density and hazard for $TPF(\beta,\lambda)$ distribution with some theorems.

Theorem 2.1. PDF of $TPF(\beta,\lambda)$ distribution is unimodal for $\beta > 2$.

Proof. $T_1(x)$ and $T_2(x)$ denote the first and second derivatives of $\log\left(f_{TPF}(x;\beta,\lambda)\right)$, respectively. They are defined as follows:

$$T_1(x) = \frac{d}{dx}\log\left(f_{TPF}(x;\beta,\lambda)\right) = \frac{2(2\beta-1)\lambda x^{\beta} - (1+\lambda)(\beta-1)}{x(2\lambda x^{\beta} - \lambda - 1)}$$

and

$$T_2(x) = \frac{d^2}{dx^2}\log\left(f_{TPF}(x;\beta,\lambda)\right) = \frac{-8\lambda^2 x^{2\beta}\left(\beta - \frac{1}{2}\right) - (1+\lambda)^2(\beta-1)}{x^2(2\lambda x^{\beta} - \lambda - 1)^2} - \frac{2\lambda(1+\lambda)(\beta-1)(\beta-2)x^{\beta}}{x^2(2\lambda x^{\beta} - \lambda - 1)^2}$$

It is observed that $T_2(x) < 0$ for $\beta > 2$. Then, the density of $TPF(\beta,\lambda)$ distribution is log-concave and unimodal for $\beta > 2$.

□

Figure 1 illustrates the possible shapes of the density of $TPF(\beta,\lambda)$ distribution.

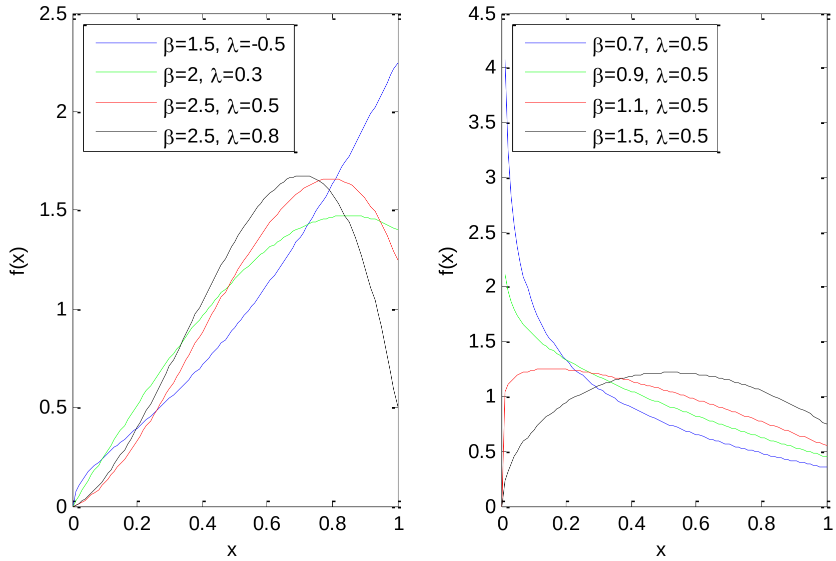


Fig. 1. The density plots of $TPF(\beta,\lambda)$ distribution for selected parameters

Theorem 2.2. The hazard function (HF) of $TPF(\beta,\lambda)$ distribution increases for $\beta > 2$.

Proof.

$$\eta(x) = -\frac{f'(x)}{f(x)} = \frac{2(1-2\beta)\lambda x^{\beta} + (1+\lambda)(\beta-1)}{x(2\lambda x^{\beta} - \lambda - 1)}$$

and the first derivative of $\eta(x)$ is defined by

$$\eta'(x) = \frac{d}{dx}\eta(x) = \frac{8\left(\beta - \frac{1}{2}\right)\lambda^2 x^{2\beta} + (1+\lambda)^2(\beta-1)}{x^2(2\lambda x^{\beta} - \lambda - 1)^2} + \frac{2\lambda(1+\lambda)(\beta-1)(\beta-2)x^{\beta}}{x^2(2\lambda x^{\beta} - \lambda - 1)^2}$$

We notice that $\eta(x) > 0$ for $\beta > 2$, and it can be concluded that the HF of $TPF(\beta,\lambda)$ distribution is increasing for $\beta > 2$ according to Glaser [28]. □

Figure 2 shows that the possible shapes of HF of $TPF(\beta,\lambda)$ distribution.

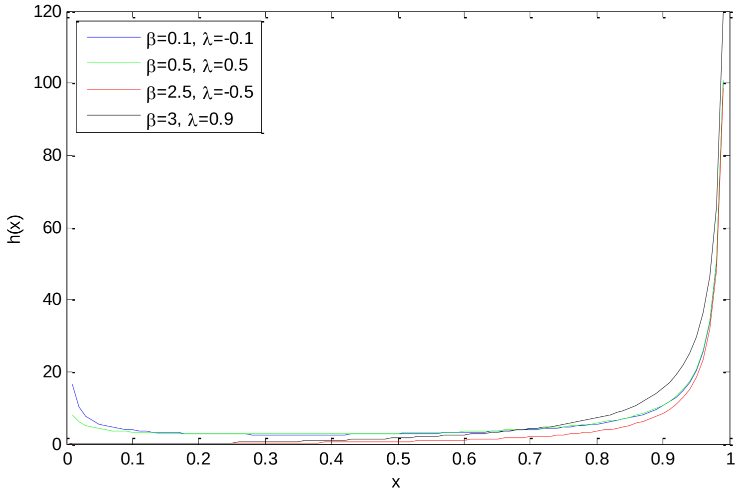


Fig. 2. The hazard plots of $TPF(\beta,\lambda)$ distribution for selected parameters

From Figure 2, we observe that the shape of HF tends to increase. Shahzad and Asghar [18] mention that the $TPF(\beta,\lambda)$ distribution is more flexible than power function distribution since it has an increasing and bathtub-shaped hazard rate.

3. Risk Measures

In this section, we discuss the theoretical and computational aspects of some essential risk measures such as value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) for the $TPF(\beta,\lambda)$ distribution.

3.1. VaR Measure

The VaR is a well-known measure of the risk of loss for investments. It is also called quantile risk measure. Firms and regulators generally use the VaR in the financial sector to determine the number of assets required to cover potential losses. The VaR of a random variable X is the q^{th} quantile of its CDF, denoted by VaR_q, and it is defined by $VaR_q = Q(q)$ [29,30].

Let X be a random variable from $TPF(\beta,\lambda)$ distribution, then its VaR can be obtained by

$$VaR_q = \left[\frac{\lambda + 1 - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]^{\frac{1}{\beta}} \tag{4}$$

where $q \in (0,1)$.

3.2. TVaR Measure

TVaR, also known as conditional tail expectation, is a significant risk measure. It measures the expected value of the loss given that an event outside a given probability level has occurred. The TVaR of $TPF(\beta,\lambda)$ distribution is

$$TVaR_q = \frac{1}{1-q}\int_{VaR_q}^{1} xf(x)\,dx = \frac{1}{1-q}\left[\frac{(1+\lambda)\beta\left(1 - {VaR_q}^{\beta+1}\right)}{\beta+1} + \frac{2\lambda\beta\left({VaR_q}^{\beta+1} - 1\right)}{2\beta+1}\right] \tag{5}$$

where VaR_q is defined in (4).

3.3. TV Measure

The TV is one of the most significant risk-quantifying measures, which pay attention to the tail variance beyond the VaR. TV was suggested by Landsman [31]. The TV of $TPF(\beta,\lambda)$ distribution is given by

$$\begin{aligned} TV_q(X) &= E\left(X^2 \mid X > x_q\right) - \left\{TVaR_q\right\}^2 \\ &= \frac{1}{1-q}\int_{VaR_q}^{1} x^2 f(x)\,dx - \left\{TVaR_q\right\}^2 \\ &= \frac{1}{1-q}\left[\frac{(1+\lambda)\left(1-VaR_q{}^{\beta+2}\right)}{\beta+2} + \frac{\lambda\beta\left(VaR_q{}^{2\beta+2}-1\right)}{\beta+1}\right] - \left\{TVaR_q\right\}^2 \end{aligned} \tag{6}$$

where $TVaR_q$ is defined in (5).

3.4. TVP Measure

The TVP is one of the most used risk measures, which essentially plays a role in insurance sciences. The TVP of $TPF(\beta,\lambda)$ distribution is given as follows:

$$TVP_q = TVaR_q + \mu TV_q \tag{7}$$

where $0 < \mu < 1$, $TVaR_q$, and TV_q are defined in (5) and (6), respectively. Tables 1-2 provide the VaR, TVaR, TV, and TVP of the $TPF(\beta,\lambda)$ distribution for some parameters.

Table 1. VaR, TVaR, TV, and TVP of the $TPF(\beta,\lambda)$ distribution for selected parameters

Parameters	μ	Significance Level	VaR	TVaR	TV	TVP
		0.7	0.226137	0.460796	0.035095	0.478343
		0.75	0.276244	0.502876	0.031448	0.5186
		0.8	0.337432	0.552153	0.027091	0.565698
$\beta=0.5,\lambda=0.9$	0.5	0.85	0.41415	0.61146	0.021889	0.622404
		0.9	0.514985	0.686193	0.015657	0.694022
		0.95	0.661611	0.789652	0.008135	0.79372
		0.99	0.876849	0.930279	0.001204	0.930881
		0.7	0.885733	0.945493	0.001083	0.945818
		0.75	0.907125	0.955287	0.000717	0.955502
		0.8	0.927441	0.964767	0.000438	0.964898
$\beta=2,\lambda=-0.5$	0.3	0.85	0.946797	0.973958	0.000236	0.974028
		0.9	0.965289	0.982881	0.0001	0.982911
		0.95	0.982999	0.991556	2.41×10^{-5}	0.991563
		0.99	0.996654	0.998329	9.33×10^{-7}	0.998329
		0.7	0.92527	0.964136	0.000463	0.96446
		0.75	0.939076	0.970518	0.000308	0.970733
		0.8	0.952255	0.976718	0.00019	0.976851
$\beta=5,\lambda=0.1$	0.7	0.85	0.96488	0.982754	0.000103	0.982826
		0.9	0.977013	0.988638	4.4×10^{-5}	0.988669
		0.95	0.988704	0.994383	1.06×10^{-5}	0.994391
		0.99	0.997771	0.998886	4.14×10^{-7}	0.998887
		0.7	0.398807	0.633135	0.026904	0.649277
		0.75	0.458578	0.674146	0.022134	0.687427
		0.8	0.526709	0.719719	0.017187	0.730031
$\beta=0.7,\lambda=0.7$	0.6	0.85	0.605911	0.771214	0.012135	0.778495
		0.9	0.700959	0.830911	0.007135	0.835192
		0.95	0.821919	0.903249	0.002592	0.904804
		0.99	0.955898	0.977402	0.000162	0.977499

Table 2. VaR, TVaR, TV, and TVP of the $TPF(\beta,\lambda)$ distribution for selected parameters

Parameters	μ	**Significance Level**	**VaR**	**TVaR**	**TV**	**TVP**
		0.7	0.82811	0.916594	0.002456	0.917576
		0.75	0.859004	0.931184	0.001654	0.931845
		0.8	0.888889	0.945473	0.001028	0.945884
$\beta = 1, \lambda = -0.9$	0.4	0.85	0.917856	0.959482	0.000562	0.959707
		0.9	0.945986	0.97323	0.000243	0.973327
		0.95	0.973348	0.986731	5.92×10^{-5}	0.986754
		0.99	0.994724	0.997364	2.32×10^{-6}	0.997365
		0.7	0.950062	0.970539	0.000164	0.970687
		0.75	0.956434	0.973997	0.000125	0.97411
		0.8	0.96285	0.977589	9.05×10^{-5}	0.97767
$\beta = 15, \lambda = 0.95$	0.9	0.85	0.96947	0.981407	6.12×10^{-5}	0.981462
		0.9	0.976577	0.985628	3.62×10^{-5}	0.98566
		0.95	0.984855	0.990692	1.55×10^{-5}	0.990706
		0.99	0.994485	0.996725	2.24×10^{-6}	0.996727
		0.7	0.289297	0.592721	0.041935	0.596915
		0.75	0.36717	0.645842	0.03329	0.649171
		0.8	0.459168	0.70432	0.024338	0.706754
$\beta = 0.3, \lambda = 0.05$	0.1	0.85	0.566811	0.768554	0.015625	0.770116
		0.9	0.691699	0.838956	0.007918	0.839747
		0.95	0.835509	0.915956	0.002255	0.916181
		0.99	0.96536	0.982606	10^{-4}	0.982616
		0.7	0.833017	0.913352	0.002269	0.915167
		0.75	0.859061	0.926815	0.001624	0.928114
		0.8	0.885264	0.940485	0.001081	0.941349
$\beta = 3, \lambda = 0.5$	0.8	0.85	0.911932	0.954465	0.00064	0.954977
		0.9	0.93947	0.968894	0.000304	0.969136
		0.95	0.968481	0.983968	8.26×10^{-5}	0.984034
		0.99	0.993418	0.996695	3.61×10^{-6}	0.996698

4. Point Estimation

In this section, we consider five estimation methods to estimate the parameters of the $TPF(\beta,\lambda)$ distribution including maximum likelihood, least squares, weighted least squares, Anderson-Darling method, and Cramér-Von Mises method.

Let $X_1, X_2, \cdots, X_n$ be a random sample from the $TPF(\beta, \lambda)$ distribution and $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the corresponding order statistics. Further, $x_{(i)}$ refers to the observed value of $X_{(i)}$. In this regard, the log-likelihood function of the $TPF(\beta, \lambda)$ distribution is

$$\ell(\theta) = n\log(\beta) + (\beta - 1)\sum_{i=1}^{n}\log\left(1 + x_i^2\right) + \sum_{i=1}^{n}\log\left(1 + \lambda - 2\lambda x_i^{\beta}\right) \tag{8}$$

where $\theta = (\beta, \lambda)$ is a parameter vector. Then, the maximum likelihood estimator (MLE) of θ is given as follows:

$$\hat{\theta}_{MLE} = \underset{\theta}{\operatorname{argmax}}\{\ell(\theta)\} \tag{9}$$

Let us define the following four functions, which are used to obtain the different type of estimates:

$$Q_{LS}(\theta) = \sum_{i=1}^{n}\left(\left[x_{(i)}{}^{\beta}\left\{1 + \lambda\left(1 - x_{(i)}{}^{\beta}\right)\right\}\right] - \frac{i}{n+1}\right)^2,$$

$$Q_{WLS}(\theta) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(\left[x_{(i)}{}^{\beta}\left\{1 + \lambda\left(1 - x_{(i)}{}^{\beta}\right)\right\}\right] - \frac{i}{n+1}\right)^2,$$

$$Q_{CvM}(\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(\left[x_{(i)}{}^{\beta}\left\{1 + \lambda\left(1 - x_{(i)}{}^{\beta}\right)\right\}\right] - \frac{2i-1}{2n}\right)^2,$$

and

$$Q_{AD}(\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2i-1)\log\left[x_{(i)}{}^{\beta}\left\{1 + \lambda\left(1 - x_{(i)}{}^{\beta}\right)\right\}\right]\right) + \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(1 - \left[x_{(i)}{}^{\beta}\left\{1 + \lambda\left(1 - x_{(i)}{}^{\beta}\right)\right\}\right]\right)\right)$$

The least squares estimators (LSEs), weighted least squares estimators (WLSEs), Cramér–von Mises estimators (CvMEs), and Anderson-Darling estimators (ADEs) of the parameters $\theta = (\beta, \lambda)$ are given, respectively, by

$$\hat{\theta}_{LSE} = \underset{\theta}{\operatorname{argmin}}\{Q_{LS}(\theta)\} \tag{10}$$

$$\hat{\theta}_{WLSE} = \underset{\theta}{\operatorname{argmin}}\{Q_{WLS}(\theta)\} \tag{11}$$

$$\hat{\theta}_{CvME} = \underset{\theta}{\operatorname{argmin}}\{Q_{CvM}(\theta)\} \tag{12}$$

$$\hat{\theta}_{ADE} = \underset{\theta}{\operatorname{argmin}}\{Q_{AD}(\theta)\} \tag{13}$$

The estimators given in (9)-(13) can be obtained by **optim ()** function in R with the BFGS algorithm.

5. Simulation Study

In this section, we perform a comprehensive Monte Carlo simulation study to compare the performances of MLEs, LSEs, WLSEs, CvMEs, and ADEs of β and λ according to biases and MSEs. The simulation study is performed based on 1000 repetitions. We consider the sample size 50, 100, 250, 500, 1000, and four-parameter settings as follows:

$$(\beta = 2, \lambda = 0.5), (\beta = 0.9, \lambda = -0.5), (\beta = 1, \lambda = 0.7), (\beta = 1.5, \lambda = -0.7)$$

BFGS algorithm is performed to get the five estimates given in (9)-(13). Tables 3 and 4 provide the biases and MSEs of five estimators for selected parameters and sample sizes.

Table 3. Average biases of MLEs, LSEs, WLSEs, ADEs, and CvMEs of β and λ parameters

		$\hat{\beta}$					$\hat{\lambda}$				
Parameters	n	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
	50	0.0777	-0.0300	-0.0071	0.0178	0.0650	-0.0592	-0.1386	-0.1235	-0.1015	-0.0573
	100	0.0518	0.0068	0.0256	0.0317	0.0550	-0.0338	-0.0655	-0.0515	-0.0456	-0.0234
$\beta = 2, \lambda = 0.5$	250	0.0388	0.0189	0.0273	0.0293	0.0370	-0.0245	-0.0388	-0.0334	-0.0316	-0.0234
	500	0.0332	0.0217	0.0274	0.0270	0.0305	-0.0136	-0.0221	-0.0174	-0.0177	-0.0145
	1000	0.0167	0.0088	0.0125	0.0123	0.0132	-0.0160	-0.0216	-0.0188	-0.0189	-0.0179
	50	0.1899	0.0331	0.1018	0.0337	0.0205	0.1487	-0.1357	-0.0043	-0.1118	-0.1762
	100	0.1330	0.0128	0.0317	0.0335	0.0027	0.0853	-0.1309	-0.0827	-0.0789	-0.1574
$\beta = 0.9, \lambda = -0.5$	250	0.0688	-0.0191	0.0354	0.0051	-0.0255	0.0321	-0.1272	-0.0313	-0.0770	-0.1418
	500	0.0496	-0.0152	0.0171	0.0138	-0.0207	0.0227	-0.0959	-0.0377	-0.0430	-0.1066
	1000	0.0222	-00022	0.0040	0.0248	-0.0032	-0.0043	-00576	-0.0410	-0.0092	-0.0603
	50	0.0234	-0.0179	-0.0082	-0.0003	0.0237	-0.1228	-0.1806	-0.1685	-0.1531	-0.1043
	100	0.0215	0.0006	0.0092	0.0098	0.0197	-0.0652	-0.0958	-0.0818	-0.0820	-0.0608
$\beta = 1, \lambda = 0.7$	250	0.0226	0.0156	0.0185	0.0185	0.0229	-0.0238	-0.0337	-0.0302	-0.0293	-0.0199
	500	0.0134	0.0091	0.0111	0.0108	0.0128	-0.0210	-0.0269	-0.0237	-0.0244	-0.0201
	1000	0.0094	0.0081	0.0090	0.0088	0.0099	-0.0137	-0.0157	-0.0142	-0.0146	-0.0123
	50	0.3421	0.2711	0.2529	0.2517	0.2234	0.1521	0.0240	0.0402	0.0493	-0.0327
	100	0.2050	0.1831	0.1782	0.1740	0.1507	0.0780	0.0069	0.0259	0.0278	-0.0303
$\beta = 1.5, \lambda = -0.7$	250	0.1492	0.1341	0.1300	0.1271	0.1134	0.0590	0.0192	0.0281	0.0272	-0.0025
	500	0.0923	0.1046	0.0884	0.0860	0.0908	0.0287	0.0204	0.0158	0.0141	0.0065
	1000	0.0712	0.0796	0.0924	0.0684	0.0693	0.0268	0.0220	0.0372	0.0183	00122

Table 4. Average MSEs of MLEs, LSEs, WLSEs, ADEs, and CvMEs of β and λ parameters

		$\hat{\beta}$					$\hat{\lambda}$				
Parameters	n	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
	50	0.1403	0.1725	0.1581	0.1431	0.1751	0.1192	0.1655	0.1556	0.1347	0.1412
	100	0.0752	0.0917	0.0794	0.0731	0.0899	0.0726	0.0837	0.0744	0.0670	0.0722
$\beta = 2, \lambda = 0.5$	250	0.0276	0.0335	0.0321	0.0302	0.0341	0.0228	0.0286	0.0295	0.0266	0.0270
	500	0.0144	0.0174	0.0149	0.0148	0.0178	0.0116	0.0142	0.0121	0.0121	0.0138
	1000	0.0069	0.0085	0.0074	0.0074	0.0086	0.0056	0.0069	0.0060	0.0060	0.0067
	50	0.1018	0.0563	0.0664	0.0506	0.0688	0.1655	0.1422	0.1289	0.1175	0.2011
	100	0.0692	0.0400	0.0380	0.0372	0.0465	0.1110	0.1164	0.0960	0.0913	0.1454
$\beta = 0.9, \lambda = -0.5$	250	0.0377	0.0273	0.0308	0.0254	0.0301	0.0768	0.0865	0.0730	0.0700	0.0982
	500	0.0258	0.0210	0.0219	0.0215	0.0224	0.0551	0.0640	0.0571	0.0563	0.0700
	1000	0.0147	0.0175	0.0149	0.0167	0.0185	0.0358	0.0460	0.0398	0.0387	0.0487
	50	0.0274	0.0354	0.0308	0.0276	0.0349	0.0951	0.1450	0.1281	0.1093	0.1141
	100	0.0129	0.0168	0.0135	0.0138	0.0172	0.0416	0.0629	0.0473	0.0513	0.0574
$\beta = 1, \lambda = 0.7$	250	0.0053	0.0064	0.0060	0.0055	0.0067	0.0149	0.0204	0.0201	0.0161	0.0194
	500	0.0025	0.0030	0.0026	0.0026	0.0031	0.0075	0.0097	0.0081	0.0080	0.0094
	1000	0.0012	0.0016	0.0013	0.0013	0.0016	0.0036	0.0050	0.0040	0.0040	0.0049
	50	0.2602	0.2566	0.2013	0.1731	0.2609	0.0762	0.0947	0.0582	0.0380	0.1220
	100	0.1212	0.1695	0.1314	0.1156	0.1741	0.0524	0.0855	0.0544	0.0414	0.1001
$\beta = 1.5, \lambda = -0.7$	250	0.0988	0.1104	0.0899	0.0838	0.1110	0.0504	0.0635	0.0454	0.0407	0.0681
	500	0.0648	0.0850	0.0638	0.0612	0.0842	0.0388	0.0515	0.0363	0.0346	0.0531
	1000	0.0481	0.0618	0.0624	0.0470	0.0604	0.0304	0.0392	0.0359	0.0285	0.0391

As a result of the simulation study, we observe that, with increasing sample sizes, the MSEs and biases decrease as expected. From Table 3 and 4, it has been general observed that the MLE has a smaller MSE compared to other estimators for both β and λ parameters. However, there are some situations that ADE and CvME have a smaller bias than MLE. As a result, we recommend MLE for point estimation of $TPF(\beta,\lambda)$ distribution. However, ADE and CvME can be good alternatives to MLE to estimate the parameters of $TPF(\beta,\lambda)$ distribution.

