Number 35 Year 2021

Journal of New Theory

ISSN: 2149-1402

Editor-in-Chief
Naim Çağman

www.dergipark.org.tr/en/pub/jnt

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J. New Theory is an open-access journal.

JNT was founded on 18 November 2014, and its first issue was published on 27 January 2015.

ISSN: 2149-1402

Editor-in-Chief: Naim Çağman

E-mail: journalofnewtheory@gmail.com

Language: JNT accepts contributions in English only.

Article Processing Charges: JNT incurs no article processing charges.

Publication Frequency: Quarterly

Review Process: Blind Peer Review

The policy of Screening for Plagiarism: As fo 2020, JNT accepts submissions for pre-review only if their reference-excluded similarity rate (obtained viaiThenticate, Turnitin, etc.) is a maximum of 30%.

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ISSN: 2149-1402

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https://dergipark.org.tr/en/pub/jnt
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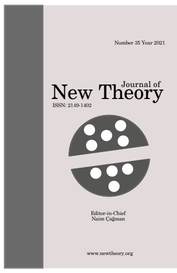

A New View of Homomorphic Properties of BCK-Algebra in Terms of Some Notions of Discrete Dynamical System

Dawood Khan[1], Abdul Rehman[2]

Article History
Received: 22 Sep 2020
Accepted: 21 May 2021
Published: 30 Jun 2021
Research Article

Abstract − In the present manuscript, we introduce the concept of a discrete dynamical system (Ƶ, Ψ) in BCK-algebra where Ƶ is a BCK-algebra and Ψ is a homomorphism from Ƶ to Ƶ and establish some of their related properties. We prove that the set of all fixed points and the set of all periodic points in BCK-algebra Ƶ are the BCK-subalgebras. We show that when a subset of BCK-algebra Ƶ is invariant concerning Ψ. We prove that the set of all fixed points and the set of all periodic points in commutative BCK-algebra Ƶ with relative cancellation property are the ideals of Ƶ. We also prove that the set of all fixed points in Ƶ is an S-invariant subset of a BCK-algebra Ƶ.

Keywords − *BCK-Algebra, discrete dynamical system, periodic points, fixed points, invariant set, strongly invariant*

Mathematics Subject Classification (2020) − 37A15, 06F35

1. Introduction

The foundation of the concept of BCK-Algebra was laid down by the famous mathematicians Imai and Iseki in their pioneering paper [1]. Their theory about BCK-algebra and related ideas and properties are nowadays utilized extensively in different areas of science like artificial intelligence information sciences, cybernetics, and computer sciences. BCK-algebra has been inspired by two considerations; one based on classical and non-classical propositional calculi of Meredith and the other based on set theory [2]. The BCK-algebra can also be considered as the algebraic formulation of Meredith's BCK-implicational calculus [3]. The concept of the ideal theory of BCK-algebra playing a fundamental role in the evolution of BCK-Algebra was first introduced by Iseki in [4]. Besides, the notion of BCK-homomorphism was also first defined by Iseki in [5]. During the past four decades, several researchers have extensively investigated this field and have produced much literature about the theory of BCK-algebra [6].

The foundation of the dynamical system was laid down by the eminent mathematician Henri Poincare in 1899 in his famous paper celestial mechanics [7]. His theories about dynamical systems and connected ideas and properties are nowadays widely used in various fields of science like physics, biology, meteorology, astronomy, economics, and many others. The main idea of the study of the dynamical system indicates the mathematical techniques for describing the eventual or asymptotic behaviour of an iterative process in different specified scientific disciplines. In 1917, Julia and Fatau diverted the concept of the dynamical system theory in the relationship of complex analysis, established a new notion, and provided the name as complex analytic dynamical system [8]. Latterly, Birkhoff enthusiastically adopted Poincare's viewpoint, realized the

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significance of the concept of mappings, and introduced a discrete dynamical system [9]. A discrete dynamical system is an interesting and active area of applied and pure mathematics that involves tools and techniques from different fields such as Number Theory, Analysis, and Geometry. According to a fixed rule, discrete dynamical systems are those dynamical systems whose states evolve over a state space in discrete steps. Birkhoff's opinion based on discrete steps was highly captivating and attracted the attention of mathematicians to use in various fields of mathematics. Many researchers applied the concept of a discrete dynamical system in their related areas. This becomes the extension of the theory of discrete dynamical systems in many branches of mathematical sciences. In 1927 Birkhoff infused the notion of discrete dynamical systems in topology and laid down the foundation of another field known as topological Dynamics [10]. Topological dynamics then flourished and further generated algebraic topology and differential topology [11]. Differential Topological techniques enabled Peixoto and Smale to understand the chaotic behaviour of a large class of dynamical systems and introduced a new area of the dynamical system known as hyperbolic dynamical system [12]. Von Neumann, Birkhoff, and Koopman introduced a discrete dynamical system in measure theory, due to which a new field emerged by the name of Ergodic theory [13]. Dikranjan and Bruno embedded the discrete dynamical system in group theory and established a new algebraic structure known as discrete dynamical systems in group theory [14]. Dawood et al. have recently infused the discrete dynamical system in BCI-algebra and obtained specific interesting properties [15].

In the present paper, we emphasize the mathematical aspects of the theory of discrete dynamical system $(Ϛ, \Psi)$. We define the concept of discrete dynamical system $(Ϛ\ \Psi)$ in BCK-algebra where Ϛ is a BCK-algebra and Ψ is a homomorphism from Ϛ to Ϛ and establish some properties of the set of periodic points and the set of fixed points of BCK-algebra Ϛ. We prove that the sets of all fixed points and periodic points of BCK-algebra Ϛ are the BCK-subalgebras of Ϛ. We show that when a subset of a BCK-algebra Ϛ is invariant concerning Ψ. We prove that the sets of all fixed and periodic points of a commutative BCK-algebra Ϛ with relative cancellation property are the ideals of Ϛ.we also prove that in the discrete dynamical system $(Ϛ, \Psi)$ the set of all fixed points is S-Invariant subset of a BCK-algebra Ϛ.

2. Preliminaries

This section consists of some preliminary definitions and basic facts about BCK-algebra, useful to prove our results. Throughout this research work, we consistently denote the BCK-algebra by Ϛ without any specification.

Here we only mention those concepts of BCK-algebra which are necessary for our treatment. For further information regarding BCK-algebra, the readers are referred to the references [16-21].

Definition 2.1. [7] A BCK-algebra $(Ϛ,*,0)$ is an algebra of the type $(2,0)$ satisfying the following five axioms for all $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in Ϛ$

(BCK-i) $((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0$

(BCK-ii) $(\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0$

(BCK-iii) $\mathfrak{a} * \mathfrak{a} = 0$

(BCK-iv) $0 * \mathfrak{a} = 0$

(BCK-v) $\mathfrak{a} * \mathfrak{b} = 0$ and $\mathfrak{b} * \mathfrak{a} = 0 \Longrightarrow \mathfrak{a} = \mathfrak{b}$

Moreover $\leq$ is a partial order on Ϛ and is defined by

$$\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow \mathfrak{a} * \mathfrak{b} = 0$$

Definition 2.2. [7] BCK-algebra $(Ϛ,*,0)$ is said to be commutative if the following condition holds in Ϛ for any $\mathfrak{a},\mathfrak{b} \in Ϛ$

$$\mathfrak{a} * (\mathfrak{a} * \mathfrak{b}) = \mathfrak{b} * (\mathfrak{b} * \mathfrak{a})$$

Definition 2.3. [7] Let $Ⱬ_s$ be a non-vacuous subset of a BCK-algebra $(Ⱬ,*,0)$. Then, $Ⱬ_s$ is said to be a BCK-subalgebra of Ⱬ if it satisfies the (s-i) and (s-ii) conditions where

(s-i) $0 \epsilon Ⱬ_s$

(s-ii) $\mathfrak{a} * \mathfrak{b} \epsilon Ⱬ_s$ for any $\mathfrak{a}, \mathfrak{b} \in Ⱬ_s$

Definition 2.4. [7] Let Ⱬ be a BCK-algebra and I_d be a non-vacuous subset of Ⱬ**.** Then, I_d is said to be an ideal of Ⱬ if it satisfies (I_d-i) and (I_d-ii) conditions, where

(I_d-i) $0 \epsilon I_d$

(I_d-ii) $\mathfrak{a} * \mathfrak{b} \epsilon I_d$ and $\mathfrak{b} \in I_d \Longrightarrow \mathfrak{a} \in I_d, \forall \mathfrak{a}, \mathfrak{b} \in Ⱬ$

Definition 2.5. [5,17] A mapping $\Psi: Ⱬ_1 \longrightarrow Ⱬ_2$ where $Ⱬ_1$ and $Ⱬ_2$ are two BCK-algebras is said to be a BCK-homomorphism if it satisfies the following condition

$$\Psi(\mathfrak{a} * \mathfrak{b}) = \Psi(\mathfrak{a}) * \Psi(\mathfrak{b}), \quad \forall \mathfrak{a}, \mathfrak{b} \in Ⱬ_1$$

Definition 2.6. [18] A commutative BCK-algebra (Ⱬ, $*$, 0) has the relative cancellation property if for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in Ⱬ$ and $\mathfrak{a} \geq \mathfrak{c}$, $\mathfrak{b} \geq \mathfrak{c}$ such that $\mathfrak{a} * \mathfrak{c} = \mathfrak{b} * \mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$.

3. Definitions of Some Notions of Discrete Dynamical System in BCK-Algebra

This section picks some terminologies of discrete dynamical systems and defines them in terms of BCK-algebra.

Definition 3.1. Let Ⱬ be a BCK-algebra and $\Psi: Ⱬ \longrightarrow Ⱬ$ be a homomorphism. Then, $(Ⱬ, \Psi)$ is called a discrete dynamical system in BCK-algebra

In the present paper, whenever we say a discrete dynamical system, it means we are taking an ordered pair $(Ⱬ, \Psi)$ where Ⱬ is a BCK-algebra and Ψ is a homomorphism from Ⱬ to Ⱬ.

Definition 3.2. In the discrete dynamical system $(Ⱬ, \Psi)$ a point $\mathfrak{a} \epsilon Ⱬ$ is a fixed point if $\Psi(\mathfrak{a}) = \mathfrak{a}$.

Definition 3.3. In the discrete dynamical system $(Ⱬ, \Psi)$ a point $\mathfrak{a} \in Ⱬ$ is a periodic point if $\Psi^m(\mathfrak{a}) = \mathfrak{a}$ for some positive integer 'm', the least value of 'm' is said to be the period of '$\mathfrak{a}$'.

Definition 3.4. In the discrete dynamical system $(Ⱬ, \Psi)$ a subset A of Ⱬ is an invariant subset of Ⱬ if $\Psi(A) \subset A$.

Definition 3.5. In the discrete dynamical system $(Ⱬ, \Psi)$ a subset A of Ⱬ is a strongly invariant subset of Ψ if $\Psi(A) = A$.

The following are simple examples regarding the definitions given in the paper.

Example 3.1. Suppose that $Ⱬ = \{0, p, q, r\}$ is a BCK-algebra.

$*$	0	p	q	r
0	0	0	0	0
p	p	0	p	p
q	q	q	0	q
r	r	r	r	0

And a mapping $\Psi: Ⱬ \longrightarrow Ⱬ$ defined by $\Psi(0) = 0, \Psi(p) = 0, \Psi(q) = q$, and $\Psi(r) = r$ is a homomorphism. Then, the points 0, q, and r are fixed points and the periodic points of period 1.

Example 3.2. Consider the BCK-algebra Ⱬ of example 3.1. Where the subset $A = \{0, q, r\}$ is a strongly invariant subset of Ⱬ because $\Psi(A) = A$ while the subset $B = \{0, p\}$ is invariant because $\Psi(B) \subset B$.

4. Basic Results

In this section, we prove some properties essential in proving the theorems in this paper.

Proposition 4.1. If a mapping $\Psi: Ꝣ \longrightarrow Ꝣ$ is a homomorphism from BCK-algebra Ꝣ to Ꝣ and $0 \in Ꝣ$, then $\Psi(0) = 0$.

PROOF. $\Psi(0) = \Psi(0 * 0)$ ∵ (BCK-iii)

$= \Psi(0) * \Psi(0) = 0$ ∵ (BCK-iii)

Proposition 4.2. If $\Psi: Ꝣ \longrightarrow Ꝣ$ is a homomorphism, then $\Psi^n: Ꝣ \longrightarrow Ꝣ$ is a homomorphism. (Here Ψ^n means $\Psi \circ \Psi \circ \Psi \ldots \circ \Psi$ (n time)).

PROOF. We prove this result by using the principle of mathematical induction. We have to show that for any two elements a, b in Ꝣ

$$\Psi^n(\mathfrak{a} * \mathfrak{b}) = \Psi^n(\mathfrak{a}) * \Psi^n(\mathfrak{b}) \quad (1)$$

where n is the positive integer.

It has been given that statement (1) is valid for $n = 1$, and we assume that it is valid for $n = k$ then we have

$$\Psi^k(\mathfrak{a} * \mathfrak{b}) = \Psi^k(\mathfrak{a}) * \Psi^k(\mathfrak{b})$$

$$\Psi(\Psi^k(\mathfrak{a} * \mathfrak{b})) = \Psi(\Psi^k(\mathfrak{a}) * \Psi^k(\mathfrak{b}))$$

$$\Psi^{k+1}(\mathfrak{a} * \mathfrak{b}) = \Psi(\Psi^k(\mathfrak{a})) * \Psi(\Psi^k(\mathfrak{b})), \quad \because \Psi \text{ is homomorphism}$$

$$\Psi^{k+1}(\mathfrak{a} * \mathfrak{b}) = \Psi^{k+1}(\mathfrak{a}) * \Psi^{k+1}(\mathfrak{b})$$

Thus, the validity of statement (1) at $n = k$ implies the validity of (1) at $n = k + 1$. Hence, (1) holds for all positive integers 'n'.

Proposition 4.3. If $\Psi: Ꝣ \longrightarrow Ꝣ$ is a homomorphism, $\Psi^n(\mathfrak{a}) = \mathfrak{a}$ and n, p are positive integers such that n divides p then $\Psi^p(\mathfrak{a}) = \mathfrak{a}$.

PROOF. Since $\Psi^n(\mathfrak{a}) = \mathfrak{a}$ (2)

Where, n is a positive integer which divides p then there exists an integer q such that $p = nq$, then we have

$$\begin{aligned}
\Psi^p(\mathfrak{a}) &= \Psi^{nq}(\mathfrak{a}) = \Psi^{n(q-1)}(\Psi^n(\mathfrak{a})) \\
&= \Psi^{n(q-1)}(\mathfrak{a}) && \because \text{using (2)} \\
&= \Psi^{n(q-2)}(\Psi^n(\mathfrak{a})) \\
&= \Psi^{n(q-2)}(\mathfrak{a}) && \because \text{using (2)} \\
&\;\;\vdots \\
&= \Psi^{n(q-(q-1))}(\mathfrak{a}) = \Psi^{n(q-q+1)}(\mathfrak{a}) = \Psi^n(\mathfrak{a}) = \mathfrak{a} && \because \text{using (2)}
\end{aligned}$$

Hence $\Psi^p(\mathfrak{a}) = \mathfrak{a}$.

Proposition 4.4. In the discrete dynamical system $(Ꝣ, \Psi)$ if '$\mathfrak{b}$' is a fixed point in Ꝣ and $\mathfrak{a} \geq \mathfrak{b}$ for any $\mathfrak{a} \in Ꝣ$ then $\Psi(\mathfrak{a}) \geq \mathfrak{b}$.

PROOF. Since $\mathfrak{a} \geq \mathfrak{b}$ can also be written as $\mathfrak{b} \leq \mathfrak{a} \Rightarrow \mathfrak{b} * \mathfrak{a} = 0 \Rightarrow \Psi(\mathfrak{b} * \mathfrak{a}) = \Psi(0)$.

$$\Rightarrow \Psi(\mathfrak{b}) * \Psi(\mathfrak{a}) = 0 \quad (3)$$

Where '$\mathfrak{b}$' is a fixed point. Therefore, (3) becomes $\mathfrak{b} * \Psi(\mathfrak{a}) = 0 \Longrightarrow \mathfrak{b} \leq \Psi(\mathfrak{a})$ or $\Psi(\mathfrak{a}) \geq \mathfrak{b}$.

Proposition 4.5. In the discrete dynamical system $(\mathfrak{Z}, \Psi)$, if '$\mathfrak{b}$' is a fixed point in $\mathfrak{Z}$ and $\mathfrak{a} \geq \mathfrak{b}$ for any $\mathfrak{a} \in \mathfrak{Z}$, then $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$. Where '$n$' is a positive integer.

Proof. We prove this result by using the principle of mathematical induction, so by Proposition 4.4, the statement $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$ is true for $n = 1$. Next, we assume that the statement $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$ is true for $n = k$ such that

$$\Psi^{k}(\mathfrak{a}) \geq \mathfrak{b} \tag{4}$$

from equation (4), we get $\mathfrak{b} * \Psi^{k}(\mathfrak{a}) = 0 \Longrightarrow \Psi(\mathfrak{b} * \Psi^{k}(\mathfrak{a})) = \Psi(0)$

$$\Longrightarrow \Psi(\mathfrak{b}) * \Psi^{k+1}(\mathfrak{a}) = 0 \tag{5}$$

where '$\mathfrak{b}$' is a fixed point.

Therefore, equation (5) becomes

$$\mathfrak{b} * \Psi^{k+1}(\mathfrak{a}) = 0 \Longrightarrow \mathfrak{b} \leq \Psi^{k+1}(\mathfrak{a}) \text{ or } \Psi^{k+1}(\mathfrak{a}) \geq \mathfrak{b}$$

Thus, the truth of the statement $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$ at $n = k$ implies the truth of $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$ at $n = k + 1$ hence the statement $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$ is true for any positive integer 'n'.

5. Main Theorems

Theorem 5.1. In the discrete dynamical system $(\mathfrak{Z}, \Psi)$, the set of all fixed points in $\mathfrak{Z}$ is a BCK-subalgebra of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a BCK-algebra and a mapping $\Psi: \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is a homomorphism. Suppose that $\mathfrak{Z}_f$ be the set of all fixed points in $\mathfrak{Z}$. We show that $\mathfrak{Z}_f$ is a BCK-subalgebra for this $\mathfrak{Z}_f$ has to satisfy the conditions of BCK-subalgebra. Since Ψ is a homomorphism therefore by Proposition 4.1, we have $\Psi(0) = 0$, which implies that 0 is a fixed point $\Longrightarrow 0 \in \mathfrak{Z}_f \Longrightarrow \mathfrak{Z}_f$ is a non-vacuous set.

Thus, condition (s-i) of BCK-sub algebra holds in $\mathfrak{Z}_f$. Now assume that $a, \mathfrak{b} \in \mathfrak{Z}_f$

Then,

$$\Psi(\mathfrak{a}) = \mathfrak{a} \tag{6}$$

and

$$\Psi(\mathfrak{b}) = \mathfrak{b} \tag{7}$$

Since Ψ is a homomorphism, therefore we get

$$\Psi(\mathfrak{a} * \mathfrak{b}) = \Psi(\mathfrak{a}) * \Psi(\mathfrak{b}) \tag{8}$$

Using (6) and (7) in (8) we get

$$\Psi(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \Longrightarrow \mathfrak{a} * \mathfrak{b} \text{ is a fixed point} \Longrightarrow \mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_f$$

Thus, for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Z}_f$ we have $\mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_f$. Hence $\mathfrak{Z}_f$ is a BCK-subalgebra.

Theorem 5.2. In the discrete dynamical system $(\mathfrak{Z}, \Psi)$, the set of all periodic points in $\mathfrak{Z}$ is a BCK-subalgebra of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a BCK-algebra and a mapping $\Psi: \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is a homomorphism Suppose that $\mathfrak{Z}_p$ be the set of all periodic points in BCK-algebra. We show that $\mathfrak{Z}_p$ is a BCK-subalgebra for this $\mathfrak{Z}_p$ has to satisfy the conditions of Definition 2.3. Since Ψ is a homomorphism therefore by Proposition 4.1, we have $\Psi(0) = 0 \Longrightarrow$

0 is a periodic point of a period 1 $\Longrightarrow\ 0 \in \mathfrak{Z}_p \Longrightarrow \mathfrak{Z}_p \neq \{\ \}$. Thus, condition (s-i) of Definition 2.3 holds in $\mathfrak{Z}_p$.

Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Z}_p$ and suppose that the periods of 'a' and '$\mathfrak{b}$' are m and n respectively.

Such that

$$\Psi^m(\mathfrak{a}) = \mathfrak{a} \tag{9}$$

and

$$\Psi^n(\mathfrak{b}) = \mathfrak{b} \tag{10}$$

Here we take r =LCM$[m, n]$, then by Proposition 4.3, equations (9) and (10) become

$$\Psi^r(\mathfrak{a}) = \mathfrak{a} \tag{11}$$

and

$$\Psi^r(\mathfrak{b}) = \mathfrak{b} \tag{12}$$

Next, by Proposition 2.2, we get

$$\Psi^r(\mathfrak{a} * \mathfrak{b}) = \Psi^r(\mathfrak{a}) * \Psi^r(\mathfrak{b}) \tag{13}$$

Using the values of (11) and (12) on the right-hand side of (13), we get

$$\Psi^r(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \Longrightarrow \mathfrak{a} * \mathfrak{b} \text{ is periodic of period '}r\text{'} \Longrightarrow \mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_p$$

Thus, for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Z}_p$ we have $\mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_p$. Hence $\mathfrak{Z}_p$ is a BCK-subalgebra.

Theorem 5.3. Let $(\mathfrak{Z}, \Psi)$ is a discrete dynamical system, then the set of all fixed points in $\mathfrak{Z}$ is a strongly invariant (S-invariant) subset of BCK-algebra $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}_f$ be the set of all fixed points in $\mathfrak{Z}$.

Let $\Psi(\mathfrak{a}) \in \Psi(\mathfrak{Z}_f)$. Where '$\mathfrak{a}$' is any element of $\mathfrak{Z}$.

$\Longrightarrow \Psi(\mathfrak{a}) = \mathfrak{a}$, $\quad\because \Psi(\mathfrak{Z}_f)$ is a set of the images of all the fixed points

$\Longrightarrow \mathfrak{a} \in \mathfrak{Z}_f$

$\Longrightarrow \Psi(\mathfrak{a}) \in \mathfrak{Z}_f$, $\quad\because \Psi(\mathfrak{a}) = \mathfrak{a}$

Thus, $\Psi(\mathfrak{a}) \in \Psi(\mathfrak{Z}_f) \Longrightarrow \Psi(\mathfrak{a}) \in \mathfrak{Z}_f$

Therefore, we have

$$\Psi(\mathfrak{Z}_f) \subseteq \mathfrak{Z}_f \tag{14}$$

Now, we suppose that

$$\mathfrak{a} \in \mathfrak{Z}_f \Longrightarrow \Psi(\mathfrak{a}) = \mathfrak{a} \tag{15}$$

where

$$\Psi(\mathfrak{a}) \in \Psi(\mathfrak{Z}_f) \Longrightarrow \mathfrak{a} \in \Psi(\mathfrak{Z}_f), \qquad \because \text{using (15)}$$

Thus, $\mathfrak{a} \in \mathfrak{Z}_f \Longrightarrow \mathfrak{a} \in \Psi(\mathfrak{Z}_f)$. Therefore, we have

$$\mathfrak{Z}_f \subseteq \Psi(\mathfrak{Z}_f) \tag{16}$$

From (14) and (16), we have $\Psi(\mathfrak{Z}_f) = \mathfrak{Z}_f$. Hence, $\mathfrak{Z}_f$ in $\mathfrak{Z}$ is S-invariant or strongly invariant.

Theorem 5.4. In the discrete dynamical system $(ʐ, \Psi)$ if '$\mathfrak{a}$' is a fixed point and '$\mathfrak{b}$' is a periodic point in ʐ, then $\mathfrak{a} * \mathfrak{b}$ is a periodic point in ʐ.

PROOF. Since '$\mathfrak{a}$' is a fixed point, therefore we have

$$\Psi(\mathfrak{a}) = \mathfrak{a} \tag{17}$$

Let the period of the point 'b' is 'k' such that

$$\Psi^k(\mathfrak{b}) = \mathfrak{b} \tag{18}$$

As 'a' is a fixed point of period 1 and 1 divides 'k'; therefore, by Proposition 4.3, we have

$$\Psi^k(\mathfrak{a}) = \mathfrak{a} \tag{19}$$

Now by using the homomorphism property of Ψ^k we can get

$$\Psi^k(\mathfrak{a} * \mathfrak{b}) = \Psi^k(\mathfrak{a}) * \Psi^k(\mathfrak{b}) \tag{20}$$

Using (18) and (19) in (20), we get

$$\Psi^k(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{21}$$

Equation (5) implies that the period of $\mathfrak{a} * \mathfrak{b}$ is 'k' hence $\mathfrak{a} * \mathfrak{b}$ is a periodic point.

Theorem 5.5. In the discrete dynamical system $(ʐ, \Psi)$ if A is a subset of ʐ such that $\Psi(ʐ) \subset A \subset ʐ$, then A is invariant concerning Ψ.

PROOF. Since

$$\Psi(ʐ) \subset A \subset ʐ \tag{22}$$

From equation (22), we have

$$A \subset ʐ \tag{23}$$

and

$$\Psi(ʐ) \subset A \tag{24}$$

From (23), we can get

$$\Psi(A) \subset \Psi(ʐ) \tag{25}$$

From (24) and (25), we get

$$\Psi(A) \subset \Psi(ʐ) \subset A \tag{26}$$

Equation (26) implies that $\Psi(A) \subset A$. Hence A is an invariant subset of BCK-algebra ʐ.

Example 5.1. Consider the BCK-algebra $ʐ = \{0, p, q, r\}$ and a mapping $\Psi: ʐ \longrightarrow ʐ$ where the mapping is a homomorphism defined by $\Psi(0) = 0, \Psi(p) = 0, \Psi(q) = 0$, and $\Psi(r) = r$. Let the subset of ʐ is $A = \{0, p, r\}$ while $\Psi(ʐ) = \{0, r\}$. Then, it is clear that $\Psi(ʐ) \subset A \subset ʐ$. Hence A is an invariant set of ʐ.

Theorem 5.6. In the discrete dynamical system $(ʐ, \Psi)$, if ʐ is a commutative BCK-algebra and for any $\mathfrak{a} * \mathfrak{b}, \mathfrak{b} \in ʐ_f$ where $ʐ_f$ is the set of all fixed points in ʐ and $\mathfrak{a} \geq \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in ʐ$, then $ʐ_f$ is an ideal of ʐ.

PROOF. Since ʐ is a commutative BCK-algebra and a mapping $\Psi: ʐ \longrightarrow ʐ$ is a homomorphism. $ʐ_f$ is the set of all fixed points in ʐ. We show that $ʐ_f$ is an ideal of ʐ for this $ʐ_f$ has to satisfy the conditions of Definition 2.4. Since Ψ is a homomorphism therefore by Proposition 4.1, we have

$$\Psi(0) = 0 \implies 0 \in ʐ_f \implies ʐ_f \neq \{\ \}$$

Thus, the first condition of ideal holds in $ʐ_f$. Next, we have $\mathfrak{a} * \mathfrak{b} \in ʐ_f$ and $\mathfrak{b} \in ʐ_f$. Then,

$$\Psi\,(\mathfrak{a} * \mathfrak{b}) \;=\; \mathfrak{a} * \mathfrak{b} \tag{27}$$

and

$$\Psi(\mathfrak{b}) \;=\; \mathfrak{b} \tag{28}$$

We know that Ψ is a homomorphism, i.e., $\Psi\,(\mathfrak{a} * b) \;=\; \Psi(\mathfrak{a})\; * \Psi(b)$ so using this in (27) we get

$$\Psi\,(\mathfrak{a}) * \Psi(\mathfrak{b}) = \;\mathfrak{a} * \mathfrak{b} \tag{29}$$

Using (28) in (29), we get

$$\Psi\,(\mathfrak{a})\; * \mathfrak{b} = \;\mathfrak{a} * \mathfrak{b} \tag{30}$$

Since $\mathfrak{a} \;\geq\; \mathfrak{b}$ and by Proposition 4.4 $\Psi\,(\mathfrak{a}) \;\geq\; \mathfrak{b}$, then by Definition 2.6 from (30), we get

$$\Psi\,(\mathfrak{a}) \;=\; \mathfrak{a} \;\Longrightarrow\; \mathfrak{a} \;\in Ȥ_f$$

Thus, $\mathfrak{a} * \mathfrak{b} \;\in Ȥ_f$ and $\mathfrak{b} \;\in Ȥ_f \;\Longrightarrow \mathfrak{a} \in Ȥ_f$. Thus, the second condition of Definition 2.4 also holds in $Ȥ_f$. Hence $Ȥ_f$ is an ideal of commutative BCK-algebra $Ȥ$.

Theorem 5.7. In the discrete dynamical system $(Ȥ, \Psi)$ if $Ȥ$ is a commutative BCK-algebra and for any $\mathfrak{a} * \mathfrak{b}, \mathfrak{b} \;\in Ȥ_p$ where $Ȥ_p$ is a set of all periodic points in $Ȥ$ and $\mathfrak{a} \;\geq\; \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \;\in Ȥ$ then $Ȥ_p$ is an ideal of BCK-algebra $Ȥ$.

PROOF. Since $Ȥ$ is a BCK-algebra and a mapping $\Psi: Ȥ \;\longrightarrow Ȥ$ is a homomorphism. $Ȥ_p$ is a set of all periodic points in $Ȥ$. We show that $Ȥ_p$ is an ideal of $Ȥ$ for this $Ȥ_p$ has to satisfy the conditions of Definition 2.4. Since Ψ is a homomorphism therefore by Proposition 4.1, we have $\Psi(0) = 0$ that implies that 0 is a periodic point of period 1 $\;\Longrightarrow\;\; 0 \;\in Ȥ_p \;\;\Longrightarrow\;\; Ȥ_p \neq \{\;\}$. Thus, the first condition of ideal holds in $Ȥ_p$. Next, we have $\mathfrak{a} * \mathfrak{b} \in Ȥ_p$ and $\mathfrak{b} \in Ȥ_p$ and let their periods are ‘ m’ and ‘n’, respectively, such that

$$\Psi^{\,m}(\mathfrak{a} * \mathfrak{b}) \;= \mathfrak{a} * \mathfrak{b} \tag{31}$$

and

$$\Psi^{\,n}(\mathfrak{b}) \;= \mathfrak{b} \tag{32}$$

Here, we take $r \;= LCM\;[m, n]$, then by Proposition 4.3 we can get

$$\Psi^{\,r}(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{33}$$

and

$$\Psi^{\,r}(\mathfrak{b}) \;= \mathfrak{b} \tag{34}$$

By Proposition 4.2 equation (33) becomes

$$\Psi^{\,r}(\mathfrak{a})\; * \Psi^{\,r}(\mathfrak{b}) = \;\mathfrak{a} * \mathfrak{b}, \because\; \Psi^{r} \text{ is homomorphism} \tag{35}$$

Using equation (34) in (35), we get

$$\Psi^{\,r}(\mathfrak{a}) * \mathfrak{b} = \;\mathfrak{a} * \mathfrak{b} \tag{36}$$

Since $\mathfrak{a} \;\geq\; b$ and by Proposition 4.5 $\Psi^{\,r}(\mathfrak{a}) \;\geq\; \mathfrak{b}$, then by Definition 2.6 from (36), we get

$$\Psi^{\,r}(\mathfrak{a}) \;=\; \mathfrak{a} \Longrightarrow\; \mathfrak{a} \in Ȥ_p$$

Thus, $\mathfrak{a} * \mathfrak{b} \in Ȥ_p$ and $\mathfrak{b} \in Ȥ_p \;\Longrightarrow\; \mathfrak{a} \;\in Ȥ_p$. Thus, the second condition of Definition 2.4 also holds in $Ȥ_p$. Hence $Ȥ_p$ is an ideal of BCK-algebra $Ȥ$.

6. Conclusion

We see that a discrete dynamical system with unique properties plays a central role in investigating the structure of an algebraic system.

We have no doubt that the research along this line can be kept up, and indeed, some results in this manuscript have already made up a foundation for further exploration concerning the further progression of a discrete dynamical system in BCK-algebra and their applications in other disciplines of algebra. The forthcoming study of a discrete dynamical system in BCK-algebras may be the following topics are worth to be taken into account.

(i) To describe other classes of BCK-algebra by using this concept.

(ii) To refer this concept to some other algebraic structures.

(iii) To consider the results of this concept to some possible applications in information systems and computer sciences.

Conflict of Interest

The authors declare no conflict of interest.

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Journal of New Theory
ISSN: 2149-1402

35 (2021) 11-19
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
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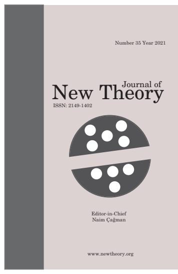

A Note on Urysohn's Lemma under $mI\alpha g$-Normal Spaces and $mI\alpha g$-Regular Spaces in Ideal Minimal Spaces

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Article History
Received: 26 Apr 2019
Accepted: 21 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract — This research article is concerned with the introduction of a new notion of normal spaces and regular spaces, namely $mI\alpha g$-normal spaces and $mI\alpha g$-regular spaces. We established their significant properties in ideal minimal spaces. Some equivalent conditions on $mI\alpha g$-normal spaces and $mI\alpha g$-regular spaces are proved. Urysohn's Lemma on $mI\alpha g$-normal spaces is also established.

Keywords — *$mI\alpha g$-closed sets, $mI\alpha g$-normal spaces, $mI\alpha g$-regular spaces, Urysohn's Lemma*

Mathematics Subject Classification (2020) — 54A05, 54C08

1. Introduction

The perception of ideals was initiated by Kuratowski [1] and Vaidyanathaswamy [2]. A subset I of the universal set X is said to be an ideal, if there exist two subsets A and B of X satisfying i) $A \in I$ and $B \subset A$ then $B \in I$ ii)$A, B \in I$ implies $A \cup B \in I$. The notion of minimal structure and minimal spaces were established by Maki et al. [3]. They have explained the minimal spaces as the generalisation of classical topological spaces. M is referred as the minimal structure of the space X, if $\phi, X \in M$. The spaces (X, M_X) is called as the minimal structure space.The introduction of m-open sets in minimal structures was initialised by Maki et al. [3]. The members of minimal structure are called m-open sets. Generalised closed sets (briefly g-closed sets)were introduced by Levine [4]. The notion of αm-open sets was introduced by Min [5]. The idea of m-normal spaces and mg-normal spaces were established by Noiri et al. [6]. m-regular spaces was elaborately studied by Popa et al. [7]. m-continuous functions and its salient features in minimal structures were instigated by Popa et al. [8]. Singha et al. [9] proved Urysohn's lemma in minimal structures. An innovative approach on ideals in minimal spaces was given by Özbakır et al. [10]. They have defined a new type of local function A^*_m named as minimal local function in ideal minimal spaces. The conception of mIg-normal spaces and its characterisations were well established by Haining et al. [11]. Also they have proved Urysohn's lemma under mIg-normal spaces ideal minimal spaces. A new notion of generalised closed sets, called $mI\alpha g$-closed sets and its features in ideal minimal spaces were studied by Parimala et al. [12]. Local closedness under $mI\alpha g$-closed sets and few separation axioms under $mI\alpha g$-closed sets are intended by Parimala et al. in [13,14]. In this article, few properties of separating sets are studied in ideal minimal spaces. Two new separations namely $mI\alpha g$-normal spaces and $mI\alpha g$-regular spaces are initiated and their significant properties are established. As an application of $mI\alpha g$-normal spaces, we have proved Urysohn's lemma on $mI\alpha g$-normal spaces.

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In the present study, Section 2 provides preliminary definitions in minimal structure spaces and in ideal minimal spaces, Section 3 $mI\alpha g$-normal spaces and its characterisations in ideal minimal spaces, Section 4 Urysohn's lemma under $mI\alpha g$-normal spaces, Section 5 $mI\alpha g$-regular spaces in ideal minimal spaces, and Section 6 conclusion and future work.

2. Preliminary

In the following sequel, the following notations are used.

(i) MSS- minimal structure spaces

(ii) IMS - ideal minimal spaces

Definition 2.1. [8] The interior and closure in an MSS are defined as follows. Let (X, M) be a MSS and $A \subset X$, then

(a) $m\text{-}int(A) = \cup\{K : K \subseteq A, K \in M\}$

(b) $m\text{-}cl(A) = \cap\{N : A \subseteq N, X - N \in M\}$

Proposition 2.2. [8] Properties of $m\text{-}cl$ and $m\text{-}int$ are listed below.

(a) $m\text{-}int(X) = X$ and $m\text{-}cl(\phi) = \phi$

(b) $m\text{-}int(A) \subseteq A$ and $A \subseteq m\text{-}cl(A)$

(c) If $A \in M$, then $m\text{-}int(A) = A$ and if $X - F \in M$, then $m\text{-}cl(F) = F$.

(d) If $A \subseteq B$, then $m\text{-}int(A) \subseteq m\text{-}int(B)$ and $m\text{-}cl(A) \subseteq m\text{-}cl(B)$.

Definition 2.3. [10] Let (X, M, I) be an IMS with an ideal I. The power set of X is denoted by $P(X)$. A mapping $(.)^*_m$ is defined from $P(X)$ to itself. For a subset $A \subset X$, the minimum local function is $A^*_m(I, M) = \{x \in X : U_m \cap A \notin I;$ for every $U_m \in U_{m(x)}\}$. The minimal $*$-closure operator $m\text{-}cl^*$ is defined as $m\text{-}cl^*(A) = A \cup A^*_m$. A minimal structure via $m\text{-}cl^*$ is termed as $M^*(I, M)$ (briefly M^*) and is described as $M^* = \{U \subset X : m\text{-}cl(X - U) = X - U\}$. The members of $M^*(I, M)$ are termed as m^*-open sets. The interior of m^*-open sets is denoted by $m\text{-}int^*$.

Theorem 2.4. [10] In an MSS (X, M), let I, J be two ideals on X. $P, Q \subset X$. Then,

(a) $P \subset Q \Rightarrow P^*_m \subset Q^*_m$

(b) $P^*_m \cup Q^*_m \subset (P \cup Q)^*_m$

(c) $(P^*_m)^*_m \subset P^*_m$

(d) $P^*_m = m\text{-}cl(P^*_m) \subset m\text{-}cl(P)$

(e) $I \subset J \Rightarrow P^*_m(J) \subset P^*_m(I)$

Remark 2.5. [10] The MSS (X, M) is said to exhibit the property $[U]$ if the union of any number of m-open sets is an m-open set and the property $[I]$ if the intersection of finite number of m-open sets is an m-open set.

Remark 2.6. [10] If (X, M) has the property $[U]$, then (b) of Theorem 2.4. can be stated as $P^*_m \cup Q^*_m = (P \cup Q)^*_m$.

Proposition 2.7. [10] Significant features of $m\text{-}cl^*$ are as follows. Let $P_1, P_2 \subseteq X$. Then,

(a) $m\text{-}cl^*(P_1) \cup m\text{-}cl^*(P_2) \subseteq m\text{-}cl^*(P_1 \cup P_2)$

(b) If $P_1 \subseteq P_2$, then $m\text{-}cl^*(P_1) \subseteq m\text{-}cl^*(P_2)$.

(c) When $A \subseteq X$, then $A \subseteq m\text{-}cl^*(A)$.

(d) $m\text{-}cl^*(\phi) = \phi$ and $m\text{-}cl^*(X) = X$

Definition 2.8. In an MSS (X, M, I), let A be a non empty subset of X. A is defined to be an

(a) m^*-closed set [10] if A_m^* is a subset of A. $(A_m^* \subset A)$.

(b) minimal generalised closed set (mg-closed) [3] if $m\text{-}cl(A) \subseteq U$, $A \subseteq U$ and U is an m-open set.

(c) minimal α-open set (αm-open set) [5] if $A \subseteq m\text{-}int(m\text{-}cl(m\text{-}int(A)))$. The complement of αm-open set is called an αm-closed set.

(d) minimal ideal α generalised closed set ($mI\alpha g$-closed set) [12] if $A_m^* \subseteq U$ whenever $A \subseteq U$ and U is an αm-open set.

Proposition 2.9. [5] In an MSS (X, M_X) if a subset A is an m-open set, then it is an αm-open set.

Definition 2.10. [5] α-closure and α-interior of a set A are defined as follows:

(a) $\alpha m\text{-}cl(A) = \cap\{F : A \subseteq F, \text{F is } \alpha m\text{-closed in } X\}$

(b) $\alpha m\text{-}int(A) = \cup\{U : U \subseteq A, U \text{ is } \alpha m\text{-open in } X\}$

Let (X, M, I) be an IMS, then we have the following theorems.

Proposition 2.11. [12] When $I = \{\phi\}$, then an $mI\alpha g$-closed set ($mI\alpha g$-open set) is an mg-closed set (mg-open set).

Proposition 2.12. [12] An m^*-closed set in an IMS is an $mI\alpha g$-closed set.

Theorem 2.13. [12] The necessary and sufficient condition for a subset to be an $mI\alpha g$-closed set in (X, M, I) is that every αm open set in X is an m^*-closed set.

Theorem 2.14. [12] Theorem 3.4(d), in an IMS X a subset A is an $mI\alpha g$-closed set if and only if every m open set is an m^*-closed set.

Proof. Obvious, since every m open set is an αm open set. □

Theorem 2.15. [12] Consider $A \subseteq X$, then A is $mI\alpha g$-open if $S \subseteq m\text{-}int^*(A)$, S is αm-closed and $S \subseteq A$. Sufficiency is also true.

Definition 2.16. [6] An MSS (X, M) is called m-normal (resp.mg-normal) if for every pair of m-closed subsets (resp. mg-closed subsets) A and B such that $A \cap B = \phi$, there exists m-open sets U and V such that $U \cap V = \phi$ and $A \subset U$, $B \subset V$.

Definition 2.17. [7] An MSS (X, M) is termed to be a m-regular space if for every m-closed set F and an element $x \notin F$, there are m-open sets U and V such that $x \in U$, $F \subseteq V$ and also $U \cap V = \phi$.

Definition 2.18. [15] A m-T_1 space we mean, for all distinct points $x_1, x_2 \in X$ there exists an m-open set X such that $x_1 \in X$, but $x_2 \notin X$ and an m-open set Y such that $x_1 \notin Y$, $x_2 \in Y$.

Theorem 2.19. [7] Consider an m-T_1 space (X, M, I) with $I = \{\phi\}$ then the following statements below given are equivalent.

(a) (X, M, I) is an m-regular space.

(b) For every m-open set V such that $x \in X$, there exists an m-open set U of X satisfying $x \in U \subseteq m\text{-}cl(U) \subseteq V$.

Proposition 2.20. [16] Every m-closed set is an $mI\alpha g$-closed set. (Every m-open set is an $mI\alpha g$-open set.)

Proposition 2.21. [16] Every mg-closed set is an $mI\alpha g$-closed set.

Definition 2.22. [8] Let (X, M_X) and (Y, M_Y) be two MSS. The function $f : (X, M_X) \to (Y, M_Y)$ is defined to be an m-continuous function, if for $x \in X$ and $V \in M(f(x))$, there exist $U \in M(x)$ satisfying $f(U) \subseteq V$.

3. $mI\alpha g$-Normal Spaces

Definition 3.1. $mI\alpha g$-normal space we mean, if for every pair of $mI\alpha g$ closed sets K_1, K_2 such that $K_1 \cap K_2 = \phi$, there exists at least a pair of m-open sets U and V of X such that $U \cap V = \phi$ satisfying $A \subseteq U$ and $B \subseteq V$.

Theorem 3.2. An $mI\alpha g$-normal space is an m-normal space (mg-normal space).

PROOF. Obvious, since every m-closed set(mg-closed set) is an $mI\alpha g$ set with references to Proposition 2.20., and Proposition 2.21. The example given below shows that the converse of the above theorem is not true. □

Example 3.3. (X, M, I) be an IMS with $X = \{a, b, c, d\}$, $M = \{\phi, X, \{b\}, \{a, b\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{c\}, \{a, c\}\}$ and $M^c = \{X, \phi, \{a, c, d\}, \{c, d\}, \{b\}\}$. Here X is an m-normal space. Since for the disjoint $mI\alpha g$-closed sets $\{a\}$ and $\{c\}$ there does not exists disjoint m open sets containing them, X is not an $mI\alpha g$-normal space.

Theorem 3.4. In an IMS (X, M, I) the equivalent statements on $mI\alpha g$-normal-spaces are given below.

(a) (X, M, I) is an $mI\alpha g$-normal space.

(b) For each $mI\alpha g$-closed set K and an $mI\alpha g$-open set F such that $K \subseteq F$, there exists an m-open set $U \subseteq X$ such that $K \subseteq U \subseteq m\text{-}cl(U) \subseteq F$.

PROOF.
(a) ⇒ (b): Assume K be an $mI\alpha g$-closed set and F be an $mI\alpha g$-open set such that $K \subset F$. Then $X - F$ is an $mI\alpha g$-closed set. Therefore $K \cap (X - F) = \phi$. Referring the hypothesis (a), for a pair of disjoint m-open sets U and V such that $K \subseteq U$ and $X - F \subseteq V$ and $U \cap V = \phi$. But $U \subseteq (X - V)$ implies $m\text{-}cl(U) \subseteq (X - V)$. Hence $K \subseteq U \subseteq m\text{-}cl(U) \subseteq (X - V) \subseteq F$ which proves (b).

(b) ⇒ (a): Let K and F be two disjoint $mI\alpha g$-closed sets such that $K \subseteq (X - F)$. Hypothesis (b) of this theorem infers the existence of the m-open subset U of X such that $K \subseteq U \subseteq m\text{-}cl(U) \subseteq (X - F)$. Let $V = X - m\text{-}cl(U)$, since $m\text{-}cl(U)$ is a m-closed set V is m-open. These U and V are the m-open sets which contains K and F which proves (a). □

Corollary 3.5. In an IMS (X, M, I) the statements below given are equivalent.

(a) (X, M, I) is an $mI\alpha g$-normal space.

(b) For every $mI\alpha g$-closed set A and $mI\alpha g$-open set B such that $A \subseteq B$, there exists an αm-open set $U \subseteq X$ satisfies $A \subseteq U \subseteq \alpha m\text{-}cl(U) \subseteq B$).

PROOF. By referring Proposition 2.9., every m-open set is an αm-open set. Apply this result in Theorem 3.4., the proof follows. □

Corollary 3.6. In an IMS (X, M, I) the following statements are equivalent on mg-normal spaces when $I = \{\phi\}$.

(a) Consider (X, M, I) be an mg-normal space

(b) For a pair of mg-closed set A and an mg-open set B such that $A \subseteq B$, there exists an m-open set $U \subseteq X$ satisfies $A \subseteq U \subseteq m\text{-}cl(U) \subseteq B$.

PROOF. When $I = \{\phi\}$, Proposition 2.11., infers that every $mI\alpha g$-open set is an mg-open set. Apply this result in Theorem 3.4., the proof follows. □

Theorem 3.7. In an IMS (X, M, I) the following statements are equivalent.

(a) (X, M, I) is an $mI\alpha g$-normal space.

(b) For every pair of $mI\alpha g$ closed subsets A and B of X, there corresponds an m-open set U of X satisfies $A \subseteq U$, then $m\text{-}cl(U) \cap B = \phi$.

(c) For every pair of $mI\alpha g$-closed subsets A and B such that $A \cap B = \phi$, there corresponds an m-open set U satisfying $A \subseteq U$ and an m-open set V satisfying $B \subseteq V$ then $m\text{-}cl(U) \cap m\text{-}cl(V)$ is an empty set.

PROOF.
(a) ⇒ (b): Consider a pair of $mI\alpha g$-closed subsets A, B such that $A \cap B = \phi$ then $A \subseteq (X - B)$ where $X - B$ is an $mI\alpha g$-open set. Referring Theorem 3.4., there corresponds an m-open set U such that $A \subseteq U \subseteq m\text{-}cl(U) \subseteq X - B$. Therefore, $m\text{-}cl(U)$ and B are disjoint sets. Hence, U is the required m-open set that satisfies (b).

(b) ⇒ (c): Hypothesis (b) of this theorem implies that $m\text{-}cl(U)$ and B are disjoint $mI\alpha g$-closed subsets of X. Therefore, there exists an m-open set V containing B such that $m\text{-}cl(U) \cap m\text{-}cl(V) = \phi$ which proves (c).

(c) ⇒ (a): Hypothesis (c) proves (a). □

Corollary 3.8. In an IMS (X, M, I) the statements given below are equivalent when $I = \{\phi\}$.

(a) The IMS X is an mg-normal space.

(b) All pairs of subsets of X consisting mg closed sets A, B there corresponds an m-open set U of X such that $A \subseteq U$, then $m\text{-}cl(U)$ and B are disjoint sets.

(c) Every pair of mg-closed sets A, B of X such that $A \cap B = \phi$ there corresponds an m-open set U such that $A \subseteq U$ and an m-open set V such that $B \subseteq V$ then $m\text{-}cl(U)$ and $m\text{-}cl(V)$ are disjoint sets.

PROOF. When $I = \{\phi\}$, every $mI\alpha g$-open set is an mg-open set by Proposition 2.11. Apply this result in Theorem 3.7., we get the proof. □

Theorem 3.9. Let (X, M, I) be an $mI\alpha g$-normal space. If A and B are $mI\alpha g$-closed sets that containing no common elements, then there exists a pair of m-open sets U and V such that $U \cap V = \phi$ and satisfies $m\text{-}cl^*(A) \subseteq U$ and $m\text{-}cl^*(B) \subseteq V$.

PROOF. Consider a pair of $mI\alpha g$-closed sets A and B. Referring Theorem 3.7 (3)., there exist m-open sets U and V such that $A \subset U$ and $B \subset V$ satisfying $m\text{-}cl(U) \cap m\text{-}cl(V) = \phi$. As A is an $mI\alpha g$-closed set, we have $A^*_m \subseteq U$ and also $A \subseteq U$. Therefore, $A \cup A^*_m = m\text{-}cl^*(A) \subseteq U$. Similarly, $m\text{-}cl^*(B) \subseteq V$. □

Corollary 3.10. Let (X, M, I) be an $mI\alpha g$-normal space with $I = \{\phi\}$ and A and B are mg-closed sets of X and $A \cap B = \phi$, then there are disjoint m-open sets U and V such that $m\text{-}cl^*(A)$ is contained in U and $m\text{-}cl^*(B)$is contained in V.

PROOF. When $I = \{\phi\}$, referring Proposition 2.11., we know that every $mI\alpha g$-open set is an mg-open set. Apply this result in Theorem 3.9., we get the proof. □

Theorem 3.11. Let (X, M, I) be an $mI\alpha g$-normal space. If A and B are $mI\alpha g$-closed and $mI\alpha g$-open sets respectively and also $A \subset B$, then there corresponds an m-open set U such that $A \subseteq m\text{-}cl^*(A) \subseteq U \subseteq m\text{-}int^*(B) \subseteq B$.

PROOF. Suppose that A is an $mI\alpha g$-closed set and B is an $mI\alpha g$-open set such that $A \subseteq B$. Then, $A \cap (X - B) = \phi$. That is, A and $X - B$ are disjoint $mI\alpha g$-closed sets. Referring Theorem 3.9., there exist disjoint m-open sets K_1 and K_2 such that $m\text{-}cl^*(A) \subseteq K_1$ and $m\text{-}cl^*(X - B) \subseteq K_2$. Also, $X - (m\text{-}int^*(B)) = m\text{-}cl^*(X - B) \subseteq K_2$. So $m\text{-}cl^*(X - B) \subseteq K_2$ implies that $X - K_2 \subseteq m\text{-}int^*(B)$. Also, since K_1 and K_2 are disjoint m-open sets, we get $A \subseteq m\text{-}cl^*(A) \subseteq K_1 \subseteq (X - K_2) \subseteq m\text{-}int^*(B) \subseteq B$. □

Corollary 3.12. Let (X, M, I) be an mg-normal space and $I = \{\phi\}$. For each mg-closed subset A and an mg-open subset U containing A there exists an m-open subset V such that $A \subseteq m\text{-}cl^*(A) \subseteq V \subseteq m\text{-}int^*(U) \subseteq U$.

Proof. Let $I = \{\phi\}$. With reference to Proposition 2.11., every $mI\alpha g$-open set is an mg-open set. Apply this result in Theorem 3.11., we get the proof. □

4. Urysohn's Lemma on $mI\alpha g$-Normal Spaces

Theorem 4.1. The necessary and sufficient condition for an IMS (X, M, I) to be an $mI\alpha g$-normal space is that, for every pair of $mI\alpha g$-closed sets A and B and $A \cap B = \phi$, it is possible to define a m-continuous mapping $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof.