6. Conclusion

In this study, $TPF(\beta,\lambda)$ distribution proposed by Shahzad and Asghar [18] is studied in terms of some characteristic properties and statistical inferences. Some critical risk measures are discussed and numerically obtained for $TPF(\beta,\lambda)$ distribution. We compare five estimators of parameters of $TPF(\beta,\lambda)$ distribution, such as MLE, LSE, WSE, ADE, and CvME via Monte Carlo simulations. In the simulation study, it is seen that MLE is the best estimator among others according to MSE criteria. We recommend MLE to estimate the parameters of $TPF(\beta,\lambda)$ distribution.

References

[1] M. R. Mahmoud, R. M. Mandouh, *On the Transmuted Fréchet Distribution*, Journal of Applied Sciences Research 9(10) (2013) 5553-5561.

[2] G. G. Hamedani, *Characterizations of Transmuted Complementary Weibull Geometric Distribution*, Pakistan Journal of Statistics and Operation Research 11(2) (2015) 153-157.

[3] A. Ahmad, S. P. Ahmad, A. Ahmed, *Characterization and Estimation of Transmuted Kumaraswamy Distribution*, Mathematical Theory and Modeling 5(9) (2015) 168-174.

[4] A. Ahmad, S. P. Ahmad, A. Ahmed, *Characterization and Estimation of Transmuted Rayleigh Distribution*, Journal of Statistics Applications & Probability 4(2) (2015) 315-321.

[5] F. A. Bhatti, M. Ahmad, A. Ali, *Some Characterizations of Transmuted Dagum Distribution*, International Conference on Statistical Sciences, Khairpur, Pakistan 31 (2017) 109-122.

[6] F. A. Bhatti, G. G. Hamedani, A. Ali, M. Ahmad, *Some Characterizations of Transmuted Modified Burr III Distribution*, Asian Journal of Probability and Statistics (2018) 1-9.

[7] F. A. Bhatti, G. G. Hamedani, M. Ç. Korkmaz, M. Ahmad, *The Transmuted Geometric-quadratic Hazard Rate Distribution: Development, Properties, Characterizations and Applications*, Journal of Statistical Distributions and Applications 5(1) (2018) 1-23.

[8] C. Tanış, M. Çokbarlı, B. Saraçoğlu, *Approximate Bayes Estimation for Log-Dagum Distribution*, Cumhuriyet Science Journal 40(2) (2019) 477-486.

[9] C. Tanış, B. Saraçoğlu, *Comparisons of Six Different Estimation Methods for log-Kumaraswamy Distribution*, Thermal Science 23 (6) (2019) 1839-1847.

[10] M. Hanif, U. Shahzad, S. Asghar, N. Koyuncu, *Goodness of Fit Testing for Rician Distribution by Using Several Estimation Methods*, International Journal of Statistics and Economics 19 (2) (2018) 17-36.

[11] M. Anas, N. Jamal, M. Hanif, U. Shahzad, *Extreme Value Distributions on Closing Quotations and Returns of Islamabad Stock Exchange*, Asian Journal of Advanced Research and Reports 5(4) (2019) 1-9.

[12] M. Hanif, U. Shahzad, S. Amin, N. Afshan, *Estimation of the Discrete Inverse Weibull Distribution Parameters Using Simple Random sampling and Ranked Set Sampling*, Asian Journal of Advanced Research and Reports (Accepted Paper) (2020).

[13] M. Hanif, U. Shahzad, I. Shahzadi N. Koyuncu, *Stochastically Increasing Grouped data Using the MLE of Mean of the Generalized Exponential Distribution*, Journal of Organizational Behavior & Analytics 1(1) (2021) 65-82.

[14] K. Karakaya, C. Tanış, *Different Methods of Estimation for the One Parameter Akash Distribution*, Cumhuriyet Science Journal 41(4) (2020) 944-950.

[15] C. Tanış, B. Saraçoğlu, *On the Record-based Transmuted Model of Balakrishnan and He Based on Weibull Distribution*, Communications in Statistics-Simulation and Computation (2020) https://doi.org/10.1080/03610918.2020.1740261

[16] K. Karakaya, C. Tanış, *Estimating the Parameters of Xgamma Weibull Distribution*, Adıyaman University Journal of Science 10(2) (2020) 557-571.

[17] C. Tanış, B. Saraçoğlu, C. Kuş, A. Pekgör, K. Karakaya, *Transmuted Lower Record Type Fréchet Distribution with Lifetime Regression Analysis Based on Type I-Censored Data*, Journal of Statistical Theory and Applications (2021) https://doi.org/10.2991/jsta.d.210115.001

[18] M. N. Shahzad, Z. Asghar, *Transmuted Power Function Distribution: A More Flexible Distribution*, Journal of Statistics and Management Systems 19(4) (2016) 519-539.

[19] A. S. Akhter, *Methods for Estimating the Parameters of the Power Function Distribution*, Pakistan Journal of Statistics and Operation Research (2013) 213-224.

[20] M. H. Tahir, M. Alizadeh, M. Mansoor, G. M. Cordeiro, M. Zubair, *The Weibull-power Function Distribution with Applications*, Hacettepe Journal of Mathematics and Statistics 45(1) (2016) 245-265.

[21] I. E. Okorie, A. C. Akpanta, J. Ohakwe, D. C. Chikezie, *The Modified Power Function Distribution*, Cogent Mathematics 4(1) (2017) https://doi.org/10.1080/23311835.2017.1319592

[22] N. Bursa, G. Özel, *The Exponentiated Kumaraswamy-power Function Distribution*, Hacettepe Journal of Mathematics and Statistics 46(2) (2017) 277-292.

[23] A. S. Hassan, S. M. Assar, *The Exponentiated Weibull Power Function Distribution*, Journal of Data Science 16 (2017) 589-614.

[24] M. A. U. Haq, M. Elgarhy, S. Hashmi, G. Ö. Kadiler, Q. U. Ain, *Transmuted Weibull Power Function Distribution: its Properties and Applications*, Journal of Data Science 16(2) (2018) 397-418.

[25] S. İ. Ansari, M. Samuh, A. Bazyari, *Cubic Transmuted Power Function Distribution*, Gazi University Journal of Science 32(4) (2019) 1322-1337.

[26] M. Z. Arshad, M. Z. Iqbal, M. Ahmad, *Exponentiated Power Function Distribution: Properties and Applications*, Journal of Statistical Theory and Applications 19(2) (2020) 297-313.

[27] R. Jabeen, A. Zaka, *Percentiles Estimation for The Parameters of Power Function Distribution*, Advances and Applications in Statistics 62 (2) (2020) 127-137.

[28] R. E. Glaser, *Bathtub and Related Failure Rate Characterizations*, Journal of the American Statistical Association 75 (371) (1980) 667-672.

[29] P. Artzner, *Application of Coherent Risk Measures to Capital Requirements in Insurance*, North American Actuarial Journal 2(2) (1999) 11-25.

[30] A. Z. Afify, A. M. Gemeay, N. A. Ibrahim, *The Heavy-tailed Exponential Distribution: Risk Measures, Estimation, and Application to Actuarial Data*, Mathematics 8(8) (2020) 1-28.

[31] Z. Landsman, *On the Tail Mean-variance Optimal Portfolio Selection*, Insurance: Mathematics and Economics 46(3) (2010) 547-553.

Journal of New Theory

ISSN: 2149-1402

34 (2021) 82-114

Journal of New Theory

https://dergipark.org.tr/en/pub/jnt

Open Access

Operability-Oriented Configurations of the Soft Decision-Making Methods Proposed between 2013 and 2016 and Their Comparisons

Serdar Enginoğlu[1], Tuğçe Aydın[2], Samet Memiş[3], Burak Arslan[4]

Article History
Received: 13 Mar 2021
Accepted: 29 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract − The concept of fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) is a mathematical tool coming into prominence with its ability to model decision-making problems. Therefore, in the present study, we configure soft decision-making (SDM) methods having been constructed with soft sets, soft matrices, and their fuzzy hybrid versions and introduced between 2013 and 2016 to operate them in *fpfs*-matrices space faithfully to the original. We then analyse the decision-making performances of the configured methods herein by using five test cases containing totally ordered alternatives. Thus, we determine the methods producing a valid ranking order according to all the test cases and apply the determined methods to a performance-based value assignment (PVA) problem in which the filters are to be ranked in terms of their image denoising performances. Therefore, we compare the performance ranking of the filters by using the methods. Finally, we discuss the need for further research.

Keywords *– Fuzzy sets, soft sets, soft matrices, fpfs-matrices, soft decision-making, PVA problem*

Mathematics Subject Classification (2020) − 03E72, 15B15

1. Introduction

The soft decision-making (SDM) methods, constructed with the concepts of soft sets [1], fuzzy soft sets [2,3], fuzzy parameterized soft sets [4], fuzzy parameterized fuzzy soft sets (*fpfs*-sets) [5], soft matrices [6], fuzzy soft matrices [7], and fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) [8], are widely used to model uncertainties mathematically. The relationship between these concepts is provided as ordered from the general to the specific in Fig. 1. Moreover, many researchers have focused on these concepts in various areas, such as algebra [9-12], topology [13-17], analysis and function theory [18,19], decision-making [3,20], and data classification [21,22]. In literature, what studies related to SDM primarily lack is usually its application to a hypothetical problem instead of a real-life problem. A limited number of studies, including methods applied to a real problem, can be summarised as follows: In [23], the authors have used soft sets to attain shoreline resources evaluation rules. [24] has attracted attention to this theory using soft set theory in the computerised classification of malignant and normal micro-calcifications on mammograms. In [25], the scholars have proposed a method via fuzzy soft sets to classify numerical data. [26] has introduced a classification method to classify medical data using fuzzy soft sets. In [21], the researchers have applied an SDM method constructed by *fpfs*-matrices to monolithic columns classification. [22] has applied a data classification problem in machine learning by using *fpfs*-matrices.

[1]serdarenginoglu@gmail.com (Corresponding Author); [2]aydinttugce@gmail.com; [3]samettmemis@gmail.com; [4]tburakarslan@gmail.com

[1,2,3,4]Department of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, Çanakkale, Turkey

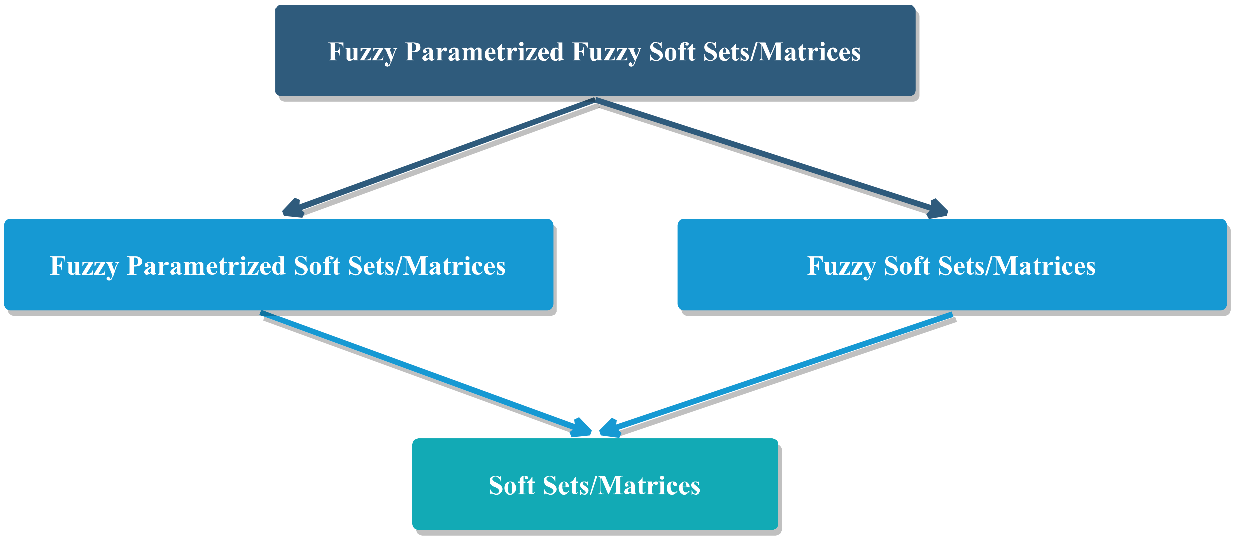


Fig. 1. Relationship between *fpfs*-sets/matrices and their substructures

Recently, the concept of *fpfs*-matrices [8] has stood out among the others due to its modelling success, the uncertainties in the decision-making problems where alternatives or parameters are fuzzy. Thus, the configurations of SDM methods constructed with the aforesaid concepts to operate them in *fpfs*-matrices space have become a popular study subject. To this end, over 50 SDM methods constructed with the aforesaid concepts have been configured [27-30] in *fpfs*-matrices space, faithfully to the original. Thereby, the configurations of the methods having been constructed with the abovementioned concepts and which were proposed between 1999 and 2012 have been completed. Furthermore, in [31-40], the authors have improved some of the configured methods to make them run faster and to simplify them mathematically. In [27,29], although some of the SDM methods proposed after 2012 have been configured, their configurations have not been completed yet. The present study aims to complete the configurations of the SDM methods having been constructed with soft sets, soft matrices, and their fuzzy hybrid versions and introduced between 2013 and 2016. To this end, we consider the SDM methods provided in [41-72].

The following tables provide some information about the preconfigured SDM methods. Table 1 explains the abbreviations used in Table 2-5. Table 2, 3, 4, and 5 show the unabbreviated forms of the previously configured SDM methods employing single, double, triple, and multiple matrices and their spaces in which they have been first put forward, respectively. Moreover, Table 6 lists the SDM methods constructed in the *fpfs*-matrices space. Lastly, Table 7 presents the SDM methods with the same configurations.

Table 1. Abbreviations of the considered spaces

SS	Soft Sets
SM	Soft Matrices
FSS	Fuzzy Soft Sets
FSM	Fuzzy Soft Matrices
FPSS	Fuzzy Parameterized Soft Sets
FPFSS	Fuzzy Parameterized Fuzzy Soft Sets
FPFSM	Fuzzy Parameterized Fuzzy Soft Matrices

Table 2. SDM methods employing single *fpfs*-matrix

Configured SDM Methods	**Original Spaces of the Configured SDM Methods**						**Descriptions**
FPFSM	FPFSS	FPSS	FSM	FSS	SM	SS	
CCE10 [27]	✓						Çağman, Çıtak, Enginoğlu 2010
CCE11 [27]		✓					Çağman, Çıtak, Enginoğlu 2011
CEC11 [29]				✓			Çağman, Enginoğlu, Çıtak 2011
F10(z) [28]				✓			Feng 2010
FJLL10 [29]				✓			Feng, Jun, Liu, Li 2010
FJLL10/2 [29]				✓			Feng, Jun, Liu, Li 2010
FJLL10/3 [29]				✓			Feng, Jun, Liu, Li 2010
FJLL10/4 [29]				✓			Feng, Jun, Liu, Li 2010
KKT13 [27]			✓				Khan, Khan, Thakur 2013
KM11 [29]				✓			Kalaichelvi, Malini 2011
KS10 [28]				✓			Kalayathankal, Singh 2010
KSM10 [28]				✓			Kuang, Shu, Mou 2010
KWW11(w, z) [28]				✓			Kong, Wang, Wu 2011
M11 [29]				✓			Mou 2011
MBR01 [27]				✓			Maji, Biswas, Roy 2001
MRB02 [27]						✓	Maji, Roy, Biswas 2002
MS10 [29]						✓	Majumdar, Samantha 2010
SM11 [28]				✓			Sun, Ma 2011
WW11 [29]						✓	Wu, Wang 2011
YE12 [37]		✓					Yılmaz, Eraslan 2012

Table 3. SDM methods employing double *fpfs*-matrices

Configured SDM Methods	**Original Spaces of the Configured SDM Methods**						**Descriptions**
FPFSM	FPFS	FPSS	FSM	FSS	SM	SS	
BMM12 [30]			✓				Basu, Mahapatra, Mondal 2012
BMM12/3 [30]					✓		Basu, Mahapatra, Mondal 2012
CD12/3 [30]		✓					Çağman, Deli 2012
CD12/4 [30]		✓					Çağman, Deli 2012
CE10 [27]						✓	Çağman, Enginoğlu 2010
CE10-2 [27]					✓		Çağman, Enginoğlu 2010
CE12 [27]			✓				Çağman, Enginoğlu 2012
CE10an [39]						✓	Çağman, Enginoğlu 2010
CE10on [39]						✓	Çağman, Enginoğlu 2010
FLC12 [30]						✓	Feng, Li, Çağman 2012
ICJ17 [29]					✓		Inthumathi, Chitra, Jayasree
KWW11/2(w, z) [30]						✓	Kong, Wang, Wu 2011
NS11 [30]			✓				Neog, Sut 2011
VR13 [27]					✓		Vijayabalaji, Ramesh 2013
Z14 [29]					✓		Zhang 2014
ZZ16 [27]	✓						Zhu, Zhan 2016
ZZ16/2 [27]	✓						Zhu, Zhan 2016

Table 4. SDM methods employing triple *fpfs*-matrices

Configured SDM Methods	**Original Spaces of the Configured SDM Methods**						**Descriptions**
FPFSM	FPFSS	FPSS	FSM	FSS	SM	SS	
BMM12/2 [30]					✓		Basu, Mahapatra, Mondal 2012
KGW09 [30]				✓			Kong, Gao, Wang 2009
QYZ12 [30]				✓			Qin, Yang, Zhang 2012
RM07a [30]				✓			Roy, Maji 2007
RM07o [30]				✓			Roy, Maji 2007
RM11 [27]						✓	Razak, Mohamad 2011
RM13 [27]				✓			Razak, Mohamad 2013

Table 5. SDM methods employing multiple *fpfs*-matrices

Configured SDM Methods	**Original Spaces of the Configured SDM Methods**						**Descriptions**
FPFSM	FPFSS	FPSS	FSM	FSS	SM	SS	
BNS12 [29]			✓				Borah, Neog, Sut 2012
CD12 [27]		✓					Çağman, Deli 2012
CD12-2 [27]		✓					Çağman, Deli 2012
DB12 [27]				✓			Das, Borgohain 2012
E15 [27]						✓	Eraslan 2015
EK15 [27]				✓			Eraslan, Karaaslan 2015
MR13 [29]			✓				Mondal, Roy 2013
MR13/2 [29]			✓				Mondal, Roy 2013
MR13/3 [29]			✓				Mondal, Roy 2013
NB14 [29]			✓				Nagarajan, Balamurugan 2014
NKY17 [29]				✓			Nagarani, Kalyani, Yookesh
S12 [29]				✓			Sut 2012
YJ11 [29]			✓				Yang, Ji 2011
YJ11/2 [29]			✓				Yang, Ji 2011

It can be seen from Table 2, 3, 4, and 5 that the fuzzy soft sets space, one of the substructures of *fpfs*-sets, is widely used in decision-making problems.

Table 6. SDM methods constructed in *fpfs*-matrices space

Proposed SDM Methods	**Number of Employed Matrices**				**Descriptions**
	Single	Double	Triple	Multiple	
EM20o [36]			✓		Enginoğlu, Memiş 2020
EMA18on [32]		✓			Enginoğlu, Memiş, Arslan 2018
EMC19o [34]		✓			Enginoğlu, Memiş, Çağman 2019
EMK19 [35]				✓	Enginoğlu, Memiş, Karaaslan 2019
EMO18o [40]		✓			Enginoğlu, Memiş, Öngel 2018
EC20 (PEM) [8]	✓				Enginoğlu, Çağman 2020 (Prevalence Effect Method)
Simplified SDM Methods					
EM20a [36]			✓		Enginoğlu, Memiş 2020
EMA18an [39]		✓			Enginoğlu, Memiş, Arslan 2018
EMC19a [34]		✓			Enginoğlu, Memiş, Çağman 2019
EMO18a [33]		✓			Enginoğlu, Memiş, Öngel 2018
sDB12 [38]				✓	Simplified DB12
sMBR01 [31]	✓				Simplified MBR01

Table 7. SDM methods with the same configurations

CE10-2	CE12
RM11	RM13
MR13	NB14

In Section 2 of the present study, we present some of the basic definitions of *fpfs*-matrices to be needed in the following sections of the paper. In Section 3, we configure the SDM methods provided in [41-72]. In Section 4, we propound five test cases to examine the consistency of the SDM methods employing fpfs-matrices. We then determine the considered SDM methods producing a valid ranking order in all the test cases. In Section 5, we apply the determined methods to a performance-based value assignment (PVA) problem in which the filters are ranked with regard to their salt-and-pepper noise (SPN) removal performances. Therefore, we compare the ranking order performances of the methods in the PVA problem. Finally, we discuss the need for further research.

2. Preliminaries

In this section, firstly, we present the concept of *fpfs*-matrices [8]. Throughout this paper, let E be a parameter set, $F(E)$ be the set of all the fuzzy sets over E, and $\mu \in F(E)$. Here, a fuzzy set is denoted by $\{{}^{\mu(x)}x \mid x \in E\}$.

Definition 2.1. [5] Let U be a universal set, $\mu \in F(E)$, and α be a function from μ to $F(U)$. Then, the set $\left\{\left({}^{\mu(x)}x, \alpha\left({}^{\mu(x)}x\right)\right) \mid x \in E\right\}$, being the graphic of α, is called a fuzzy parameterized fuzzy soft set (*fpfs*-set) parameterized via E over U (or briefly over U).

In the present paper, the set of all the *fpfs*-sets over U is denoted by $FPFS_E(U)$. In $FPFS_E(U)$, since the graph(α) and α generate each other uniquely, the notations are interchangeable. Therefore, as long as it causes no confusion, we denote an *fpfs*-set graph(α) by α.

Example 2.2. Let $E = \{x_1, x_2, x_3\}$ and $U = \{u_1, u_2, u_3, u_4\}$. Then,

$$\alpha = \{({}^{0.4}x_1, \{{}^{0.2}u_2, {}^{0.4}u_3, {}^{0.7}u_4\}), ({}^{0.9}x_2, \{{}^{0.5}u_1, {}^{0.3}u_2, {}^{0.6}u_3, {}^{0.4}u_4\}), ({}^{0.7}x_3, \{{}^{0.2}u_1, {}^{0.9}u_3, {}^{1}u_4\})\}$$

is an *fpfs*-set over U.

Definition 2.3. [8] Let $\alpha \in FPFS_E(U)$. Then, $[a_{ij}]$ is called *fpfs*-matrix of α and is defined by

$$[a_{ij}] = \begin{bmatrix} a_{01} & a_{02} & a_{03} & \cdots & a_{0n} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{0,1,2,\cdots\}$ and $j \in \{1,2,\cdots\}$,

$$a_{ij} := \begin{cases} \mu(x_j), & i = 0 \\ \alpha\left({}^{\mu(x_j)}x_j\right)(u_i), & i \neq 0 \end{cases}$$

Here, if $|U| = m - 1$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

From now on, the set of all the *fpfs*-matrices parameterized via E over U is denoted by $FPFS_E[U]$.

Example 2.4. The *fpfs*-matrix of α provided in Example 2.2 is as follows:

$$[a_{ij}] = \begin{bmatrix} 0.4 & 0.9 & 0.7 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0 \\ 0.4 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.1 \end{bmatrix}$$

Definition 2.5. [8] Let $[a_{ij}]_{m\times n_1} \in FPFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in FPFS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in FPFS_{E_1\times E_2}[U]$ such that $p = n_2(j-1) + k$. For all i and p, if $c_{ip} := \min\{a_{ij}, b_{ik}\}$, then $[c_{ip}]$ is called and-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \wedge [b_{ik}]$.

Definition 2.6. Let $[a_{ij}]_{m\times n_1} \in FPFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in FPFS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in FPFS_{E_1\times E_2}[U]$ such that $p = n_2(j-1)+k$. For all i and p, if $c_{ip} := \frac{a_{ij}+b_{ik}}{2}$, then $[c_{ip}]$ is called mean-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \times_m [b_{ik}]$.

Definition 2.7. Let $[s_{i1}] \in M_{(m-1)\times 1}(\mathbb{R})$ such that $m \geq 2$. Then, normalisation $[\hat{s}_{i1}]$ of $[s_{i1}]$ is defined by

$$\hat{s}_{i1} := \begin{cases} \dfrac{s_{i1} - \min\limits_k s_{k1}}{\max\limits_k s_{k1} - \min\limits_k s_{k1}}, & \max\limits_k s_{k1} \neq \min\limits_k s_{k1} \\ 1, & \max\limits_k s_{k1} = \min\limits_k s_{k1} \end{cases}$$

To obtain an increasing sequence consisting of all the elements of an index set, being a subset of $\mathbb{N}^n$, we present a linear ordering relation over $\mathbb{N}^n$ as follows:

Definition 2.8. [30] Let $(j_1, j_2, \dots, j_n), (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$. Then, the relation "≤" is called a linear ordering relation and is defined by

$$(j_1, j_2, \dots, j_n) \leq (k_1, k_2, \dots, k_n) \Leftrightarrow [j_1 < k_1 \vee (j_1 = k_1 \wedge j_2 < k_2) \vee \dots \vee (j_1 = k_1 \wedge j_2 = k_2 \wedge \dots \wedge j_{n-1} = k_{n-1} \wedge j_n \leq k_n)]$$

3. Configurations of Soft Decision-Making Methods

In this section, we configure the SDM methods constructed by soft sets [41-47], fuzzy soft sets [41,46,48-63], fuzzy parameterized soft sets [64,65], *fpfs*-sets [66,67], soft matrices [47,68], and fuzzy soft matrices [43,69-72]. From now on, $I_n = \{1,2,\cdots,n\}$ and $I_n^* = \{0,1,2,\cdots,n\}$.

[69] has employed fuzzy soft matrices to determine an eligible candidate in the recruitment process just as [70] has utilised them to select an environment with healthy living conditions. We configure the proposed methods therein as follows:

Algorithm 3.1. BSD13 and SR15

BSD13, SR15, and NS11 [30] are the same. Therefore, we prefer the notation NS11.

In [41], the authors have suggested a new method based on spatial distance and fuzzy soft sets. Moreover, they have presented the method for two soft sets. We configure the proposed methods therein as follows:

Algorithm 3.2. CXL13($\boldsymbol{\lambda}$)

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Construct the parameters' optimum solution matrix $\lambda := [\lambda_{1j}]_{1\times n}$ such that $0 \leq \lambda_{1j} \leq 1$, for all $j \in I_n$

Step 3. Obtain $[b_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} := \sqrt{\sum_{j=1}^{n} \left(\lambda_{1j} - a_{0j}a_{ij}\right)^2}, \quad i \in I_{m-1}$$

Step 4. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \max\limits_k b_{k1} - b_{i1}$ such that $i \in I_{m-1}$

Step 5. Obtain the decision set $\{^{\hat{s}_{k1}}u_k | u_k \in U\}$

Algorithm 3.3. CXL13/2($\boldsymbol{\lambda}$)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{ij}]_{m\times n}$ defined by $c_{ij} := \frac{a_{ij}+b_{ij}}{2}$ such that $i \in I_{m-1}^*$ and $j \in I_n$

Step 3. Apply CXL13 to $\left[c_{ij}\right]$

[48] has studied the selection of a suitable house with fuzzy soft sets. We configure the proposed method therein as follows:

Algorithm 3.4. GLF13($\boldsymbol{R}$)

Step 1. Construct *fpfs*-matrices $\left[a_{ij}^1\right]_{m\times n}, \left[a_{ij}^2\right]_{m\times n}, \cdots, \left[a_{ij}^t\right]_{m\times n}$

Step 2. Determine a set R of indices such that $R \subseteq I_n$. Moreover, let (r_k) denote the increasing sequence of the elements of R.

Step 3. Obtain $K_r = \{v \in I_t: \exists i \ni a_{0r}^v a_{ir}^v \neq 0\}$, for all $r \in R$. For $r \in R$, if $K_r = \emptyset$, then K_r is chosen as $\{0\}$. Furthermore, let (u_k^r) stand for the increasing sequence of the elements of K_r, for all $r \in R$.

Step 4. Obtain $[b_{ik}^z]_{m\times|K_r|}$ defined by

$$b_{ik}^z := \begin{cases} a_{ir_z}^{u_k^{r_z}}, & \forall z \in I_{|R|}, u_k^{r_z} \neq 0 \\ 1, & otherwise \end{cases}$$

such that $i \in I_{m-1}^*$, $z \in I_{|R|}$, and $k \in I_{|K_r|}$

Here, $|R|$ and $|K_r|$ denote the cardinality of R and K_r, respectively.

Step 5. Obtain $\left[c_{ij}\right]_{m\times|R|}$ defined by

$$c_{ij} := \min_k \{b_{ik}^j\}$$

such that $i \in I_{m-1}^*$and $j \in I_{|R|}$

Step 6. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \min_{j\in I_{|R|}} \{c_{0j} c_{ij}\}, \quad i \in I_{m-1}$$

Step 7. Obtain the decision set $\left\{{}^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

In [42], the authors have developed a pruning method using soft sets. We configure it as follows:

Algorithm 3.5. HG13

Step 1. Construct two *fpfs*-matrices $\left[a_{ij}\right]_{m\times n}$ and $\left[b_{ij}\right]_{m\times n}$

Step 2. Obtain $[c_{i1}]_{(m-1)\times 1}$, $[d_{i1}]_{(m-1)\times 1}$, $[e_{i1}]_{(m-1)\times 1}$, and $V = \left\{u_i : \; e_{i1} = \max_{k\in I_{m-1}} e_{k1}\right\}$ such that

$$c_{i1} := \sum_{j=1}^{n} a_{0j} a_{ij}, \quad d_{i1} := \sum_{j=1}^{n} b_{0j} b_{ij}, \quad \text{and} \quad e_{i1} := c_{i1} + d_{i1}, \quad i \in I_{m-1}$$

Step 3. For all $u_i \in V$, obtain $\bar{u}_i := \left\{u_j \in V : \; (c_{i1}, d_{i1}) = \left(c_{j1}, d_{j1}\right) \vee (c_{i1}, d_{i1}) = \left(d_{j1}, c_{j1}\right)\right\}$

Step 4. Obtain $W = \left\{ u_i \in V : \; |\bar{u}_i| = \min_{u_k \in V} |\bar{u}_k|\right\}$

Here, $|\bar{u}_i|$ denotes the cardinality of $\bar{u}_i$.

Step 5. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \begin{cases} \frac{1+e_{k1}}{1+\sum_j (a_{0j}+b_{0j})}, & u_k \in W \\ \frac{e_{k1}}{\sum_j (a_{0j}+b_{0j})}, & u_k \in U - W \end{cases}, \quad i \in I_{m-1}$

Step 6. Obtain the decision set $\left\{{}^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

[49] has proposed a method based on group decision-making and applied it to a company's staff selection problem. We configure the proposed method therein as follows:

Algorithm 3.6. SM13($\boldsymbol{w}, \boldsymbol{\alpha}$)

Step 1. Construct *fpfs*-matrices $[a_{ij}^1]_{m\times n}, [a_{ij}^2]_{m\times n}, \cdots, [a_{ij}^t]_{m\times n}$ such that $a_{0j}^1 = a_{0j}^2 = \cdots = a_{0j}^t = a_{0j}$

Step 2. Construct $w := [w_{1k}]_{1\times t}$ such that $0 \le w_{1k} \le 1$ and $\sum_{k=1}^t w_{1k} = 1$, for $k \in I_t$

Step 3. Obtain $[b_{1j}]_{1\times n}$ defined by

$$b_{1j} := \begin{cases} \dfrac{a_{0j}}{\sum_{k=1}^n a_{0k}}, & \displaystyle\sum_{k=1}^n a_{0k} \neq 0 \\ \dfrac{1}{n}, & otherwise \end{cases}, \quad j \in I_n$$

Step 4. Obtain $[c_{kr}]_{t\times t}$ defined by

$$c_{kr} := \sum_{j=1}^n b_{1j}\, z_{kr}^j, \quad k, r \in I_t$$

such that

$$z_{kr}^j := \begin{cases} \dfrac{\sum_{i=1}^{m-1} \min\{a_{ij}^k, a_{ij}^r\}}{\sum_{i=1}^{m-1} \max\{a_{ij}^k, a_{ij}^r\}}, & \displaystyle\sum_{i=1}^{m-1} \max\{a_{ij}^k, a_{ij}^r\} \neq 0 \\ 1, & otherwise \end{cases}$$

Step 5. Obtain $[d_{1k}]_{1\times t}$ defined by

$$d_{1k} := \frac{1}{t-1} \sum_{r=1, r\neq k}^{t} c_{kr}, \quad k \in I_t$$

Step 6. Obtain $[e_{1k}]_{1\times t}$ defined by

$$e_{1k} := \begin{cases} \dfrac{d_{1k}}{\sum_{l=1}^t d_{1l}}, & \displaystyle\sum_{l=1}^t d_{1l} \neq 0 \\ \dfrac{1}{t}, & otherwise \end{cases}, \quad k \in I_t$$

Step 7. For $\alpha \in [0,1]$, obtain $[\lambda_{1k}]_{1\times t}$ defined by

$$\lambda_{1k} := \alpha w_{1k} + (1-\alpha) e_{1k}, \quad k \in I_t$$

Step 8. Obtain $[f_{ij}]_{(m-1)\times n}$ defined by

$$f_{ij} := \sum_{k=1}^t \lambda_{1k} a_{ij}^k$$

such that $i \in I_{m-1}$, $j \in I_n$, and $k \in I_t$

Step 9. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := 1 - \sqrt{\sum_{j=1}^n b_{1j} \left(f_{ij} - \max_{k\in I_{m-1}} f_{kj}\right)^2}, \quad i \in I_{m-1}$$

Step 10. Obtain the decision set $\{{}^{\hat{s}_{k1}}u_k | u_k \in U\}$

[43] has proposed two SDM methods via soft sets and fuzzy soft matrices. Moreover, [67] has suggested an SDM method constructed with *fpfs*-sets. We configure the proposed methods therein as follows:

Algorithm 3.7. GDC14 and RH16/2

GDC14, RH16/2, and MRB02 [27] are the same. Therefore, we prefer the notation MRB02.

Algorithm 3.8. GDC14/2(λ)

GDC14/2 and NKY17(λ) [29] are the same. Therefore, we prefer the notation NKY17(λ).

In [44], the author has utilised soft sets to determine an optimal alternative. We configure the proposed method therein as follows:

Algorithm 3.9. K14

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{ij}]_{(m-1)\times n}$ defined by $c_{ij} := \min\{a_{0j}a_{ij}, b_{0j}b_{ij}\}$ such that $i \in I_m$ and $j \in I_n$

Step 3. Obtain $V = \{u_i \in U : \sum_{j=1}^{n} c_{ij} \neq 0\}$

Step 4. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \begin{cases} \left(\sum_{j=1}^{n} a_{0j}a_{ij}\right)\left(\sum_{j=1}^{n} b_{0j}b_{ij}\right) - \sum_{j=1}^{n} a_{0j}a_{ij} - \sum_{j=1}^{n} b_{0j}b_{ij}, & u_i \in V \\ 0, & u_i \in U - V \end{cases}$$

such that $i \in I_{m-1}$

Step 5. Obtain the decision set $\{^{\hat{s}_{k1}}u_k | u_k \in U\}$

[64] has studied financial decision-making problems using fuzzy parameterized soft sets, although it is stated that fuzzy soft sets are used. We configure the proposed method as follows:

Algorithm 3.10. MM14

MM14 and CCE10 [27] are the same. Therefore, we prefer the notation CCE10.

In [50], the authors have proposed an algorithm using fuzzy soft sets to determine the optimal decision program. We configure the proposed method therein as follows:

Algorithm 3.11. WQ14(κ)

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ik}]_{(m-1)\times(m-1)}$ defined by

$$b_{ik} := \frac{1}{n}\sum_{j=1}^{n}\left(1 - a_{0j}|a_{ij} - a_{kj}|\right), \quad i,k \in I_{m-1}$$

Step 3. Obtain $[c_{ik}]_{(m-1)\times(m-1)}$ defined by

$$c_{ik} := \max_{j}\left\{\min\{b_{ij}, b_{jk}\}\right\}, \quad i,j,k \in I_{m-1}$$

Step 4. Obtain the set D of all the entries of $[c_{ik}]$

Step 5. Obtain the descending-sorted matrix $[e_{1j}]_{1\times|D|}$ of the D's elements such that $j \in I_{|D|}$

Step 6. Obtain $[f_{ik}^{j}]_{(m-1)\times(m-1)}$ defined by

$$f_{ik}^{j} := \begin{cases} 1, & c_{ik} \geq e_{1j} \\ 0, & c_{ik} < e_{1j} \end{cases}$$

such that $i,k \in I_{m-1}$ and $j \in I_{|D|}$

Step 7. Obtain $[g_{1j}]_{1\times|D|}$ defined by

$$g_{1j} := \sum_{k=1}^{m-1} \chi(j,k)\,, \quad j \in I_{|D|}$$

such that

$$\chi(j,k) := \begin{cases} 1, & 1 < \sum_{i=1}^{m-1} f_{ik}^{j} < m-1 \\ 0, & otherwise \end{cases}$$

Step 8. Obtain $[h_{1j}]_{1\times(|D|-1)}$ defined by

$$h_{1j} := \begin{cases} \dfrac{e_{1j} - e_{1(j+1)}}{g_{1(j+1)} - g_{1j}}, & g_{1(j+1)} > g_{1j} \\ 0, & g_{1(j+1)} \leq g_{1j} \end{cases}, \quad j \in I_{(|D|-1)}$$

Step 9. Obtain $\lambda := e_{1p}$ such that $p := 1 + \underset{j}{\operatorname{argmax}}\, h_{1j}$

Here, $\underset{j}{\operatorname{argmax}}\, h_{1j}$ is an index of the h_{1j} being maximum for all $j \in I_{(|D|-1)}$.

Step 10. For all $u_i \in U$, obtain $\bar{u}_i := \{u_s \in U : \forall k \in I_n, j \in I_{|D|}(f_{ik}^{j} = f_{sk}^{j})\}$

Step 11. Obtain the clustering set $C = \{\bar{u}_i : u_i \in U\}$

Step 12. For all $r \in I_n$, obtain $[a_{ij}^{r}]_{m\times(n-1)}$ deleting r^{th} column of $[a_{ij}]$

Step 13. For all $r \in I_n$, obtain $[b_{ik}^{r}]_{(m-1)\times(m-1)}$ applying Step 3 to $[a_{ij}^{r}]$

Step 14. For all $r \in I_n$, obtain $[c_{ik}^{r}]_{(m-1)\times(m-1)}$ applying Step 4 to $[b_{ik}^{r}]$

Step 15. Obtain $[\tilde{f}_{ik}^{r}]_{(m-1)\times(m-1)}$ defined by

$$\tilde{f}_{ik}^{r} := \begin{cases} 1, & c_{ik}^{r} \geq \lambda \\ 0, & c_{ik}^{r} < \lambda \end{cases}$$

such that $i,k \in I_{m-1}$ and $r \in I_n$

Step 16. For all $u_i \in U$, obtain $\bar{u}_i^{r} := \{u_s \in U : \forall k \in I_n, (\tilde{f}_{ik}^{r} = \tilde{f}_{sk}^{r})\}$

Step 17. For all $r \in I_n$, obtain the clustering set $C^{r} = \{\bar{u}_i^{r} : u_i \in U\}$

Step 18. Obtain $[\sigma_{1j}]_{1\times n}$ defined by

$$\sigma_{1j} := 1 - \frac{|C \cap C^{j}|}{m-1}, \quad j \in I_n$$

Step 19. Obtain $[\beta_{1j}]_{1\times n}$ defined by

$$\beta_{1j} := \begin{cases} \dfrac{\sigma_{1j}}{\sum_{k=1}^{n} \sigma_{1k}}, & \displaystyle\sum_{k=1}^{n} \sigma_{1k} \neq 0 \\ \dfrac{1}{n}, & otherwise \end{cases}, \quad j \in I_n$$

Step 20. Obtain $[w_{1j}]_{1\times n}$ defined by

$$w_{1j} := \kappa a_{0j} + (1-\kappa)\beta_{1j}, \quad j \in I_n$$

Here, κ is Bias coefficient chosen by decision-maker and $\kappa \in [0,1]$

Step 21. Obtain $[\tilde{a}_{ij}]_{m\times n}$ defined by $\tilde{a}_{0j} := w_{1j}$ and $\tilde{a}_{ij} := a_{ij}$, for all $i \in I_{m-1}$ and $j \in I_n$

Step 22. Apply MRB02 [27] to $[\tilde{a}_{ij}]$

[51,55,56] have introduced the same methods for fuzzy soft sets by combining grey relational analysis with the Dempster-Shafer theory of evidence and applied them to medical diagnosis. We configure the proposed methods therein as follows:

Algorithm 3.12. XWL14(α, q), LWX15(α, q), and T15(α, q)

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} := \frac{1}{n}\sum_{j=1}^{n} a_{0j}a_{ij}, \quad i \in I_{m-1}$$

Step 3. Obtain $[c_{ij}]_{(m-1)\times n}$ defined by

$$c_{ij} := |a_{0j}a_{ij} - b_{i1}|$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 4. For $\alpha \in [0,1]$, obtain $[d_{ij}]_{(m-1)\times n}$ defined by

$$d_{ij} := \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}} c_{kj} + \alpha \max\limits_{k\in I_{m-1}} c_{kj}}{c_{ij} + \alpha \max\limits_{k\in I_{m-1}} c_{kj}}, & \max\limits_{k\in I_{m-1}} c_{kj} \neq 0 \\ 1, & otherwise \end{cases}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 5. For $q \in \mathbb{N}^+$, obtain $[e_{1j}]_{1\times n}$ defined by

$$e_{1j} := \frac{1}{m-1}\left(\sum_{i=1}^{m-1} (d_{ij})^q\right)^{\frac{1}{q}}, \quad j \in I_n$$

Step 6. Obtain $[f_{ij}]_{(m-1)\times n}$ defined by

$$f_{ij} := \begin{cases} \dfrac{a_{0j}a_{ij}}{\sum_{k=1}^{m-1} a_{0j}a_{kj}}, & \displaystyle\sum_{k=1}^{m-1} a_{0j}a_{kj} \neq 0 \\ \dfrac{1}{m-1}, & otherwise \end{cases}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 7. Obtain $[g_{ij}]_{(m-1)\times n}$ and $[h_{1j}]_{1\times n}$ defined by

$$g_{ij} := (1 - e_{1j})f_{ij}$$

and

$$h_{1j} := 1 - \sum_{i=1}^{m-1} g_{ij}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 8. Obtain $[Bel_{i1}^{n-1}]_{m\times 1}$ defined by

$$Bel_{i1}^{j} := \begin{cases} \dfrac{g_{i1}g_{i2} + g_{i1}h_{12} + h_{11}g_{i2}}{\left|1 - \sum_{k=1}^{m-1}\sum_{l=1,k\neq l}^{m-1} g_{k1}g_{l2}\right|}, & i \in I_{m-1} \text{ and } j = 1 \\ \dfrac{h_{11}h_{12}}{\left|1 - \sum_{k=1}^{m-1}\sum_{l=1,k\neq l}^{m-1} g_{k1}g_{l2}\right|}, & i = m \text{ and } j = 1 \\ \dfrac{Bel_{i1}^{j-1}g_{i(j+1)} + Bel_{i1}^{j-1}h_{1(j+1)} + Bel_{m1}^{j-1}g_{i(j+1)}}{\left|1 - \sum_{k=1}^{m-1}\sum_{l=1,k\neq l}^{m-1} Bel_{k1}^{j-1}g_{l(j+1)}\right|}, & i \in I_{m-1}, j \in I_{n-1}, \text{and } j \neq 1 \\ \dfrac{Bel_{m1}^{j-1}h_{1(j+1)}}{\left|1 - \sum_{k=1}^{m-1}\sum_{l=1,k\neq l}^{m-1} Bel_{k1}^{j-1}g_{l(j+1)}\right|}, & i = m, j \in I_{n-1}, \text{and } j \neq 1 \end{cases}$$

Step 9. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := Bel_{i1}^{n-1}$ such that $i \in I_{m-1}$

Step 10. Obtain the decision set $\{{}^{\hat{s}_{k1}}u_k | u_k \in U\}$

XWL14(α, q), LWX15(α, q), and T15(α, q) are the same. Therefore, we prefer the notation XWL14(α, q).

Algorithm 3.13. XWL14/2$(\boldsymbol{\alpha}, \boldsymbol{q})$, LWX15/2$(\boldsymbol{\alpha}, \boldsymbol{q})$, and T15/2$(\boldsymbol{\alpha}, \boldsymbol{q})$

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n_1}$ and $[b_{ik}]_{m\times n_2}$

Step 2. Find and-product *fpfs*-matrix $[c_{ip}]_{m\times n_1 n_2}$ of $[a_{ij}]$ and $[b_{ik}]$

Step 3. Apply XWL14 to $[c_{ip}]$

XWL14/2(α, q), LWX15/2(α, q), and T15/2(α, q) are the same. Therefore, we prefer the notation XWL14/2(α, q).