Necessary Part: Consider a $mI\alpha g$-normal space (X, M, I) and let $A, B \subset X$ be a pair of $mI\alpha g$-closed sets such that $A \cap B = \phi$. As B is an $mI\alpha g$-closed set, $X - B$ is $mI\alpha g$-closed and also $A \subset (X - B)$. Here, A is an $mI\alpha g$-closed set and $X - B$ is an $mI\alpha g$-open set in X. Referring Theorem 3.4., there exists an m-open set namely $U_{1/2}$ satisfies $A \subseteq U_{1/2} \subseteq m\text{-}cl(U_{1/2}) \subseteq (X - B)$. With reference to Theorem 2.19., $U_{1/2}$ is a $mI\alpha g$-open set. That is, $U_{1/2}$ and $X - B$ are the $mI\alpha g$-open sets such that $A \subseteq U_{1/2}$ and $m\text{-}cl(U_{1/2}) \subseteq (X - B)$, where A and $m\text{-}cl(U_{1/2})$ are $mI\alpha g$-closed sets. Therefore, with reference to Theorem 3.4., there exist m-open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subseteq U_{1/4} \subseteq m\text{-}cl(U_{1/4}) \subseteq U_{1/2} \quad \text{and} \quad m\text{-}cl(U_{1/2}) \subseteq U_{3/4} \subseteq m\text{-}cl(U_{3/4}) \subseteq (X - B)$$

Continuing in this way, for every rational number in the open interval $(0, 1)$ of the form $t = \frac{m}{2^n}, n = 1, 2, 3, ...$ and $m = 1, 2, 3, ...2^{n-1}$, we obtain m-open sets of the form U_t such that for each $s < t$,

$$A \subseteq U_s \subseteq m\text{-}cl(U_s) \subseteq U_t \subseteq m\text{-}cl(U_t) \subseteq (X - B)$$

Let us denote the set of all rational number by $\mathcal{Q}$. Also, $Q(x) = \{t : t \in \mathcal{Q}$ and $x \in U_t\}$, this set contains no number less than 0, since no x is in U_t for $t < 0$ and it contains every number greater than 1. Let us define $f : X \to [0, 1]$ as $f(x) = 1$, if $x \notin U_t$ and $f(x) = inf\{t : t \in \mathcal{Q}$ and $x \in U_t\}$. For each $x \in B$, $x \notin X - B$ implies $x \notin U_t$. Therefore, $f(B) = \{1\}$. For each $x \in A$, $x \in U_t$ and $t \in \mathcal{Q}$. By definition $f(x) = inf\{t : t \in \mathcal{Q}$ and $x \in U_t\} = inf\mathcal{Q} = 0$. Hence, $f(A) = 0$. To prove f is an m-continuous mapping, let the intervals of the form $[0, a)$ and $(b, 1]$ where $a, b \in (0, 1)$ forms an open subbase in the space $[0, 1]$. Therefore our aim is to prove that $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are m-open sets in X. To prove $f^{-1}([0, a))$ is an m-open set in X. Let $x \in U_t$ for some $t < a$, then by definition $f(x) = inf\{s : s \in \mathcal{Q}$ and $x \in U_s\} = r \leq t < a$. That is, $f(x) < a$. Thus $0 \leq f(x) < a$. Conversely, if $f(x) = 0$, then $x \in U_t$ for all $t \in \mathcal{Q}$, hence $x \in U_t$ for some $t < a$. If $0 < f(x) < a$, by definition of we have $f(x) = \{s : s \in \mathcal{Q}$ and $x \in U_t\} < a$. Since $a < 1$ we get $f(x) = t$ for some $t < a$ and hence $x \in U_t$ for some $t < a$. Therefore, we conclude that $0 \leq f(x) < a$ if and only if $x \in U_t$ for some $t < a$. Hence, $f^{-1}([0, a)) = \cup\{U_t; t \in \mathcal{Q}$ and $x \in U_t\}$ which is an m-open set of X. To prove $f^{-1}((b, 1])$ is an m-open set in X. We need to prove $X - f^{-1}([0, b])$ is an m-open subset of X. For that we have to prove $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$ to get union of m-open subsets U_t. Let $x \in X$ such that $0 \leq f(x) \leq b$ when $t > b$, It is evident that $f(x) < t$ implies $x \in U_t$ for $t > b$. Conversely, let $x \in U_t$ for all $t > b$, then by definition $f(x) = inf\{t : t \in \psi$ and $x \in U_t\} \leq t$. Since $t > b$, $f(x) \leq b$ for all $t > b$. From the definition of f, it is clear that $f(x) \geq 0$. Therefore, we get $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$. Also, $t > b$ implies that there is $r \in \mathcal{Q}$ such that $t > r > b$. then $m - cl(U_t) \subseteq U_t$. Consequently, we have $\cap\{U_t; t \in \mathcal{Q}$ and $t > b\}$=$\cap\{m\text{-}cl(U_t); r \in \mathcal{Q}$ and $r > b\}$. Therefore, $f^{-1}([0, b]) = \{x : 0 \leq f(x) \leq b\} = \cap\{U_t; t \in \mathcal{Q}$ and $t > b\}$=$\cap\{m\text{-}cl(U_t; r \in \psi$ and $r > b)\}$. Since, $f^{-1}((0, 1]) = f^{-1}(X - ([0, b]))$=$X - f^{-1}([0, b]) = \cup\{X - m\text{-}cl(U_t); r \in \mathcal{Q}$ and $r > b\}$, which is m-open in X. Therefore, $f : X \to [0, 1]$ is m-continuous.

Sufficient Part: Consider a pair of $mI\alpha g$-closed sets A and B such that $A \cap B = \phi$. Referring the sufficient condition, there exists an m-continuous mapping $f : X \to [0, 1]$ satisfying $f(A) = \{0\}$ and $f(B) = \{1\}$. Moreover, $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$ are disjoint m-open subsets of X. Clearly $A \subset U$ and $B \subset V$. Hence, X is an $mI\alpha g$-normal space. □

5. $mI\alpha g$-Regular Spaces

Definition 5.1. An IMS (X, M, I) is referred to be an $mI\alpha g$-regular space, if for every pair consisting a point $x \in X$ and an m-closed set B such that $x \notin B$ there exists at least one pair of $mI\alpha g$-open sets U and V with $U \cap V = \phi$ satisfying $x \in U$ and $B \subset V$.

Example 5.2. Consider an IMS (X, M, I) with $X = \{a, b, c\}$, $M = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ $M^c = \{X, \phi, \{a, c\}, \{c\}, \{a\}\}$ and the ideal $I = \{\phi, \{b\}\}$. Here $mI\alpha g$ closed sets are the elements of the power set $P(X)$ and X is $mI\alpha g$-regular.

Theorem 5.3. If (X, M, I) is an $mI\alpha g$-regular space, then it is an m-regular space, but the converse of this theorem may not be true.

PROOF. Obvious, since every m-closed set is an $mI\alpha g$-closed set. □

Example 5.4. In Example 4.2., X is an $mI\alpha g$-regular space, but not a m-regular space, since for the point $x = a \in X$ and an m-closed set $B = \{c\}$, there does not exist m-open sets containing x and B.

Theorem 5.5. In an IMS (X, M, I), if every m-open set is m^*-closed, then the minimal space is an $mI\alpha g$-regular space.

PROOF. Suppose every m-open subset of X is m^*-closed, then by Theorem 2.11., every subset of X is an $mI\alpha g$-open set. If B is an m-closed set such that $x \notin B$, then $\{x\}$ and B are the two $mI\alpha g$-open sets such that $\{x\} \cap B = \phi$ and also $x \in \{x\}$ and $B \subseteq B$. Therefore, X is an $mI\alpha g$-regular space. □

Definition 5.6. A subset K of (X, M, I) is termed as an $mI\alpha g$-neighbourhood of $B \subseteq X$, if there exists an $mI\alpha g$-open set U such that $B \subseteq U \subseteq K$.

Definition 5.7. A subset K of (X, M, I) is termed to be an $mI\alpha g$-closed neighbourhood of $B \subseteq X$, if there exists an $mI\alpha g$-closed set U such that $B \subseteq U \subseteq K$.

Theorem 5.8. In an IMS (X, M, I) the following are statements equivalent.

(a) (X, M, I) is an $mI\alpha g$-regular space.

(b) For each m-open se)t U and let $x \in U$, there corresponds an $mI\alpha g$-open set V satisfying $x \in V \subseteq m\text{-}cl^*(V) \subset U$.

(c) For each m-closed set A, $\cap A_i = A$ where A_i are $mI\alpha g$-closed neighbourhoods of A.

(d) For any set A and an m-open set B such that $A \cap B$ contains at least one element, there exists an $mI\alpha g$-open set U such that $A \cap U \neq \phi$ and $m\text{-}cl^*(U) \subseteq B$.

(e) For any non empty set A and an m-closed set B such that A and B are disjoint, there exists atleast a pair of $mI\alpha g$-open sets U, V satisfies $A \cap U \neq \phi$ and $B \subseteq V$.

PROOF.
(a) ⇒ (b): Consider an m-open set V and let $x \in V$. Hence $X - V$ is m-closed such that $x \notin (X - V)$. Since X is an $mI\alpha g$-regular space, there exists a pair of $mI\alpha g$-open sets U and W such that $U \cap W = \phi$ satisfying $x \in U$ and $X - V \subseteq W$. Observing Theorem 2.8., $X - V$ is αm closed. Theorem 2.14., infers that $X - V \subseteq m\text{-}int^*(W)$. Therefore, $X - (m\text{-}int^*(W)) \subseteq V$. Hence, $U \cap W = \phi$ implies $U \cap m\text{-}int^*(W) = \phi$ and so $m\text{-}cl^*(U) \subseteq X - (m\text{-}int^*(W))$. Which implies $x \in U \subseteq m\text{-}cl^*(U) \subseteq V$.

(b) ⇒ (c): Let A be an m-closed set and $x \notin A$ then $X - A$ is an m-open set containing x. By hypothesis (b), there exists an $mI\alpha g$-open set V satisfying $x \in V \subseteq m\text{-}cl^*(V) \subseteq (X - A)$. Thus, $A \subseteq X - (m\text{-}cl^*(V)) \subseteq (X - V)$. Since $X - (m\text{-}cl^*(V))$ is $mI\alpha g$-open, we get $X - V$ is $mI\alpha g$-closed neighbourhood of A and $x \notin (X - V)$. This shows that A is the intersection of all $mI\alpha g$ neighbourhood of A.

(c) ⇒ (d): Assume a non empty set A and an m-closed set B such that $A \cap B \neq \phi$. Consider an element x of $A \cap B$ then, $X - B$ is m-closed and $x \notin (X - B)$. Observing the hypothesis (c), there exists an $mI\alpha g$-closed neighbourhood V of $X - B$ such that, $x \notin V$. Let $(X - V) \subseteq G \subseteq V$ and G be an $mI\alpha g$-open then, $U = (X - V)$ is an $mI\alpha g$-open set satisfying $x \in U$ and $A \cap U \neq \phi$. Further, $X - G$ is $mI\alpha g$-closed and $m\text{-}cl^*(U) = m\text{-}cl^*(X - V) \subseteq m\text{-}cl^*(X - G) \subseteq B$.

(d) ⇒ (e): Consider a non empty set A and m-closed set B, $A \cap B$ contains no element. $X - B$ is an m-open set and so $A \cap (X - B) \neq \phi$. Observing hypothesis (d), there exists an $mI\alpha g$-open set U such that the sets A and U contains at least one common element. Also, $U \subseteq m\text{-}cl^*(U) \subseteq (X - B)$. Assume that $V = X - (m\text{-}cl^*(U))$. Then, U and V are $mI\alpha g$-open sets satisfying $B \subseteq X - (m\text{-}cl^*(U) = V \subseteq (X - U))$ which implies (e).

(e) ⇒ (a): Let A be an m-closed set and $x \notin A$. Let the set $B = \{x\}$. Then there exist disjoint $mI\alpha g$-open sets U and V such that $\{x\} \cap U = B \cap U \neq \phi$ and $A \subseteq V$. Thus, $x \in U$. □

Definition 5.9. A subset K of (X, M, I) is referred as an mg-neighbourhood of set $B \subseteq X$, if there exists an mg-open set U such that $B \subseteq U \subseteq K$.

Definition 5.10. A subset K of (X, M, I) is referred as an mg-closed neighbourhood of set $B \subseteq X$, if there exists an mg-closed set U such that $B \subseteq U \subset K$.

Corollary 5.11. Let (X, M, I) be an IMS such that $I = \{\phi\}$. Then, the following statements on mg-regular spaces are equivalent.

(a) (X, M, I) is an mg-regular space.

(b) Let U be an m-open set containing x, then there exists an mg-open set V satisfying $x \in V \subseteq m\text{-}cl^*(V) \subseteq U$.

(c) For any m-closed set A, $\cap A_i = A$ where A_i are mg-closed neighbourhoods of A.

(d) For any set A and an m-open set B, $A \cap B$ is non empty, then there exists an mg-open set U such that $A \cap U \neq \phi$ and $m\text{-}cl^*(U) \subseteq B$.

(e) For any non empty set A and an m-closed set B such that A and B are disjoint, then there exists disjoint mg-open sets U, V satisfies $A \cap U$ is non empty and $B \subseteq V$

Proof. When $I = \{\phi\}$, observing Theorem 2.10., we have inferred that every $mI\alpha g$-open set is mg-open set. Apply this result in Theorem 5.8., the proof follows. □

If $I = \{\phi\}$ in Theorem 2.18., then we have the following Corollary.

Corollary 5.12. If (X, M, I) is an m-T_1 space with $I = \{\phi\}$ then the statements given below are equivalent.

(a) (X, M, I) is an m-regular space.

(b) Consider an m-open set V and let $x \in X$, there exists an $mI\alpha g$-open set U of X such that $x \in U \subseteq m\text{-}cl(U) \subseteq V$.

Proof. Observing Theorem 2.19., every m-closed set is an $mI\alpha g$-closed set, the proof is obvious by Theorem 2.18. □

6. Conclusion

In this work, we discussed about two separations called $mI\alpha g$-normal and $mI\alpha g$-regular spaces in ideal minimal spaces. The famous separation lemma called Urysohn's has been proved under $mI\alpha g$-normal spaces. Few equivalent statements on $mI\alpha g$-normal and $mI\alpha g$-regular spaces were established. In future, this work will be extended to discuss about Tietze extension theorem and Hausdorff spaces in ideal minimal spaces.

Conflicts of Interest

The authors declare no conflict of interest.

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New Theory Journal of
ISSN: 2149-1402

35 (2021) 20-31
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Fuzziness on Hopf Algebraic Structures with Its Application

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Article History
Received: 01 May 2019
Accepted: 25 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract − The objective of writing this manuscript is to apply the concept of fuzzy set on some basic Hopf algebraic structures. In this manuscript, the novel concepts of fuzzy Hopf subalgebra, fuzzy Hopf ideal, and fuzzy H-submodule are proposed. Some properties of these concepts are discussed, and some significant results are also proved in it. The advantages of the proposed work are also studied in it. The application of the proposed work is also discussed in it.

Keywords − *Fuzzy Hopf subalgebra, fuzzy Hopf ideal, fuzzy H-submodule*

Mathematics Subject Classification (2020) − 03E72, 18M80

1. Introduction

Heinz Hopf did much work in the field of the algebraic topology of Lie groups [1,2]. Later on, mathematicians worked on it and named it after Heinz Hopf as Hopf algebra (HA) [3-5]. Vast applications of HA are in physics, quantum groups, non-commutative geometry and representation theory etc. [6,7], e.g., a particle move in space-time has HA structures [8]. So, it is necessary to extend these concepts in uncertainty. In 1969, Sweedler [9] wrote the first book on HA. After that, much research was done in this algebra [10,11].

HA is an algebra with dual structure coalgebra and has an endomorphism called antipode. In other words, we can also say that an algebra with the structures of cohomology and homology of a topological group is called HA. In 1939, such algebras were introduced and linked with Lie groups. An example of HA is the Steenrod algebra introduced in the 1960s by Milnor and Moore [12], the cohomology algebra.

Zadeh gave the approach to fuzzy sets (FS) in 1965 [13]. Rosenfeld worked on the notion of fuzzy groups [14]. Basically, in fuzzy sets, the range of membership degree is $[0,1]$, which tells us up to which degree the element belongs to a set. The fuzzy set theory deals with uncertainties. Fuzzy set theory is the door to developing an intelligent system for identification and decision making etc. There are vast applications of fuzzy sets in decision-making, pattern recognition, control theory, and optimization. A fuzzy set is also known as a model that represents uncertainty in the universe.

The thought of fuzzy submodule was given by Zahedi and Ameri [15]. Many authors used the concept of fuzzy submodules in different fields of mathematics and physics [16-18]. The thought of fuzzy subcomodule was offered by Chen and Akram [19] in 2012. Notions of comodule and coalgebra are generalizations and dualizations of module and algebra, respectively.

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The thought of fuzzy algebra was offered by Abdulkhalikov et al. [20] in 1988. The thought of fuzzy subcoalgebra was given by Chen [21] in 2009, and the thought of fuzzy subbialgebra was also given by Chen and Wenxu [22] in 2012. The concept of fuzziness was also applied in some other algebras in [23-25].

The main problem in mathematics containing uncertainty is how to carry out the ordinary concepts to the uncertainty case. The proposed work will help in dealing with uncertainty problems in HA and quantum groups. Our obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, expert systems, medical diagnosis, and engineering.

In this study, we introduce the concepts of fuzzy Hopf subalgebra (FHA), fuzzy Hopf ideal (FHI) and fuzzy H-submodule (FHM). Some basic and essential definitions that are helpful in this paper are recalled in Section 2. The notion of fuzzy Hopf subalgebra is introduced in Section 3. In this section, we also investigate some results and examples about fuzzy Hopf subalgebra. The concept of fuzzy Hopf ideal is introduced in Section 4. In this section, we also investigate some examples and results about fuzzy Hopf ideals. The thought of fuzzy H-submodule is offered in Section 5. In this section, we also investigate some examples and results about fuzzy H-submodule. Section 6 is about the advantages of our article, and Section 7 have application.

2. Preliminaries

In this section, we include some relevant definitions which are helpful for the reader. From now onward, except if stated otherwise, we use H for Hopf algebra, k for field and Q for H-module.

Definition 2.1. [5] A HA $(H, m, u, \Delta, \varepsilon)$ is a bialgebra with antipode S, where S is an inverse map under convolution operation.

Definition 2.2. [5] A subspace J of H is called Hopf subalgebra if it is subalgebra and $S(J) \subseteq J$.

Definition 2.3. [5] A B^{cop}(bialgebra having opposite comultiplication) is a co-opposite HA H^{cop} with antipode $\tilde{S}$.

Definition 2.4. [5] A subspace I of H is a Hopf ideal if it is a bi-ideal and $SI \subseteq I$

Definition 2.5. [5] A right k-space Q is H-Hopf module of H if

i. Q is a right H-module and right H-comodule, via $\rho: Q \to Q \otimes H$.

ii. ρ is a right H-module map.

Definition 2.6. [13] An FS $\xi: X \to [0,1]$ is a function of a non-empty set X.

Definition 2.7. If ξ is an FS of X, then the subset $\xi_{t'} = \{x \in X: \xi(x) \geq t', t' \in [0,1]\}$ is called level subsets.

Definition 2.8. [14] The intersection of two FSs ξ and σ of X is also an FS of X and is defined by

$$(\xi \cap \sigma)(x) = \xi(x) \wedge \sigma(x), \text{ for all } x \in X$$

Definition 2.9. [14] The sum of two FSs ξ and σ of X is defined as follows:

$$(\xi + \sigma)(x) = sup\{\xi(a) \wedge \sigma(b)\}, \text{ for all } x \in X$$

Definition 2.10. [20] An FS ξ of k-vector space V is called fuzzy subspace if for any $v, v' \in V$ and $\alpha, \beta \in k$

$$\xi(\alpha v + \beta v') \geq \xi(v) \wedge \xi(v')$$

Definition 2.11. [22] An FS ξ of a bialgebra B is called fuzzy subbialgebra of B, if for any b, $b' \in B$ and $\alpha, \beta \in k,$ the following conditions are satisfied:

i. $\xi(\alpha b + \beta b') \geq \xi(b) \wedge \xi(b')$

ii. $\xi(by) \geq \xi(b) \wedge \xi(bb')$

iii. $\xi(b) \leq \xi(b_{i1}) \wedge \xi(b_{i2})$

Definition 2.12. [16] An FS ξ of a P-module Q is said to be a fuzzy submodule of Q if for any $q, q' \in Q$ and $p \in P$,

i. $\xi(0) = 1$

ii. $\xi(\alpha q + \beta q') \geq \xi(q) \wedge \xi(q')$

iii. $\xi(pq) \geq \xi(q)$

Definition 2.13. [19] An FS ξ of a C-comodule Q and a left comodule map $\rho: Q \rightarrow C \otimes Q$ is said to be fuzzy subcomodule of Q, where $\rho(q) = \sum_{i=1,n} q_{i0} \otimes q_{i1}$ if for any $q, q' \in Q$, $c \in C$, and $\alpha, \beta \in k$,

i. $\xi(0) = 1$

ii. $\xi(\alpha q + \beta q') \geq \xi(q) \wedge \xi(q')$

iii. $\xi(q) \leq \xi(q_{i0})$, for all i.

3. Fuzzy Hopf Subalgebra

In this section, the notion of FHA is proposed. Some significant results related to this concept are also discussed in it.

Definition 3.1. A FS ξ of H is called FHA, if for any $h, h' \in H$ and $\alpha, \beta \in k$ it satisfies:

i. $\xi(\alpha h + \beta h') \geq \xi(h) \wedge \xi(h')$

ii. $\xi(hh') \geq \xi(h) \wedge \xi(h')$

iii. $\xi(h) \leq \xi(h_{i1}) \wedge \xi(h_{i2})$

iv. $\xi(h') \geq \begin{cases} \sup\{\xi(h)\}, & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$

Example 3.2. Consider the 4-dimensional HA

$$H_4 = \langle 1, h_1, h_2, h_1 h_2 \mid {h_1}^2 = 1, {h_2}^2 = 0, h_2 h_1 = -h_1 h_2 \rangle$$

and $\xi: H_4 \rightarrow [0,1]$ defined by

$$\xi(a) = \begin{cases} 0.4, & \text{if } a \in H_4 \backslash \{0\} \\ 0.8, & \text{if } a = 0 \end{cases}$$

Then, ξ becomes an FHA of H_4.

Remark 3.3. There is no difference between FHA and fuzzy co-opposite HA because $\xi(h) \wedge \xi(h') = \xi(h') \wedge \xi(h)$.

Theorem 3.4. Let ξ and σ be FHA of H such that $\xi(0) = \sigma(0)$. Then, $\xi + \sigma$ is also fuzzy FHA of H.

Proof. Let $h, h' \in H$ and $\alpha, \beta \in k$. Now, we first have to show that $\xi + \sigma$ is a fuzzy subspace.

Suppose on the contrary

$$(\xi + \sigma)(\alpha h + \beta h') < (\xi + \sigma)(h) \wedge (\xi + \sigma)(h')$$

$$(\xi + \sigma)(\alpha h + \beta h') < (\xi + \sigma)(h)$$

and

$$(\xi + \sigma)(\alpha h + \beta h') < (\xi + \sigma)(h')$$

Let $\exists t' \in [0,1]$ such that

$$(\xi+\sigma)(\alpha h+\beta h') < t' < (\xi+\sigma)(h)$$

and

$$(\xi+\sigma)(\alpha h+\beta h') < t' < (\xi+\sigma)(h')$$

Then, $\exists h_1, h_2, h_3, h_4 \in H$ with $\alpha h = h_1 + h_2$

$$h = (\frac{h_1+h_2}{\alpha}) \text{ and } \beta h' = h_3 + h_4$$

$$h' = (\frac{h_3+h_4}{\beta}) \text{ such that } \xi((\frac{h_1+h_2}{\alpha})) > t', \sigma((\frac{h_3+h_4}{\beta})) > t'$$

Now, we have

$$\begin{aligned}(\xi+\sigma)(\alpha h+\beta h') &= \sup_{h+h'=m+n}\{\xi(m)\wedge\sigma(n)\} \\ &\geq \sup_{h+h'=m+n}\left\{\xi(\frac{h_1+h_2}{\alpha})\wedge\sigma(\frac{h_3+h_4}{\beta})\right\} \\ &> t' > (\xi+\sigma)(\alpha h+\beta h')\end{aligned}$$

which is contradiction. Therefore,

$$(\xi+\sigma)(\alpha h+\beta h') \geq (\xi+\sigma)(h)\wedge(\xi+\sigma)(h')$$

For all $h, h' \in H$, let $h = h_1 + h_2$ and $h' = h_3 + h_4$. Then, $hh' = h_1h_3 + h_1h_4 + h_2h_3 + h_2h_4$. Moreover,

$$\begin{aligned}(\xi+\sigma)(hh') &= \sup_{hh'=m+n}\{\xi(m)\wedge\sigma(n)\} \\ &\geq \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4}\{\xi(h_1h_3+h_1h_4)\wedge\sigma(h_2h_3+h_2h_4)\} \\ &\geq \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4}\{\xi(h_1h_3)\wedge\xi(h_1h_4)\wedge\sigma(h_2h_3)\wedge\sigma(h_2h_4)\} \\ &\geq \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4}\{\xi(h_1)\wedge\xi(h_3)\wedge\xi(h_4)\wedge\sigma(h_2)\wedge\sigma(h_3)\wedge\sigma(h_4)\} \\ &\geq \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4}\{(\xi(h_1)\wedge\sigma(h_2))\wedge(\xi(h_3)\wedge\sigma(h_4))\wedge(\xi(h_4)\wedge\sigma(h_3))\} \\ &= (\xi+\sigma)(h)\wedge(\xi+\sigma)(h')\wedge(\xi+\sigma)(h') \\ &= (\xi+\sigma)(h)\wedge(\xi+\sigma)(h')\end{aligned}$$

Furthermore, let $h = h_1 + h_2 \in H$. Then,

$$\sum_{i=1,n} h_{i1}\otimes h_{i2} = \Delta(h) = \Delta(h_1+h_2) = \Delta(h_1)+\Delta(h_2)$$

Here, $h_1 = \sum_{s=1,n} h_{1(s1)} \otimes h_{1(s2)}$ and $h_2 = \sum_{t=1,n} h_{2(t1)} \otimes h_{2(t2)}$.

Since

$$\begin{aligned}(\xi+\sigma)(h) &= \sup_{h=h_1+h_2}\{\xi(h_1)\wedge\sigma(h_2)\} \\ &\leq \sup_{h=h_1+h_2}\left\{(\xi(h_{1(s1)})\wedge\xi(h_{1(s2)}))\wedge(\sigma(h_{2(t1)})\wedge\sigma(h_{2(t2)}))\right\} \\ &= \sup_{h=h_1+h_2}\left\{\xi(h_{1(s1)})\wedge\sigma(0)\wedge\xi(0)\wedge\sigma(h_{2(t1)})\right\}\wedge\sup_{h=h_1+h_2}\left\{\xi(h_{1(s2)})\wedge\sigma(0)\wedge\xi(0)\wedge\sigma(h_{2(t2)})\right\}\end{aligned}$$

$$\leq \left((\xi+\sigma)(h_{1(s1)}) \wedge (\xi+\sigma)(h_{2(t1)})\right) \wedge \left((\xi+\sigma)(h_{1(s2)}) \wedge (\xi+\sigma)(h_{2(t2)})\right)$$

$$\leq (\xi+\sigma)(h_{i1}) \wedge (\xi+\sigma)(h_{i2})$$

$$(\xi+\sigma)(h') \geq \begin{cases} \sup\limits_{h\in S^{-1}(h')} \left\{ \sup\limits_{h=h_1+h_2} \xi(h^1) \wedge \sigma(h^2) \right\}, & \text{if } \mathrm{h}' \in \mathrm{S(H)} \\ 0, & \text{if } h' \notin S(H) \end{cases}$$

and

$$(\xi+\sigma)(h') \geq \begin{cases} \sup\limits_{h\in S^{-1}(h')} (\xi+\sigma)(h), & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$$

Then $\xi+\sigma$ is FHA.

Theorem 3.5. Let ξ and σ be two FHA of H then $\xi \cap \sigma$ is also FHA.

Theorem 3.6. Let $\{\xi_i : i \in N\}$ be any collection of an FHA of H then $\cap\, \xi_i$ is also FHA.

Theorem 3.7. An FS ξ of H is an FHA iff the level sets $\xi_{t'}$ are Hopf subalgebras of H.

Proof. Assume that ξ is an FHA. Since,

$$\xi(h) \geq 0 = t', \forall h \in H$$

$$\xi_{t'} \neq \phi$$

Let $h, h' \in \xi_{t'}$, $\xi(h) \geq t'$, and $\xi(h') \geq t'$. Since

$$\xi(\alpha h + \beta h') \geq \xi(h) \wedge \xi(h') \geq t' \wedge t' = t'$$

then $\alpha h + \beta h' \in \xi_{t'}$. Similarly, since

$$\xi(hh') \geq \xi(h) \wedge \xi(h') \geq t' \wedge t' = t'$$

then $hh' \in \xi_{t'}$.