In [52], the scholars have suggested a new SDM method based on grey relational analysis and fuzzy soft sets. We configure the proposed method therein as follows:

Algorithm 3.14. YHX14$(\boldsymbol{\alpha}, \boldsymbol{\beta})$

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[\lambda_{1j}]_{1\times n}$ defined by

$$\lambda_{1j} := \frac{1}{m-1}\sum_{i=1}^{m-1} a_{ij}, \quad i \in I_n$$

Step 3. Obtain $[b_{ij}]_{(m-1)\times n}$ defined by

$$b_{ij} := \begin{cases} a_{0j}, & a_{ij} \geq \lambda_{1j} \\ 0, & a_{ij} < \lambda_{1j} \end{cases}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 4. Obtain $[c_{i1}]_{(m-1)\times 1}$ defined by

$$c_{i1} \coloneqq \sum_{j=1}^{n} b_{ij}\,, \quad i \in I_{m-1}$$

Step 5. Obtain $\left[d_{1j}\right]_{1\times n}$ defined by

$$d_{1j} \coloneqq \max_{i\in I_{m-1}} \left\{a_{0j}a_{ij}\right\}, \quad j \in I_n$$

Step 6. Obtain $[e_{i1}]_{(m-1)\times 1}$ defined by

$$e_{i1} \coloneqq -\frac{1}{n\ln(2)}\sum_{j=1}^{n}\left(a_{0j}a_{ij}\ln\left(\frac{\varepsilon + a_{0j}a_{ij}}{\varepsilon + \frac{1}{2}\left(a_{0j}a_{ij} + d_{1j}\right)}\right) + \left(1 - a_{0j}a_{ij}\right)\ln\left(\frac{1 + \varepsilon - a_{0j}a_{ij}}{1 + \varepsilon - \frac{1}{2}\left(a_{0j}a_{ij} + d_{1j}\right)}\right)\right.$$
$$\left. + d_{1j}\ln\left(\frac{\varepsilon + d_{1j}}{\varepsilon + \frac{1}{2}\left(d_{1j} + a_{0j}a_{ij}\right)}\right) + \left(1 - d_{1j}\right)\ln\left(\frac{1 + \varepsilon - d_{1j}}{1 + \varepsilon - \frac{1}{2}\left(d_{1j} + a_{0j}a_{ij}\right)}\right)\right), \quad i \in I_{m-1}$$

Here, if $a_{0j}a_{ij} = 0$, $d_{1j} = 0$, $a_{0j}a_{ij} = 1$, or $d_{1j} = 1$, then $\ln\left(\frac{a_{0j}a_{ij}}{\frac{1}{2}(a_{0j}a_{ij}+d_{1j})}\right)$, $\ln\left(\frac{d_{1j}}{\frac{1}{2}(d_{1j}+a_{0j}a_{ij})}\right)$, $\ln\left(\frac{1-a_{0j}a_{ij}}{1-\frac{1}{2}(a_{0j}a_{ij}+d_{1j})}\right)$, or $\ln\left(\frac{1-d_{1j}}{1-\frac{1}{2}(d_{1j}+a_{0j}a_{ij})}\right)$ are undefined, respectively. To cope with these drawbacks, we modify them as $\ln\left(\frac{\varepsilon+a_{0j}a_{ij}}{\varepsilon+\frac{1}{2}(a_{0j}a_{ij}+d_{1j})}\right)$, $\ln\left(\frac{\varepsilon+d_{1j}}{\varepsilon+\frac{1}{2}(d_{1j}+a_{0j}a_{ij})}\right)$, $\ln\left(\frac{1+\varepsilon-a_{0j}a_{ij}}{1+\varepsilon-\frac{1}{2}(a_{0j}a_{ij}+d_{1j})}\right)$, and $\ln\left(\frac{1+\varepsilon-d_{1j}}{1+\varepsilon-\frac{1}{2}(d_{1j}+a_{0j}a_{ij})}\right)$, respectively, such that $\varepsilon \ll 1$ is a positive constant, e.g., $\varepsilon = 0.0001$.

Step 7. For $\alpha \in [0,1]$, obtain $[f_{i1}]_{(m-1)\times 1}$ and $[g_{i1}]_{(m-1)\times 1}$ defined by

$$f_{i1} \coloneqq \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}}\{n - c_{k1}\} + \alpha \max\limits_{k\in I_{m-1}}\{n - c_{k1}\}}{n - c_{i1} + \alpha \max\limits_{k\in I_{m-1}}\{n - c_{k1}\}}, & \max\limits_{k\in I_{m-1}}\{n - c_{k1}\} \neq 0 \\ 1, & otherwise \end{cases}, \quad i \in I_{m-1}$$

and

$$g_{i1} \coloneqq \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}} e_{k1} + \alpha \max\limits_{k\in I_{m-1}} e_{k1}}{e_{i1} + \alpha \max\limits_{k\in I_{m-1}} e_{k1}}, & \max\limits_{k\in I_{m-1}} e_{k1} \neq 0 \\ 1, & otherwise \end{cases}, \quad i \in I_{m-1}$$

Step 8. For $\beta \in [0,1]$, obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} \coloneqq \beta f_{i1} + (1 - \beta)g_{i1}, \quad i \in I_{m-1}$$

Step 9. Obtain the decision set $\left\{^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

[45] has applied an SDM method constructed via soft sets to a problem related to a company's recruitment scenario. Moreover, the following method is a version constructed with t *fpfs*-matrices of Z14 provided in [29]. We configure the proposed method therein as follows:

Algorithm 3.15. Z14/2

Step 1. Construct *fpfs*-matrices $\left[a_{ij}^{1}\right]_{m\times n}, \left[a_{ij}^{2}\right]_{m\times n}, \cdots, \left[a_{ij}^{t}\right]_{m\times n}$

Step 2. Obtain $\left[b_{ij}\right]_{m\times n}$ defined by $b_{ij} \coloneqq \max\limits_{k\in I_t} a_{ij}^{k}$ such that $i \in I_{m-1}^{*}$ and $j \in I_n$

Step 3. Apply MRB02 [27] to $\left[b_{ij}\right]$

Z14 [29] is a special version of Z14/2.

In [53], the researcher has availed of the concept of fuzzy soft sets. We configure the proposed method therein as follows:

Algorithm 3.16. A15

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ik}]_{(m-1)\times(m-1)}$ defined by

$$b_{ik} := \begin{cases} \displaystyle\sum_{j=1}^{n} a_{0j}\chi(a_{ij}, a_{kj}), & i \neq k \\ 0, & i = k \end{cases}$$

such that

$$\chi(a_{ij}, a_{kj}) := \begin{cases} 1, & a_{ij} > a_{kj} \\ 0, & a_{ij} \leq a_{kj} \end{cases}, \quad i,k \in I_{m-1}$$

Step 3. Obtain $[c_{ik}]_{(m-1)\times(m-1)}$ defined by

$$c_{ik} := \begin{cases} b_{ik}, & i \neq k \\ n(m-2) - \displaystyle\sum_{l=1,}^{m-1} b_{lk}, & i = k \end{cases}, \quad i,k \in I_{m-1}$$

Step 4. Obtain sum of the eigenvectors $[s_{i1}]_{(m-1)\times 1}$ associated with the dominant eigenvalues $\lambda := n(m-2)$

Step 5. Obtain the decision set $\{^{\hat{s}_{k1}}u_k | u_k \in U\}$

[65] has modelled a car purchasing problem through fuzzy parameterized soft sets. We configure the proposed method therein as follows:

Algorithm 3.17. DC15($\boldsymbol{\alpha}$)

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain increasing sequence (r_t) consisting of all the elements of $R := \{j : a_{0j} \geq \alpha\}$ such that $\alpha \in [0,1]$. If $R = \emptyset$, then $\alpha := \frac{1}{n}\sum_{j=1}^{n} a_{0j}$.

Step 3. Obtain $[b_{ip}]_{m\times|R|^2}$ defined by $b_{ip} = \min\{a_{ir_j}, a_{ir_k}\}$ such that $p = n(j-1)+k$, $i \in I_{m-1}^*$, and $j,k \in I_{|R|}$. Here, $|R|$ denote the cardinality of R.

Step 4. Obtain the score matrix $[s_{i1}]_{m\times 1}$ defined by

$$s_{i1} := \frac{1}{|R|^2}\sum_{p=1}^{|R|^2} b_{0p}b_{ip}, \quad i \in I_{m-1}$$

Step 5. Obtain the decision set $\{^{\hat{s}_{k1}}u_k | u_k \in U\}$

[54,58] have proposed an SDM method based on fuzzy soft sets. We configure the proposed methods therein as follows:

Algorithm 3.18. HJ15($\boldsymbol{\lambda}$) and H16($\boldsymbol{\lambda}$)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. For $\lambda \in [0,1]$, obtain $[c_{ij}]_{m\times n}$ and $[d_{ij}]_{m\times n}$ defined by

$$c_{ij} \coloneqq \begin{cases} a_{ij}, & a_{ij} \geq \lambda \\ 0, & a_{ij} < \lambda \end{cases} \text{ and } d_{ij} \coloneqq \begin{cases} b_{ij}, & b_{ij} \geq \lambda \\ 0, & b_{ij} < \lambda \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

Step 3. Apply CE10a [27] to $[c_{ij}]$ and $[d_{ij}]$

In [71], the scholars have suggested two SDM methods based on grey relational analysis and fuzzy soft matrices. We configure the proposed methods therein as follows:

Algorithm 3.19. XHL15$(\boldsymbol{\alpha}, \boldsymbol{q})$

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{(m-1)\times n}$ defined by

$$b_{ij} \coloneqq (2a_{0j} - 1)(2a_{ij} - 1)$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 3. Obtain $[c_{i1}]_{(m-1)\times 1}$ defined by

$$c_{i1} \coloneqq \frac{1}{n}\sum_{j=1}^{n} b_{ij}\,, \quad i \in I_{m-1}$$

Step 4. Obtain $[d_{ij}]_{(m-1)\times n}$ defined by

$$d_{ij} \coloneqq |b_{ij} - c_{i1}|$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 5. For $\alpha \in [0,1]$, obtain $[e_{ij}]_{(m-1)\times n}$ defined by

$$e_{ij} \coloneqq \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}} d_{kj} + \alpha \max\limits_{k\in I_{m-1}} d_{kj}}{d_{ij} + \alpha \max\limits_{k\in I_{m-1}} d_{kj}}, & \max\limits_{k\in I_{m-1}} d_{kj} \neq 0 \\ 1, & otherwise \end{cases}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 6. For $q \in \mathbb{N}^+$, obtain $[f_{1j}]_{1\times n}$ defined by

$$f_{1j} \coloneqq \frac{1}{m-1}\left(\sum_{i=1}^{m-1} (e_{ij})^q\right)^{\frac{1}{q}}, \quad j \in I_n$$

Step 7. Obtain $[g_{1j}]_{1\times n}$ defined by

$$g_{1j} \coloneqq 1 - f_{1j}, \quad j \in I_n$$

Step 8. Obtain $[h_{ij}]_{(m-1)\times n}$ defined by

$$h_{ij} \coloneqq b_{ij} g_{1j}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 9. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$

$$s_{i1} \coloneqq \delta_i(1,2,\dots,n), \quad i \in I_{m-1}$$

such that

$$\delta_i(1,2,\dots,n) \coloneqq \frac{\delta_i(1,2,\dots,n-1)+h_{in}}{1+\delta_i(1,2,\dots,n-1)h_{in}}, \quad i \in I_{m-1} \text{ and } n \geq 2$$

and

$$\delta_i(1,2) \coloneqq \frac{h_{i1}+h_{i2}}{1+h_{i1}h_{i2}}, \quad i \in I_{m-1}$$

Step 10. Obtain the decision set $\{{}^{\hat{s}_{k1}}u_k | u_k \in U\}$

Algorithm 3.20. XHL15/2$(\boldsymbol{\alpha},\boldsymbol{q})$

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n_1}$ and $[b_{ik}]_{m\times n_2}$

Step 2. Find and-product *fpfs*-matrix $[c_{ip}]_{m\times n_1 n_2}$ of $[a_{ij}]$ and $[b_{ik}]$

Step 3. Apply XHL15(α,q) to $[c_{ip}]$

[66] has benefited *fpfs*-sets to fill an announced position in a company. We configure the proposed methods therein as follows:

Algorithm 3.21. ZXZ15$(\boldsymbol{\alpha})$

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{ij}]_{m\times n}$ defined by

$$c_{0j} \coloneqq \sqrt{\frac{(a_{0j})^2+(b_{0j})^2}{2}} \text{ and } c_{ij} \coloneqq \max\{a_{ij},b_{ij}\}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 3. Obtain $[d_{ij}]_{m\times n}$ defined by

$$d_{0j} \coloneqq c_{0j} \text{ and } d_{ij} \coloneqq \begin{cases} c_{ij}, & c_{ij} \geq \alpha \\ 0, & c_{ij} < \alpha \end{cases}$$

such that $\alpha \in [0,1]$, $i \in I_{m-1}$, and $j \in I_n$

Step 4. Apply CCE10 [27] to $[d_{ij}]$

Algorithm 3.22. ZXZ15/2$(\boldsymbol{\alpha})$

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{ij}]_{m\times n}$ defined by

$$c_{0j} \coloneqq \sqrt{\frac{(a_{0j})^2+(b_{0j})^2}{2}} \text{ and } c_{ij} \coloneqq \min\{a_{ij},b_{ij}\}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 3. Obtain $[d_{ij}]_{m\times n}$ defined by

$$d_{0j} \coloneqq c_{0j} \text{ and } d_{ij} \coloneqq \begin{cases} c_{ij}, & c_{ij} \geq \alpha \\ 0, & c_{ij} < \alpha \end{cases}$$

such that $\alpha \in [0,1]$, $i \in I_{m-1}$, and $j \in I_n$

Step 4. Apply CCE10 [27] to $[d_{ij}]$

In [46], the researchers have revised two SDM methods based on soft sets and fuzzy soft sets. We configure the proposed methods therein as follows:

Algorithm 3.23. ZZ15

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{m\times n}$defined by $b_{0j} := \begin{cases} \frac{a_{0j}}{\sum_{k=1}^{n} a_{0k}}, & \sum_{k=1}^{n} a_{0k} \neq 0 \\ \frac{1}{n}, & otherwise \end{cases}$ and $b_{ij} := a_{ij}$ such that $i \in I_{m-1}$ and $j \in I_n$

Step 3. Apply MRB02 [27] to $[b_{ij}]$

Algorithm 3.24. ZZ15/2(λ)

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Construct $\lambda := [\lambda_{1j}]_{1\times n}$ such that $0 \leq \lambda_{1j} \leq 1$ for $j \in I_n$

Step 3. Obtain $[b_{ij}]_{m\times n}$ defined by

$$b_{ij} := \begin{cases} a_{0j}, & i = 0 \\ 1, & i \neq 0 \text{ and } a_{ij} \geq \lambda_{1j} \\ 0, & i \neq 0 \text{ and } a_{ij} < \lambda_{1j} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

Step 4. Apply ZZ15 to $[b_{ij}]$

[57] has propounded a novel approach associated with fuzzy soft sets. We configure the proposed method therein as follows:

Algorithm 3.25. A16

Step 1. Construct *fpfs*-matrices $[a_{ij_1}^1]_{m\times n_1}, [a_{ij_2}^2]_{m\times n_2}, \cdots, [a_{ij_t}^t]_{m\times n_t}$

Step 2. Find and-product *fpfs*-matrix $[b_{ij}]_{m\times n}$ of $[a_{ij_1}^1], [a_{ij_2}^2], \cdots, [a_{ij_t}^t]$ such that $n = n_1 n_2 \dots n_t$

Step 3. Obtain $[c_{ik}]_{(m-1)\times(m-1)}$ defined by

$$c_{ik} := \sum_{j=1}^{n} b_{0j}\chi(b_{ij}, b_{kj}), \quad i,k \in I_{m-1}$$

such that

$$\chi(b_{ij}, b_{kj}) := \begin{cases} \frac{b_{ij} - b_{kj}}{\max_{t\in I_{m-1}} \{b_{tj}\}}, & b_{ij} > b_{kj} \\ 0, & b_{ij} \leq b_{kj} \end{cases}$$

Step 4. Apply Step 3-6 of MBR01 [27] to $[c_{ik}]$

[47] has assessed the eligibility of a group of students for a scholarship using soft sets and soft matrices. Moreover, [67] has revised the SDM method in [3] via *fpfs*-sets. We configure the proposed method therein as follows:

Algorithm 3.26. AC16, AC16/2, RH16

AC16, AC16/2, RH16, and CEC11 [29] are the same. Therefore, we prefer the notation CEC11.

In [72], the researchers have studied the fuzzy soft matrices by using different t-norms. We configure the proposed methods therein as follows:

Algorithm 3.27. AM16

AM16 is the same as MR13 [29]. Therefore, we prefer notation MR13.

Algorithm 3.28. AM16/2

AM16/2 is the same as MR13/2 [29]. Therefore, we prefer notation MR13/2.

Algorithm 3.29. AM16/3

AM16/3 is the same as MR13/3 [29]. Therefore, we prefer notation MR13/3.

In [59], the authors have applied the concept of fuzzy soft sets to a problem concerning selecting an investing area. We configure the proposed method therein as follows:

Algorithm 3.30. NRM16($\boldsymbol{R}$)

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Determine a set R of indices such that $R \subseteq I_n$.

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \prod_{j\in R} a_{0j}a_{ij}\,, \quad i \in I_{m-1}$$

Step 4. Obtain the decision set $\{{}^{\hat{s}_{k1}}u_k | u_k \in U\}$

[67] has used *fpfs*-sets to decide on a car purchase problem. We configure the proposed method therein as follows:

Algorithm 3.31. RH16/3($\boldsymbol{R}$)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n_1}$ and $[b_{ik}]_{m\times n_2}$

Step 2. Determine a set R of indices such that $R \subseteq I_{n_1} \times I_{n_2}$

Step 3. Obtain increasing sequence (r_t) consisting of all the elements of R such that $r_t := (u_t, v_t)$

Step 4. Obtain $[d_{it}]_{m\times|R|}$ defined by

$$d_{it} := \min_{r_t\in R}\{a_{iu_t}, b_{iv_t}\}, \quad i \in I^*_{m-1}$$

Here, $|R|$ denotes the cardinality of R.

Step 5. Obtain $[e_{it}]_{m\times|R|}$ defined by

$$e_{0t} := d_{0t} \text{ and } e_{it} := d_{0t}d_{it}$$

such that $i \in I_{m-1}$ and $t \in I_{|R|}$

Step 6. Apply MBR01 [27] to $[e_{it}]$

In [60], the researchers have employed fuzzy soft sets to choose practical and reliable social network sites. We configure the proposed method therein as follows:

Algorithm 3.32. RK16

RK16 is the same as MBR01 [27]. Therefore, we prefer the notation MBR01.

[61] has constructed an SDM method being a generalisation of MRB02 by using multiple fuzzy soft sets. We configure the proposed method therein as follows:

Algorithm 3.33. RS16

Step 1. Construct *fpfs*-matrices $\left[a_{ij}^1\right]_{m\times n}, \left[a_{ij}^2\right]_{m\times n}, \cdots, \left[a_{ij}^t\right]_{m\times n}$

Step 2. Obtain $\left[b_{i1}^1\right]_{(m-1)\times 1}, \left[b_{i1}^2\right]_{(m-1)\times 1}, \dots, \left[b_{i1}^t\right]_{(m-1)\times 1}$ defined by

$$b_{i1}^k \coloneqq \sum_{j=1}^{n} a_{0j}^k a_{ij}^k$$

such that $i \in I_{m-1}$ and $k \in I_t$

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \sum_{k=1}^{t} b_{i1}^k\,, \quad i \in I_{m-1}$$

Step 4. Obtain the decision set $\left\{{}^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

In [62], the authors have drawn on fuzzy soft sets to solve a group decision-making problem. We configure the proposed method therein as follows:

Algorithm 3.34. SMT16

Step 1. Construct *fpfs*-matrices $\left[a_{ij}^1\right]_{m\times n}, \left[a_{ij}^2\right]_{m\times n}, \dots, \left[a_{ij}^t\right]_{m\times n}$

Step 2. Obtain $\left[b_{ik}^1\right]_{(m-1)\times(m-1)}, \left[b_{ik}^2\right]_{(m-1)\times(m-1)}, \dots, \left[b_{ik}^t\right]_{(m-1)\times(m-1)}$ defined by

$$b_{ik}^r \coloneqq \sum_{j=1}^{n} a_{0j}^r \chi\left(a_{ij}^r, a_{kj}^r\right), \quad i,k \in I_{m-1}, r \in I_t$$

such that

$$\chi\left(a_{ij}^r, a_{kj}^r\right) \coloneqq \begin{cases} 1, & a_{ij}^r \geq a_{kj}^r \\ 0, & a_{ij}^r < a_{kj}^r \end{cases}$$

Step 3. Obtain $\left[c_{i1}^1\right]_{(m-1)\times 1}, \left[c_{i1}^2\right]_{(m-1)\times 1}, \dots, \left[c_{i1}^t\right]_{(m-1)\times 1}$ defined by

$$c_{i1}^r \coloneqq \sum_{k=1}^{m-1} b_{ik}^r, \quad i \in I_{m-1}$$

Step 4. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} \coloneqq \sum_{r=1}^{t} c_{i1}^r\,, \quad i \in I_{m-1}$$

Step 5. Obtain the decision set $\left\{{}^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

[68] has developed an SDM method based on mean-product and max-max decision-making via soft matrices. We configure the proposed method therein as follows:

Algorithm 3.35. VMH16

Step 1. Construct two *fpfs*-matrices $\left[a_{ij}\right]_{m\times n_1}$ and $[b_{ik}]_{m\times n_2}$

Step 2. Find the mean-product matrix $[c_{ip}]_{m\times n_1 n_2}$ of $[a_{ij}]$ and $[b_{ik}]$

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \max_{k\in I_{n_1}} \begin{cases} \max\limits_{p\in I_k}\left(c_{0p}c_{ip}\right), & I_k \neq \emptyset \\ 0, & I_k = \emptyset \end{cases}$$

such that $i \in I_{m-1}$ and $I_k := \{p \mid \exists i \ni c_{0p}c_{ip} \neq 0, (k-1)n_2 < p \leq kn_2\}$

Step 4. Obtain the decision set $\{^{\hat{s}_{k1}}u_k | u_k \in U\}$

[63] has propounded two novel methods based upon ambiguity measure and Dempster-Shafer theory of evidence in the fuzzy soft sets space. We configure the proposed methods therein as follows:

Algorithm 3.36. WHXDD16

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{(m-1)\times n}$ defined by

$$b_{ij} := \begin{cases} \dfrac{a_{0j}a_{ij}}{\sum_{k=1}^{m-1} a_{0j}a_{kj}}, & \sum\limits_{k=1}^{m-1} a_{0j}a_{kj} \neq 0 \\ \dfrac{1}{m-1}, & otherwise \end{cases}$$

such that $i \in I_{m-1}$ and $j \in I_n$.

Step 3. Obtain $[c_{1j}]_{1\times n}$ defined by

$$c_{1j} := -\sum_{i=1}^{m-1} b_{ij}\log_2\left(\varepsilon + b_{ij}\right), \quad j \in I_n$$

Here, if $b_{ij} = 0$, then $c_{1j} := -\sum_{i=1}^{m-1} b_{ij}\log_2 b_{ij}$ is undefined. To cope with this drawback, we modify it as $c_{1j} := -\sum_{i=1}^{m-1} b_{ij}\log_2\left(\varepsilon + b_{ij}\right)$ such that $\varepsilon \ll 1$ is a positive constant, e.g., $\varepsilon = 0.0001$.

Step 4. Obtain $[d_{1j}]_{1\times n}$ defined by

$$d_{1j} := \begin{cases} \dfrac{c_{1j}}{\sum_{k=1}^{n} c_{1k}}, & \sum\limits_{k=1}^{n} c_{1k} \neq 0 \\ \dfrac{1}{n}, & otherwise \end{cases}, \quad j \in I_n$$

Step 5. Obtain $[e_{ij}]_{(m-1)\times n}$ defined by

$$e_{ij} := b_{ij}\left(1 - d_{1j}\right)$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 6. Obtain $[f_{1j}]_{1\times n}$ defined by

$$f_{1j} := 1 - \sum_{i=1}^{m-1} e_{ij}, \quad j \in I_n$$

Step 7. Apply Step 7-10 of XWL14 to $[f_{1j}]$

Algorithm 3.37. WHXDD16/2

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n_1}$ and $[b_{ik}]_{m\times n_2}$

Step 2. Find and-product *fpfs*-matrix $[c_{ip}]_{m\times n_1 n_2}$ of $[a_{ij}]$ and $[b_{ik}]$

Step 3. Apply WHXDD16 to $[c_{ip}]$

4. Test Cases for the Comparison of the SDM Methods

This section proposes five test cases to compare decision-making performances of SDM methods. SDM methods employ single, double, or multiple *fpfs*-matrices. Therefore, each test case consists of t *fpfs*-matrices $[a_{ij}^1],[a_{ij}^2],\dots,[a_{ij}^t]$, which has order $m\times n$ and manifest the same ranking order of alternatives without employing SDM methods. If an SDM method employs a single *fpfs*-matrix, we only use $[a_{ij}^1]$. Similarly, if double, we use $[a_{ij}^1]$ and $[a_{ij}^2]$. If an SDM method produces the ranking order provided in a test case, then it is said to accomplish the test case. In this section, let $t=3$, $m=5$, $n=4$, $U=\{u_1,u_2,u_3,u_4\}$ be the set of alternatives, and $E=\{x_1,x_2,x_3,x_4\}$ be the set of parameters.

4.1. Test Case 1

Test Case 1 constructs three *fpfs*-matrices $[a_{ij}^1]_{5\times4}$, $[a_{ij}^2]_{5\times4}$, and $[a_{ij}^3]_{5\times4}$ such that for all $j\in I_4$ and $k\in I_3$, $a_{01}^k=a_{02}^k=a_{03}^k=a_{04}^k$ and $a_{1j}^k<a_{2j}^k<a_{3j}^k<a_{4j}^k$. Therefore, $a_{0j}^k a_{1j}^k<a_{0j}^k a_{2j}^k<a_{0j}^k a_{3j}^k<a_{0j}^k a_{4j}^k$, for all $j\in I_4$ and $k\in I_3$. For each *fpfs*-matrix herein, the ranking order of alternatives is $u_1\prec u_2\prec u_3\prec u_4$. For example,

$$[a_{ij}^1]:=\begin{bmatrix}0.4&0.4&0.4&0.4\\0.7&0.6&0.5&0.4\\0.8&0.7&0.6&0.5\\0.9&0.8&0.7&0.6\\1&0.9&0.8&0.7\end{bmatrix},\quad [a_{ij}^2]:=\begin{bmatrix}0.4&0.4&0.4&0.4\\0.3&0.2&0.1&0\\0.4&0.3&0.2&0.1\\0.5&0.4&0.3&0.2\\0.6&0.5&0.4&0.3\end{bmatrix},\quad \text{and}\quad [a_{ij}^3]:=\begin{bmatrix}0.4&0.4&0.4&0.4\\0.5&0.4&0.3&0.2\\0.6&0.5&0.4&0.3\\0.7&0.6&0.5&0.4\\0.8&0.7&0.6&0.5\end{bmatrix}$$

4.2. Test Case 2

Test Case 2 constructs three *fpfs*-matrices $[b_{ij}^1]_{5\times4}$, $[b_{ij}^2]_{5\times4}$, and $[b_{ij}^3]_{5\times4}$ such that for all $j\in I_4$ and $k\in I_3$, $b_{01}^k=b_{02}^k=b_{03}^k=b_{04}^k$ and $b_{4j}^k<b_{3j}^k<b_{2j}^k<b_{1j}^k$. Therefore, $b_{0j}^k b_{4j}^k<b_{0j}^k b_{3j}^k<b_{0j}^k b_{2j}^k<b_{0j}^k b_{1j}^k$, for all $j\in I_4$ and $k\in I_3$. For each *fpfs*-matrix herein, the ranking order of alternatives is $u_4\prec u_3\prec u_2\prec u_1$. For example,

$$[b_{ij}^1]:=\begin{bmatrix}0.7&0.7&0.7&0.7\\1&0.9&0.8&0.7\\0.9&0.8&0.7&0.6\\0.8&0.7&0.6&0.5\\0.7&0.6&0.5&0.4\end{bmatrix},\quad [b_{ij}^2]:=\begin{bmatrix}0.7&0.7&0.7&0.7\\0.6&0.5&0.4&0.3\\0.5&0.4&0.3&0.2\\0.4&0.3&0.2&0.1\\0.3&0.2&0.1&0\end{bmatrix},\quad \text{and}\quad [b_{ij}^3]:=\begin{bmatrix}0.7&0.7&0.7&0.7\\0.8&0.7&0.6&0.5\\0.7&0.6&0.5&0.4\\0.6&0.5&0.4&0.3\\0.5&0.4&0.3&0.2\end{bmatrix}$$

4.3. Test Case 3

Test Case 3 constructs three *fpfs*-matrices $[c_{ij}^1]_{5\times4}$, $[c_{ij}^2]_{5\times4}$, and $[c_{ij}^3]_{5\times4}$ such that for all $i,j\in I_4$ and $k\in I_3$, $c_{01}^k<c_{02}^k<c_{03}^k<c_{04}^k$, $c_{ii}^k=\lambda\in[0,1]$, and if $i\neq j$, then $c_{ij}^k=0$. Therefore, $c_{01}^k c_{11}^k<c_{02}^k c_{22}^k<c_{03}^k c_{33}^k<c_{04}^k c_{44}^k$ and if $i\neq j$, then $c_{0j}^k c_{ij}^k=0$, for all $j\in I_4$ and $k\in I_3$. For each *fpfs*-matrix herein, the ranking order of alternatives is $u_1\prec u_2\prec u_3\prec u_4$. For example,

$$[c_{ij}^1] := \begin{bmatrix} 0.6 & 0.7 & 0.8 & 0.9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [c_{ij}^2] := \begin{bmatrix} 0.4 & 0.5 & 0.6 & 0.7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad [c_{ij}^3] := \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.4. Test Case 4

Test Case 4 constructs three *fpfs*-matrices $[d_{ij}^1]_{5\times4}$, $[d_{ij}^2]_{5\times4}$, and $[d_{ij}^3]_{5\times4}$ such that for all $i,j \in I_4$ and $k \in I_3$, $d_{04}^k < d_{03}^k < d_{02}^k < d_{01}^k$, $d_{ii}^k = \lambda \in [0,1]$, and if $i \neq j$, then $d_{ij}^k = 0$. Therefore, $d_{04}^k d_{44}^k < d_{03}^k d_{33}^k < d_{03}^k d_{33}^k < d_{01}^k d_{11}^k$ and if $i \neq j$, then $d_{0j}^k d_{ij}^k = 0$, for all $j \in I_4$ and $k \in I_3$. For each *fpfs*-matrix herein, the ranking order of alternatives is $u_4 \prec u_3 \prec u_2 \prec u_1$. For example,

$$[d_{ij}^1] := \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [d_{ij}^2] := \begin{bmatrix} 0.7 & 0.6 & 0.5 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad [d_{ij}^3] := \begin{bmatrix} 0.5 & 0.4 & 0.3 & 0.2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.5. Test Case 5

Test Case 4 constructs three *fpfs*-matrices $[e_{ij}^1]_{5\times4}$, $[e_{ij}^2]_{5\times4}$, and $[e_{ij}^3]_{5\times4}$ such that for all $i,j \in I_4$ and $k \in I_3$, $e_{ij}^k = \lambda \in [0,1]$. For each *fpfs*-matrix herein, the ranking order of alternatives is $u_1 \approx u_2 \approx u_3 \approx u_4$. Here, $\approx$ denotes the same ranking order. For example,

$$[e_{ij}^1] := \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}, \quad [e_{ij}^2] := \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}, \quad \text{and} \quad [e_{ij}^3] := \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}$$

4.6. Results of Test Cases

In this subsection, we test the configured SDM methods using the aforesaid five test cases. Here, the methods working with a single matrix employ the first *fpfs*-matrices in each test case. Similarly, the methods working with double matrices utilise the first two *fpfs*-matrices. Moreover, the other methods use all the *fpfs*-matrices. For example, in Test Case 1, the methods employing single matrix, double matrices, and multiple matrices use the first *fpfs*-matrix $[a_{ij}^1]$, the first two *fpfs*-matrices $[a_{ij}^1]$ and $[a_{ij}^2]$, and all the *fpfs*-matrices $[a_{ij}^1]$, $[a_{ij}^2]$, and $[a_{ij}^3]$, respectively.

Table 8 indicates in which test cases the methods are successful. It can be seen from Table 8 that 20 of 37 methods, namely MBR01, MRB02, CCE10, CEC11, CXL13(λ_1), WQ14(κ), YHX14(α,β), DC15(α), ZZ15, CXL13/2(λ_1), HG13, ZXZ15(α), VMH16, MR13, MR13/2, SM13(w,α), Z14/2, RS16, SMT16, and NKY17(λ_2), pass all the tests. Moreover, the numbers of the passed tests are provided in the last column of Table 8. Here, $\alpha = 0.5$, $\beta = 0.5$, $\kappa = 0.4$, $\lambda = 0.5$, $\lambda_1 = [1\ 1\ 1\ 1]$, $\lambda_2 = [0.25\ 0.25\ 0.25\ 0.25]$, $\lambda_3 = [0.5\ 0.5\ 0.5\ 0.5]$, $q = 2$, $R = \{1, 2, 3, 4\}$, and $w = [0.34\ 0.34\ 0.34]$.

Table 8. Success of the methods in the test cases

	Algorithms\Test Cases	Test Case 1	Test Case 2	Test Case 3	Test Case 4	Test Case 5	Passed Test's Numbers
1	NS11 [BSD13, SR15]		✓	✓	✓	✓	4
2	CXL13(λ_1)	✓	✓	✓	✓	✓	5
3	CXL13/2(λ_1)	✓	✓	✓	✓	✓	5
4	GLF13(R)	✓	✓			✓	3
5	HG13	✓	✓	✓	✓	✓	5
6	SM13(w, α)	✓	✓	✓	✓	✓	5
7	MRB02 [GDC14, RH16/2]	✓	✓	✓	✓	✓	5
8	NKY17(λ_2) [GDC14/2]	✓	✓	✓	✓	✓	5
9	K14		✓			✓	2
10	CCE10 [MM14]	✓	✓	✓	✓	✓	5
11	WQ14(κ)	✓	✓	✓	✓	✓	5
12	XWL14(α, q) [LWX15(α, q), T15(α, q)]	✓	✓			✓	3
13	XWL14/2(α, q) [LWX15/2(α, q), T15/2(α, q)]	✓	✓			✓	3
14	YHX14(α, β)	✓	✓	✓	✓	✓	5
15	Z14/2	✓	✓	✓	✓	✓	5
16	A15			✓	✓	✓	3
17	DC15(α)	✓	✓	✓	✓	✓	5
18	HJ15(λ) [H16(λ)]		✓			✓	2
19	XHL15(α, q)		✓	✓	✓	✓	4
20	XHL15/2(α, q)		✓	✓	✓	✓	4
21	ZXZ15(α)	✓	✓	✓	✓	✓	5
22	ZXZ15/2(α)			✓	✓	✓	3
23	ZZ15	✓	✓	✓	✓	✓	5
24	ZZ15/2(λ_3)			✓	✓	✓	3
25	A16		✓		✓	✓	3
26	CEC11 [AC16, AC16/2, RH16]	✓	✓	✓	✓	✓	5
27	MR13 [AM16]	✓	✓	✓	✓	✓	5
28	MR13/2 [AM16/2]	✓	✓	✓	✓	✓	5
29	MR13/3 [AM16/3]					✓	1
30	NRM16(R)	✓	✓			✓	3
31	RH16/3(R)	✓	✓			✓	3
32	MBR01 [RK16]	✓	✓	✓	✓	✓	5
33	RS16	✓	✓	✓	✓	✓	5
34	SMT16	✓	✓	✓	✓	✓	5
35	VMH16	✓	✓	✓	✓	✓	5
36	WHXDD16					✓	1
37	WHXDD16/2	✓				✓	2
	Total	26	31	26	27	37	

5. An Application of Some of the Configured Methods to a PVA Problem

This section applies the configured methods herein to a PVA problem concerning the salt-and-pepper noise (SPN) removal performance of the filters provided in [73]. Therefore, firstly, we present the results of the filters in [73] produced by the quality metrics Peak Signal-to-Noise Ratio (PSNR), Structural Similarity (SSIM) [74], and Visual Information Fidelity (VIF) [75] for 20 traditional images at noise density occurring between 10% and 90% in Table 9, 10, and 11, respectively. Moreover, the bold values in the tables signify the filters with the best performance.

Table 9. Mean-PSNR results for the 20 traditional images with different noise densities

Filters/Noise Densities	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBAIN	37.52	34.29	31.96	29.83	27.86	25.89	23.90	21.55	18.55
MDBUTMF	36.80	32.18	29.02	28.48	28.81	28.34	26.95	23.42	15.29
BPDF	36.98	33.54	31.03	28.88	26.82	24.60	21.98	17.74	10.51
NAFSMF	36.08	33.27	31.49	30.15	29.02	27.96	26.82	25.47	22.34
AWMF	36.34	35.00	33.83	32.69	31.47	30.14	28.68	26.99	24.70
DAMF	39.58	36.33	34.14	32.45	30.99	29.64	28.28	26.69	24.35
ARmF	**40.04**	**37.12**	**35.14**	**33.53**	**31.99**	**30.45**	**28.86**	**27.08**	**24.74**

Table 10. Mean-SSIM results for the 20 traditional images with different noise densities

Filters/Noise Densities	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBAIN	0.9796	0.9584	0.9315	0.8968	0.8520	0.7949	0.7213	0.6265	0.4966
MDBUTMF	0.9774	0.9197	0.8117	0.7973	0.8399	0.8410	0.8025	0.7023	0.3566
BPDF	0.9783	0.9536	0.9229	0.8838	0.8323	0.7634	0.6680	0.5096	0.2585
NAFSMF	0.9748	0.9504	0.9248	0.8973	0.8666	0.8320	0.7910	0.7357	0.6190
AWMF	0.9728	0.9622	0.9484	0.9315	0.9098	0.8816	0.8437	0.7904	0.7028
DAMF	0.9854	0.9699	0.9516	0.9303	0.9051	0.8748	0.8368	0.7846	0.6964
ARmF	**0.9868**	**0.9735**	**0.9581**	**0.9400**	**0.9173**	**0.8880**	**0.8491**	**0.7947**	**0.7056**

Table 11. Mean-VIF results for the 20 traditional images with different noise densities

Filters/Noise Densities	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
DBAIN	0.8548	0.7319	0.6179	0.5119	0.4095	0.3128	0.2229	0.1365	0.0635
MDBUTMF	0.8272	0.6713	0.5044	0.4420	0.4310	0.3978	0.3302	0.2212	0.0730
BPDF	0.8188	0.6858	0.5659	0.4564	0.3529	0.2541	0.1614	0.0783	0.0334
NAFSMF	0.7902	0.6751	0.5828	0.5030	0.4307	0.3604	0.2897	0.2129	0.1226
AWMF	0.7896	0.7366	0.6789	0.6181	0.5533	0.4833	0.4066	0.3129	0.1928
DAMF	0.8787	0.7816	0.6943	0.6162	0.5437	0.4731	0.3998	0.3096	0.1913
ARmF	**0.8832**	**0.7975**	**0.7210**	**0.6474**	**0.5741**	**0.4974**	**0.4158**	**0.3182**	**0.1955**

In this PVA problem, the alternatives are indicated as $u_1 :=$ "DBAIN", $u_2 :=$ "MDBUTMF", $u_3 :=$ "BPDF", $u_4 :=$ "NAFSMF", $u_5 :=$ "AWMF", $u_6 :=$ "DAMF", and $u_7 :=$ "ARmF" such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$. Moreover, the parameters are denoted by $x_1 :=$ "SPN ratio 10%", $x_2 :=$ "SPN ratio 20%", $x_3 :=$ "SPN ratio 30%", $x_4 :=$ "SPN ratio 40%", $x_5 :=$ "SPN ratio 50%", $x_6 :=$ "SPN ratio 60%", $x_7 :=$ "SPN ratio 70%", $x_8 :=$ "SPN ratio 80%", and $x_9 :=$ "SPN ratio 90%" such that $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

Suppose that the noise removal performances of the filters at high noise densities are more significant than at the other densities. In such a case, it is anticipated that the membership degrees at high noise densities are greater than at the other noise densities. In other words, the first rows of the *fpfs*-matrices are considered to be [0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9] herein. Furthermore, while the SSIM and VIF values are in the interval [0,1], the PSNR values are not. Hence, the PSNR values are normalised via the maximum value provided in Table 9 to construct the *fpfs*-matrix $[a_{ij}]$. Thus, Table 9, 10, and 11 can be represented with *fpfs*-matrices $[a_{ij}]_{8\times 9}$, $[b_{ij}]_{8\times 9}$, and $[c_{ij}]_{8\times 9}$ as follows:

$$[a_{ij}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9371 & 0.8564 & 0.7982 & 0.7450 & 0.6958 & 0.6466 & 0.5969 & 0.5382 & 0.4633 \\ 0.9191 & 0.8037 & 0.7248 & 0.7113 & 0.7195 & 0.7078 & 0.6731 & 0.5849 & 0.3819 \\ 0.9236 & 0.8377 & 0.7750 & 0.7213 & 0.6698 & 0.6144 & 0.5490 & 0.4431 & 0.2625 \\ 0.9011 & 0.8309 & 0.7865 & 0.7530 & 0.7248 & 0.6983 & 0.6698 & 0.6361 & 0.5579 \\ 0.9076 & 0.8741 & 0.8449 & 0.8164 & 0.7860 & 0.7527 & 0.7163 & 0.6741 & 0.6169 \\ 0.9885 & 0.9073 & 0.8526 & 0.8104 & 0.7740 & 0.7403 & 0.7063 & 0.6666 & 0.6081 \\ 1.0000 & 0.9271 & 0.8776 & 0.8374 & 0.7990 & 0.7605 & 0.7208 & 0.6763 & 0.6179 \end{bmatrix}$$

$$[b_{ij}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9796 & 0.9584 & 0.9315 & 0.8968 & 0.8520 & 0.7949 & 0.7213 & 0.6265 & 0.4966 \\ 0.9774 & 0.9197 & 0.8117 & 0.7973 & 0.8399 & 0.8410 & 0.8025 & 0.7023 & 0.3566 \\ 0.9783 & 0.9536 & 0.9229 & 0.8838 & 0.8323 & 0.7634 & 0.6680 & 0.5096 & 0.2585 \\ 0.9748 & 0.9504 & 0.9248 & 0.8973 & 0.8666 & 0.8320 & 0.7910 & 0.7357 & 0.6190 \\ 0.9728 & 0.9622 & 0.9484 & 0.9315 & 0.9098 & 0.8816 & 0.8437 & 0.7904 & 0.7028 \\ 0.9854 & 0.9699 & 0.9516 & 0.9303 & 0.9051 & 0.8748 & 0.8368 & 0.7846 & 0.6964 \\ 0.9868 & 0.9735 & 0.9581 & 0.9400 & 0.9173 & 0.8880 & 0.8491 & 0.7947 & 0.7056 \end{bmatrix}$$

and

$$[c_{ij}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.8548 & 0.7319 & 0.6179 & 0.5119 & 0.4095 & 0.3128 & 0.2229 & 0.1365 & 0.0635 \\ 0.8272 & 0.6713 & 0.5044 & 0.4420 & 0.4310 & 0.3978 & 0.3302 & 0.2212 & 0.0730 \\ 0.8188 & 0.6858 & 0.5659 & 0.4564 & 0.3529 & 0.2541 & 0.1614 & 0.0783 & 0.0334 \\ 0.7902 & 0.6751 & 0.5828 & 0.5030 & 0.4307 & 0.3604 & 0.2897 & 0.2129 & 0.1226 \\ 0.7896 & 0.7366 & 0.6789 & 0.6181 & 0.5533 & 0.4833 & 0.4066 & 0.3129 & 0.1928 \\ 0.8787 & 0.7816 & 0.6943 & 0.6162 & 0.5437 & 0.4731 & 0.3998 & 0.3096 & 0.1913 \\ 0.8832 & 0.7975 & 0.7210 & 0.6474 & 0.5741 & 0.4974 & 0.4158 & 0.3182 & 0.1955 \end{bmatrix}$$

Nine of the SDM methods having passed all the test cases, namely MBR01, MRB02, CCE10, CEC11, CXL13(λ_1), WQ14(κ), YHX14(α,β), DC15(α), and ZZ15, employ only an *fpfs*-matrix. Similarly, CXL13/2(λ_1), HG13, ZXZ15(α), and VMH16 utilise two *fpfs*-matrices, and the others work with multiple *fpfs*-matrices. Moreover, we consider the variables $\alpha = 0.5$, $\beta = 0.5$, $\kappa = 0.4$, $\lambda_1 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$, $\lambda_2 = \left[\frac{1}{7}\ \frac{1}{7}\ \frac{1}{7}\ \frac{1}{7}\ \frac{1}{7}\ \frac{1}{7}\ \frac{1}{7}\right]^T$, and $w = [1\ 1\ 1]$.

Secondly, we apply the SDM methods to the aforesaid *fpfs*-matrices $[a_{ij}]_{8\times 9}$, $[b_{ij}]_{8\times 9}$, and $[c_{ij}]_{8\times 9}$. The decision sets and ranking orders produced by these SDM methods are manifested in Table 12 and 13, respectively. The last column in Table 13 demonstrates the number of the methods producing the same ranking order.