Let $h \in H$ such that $\xi(h) = t' \Rightarrow h \in \xi_{t'}$ and $\Delta(h) = \sum_{i=1,n} \Sigma\, h_{i0} \otimes h_{i1} \Rightarrow \xi(h_{i0}) \geq t'$ and $\xi(h_{i1}) \geq t'$. Then,

$$\xi(h_{i0}) \wedge \xi(h_{i1}) \geq t' \wedge t' = t' = \xi(h)$$

Let $h \in H$ such that $\xi(h) = t' \Rightarrow h \in \xi_{t'}$. If $h' \in S(H)$, then,

$$\xi(h') \geq \sup_{h\in S^{-1}(h')} \xi(h) = t'$$

$$\Rightarrow h' \in \xi_{t'}$$

Hence, $\xi_{t'}$ is Hopf subalgebra for all t'.

Conversely, assume that all $\xi_{t'}$ are Hopf subalgebras.

Let $h, h' \in \xi_{t'}$ and $\alpha, \beta \in k$. We may assume that

$$\xi(h') \geq \xi(h) = t'$$

Since $h, h' \in \xi_{t'}$, then $\alpha h + \beta h' \in \xi_{t'}$. Thus,

$$\xi(\alpha h + \beta h') \geq t' = \xi(h) \wedge \xi(h')$$

Since $h, h' \in \xi_{t'}$, then $hh' \in \xi_{t'}$. Thus,

$$\xi(hh') \geq t' = \xi(h) \wedge \xi(h')$$

Let $h \in \xi_{t'}$ and $\Delta(h) = \sum_{i=1,n} h_{i0} \otimes h_{i1}$ where $h_{i0}, h_{i1} \in \xi_{t'}$. Therefore,

$$\xi(h_{i0}) \wedge \xi(h_{i1}) \geq t' \wedge t' = t' = \xi(h)$$

Let $h' \in S(H)$ and $\xi(h') \geq t'$. Since $h' \in \xi_{t'}$ and

$$\xi(h') \geq t' = \sup_{h \in S^{-1}(h')} \xi(h)$$

Then, ξ is FHA.

4. Fuzzy Hopf Ideal

In this section, the concept of fuzzy Hopf Ideal is proposed. Some significant results of the fuzzy Hopf ideal are also studied in it.

Definition 4.1. A fuzzy subspace ξ of H is called fuzzy Hopf left (right) ideal if for any $h, h' \in H$

i. $\xi(hh') \geq \xi(h'), (\xi(hh') \geq \xi(h))$

ii. $\xi(h) \leq \xi(h_{i2})$, $(\xi(h) \leq \xi(h_{i1}))$

iii. $\xi(h') \geq \begin{cases} \sup_{h \in S^{-1}(h')} \xi(h), & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$

Example 4.2. Consider $\xi: H_4 \to [0,1]$ defined by

$$\xi(a) = \begin{cases} 0.5, & \text{if } a \in H_4 \backslash \{0,1\} \\ 0.3, & \text{if } a = 1 \\ 0.8, & \text{if } a = 0 \end{cases}$$

Then, ξ becomes a fuzzy Hopf left ideal of H_4.

Remark 4.3. If ξ is both right and left fuzzy Hopf ideal of H then ξ is called an FHI of H.

Theorem 4.4. Let ξ and σ be two fuzzy Hopf left (right) ideals of H such that $\xi(0) = \sigma(0)$. Then, $\xi + \sigma$ is also a fuzzy Hopf left (right) ideal of H.

Proof. $\xi + \sigma$ is a fuzzy subspace. Let $h = h_1 + h_2$, $h' = h_3 + h_4$. Then,

$$hh' = h_1h_3 + h_1h_4 + h_2h_3 + h_2h_4$$

Moreover,

$$\begin{aligned}
(\xi + \sigma)(hh') &= \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4} \{\xi(h_1h_3 + h_1h_4) \wedge \sigma(h_2h_3 + h_2h_4)\} \\
&\geq \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4} \{\xi(h_1h_3) \wedge \xi(h_1h_4) \wedge \sigma(h_2h_3) \wedge \sigma(h_2h_4)\} \\
&\geq \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4} \{\xi(h_3) \wedge \xi(h_4) \wedge \sigma(h_3) \wedge \sigma(h_4)\} \\
&= \sup_{hh'=h_1h_3+h_1h_4+h_2h_3+h_2h_4} \{\xi(h_3) \wedge \sigma(h_4) \wedge \sigma(h_3) \wedge \xi(h_4)\} \\
&= (\xi + \sigma)(h') \wedge (\xi + \sigma)(h') \\
&= (\xi + \sigma)(h')
\end{aligned}$$

Let $h = h_1 + h_2 \in H$. Then,

$$\sum_{i=1,n} \mathrm{h}_{i1} \otimes \mathrm{h}_{i2} = \Delta(\mathrm{h}) = \Delta(\mathrm{h}_1 + \mathrm{h}_2) = \Delta(\mathrm{h}_1) + \Delta(\mathrm{h}_2)$$

where, $h_1 = \sum_{s=1,n} h_{1(s1)} \otimes h_{1(s2)}$ and $h_2 = \sum_{t=1,n} h_{2(t1)} \otimes h_{2(t2)}$ such that

$$\begin{aligned}(\xi+\sigma)(h) &= \sup_{h=h_1+h_2}\{\xi(h_1)\wedge\sigma(h_2)\}\\ &\le \sup_{h=h_1+h_2}\{\xi(h_{1(s2)})\wedge\sigma(h_{2(t2)})\}\\ &= \sup_{h=h_1+h_2}\{\xi(h_{1(s2)})\wedge\sigma(0)\wedge\xi(0)\wedge\sigma(h_{2(t2)})\}\\ &\le (\xi+\sigma)(h_{1(s2)})\wedge(\xi+\sigma)(h_{2(t2)})\\ &\le (\xi+\sigma)(h_{i2})\end{aligned}$$

$$(\xi+\sigma)(h') \ge \begin{cases}\sup\limits_{h\in S^{-1}(h')}\left\{\sup\limits_{h=h_1+h_2} \xi(h_1)\wedge\sigma(h_2)\right\}, & \text{if } h' \in S(H)\\ 0, & \text{if } h' \notin S(H)\end{cases}$$

and

$$(\xi+\sigma)(h') \ge \begin{cases}\sup\limits_{h\in S^{-1}(h')} (\xi+\sigma)(h), & \text{if } h' \in S(H)\\ 0, & \text{if } h' \notin S(H)\end{cases}$$

Hence, $\xi+\sigma$ is fuzzy Hopf left ideal.

Theorem 4.5. Let ξ and σ be two fuzzy Hopf left (right) ideals then $\xi \cap \sigma$ is also a fuzzy Hopf left (right) ideal.

Theorem 4.6. Let $\{\xi_i : i \in N\}$ be a collection of fuzzy Hopf left (right) ideals then $\cap\, \xi_i$ is also a fuzzy Hopf left (right) ideal.

5. Fuzzy H-Submodule

In this section, the concept of fuzzy H-Submodule is proposed. Some significant results of fuzzy H-Submodule are also studied in it.

Definition 5.1. A FS ξ of Q is right FHM if for any $q, q' \in Q$, $h \in H$ and $\alpha, \beta \in k$

i. $\xi(0) = 1$

ii. $\xi(\alpha q + \beta q') \ge \xi(q) \wedge \xi(q')$

iii. $\xi(qh) \ge \xi(q)$

iv. $\xi(q) \le \xi(q_{i0})$

v. $\xi(q') \ge \begin{cases}\sup\limits_{q\in\rho^{-1}(q')} \xi(q), & \text{if } q' \in \rho(Q)\\ 0, & \text{if } q' \notin \rho(Q)\end{cases}$

(vi). $\xi(\rho(q)h) \ge \xi(\rho(q))$

Example 5.2. Consider $\xi: H_4 \to [0,1]$ defined by

$$\xi(a) = \begin{cases}0.4, & \text{if } a \in H_4\backslash\{0\}\\ 1, & \text{if } a = 0\end{cases}$$

Then, ξ becomes an FHM of H_4.

Theorem 5.3. Let ξ and σ be two FHM of Q such that $\xi(0) = \sigma(0)$. Then, $\xi+\sigma$ is also an FHM of Q.

PROOF. Let $q, q' \in Q$ and $\alpha, \beta \in k$. Now, we have to show that $\xi + \sigma$ is a fuzzy subspace. On the contrary, suppose that

$$(\xi + \sigma)(\alpha q + \beta q') < (\xi + \sigma)(q) \wedge (\xi + \sigma)(q'),$$

$$(\xi + \sigma)(\alpha q + \beta q') < (\xi + \sigma)(q)$$

and

$$(\xi + \sigma)(\alpha q + \beta q') < (\xi + \sigma)(q')$$

If $\exists t' \in [0,1]$ such that

$$(\xi + \sigma)(\alpha q + \beta q') < t' < (\xi + \sigma)(q)$$

and

$$(\xi + \sigma)(\alpha q + \beta q') < t' < (\xi + \sigma)(q')$$

If $\exists\ q_1, q_2, q_3, q_4 \in Q$ with $\alpha q = q_1 + q_2$, then

$$q = ((q_1 + q_2)/\alpha) \text{ and } \beta q' = q_3 + q_4$$

$$q' = (\frac{q_3+q_4}{\beta}) \text{ such that } \xi((\frac{q_1+q_2}{\alpha})) > t', \sigma((\frac{q_3+q_4}{\beta})) > t'$$

Now, we have

$$\begin{aligned}(\xi + \sigma)(\alpha q + \beta q') &= \sup_{q+q'=n+n'} \{\xi(n) \wedge \sigma(n')\} \\ &\geq \sup_{q+q'=\frac{q_1+q_2}{\alpha}+\frac{q_3+q_4}{\beta}} \left\{\xi((\frac{q_1+q_2}{\alpha})) \wedge \sigma((\frac{q_3+q_4}{\beta}))\right\} \\ &> t' > (\xi + \sigma)(\alpha q + \beta q')\end{aligned}$$

which is a contradiction. Therefore,

$$(\xi + \sigma)(\alpha q + \beta q') \geq (\xi + \sigma)(q) \wedge (\xi + \sigma)(q')$$

Now, let $q \in Q$ and $h \in H$ such that

$$\begin{aligned}(\xi + \sigma)(qh) &= \sup_{qh=q_1+q_2} \{\xi(q_1) \wedge \sigma(q_2)\} \\ &\geq \sup_{q=\frac{q_1}{h}+\frac{q_2}{h}} \left\{\xi(\frac{q_1}{h}) \wedge \sigma(\frac{q_2}{h})\right\} \\ &= (\xi + \sigma)(q)\end{aligned}$$

Let $q = q_1 + q_2 \in Q$. Then,

$$\begin{aligned}\sum_{i=1,n} q_{i0} \otimes q_{i1} = \rho(q) &= \rho(q_1 + q_2) \\ &= \rho(q_1) + \rho(q_2) \\ &= \sum_{s=1,n} q_{1(s0)} \otimes q_{1(s1)} + \sum_{t=1,n} q_{2(t0)} \otimes q_{2(t1)}\end{aligned}$$

Moreover,

$$
\begin{aligned}
(\xi+\sigma)(q) &= \sup_{q=q_1+q_2}\{\xi(q_1)\wedge\sigma(q_2)\} \\
&\leq \sup_{q=q_1+q_2}\left\{\xi(q_{1(s0)})\wedge\sigma(q_{2(t0)})\right\} \\
&= \sup_{q=q_1+q_2}\left\{\xi(q_{1(s0)})\wedge\sigma(0)\wedge\xi(0)\wedge\sigma(q_{2(t0)})\right\} \\
&= \sup_{q=q_1+q_2}\left\{\xi(q_{1(s0)})\wedge\sigma(0)\right\}\wedge \sup_{q=q_1+q_2}\xi(0)\wedge\sigma(q_{2(t0)}) \\
&\leq (\xi+\sigma)(q_{1(s0)})\wedge(\xi+\sigma)(q_{2(t0)}) \\
&\leq (\xi+\sigma)(q_{i0})
\end{aligned}
$$

and

$$
(\xi+\sigma)(0) = \sup_{0=q_1+q_2}\{\xi(q_1)\wedge\sigma(q_2)\} \geq \xi(0)\wedge\sigma(0) = 1\wedge 1 = 1
$$

Thus,

$$
\begin{aligned}
(\xi+\sigma)(\rho(q)h) &= \sup_{\rho(q)h=q_1+q_2}\{\xi(q_1)\wedge\sigma(q_2)\} \\
&\geq \sup_{\rho(q)=\frac{q_1}{h}+\frac{q_2}{h}}\left\{\xi(\frac{q_1}{h})\wedge\sigma(\frac{q_2}{h})\right\} \\
&= (\xi+\sigma)(\rho(q))
\end{aligned}
$$

Hence, $\xi+\sigma$ is FHM of Q.

Theorem 5.4. Let ξ and σ be two FHMs of Q then $\xi\cap\sigma$ is also an FHM.

Theorem 5.5. Let $\{\xi_i : i\in N\}$ be a collection of FHMs of Q. Then $\cap\,\xi_i$ is also an FHM.

Theorem 5.6. An FS ξ of Q is an FHM iff the level sets $\xi_{t'}$ are H −submodules of Q.

Proof. Assume that ξ is an FHM. As $\xi(0)=1$. Therefore, $\xi_t\neq\varphi$. Let $q,q'\in\xi_{t'}$. Then,

$$
\xi(q)\geq t' \text{ and } \xi(q')\geq t'
$$

Since

$$
\xi(\alpha q+\beta q')\geq\xi(q)\wedge\xi(q')\geq t'\wedge t'=t'
$$

then $\alpha q+\beta q'\in\xi_{t'}$. Similarly, since $\xi(qh)\geq\xi(q)=t'$ then $qh\in\xi_{t'}$. Moreover, since $\xi(q_{i0})\geq\xi(q)\geq t'$ then $x_{i0}\in\xi_{t'}$. Let $\rho(x)\in\xi_{t'}$ and $h\in H$. Since

$$
\xi(\rho(q)h)\geq\xi(\rho(q))\geq t'
$$

then $\rho(q)h\in\xi_{t'}$. Therefore, each $\xi_{t'}$ is H −submodule of Q.

Conversely, assume that each $\xi_{t'}$ is a fuzzy H-submodules. Let $q,q'\in Q$ and $\alpha,\beta\in k$. We may assume that $\xi(q')\geq\xi(q)=t'$. Therefore, $q,q'\in\xi_{t'}$ Since, $\xi_{t'}$ is H −submodule of Q, then $\alpha q+\beta q'\in\xi_{t'}$. Therefore,

$$
\xi(\alpha q+\beta q')\geq t'=\xi(q)\wedge\xi(q')
$$

Let $q\in Q$ and $h\in H$. Let $\xi(q)=t'$. Therefore, $q\in\xi_{t'}$ and $qh\in\xi_{t'}$. Thus, $\xi(qh)\geq t'=\xi(q)$.

Since $\rho(q)=\sum_{i=1,n}q_{i0}\otimes q_{i1}$ where $q_{i0}\in\xi_{t'}$, then

$$
\xi(q_{i0})\geq t'=\xi(q)
$$

Moreover,

$$\xi(0) = \sup_{q \in Q}\{\xi(q)\} = 1$$

Let $\rho(q) \in Q$ and $h \in H$, such that $\xi(\rho(q)) = t'$. Then, $\rho(q)h \in \xi_{t'}$

$$\xi(\rho(q)h) \geq t' = \xi(\rho(q))$$

Thus, ξ is an FHM.

6. Advantages

The importance of Hopf algebra in Quantum physics and physics cannot be rejected. Shahn Majid and other physicist and mathematician use Hopf algebra to solve many problems. By using fuzzy theory, we can solve these problems with easier. So, it is crucial to extend these concepts in uncertainty. So, in this manuscript, we proposed fuzziness on Hopf algebraic structures.

7. Application

Discrete gauge theory in the $2 + 1$ dimension arises by breaking a gauge symmetry with gauge group G to discrete subgroup H due to photons becoming massive w, making the guage force ultra-short ranged. Some particles are charged, and some carry flux. Charged particles carry a representation of H, and massive charges are ultra-short ranged. By using fuzziness, we can deal with these particles easily. Here is a simple example,

Example 7.1. Let $H = \{h_1, h_2, h_3, h_4\}$ be a Hopf algebra with antipode S, $k = \mathbb{R}$, $\Delta(h) = h \otimes h$, $\varepsilon(h) = 1$, $S(h) = h^{-1}$,

$+$	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\boldsymbol{h_3}$	$\boldsymbol{h_4}$
$\boldsymbol{h_1}$	h_1	h_1	h_1	h_1
$\boldsymbol{h_2}$	h_1	h_1	h_1	h_1
$\boldsymbol{h_3}$	h_1	h_1	h_1	h_1
$\boldsymbol{h_4}$	h_1	h_1	h_1	h_1

and

$\cdot$	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\boldsymbol{h_3}$	$\boldsymbol{h_4}$
$\boldsymbol{h_1}$	h_1	h_2	h_3	h_4
$\boldsymbol{h_2}$	h_2	h_1	h_4	h_3
$\boldsymbol{h_3}$	h_3	h_4	h_2	h_1
$\boldsymbol{h_4}$	h_4	h_3	h_1	h_2

Then, $\xi: H \to [0,1]$ defined by

$$\xi(d) = \begin{cases} 0.6, & \text{if } h = h_1 \\ 0.4, & \text{if } h \in H/\{h_1\} \end{cases}$$

is an FHA. Therefore, h_1 is the short, ranged particle.

Similarly, we can apply this uncertainty in other fields of physics, like to know the energy level of photons in non-ideal lasers etc.

8. Conclusion

This manuscript introduces the concept of fuzzy Hopf subalgebra, fuzzy Hopf ideal, and fuzzy H-submodule. Some properties and significant results are also studied in it. Our obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, expert systems, medical diagnosis, and engineering. Weak Hopf algebra is a Hopf algebra with a linear map that satisfy some specific conditions. In the future study, we aim to extend this concept in the field of weak Hopf algebra.

Conflict of Interest

The authors declare no conflict of interest.

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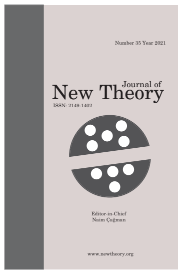
Journal of New Theory
ISSN: 2149-1402

35 (2021) 32-47
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Intuitionistic Fuzzy Magnified Translation of PS-Algebra

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Article History
Received: 11 Jul 2019
Accepted: 04 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract — In this paper, the concepts of intuitionistic fuzzy α-translation (IFAT), intuitionistic fuzzy α-multiplication (IFAM), and intuitionistic fuzzy magnified $\beta\alpha$-translation (IFMBAT) are introduced in the setup of PS-algebra. Some properties of PS-ideal and PS-subalgebra are investigated by applying the concepts of IFAT, IFAM, and IFMBAT. Intersection and union of intuitionistic fuzzy PS-ideals are explained through results and examples.

Keywords — *Intuitionistic fuzzy α-translation, intuitionistic fuzzy α-multiplication, intuitionistic fuzzy PS-ideal, intuitionistic fuzzy PS-subalgebra, intuitionistic fuzzy magnified $\beta\alpha$-translation*

Mathematics Subject Classification (2020) — 03B20, 06F99

1. Introduction

Zadeh [1] introduced the idea of fuzzy set in 1965. The deep study of fuzzy subsets and its applications to different mathematical structures developed the fuzzy mathematics. Fuzzy algebra is a significant branch of fuzzy mathematics. Idea of Fuzzy set has been applied to different algebraic structures like groups, rings, modules, vector spaces and topologies. In this way, Iseki and Tanaka [2] introduced the idea of BCK-algebra in 1978. Iseki [3] introduced the idea of BCI-algebra in 1980 and it is clear that the class of BCK-algebra is a proper sub class of the class of BCI-algebra. Lee et al. [4] discussed the fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication of fuzzy subalgebra in BCK/BCI-algebra. Relationship among fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication are also investigated. Ansari and Chandramouleeswaran [5] introduced the notion of fuzzy translation, fuzzy extension and fuzzy multiplication of fuzzy β ideals of β-algebra and investigated some of their properties. Priya and Ramachandran [6, 7] introduced the class of PS-algebra. Lekkoksung [8] concentrated on fuzzy magnified translation in ternary hemirings, which is a generalization of BCI / BCK/Q / KU / d-algebra. Senapati et al. [9] have done much work on intuitionistic fuzzy H-ideals in BCK/BCI-algebra. Jana et al. [10] wrote on intuitionistic fuzzy G-algebra. Senapati et al. [11] discussed fuzzy translations of fuzzy H-ideals in BCK/BCI-algebra. Atanassov [12] introduced intuitionistic fuzzy set. Senapati [13] investigated the relationship among intuitionistic fuzzy translation, intuitionistic fuzzy extension and intuitionistic fuzzy multiplication in B-algebra. Kim and Jeong [14] introduced the intuitionistic fuzzy structure of B-algebra. Senapati et al. [15] introduced the cubic subalgebras and cubic closed ideals of B-algebras. Senapati et al. [16] discussed the fuzzy dot subalgebra and fuzzy dot ideal of B-algebras. Priya and Ramachandran [17] worked on fuzzy translation and fuzzy multiplication in PS-algebra. Chandramouleeswaran et al. [18]

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worked on fuzzy translation and fuzzy multiplication in BF/BG-algebra. Jun and Kim [19] worked on intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras.

Purpose of this paper is to introduce the idea of intuitionistic fuzzy α translation (IFAT), intuitionistic fuzzy α multiplication (IFAM) and intuitionistic fuzzy magnified $\beta\alpha$ translation (IFMBAT) in PS-algebra. Some of their properties are investigated in depth by using the idea of intuitionistic fuzzy PS ideal (IFID) and intuitionistic fuzzy PS subalgebra (IFSU).

2. Preliminaries

In this section, we present some basic definitions, that are helpful to understand the paper.

Definition 2.1. [3] An algebra $(Y;*,0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions:

i. $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$

ii. $t_1 * (t_1 * t_2) \leq t_2$

iii. $t_1 \leq t_1$

iv. $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$

v. $t_1 \leq 0 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, $\forall\ t_1, t_2, t_3 \in Y$

Definition 2.2. [1] An algebra $(Y;*,0)$ of type (2,0) is called a BCK-algebra if it satisfies the following conditions:

i. $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$

ii. $t_1 * (t_1 * t_2) \leq t_2$

iii. $t_1 \leq t_1$

iv. $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$

v. $0 \leq t_1 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$

Definition 2.3. [7] A nonempty set S with a constant 0 and having binary operation $*$ is called PS-algebra if it satisfies the following conditions:

i. $t_1 * t_1 = 0$

ii. $t_1 * 0 = 0$

iii. $t_1 * t_2 = 0$ and $t_2 * t_1 = 0 \Rightarrow t_1 = t_2$, $\forall\ t_1, t_2 \in Y$

Definition 2.4. [7] Let S be a nonempty subset of PS-algebra Y, then S is called a PS subalgebra of Y if $t_1 * t_2 \in S$, $\forall\ t_1, t_2 \in S$.

Definition 2.5. [7] Let Y ba a PS-algebra and I is a subset of Y, then I is called a PS ideal of Y if it satisfies following conditions:

i. $0 \in I$

ii. $t_2 * t_1 \in I$ and $t_2 \in I \rightarrow t_1 \in I$

Definition 2.6. [6] Let Y be a PS-algebra. A fuzzy set B of Y is called a fuzzy PS ideal of Y if it satisfies the following conditions:

i. $\mu(0) \geq \mu(t_1)$

ii. $\mu(t_1) \geq \min\{\mu(t_2 * t_1), \mu(t_2)\}$, for all $t_1, t_2 \in Y$

2.1. Fuzzy and Intuitionistic Fuzzy Logics

Definition 2.7. [1] Let Y be the group of objects denoted generally by t_1. Then, a fuzzy set B of Y is defined as $B = \{< t_1, \mu_B(t_1) > \mid t_1 \in Y\}$, where $\mu_B(t_1)$ is called the membership value of t_1 in B and $\mu_B(t_1) \in [0,1]$.

Definition 2.8. [6] A fuzzy set B of PS-algebra Y is called a fuzzy PS subalgebra of Y if $\mu(t_1 * t_2) \geq \min\{\mu(t_1), \mu(t_2)\}$, $\forall\ t_1, t_2 \in Y$.

Definition 2.9. [4, 5] Let a fuzzy subset B of Y and $\alpha \in [0, 1 - sup\{\mu_B(t_1) \mid t_1 \in Y\}]$. A mapping $(\mu_B)^T_\alpha \mid Y \in [0,1]$ is said to be a fuzzy α translation of μ_B if it satisfies $(\mu_B)^T_\alpha(t_1) = \mu_B(t_1) + \alpha$, $\forall\ t_1 \in Y$.

Definition 2.10. [4, 5] Let a fuzzy subset B of Y and $\alpha \in [0,1]$. A mapping $(\mu_B)^M_\alpha \mid Y \to [0,1]$ is said to be a fuzzy α multiplication of B if it satisfies $(\mu_B)^M_\alpha(t_1) = \alpha.(\mu_B)(t_1)$, $\forall\ t_1 \in Y$.

Definition 2.11. [12] An intuitionistic fuzzy set (IFS) B over Y is an object having the form $B = \{\langle t_1, \mu_B(t_1), \nu_B(t_1)\rangle \mid t_1 \in Y\}$, where $\mu_B(t_1) \mid Y \to [0,1]$ and $\nu_B(t_1) \mid Y \to [0,1]$, with the condition $0 \leq \mu_B(t_1) + \nu_B(t_1) \leq 1$, $\forall\ t_1 \in Y$. $\mu_B(t_1)$ and $\nu_B(t_1)$ represent the degree of membership and the degree of non-membership of the element t_1 in the set B respectively.

Definition 2.12. [12] Let $A = \{\langle t_1, \mu_A(t_1), \nu_A(t_1)\rangle \mid t_1 \in Y\}$ and $B = \{\langle t_1, \mu_B(t_1), \nu_B(t_1)\rangle \mid t_1 \in Y\}$ are two IFSs on Y. Then, intersection and union of A and B are indicated by $A \cap B$ and $A \cup B$ respectively and are given by

$$A \cap B = \{\langle t_1, \min(\mu_A(t_1), \mu_B(t_1)), \max(\nu_A(t_1), \nu_B(t_1))\rangle \mid t_1 \in Y\}$$

$$A \cup B = \{\langle t_1, \max(\mu_A(t_1), \mu_B(t_1)), \min(\nu_A(t_1), \nu_B(t_1))\rangle \mid t_1 \in Y\}$$

Definition 2.13. [14] An IFS $B = \{\langle t_1, \mu_B(t_1), \nu_B(t_1)\rangle \mid t_1 \in Y\}$ of Y is called an IFSU of Y if it satisfies these two conditions:

i. $\mu_B(t_1 * t_2) \geq \min\{\mu_B(t_1), \mu_B(t_2)\}$

ii. $\nu_B(t_1 * t_2) \leq \max\{\nu_B(t_1), \nu_B(t_2)\}$, $\forall\ t_1, t_2 \in Y$

Definition 2.14. [19] An IFS $B = \{\langle t_1, \mu_B(t_1), \nu_B(t_1)\rangle \mid t_1 \in Y\}$ of Y is said to be an IFID of Y if satisfies these conditions:

i. $\mu_B(0) \geq \mu_B(t_1), \nu_B(0) \leq \nu_B(t_1)$

ii. $\mu_B(t_1) \geq \min\{\mu_B(t_1 * t_2), \mu_B(t_2)\}$

iii. $\nu_B(t_1) \leq \max\{\nu_B(t_1 * t_2), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$

Definition 2.15. [8] Let μ be a fuzzy subset of Y, $\alpha \in$ [0,T] and $\beta \in [0,1]$. A mapping $\mu^{MT}_{\beta\alpha} \mid Y \to$[0,1] is said to be fuzzy magnified $\beta\alpha$ translation of μ if it satisfies $\mu^{MT}_{\beta\alpha}(t_1) = \beta.\mu(t_1) + \alpha$, for all $t_1 \in Y$.

3. Intuitionistic Fuzzy Translation and Multiplication

For simplicity, we use the notion $B = (\mu_B, \nu_B)$ for the IFS $B = \{\langle t_1, \mu_B(t_1), \nu_B(t_1)\rangle \mid t_1 \in Y\}$. In this paper, we use $¥ = \inf\{\nu_B(t_1) \mid t_1 \in Y\}$ for any IFS $B = (\mu_B, \nu_B)$ of Y.

3.1. Intuitionistic Fuzzy Translation and Multiplication of PS Subalgebra

Definition 3.1. Let $B = (\mu_B, \nu_B)$ be an IFS of Y and let $\alpha \in [0, ¥]$. An object of the form $B^T_\alpha = ((\mu_B)^T_\alpha, (\nu_B)^T_\alpha)$ is called an IFAT of B, when $(\mu_B)^T_\alpha(t_1) = \mu_B(t_1)+\alpha$ and $(\nu_B)^T_\alpha(t_1) = \nu_B(t_1)-\alpha$, for all $t_1 \in Y$.

Example 3.2. Let $Y = \{0, 1, 2\}$ be a PS-algebra with the following Cayley table:

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

thus $(Y; *, 0)$ is a PS-algebra. Now, IFS $B = (\mu_B, \nu_B)$ is defined as

$$\mu_B(t_1) = \begin{cases} 0.2 & \text{if } t_1 \neq 1 \\ 0.4 & \text{if } t_1 = 1, \end{cases}$$

$$\nu_B(t_1) = \begin{cases} 0.6 & \text{if } t_1 \neq 1 \\ 0.3 & \text{if } t_1 = 1 \end{cases}$$

is an IFSU. HereÃ¼ ¥ $= inf\{\nu_B(t_1) \mid t_1 \in Y\} = 0.3$, choose $\alpha = 0.2$, then the mapping $B^T_{0.2} \mid Y \to [0, 1]$ is defined as

$$(\mu_B)^T_{0.2}(t_1) = \begin{cases} 0.4 & \text{if } t_1 \neq 1 \\ 0.6 & \text{if } t_1 = 1 \end{cases}$$

and

$$(\nu_B)^T_{0.2}(t_1) = \begin{cases} 0.4 & \text{if } t_1 \neq 1 \\ 0.1 & \text{if } t_1 = 1 \end{cases}$$

which imply that, $(\mu_B)^T_{0.2}(t_1) = \mu_B(t_1)+0.2$ and $(\nu_B)^T_{0.2}(t_1) = \nu_B(t_1)-0.2$, $\forall\, t_1 \in Y$ is an intuitionistic fuzzy (0.2) translation.

Theorem 3.3. Let B be an IFSU of Y and $\alpha \in [0, ¥]$. Then, IFAT B^T_α of B is an IFSU of Y.

Proof. Assume that, $t_1, t_2 \in Y$. Then,

$$\begin{aligned} (\mu_B)^T_\alpha(t_1 * t_2) &= \mu_A(t_1 * t_2) + \alpha \\ &\geq \min\{\mu_B(t_1), \mu_B(t_2)\} + \alpha \\ &= \min\{\mu_B(t_1) + \alpha, \mu_B(t_2) + \alpha\} \\ &= \min\{(\mu_B)^T_\alpha(t_1), (\mu_B)^T_\alpha(t_2)\} \end{aligned}$$

and

$$\begin{aligned} (\nu_B)^T_\alpha(t_1 * t_2) &= \nu_B(t_1 * t_2) - \alpha \\ &\leq \max\{\nu_B(t_1), \nu_B(t_2)\} - \alpha \\ &= \max\{\nu_B(t_1) - \alpha, \nu_B(t_2) - \alpha\} \\ &= \max\{(\nu_B)^T_\alpha(t_1), (\nu_B)^T_\alpha(t_2)\} \end{aligned}$$

Hence, IFAT B^T_α of B is an IFSU of Y. □

Theorem 3.4. Let B be an IFS of Y such that IFAT B^T_α of B is an IFSU of Y for some $\alpha \in [0, ¥]$. Then, B is an IFSU of Y.

PROOF. Let $B_\alpha^T = ((\mu_B)_\alpha^T, (\nu_B)_\alpha^T)$ be an IFSU of Y for some $\alpha \in [0, ¥]$ and $t_1, t_2 \in Y$. So, we have

$$\begin{aligned}\mu_B(t_1 * t_2) + \alpha &= (\mu_B)_\alpha^T(t_1 * t_2)\\ &\geq \min\{(\mu_B)_\alpha^T(t_1), (\mu_B)_\alpha^T(t_2)\}\\ &= \min\{\mu_B(t_1) + \alpha, \mu_B(t_2) + \alpha\}\\ &= \min\{\mu_B(t_1), \mu_B(t_2)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\nu_B(t_1 * t_2) - \alpha &= (\nu_B)_\alpha^T(t_1 * t_2)\\ &\leq \max\{(\nu_B)_\alpha^T(t_1), (\nu_B)_\alpha^T(t_2)\}\\ &= \max\{\nu_B(t_1) - \alpha, \nu_B(t_2) - \alpha\}\\ &= \max\{\mu_B(t_1), \nu_B(t_2)\} - \alpha\end{aligned}$$

which imply that, $\mu_B(t_1 * t_2) \geq \min\{\mu_B(t_1), \mu_B(t_2)\}$ and $\nu_B(t_1 * t_2) \leq \max\{\nu_B(t_1), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$. Hence, B is an IFSU of Y. □

Definition 3.5. Let B be an IFS of Y and $\alpha \in [0, 1]$. An object having the form $B_\alpha^M = (\mu_B)_\alpha^M, (\nu_B)_\alpha^M$ is called an IFAM of B. If $(\mu_B)_\alpha^M(t_1) = \alpha.\mu_B(t_1)$ and $(\nu_B)_\alpha^M(t_1) = \alpha.\nu_B(t_1)$, for all $t_1 \in Y$.

Example 3.6. Let $Y = \{0, 1, 2\}$ be a PS-algebra with the following Cayley table:

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

thus $(Y; *, 0)$ is a PS-algebra. Now IFS $B = (\mu_B, \nu_B)$ is defined as

$$\mu_B(t_1) = \begin{cases} 0.5 & \text{if } t_1 \neq 1 \\ 0.4 & \text{if } t_1 = 1, \end{cases}$$

$$\nu_B(t_1) = \begin{cases} 0.2 & \text{if } t_1 \neq 1 \\ 0.3 & \text{if } t_1 = 1 \end{cases}$$

is an IFSU, choose $\alpha = 0.2$, then the mapping $B_{(0.2)}^M \mid Y \to [0, 1]$ is defined by

$$(\mu_B)_{0.2}^M(t_1) = \begin{cases} 0.10 & \text{if } t_1 \neq 1 \\ 0.08 & \text{if } t_1 = 1 \end{cases}$$

and

$$(\mu_B)_{0.2}^M(t_1) = \begin{cases} 0.04 & \text{if } t_1 \neq 1 \\ 0.06 & \text{if } t_1 = 1 \end{cases}$$

which imply that, $(\mu_B)_{0.2}^M(t_1) = \mu_B(t_1).(0.2)$, $(\nu_B)_{0.2}^M(t_1) = \nu_B(t_1).(0.2)$, $\forall\ t_1 \in Y$ is an intuitionistic fuzzy (0.2) multiplication.

Theorem 3.7. Let IFS $B = (\mu_B, \nu_B)$ of Y and $\alpha \in [0, 1]$, if the IFAM B_α^M of B be an IFSU of Y. Then, B is an IFSU of Y.

PROOF. Assume that, B_α^M of B is an IFSU of Y for some $\alpha \in [0, 1]$. Now, for all $t_1, t_2 \in Y$, we have

$$\begin{aligned}\mu_B(t_1 * t_2).\alpha &= (\mu_B)_\alpha^M(t_1 * t_2)\\ &\geq \min\{(\mu_B)_\alpha^M(t_1), (\mu_B)_\alpha^M(t_2)\}\\ &= \min\{\mu_B(t_1).\alpha, \mu_B(t_2).\alpha\}\\ &= \min\{\mu_B(t_1), \mu_B(t_2)\}.\alpha\end{aligned}$$

and

$$\begin{aligned}\nu_B(t_1 * t_2).\alpha &= (\nu_B)^M_\alpha(t_1 * t_2)\\ &\le \max\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\alpha(t_2)\}\\ &= \max\{\nu_B(t_1).\alpha, \nu_B(t_2).\alpha\}\\ &= \max\{\nu_B(t_1), \nu_B(t_2)\}.\alpha\end{aligned}$$

which imply that, $\mu_B(t_1 * t_2) \ge \min\{\mu_B(t_1), \mu_B(t_2)\}$ and $\nu_B(t_1 * t_2) \le \max\{\nu_B(t_1), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$. Hence, B is an IFSU of Y. □

Theorem 3.8. Let IFS $B = (\mu_B, \nu_B)$ of Y is an IFSU of Y and $\alpha \in [0, 1]$, then IFAM B^M_α of B is an IFSU of Y.

Proof. Suppose that, $B = (\mu_B, \nu_B)$ be an IFSU of Y. Then, for all $t_1, t_2 \in Y$, we have

$$\begin{aligned}(\mu_B)^M_\alpha(t_1 * t_2) &= \alpha.\mu(t_1 * t_2)\\ &\ge \alpha.\min\{(\mu_B)(t_1), (\mu_B)(t_2)\}\\ &= \min\{\alpha.\mu_B(t_1), \alpha.\mu_B(t_2)\}\\ &= \min\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\alpha(t_2)\}\\ &\ge \min\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\alpha(t_2)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^M_\alpha(t_1 * t_2) &= \alpha.\nu(t_1 * t_2)\\ &\le \alpha.\max\{(\nu_B)(t_1), (\nu_B)(t_2)\}\\ &= \max\{\alpha.\nu_B(t_1), \alpha.\nu_B(t_2)\}\\ &= \max\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\alpha(t_2)\}\\ &\le \max\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\alpha(t_2)\}\end{aligned}$$

which imply that, $\mu_B(t_1 * t_2) \ge \min\{\mu_B(t_1), \mu_B(t_2)\}$ and $\nu_B(t_1 * t_2) \le \max\{\nu_B(t_1), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$. Hence, B^M_α is an IFSU of Y. □

3.2. Intuitionistic Fuzzy Translation and Multiplication of PS Ideal

In this section, intuitionistic fuzzy α translation of IFID, intuitionistic fuzzy α multiplication of IFID, union and intersection of intuitionistic fuzzy translation of IFID are investigated through some results.

Theorem 3.9. If IFAT B^T_α of B is an intutionistic fuzzy PS ideal, then it fulfills the condition $(\mu_B)^T_\alpha(t_1 * (t_2 * t_1)) \ge (\mu_B)^T_\alpha(t_2)$ and $(\nu_B)^T_\alpha(t_1 * (t_2 * t_1)) \le (\nu_B)^T_\alpha(t_2)$.

Proof. Let IFAT B^T_α of B is an intutionistic fuzzy PS ideal. Then,

$$\begin{aligned}(\mu_B)^T_\alpha(t_1 * (t_2 * t_1)) &= \mu_B(t_1 * (t_2 * t_1)) + \alpha\\ &\ge \min\{\mu_B(t_2 * (t_1 * (t_2 * t_1))) + \alpha, \mu_B(t_2) + \alpha\}\\ &= \min\{\mu_B(0) + \alpha, \mu_B(t_2) + \alpha\}\\ &= \min\{(\mu_B)^T_\alpha(0), (\mu_B)^T_\alpha(t_2)\}\\ &= (\mu_B)^T_\alpha(t_2)\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^T_\alpha(t_1 * (t_2 * t_1)) &= \nu_B(t_1 * (t_2 * t_1)) - \alpha\\ &\le \max\{\nu_B(t_2 * (t_1 * (t_2 * t_1))) - \alpha, \nu_B(t_2) - \alpha\}\\ &= \max\{\nu_B(0) - \alpha, \nu_B(t_2) - \alpha\}\\ &= \max\{(\nu_B)^T_\alpha(0), (\nu_B)^T_\alpha(t_2)\}\\ &= (\nu_B)^T_\alpha(t_2)\end{aligned}$$

Hence, $(\mu_B)^T_\alpha(t_1 * (t_2 * t_1)) \geq (\mu_B)^T_\alpha(t_2)$ and $(\nu_B)^T_\alpha(t_1 * (t_2 * t_1)) \leq (\nu_B)^T_\alpha(t_2)$. □

Theorem 3.10. If B is an IFID of Y, then IFAT B^T_α of B is an IFID of Y, for all $\alpha \in [0, ¥]$.

PROOF. Let B be an IFID of Y and $\alpha \in [0, ¥]$. Then, $(\mu_B)^T_\alpha(0) = \mu_B(0)+\alpha \geq \mu_B(t_1)+\alpha = (\mu_B)^T_\alpha(t_1)$ and $(\nu_B)^T_\alpha(0) = \nu_B(0) - \alpha \leq \nu_B(t_1) - \alpha = (\nu_B)^T_\alpha(t_1)$. Therefore,

$$\begin{aligned}(\mu_B)^T_\alpha(t_1) &= \mu_B(t_1) + \alpha,\\ &\geq \min\{\mu_B(t_1 * t_2), \mu_B(t_2)\} + \alpha\\ &= \min\{\mu_B(t_1 * t_2) + \alpha, \mu_B(t_2) + \alpha\}\\ &= \min\{(\mu_B)^T_\alpha(t_1 * t_2), (\mu_B)^T_\alpha(t_2)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^T_\alpha(t_1) &= \nu_B(t_1) - \alpha,\\ &\leq \max\{\nu_B(t_1 * t_2), \nu_B(t_2)\} - \alpha\\ &= \max\{\nu_B(t_1 * t_2) - \alpha, \nu_B(t_2) - \alpha\}\\ &= \max\{(\nu_B)^T_\alpha(t_1 * t_2), (\nu_B)^T_\alpha(t_2)\}\end{aligned}$$

for all $t_1, t_2 \in Y$ and $\alpha \in [0, ¥]$. Hence, B^T_α of B is an IFID of Y. □

Theorem 3.11. If B is an intutionistic fuzzy set of Y, such that IFAT B^T_α of B is an IFID of Y, for all $\alpha \in [0, ¥]$. Then, B is an IFID of Y.

PROOF. Suppose B^T_α is an IFID of Y, where $\alpha \in [0, ¥]$ and $t_1, t_2 \in Y$ then,

$$\begin{aligned}\mu_B(0) + \alpha = (\mu_B)^T_\alpha(0) \geq (\mu_B)^T_\alpha(t_1) = \mu_B(t_1) + \alpha\\ \nu_B(0) - \alpha = (\nu_B)^T_\alpha(0) \leq (\nu_B)^T_\alpha(t_1) = \nu_B(t_1) - \alpha\end{aligned}$$

which imply, $\mu_B(0) \geq \mu_B(t_1)$ and $\nu_B(0) \leq \nu_B(t_1)$ now,

$$\begin{aligned}\mu_B(t_1) + \alpha = (\mu_B)^T_\alpha(t_1) &\geq \min\{(\mu_B)^T_\alpha(t_1 * t_2), (\mu_B)^T_\alpha(t_2)\}\\ &= \min\{\mu_B(t_1 * t_2) + \alpha, \mu_B(t_2) + \alpha\}\\ &= \min\{\mu_B(t_1 * t_2), \mu_B(t_2)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\nu_B(t_1) - \alpha = (\nu_B)^T_\alpha(t_1) &\leq \max\{(\nu_B)^T_\alpha(t_1 * t_2), (\nu_B)^T_\alpha(t_2)\}\\ &= \max\{\nu_B(t_1 * t_2) - \alpha, \nu_B(t_2) - \alpha\}\\ &= \max\{\nu_B(t_1 * t_2), \nu_B(t_2)\} - \alpha\end{aligned}$$

which imply that, $\mu_B(t_1) \geq \min\{\mu_B(t_1 * t_2), \mu_B(t_2)\}$ and $\nu_B(t_1) \leq \max\{\nu_B(t_1 * t_2), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$. Hence, B is an IFID of Y. □

Theorem 3.12. Let B be an IFID of Y for some $\alpha \in [0, ¥]$. Then, IFAT B^T_α of B is an IFSU of Y.

PROOF. Assume that, $t_1, t_2 \in Y$, then

$$\begin{aligned}(\mu_B)^T_\alpha(t_1 * t_2) &= \mu_B(t_1 * t_2) + \alpha\\ &\geq \min\{\mu_B(t_2 * (t_1 * t_2)), \mu_B(t_2)\} + \alpha\\ &= \min\{\mu_B(0), \mu_B(t_2)\} + \alpha\\ &\geq \min\{\mu_B(t_1), \mu_B(t_2)\} + \alpha\\ &= \min\{\mu_B(t_1) + \alpha, \mu_B(t_2) + \alpha\}\\ &= \min\{(\mu_B)^T_\alpha(t_1), (\mu_B)^T_\alpha(t_2)\}\\ &\geq \min\{(\mu_B)^T_\alpha(t_1), (\mu_B)^T_\alpha(t_2)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_B)_\alpha^T(t_1 * t_2) &= \nu_B(t_1 * t_2) - \alpha \\ &\leq \max\{\nu_B(t_2 * (t_1 * t_2)), \nu_B(t_2)\} - \alpha \\ &= \max\{\nu_B(0), \nu_B(t_2)\} - \alpha \\ &\leq \max\{\nu_B(t_1), \nu_B(t_2)\} - \alpha \\ &= \max\{\nu_B(t_1) - \alpha, \nu_B(t_2) - \alpha\} \\ &= \max\{(\nu_B)_\alpha^T(t_1), (\nu_B)_\alpha^T(t_2)\} \\ &\leq \max\{(\nu_B)_\alpha^T(t_1), (\nu_B)_\alpha^T(t_2)\}\end{aligned}$$

Hence, B_α^T is an IFSU of Y. □

Theorem 3.13. If IFAT B_α^T of B is an IFID of Y and $\alpha \in [0, ¥]$, then B is an IFSU of Y.

PROOF. Suppose that, B_α^T of B is an IFID of Y. Since

$$\begin{aligned}(\mu_B)(t_1 * t_2) + \alpha &= (\mu_B)_\alpha^T(t_1 * t_2) \\ &\geq \min\{(\mu_B)_\alpha^T(t_2 * (t_1 * t_2)), (\mu_B)_\alpha^T(t_2)\} \\ &= \min\{(\mu_B)_\alpha^T(0), (\mu_B)_\alpha^T(t_2)\} \\ &\geq \min\{(\mu_B)_\alpha^T(t_1), (\mu_B)_\alpha^T(t_2)\} \\ &= \min\{\mu_B(t_1) + \alpha, \mu_B(t_2) + \alpha\} \\ &= \min\{\mu_B(t_1), \mu_B(t_2)\} + \alpha\end{aligned}$$

then $\mu_B(t_1 * t_2) \geq \min\{\mu_B(t_1), \mu_B(t_2)\}$. Similarly, since

$$\begin{aligned}(\nu_B)(t_1 * t_2) - \alpha &= (\nu_B)_\alpha^T(t_1 * t_2) \\ &\leq \max\{(\nu_B)_\alpha^T(t_2 * (t_1 * t_2)), (\nu_B)_\alpha^T(t_2)\} \\ &= \max\{(\nu_B)_\alpha^T(0), (\nu_B)_\alpha^T(t_2)\} \\ &\leq \max\{(\nu_B)_\alpha^T(t_1), (\nu_B)_\alpha^T(t_2)\} \\ &= \max\{\nu_B(t_1) - \alpha, \nu_B(t_2) - \alpha\} \\ &= \max\{\nu_B(t_1), \nu_B(t_2)\} - \alpha\end{aligned}$$

then $\nu_B(t_1 * t_2) \leq \max\{\nu_B(t_1), \nu_B(t_2)\}$. Hence, B is an IFSU of Y. □

Theorem 3.14. Intersection of any two intuitionistic fuzzy translations of an intuitionistic fuzzy PS ideal B of Y is an intuitionistic fuzzy PS ideal of Y.

PROOF. Suppose, B_α^T and B_β^T are intuitionistic fuzzy translations of intuitionistic fuzzy PS ideal B of Y, where $\alpha, \beta \in [0, ¥]$ and $\alpha \leq \beta$, as we know that, B_α^T and B_β^T are intuitionistic fuzzy PS ideals of Y. Then,

$$\begin{aligned}((\mu_B)_\alpha^T \cap (\mu_B)_\beta^T)(t_1) &= \min\{(\mu_B)_\alpha^T(t_1), (\mu_B)_\beta^T(t_1)\} \\ &= \min\{\mu_B(t_1) + \alpha, \mu_B(t_1) + \beta\} \\ &= \mu_B(t_1) + \alpha \\ &= (\mu_B)_\alpha^T(t_1)\end{aligned}$$

and

$$\begin{aligned}((\nu_B)_\alpha^T \cap (\nu_B)_\beta^T)(t_1) &= \max\{(\nu_B)_\alpha^T(t_1), (\nu_B)_\beta^T(t_1)\} \\ &= \max\{\nu_B(t_1) - \alpha, \nu_B(t_1) - \beta\} \\ &= \nu_B(t_1) - \alpha \\ &= (\nu_B)_\alpha^T(t_1)\end{aligned}$$

Hence, $B_\alpha^T \cap B_\beta^T$ is an intuitionistic fuzzy PS ideal of Y. □

Theorem 3.15. Union of any two intuitionistic fuzzy translations of an IFID B of Y is an IFID of Y.

Proof. Suppose B_α^T and B_β^T are intuitionistic fuzzy translations of an IFID B of Y, where $\alpha, \beta \in [0, ¥]$ and $\alpha \leq \beta$, as we know that, B_α^T and B_β^T are intuitionistic fuzzy PS ideals of Y. Then,

$$\begin{aligned}((\mu_B)_\alpha^T \cup (\mu_B)_\beta^T)(t_1) &= \max\{(\mu_B)_\alpha^T(t_1), (\mu_B)_\beta^T(t_1)\} \\ &= \max\{\mu_B(t_1) + \alpha, \mu_B(t_1) + \beta\} \\ &= \mu_B(t_1) + \beta \\ &= (\mu_B)_\beta^T(t_1)\end{aligned}$$

and

$$\begin{aligned}((\nu_B)_\alpha^T \cup (\nu_B)_\beta^T)(t_1) &= \min\{(\nu_B)_\alpha^T(t_1), (\nu_B)_\beta^T(t_1)\} \\ &= \min\{\nu_B(t_1) - \alpha, \nu_B(t_1) - \beta\} \\ &= \nu_B(t_1) - \beta \\ &= (\nu_B)_\beta^T(t_1)\end{aligned}$$

Hence, $B_\alpha^T \cup B_\beta^T$ is an intuitionistic fuzzy PS ideal of Y. □

Theorem 3.16. Let B be an IFS of Y such that IFAM B_α^M of B is an IFID of Y for $\alpha \in (0, 1]$, then B is an IFID of Y.

Proof. Suppose that, B_α^M is an IFID of Y for $\alpha \in (0, 1]$ and $t_1, t_2 \in Y$. Then, $\alpha.\mu_B(0) = (\mu_B)_\alpha^M(0) \geq (\mu_B)_\alpha^M(t_1) = \alpha.\mu_B(t_1)$, so $\mu_B(0) \geq \mu_B(t_1)$ and $\alpha.\nu_B(0) = (\nu_B)_\alpha^M(0) \leq (\nu_B)_\alpha^M(t_1) = \alpha.\nu_B(t_1)$, so $\nu_B(0) \leq \nu_B(t_1)$. Since

$$\begin{aligned}\alpha.\mu_B(t_1) &= (\mu_B)_\alpha^M(t_1) \\ &\geq \min\{(\mu_B)_\alpha^M(t_1 * t_2), (\mu_B)_\alpha^M(t_2)\} \\ &= \min\{\alpha.\mu_B(t_1 * t_2), \alpha.\mu_B(t_2)\} \\ &= \alpha.\min\{\mu_B(t_1 * t_2), \mu_B(t_2)\}\end{aligned}$$

then $\mu_B(t_1) \geq \min\{\mu_B(t_1 * t_2), \mu_B(t_2)\}$. Similarly, since

$$\begin{aligned}\alpha.\nu_B(t_1) &= (\nu_B)_\alpha^M(t_1) \\ &\leq \max\{(\nu_B)_\alpha^M(t_1 * t_2), (\nu_B)_\alpha^M(t_2)\} \\ &= \max\{\alpha.\nu_B(t_1 * t_2), \alpha.\nu_B(t_2)\} \\ &= \alpha.\max\{\nu_B(t_1 * t_2), \nu_B(t_2)\}\end{aligned}$$

then $\nu_B(t_1) \leq \max\{\nu_B(t_1 * t_2), \nu_B(t_2)\}$. Hence, B is an IFID of Y. □

Theorem 3.17. If B is an IFID of Y, then IFAM B_α^M of B is an IFID of Y, for all $\alpha \in (0, 1]$.

Proof. Let B be an IFID of Y and $\alpha \in (0, 1]$, we have

$$\begin{aligned}(\mu_B)_\alpha^M(0) &= \alpha.\mu_B(0) \\ &\geq \alpha.\mu_B(t_1) \\ &= (\mu_B)_\alpha^M(t_1)\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^M_\alpha(0) &= \alpha.\nu_B(0)\\ &\le \alpha.\nu_B(t_1)\\ &= (\nu_B)^M_\alpha(t_1)\end{aligned}$$

Moreover,

$$\begin{aligned}(\mu_B)^M_\alpha(t_1) &= \alpha.\mu_B(t_1)\\ &\ge \alpha.\min\{\mu_B(t_1 * t_2), \mu_B(t_2)\}\\ &= \min\{\alpha.\mu_B(t_1 * t_2), \alpha.\mu_B(t_2)\}\\ &= \min\{(\mu_B)^M_\alpha(t_1 * t_2), (\mu_B)^M_\alpha(t_2)\}\\ &\ge \min\{(\mu_B)^M_\alpha(t_1 * t_2), (\mu_B)^M_\alpha(t_2)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^M_\alpha(t_1) &= \alpha.\nu_B(t_1)\\ &\le \alpha.\max\{\nu_B(t_1 * t_2), \nu_B(t_2)\}\\ &= \max\{\alpha.\nu_B(t_1 * t_2), \alpha.\nu_B(t_2)\}\\ &= \max\{(\nu_B)^M_\alpha(t_1 * t_2), (\nu_B)^M_\alpha(t_2)\}\\ &\le \max\{(\nu_B)^M_\alpha(t_1 * t_2), (\nu_B)^M_\alpha(t_2)\}\end{aligned}$$

Hence, B^M_α of B is an IFID of Y, $\forall\ \alpha \in (0, 1]$. □

Theorem 3.18. Let B be an IFID of Y and $\alpha \in [0, 1]$. Then, IFAM B^M_α of B is an IFSU of Y.