Table 12. Decision sets produced by SDM methods (in the event of more-importance-attached noise removal performance at high noise densities)

Algorithms	**Matrices**	**Decision Sets**
MBR01	$[a_{ij}]$	$\{^{0.2227}$DBAIN, $^{0.2422}$MDBUTMF, ^{0}BPDF, $^{0.3828}$NAFSMF, $^{0.7891}$AWMF, $^{0.6719}$DAMF, ^{1}ARmF$\}$
MRB02	$[a_{ij}]$	$\{^{0.8391}$DBAIN, $^{0.8446}$MDBUTMF, $^{0.7361}$BPDF, $^{0.9143}$NAFSMF, $^{0.9835}$AWMF, $^{0.9776}$DAMF, ^{1}ARmF$\}$
CCE10	$[a_{ij}]$	$\{^{0.8391}$DBAIN, $^{0.8446}$MDBUTMF, $^{0.7361}$BPDF, $^{0.9143}$NAFSMF, $^{0.9835}$AWMF, $^{0.9776}$DAMF, ^{1}ARmF$\}$
CEC11	$[a_{ij}]$	$\{^{0.8450}$DBAIN, $^{0.8523}$MDBUTMF, $^{0.7523}$BPDF, $^{0.9137}$NAFSMF, $^{0.9818}$AWMF, $^{0.9772}$DAMF, ^{1}ARmF$\}$
CXL13(λ_1)	$[a_{ij}]$	$\{^{0.4077}$DBAIN, $^{0.4027}$MDBUTMF, ^{0}BPDF, $^{0.6642}$NAFSMF, $^{0.9219}$AWMF, $^{0.9125}$DAMF, ^{1}ARmF$\}$
WQ14(κ)	$[a_{ij}]$	$\{^{0.8483}$DBAIN, $^{0.8480}$MDBUTMF, $^{0.7550}$BPDF, $^{0.9124}$NAFSMF, $^{0.9788}$AWMF, $^{0.9776}$DAMF, ^{1}ARmF$\}$
YHX14(α,β)	$[a_{ij}]$	$\{^{0.2630}$DBAIN, $^{0.2779}$MDBUTMF, ^{0}BPDF, $^{0.6484}$NAFSMF, $^{0.9819}$AWMF, $^{0.9750}$DAMF, ^{1}ARmF$\}$
DC15(α)	$[a_{ij}]$	$\{^{0.4628}$DBAIN, $^{0.4727}$MDBUTMF, ^{0}BPDF, $^{0.7927}$NAFSMF, $^{0.9880}$AWMF, $^{0.9526}$DAMF, ^{1}ARmF$\}$
ZZ15	$[a_{ij}]$	$\{^{0.8391}$DBAIN, $^{0.8446}$MDBUTMF, $^{0.7361}$BPDF, $^{0.9143}$NAFSMF, $^{0.9835}$AWMF, $^{0.9776}$DAMF, ^{1}ARmF$\}$
CXL13/2(λ_1)	$[a_{ij}]$, $[b_{ij}]$	$\{^{0.4358}$DBAIN, $^{0.3630}$MDBUTMF, ^{0}BPDF, $^{0.6997}$NAFSMF, $^{0.9437}$AWMF, $^{0.9327}$DAMF, ^{1}ARmF$\}$
HG13	$[a_{ij}]$, $[b_{ij}]$	$\{^{0.3674}$DBAIN, $^{0.3409}$MDBUTMF, ^{0}BPDF, $^{0.6317}$NAFSMF, $^{0.8651}$AWMF, $^{0.8459}$DAMF, ^{1}ARmF$\}$
ZXZ15(α)	$[a_{ij}]$, $[b_{ij}]$	$\{^{0.7437}$DBAIN, $^{0.7582}$MDBUTMF, $^{0.6995}$BPDF, $^{0.9321}$NAFSMF, $^{0.9925}$AWMF, $^{0.9878}$DAMF, ^{1}ARmF$\}$
VMH16	$[a_{ij}]$, $[b_{ij}]$	$\{^{0.2722}$DBAIN, $^{0.5393}$MDBUTMF, ^{0}BPDF, $^{0.7090}$NAFSMF, $^{0.9870}$AWMF, $^{0.9513}$DAMF, ^{1}ARmF$\}$
MR13	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.6585}$DBAIN, $^{0.7256}$MDBUTMF, $^{0.5448}$BPDF, $^{0.7414}$NAFSMF, $^{0.9616}$AWMF, $^{0.9627}$DAMF, ^{1}ARmF$\}$
MR13/2	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.4119}$DBAIN, $^{0.5444}$MDBUTMF, $^{0.2777}$BPDF, $^{0.6026}$NAFSMF, $^{0.9569}$AWMF, $^{0.9290}$DAMF, ^{1}ARmF$\}$
SM13(w,α)	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.3799}$DBAIN, $^{0.3521}$MDBUTMF, ^{0}BPDF, $^{0.6412}$NAFSMF, $^{0.9208}$AWMF, $^{0.9287}$DAMF, ^{1}ARmF$\}$
Z14/2	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.8608}$DBAIN, $^{0.8483}$MDBUTMF, $^{0.7614}$BPDF, $^{0.9321}$NAFSMF, $^{0.9925}$AWMF, $^{0.9878}$DAMF, ^{1}ARmF$\}$
RS16	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.3523}$DBAIN, $^{0.3832}$MDBUTMF, ^{0}BPDF, $^{0.6067}$NAFSMF, $^{0.9411}$AWMF, $^{0.9278}$DAMF, ^{1}ARmF$\}$
SMT16	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.2237}$DBAIN, $^{0.2666}$MDBUTMF, ^{0}BPDF, $^{0.3641}$NAFSMF, $^{0.7867}$AWMF, $^{0.6723}$DAMF, ^{1}ARmF$\}$
NKY17(λ_2)	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	$\{^{0.2412}$DBAIN, $^{0.2490}$MDBUTMF, ^{0}BPDF, $^{0.3696}$NAFSMF, $^{0.7899}$AWMF, $^{0.6732}$DAMF, ^{1}ARmF$\}$

The ranking orders in Table 13 manifest that MBR01, MRB02, CCE10, CEC11, YHX14(α,β), DC15(α), ZZ15, ZXZ15(α), VMH16, MR13/2, RS16, SMT16, and NKY17(λ_2) produce the same ranking order just as CXL13(λ_1), WQ14(κ), CXL13/2(λ_1), HG13, and Z14/2 do. Moreover, the ranking orders produced by MR13 and SM13(w,α) except for those of MDBUTMF and DBAIN tend to generate the same pattern. The results show that the decision-making abilities of SDM methods herein agree that ARmF outperforms the other filters and BPDF exhibits the minimum performance compared to the others.

Table 13. Ranking orders produced by SDM methods (in the event of more-importance-attached noise removal performance at high noise densities)

Algorithms	**Matrices**	**Ranking Orders**	**Frequency**
MBR01	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
MRB02	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
CCE10	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
CEC11	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
CXL13(λ_1)	$[a_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺DAMF≺AWMF≺ARmF	5
WQ14(κ)	$[a_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺DAMF≺AWMF≺ARmF	5
YHX14(α,β)	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
DC15(α)	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
ZZ15	$[a_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
CXL13/2(λ_1)	$[a_{ij}]$, $[b_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺DAMF≺AWMF≺ARmF	5
HG13	$[a_{ij}]$, $[b_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺DAMF≺AWMF≺ARmF	5
ZXZ15(α)	$[a_{ij}]$, $[b_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
VMH16	$[a_{ij}]$, $[b_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
MR13	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺AWMF≺DAMF≺ARmF	1
MR13/2	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
SM13(w,α)	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	1
Z14/2	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺DAMF≺AWMF≺ARmF	5
RS16	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
SMT16	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13
NKY17(λ_2)	$[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$	BPDF≺ DBAIN ≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	13

On the other hand, assume that the noise removal performances of the filters at low noise densities are more significant than at the higher densities. In such a case, it is anticipated that the membership degrees at low noise densities are greater than at the higher noise densities. In other words, the first rows of the *fpfs*-matrices are considered to be $[0.9\ 0.8\ 0.7\ 0.6\ 0.5\ 0.4\ 0.3\ 0.2\ 0.1]$ herein. Therefore, Table 9, 10, and 11 can be represented with *fpfs*-matrices $[d_{ij}]_{8\times 9}$, $[e_{ij}]_{8\times 9}$, and $[f_{ij}]_{8\times 9}$ as follows:

$$[d_{ij}] := \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.9371 & 0.8564 & 0.7982 & 0.7450 & 0.6958 & 0.6466 & 0.5969 & 0.5382 & 0.4633 \\ 0.9191 & 0.8037 & 0.7248 & 0.7113 & 0.7195 & 0.7078 & 0.6731 & 0.5849 & 0.3819 \\ 0.9236 & 0.8377 & 0.7750 & 0.7213 & 0.6698 & 0.6144 & 0.5490 & 0.4431 & 0.2625 \\ 0.9011 & 0.8309 & 0.7865 & 0.7530 & 0.7248 & 0.6983 & 0.6698 & 0.6361 & 0.5579 \\ 0.9076 & 0.8741 & 0.8449 & 0.8164 & 0.7860 & 0.7527 & 0.7163 & 0.6741 & 0.6169 \\ 0.9885 & 0.9073 & 0.8526 & 0.8104 & 0.7740 & 0.7403 & 0.7063 & 0.6666 & 0.6081 \\ 1.0000 & 0.9271 & 0.8776 & 0.8374 & 0.7990 & 0.7605 & 0.7208 & 0.6763 & 0.6179 \end{bmatrix}$$

$$[e_{ij}] := \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.9796 & 0.9584 & 0.9315 & 0.8968 & 0.8520 & 0.7949 & 0.7213 & 0.6265 & 0.4966 \\ 0.9774 & 0.9197 & 0.8117 & 0.7973 & 0.8399 & 0.8410 & 0.8025 & 0.7023 & 0.3566 \\ 0.9783 & 0.9536 & 0.9229 & 0.8838 & 0.8323 & 0.7634 & 0.6680 & 0.5096 & 0.2585 \\ 0.9748 & 0.9504 & 0.9248 & 0.8973 & 0.8666 & 0.8320 & 0.7910 & 0.7357 & 0.6190 \\ 0.9728 & 0.9622 & 0.9484 & 0.9315 & 0.9098 & 0.8816 & 0.8437 & 0.7904 & 0.7028 \\ 0.9854 & 0.9699 & 0.9516 & 0.9303 & 0.9051 & 0.8748 & 0.8368 & 0.7846 & 0.6964 \\ 0.9868 & 0.9735 & 0.9581 & 0.9400 & 0.9173 & 0.8880 & 0.8491 & 0.7947 & 0.7056 \end{bmatrix}$$

and

$$[f_{ij}] := \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.8548 & 0.7319 & 0.6179 & 0.5119 & 0.4095 & 0.3128 & 0.2229 & 0.1365 & 0.0635 \\ 0.8272 & 0.6713 & 0.5044 & 0.4420 & 0.4310 & 0.3978 & 0.3302 & 0.2212 & 0.0730 \\ 0.8188 & 0.6858 & 0.5659 & 0.4564 & 0.3529 & 0.2541 & 0.1614 & 0.0783 & 0.0334 \\ 0.7902 & 0.6751 & 0.5828 & 0.5030 & 0.4307 & 0.3604 & 0.2897 & 0.2129 & 0.1226 \\ 0.7896 & 0.7366 & 0.6789 & 0.6181 & 0.5533 & 0.4833 & 0.4066 & 0.3129 & 0.1928 \\ 0.8787 & 0.7816 & 0.6943 & 0.6162 & 0.5437 & 0.4731 & 0.3998 & 0.3096 & 0.1913 \\ 0.8832 & 0.7975 & 0.7210 & 0.6474 & 0.5741 & 0.4974 & 0.4158 & 0.3182 & 0.1955 \end{bmatrix}$$

Thirdly, we apply the SDM methods to the *fpfs*-matrices $[d_{ij}]_{8\times 9}$, $[e_{ij}]_{8\times 9}$, and $[f_{ij}]_{8\times 9}$. The decision sets and ranking orders generated by the SDM methods are provided in Table 14 and 15, respectively. The last column in Table 15 demonstrates the number of the methods producing the same ranking order.

Table 14. Decision sets produced by SDM methods (in the event of more-importance-attached noise removal performance at low noise densities)

Algorithms	**Matrices**	**Decision Sets**
MBR01	$[d_{ij}]$	$\{^{0.2546}$DBAIN, ^{0}MDBUTMF, $^{0.0093}$BPDF, $^{0.1111}$NAFSMF, $^{0.5556}$AWMF, $^{0.6944}$DAMF, ^{1}ARmF$\}$
MRB02	$[d_{ij}]$	$\{^{0.8964}$DBAIN, $^{0.8784}$MDBUTMF, $^{0.8610}$BPDF, $^{0.9041}$NAFSMF, $^{0.9555}$AWMF, $^{0.9774}$DAMF, ^{1}ARmF$\}$
CCE10	$[d_{ij}]$	$\{^{0.8964}$DBAIN, $^{0.8784}$MDBUTMF, $^{0.8610}$BPDF, $^{0.9041}$NAFSMF, $^{0.9555}$AWMF, $^{0.9774}$DAMF, ^{1}ARmF$\}$
CEC11	$[d_{ij}]$	$\{^{0.9023}$DBAIN, $^{0.9805}$MDBUTMF, $^{0.8717}$BPDF, $^{0.9030}$NAFSMF, $^{0.9512}$AWMF, $^{0.9779}$DAMF, ^{1}ARmF$\}$
CXL13(λ_1)	$[d_{ij}]$	$\{^{0.2896}$DBAIN, $^{0.2641}$MDBUTMF, ^{0}BPDF, $^{0.4731}$NAFSMF, $^{0.8380}$AWMF, $^{0.8623}$DAMF, ^{1}ARmF$\}$
WQ14(κ)	$[d_{ij}]$	$\{^{0.8894}$DBAIN, $^{0.8760}$MDBUTMF, $^{0.8450}$BPDF, $^{0.9057}$NAFSMF, $^{0.9576}$AWMF, $^{0.9780}$DAMF, ^{1}ARmF$\}$
YHX14(α,β)	$[d_{ij}]$	$\{^{0.1511}$DBAIN, $^{0.0652}$MDBUTMF, ^{0}BPDF, $^{0.1412}$NAFSMF, $^{0.5084}$AWMF, $^{0.9463}$DAMF, ^{1}ARmF$\}$
DC15(α)	$[d_{ij}]$	$\{^{0.2460}$DBAIN, ^{0}MDBUTMF, $^{0.0489}$BPDF, $^{0.2624}$NAFSMF, $^{0.7380}$AWMF, $^{0.7937}$DAMF, ^{1}ARmF$\}$
ZZ15	$[d_{ij}]$	$\{^{0.8964}$DBAIN, $^{0.8784}$MDBUTMF, $^{0.8610}$BPDF, $^{0.9041}$NAFSMF, $^{0.9555}$AWMF, $^{0.9774}$DAMF, ^{1}ARmF$\}$
CXL13/2(λ_1)	$[d_{ij}]$, $[e_{ij}]$	$\{^{0.3143}$DBAIN, $^{0.2127}$MDBUTMF, ^{0}BPDF, $^{0.5349}$NAFSMF, $^{0.8828}$AWMF, $^{0.9884}$DAMF, ^{1}ARmF$\}$
HG13	$[d_{ij}]$, $[e_{ij}]$	$\{^{0.2511}$DBAIN, $^{0.0346}$MDBUTMF, ^{0}BPDF, $^{0.3467}$NAFSMF, $^{0.6800}$AWMF, $^{0.7787}$DAMF, ^{1}ARmF$\}$
ZXZ15(α)	$[d_{ij}]$, $[e_{ij}]$	$\{^{0.9318}$DBAIN, $^{0.9022}$MDBUTMF, $^{0.9126}$BPDF, $^{0.9583}$NAFSMF, $^{0.9871}$AWMF, $^{0.9901}$DAMF, ^{1}ARmF$\}$
VMH16	$[d_{ij}]$, $[e_{ij}]$	$\{^{0.3679}$DBAIN, $^{0.1858}$MDBUTMF, $^{0.2334}$BPDF, ^{0}NAFSMF, $^{0.0406}$AWMF, $^{0.8737}$DAMF, ^{1}ARmF$\}$
MR13	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.8361}$DBAIN, $^{0.8030}$MDBUTMF, $^{0.7619}$BPDF, $^{0.8153}$NAFSMF, $^{0.9337}$AWMF, $^{0.9715}$DAMF, ^{1}ARmF$\}$
MR13/2	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.8222}$DBAIN, $^{0.7156}$MDBUTMF, $^{0.7516}$BPDF, $^{0.7477}$NAFSMF, $^{0.8485}$AWMF, $^{0.9575}$DAMF, ^{1}ARmF$\}$
SM13(w,α)	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.3023}$DBAIN, $^{0.1238}$MDBUTMF, ^{0}BPDF, $^{0.3911}$NAFSMF, $^{0.7184}$AWMF, $^{0.8759}$DAMF, ^{1}ARmF$\}$
Z14/2	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.9435}$DBAIN, $^{0.9112}$MDBUTMF, $^{0.9188}$BPDF, $^{0.9583}$NAFSMF, $^{0.9871}$AWMF, $^{0.9901}$DAMF, ^{1}ARmF$\}$
RS16	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.2930}$DBAIN, $^{0.0990}$MDBUTMF, ^{0}BPDF, $^{0.3115}$NAFSMF, $^{0.7384}$AWMF, $^{0.8701}$DAMF, ^{1}ARmF$\}$
SMT16	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.2734}$DBAIN, $^{0.0230}$MDBUTMF, ^{0}BPDF, $^{0.1229}$NAFSMF, $^{0.5300}$AWMF, $^{0.6959}$DAMF, ^{1}ARmF$\}$
NKY17(λ_2)	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	$\{^{0.3049}$DBAIN, $^{0.0717}$MDBUTMF, ^{0}BPDF, $^{0.1121}$NAFSMF, $^{0.5995}$AWMF, $^{0.7040}$DAMF, ^{1}ARmF$\}$

The ranking orders in Table 15 show that MRB02, CCE10, CXL13(λ_1), WQ14(κ), ZZ15, CXL13/2(λ_1), HG13, SM13(w,α), and RS16 produce the same ranking order. The ranking orders produced by MBR01 and MR13/2 except for those of NAFMSF and BPDF tend to exhibit the same pattern. Moreover, YHX14(α,β), MR13, SMT16, and NKY17(λ_2) have the same ranking order just as DC15(α), ZXZ15, and Z14/2 do even though CEC11 and VMH16 have more incoherent ranking order than the others. Despite the different decision-making skills of all the SDM methods herein, all the methods validate that ARmF performs better than the other filters, and all the SDM methods except for MBR01, DC15(α), ZXZ15, VMH16, MR13/2, and Z14/2 indicate that BPDF displays the minimum performance.

Table 15. Ranking orders produced by SDM methods (in the event of more-importance-attached noise removal performance at low noise densities)

Algorithms	**Matrices**	**Ranking Orders**	**Frequency**
MBR01	$[d_{ij}]$	MDBUTMF≺BPDF≺NAFMSF≺ DBAIN ≺AWMF≺DAMF≺ARmF	1
MRB02	$[d_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
CCE10	$[d_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
CEC11	$[d_{ij}]$	BPDF≺ DBAIN ≺NAFMSF≺AWMF≺DAMF≺MDBUTMF≺ARmF	1
CXL13(λ_1)	$[d_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
WQ14(κ)	$[d_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
YHX14(α,β)	$[d_{ij}]$	BPDF≺MDBUTMF≺NAFMSF≺ DBAIN ≺AWMF≺DAMF≺ARmF	4
DC15(α)	$[d_{ij}]$	MDBUTMF≺BPDF≺ DBAIN ≺NAFMSF≺AWMF≺DAMF≺ARmF	3
ZZ15	$[d_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
CXL13/2(λ_1)	$[d_{ij}]$, $[e_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
HG13	$[d_{ij}]$, $[e_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
ZXZ15(α)	$[d_{ij}]$, $[e_{ij}]$	MDBUTMF≺BPDF≺ DBAIN ≺NAFMSF≺AWMF≺DAMF≺ARmF	3
VMH16	$[d_{ij}]$, $[e_{ij}]$	NAFSMF≺AWMF≺MDBUTMF≺BPDF≺ DBAIN ≺DAMF≺ARmF	1
MR13	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	BPDF≺MDBUTMF≺NAFMSF≺ DBAIN ≺AWMF≺DAMF≺ARmF	4
MR13/2	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	MDBUTMF≺NAFMSF≺BPDF≺ DBAIN ≺AWMF≺DAMF≺ARmF	1
SM13(w,α)	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
Z14/2	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	MDBUTMF≺BPDF≺ DBAIN ≺NAFMSF≺AWMF≺DAMF≺ARmF	3
RS16	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	BPDF≺MDBUTMF≺ DBAIN ≺NAFSMF≺AWMF≺DAMF≺ARmF	9
SMT16	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	BPDF≺MDBUTMF≺NAFMSF≺ DBAIN ≺AWMF≺DAMF≺ARmF	4
NKY17(λ_2)	$[d_{ij}]$, $[e_{ij}]$, $[f_{ij}]$	BPDF≺MDBUTMF≺NAFMSF≺ DBAIN ≺AWMF≺DAMF≺ARmF	4

6. Conclusion

In this study, we configured SDM methods constructed with the concepts of soft sets, fuzzy soft sets, fuzzy parameterized soft sets, *fpfs*-sets, soft matrices, fuzzy soft matrices, and fuzzy parameterized soft matrices, faithfully to the original. Hereby, in 2013 and 2016, we completed the configurations of the methods proposed via these concepts to the *fpfs*-matrices space. Afterwards, we implemented the configured methods to five test cases. By doing so, we determined the methods passing all the test cases. We then applied them to a PVA problem to order the state-of-the-art filters with respect to their noise removal performance.

SDM methods constructed by the superstructures of *fpfs*-sets were not included in this study. In the future, researchers can also configure methods constructed via these superstructures to convenient spaces, such as intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices space [76] and interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets/matrices space [77]. Furthermore, the configured methods can be compared by applying them to decision-making problems in different fields, such as medical diagnosis. Besides, it will be possible to compare all the SDM methods put forward via the aforesaid concepts in the literature and apply them to different decision-making problems once these methods have been configured.

Acknowledgement

This work was supported by the Office of Scientific Research Projects Coordination at Çanakkale Onsekiz Mart University, Grant Number: FBA-2020-3259.

References

[1] D. Molodtsov, *Soft Set Theory-First Results*, Computers Mathematics with Applications 37(4-5) (1999) 19-31.

[2] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy Soft Sets*, The Journal of Fuzzy Mathematics 9(3) (2001) 589-602.

[3] N. Çağman, S. Enginoğlu, F. Çıtak, *Fuzzy Soft Set Theory and Its Applications*, Iranian Journal of Fuzzy Systems 8(3) (2011) 137-147.

[4] N. Çağman, F. Çıtak, S. Enginoğlu, *FP-Soft Set Theory and Its Applications*, Annals of Fuzzy Mathematics and Informatics 2(2) (2011) 219-226.

[5] N. Çağman, F. Çıtak, S. Enginoğlu, *Fuzzy Parameterized Fuzzy Soft Set Theory and Its Applications*, Turkish Journal of Fuzzy Systems 1(1) (2010) 21-35.

[6] N. Çağman, S. Enginoğlu, *Soft Matrix Theory and Its Decision Making*, Computers and Mathematics with Applications 59(10) (2010) 3308-3314.

[7] N. Çağman, S. Enginoğlu, *Fuzzy Soft Matrix Theory and Its Application in Decision Making*, Iranian Journal of Fuzzy Systems 9(1) (2012) 109-119.