Proof. Suppose that, $t_1, t_2 \in Y$, we have

$$\begin{aligned}(\mu_B)^M_\alpha(t_1 * t_2) &= \alpha.\mu_B(t_1 * t_2)\\ &\ge \alpha.\min\{\mu_B(t_2 * (t_1 * t_2)), \mu_B(t_2)\}\\ &= \alpha.\min\{\mu_B(0), \mu_B(t_2)\}\\ &\ge \alpha.\min\{\mu_B(t_1), \mu_B(t_2)\}\\ &= \min\{\alpha.\mu_B(t_1), \alpha.mu_B(t_2)\}\\ &= \min\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\alpha(t_2)\}\\ &\ge \min\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\alpha(t_2)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^M_\alpha(t_1 * t_2) &= \alpha.\nu_B(t_1 * t_2)\\ &\le \alpha.\max\{\nu_B(t_2 * (t_1 * t_2)), \nu_B(t_2)\}\\ &= \alpha.\max\{\nu_B(0), \nu_B(t_2)\}\\ &\le \alpha.\max\{\nu_B(t_1), \nu_B(t_2)\}\\ &= \max\{\alpha.\nu_B(t_1), \alpha.\nu_B(t_2)\}\\ &= \max\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\alpha(t_2)\}\\ &\le \max\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\alpha(t_2)\}\end{aligned}$$

Hence, B^M_α is an IFSU of Y. □

Theorem 3.19. If the IFAM B^M_α of B is an IFID of Y, for $\alpha \in (0, 1]$. Then, B is an intuitionistic fuzzy PS-subalgebra of Y.

PROOF. Assume that, B^M_α of B is an IFID of Y. Since

$$\begin{aligned}\alpha.(\mu_B)(t_1 * t_2) &= (\mu_B)^M_\alpha(t_1 * t_2)\\ &\geq \min\{(\mu_B)^M_\alpha(t_2 * (t_1 * t_2)), (\mu_B)^M_\alpha(t_2)\}\\ &= \min\{(\mu_B)^M_\alpha(0), (\mu_B)^M_\alpha(t_2)\}\\ &\geq \min\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\alpha(t_2)\}\\ &= \min\{\alpha.\mu_B(t_1), \alpha.\mu_B(t_2)\}\\ &= \alpha.\min\{\mu_B(t_1), \mu_B(t_2)\}\end{aligned}$$

then $\mu_B(t_1 * t_2) \geq \min\{\mu_B(t_1), \mu_B(t_2)\}$. Similarly, since

$$\begin{aligned}\alpha.(\nu_B)(t_1 * t_2) &= (\nu_B)^M_\alpha(t_1 * t_2)\\ &\leq \max\{(\nu_B)^M_\alpha(t_2 * (t_1 * t_2)), (\nu_B)^M_\alpha(t_2)\}\\ &= \max\{(\nu_B)^M_\alpha(0), (\nu_B)^M_\alpha(t_2)\}\\ &\leq \max\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\alpha(t_2)\}\\ &= \max\{\alpha.\nu_B(t_1), \alpha.\nu_B(t_2)\}\\ &= \alpha.\max\{\nu_B(t_1), \nu_B(t_2)\}\end{aligned}$$

then $\nu_B(t_1 * t_2) \leq \max\{\nu_B(t_1), \nu_B(t_2)\}$. Hence, B is an IFSU of Y. □

Theorem 3.20. Intersection of any two intuitionistic fuzzy multiplications of an IFID B of Y is an IFID of Y.

PROOF. Suppose that, B^M_α and B^M_β are intuitionistic fuzzy multiplications of IFID B of Y, where $\alpha, \beta \in [0,1]$ and $\alpha \leq \beta$, as we know that B^M_α and B^M_β are IFIDs of Y. Then,

$$\begin{aligned}((\mu_B)^M_\alpha \cap (\mu_B)^M_\beta)(t_1) &= \min\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\beta(t_1)\}\\ &= \min\{\mu_B(t_1).\alpha, \mu_B(t_1).\beta\}\\ &= \mu_B(t_1).\alpha\\ &= (\mu_B)^M_\alpha(t_1)\end{aligned}$$

and

$$\begin{aligned}((\nu_B)^M_\alpha \cap (\nu_B)^M_\beta)(t_1) &= \max\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\beta(t_1)\}\\ &= \max\{\nu_B(t_1).\alpha, \nu_B(t_1).\beta\}\\ &= \nu_B(t_1).\alpha\\ &= (\nu_B)^M_\alpha(t_1)\end{aligned}$$

Hence, $B^M_\alpha \cap B^M_\beta$ is IFID of Y. □

Theorem 3.21. Union of any two intuitionistic fuzzy multiplications of an IFID B of Y is an IFID of Y.

PROOF. Suppose that, B^M_α and B^M_β are intuitionistic fuzzy multiplications of an IFID B of Y, where $\alpha, \beta \in [0,1]$ and $\alpha \leq \beta$ and B^M_α and B^M_β are IFIDs of Y. Then,

$$\begin{aligned}((\mu_B)^M_\alpha \cup (\mu_B)^M_\beta)(t_1) &= \max\{(\mu_B)^M_\alpha(t_1), (\mu_B)^M_\beta(t_1)\}\\ &= \max\{\mu_B(t_1).\alpha, \mu_B(t_1).\beta\}\\ &= \mu_B(t_1).\beta\\ &= (\mu_B)^M_\beta(t_1)\end{aligned}$$

and

$$\begin{aligned}((\nu_B)^M_\alpha \cup (\nu_B)^M_\beta)(t_1) &= \min\{(\nu_B)^M_\alpha(t_1), (\nu_B)^M_\beta(t_1)\} \\ &= \min\{\nu_B(t_1).\alpha, \nu_B(t_1).\beta\} \\ &= \nu_B(t_1).\beta \\ &= (\nu_B)^M_\beta(t_1)\end{aligned}$$

Hence, $B^M_\alpha \cup B^M_\beta$ is IFID of Y. □

3.3. Intuitionistic Fuzzy Magnified $\beta\alpha$ Translation

In this section, the notion of intuitionistic fuzzy magnified $\beta\alpha$ translation IFMBAT is presented and investigated.

Definition 3.22. Let $B = (\mu_B, \nu_B)$ be an IFS of Y and $\alpha \in [0, ¥]$, $\beta \in [0, 1]$. An object having the form $B^{MT}_{\beta\,\alpha} = \{(\mu_B)^{MT}_{\beta\,\alpha}, (\nu_B)^{MT}_{\beta\,\alpha}\}$ is said to be an IFMBAT of B if it satisfies $(\mu_B)^{MT}_{\beta\,\alpha}(t_1) = \beta.\mu_B(t_1) + \alpha$ and $(\nu_B)^{MT}_{\beta\,\alpha}(t_1) = \beta.\nu_B(t_1) - \alpha$, $\forall\ t_1 \in Y$.

Example 3.23. Let $Y = \{0, 1, 2\}$ be a PS-algebra defined in example 2.1. A IFS $B = (\mu_B, \nu_B)$ of Y is defined as:

$$\mu_B(t_1) = \begin{cases} 0.3 & \text{if } t_1 \neq 2 \\ 0.5 & \text{if } t_1 = 2 \end{cases}$$

$$\nu_B(t_1) = \begin{cases} 0.6 & \text{if } t_1 \neq 2 \\ 0.4 & \text{if } t_1 = 2 \end{cases}$$

is an IFSU and $¥ = inf\{\nu_B(t_1) \mid t_1 \in Y\} = 0.4$, choose $\alpha = 0.1 \in [0, ¥]$ and $\beta = 0.3 \in [0, 1]$, then the mapping $B^{MT}_{(0.3)\,(0.1)} \mid Y \to [0, 1]$ is given as

$$(\mu_B)^{MT}_{(0.3)\,(0.1)}(t_1) = \begin{cases} (0.3)(0.3) + (0.1) = 0.19 & \text{if } t_1 \neq 2 \\ (0.3)(0.5) + (0.1) = 0.25 & \text{if } t_1 = 2 \end{cases}$$

and

$$(\nu_B)^{MT}_{(0.3)\,(0.1)}(t_1) = \begin{cases} (0.3)(0.6) - (0.1) = 0.08 & \text{if } t_1 \neq 2 \\ (0.3)(0.4) - (0.1) = 0.02 & \text{if } t_1 = 2 \end{cases}$$

which imply that, $(\mu_B)^{MT}_{(0.3)\,(0.1)}(t_1) = (0.3).\mu_B(t_1) + 0.1$ and $(\nu_B)^{MT}_{(0.3)\,(0.1)}(t_1) = (0.3).\nu_B(t_1) - 0.1$, $\forall\ t_1 \in Y$. Hence, $B^{MT}_{(0.3)\,(0.1)}$ is an intuitionistic fuzzy magnified (0.3)(0.1) translation.

Theorem 3.24. Let B be an intuitionistic fuzzy subset of Y, such that $\alpha \in [0, ¥]$, $\beta \in [0, 1]$ and a mapping $B^{MT}_{\beta\,\alpha} \mid Y \to [0, 1]$ is IFMBAT of B, if B is IFSU of Y. Then, $B^{MT}_{\beta\,\alpha}$ is IFSU of Y.

Proof. Let B be an IFS of Y, $\alpha \in [0, ¥]$, $\beta \in [0, 1]$ and a mapping $B^{MT}_{\beta\,\alpha} \mid Y \to [0, 1]$ is IFMBAT of B. Suppose B is an IFSU of Y. Then,

$$\mu_B(t_1 * t_2) \geq \min\{\mu_B(t_1), \mu_B(t_2)\}$$
$$\nu_B(t_1 * t_2) \leq \max\{\nu_B(t_1), \nu_B(t_2)\}$$

Moreover,

$$\begin{aligned}(\mu_B)^{MT}_{\beta\,\alpha}(t_1 * t_2) &= \beta.\mu_B(t_1 * t_2) + \alpha \\ &\geq \beta.\min\{\mu_B(t_1), \mu_B(t_2)\} + \alpha \\ &= \min\{\beta.\mu_B(t_1) + \alpha, \beta.\mu_B(t_2) + \alpha\} \\ &= \min\{(\mu_B)^{MT}_{\beta\,\alpha}(t_1), (\mu_B)^{MT}_{\beta\,\alpha}(t_2)\} \\ &\geq \min\{(\mu_B)^{MT}_{\beta\,\alpha}(t_1), (\mu_B)^{MT}_{\beta\,\alpha}(t_2)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^{MT}_{\beta\alpha}(t_1 * t_2) &= \beta.\nu_B(t_1 * t_2) - \alpha \\ &\leq \beta.\max\{\nu_B(t_1), \nu_B(t_2)\} - \alpha \\ &= \max\{\beta.\nu_B(t_1) - \alpha, \beta.\nu_B(t_2) - \alpha\} \\ &= \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1), (\nu_B)^{MT}_{\beta\alpha}(t_2)\} \\ &\leq \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1), (\nu_B)^{MT}_{\beta\alpha}(t_2)\}\end{aligned}$$

Hence, IFMBAT $B^{MT}_{\beta\alpha}$ is an IFSU of Y. □

Theorem 3.25. Let B be an IFS of Y, such that $\alpha \in [0, ¥]$, $\beta \in [0,1]$ and a mapping $B^{MT}_{\beta\alpha} \mid Y \to [0,1]$ is IFMBAT of B, if $B^{MT}_{\beta\alpha}$ is IFSU of Y. Then, B is an IFSU of Y.

Proof. Let B be an intuitionistic fuzzy subset of Y, where $\alpha \in [0, ¥]$, $\beta \in [0,1]$ and a mapping $B^{MT}_{\beta\alpha} \mid Y \to [0,1]$ is IFMBAT of B. Let $B^{MT}_{\beta\alpha} = \{(\mu_B)^{MT}_{\beta\alpha}, (\nu_B)^{MT}_{\beta\alpha}\}$ is an IFSU of Y, we have

$$\begin{aligned}\beta.\mu_B(t_1 * t_2) + \alpha &= (\mu_B)^{MT}_{\beta\alpha}(t_1 * t_2) \\ &\geq \min\{(\mu_B)^{MT}_{\beta\alpha}(t_1), (\mu_B)^{MT}_{\beta\alpha}(t_2)\} \\ &= \min\{\beta.\mu_B(t_1) + \alpha, \beta.\mu_B(t_2) + \alpha\} \\ &= \beta.\min\{\mu_B(t_2), \mu_B(t_1)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\beta.\nu_B(t_1 * t_2) - \alpha &= (\nu_B)^{MT}_{\beta\alpha}(t_1 * t_2) \\ &\leq \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1), (\nu_B)^{MT}_{\beta\alpha}(t_2)\} \\ &= \max\{\beta.\nu_B(t_2) - \alpha, \beta.\nu_B(t_1) - \alpha\} \\ &= \beta.\max\{\nu_B(t_1), \nu_B(t_1)\} - \alpha\end{aligned}$$

which imply that, $\mu_B(t_1 * t_2) \geq \min\{\mu_B(t_1), \mu_B(t_2)\}$ and $\nu_B(t_1 * t_2) \leq \max\{\nu_B(t_1), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$. Hence, B is an IFSU of Y. □

Theorem 3.26. If B is an IFID of Y, then IFMBAT $B^{MT}_{\beta\alpha}$ of B is an IFID of Y, for all $\alpha \in [0, ¥]$ and $\beta \in (0,1]$.

Proof. Suppose that $B = (\mu_B, \nu_B)$ be an IFID of Y. Then,

$$\begin{aligned}(\mu_B)^{MT}_{\beta\alpha}(0) &= \beta.\mu_B(0) + \alpha \\ &\geq \beta.\mu_B(t_1) + \alpha \\ &= (\mu_B)^{MT}_{\beta\alpha}(t_1)\end{aligned}$$

and

$$\begin{aligned}(\nu_B)^{MT}_{\beta\alpha}(0) &= \beta.\nu_B(0) - \alpha \\ &\leq \beta.\nu_B(t_1) - \alpha \\ &= (\nu_B)^{MT}_{\beta\alpha}(t_1)\end{aligned}$$

Moreover, since

$$\begin{aligned}(\mu_B)^{MT}_{\beta\alpha}(t_1) &= \beta.\mu_B(t_1) + \alpha \\ &\geq \beta.\min\{\mu_B(t_1 * t_2), \mu_B(t_2)\} + \alpha \\ &= \min\{\beta.\mu_B(t_1 * t_2) + \alpha, \beta.\mu_B(t_2) + \alpha\} \\ &= \min\{(\mu_B)^{MT}_{\beta\alpha}(t_1 * t_2), (\mu_B)^{MT}_{\beta\alpha}(t_2)\}\end{aligned}$$

then $(\mu_B)^{MT}_{\beta\alpha}(t_1) \geq \min\{(\mu_B)^{MT}_{\beta\alpha}(t_1 * t_2), (\mu_B)^{MT}_{\beta\alpha}(t_2)\}$, for all $t_1, t_2 \in Y$ and $\forall\ \alpha \in [0, ¥]$, $\beta \in (0, 1]$. Similarly, since

$$\begin{aligned}(\nu_B)^{MT}_{\beta\alpha}(t_1) &= \beta.\nu_B(t_1) - \alpha\\ &\leq \beta.\max\{\nu_B(t_1 * t_2), \nu_B(t_2)\} - \alpha\\ &= \max\{\beta.\nu_B(t_1 * t_2) - \alpha, \beta.\nu_B(t_2) - \alpha\}\\ (\nu_B)^{MT}_{\beta\alpha}(t_1) &= \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1 * t_2), (\nu_B)^{MT}_{\beta\alpha}(t_2)\}\end{aligned}$$

then $(\nu_B)^{MT}_{\beta\alpha}(t_1) \leq \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1 * t_2), (\nu_B)^{MT}_{\beta\alpha}(t_2)\}$, for all $t_1, t_2 \in Y$ and $\forall\ \alpha \in [0, ¥]$, $\beta \in (0, 1]$. Hence, $B^{MT}_{\beta\alpha}$ of B is an IFID of Y. □

Theorem 3.27. If B is an intuitionistic fuzzy set of Y, such that IFMBAT $B^{MT}_{\beta\alpha}$ of B is an IFID of Y, for all $\alpha \in [0, ¥]$ and $\beta \in (0, 1]$. Then, B is an IFID of Y.

Proof. Suppose that IFMBAT $B^{MT}_{\beta\alpha}$ is an IFID of Y for some $\alpha \in [0, ¥]$, $\beta \in (0, 1]$ and $t_1, t_2 \in Y$, then

$$\begin{aligned}\beta.\mu_B(0) + \alpha &= (\mu_B)^{MT}_{\beta\alpha}(0)\\ &\geq (\mu_B)^{MT}_{\beta\alpha}(t_1)\\ &= \beta.\mu_B(t_1) + \alpha\end{aligned}$$

and

$$\begin{aligned}\beta.\nu_B(0) - \alpha &= (\nu_B)^{MT}_{\beta\alpha}(0)\\ &\leq (\nu_B)^{MT}_{\beta\alpha}(t_1)\\ &= \beta.\nu_B(t_1) - \alpha\end{aligned}$$

which imply that, $\mu_B(0) \geq \mu_B(t_1)$ and $\nu_B(0) \leq \nu_B(t_1)$. Now, we have

$$\begin{aligned}\beta.\mu_B(t_1) + \alpha &= (\mu_B)^{MT}_{\beta\alpha}(t_1)\\ &\geq \min\{(\mu_B)^{MT}_{\beta\alpha}(t_1 * t_2), (\mu_B)^{MT}_{\beta\alpha}(t_2)\}\\ &= \min\{\beta.\mu_B(t_1 * t_2) + \alpha, \beta.\mu_B(t_2) + \alpha\}\\ &= \beta.\min\{\mu_B(t_1 * t_2), \mu_B(t_2)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\beta.\nu_B(t_1) - \alpha &= (\nu_B)^{MT}_{\beta\alpha}(t_1)\\ &\leq \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1 * t_2), (\nu_B)^{MT}_{\beta\alpha}(t_2)\}\\ &= \max\{\beta.\nu_B(t_1 * t_2) - \alpha, \beta.\nu_B(t_2) - \alpha\}\\ &= \beta.\max\{\nu_B(t_1 * t_2), \nu_B(t_2)\} - \alpha\end{aligned}$$

which imply that, $\mu_B(t_1) \geq \min\{\mu_B(t_1 * t_2), \mu_B(t_2)\}$ and $\nu_B(t_1) \leq \max\{\nu_B(t_1 * t_2), \nu_B(t_2)\}$, for all $t_1, t_2 \in Y$. Hence, B is an IFID of Y.

□

Theorem 3.28. Intersection of any two IFMBATs $B^{MT}_{\beta\alpha}$ of an IFID B of Y is an IFID of Y.

Proof. Suppose that, $B^{MT}_{\beta\alpha}$ and $B^{MT}_{\acute{\beta}\acute{\alpha}}$ are two IFMBATs of IFID B of Y, where $\alpha, \acute{\alpha} \in [0, ¥]$ and $\beta, \acute{\beta} \in (0, 1]$. Assume $\alpha \leq \acute{\alpha}$, and $\beta = \acute{\beta}$, then by Theorem 3.26, $B^{MT}_{\beta\alpha}$ and $B^{MT}_{\acute{\beta}\acute{\alpha}}$ are IFIDs of Y.

Therefore,

$$\begin{aligned}((\mu_B)^{MT}_{\beta\alpha} \cap (\mu_B)^{MT}_{\acute{\beta}\acute{\alpha}})(t_1) &= \min\{(\mu_B)^{MT}_{\beta\alpha}(t_1), (\mu_B)^{MT}_{\acute{\beta}\acute{\alpha}}(t_1)\} \\ &= \min\{\beta.\mu_B(t_1) + \alpha, \acute{\beta}.\mu_B(t_1) + \acute{\alpha}\} \\ &= \beta.\mu_B(t_1) + \alpha \\ &= (\mu_B)^{MT}_{\beta\alpha}(t_1)\end{aligned}$$

and

$$\begin{aligned}((\nu_B)^{MT}_{\beta\alpha} \cap (\nu_B)^{MT}_{\acute{\beta}\acute{\alpha}})(t_1) &= \max\{(\nu_B)^{MT}_{\beta\alpha}(t_1), (\nu_B)^{MT}_{\acute{\beta}\acute{\alpha}}(t_1)\} \\ &= \max\{\beta.\nu_B(t_1) - \alpha, \acute{\beta}.\nu_B(t_1) - \acute{\alpha}\} \\ &= \beta.\nu_B(t_1) - \alpha \\ &= (\nu_B)^{MT}_{\beta\alpha}(t_1)\end{aligned}$$

Hence, $B^{MT}_{\beta\alpha} \cap B^{MT}_{\acute{\beta}\acute{\alpha}}$ is IFID of Y. □

Theorem 3.29. Union of any two IFMBATs $B^{MT}_{\beta\alpha}$ of an IFID B of Y is an IFID of Y.

PROOF. Suppose that, $B^{MT}_{\beta\alpha}$ and $B^{MT}_{\acute{\beta}\acute{\alpha}}$ are two IFMBATs of IFID B of Y, where $\alpha, \acute{\alpha} \in [0, ¥]$ and $\beta, \acute{\beta} \in (0, 1]$. Assume $\alpha \leq \acute{\alpha}$, and $\beta = \acute{\beta}$, then by Theorem 3.26, $B^{MT}_{\beta\alpha}$ and $B^{MT}_{\acute{\beta}\acute{\alpha}}$ are IFIDs of Y. Therefore,

$$\begin{aligned}((\mu_B)^{MT}_{\beta\alpha} \cup (\mu_B)^{MT}_{\acute{\beta}\acute{\alpha}})(t_1) &= \max\{(\mu_B)^{MT}_{\beta\alpha}(t_1), (\mu_B)^{MT}_{\acute{\beta}\acute{\alpha}}(t_1)\} \\ &= \max\{\beta.\mu_B(t_1) + \alpha, \acute{\beta}.\mu_B(t_1) + \acute{\alpha}\} \\ &= \acute{\beta}.\mu_B(t_1) + \acute{\alpha} \\ &= (\mu_B)^{MT}_{\acute{\beta}\acute{\alpha}}(t_1)\end{aligned}$$

and

$$\begin{aligned}((\nu_B)^{MT}_{\beta\alpha} \cup (\nu_B)^{MT}_{\acute{\beta}\acute{\alpha}})(t_1) &= \min\{(\nu_B)^{MT}_{\beta\alpha}(t_1), (\nu_B)^{MT}_{\acute{\beta}\acute{\alpha}}(t_1)\} \\ &= \min\{\beta.\nu_B(t_1) - \alpha, \acute{\beta}.\nu_B(t_1) - \acute{\alpha}\} \\ &= \acute{\beta}.\nu_B(t_1) - \acute{\alpha} \\ &= (\nu_B)^{MT}_{\acute{\beta}\acute{\alpha}}(t_1)\end{aligned}$$

Hence, $B^{MT}_{\beta\alpha} \cup B^{MT}_{\acute{\beta}\acute{\alpha}}$ is IFID of Y. □

4. Conclusion

In this paper, IFAT, IFAM and IFMBAT of PS-algebra are discussed with the help of subalgebras and ideals. Moreover, IFMBAT of PS-algebra is studied, which gave us new line of thought to apply PS-algebra on some other sets. For future work, PS-algebra can be applied on interval valued intuitionistic fuzzy magnified translation, neutrosophic cubic magnified translation and T-neutrosophic cubic magnified translation.

Conflicts of Interest

The authors declare no conflict of interest.

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Journal of New Theory
ISSN: 2149-1402

35 (2021) 48-61
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
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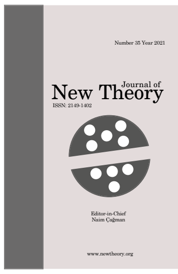

Models for Overdispersion Count Data with Generalized Distribution: An Application to Parasites Intensity

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Article History
Received: 23 Mar 2021
Accepted: 04 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract − The Poisson regression model is widely used for count data. This model assumes equidispersion. In practice, equidispersion is seldom reflected in data. However, in real-life data, the variance usually exceeds the mean. This situation is known as overdispersion. Negative binomial distribution and other Poisson mix models are often used to model overdispersion count data. Another extension of the negative binomial distribution in another model for count data is the univariate generalized Waring. In addition, the model developed by Famoye can be used in the analysis of count data. When the count data contains a large number of zeros, it is necessary to use zero-inflated models. In this study, different generalized regression models are emphasized for the analysis of excessive zeros count data. For this purpose, a real data set was analysed with the generalized Poisson model, generalized negative binomial model, generalized negative binomial Famoye, generalized Waring model, and the foregoing zero-inflated models. Log-likelihood, Akaike information criterion, Bayes information criterion, Vuong statistics were used for model comparisons.

Keywords − *Count data, overdispersion, generalized distribution models, zero-inflated*

Mathematics Subject Classification (2020) − 62P10, 62J05

1. Introduction

In regression analysis, the relationship between a dependent variable and one or more independent variables is examined. When the dependent variable consists of count data, count regression models are used instead of classical regression analysis. Count data can be expressed as observations consisting of nonnegative integers that can take the value of zero or a value greater than zero and show a discrete distribution [1]. Count data are generally right skewed and do not show normal distribution [2]. Different count regression models are used according to the mean and variance in modelling this type of data. Poisson regression analysis is based on the assumption of equidispersion, in other words, equality of mean and variance(mean=variance) in cases where the dependent variable is count data. In practice, however, even the distribution of data is rare. In practice, however, the variance usually exceeds the mean. This occurrence of non-Poisson variation is known as overdispersion (mean>variance) [3]. In cases where the variance is smaller than the mean, the data is considered underdispersion (mean<variance). Modelling overdispersion or underdispersion count data with inappropriate models can lead to overestimated standard errors and misleading inferences [4]. In this case, regression models containing the dispersion parameter should be used in the dataset instead of Poisson

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regression. Two approaches explain the overdispersion that occurs in Poisson regression. One of these approaches is the semi-likelihood approach, and the other approach is the mixed Poisson model approach [5]. Besides several models for modelling overdispersion count data, such as negative binomial distributions, quasi-Poisson, and other Poisson mixes, there are several models for underdispersion count data [6,7]. Harris, Yang, and Hardin (2012) proposed a generalized Poisson (GP) regression model for underdispersion count data [8].

Count data can be expressed as observations made up of nonnegative integers and showing a discrete distribution. For this reason, count data are generally right-skewed and do not show normal distribution. Poisson regression or alternative models are used in modelling this type of data, depending on the state of mean and variance. Count data can be analysed using regression models based on the Poisson distribution in the case of equidispersion. However, other discrete regression models, such as the generalized negative binomial distribution (GNB), can be used in case of overdispersion [9,10]. Also, a model was investigated by Famoye (1995) to show its use for analysing grouped binomial data [11]. In case of overdispersion, GNB is recommended besides the negative binomial distribution. GP has been found to be useful in fitting overdispersion and under dispersion count data to a model [12]. GNB is a simplification based on the generalized negative binomial distribution.

The generalized Waring distribution is an extension of the negative binomial distribution. This distribution is known as the beta negative binomial distribution. Waring distribution was first proposed and used by Irwin (1968) to model accident number data [13]. One advantage of this model over the negative binomial model is that researchers can distinguish unobserved heterogeneity from internal factors of each individual's characteristics and covariates that can affect the variability of the data. These models obtain parameter estimates by including the effect from overdispersion into the model.