[8] S. Enginoğlu, N. Çağman, *Fuzzy Parameterized Fuzzy Soft Matrices and Their Application in Decision-Making*, TWMS Journal of Applied and Engineering Mathematics 10(4) (2020) 1105-1115.

[9] F. Çıtak, N. Çağman, *Soft Int-Rings and Its Algebraic Applications*, Journal of Intelligent and Fuzzy Systems 28(3) (2015) 1225-1233.

[10] A. Sezgin, *A New Approach to Semigroup Theory I: Soft Union Semigroups, Ideals and bi-ideals*, Algebra Letters 2016(3) (2016) 1-46.

[11] A. Ullah, F. Karaaslan, I. Ahmad, *Soft Uni-Abel-Grassmann's Groups*, European Journal of Pure and Applied Mathematics 11(2) (2018) 517-536.

[12] A. Sezgin, N. Çağman, F. Çıtak, *α-Inclusions Applied to Group Theory via Soft Set and Logic*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 68(1) (2019) 334-352.

[13] S. Enginoğlu, N. Çağman, S. Karataş, T. Aydın, *On Soft Topology*, El-Cezerî Journal of Science and Engineering 2(3) (2015) 23-38.

[14] G. Şenel, *A New Approach to Hausdorff Space Theory via the Soft Sets*, Mathematical Problems in Engineering Article ID 2196743, (2016) 6 pages.

[15] J. Thomas, S. J. John, *A Note on Soft Topology*, Journal of New Results Science 5(11) (2016) 24-29.

[16] S. Atmaca, *Relationship between Fuzzy Soft Topological Spaces and (X,τ_e) Parameter Spaces*, Cumhuriyet Sciences Journal 38(4) (2017) 77-85.

[17] M. Riaz, M. R. Hashmi, *Fuzzy Parameterized Fuzzy Soft Topology with Applications*, Annals of Fuzzy Mathematics and Informatics 13(5) (2017) 593-613.

[18] M. Riaz, M. R. Hashmi, A. Farooq, *Fuzzy Parameterized Fuzzy Soft Metric Spaces*, Journal of Mathematical Analysis 9(2) (2018) 25-36.

[19] G. Şenel, 2018. *Analysing the Locus of Soft Spheres: Illustrative Cases and Drawings*, European Journal of Pure and Applied Mathematics 11(4) (2018) 946-957.

[20] N. Çağman, S. Enginoğlu, *Soft Sets Theory and uni-int Decision-Making*, European Journal of Operational Research 207(2) (2010) 848-855.

[21] S. Enginoğlu, M. Ay, N. Çağman, V. Tolun, *Classification of The Monolithic Columns Produced in Troad and Mysia Region Ancient Granite Quarries in Northwestern Anatolia via Soft Decision-Making*, Bilge International Journal of Science and Technology Research 3(Special Issue) (2019) 21-34.

[22] S. Memiş, S. Enginoğlu, U. Erkan, *A Data Classification Method in Machine Learning Based on Normalised Hamming Pseudo-Similarity of Fuzzy Parameterized Fuzzy Soft Matrices*, Bilge International Journal of Science and Technology Research 3(Special Issue) (2019) 1-8.

[23] K. Gong, P. Wang, Y. Peng, Fault-Tolerant Enhanced Bijective Soft Set with Applications, Applied Soft Computing 54 (2017) 431-439.

[24] S. Sreedevi, T. J. Mathew, E. Sherly, *Computerized Classification of Malignant and Normal Microcalcifications on Mammograms: Using Soft Set Theory*, International Conference in Information Science (ICIS) (2016) 131-137.

[25] B. Handaga, H. Onn, T. Herawan, *FSSC: An Algorithm for Classifying Numerical Data Using Fuzzy Soft Set Theory*, International Journal of Fuzzy System Applications 3 (2012) 29-46.

[26] S. A. Lashari, R. Ibrahim, N. Senan, *Medical Data Classification Using Similarity Measure Of Fuzzy Soft Set Based Distance Measure*, Journal of Telecommunication, Electronic and Computer Engineering 9 (2017) 95-99.

[27] S. Enginoğlu, S. Memiş, *A Configuration of Some Soft Decision-Making Algorithms via fpfs-matrices,* Cumhuriyet Science Journal 39(4) (2018) 871-881.

[28] T. Aydın, S. Enginoğlu, *A Configuration of Five of the Soft Decision-Making Methods via Fuzzy Parameterized Fuzzy Soft Matrices and Their Application to a Performance-Based Value Assignment Problem*, M. Kılıç, K. Özkan, M. Karaboyacı, K. Taşdelen, H. Kandemir, A. Beram, (Ed.), International Conferences on Science and Technology, Natural Science and Technology (2019) 56-67 Prizren, Kosovo.

[29] S. Enginoğlu, T. Öngel, *Configurations of Several Soft Decision-Making Methods to Operate in Fuzzy Parameterized Fuzzy Soft Matrices Space*, Eskişehir Technical University Journal of Science and Technology A-Applied Sciences and Engineering 21(1) (2020) 58-71.

[30] T. Aydın, S. Enginoğlu, *Configurations of SDM methods Proposed between 1999 and 2012: A Follow-Up Study*, K. Yıldırım, (Ed.) International Conference on Mathematics: "An Istanbul Meeting for World Mathematicians" (2020) 192-211 Istanbul.

[31] S. Enginoğlu, S. Memiş, *Comment on "Fuzzy soft sets" [The Journal of Fuzzy Mathematics, 9(3), 2001, 589-602]*, International Journal of Latest Engineering Research and Applications 3(9) (2018) 1-9.

[32] S. Enginoğlu, S. Memiş, B. Arslan, *Comment (2) on Soft Set Theory and uni-int Decision Making [European Journal of Operational Research, (2010) 207, 848-855]*, Journal of New Theory (25) (2018) 84-102.

[33] S. Enginoğlu, S. Memiş, T. Öngel, *Comment on Soft Set Theory and Uni-Int Decision Making [European Journal of Operational Research, (2010) 207, 848-855]*, Journal of New Results in Science 7(3) (2018) 28-43.

[34] S. Enginoğlu, S. Memiş, N. Çağman, *A Generalisation of Fuzzy Soft Max-Min Decision-Making Method and Its Application to A Performance-Based Value Assignment in Image Denoising*, El-Cezerî Journal of Science and Engineering 6(3) (2019) 466-481.

[35] S. Enginoğlu, S. Memiş, F. Karaaslan, *A New Approach to Group Decision-Making Method Based on TOPSIS under Fuzzy Soft Environment*, Journal of New Results in Science 8(2) (2019) 42-52.

[36] S. Enginoğlu, S. Memiş, *A New Approach to the Criteria-Weighted Fuzzy Soft Max-Min Decision-Making Method and Its Application to a Performance-Based Value Assignment Problem*, Journal of New Results in Science 9(1) (2020) 19-36.

[37] S. Enginoğlu, S. Memiş, *A Review on Some Soft Decision-Making Methods*. Mathematical Studies and Applications (2018) 437-442 Karaman, Turkey.

[38] S. Enginoğlu, S. Memiş, *A Review on An Application of Fuzzy Soft Set in Multicriteria Decision Making Problem [P. K. Das, R. Borgohain, International Journal of Computer Applications 38(12) (2012) 33-37],* Mathematical Studies and Applications (2018) 173-178 Karaman, Turkey.

[39] S. Enginoğlu, S. Memiş, B. Arslan, *A Fast and Simple Soft Decision-Making Algorithm: EMA18an*, Mathematical Studies and Applications (2018) 428-436 Karaman, Turkey.

[40] S. Enginoğlu, S. Memiş, T. Öngel, *A Fast and Simple Soft Decision-Making Algorithm: EMO18o*, Mathematical Studies and Applications (2018) 179-187 Karaman, Turkey.

[41] Y. Cheng, Z. Xiao, H. Lin, *A Fuzzy Soft Set Approach to Decision Making Problems Based on The Optimum Object for Ideal and Spatial Distance*, Advances in Information Sciences and Service Sciences 5(10) (2013) 965-974.

[42] B. H. Han, S. L. Geng, *Pruning Method for Optimal Solutions of Int^m-Int^n Decision Making Scheme*, European Journal of Operational Research 231(3) (2013) 779-783.

[43] K. Gogoi, A. Kr. Dutta, C. Chutia, *Application of Fuzzy Soft Set Theory in Day to Day Problems*, International Journal of Computer Applications 85(7) (2014) 27-31.

[44] A. Kharal, *Soft Approximations and Uni-Int Decision Making*, The Scientific World Journal Article ID 327408 (2014) 7 pages.

[45] Z. Zhang, *A Note on Soft Set Theory and uni-int Decision Making*, Advances in Research 2(8) (2014) 469-477.

[46] J. Zhan, K. Zhu, *Reviews on Decision Making Methods Based on (Fuzzy) Soft Sets and Rough Soft Sets*, Journal of Intelligent and Fuzzy Systems 29 (2015) 1169-1176.

[47] H. Aktaş, N. Çağman, *Soft Decision-Making Methods Based on Fuzzy Sets and Soft Sets*, Journal of Intelligent and Fuzzy Systems 30 (2016) 2797-2803.

[48] X. Guan, Y. Li, F. Feng, *A New Order Relation on Fuzzy Soft Sets and Its Application*, Soft Computing 17(1) (2013) 63-70.

[49] N. H. Sulaiman, D. Mohamad, *A Set Theoretic Similarity Measure for Fuzzy Soft Sets and Its Application in Group Decision Making*, Proceedings of The National Symposium on Mathematical Sciences (2013) 237-244.

[50] L.Wenjun, L. Qingguo, *An Algorithm to Select the Optimal Program Based on Rough Sets and Fuzzy Soft Sets*, The Scientific World Journal Article ID 514369 (2014) 6 pages.

[51] N. Xie, G. Wen, Z. Li, *A Method for Fuzzy Soft Sets in Decision Making Based on Grey Relational Analysis and D-S Theory of Evidence: Application to Medical Diagnosis*, Computational and Mathematical Methods in Medicine Article ID 581316 (2014) 12 pages.

[52] Y. W. Yang, P. F. He, X. L. Xin, *A Decision-Making Method for The Quality Identification of Wines Based on Fuzzy Soft Sets*, Fuzzy Information and Engineering 6(1) (2014) 19-27.

[53] J. C. R. Alcantud, *Fuzzy Soft Set Based Decision Making: A Novel Alternative Approach*, Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (2015) 106-111.

[54] Z. Haiyan, J. Jingjing, Fuzzy Soft Relation and Its Application in Decision Making, International Conference on Modelling, Identification and Control (ICMIC) (2015) Sousse, Tunisia.

[55] Z. Li, G. Wen, N. Xie, *An Approach to Fuzzy Soft Sets in Decision Making Based on Grey Relational Analysis and Dempster–Shafer Theory of Evidence: An Application in Medical* Diagnosis, Artificial Intelligence in Medicine 64 (2015) 161-171.

[56] H. Tang, *A Novel Fuzzy Soft Set Approach in Decision Making Based on Grey Relational Analysis and Dempster–Shafer Theory of Evidence*, Applied Soft Computing 31 (2015) 317-325.

[57] J. C. R. Alcantud, *A Novel Algorithm for Fuzzy Soft Set Based Decision Making from Multiobserver Input Parameter Data Set*, Information Fusion 29 (2016) 142-148.

[58] Z. Haiyan, *A Multi-Attribute Decision-Making Method Based on Fuzzy Soft Set Theory,* Journal of Research in Engineering and Applied Sciences 1(4) (2016) 162-165.

[59] K. C. Nayak, B. S. Rath, S. C. Mohapatra, *A Soft Set Approach in Investment Decision*, International Journal of Latest Technology in Engineering, Management and Applied Science 5(2) (2016) 58-60.

[60] D. S. Rajput, N. Khare, *Fuzzy Soft Set Decision-Making Model for Social Networking Sites*, International Journal of Social Network Mining 2(3) (2016) 256-266.

[61] V. Renukadevi, G. Sangeetha, *A Novel Approach to Decision Making via Fuzzy Soft Sets*, Turkish Journal of Fuzzy Systems 7(1) (2016) 41-54.

[62] T. R. Sooraj, R. K. Mohanty, B. K. Tripathy, *Fuzzy Soft Set Theory and Its Application in Group Decision Making*, Advances in Intelligent Systems and Computing 452 (2016) 171-178.

[63] J. Wang, Y. Hu, F. Xiao, X. Deng, Y. Deng, *A Novel Method to Use Fuzzy Soft Sets in Decision Making Based on Ambiguity Measure and Dempster-Shafer Theory of Evidence: An Application in Medical Diagnosis*, Artificial Intelligence in Medicine 69 (2016) 1-11.

[64] M. Muthumeenakshi, P. Muralikrishna, *A Study on SFPM Analysis Using Fuzzy Soft Set*, International Journal of Pure and Applied Mathematics 94(2) (2014) 207-213.

[65] İ. Deli, N. Çağman, *Relations on FP-Soft Sets Applied to Decision Making Problems*, Journal of New Theory (3) (2015) 98-107.

[66] X. Zhou, D. Xiang, J. Zhan, *Means of FP-Fuzzy Soft Sets and Decision Making*, Express Letters 9(10) (2015) 2765-2770.

[67] M. Riaz, M. R. Hashmi, *Certain Applications of Fuzzy Parameterized Fuzzy Soft Sets in Decision-Making Problems*, International Journal of Algebra and Statistics 5(2) (2016) 135-146.

[68] S. S. Vani, S. V. D. R. Mohana, J. Haripriya, *On Soft Matrices and Their Application in Decision Making*, Mathematical Sciences International Research Journal 5(Special Issue) (2016) 1-5.

[69] S. Broumi, F. Smarandache, M. Dhar, *On Fuzzy Soft Matrix Based on Reference Function*, International Journal of Information Engineering and Electronic Business 2(2) (2013) 52-59.

[70] N. Sarala, S. Rajkumari, Advantages and Characteristic Features of Human Life in Villages by using Fuzzy Soft Matrices, International Journal Fuzzy Mathematical Archive 9(2) (2015) 251-260.

[71] N. Xie, Y. Han, Z. Li, *A Novel Approach to Fuzzy Soft Sets in Decision Making Based on Grey Relational Analysis and MYCIN Certainty Factor*, International Journal of Computational Intelligence Systems 8(5) (2015) 959-976.

[72] J. J. Arockiaraj, S. Madhanraj, *Fuzzy Soft Matrix Theory and Its Multi Criteria Decision Making on Radio Frequency*, International Journal of Mathematics and Its Applications 4(4) (2016) 137-143.

[73] S. Enginoğlu, U. Erkan, S. Memiş, *Pixel Similarity-based Adaptive Riesz Mean Filter for Salt-and-Pepper Noise Removal*, Multimedia Tools and Applications 78 (2019) 35401-35418.

[74] Z. Wang, A. Bovik, H. Sheikh, E. Simoncelli, *Image Quality Assessment: From Error Visibility to Structural Similarity*, IEEE Transactions on Image Processing 13(4) (2004) 600-612.

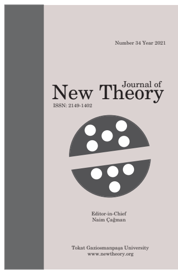
[75] H. R. Sheikh, A. C. Bovik, *Image Information and Visual Quality*, IEEE Transactions on Image Processing 15(2) (2006) 430-444.

[76] S. Enginoğlu, B. Arslan, *Intuitionistic Fuzzy Parameterized Intuitionistic Fuzzy Soft Matrices and Their Application in Decision-Making*, Computational and Applied Mathematics 39 (2020) Article No. 325.

[77] T. Aydın, S. Enginoğlu, *Interval-Valued Intuitionistic Fuzzy Parameterized Interval-Valued Intuitionistic Fuzzy Soft Sets and Their Application in Decision-Making*, Journal of Ambient Intelligence and Humanized Computing 12 (2021) 1541-1558.

Journal of New Theory
ISSN: 2149-1402

34 (2021) 115-122
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



On the Ricci Curvature of Normal-Metric Contact Pair Manifolds

İnan Ünal[1], Ramazan Sarı[2]

Article History
Received: 22 Mar 2021
Accepted: 28 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — In this study, we work on normal-metric contact pair manifolds under certain conditions related to the Ricci curvature. We obtain some results for generalized quasi-Einstein normal-metric contact pair manifolds. We prove that such manifolds are not pseudo-Ricci symmetric. Finally, we investigate Ricci solitons on normal-metric contact pair manifolds.

Keywords — *Normal-metric contact pair manifold, generalized quasi-Einstein, Ricci curvature, Ricci symmetric*

Mathematics Subject Classification (2020) — 53D10, 53C15

1. Introduction

A real contact manifold is defined by a contact form η which is a volume form on a real $(2p+1)-$dimensional differentiable manifold M. The kernel of η defines $2p-$dimensional a non-integrable distribution of TM:

$$\mathcal{D} = \{X : \eta(X) = 0,\ X \in \Gamma(TM)\}$$

We also recall $\mathcal{D}$ contact or horizontal distribution. Let ξ be a vector field on M, which is dual vector of η. Then, for $(1,1)-$tensor field ϕ, M is called an almost-contact metric manifold if following conditions are satisfied:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$, where I is identity map on TM and g is a Riemannian metric [1]. Moreover, we call g compatible metric. Similar to the Kähler manifold, we have a second fundamental form on an almost-contact metric manifold $\Omega(X_1, X_2) = d\eta(X_1, X_2)$. Fruthermore, $d\eta(X_1, X_2) = g(X_1, \phi X_2)$ and in this case we recall g is an associated metric. An almost-contact structure is normal if $N(\phi X_1, \phi X_2) + 2d\eta(X_1, X_2)\xi = 0$, where $N(\phi X_1, \phi X_2)$ is the Nijenhuis tensor field of ϕ. A normal almost-contact metric manifold is called a Sasakian manifold.

In 1959, Kobayashi [2] defined the complex analogue of a real contact manifold. Later, in the 1980s Ishihara and Konishi [3] proved that a complex contact manifold carried an almost-contact structure. A complex almost-contact metric manifold is a complex odd $(2p+1)-$dimensional complex manifold with $(J, \phi, \phi \circ J, \xi, -J \circ \xi, \eta, \eta \circ J, g)$ structure such that

$$\phi^2 = (\phi J)^2 = -I + \eta \otimes \xi - (\eta \circ J) \otimes (J \circ \xi),$$
$$\eta(\xi) = 1 \ \ \eta(-J \circ \xi) = 0, (\eta \circ J)(-J \circ \xi) = 1,\ (\eta \circ J)(\xi) = 0,$$
$$g(\phi X_1, X_2) = -g(X_1, \phi X_2), \quad g((\phi \circ J)X_1, X_2) = -g(X_1, (\phi \circ J)X_2)$$

[1]inanunal@munzur.edu.tr (Corresponding Author); [2]ramazan.sari@amasya.edu.tr
[1]Department of Computer Engineering, Faculty of Engineering, Munzur University, Tunceli, Turkey
[2]Gümüşhacıköy Hasan Duman Vocational School, Amasya University, Amasya, Turkey

where g is an Hermitian metric on M, J is a natural almost complex structure. The normality of complex almost-contact metric manifolds was given by Ishihara and Konishi [3] and Korkmaz [4]. Normal complex contact metric manifolds have been studied by several authors [4–7]. A normal complex contact manifold with a globally defined holomorphic 1-form is called complex Sasakian manifolds. This type of manifolds have been worked in [8–11]

It is wel-known that an odd-dimensional sphere S^{2p+1} carries a contact structure. Calabi and Eckmann showed that the product of two odd-dimensional spheres $M = S^{2p+1} \times S^{2q+1}$ is a complex manifold [12]. These kinds of manifolds recall Calabi-Eckman manifolds. These manifolds have some significant properties in complex geometry. Blair, Ludden, and Yano [13] studied complex manifolds whose complex structures are similar to the complex structure on M. In [13], the authors defined a new structure on Hermitian manifolds called bicontact manifolds. They proved that *"A Hermitian bicontact manifold is locally the product of two normal contact manifolds M^{2p+1} and M^{2q+1}."* Hermitian bicontact manifolds were studied by Abe [14]. Abe obtained many useful results for complex manifolds by using the notion of Hermitian bicontact manifolds.

In 2005, Bande and Hadjar [15] gave the definition of a contact pair manifold, and this definition was similar to bicontact manifolds. Then, they constructed an almost-contact structure on a contact pair manifold and defined the associated metric [16]. In 2013, the normality of almost-contact metric pair structure was studied [17]. Later, some details of the normal contact metric pair (NMCP) manifolds were studied by Bande, Hadjar and Blair in [18–20]. In 2020, one of the authors [21] defined the notion of generalized quasi-Einstein normal-metric contact pair manifold and obtained some results on curvature relations. Besides, same author worked on certain flatness conditions [22] and some semi-symmetry conditions [23]. In [24], NMCP manifolds were studied under conditions of the generalized quasi-conformal curvature tensor.

In this study, we work on NMCP manifolds under certain conditions related to the Ricci curvature. We obtain some results for generalized quasi-EinsteinNMCP manifolds. We prove that such manifolds are not Ricci pseudo-symmetric. Finally, we work on the notion of Ricci solitons.

2. Preliminary

In this section, we give a brief survey on normal metric contact pair manifolds. For details see [15–17].

Definition 2.1. A differentiable manifold $M^{2p+2q+2}$ is called a contact pair manifold if we have

- $\alpha_1 \wedge (d\alpha_1)^p \wedge \alpha_2 \wedge (d\alpha_2)^q \neq 0$,
- $(d\alpha_1)^{p+1} = 0$ and $(d\alpha_2)^{q+1} = 0$.

for two 1-form α_1, α_2 [15]. We recall (α_1, α_2) as (p, q)-type contact pairs.

Two canonical examples of contact pair manifolds are given below.

Example 2.2. Let $x_1, ..., x_{2p+1}, y_1, ..., y_{2q+1}$ be the coordinate functions on $\mathbb{R}^{2p+2q+2}$. Then, two 1-form

$$\alpha_1 = dx_{2p+1} + \sum_{i=1}^{p} x_{2i-1} dx_{2i}, \quad \alpha_2 = dy_{2q+1} + \sum_{j=1}^{q} y_{2i-1} dy_{2i}$$

defines a (p, q)-type contact pairs. $(\mathbb{R}^{2p+2q+2}, \alpha_1, \alpha_2)$ is an example of contact pair manifolds.

Example 2.3. Let (M_1^{2p+1}, α_1) and $M_2^{2q+1}, \alpha_2)$ be two contact manifolds and M be the product of M_1^{2p+1} and M_2^{2q+1}. Then, (α_1, α_2) is a (p, q)-type contact pairs. $(M = M_1^{2p+1} \times M_2^{2q+1}, \alpha_1, \alpha_2)$ is called as product contact pairs.

As we know, the kernel of contact form defines a distribution which we recall *contact distribution.* For contact pairs, since we have two 1−forms α_1 and α_2, we have two integrable subbundle of TM as $\mathcal{D}_1 = \ker \alpha_1$, $\mathcal{D}_2 = \ker \alpha_2$. We can naturally associate it to the distribution of vectors on which α_1 and

$d\alpha_1$ vanish, and the one of vectors on which α_2 and $d\alpha_2$ vanish. (α_1, α_2) of Pfaffian forms of constant classes $2p+1$ and $2q+1$, whose characteristic foliations are transverse and complementary, such that α_1 and α_2 restrict to contact forms on the leaves of the characteristic foliations of α_1 and α_2, respectively. We determine $\mathcal{F}_1$ and $\mathcal{F}_2$ of these foliations. These distributions are involutive. Moreover, they are of codimension $2p+1$ and $2q+1$, respectively, and their leaves are contact manifolds [15]. This allow us to use the name of contact pairs. These two characteristic foliations of M are denoted by

$$\mathcal{F}_1 = \mathcal{D}_1 \cap kerd\alpha_1 \text{ and } \mathcal{F}_2 = \mathcal{D}_2 \cap kerd\alpha_2$$

The Reeb vector fields of contact pair (α_1, α_2) are determined by the following equations:

$$\alpha_1(Z_1) = \alpha_2(Z_2) = 1, \; \alpha_1(Z_2) = \alpha_2(Z_1) = 0$$
$$i_{Z_1} d\alpha_1 = i_{Z_1} d\alpha_2 = i_{Z_2} d\alpha_2 = 0$$

where i_X is the contraction with the vector field X.

Let's define two subbundle of TM by

$$T\mathcal{G}_i = kerd\alpha_i \cap ker\alpha_1 \cap ker\alpha_2, \; i = 1, 2$$

then we can write

$$T\mathcal{F}_i = T\mathcal{G}_i \oplus \mathbb{R}Z_1$$

and so

$$TM = T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$$

Thus, the horizontal and vertical subbundles are defined by $\mathcal{H} = T\mathcal{G}_1 \oplus T\mathcal{G}_2$ and $\mathcal{V} = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, respectively. Finally, we have $TM = \mathcal{H} \oplus \mathcal{V}$ [16].

Any $X \in \Gamma(TM)$ could be written as $X = X^{\mathcal{H}} + X^{\mathcal{V}}$, where $X^{\mathcal{H}} \in \mathcal{H}, \; X^{\mathcal{V}} \in \mathcal{V}$. In another way, we can write $X = X^1 + X^2$ for $X^1 \in T\mathcal{F}_1$ and $X^2 \in T\mathcal{F}_2$. Furthermore, we can state $X^1 = X^{1^h} + \alpha_2(X^1)Z_2$ and $X^2 = X^{2^h} + \alpha_1(X^2)Z_1$, where X^{1^h} and X^{2^h} are horizontal parts of X^1 and X^2, respectively. From all these decomposition of X finally we get

$$X = X^{1^h} + X^{2^h} + \alpha_1(X^2)Z_1 + \alpha_2(X^1)Z_2$$
$$\alpha_1(X^{1^h}) = \alpha_1(X^{2^h}) = 0, \quad \alpha_2(X^{1^h}) = \alpha_2(X^{2^h}) = 0$$

Let's define $(1,1)-$tensor field ϕ such as

$$\phi^2 = -I + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \quad \phi Z_1 = \phi Z_2 = 0, \; \alpha_1(\phi) = \alpha_2(\phi) = 0$$

If $\phi T\mathcal{F}_i = T\mathcal{F}_i$, then ϕ is said to be decomposable, i.e $\phi = \phi_1 + \phi_2$. With the decomposability of ϕ, we have that (α_1, Z_1, ϕ_1) (resp. (α_2, Z_2, ϕ_2)) induces an almost-contact structure on the leaves of $\mathcal{F}_2$ (resp.$\mathcal{F}_1$) [16]. Throughout this study, it is assume that ϕ is decomposable. We recall $(\phi_1, \phi_2, g, \alpha_2, Z_2, \phi_2)$ the contact pair structure

A Riemannian metric g on $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2)$ is called compatible if $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in TM$, and associated if $g(X_1, \phi X_2) = (d\alpha_1 + d\alpha_2)(X_1, X_2)$ and $g(X_1, Z_i) = \alpha_i(X_1)$, for $i = 1, 2$. 4-tuple $(\alpha_1, \alpha_2, \phi, g)$ is called metric contact pair structure on M.

Normality of almost-contact structure is an important notion in contact geometry. As we know a normal contact metric manifold is called as Sasakian manifold. A Sasakian manifold can be seen as odd-dimensional Kähler manifolds. Similarly, we have many subclasses of complex contact manifolds which are normal. A complex Sasakian manifold is also a normal complex contact metric manifold [11]. The normality of a MCP manifold was studied in [17]. We have two almost complex structures:

$$\mathcal{J} = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2, \quad \mathcal{T} = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2$$

$\mathcal{J}$ and $\mathcal{T}$ are called almost complex structure associated $(\alpha_1, \alpha_2, \phi)$. If $\mathcal{J}$ and $\mathcal{T}$ are integrable, then M is normal. On the other hand, the integrability of $\mathcal{J}$ and $\mathcal{T}$ is determined by the following condition

$$[\phi, \phi](X_1, X_2) + 2d\alpha_1(X_1, X_2)Z_1 + 2d\alpha_2(X_1, X_2)Z_2 = 0,$$

for all $X_1, X_2 \in \Gamma(TM)$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [17]. For the sake of brevity, we use the abbreviation of NMCP instead of the term of *normal metric contact pair*.

The curvature properties of a NMCP manifold are given by

$$\begin{aligned} R(X_1, Z)X_2 &= -g(\phi X_1, \phi X_2)Z, \\ R(X_1, X_2, Z, X_3) &= d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) + d\alpha_2(\phi X_3 X_1)\alpha_2(X_2) \\ &\quad -d\alpha_1(\phi X_3, X_2)\alpha_1(X_1) - d\alpha_2(\phi X_3, X_2)\alpha_2(X_1) \\ R(X_1, Z)Z &= -\phi^2 X_1 \end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(TM)$ and $Z = Z_1 + Z_2$ for the Reeb vector fields Z_1, Z_2, R is the Riemannian curvature tensor [18]. Moreover, the Ricci curvature of M has the following properties [18];

$$Ric(X_1, Z) = 0, \quad \text{for } X_1 \in \Gamma(\mathcal{H}) \tag{1}$$

$$Ric(Z, Z) = 2p + 2q. \tag{2}$$

$$Ric(Z_1, Z_1) = 2p, \;\; Ric(Z_2, Z_2) = 2q, \;\; Ric(Z_1, Z_2) = 0 \tag{3}$$

Definition 2.4. An NMCP manifold is called a generalized quasi-Einstein (GQE) manifold if the Ricci curvature of M has the following form:

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + \beta\alpha_1(X_1)\alpha_1(X_2) + \gamma\alpha_2(X_1)\alpha_2(X_2)$$

where λ, β, and γ are scalar fields on M and $X_1, X_2 \in \Gamma(TM)$ [21].

Thus, from (2) and (3) we have

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + (2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) + (2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$.

3. Certain Conditions on the Ricci Curvature of Nomral-metric Contact Pair Manifolds

Ricci curvature *Ric*, which is defined as the trace of Riemannian curvature tensor, has a major role in the Riemannian geometry. In this section, we work on NMCP manifolds with certain conditions related to the Ricci curvature.

We recall a Riemannian manifold as flat if it has zero curvature. Furthermore, a Riemannian manifold is said to be Ricci-flat if $Ric = 0$.

Theorem 3.1. An NMCP manifold could not be Ricci-flat.

Proof. Let M be an NMCP manifold. Suppose that it is Ricci-flat, i.e for every X_1, X_2 vector fields we have $Ric(X_1, X_2) = 0$. Then, from (2) we get $2p + 2q = 0$, which is impossible. Thus, there is a contradiction. The manifold could not be Ricci-flat. □

An normal-metric contact pair manifoldmanifold is Ricci symmetric if $\nabla Ric = 0$. Let M be a GQE NMCP manifold with constant λ. From the Riemannian geometry we have following wel-known relation;

$$(\nabla_X Ric)(X_1, X_2) = \nabla_X Ric(X_1, X_2) - Ric(\nabla_X X_1, X_2) - Ric(X_1, \nabla_X X_2)$$

for all $X, X_1, X_2 \in \Gamma(TM)$. Then, using (2) we obtain

$$(\nabla_X Ric)(X_1, X_2) = (2p - \lambda)g(\phi_1 X, X_1)\alpha_1(X_2) + (2q - \lambda)g(\phi_2 X, X_2)\alpha_1(X_1)$$

If X_1 and X_2 are horizontal vector fields, we get $(\nabla_X Ric)(X_1, X_2) = 0$.

Corollary 3.2. On the horizontal bundle of a GQE NMCP manifold with constant λ, $\nabla Ric = 0$.

If we take $X_1 = a_1 Z_1 + a_2 Z_2, X_2 = b_1 Z_1 + b_2 Z_2$ for coefficients $a_i, b_i,\ i = 1, 2$ since $g(\phi_1 X, X_1) = -g(X, \phi_1 X_1)$ and $g(\phi_2 X, X_2) = -g(X, \phi_2 X_2)$, we get $(\nabla_X Ric)(X_1, X_2) = 0$.

Corollary 3.3. On the vertical bundle of a GQE NMCP manifold with constant λ, $\nabla Ric = 0$.

Let $X_1 = X_1^{1^h} + X_1^{2^h} + \alpha_1(X_1^2)Z_1 + \alpha_2(X_1^1)Z_2$ and $X_2 = X_2^{1^h} + X_2^{2^h} + \alpha_1(X_2^2)Z_1 + \alpha_2(X_2^1)Z_2$. Then, we obtain

$$(2p - \lambda)g(\phi_1 X, X_1^{1^h} + X_1^{2^h})\alpha_1(X_1^2) - (2q - \lambda)g(\phi_2 X, X_2^{1^h} + X_2^{2^h})\alpha_1(X_2^1) = 0$$

Thus, we state the following theorem.

Proposition 3.4. On a GQE NMCP manifold with constant λ, $(\nabla_X Ric)(X_1, X_2) = 0$ if and only if $(2p-\lambda)g(\phi_1 X, X_1^{1^h} + X_1^{2^h})\alpha_1(X_1) - (2q-\lambda)g(\phi_2 X, X_2^{1^h} + X_2^{2^h})\alpha_1(X_2) = 0$ for all $X_1 = X_1^{1^h} + X_1^{2^h} + \alpha_1(X_1^2)Z_1 + \alpha_2(X_1^1)Z_2, X_2 = X_2^{1^h} + X_2^{2^h} + \alpha_1(X_2^2)Z_1 + \alpha_2(X_2^1)Z_2$ and $X \in \Gamma(TM)$.

An NMCP manifold M satisfies cyclic parallel Ricci tensor if we have

$$(\nabla_{X_1} Ric)(X_2, X_3) + (\nabla_{X_2} Ric)(X_3, X_1) + (\nabla_{X_3} Ric)(X_1, X_2) = 0$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$. M is also satisfies Codazzi type of Ricci tensor if we have

$$(\nabla_{X_1} Ric)(X_2, X_3) - (\nabla_{X_2} Ric)(X_1, X_3) = 0$$

for all X_1 and X_2 vector fields on M. In [21], one of the presented authors proved the following results.

Theorem 3.5. A GQE NMCP manifold with constant λ satisfies cyclic parallel Ricci tensor [21].

Theorem 3.6. A GQE NMCP manifold with constant λ does not satisfy Codazzi type of Ricci tensor [21].

In [25], the authors proved that *if the generators of GQE manifolds are Killing then the manifold satisfies cyclic parallel Ricci tensor.* Since Z_1 and Z_2 are Killing, Theorem 3.5 is compatible with this result. The same authors proved that *if a GQE manifold is Codazzi type of Ricci tensor then the integral curves of the generator vector fields are geodesic.* It is known that Z_1 and Z_2 are geodesics. But the manifold is not the Codazzi type of Ricci tensor. Theorem 3.6 is guaranteed that the converse of the second result in [25] is not satisfied.

In [26], the authors proved that *in a GQE manifold, if the associated scalars are constant and the Ricci tensor is of Codazzi type, then the associated* 1*-form are closed.* As we know, α_1 and α_2 are not closed. Thus, Theorem 3.6 is compatible with this result.

A generalization of Ricci symmetry was pointed out by the name of Ricci semi-symmetry. If $R \cdot Ric = 0$ we recall the manifold as Ricci semi-symmetric manifold. In [23], we proved following theorem

Theorem 3.7. A Ricci semi-symmetric NMCP manifold is a GQE manifold [23].

A non-flat NMCP manifold M is called a Chaki pseudo-Ricci symmetric manifold if the Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the condition

$$(\nabla_X Ric)(X_1, X_2) = 2A(X)Ric(X_1, X_2) + A(X_2)Ric(X, X_3) + A(X_3)Ric(X_2, X)$$

where A is non-zero $1-$form such that $g(X, \rho) = A(X)$ for all vector fields X; ρ being the vector field corresponding to the associated 1-form [27]. If $A = 0$, then the manifold is called Ricci symmetric.

The another study on GQE manifolds was presented in [28], the authors proved that *a pseudo-Ricci symmetric manifold cannot cyclic parallel Ricci tensor; otherwise, this manifold reduces to a Ricci symmetric manifolds.* Thus, with the considered Theorem 3.5, we can state,

Theorem 3.8. A GQE NMCP manifold with constant λ cannot be pseudo-Ricci symmetric.

A Riemannian manifold (M, g) is called a *Ricci soliton* if there is a smooth vector field V and a scalar $\nu \in \mathbb{R}$ such that

$$\mathcal{L}_V g + 2Ric = 2\nu g \tag{4}$$

on M, where Ric is the Ricci tensor and $\mathcal{L}_V g$ is the Lie derivative of the metric g. The Ricci soliton is called *shriking, steady, or expanding* according to $\nu < 0$, $\nu = 0$, $\nu > 0$, respectively [29]. Contact manifolds have been studied as the solution of Ricci soliton equations. For different structures, see [30–33]

Suppose that a NMCP manifold satisfies (4) with the potential vector fields Z_1 and Z_2. Since Z_1 and Z_2 are the Killing vector fields, we get

$$Ric(X_1, X_2) = \nu g(X_1, X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$. Thus, M is Einstein manifold. Moreover, from (2), we get $\nu = 2p + 2q$. Thus, we state the following result:

Theorem 3.9. Let a NMCP manifold satisfy the Ricci soliton equation with the potential vector fields Z_1 (and Z_2). Then, the Ricci soliton is expanding.

Let M be a GQE NMCP manifold which satisfies the Ricci soliton equation. Thus, from (2), we obtain

$$(\mathcal{L}_V g)(X_1, X_2) = -(2\nu + 2\lambda)g(X_1, X_2) - 2(2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) - 2(2q - \lambda)\alpha_2(X_1)\alpha_2(X_2) \tag{5}$$

Corollary 3.10. Let M be a GQE NMCP manifold which satisfies the Ricci soliton equation. The potential vector field is Killing if and only if $g(X_1, X_2) = \frac{2(2p-\lambda)}{(2\nu+2\lambda)}\alpha_1(X_1)\alpha_1(X_2) + \frac{2(2q-\lambda)}{(2\nu+2\lambda)}\alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in \Gamma(TM)$.

Let $V = Z_1$ in (5), then we have

$$0 = -(2\nu + 2\lambda)g(X_1, X_2) - 2(2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) - 2(2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

Choose $X_1 = X_2 = Z$, then we get $\nu = p + q$. Thus, we state,

Theorem 3.11. Let M be a GQE NMCP manifold, which satisfies the Ricci soliton equation. If $V = Z_1(or Z_2)$, then the Ricci soliton is expanding with $\nu = p + q$.

There are many interesting vector fields (sometimes called collineations), considered infinitesimal symmetries of geometric structure or physical quantities such as metric, curvature, energy-momentum tensors, geodesics, and light cones. These vector fields have many applications in Riemannian geometry and general relativity. One of them is Killing vectors, which are named after a Norwegian mathematician Killing, who first described these notions in 1892. The Killing vectors preserve the metric and all the derived structures. Another type is the conformal Killing vector field. A vector field X recall conformal Killing if its Lie derivative is proportional to itself $\mathcal{L}_X g = 2\mu g$, for some scalar field μ. If μ is zero, X is the Killing vector fields and if μ is constant, but not zero, the vector field is said to be homothetic, and the metric is changed by a (constant) scale factor as it moves along.

Suppose that V is conformal Killing in (5). Then, we obtain

$$-2(\nu + \lambda + \mu)g(X_1, X_2) = 2(2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) + 2(2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

By taking $X_1 = X_2 = Z$, we get $-4(\nu + \lambda + \mu) = 4(p + q) - 4\lambda$ and so $\mu + \nu = -(p + q)$. Since ν is constant, we state the following result.

Theorem 3.12. Let M be a GQE NMCP manifold which satisfies the Ricci soliton equation. If the potential vector field is conformal Killing, then it reduces to the homothetic vector field.

4. Conclusion

Normal-metric contact pair manifolds are an important class of contact manifolds. These manifolds have many significant properties that differ from the classical contact structures. Moreover, a normal contact metric pair manifold could be a special solution to Einstein's field equations. Furthermore, we have the applications of GQE manifolds in the contact geometry thanks to normal-metric contact pair manifolds. In this paper, we study normal-metric contact pair manifolds from the Riemannian geometric perspective. We obtain some results on the Ricci curvature and the Ricci solitons. The results of the paper will be a reference for future works on contact manifolds and general relativity.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Springer Science Business Media, 2010.

[2] S. Kobayashi, *Remarks on Complex Contact Manifolds*, Proceedings of the American Mathematical Society 10 (1959) 164–167.

[3] S. Ishihara, M.Konishi, *Complex Almost-contact Structures in a Complex Contact Manifold*, Kodai Mathematical Journal 5 (1982) 30–37.

[4] B. Korkmaz, *Normality of Complex Contact Manifolds*, Rocky Mountain Journal of Mathematics 30 (2000) 1343–1380.

[5] A. T. Vanli, D. E. Blair, *The Boothby-Wang Fibration of the Iwasawa Manifold as a Critical Point of the Energy*, Monatshefte für Mathematik 147 (2006) 75–84.

[6] A. T. Vanli, İ. Ünal, *Conformal, Concircular, Quasi-conformal and Conharmonic Flatness on Normal Complex Contact Metric Manifolds*, International Journal of Geometric Methods in Modern Physics 14(05) (2017).

[7] A. T. Vanli, İ. Ünal, *On Complex η-Einstein Normal Complex Contact Metric Manifolds*, Communications In Mathematics And Application 8(3) (2017) 301–313.

[8] D. Fetcu, *Harmonic Maps between Complex Sasakian Manifolds*, Rendiconti del Seminario Matematico Universita e Politecnico di Torino 64 (2006) 319–329.

[9] B. J. Foreman, *Complex Contact Manifolds and Hyperkähler Geometry*, Kodai Mathematical Journal 23(1) (2000) 12–26.

[10] D. Fetcu, *An Adapted Connection on a Strict Complex Contact Manifold*, in Proceedings of the 5th Conference of Balkan Society of Geometers (2006) 54–61.

[11] A. T. Vanli, İ. Ünal, K. Avcu, *On Complex Sasakian Manifolds*, Afrika Matematika (2020) 1–10.

[12] E. Calabi, B. Eckmann, *A Class of Compact Complex Manifolds Which are Not Algebraic*, Annals of Mathematics 58 (1953) 494–500.

[13] D. E. Blair, G. D. Ludden, K. Yano, *Geometry of Complex Manifolds Similar to the Calabi-Eckmann Manifolds*, Journal of Differential Geometry 9(2) (1974) 263–274.

[14] K. Abe, *On a Class of Hermitian Manifolds*, Inventiones Mathematicae 51(2) (1979) 103–121.

[15] G.Bande, A. Hadjar, *Contact Pairs*, Tohoku Mathematical Journal 57(2) (2005) 247–260.

[16] G.Bande, A. Hadjar, *Contact Pair Structures and Associated Metrics*, Differential Geometry-Proceedings of the $VIII$ International Colloquium (2009) 266–275.

[17] G. Bande, A. Hadjar, *On Normal Contact Pairs*, International Journal of Mathematics 21(06) (2010) 737–754.

[18] G. Bande, D. E. Blair, A. Hadjar, *Bochner and Conformal Flatness of Normal Metric Contact Pairs*, Annals of Global Analysis and Geometry 48(1) (2015) 47–56.

[19] G. Bande, A. Hadjar, *On the Characteristic Foliations of Metric Contact Pairs*, Harmonic Maps and Differential Geometry, Contemporary Mathematics 542 (2011) 255–259.

[20] G. Bande, D. E. Blair, A. Hadjar, *On the Curvature of Metric Contact Pairs*, Mediterranean Journal of Mathematics 10(2) (2013) 989–1009.

[21] İ. Ünal, *Generalized Quasi-Einstein Manifolds in Contact Geometry*, Mathematics 8(9) (2020) 1–14.

[22] İ. Ünal, *Some Flatness Conditions on Normal Metric Contact Pairs*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 69(2) (2020) 262–271.

[23] İ. Ünal, *On Metric Contact Pairs with Certain Semi-symmetry Conditions*, Journal of Polytechnic 24(1) (2021) 333–338.

[24] İ. Ünal, *Generalized Quasi-Conformal Curvature Tensor On Normal Metric Contact Pairs*, International Journal of Pure and Applied Sciences 6(2) (2020) 194–199.

[25] U. C. De, S. Mallick, *On Generalized Quasi-Einstein Manifolds Admitting Certain Vector Fields*, Filomat 29(3) (2015) 599–609.

[26] D. G. Prakasha, H. Venkatesha *Some Results on Generalized Quasi-Einstein Manifolds*, Chinese Journal of Mathematics Article ID 563803 (2014) 5 pages.

[27] M. C. Chaki *On Pseudo Ricci Symmetric Manifolds*, Bulgarian Journal of Physics 15(6) (1988) 526–531.

[28] B. Kirik, *Generalized Quasi-Einstein Manifolds Admitting Special Vector Fields*, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 31(1) (2015) 61–69.

[29] R. Hamilton, *The Ricci flow on Surfaces*, Contemporary Mathematics 71 (1988) 237–261.

[30] G. Ayar, M. Yıldırım, *Ricci solitons and Gradient Ricci solitons on Nearly Kenmotsu Manifolds*, Facta Universitatis, Series: Mathematics and Informatics 34(3) (2019) 503–510.

[31] C. S. Bagewadi, G. Ingalahalli, S. R. Ashoka, *A Study on Ricci Solitons in Kenmotsu Manifolds*, International Scholarly Research Notices Geometry Article ID 412593 (2013) 6 pages.

[32] G. Ayar, D. Demirhan, *Ricci Solitons on Nearly Kenmotsu Manifolds with Semi-symmetric Metric Connection*, Journal of Engineering Technology and Applied Sciences 4(3) (2019) 131–140.

[33] H. İ. Yoldaş, E. Yaşar *On Submanifolds of Kenmotsu Manifold with Torqued Vector Field*, Hacettepe Journal of Mathematics and Statistics 49(2) (2020) 843–853.