Generalized Famoye (GNB-F) and generalized Waring (GNB-W) models have different applications in the literature. Various applications of GNB-F have been demonstrated in physics, ecology, medicine, etc. [14-16]. Another issue to be considered in the analysis of count data is the zeros' density in the dependent variable. Count data have zero values by nature, and the classical ordinary least squares (OLS) method does not give good estimates because it does not show a normal distribution.

The presence of more than expected zero values in the data set is defined as zero inflation [17,18]. It is more appropriate to analyse such data sets with zero-inflated models that take into account zeros [19]. Zero-inflated models are used in different fields such as econometrics, demography, medicine, public health, biology, agriculture, etc. Failure to use appropriate methods to analyse zero-inflated data may result in biased parameter estimates, smaller standard errors, and inconsistent results [20]. Zero-inflated count data may lack equality of mean and variance. In such a case, overdispersion or underdispersion must be taken into account.

The zero-inflated generalized Poisson (ZIGP) model is an extension of the generalized Poisson distribution [21]. Other widely used methods are the zero-inflated negative binomial (ZINB) model and Hurdle models in case of excess zeros in the data [22,23]. There are two types of zeros in the zero-inflated model: "real zeros" and "excess zeros". There are situations where a zero-inflation model makes sense in terms of theory or common sense. Altun (2018) proposed Poisson-Lindley distribution for overdispersion data. The Poisson-Lindley distribution arises when the parameter of the Poisson distribution has the Lindley distribution [24]. Unlike the Poisson distribution, the Poisson-Lindley distribution allows for overdispersion. Therefore, this model is a good option for modelling datasets that are overdispersion and zero-inflated.

In this study, some generalized models used for count data with overdispersion are discussed. These models are generalized Poisson (GP), generalized negative binomial Famoye (GNB-F), generalized negative binomial (GNB), generalized negative binomial Waring (GNB-W), zero-inflated negative binomial (ZINB), zero-inflated negative binomial Waring (ZINB-W) and zero-inflated negative binomial Famoye (ZINB-F) regression models.

2. The Generalized Models

2.1. Generalized Poisson Regression Model (GP)

The most widely used regression model for count data sets is the Poisson regression model with the log-link function. The most prominent feature of the Poisson model is its equidispersion. Still, in implementations, data sets often have a variance that exceeds the mean. When there is overdispersion in the data set, the generalized Poisson distribution is as follows [25];

$$f(y_i, \theta_i, k) = \frac{\theta_i(\theta_i + ky_i)^{y_i-1}e^{-\theta_i-ky_i}}{y_i!} \quad y_i = 0, 1,2, \ldots. \tag{1}$$

here $\theta_i > 0$ and $\max\left(-1, \frac{-\theta_i}{4}\right) < k < 1$. Also, the expected value and variance of the generalized Poisson distribution can be written as:

$$\mu_i = E(Y_i) = \frac{\theta_i}{1-k}$$
$$var(Y_i) = \frac{\theta_i}{(1-k)^3} = \frac{\theta_i}{(1-k)^2} \tag{2}$$
$$E(Y_i) = \emptyset E(Y_i)$$

In particular, the term $\emptyset = 1(1-k)^2$ plays the role of a dispersed factor. It is clear that the generalized Poisson distribution for $k = 0$ is the general Poisson distribution with the parameter of θ_i. When $k < 0$, under dispersion, occurs, while when $k > 0$, overdispersion occurs [26]. The presence of overdispersion will cause the standard error to be below estimate and misinterpretation of the regression parameters. As a result, a number of estimation methods have been proposed to model data in the occurrence of overdispersion. These models include the quasi-Poisson or quasi-binomial regression model and the negative binomial distribution. Parameter estimates of these models are similar to the simple Poisson approach, but confidence intervals are larger [27]. As a result, the models will give different results in terms of the significance of the coefficients.

2.2. Generalized Negative Binomial: Famoye (GNB-F)

The GNB-F model assumes that the value of θ is an unknown scalar parameter. So, the probability mass function of the distribution, mean, and variance are given as:

$$P(Y = y) = \frac{\theta}{\theta + \emptyset y}\binom{\theta + \emptyset y}{y}\mu^y(1-\mu)^{\theta - y - \emptyset y}$$
$$0 < \mu < 1, \qquad 1 \leq \emptyset < \mu^{-1}\ , \qquad \theta > 0 \text{ and } y_i \in (0,1,2, \ldots) \tag{3}$$
$$E(Y) = \theta_i\mu(1-\emptyset\mu)^{-1}$$
$$var(Y_i) = \theta_i\mu(1-\emptyset\mu)^{-1}(1-\emptyset\mu)^{-3}$$

Its main difference from the negative binomial model is that the θ parameter is unknown in Equation 2, but a known parameter in Equation 3 $\sigma = \emptyset > 1$. As the $\emptyset$ value approaches 1, the variance approaches the negative binomial. Thus, the parameter is generalized to have greater variance than is allowed in the GNB-F model. To compare the results of the Poisson and negative binomial distribution, the log link is as follows:

$$log(\mu) = x\beta \tag{4}$$

2.3. Generalized Negative Binomial Regression Model (GNB)

The negative binomial distribution is the first distribution to consider when the variance is greater than the mean. Negative binomial regression is used as an alternative to Poisson regression because these two methods fit the model by using the same connection log link function [28]. NB model is often used to model overdispersion count result variables. The assumption that the Poisson parameter changes proportionally to the chi-square leads to the negative binomial distribution.

The GNB model is based on the simplification of the generalized negative binomial distribution. If $\sigma = \emptyset$ and $\mu = \pi/(1+\emptyset\pi)$ expressions are substituted for σ and μ expressions given in Equation 3, the parameter becomes a vector of observation-specific known constants. When the θ parameter is known, while $\emptyset > 1$, the σ parameter is not negative in the generalized negative binomial distribution. Thus, under these conditions, the probability mass function, mean, and variance are given by:

$$P(Y=y) = \frac{n}{n+\sigma y}\binom{n+\sigma y}{y}\left(\frac{\pi}{1+\sigma\pi}\right)^{y}\left(1-\frac{\pi}{1+\sigma\pi}\right)^{n-y-\sigma y}$$

$$E(Y) = n\frac{\pi}{1+\sigma\pi}\left(1-\frac{\pi}{1+\sigma\pi}\right)\sigma^{-1} \quad (5)$$

$$var(Y) = n\frac{\pi}{1+\sigma\pi}\left(1-\frac{\pi}{1+\sigma\pi}\sigma\right)(1+\sigma\pi)^{-3}$$

$$= n\pi(1+\sigma\pi)(1+\sigma\pi-\pi)$$

Therefore, the variance is equal to binomial variance, $= 0$. It is equal to negative binomial variance $if\ \sigma = 1$. Here, if $\sigma > 0$, GNB generalizes the binomial distribution in the regression model.

2.3.1. Generalized Waring Regression Model (GNB-W)

The negative binomial distribution is a limiting case of the generalized Waring distribution. This distribution provides a model for the distribution of accidents. Here the variance is divided into three components. The first of these is the usual random component in classical accident theory; the other two can often be described as separate variances due to "liability" and "proneness". The sum of the last two components is the only component defined by the variation in sensitivity in classical theory [13]. The generalized Waring distribution (the number of crashes A) depends on three parameters: $\pi, a,$ and k. However, the "three-component distribution" is not necessarily a generalized Waring distribution. The generalized Waring distribution must satisfy the following conditions:

i. $Y \,/\, x, \lambda_x, v$ ~ Poisson (μ_x)
ii. $\lambda_x, /v$ ~ Gamma (a_x, v)
iii. v ~ Beta (ρ, k)

In Irwin's study on accident data, λ/v is specified as "accident liability" and v as "accident proneness". Thus, the mass density function is given by:

$$P(Y=y) = \frac{\Gamma(a_x+\rho)\Gamma(k+\rho)}{\Gamma(\rho)\Gamma(a_{x+k+\rho})}\frac{(a_x)_y(k)_y}{(a_x+k+\rho)_y}\frac{1}{y!} \quad (6)$$

where $a_x, k, \rho > 0$; $a_x = \mu(\rho-1)/k$ and $(a)_w$ is the Pochhammer notation $\Gamma(a+w)/\Gamma(w)$, if a>0.

$$E(Y) = \mu = \frac{a_x k}{\rho - 1} \quad (7)$$

$$var(Y) = \mu + \mu\left(\frac{k+1}{\rho-2}\right) + \mu^2\left\{\frac{k+\rho-1}{k(\rho-2)}\right\}$$

2.3.2. Zero-Inflation Model

Count models can also take zero value due to their nature. However, having more than the expected number of zero values in the data set is defined as zero inflation. In the datasets where most of the observations are zero, excluding the zero values from the analysis leads to incorrect results. Zero-inflated count data may lack equality of mean and variance. Therefore, when there are too many zeros, it may not be appropriate to use Poisson and other models. It is more appropriate to use zero-inflated Poisson (ZIP), zero-inflated negative binomial (ZINB), Poisson Hurdle (PH) or negative binomial Hurdle (NBH) regression methods in modelling dependent variables with more than the expected number of zero values [29].

Hardin and Hilbe (2012) describe two origins of the zero result [15]: Those who do not enter the counting process and those who enter the counting process and have a zero result. Therefore, the model should be divided into different parts, one being zero count $y = 0$ and the other being nonzero $y > 0$. The zero-inflated model can be given as:

$$P(Y = y) = \begin{cases} p + (1-p)f(y), & y = 0 \\ (1-p)f(y), & y > 0 \end{cases} \tag{8}$$

In the above equation, p is the probability that the binary process will result in zero results. Here $0 \leq p < 1$ and $f(y)$ is the probability function. Famoye and Singh (2006) proposed the zero-inflated generalized Poisson (ZIGP) model, an extension of the generalized Poisson distribution. In another widely used method, the negative binomial model may be preferred where the Poisson mean has a gamma distribution [21]. A natural extension of the negative binomial model, the zero-inflated negative binomial (ZINB) model, is used in case of excess zeros in the data [23].

For the Waring distribution and Famoye's proposed models, it is more appropriate to use zero-inflated versions if there are too many zeros in the data. In this context, ZINB-W and ZINB-F distributions have been proposed for models based on zero.

3. Model Selection

The fact that all p values in the model selection are less than 0.05 means that all explanatory variables are suitable for the model. However, the fact that all the explanatory variables are significant does not mean that the regression model applied will be suitable for the data. Various tests are used to determine which model is more suitable for count data. In this study, Akaike Information Criterion (AIC), Bayes Information Criterion (BIC), log-likelihood (LL) value and Vuong statistics were used. The interpretation that a model is good can be made when the AIC and BIC value is the smallest, or the LL value is the largest. Vuong test is one of the tests used to compare non-nested models. Apart from nested model comparisons, possible binary models can also be compared with the Vuong test. It is a widely used test, especially in zero-inflated model comparisons. In this way, it can be determined which models are suitable for models with excessive zeros.

3.1. Log-Likelihood (LL)

The advantage of using the maximum likelihood method (ML) is that the log-likelihood (LL) test can be used for model comparisons. The LL test can be used to test for the presence of overdispersion. To test the Poisson model against the GP model, where α is the overdispersion parameter, the hypothesis is expressed as $H_0: \alpha = 0$ and $H_0: \alpha \neq 0$. Probability ratio statistics is calculated as;

$$LL = 2(lnL_1 - lnL_0) \tag{9}$$

Where L_1 and L_0 are the log-likelihood under the respective hypothesis. LL has an asymptotic chi-square distribution with one degree of freedom [30]. When choosing the model over the LL value, the model with the largest log-likelihood value is determined as the appropriate model.

3.2. Akaike Information Criterion (AIC)

This criterion, which is widely used to compare different models, can be expressed as follows [31];

$$AIC = 2k - 2log(L) \tag{10}$$

In this equation, L represents the maximum value of the log-likelihood function, and k represents the number of explanatory variables. Among the existing models, the model with the lowest AIC value is selected as the appropriate model. In cases where the number of parameters is larger than the sample size, the AICc proposed by Hurvich and Tsai should be used instead of AIC [32]. This value can be written as follows [31-33];

$$AICc = AIC + \frac{2k(k+1)}{n-k-1} = \frac{2kn}{n-k-1} - 2\ln(L) \tag{11}$$

3.3. Bayes Information Criteria (BIC)

Akaike derived the BIC (Bayesian Information Criterion) model selection criteria for selected model problems in linear regression [34]. The equation regarding the Bayesian measure of knowledge is as follows:

$$BIC = -2log(\mathcal{L}) + klog(n) \tag{12}$$

As in the Akaike information criterion, the model with the smallest BIC value among the available models is selected as the appropriate model.

3.4. Vuong Test

The Vuong statistic is used to compare non-nested models such as ZIP, NB and ZINB. This test is a statistic used when there is no missing observation in the data set [35,36]. Equations used for the Vuong test are given in Equation 13 and Equation 14.

$$Vuong = \frac{\overline{m}\sqrt{n}}{\sum\sqrt{\frac{m-\overline{m}}{n-1}}} = \frac{\overline{m}\sqrt{n}}{s_m} \tag{13}$$

Here, m_i is a random variable, $\overline{m}$ is the mean of m_i, s_m is the standard deviation, and n is the sample size. Suppose we want to compare the probability density functions of the ZIP and ZINB models. The H_o and H_1 hypotheses are as follows:

H_0: ZIP and ZINB distribution functions are equal

H_1: ZIP and ZINB distribution functions are not equal

Probability density functions with f_1 and f_2 , the representation way of m_i is as follows;

$$m_i = \log\left(\frac{f_1(y_i/x_i)}{f_2(y_i/x_i)}\right) \tag{14}$$

Within the family of ZIP models, testing if a Poisson model is adequate corresponds to testing: $H_0 = \emptyset_i = 0$ vs. $H_0 = \emptyset_i > 0$. In the interpretation of the Vuong test value having a normal distribution (e.g. for $\alpha = 0.05$ significance level), if the Vuong value is greater than 1.96, the first model is interpreted as "closer" to the real model; if the Vuong value is less than -1.96, the second model can be interpreted as "closer" to the real model. If the calculated value is not between $(-1.96;\ 1.96)$, it is interpreted as "there is no difference between using the first or the second model" [37].

4. Experimental Results

In this study, the data from a three-year study conducted by Hemmingsen et al. (2005) in four regions off the Norwegian coast to count parasites [38] were used. The dependent variable is the number of parasites (Intensity), while the independent variables are depth, length of the fish, and area. In addition, missing observations in the original data were removed from the model. Since the data set contains a large number of zeros, it was tested in zero-inflated models and generalized models. Statistical analysis of the study was made using Stata 14 software program. The frequency distribution showing the parasite density is given in Figure 1.

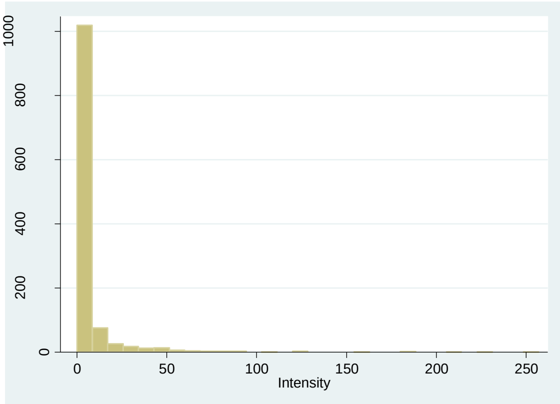

Fig 1. Frequency distribution of the number of parasites

4.1. Generalized Poisson Regression Model (GP)

The Poisson regression model, one of the most widely used generalized models, was first tried because overdispersion was detected in observation values. The results obtained are given in Table 1. IRR values show exp β values in Tables.

Table 1. Generalized Poisson regression model (GP)

Generalized Poisson regression	Number of obs	=	1191
	LR chi2(3)	=	89.68
Dispersion = .8910582	Prob > chi2	=	0.0000
Log-likelihood = −2566.7445	Pseudo R2	=	0.0172

Intensity	IRR	Std. Err.	z	$P > z$	[95% Conf.	Interval]
Depth	1.004756	.0006329	7.50	0.000	1.00351	1.006003
Length	.9981571	.0028743	−0.64	0.521	.9925496	1.003796
Area	1.123788	.047966	2.43	0.015	1.022953	1.234562
_cons	2.024088	.2408844	2.93	0.003	1.262374	3.245417
/atanhdelta	1.427039	.0548193			1.319595	1.534483
delta	.8910582	.0112936			.8666833	.9111886

Likelihood-ratio test of delta=0: chi2(1) = $1.8e+04$ Prob>=chi2 = 0.0000

According to the results presented in Table 1, the length was found to be insignificant in terms of the number of parasites. The area and depth variables were found to be significant. According to the GP model, the length variable was found insignificant ($p > 0.05$). Area and depth variables are significant ($p < 0.05$). Accordingly, a one-unit increase in depth increases the parasite intensity approximately 1.005 times. When the area changes, the parasites density increases approximately 1.124 times. When the model is evaluated as a whole, it is significant according to the chi-square test.

4.2. Generalized Negative Binomial: Famoye (GNB-F)

Results for GNB-F are shown in Table 2. According to this model, the depth, length, and area variables are significant ($p < 0.05$). The model was again found to be statistically significant ($Prob > chi2 = 0.0000$).

Table 2. Generalized negative binomial: Famoye (GNB-F)

Generalized negative binomial-F regression		Number of obs	=	1191		
		LR chi2(3)	=	120.96		
Log likelihood = -2550.873		Prob > chi2	=	0.0000		
Intensity	**IRR**	**Std. Err.**	***z***	***P* > *z***	**[95% Conf.**	**Interval]**
Depth	1.006702	.0013168	5.11	0.000	1.004.125	1.009286
Length	.9756556	.0050295	−4.78	0.000	.9658476	.9855632
Area	1.245722	.0802515	3.41	0.001	1.097957	1.413373
_cons	15.753	6.412282	6.77	0.000	7.093826	34.98212
/lnphim1	−6.154845	2.049813			−101.724	−2.137286
/lntheta	−1.64351	.0809544			−1.802178	−1.484842
phi	1.002123	.0043521			1.000038	1.117975
theta	.1933004	.0156485			.1649393	.2265381

4.3. Generalized Negative Binomial Regression Model (GNB)

The GNB distribution is one of the most widely used models in cases where the variance is greater than the mean; that is, in the case of overdispersion. The results obtained are given in Table 3. The fact that $\alpha = 5.34$ dispersion parameter is greater than zero indicates that it is overdispersion. According to the GNB model, the depth, length, and area variables are significant ($p < 0.05$). One unit increase in depth increases the parasite intensity nearly 1.006 times. When the area changes, the parasite intensity increases approximately 1.258 times.

Table 3. Generalized negative binomial regression model (GNB)

Generalized Poisson regression		Number of obs	=	1191		
		LR chi2(3)	=	114.65		
		Prob > chi2	=	0.0000		
Log likelihood = -2551.0291		Pseudo R2	=	0.0220		
Intensity	**IRR**	**Std. Err.**	***z***	***P* > *z***	**[95% Conf.**	**Interval]**
Depth	1.006555	.0012471	5.27	0.000	1.004114	1.009002
Length	.9753681	.004905	−4.96	0.000	.9658017	.9850293
Area	1.258504	.0772588	3.75	0.000	1.115834	1.419414
_cons	3.096207	1.192705	2.93	0.003	1.45524	6.587573
/lnalpha	1.675206	.0539381			1.56949	1.780923
alpha	5.339897	.288024			4.804195	5.935333
Likelihood-ratio test of delta= 0: chibar2(01) = $1.8e+04$ Prob>=chibar2 = 0.0000						

4.3.1. Generalized Waring Regression Model (GNB-W)

The results obtained for the GNB-W model are given in Table 4. The depth, length and area variables are significant according to the GNB-W model ($p < 0.05$).

Table 4. Generalized Waring regression model (GNB-W)

Generalized negative binomial-W regression		Number of obs	=	1191		
		LR chi2(3)	=	106.77		
Log likelihood = −2544.401		Prob > chi2	=	0.0000		

Intensity	**IRR**	**Std. Err.**	***z***	***P* > *z***	**[95% Conf.**	**Interval]**
Depth	1.008.675	.0014077	6.19	0.000	1.005919	1.011437
Length	.9783038	.0058167	−3.69	0.000	.9669695	.9897709
Area	1.204.492	.0851747	2.63	0.009	1.048605	1.383554
_cons	2.089.578	.9044374	1.70	0.089	.8946043	4.880748
/lnrhom2	−.4083326	.4713641			−1.332189	.5155241
/lnk	−1.508367	.0772628			−1.659799	−1.356935
rho	2.664758	.3133429			2.263899	3.674516
k	.221271	.017096			.1901771	.2574488

One unit increase in depth increases the parasites intensity approximately 1.008 times. When the area changes, the parasite intensity increases about 1.204 times.

4.3.2. Zero-Inflation

In the study, 651 observations within 1191 observations were found to contain zero values. In other words, approximately 55% of the parasite count data consists of zero observation. For this reason, analyses have also been made with zero-inflated models. The results of ZINB, ZINB-W, and ZINB-F models are given below.

4.3.2.1. Zero-Inflated Negative Binomial Regression (ZINB)

Zero-inflated models consist of two parts. ZINB model results are given in Table 5. The length variable is specified as the inflate variable. One unit increase in depth increases the parasites intensity approximately 1.006 times. When the area changes, the parasite intensity increases approximately 1.238 times. As the length decreases, the parasite density decreases (−0.223).

Table 5. Zero-inflated negative binomial regression (ZINB)

Zero-inflated negative binomial regression		Number of obs	=	1191		
		Nonzero obs	=	540		
		Zero obs	=	651		
Inflation model = logit		LR chi2(3)	=	128.03		
Log likelihood = −2539.562		Prob > chi2	=	0.0000		

Intensity	**IRR**	**Std. Err.**	***z***	***P* > *z***	**[95% Conf.**	**Interval]**
intensity						
Depth	1.006497	.001221	5.34	0.000	1.004107	1.008893
Length	.9693981	.00494	−6.10	0.000	.9597642	.9791287
Area	1.238253	.0743365	3.56	0.000	1.100801	1.392868
_cons	4.755356	1.888854	3.93	0.000	2.183137	10.35822
inflate						
Length	−.2229618	.0809777	−2.75	0.006	−.3816751	−.0642484
_cons	5.721762	2.252244	2.54	0.011	1.307444	10.13608
/lnalpha	1.584263	.061479	25.77	0.000	1.463767	1.70476
alpha	4.875699	.2997532			4.322209	5.500066

Vuong test of zinb vs. standard negative binomial: $z = 2.58$ Pr> $z = 0.0049$

Using the ZINB distribution is more meaningful than the standard negative binomial distribution according to the Vuong test ($z = 2.58$ Pr $> z = 0.0049$). The variables are significant for both parts of the model.

4.3.2.2. Zero-Inflated Negative Binomial Regression-W (ZINB-W)

The ZINB-W model was compared with the standard Waring model. Table 6 shows the results. One unit increase in depth increases the parasites intensity approximately 1.008 times. When the area changes, the parasite intensity increases approximately 1.164 times. As the length decreases, the parasite density decreases (-0.175).

Table 6. Zero-inflated negative binomial regression-W (ZINB-W)

Zero-inflated gen neg binomial-W regression		Number of obs	= 1191			
Regression link :		Nonzero obs	= 540			
Inflation link: logit		Zero obs	= 651			
		Wald chi2(3)	= 126.93			
Log likelihood = -2530.653		Prob > chi2	= 0.0000			
Intensity	**IRR**	**Std. Err.**	z	$P > z$	**[95% Conf.**	**Interval]**
intensity						
Depth	1.008.867	.0013606	6.55	0.000	1.006204	1.011537
Length	.9667968	.0063723	−5.12	0.000	.9543876	.9793674
Area	1.163945	.0820843	2.15	0.031	1.013686	1.336477
_cons	4.490452	2.178.058	3.10	0.002	1.735488	1.161873
inflate						
Lengt	−.1750797	.0808031	−2.17	0.030	−.3334509	−.0167085
_cons	4.615169	2.244926	2.6	0.040	.2151952	9.015143
/lnrhom2	−.9190866	.4479906			−1.797132	−.0410411
/lnk	−1.337637	.1120339			−1.55722	−1.118055
rho	2.398883	.1786959			2.165774	2.95979
k	.262465	.029405			.2107211	.3269151
Vuong test of zinbregw vs. gen neg binomial(W): $z = 29.81$ Pr $> z = 0.0000$						
Bias-corrected (AIC) Vuong test: $z = 29.81$ Pr $> z = 0.0000$						
Bias-corrected (BIC) Vuong test: $z = 29.79$ Pr $> z = 0.0000$						

According to the Vuong test, this model is more significant than the generalized negative binomial distribution. The number of parasites increases as depth, length, and area change.

4.3.2.3. Zero-Inflated Negative Binomial Regression-F (ZINB-F)

The results found for ZINB-F are as in Table 7. When this model was compared with the results obtained with GNB-F, the Vuong test was found to be significant.

Table 7. Zero-inflated negative binomial regression-F (ZINB-F)

Zero-inflated gen neg binomial-F regression		Number of obs	=	1191		
Regression link :		Nonzero obs	=	540		
Inflation link: logit		Zero obs	=	651		
		Wald chi2(3)	=	139.87		
Log likelihood = −2539.562		Prob > chi2	=	0.0000		

Intensity_	**IRR**	**Std. Err.**	***z***	***P* > *z***	**[95% Conf.**	**Interval]**
Intensity_						
Depth	1.006497	.0012211	5.34	0.000	1.004107	1.008893
Length	.9693989	.0049403	−6.10	0.000	.9597644	.9791302
Area	1.238.237	.0743426	3.56	0.000	1.100775	1.392866
_cons	2.318.361	9.117.887	7.99	0.000	10.72536	50.11299
inflate						
Length	−.2229628	.0809865	−2.75	0.006	−.3816934	−.0642322
_cons	5.721764	2.252445	2.54	0.011	1.307053	10.13647
/lnphim1	−15.16193	695.4499			−1378.219	1347.895
/lntheta	−1.584.224	.061541			−1.704842	−1.463606
phi	1	.0001809			1	.
theta	.2051068	.0126225			.181801	.2314003

Vuong test of zinbregf vs. gen neg binomial(F): $z = 3.97$ Pr> $z = 0.0000$

Bias-corrected (AIC) Vuong test: $z = 3.74$ Pr > $z = 0.0001$

Bias-corrected (BIC) Vuong test: $z = 3.14$ Pr > $z = 0.0008$

5. Conclusion

Modelling discrete data is a special type of regression. As is known, linear regression analysis can be used in cases where the dependent variable is continuous. However, the data to be used in the analysis may not always be available continuously. In such cases, if the data are discontinuous, analyses using linear regression models will give ineffective, inconsistent, and contradictory results. Therefore, count data models should be used when the dependent variable consists of nonnegative discrete values. One of the most common models used in count data analysis is the Poisson regression model. The most important feature of the Poisson regression model is that the variance and mean are equal. Generally, this feature cannot be provided in practice. In this case, negative binomial regression analysis or generalized Poisson regression analysis is widely used. In addition, cases where count data contain too many zero values are encountered in many areas. In such cases, the zero-inflated Poisson, zero-inflated Negative Binomial, Poisson Hurdle and Negative Binomial Hurdle regression models can be preferred.

Table 8. Model Selection

Count Models	**LL**	**AIC**	**BIC**
GP	−2566.7445	5143.489	5168.902
GNB-F	−2550.873	5113.747	5144.242
GNB	−2551.0291	5112.058	5137.471
GNB-W	−2544.401	5100.802	5131.298
ZINB	−2539.562	5093.124	5128.702
ZINB-W	**−2530.653**	**5077.306**	**5117.967**
ZINB-F	−2539.562	5095.124	5135.785

This study aims to compare the generalized Famoye and Waring models with classical methods apart from the commonly used methods. Thus, GP, GNB-F, GNB and GNB-W models were examined by considering an overdispersion data set. Since approximately 55% of the data set consists of zero, the zero-inflated models of these models were also tested, and a model comparison was made. As a result, LL, AIC, and BIC values for the six count models are given in Table 8. Due to the large number of zeros in the data set we were using, zero-inflated models yielded better results. Among these, the highest LL value and the lowest AIC and BIC values were obtained for the ZINB-W model.

The study focused on generalized models, especially on count regression models. For these models, their performances can also be investigated by conducting a simulation study. In the case of different rates of zero values and outliers in the data set, the models' performances can be compared. Thus, the reliability of the obtained results can be increased by selecting the appropriate model for the data structure.

Conflict of Interest

The authors declare no conflict of interest.

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Journal of New Theory

ISSN: 2149-1402

35 (2021) 62-71
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Semi-Invariant Riemannian Submersions with Semi-Symmetric Non-Metric Connection

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Article History
Received: 28 Mar 2021
Accepted: 07 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract — In this paper, we investigate semi-invariant Riemannian submersion from a Kaehler manifold with semi-symmetric non-metric connection to a Riemannian manifold. We study the geometry of foliations with semi-symmetric non-metric connection. Later, we introduce base manifold to be a local product manifold with semi-symmetric non-metric connection.

Keywords — *Riemannian submersions, semi-invariant submersions, semi-symmetric non-metric connection, Kaehler manifold*

Mathematics Subject Classification (2020) — 53C15, 53C40

1. Introduction

A conventional way to compare two manifolds is by defining smooth maps from one manifold to another. One such map is submersion, whose rank equals to the dimension of the target manifold. Riemannian submersion between Riemannian submanifolds were first introduced by O' Neill and Gray [1, 2]. Later many authors studied different geometric properties of the Riemannian submersions [3], semi-slant submersions [4–6], hemi-slant submersions [7–9], semi-invariant submersions [10–12], anti-invariant submersions [13–15].

On the other hand, Friedmann et al. defined the concept of the semi-symmetric non-metric connection in a differential manifold [16]. Hayden studied metric connection with torsion a Riemannian manifold [17]. Later, Yano investigated a Riemannian manifold with new connection, which is called a semi-symmetric metric connection [18]. Afterwards, Agashe et al. studied semi-symmetric non-metric connection (SSNMC) on a Riemannian manifold [19]. Many author have studied semi-symmetric connection [20–26].

Let M be differentiable manifold with linear connection ∇. Therefore, for all $K, L \in \Gamma(TN)$, we get

$$T(K,L) = \nabla_K L - \nabla_L K - [K,L],$$

where T is torsion tensor of ∇. If the torsion tensor $T = 0$, then the connection ∇ is said to be symmetric, otherwise it is called non-symmetric. Moreover, for all $K, L \in \Gamma(TN)$, the connection ∇ is said to be semi-symmetric if

$$T(K,L) = \eta(L)K - \eta(K)L$$

where η is a 1-form on N. However, ∇ is called metric connection if $\nabla g = 0$ with Riemannian metric g, otherwise it is said to be non-metric.

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In [27], Akyol and Beyendi studied the idea of Riemannian submersion with SSNMC. They investigated O'Neill's tensor fields, obtain derivatives of those tensor fields and compare curvatures of the total manifold, the base manifold and the fibres by computing curvatures.

The main purpose of this paper is to investigate geometry of semi-invariant Riemannian submersion from a Kaehler manifold with SSNMC to a Riemannian manifold.

2. Preliminaries

Definition 2.1. Let $F : (N^n, g_N) \to (B^b, g_B)$ be a submersion between two Riemannian manifolds. Then, F said to be Riemannian submersion if

i. F has maximal rank.

ii. The differential F_* preserves the lenghts of horizontal vectors.

On the other hand, $F^{-1}(k)$ is an $(n-b)$ dimensional submanifold of N, for each $k \in N$ The submanifolds $F^{-1}(k)$ are called fibers. Moreover, vector fields tangent to fibers are called vertical and vector fields orthogonal to fibers are horizantal. A vector field X on N is called basic if X is horizontal and $F_*X_q = X_{\pi_*(q)}$ for all $q \in N$. We determine that $\mathcal{V}$ and $\mathcal{H}$ define projections $kerF_*$ and $(kerF_*)^{\perp}$, respectively.

On the other hand, a Riemannian submersion $F : N \to B$ determines tensor fields T and A on N such that,

$$T(E,F) = T_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{H}F, \tag{1}$$

$$A(E,F) = A_E F = \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{V}F \tag{2}$$

for any $E, F \in \Gamma(TM)$ (see [1]). By virtue of (1) and (2), one can obtain

$$\overset{M}{\nabla}_V W = T_V W + \hat{\nabla}_V W \tag{3}$$

$$\overset{M}{\nabla}_V X = T_V X + \mathcal{H}(\overset{M}{\nabla}_V X) \tag{4}$$

$$\overset{M}{\nabla}_X V = \mathcal{V}(\overset{M}{\nabla}_X V) + A_X V \tag{5}$$

$$\overset{M}{\nabla}_X Y = A_X Y + \mathcal{H}(\overset{M}{\nabla}_X Y) \tag{6}$$

for all $V, W \in \Gamma(kerF_*)$ and $X, Y \in \Gamma((\ker F_*)^{\perp})$. Further, if X is basic, then

$$\mathcal{H}(\overset{M}{\nabla}_V X) = A_X V \tag{7}$$

On the other hand, let N, B be two Rieamannian manifold and $F : N \to B$ is a smooth map. Therefore, the second fundamental form of F is expressed by

$$(\nabla F_*)(K,L) = \overset{B}{\nabla}_K F_* L - F_*(\overset{N}{\nabla}_K L) \tag{8}$$

for $K, L \in \Gamma(TN)$. Moreover, π is said to be a *totally geodesic* map if $(\nabla F_*)(K,L) = 0$ for $K, L \in \Gamma(TN)$ [28].

Now, we recall the definition of Kaehler manifold. Let N be a Hermitian manifold with respect Hermitian structure (J, g) such that

$$J^2 = -I \tag{9}$$

and

$$g(E,F) = g(JE, JF) \tag{10}$$

for all $E, F \in \Gamma(TN)$,where $g(JE,F) = -g(E,JF)$.

A Hermitian manifold is called Kaehler manifold if

$$\nabla J = 0 \tag{11}$$

On the other hand, we define a linear connection $\widetilde{\nabla}$ on Kaehler manifold N such that

$$\widetilde{\nabla}_E F = \nabla_E F + \eta(F)E \tag{12}$$

where $E, F \in \Gamma(TN)$, ∇ is a Levi-Civita connection on N and η is a 1-form with the vector field P on N by

$$\eta(E) = g(E, P)$$

By virute of (12), we arrive that

$$\widetilde{T}(E, F) = \eta(F)E - \eta(E)F$$

and

$$(\widetilde{\nabla}_E g)(F, K) = -\eta(F)g(E, K) - \eta(K)g(E, F)$$

where $\widetilde{T}$ is torsion tensor of $\widetilde{\nabla}$. Then, $\widetilde{\nabla}$ defined a semi-symmetric non metric conection with (12).

Let N be a Kaehler manifold. We using (12), for all $K, L \in \Gamma(TN)$, we get,

$$\begin{aligned}(\nabla_K J)L &= \nabla_K JL - J\nabla_K L \\ &= \widetilde{\nabla}_K JL - \eta(L)K - J\widetilde{\nabla}_K L + \eta(L)JK\end{aligned}$$

Then, using (11) we obtain,

$$(\widetilde{\nabla}_K J)L = \eta(L)JK - \eta(L)K \tag{13}$$

Now, we call O'Neill's tensor fields for SSNMC [27]. For all $K, L \in \Gamma(TN)$, we have,

$$\widetilde{T}_K L = T_K L + \eta(hL)vK$$

and

$$\widetilde{A}_K L = A_K L + \eta(vL)hK$$

Then, using last two equations, we obtain

$$\widetilde{\nabla}_K L = T_K L + v\widetilde{\nabla}_K L \tag{14}$$

$$\widetilde{\nabla}_K X = T_K X + h\widetilde{\nabla}_K X + \eta(X)K \tag{15}$$

$$\widetilde{\nabla}_X K = A_X K + v\widetilde{\nabla}_X K + \eta(K)X \tag{16}$$

$$\widetilde{\nabla}_X Y = A_X Y + h\widetilde{\nabla}_X Y \tag{17}$$

where for all $K, L \in \Gamma(kerF_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$.

3. Semi-Invariant Riemannian Submersion

Definition 3.1. Let N and B be a Kaehler manifold and Riemannian manifold, respectively. Let us assume that $F : N \to B$ be a Riemannian submersion. Therefore, F is called semi-invariant Riemannian submersion if there is a distribution $D_1 \subseteq kerF_*$ such that

$$kerF_* = D_1 \oplus D_2$$

and

$$JD_1 = D_1, \quad JD_2 \subseteq (\ker F_*)^\perp$$

where D_2 is orthogonal complementary to D_1 in $kerF_*$ ([12]).

Example 3.2. Let F be a submersion. We denote that

$$F : \begin{array}{ccc} \mathbb{R}^6 & \longrightarrow & \mathbb{R}^3 \\ (x_1, x_2, x_3, x_4, x_5, x_6) & & (\frac{x_1+x_2}{\sqrt{2}}, \frac{x_3+x_6}{\sqrt{2}}, \frac{x_4+x_5}{\sqrt{2}}) \end{array}$$

Then, it follows that

$$kerF_* = span\{V_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2},\ V_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_6}, V_3 = -\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}\}$$

and

$$(kerF_*)^{\perp} = span\{H_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\ H_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6},\ H_3 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$$

Hence we have $JV_1 = -V_1$, $JV_2 = H_3$ and $JV_3 = -H_2$. Thus it follows that $D_1 = span\{V_1\}$ and $D_2 = span\{H_2, H_3\}$. On the other hand, we arrive that,

$$g_{\mathbb{R}^3}(F_*H_2, F_*H_2) = g_{\mathbb{R}^6}(H_2, H_2), \quad g_{\mathbb{R}^3}(F_*H_3, F_*H_3) = g_{\mathbb{R}^6}(H_3, H_3)$$

where $g_{\mathbb{R}^3}$ and $g_{\mathbb{R}^6}$ determine metrics of $\mathbb{R}^3$ and $\mathbb{R}^6$, respectively. Then, F is semi-invariant Riemannian submersion.

Let $F : (N, J, g) \to (B, g)$ be a semi-invariant Riemannian submerion such that N and B are Kaehler manifold and Riemannian manifold respectively. For all $K \in \Gamma(TN)$, we write

$$E = \mathcal{V}E + \mathcal{H}E$$

where $\mathcal{V}E \in \Gamma(kerF_*)$ and $\mathcal{H}E \in \Gamma((\ker F_*)^{\perp})$. Then, for all $K \in \Gamma(kerF_*)$, we write

$$JK = \phi K + \omega K \tag{18}$$

where $\phi K \in \Gamma(D_1)$ and $\omega K \in \Gamma(JD_2)$.

Since F is a semi-invariant Riemannian submersion, we can determine

$$(\ker F_*)^{\perp} = JD_2 \oplus \mu$$

where JD_2 and μ are complementary to each other. Similarly, $x \in \Gamma((\ker F_*)^{\perp})$, we get

$$JX = BX + CX \tag{19}$$

where $BX \in \Gamma(D_2)$ and $CX \in \Gamma(\mu)$.

4. Geometry of Distributions

We note that, for brevity we use a abbreviation "F is a semi-invariant Riemannian submersion with SSNMC" for $F : (N, J, g) \to (B, g)$ be a semi-invariant Riemannian submersion from Kaehler manifold with SSNMC M and Riemannian manifold N.

Theorem 4.1. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution D_1 is integrable if and only if we have

$$\begin{aligned} g_B(F_*(T_V BZ + h\widetilde{\nabla}_V CZ), F_*(\omega U)) - g_B(F_*(T_U BZ + h\widetilde{\nabla}_U CZ), F_*(\omega V)) \quad = \quad & g_N(v\widetilde{\nabla}_U BZ + T_U CZ, \phi V) \\ & -g_N(v\widetilde{\nabla}_V BZ + T_V CZ, \phi U) \\ & +2g_N(\phi U, V)\eta(Z) \end{aligned}$$

for all $U, V \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. Firstly, we using (10) and (13). For all $U, V \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we arrive that

$$g_N(\widetilde{\nabla}_U V, Z) = g_N(\widetilde{\nabla}_U JV, JZ) \tag{20}$$

By virtue of (13) and (20), we get,

$$g_N([U,V], Z) = g_N(\widetilde{\nabla}_U JV, JZ) - g_N(\widetilde{\nabla}_V JU, JZ)$$

After some calculations, we conclude,

$$g_N([U,V], Z) = -g_N(JV, \widetilde{\nabla}_U JZ) + g_N(JU, \widetilde{\nabla}_V JZ)$$

We know that F is a semi-invariant Riemannian submersion, by virtue of (19), (14), (15) and (18), we conclude that,

$$\begin{aligned} g_N([U,V], Z) &= -g_N(\phi V, v\widetilde{\nabla}_U BZ + T_U CZ) - g_N(wV, T_U BZ + h\widetilde{\nabla}_U CZ) - \eta(CZ) g_N(JV, U) \\ &\quad + g_N(\phi U, v\widetilde{\nabla}_V BZ + T_V CZ) + g_N(wU, T_V BZ + h\widetilde{\nabla}_V CZ) + \eta(CZ) g_N(JU, V) \end{aligned}$$

which gives proof. □

Theorem 4.2. Let F be a semi-invariant Riemannian submersion with SSNMC . Therefore, the distribution D_2 is integrable if and only if we have

$$g_B(F_*(T_Z\phi U + h\widetilde{\nabla}_Z wU), F_*(JW)) = g_B(F_*(T_W\phi U + h\widetilde{\nabla}_W wU), F_*(JZ))$$

for all $U \in \Gamma(D_1)$ and $Z, W \in \Gamma(D_2)$.

Proof. By virtue of (12), (10) and (13), we get

$$g_N([Z,W], U) = g_N(\widetilde{\nabla}_Z JW, JU) - g_N(\widetilde{\nabla}_W JZ, JU)$$

for all $U \in \Gamma(D_1)$ and $Z, W \in \Gamma(D_2)$. Therefore, we conclude

$$g_N([Z,W], U) = -g_N(JW, \widetilde{\nabla}_Z JU) + g_N(JZ, \widetilde{\nabla}_W JU)$$

Then, using (18), (14) and (15), we arrive,

$$g_N([Z,W], U) = -g_N(JW, T_Z\phi U + h\widetilde{\nabla}_Z wU) + g_N(JZ, T_W\phi U + h\widetilde{\nabla}_W wU)$$

which proves assertion. □

Theorem 4.3. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution D_1 defines a totally geodesic foliation on N if and only if we have

$$F_*(\widetilde{\nabla}_U JV) \in \Gamma(\mu)$$

and

$$\begin{aligned} g_B(F_*(T_U\phi V), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_U wV), F_*(CX)) &= g_N(v\widetilde{\nabla}_U \phi V + T_U wV, BX) \\ &\quad + g_N(U, BX)\eta(wV) \end{aligned}$$

for all $U, V \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*^{\perp})$.

Proof. We know that, D_1 defines a totally geodesic foliation on M if and only if $g_N(\widetilde{\nabla}_U V, Z) = 0$ and $g_N(\widetilde{\nabla}_U V, X) = 0$, for all $U, V \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*^{\perp})$.

Then, using (13) and (10), we get,

$$g_N(\widetilde{\nabla}_U V, Z) = g_N(\widetilde{\nabla}_U JV, JZ)$$

Since $E = \mathcal{V}E + \mathcal{H}E$, for all $E \in \Gamma(TM)$, we have

$$g_N(\widetilde{\nabla}_U V, Z) = g_N(\mathcal{H}\widetilde{\nabla}_U JV, JZ)$$

Therefore, F is a semi-invariant Riemannian submersion and character of F, we arrive that,

$$g_N(\widetilde{\nabla}_U V, Z) = g_B(F_*(\mathcal{H}\widetilde{\nabla}_U JV), F_*(JZ))$$

Moreover, using (13) and (10), we get

$$g_N(\widetilde{\nabla}_U V, X) = g_N(\widetilde{\nabla}_U JV, JX)$$

By virtue of (14) and (15), we get

$$g_N(\widetilde{\nabla}_U V, X) = g_N(T_U \phi V + h\widetilde{\nabla}_U wV, CX) + g_N(v\widetilde{\nabla}_U \phi V, BX) + \eta(wV) g_N(U, BX)$$

or

$$g_N(\widetilde{\nabla}_U V, X) = g_B(F_*(T_U \phi V + h\widetilde{\nabla}_U wV), F_*(CX)) + g_M(v\widetilde{\nabla}_U \phi V, BX) + \eta(wV) g_N(U, BX)$$

which gives our assertion. □

Theorem 4.4. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution D_2 defines a totally geodesic foliation on N if and only if we have

$$g_B(F_*(T_Z BX), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_Z CX), F_*(CX)) = -g_N(v\widetilde{\nabla}_Z BX + T_Z CX, BX) - \eta(CX) g_N(Z, BX)$$

and

$$\begin{aligned} g_B(F_*(T_Z \phi U), F_*(CW)) + g_B(F_*(h\widetilde{\nabla}_Z wU), F_*(CW)) &= g_N(v\widetilde{\nabla}_Z \phi U + T_Z wU, BW) \\ &\quad + g_N(Z, BW)\eta(wU) - g_N(Z, \phi U)\eta(wW) \end{aligned}$$

for all $Z, W \in \Gamma(D_2), U \in \Gamma(D_1)$ and $X \in \Gamma(\ker F_*^{\perp})$.

Proof. For all $Z, W \in \Gamma(D_2), X \in \Gamma(\ker F_*^{\perp})$, using (10), and (13), we conclude,

$$g_N(\widetilde{\nabla}_Z W, X) = g_N(\widetilde{\nabla}_Z JW, JX)$$

Then, from (19), (14) and (15), we have,

$$g_N(\widetilde{\nabla}_Z W, X) = g_N(T_Z BX + v\widetilde{\nabla}_Z BX + T_Z CX + h\widetilde{\nabla}_Z CX + \eta(CX)Z, BX + CX)$$

We know that F is semi-invariant Riemannian submersion, we conclude,

$$g_N(\widetilde{\nabla}_Z W, X) = g_B(F_*(T_Z BX + h\widetilde{\nabla}_Z CX), F_*(CX)) + g_N(v\widetilde{\nabla}_Z BX + T_Z CX, BX) + \eta(CX) g_N(Z, BX)$$

Moreover, for all $Z, W \in \Gamma(D_2), U \in \Gamma(D_1)$, using (10), and (13),

$$g_N(\widetilde{\nabla}_Z W, U) = -g_N(JW, \widetilde{\nabla}_Z JU)$$

By virtue of (19), (14) and (15), imply that

$$g_N(\widetilde{\nabla}_Z W, U) = -g_N(BW + CW, T_Z \phi U + v\widetilde{\nabla}_Z \phi U + T_Z wU + h\widetilde{\nabla}_Z wU + \eta(wU)Z) - \eta(JW) g_N(Z, JU)$$

Since F is semi-invariant Riemannian submersion, we arrive,

$$\begin{aligned} g_N(\widetilde{\nabla}_Z W, U) &= -g_B(F_*(T_Z \phi U + h\widetilde{\nabla}_Z wU), F_*(CW)) - g_N(BW, v\widetilde{\nabla}_Z \phi U + T_Z wU) \\ &\quad - \eta(wU) g_N(BW, Z) - \eta(wW) g_N(Z, \phi U) \end{aligned}$$

which give proof. □

Corollary 4.5. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the fibers of F are the locally product Riemannian manifold of leaves of D_1 and D_2 if and only if

$$F_*(\widetilde{\nabla}_U JV) \in \Gamma(\mu),$$

$$\begin{aligned} g_B(F_*(T_U\phi V), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_U wV), F_*(CX)) &= g_N(v\widetilde{\nabla}_U\phi V + T_U wV, BX) \\ &+ g_N(U, BX)\eta(wV) \end{aligned}$$

and

$$g_B(F_*(T_Z BX), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_Z CX), F_*(CX)) = -g_N(v\widetilde{\nabla}_Z BX + T_Z CX, BX) - \eta(CX)g_N(Z, BX)$$

$$\begin{aligned} g_B(F_*(T_Z\phi U), F_*(CW)) + g_B(F_*(h\widetilde{\nabla}_Z wU), F_*(CW)) &= g_N(v\widetilde{\nabla}_Z\phi U + T_Z wU, BW) \\ &+ g_N(Z, BW)\eta(wU) - g_M(Z, \phi U)\eta(wW) \end{aligned}$$

for all $U, V \in \Gamma(D_1)$, $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*^{\perp})$.

Theorem 4.6. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution $\ker F_*^{\perp}$ is integrable if and only if we have

$$A_Y CX - A_X CY + v\widetilde{\nabla}_Y BX - v\widetilde{\nabla}_X BY \notin \Gamma(D_1)$$

and

$$\begin{aligned} g_B(F_*(A_Y BX), F_*(wZ)) + g_B(F_*(h\widetilde{\nabla}_Y CX, F_*(wZ)) &= -g_N(v\widetilde{\nabla}_Y BX + A_Y CX, \phi Z) \\ &+ \eta(X)g_N(Y - wY, wZ) - \eta(Y)g_N(X - wX, wZ) \end{aligned}$$

for all $X \in \Gamma(\ker F_*^{\perp}), Z \in \Gamma(D_2)$ and $U \in \Gamma(D_1)$.

PROOF. We using (12), (10) and (13), for all $X, Y \in \Gamma(\ker F_*^{\perp}), U \in \Gamma(D_1)$, we have

$$g_N([X, Y], U) = g_N(\widetilde{\nabla}_X JY, JU) - g_N(\widetilde{\nabla}_Y JX, JU)$$

Then, using (19), (16) and (17), we arrive,

$$g_N([X, Y], U) = -g_N(-v\widetilde{\nabla}_X BY - A_X CY + v\widetilde{\nabla}_Y BX + A_Y CX, U)$$

Moreover, for $Z \in \Gamma(D_2)$, by (12), (10) and (13), we get

$$\begin{aligned} g_N([X, Y], Z) &= g_N(\widetilde{\nabla}_X JY, JZ) - \eta(Y)g_N(X - JX, JZ) \\ &- g_N(\widetilde{\nabla}_Y JX, JZ) + \eta(X)g_N(Y - JY, JZ) \end{aligned}$$

Therefore, by virtue of (18), (15) and (16), we conclude that

$$\begin{aligned} g_N([X, Y], Z) &= -g_N(A_Y BX + h\widetilde{\nabla}_Y CX, wZ) - g_N(v\widetilde{\nabla}_Y BX + A_Y CX, \phi Z) \\ &- \eta(Y)g_N(X - wX, wZ) + \eta(X)g_N(Y - wY, wZ) \end{aligned}$$

On the other hand, F is semi-invariant Riemannian submersion, therefore we get,

$$\begin{aligned} g_N([X, Y], Z) &= -g_B(F_*(A_Y BX + h\widetilde{\nabla}_Y CX), F_*(wZ)) - g_N(v\widetilde{\nabla}_Y BX + A_Y CX, \phi Z) \\ &- \eta(Y)g_N(X - wX, wZ) + \eta(X)g_N(Y - wY, wZ) \end{aligned}$$

which conclude proof. □

Theorem 4.7. Let F be a semi-invariant Riemannian submersion with SSNMC . Therefore, the distribution $\ker F_*^\perp$ defines a totally geodesic foliation on N if and only if we have

$$v\widetilde{\nabla}_X BY + A_X CY \in \Gamma(D_2)$$

and

$$g_B(F_*(h\widetilde{\nabla}_X CY), F_*(wZ)) = -g_N(A_X BY, JZ)$$

for all $X, Y \in \Gamma(\ker F_*^\perp), U \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. We using (10), (13) and (19), for all $X, Y \in \Gamma(\ker F_*^\perp), U \in \Gamma(D_1)$, we have

$$g_N(\widetilde{\nabla}_X Y, U) = g_N(\widetilde{\nabla}_X BY, JU) + g_M(\widetilde{\nabla}_X CY, JU)$$

Therefore, using (15), (16) and (10), we arrive

$$g_N(\widetilde{\nabla}_X Y, U) = -g_N(J(v\widetilde{\nabla}_X BY + A_X CY), U)$$

Moreover, for $Z \in \Gamma(D_2)$, using (10), (19),(16) and (17), we conclude,

$$g_N(\widetilde{\nabla}_X Y, Z) = g_N(A_X BY + v\widetilde{\nabla}_X BY + \eta(BY)X + A_X CY + h\widetilde{\nabla}_X CY, JZ)$$

Also, character of F, we obtain

$$g_N(\widetilde{\nabla}_X Y, Z) = g_B(F_*(h\widetilde{\nabla}_X CY), F_*(wZ)) + g_N(A_X BY, JZ)$$

which completes proof. □

Theorem 4.8. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution $\ker F_*$ defines a totally geodesic foliation on N if and only if we have

$$v\widetilde{\nabla}_K L + T_K wL + \eta(wL)E \in \Gamma(D_1)$$

and

$$\begin{aligned} g_B(F_*(T_K\phi L), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_K wL), F_*(CX)) &= -g_N(v\widetilde{\nabla}_K \phi L, BX) - g_N(T_K wL, BX) \\ &\quad -\eta(\phi L) g_N(K, BX) \end{aligned}$$

for all $K, L \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu)$.

Proof. We know that $\ker F_*$ denote a totally geodesic foliation on N if and only if $g_N(\widetilde{\nabla}_K L, X) = 0$ and $g_N(\widetilde{\nabla}_K L, JZ) = 0$ for all $Z \in \Gamma(D_2)$, $K, L \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu)$.

Then, by (10), (13) and (18), we get

$$g_N(\widetilde{\nabla}_K L, X) = g_N(\widetilde{\nabla}_K \phi L, JX) + g_N(\widetilde{\nabla}_K wL, JX)$$

Therefore, using (14), (15) and (19), we have

$$g_N(\widetilde{\nabla}_K L, X) = g_N(v\widetilde{\nabla}_K \phi L + T_K wL + \eta(\phi L)K, BX) + g_N(T_K \phi L + h\widetilde{\nabla}_K wL, CX)$$

We know that F is semi-invariant Riemannian submersion, we arrive

$$g_N(\widetilde{\nabla}_K L, X) = g_N(v\widetilde{\nabla}_K \phi L + T_K wL + \eta(\phi L)K, BX) + g_B(F_*(T_K \phi L + h\widetilde{\nabla}_K wL), F_*(CX))$$

Moreover, from (10), (13) ,(18), (14) and (15), we obtain,

$$g_N(\widetilde{\nabla}_K L, JZ) = g_N(v\widetilde{\nabla}_K L + T_K wL + \eta(wL)E, Z)$$

which give proof. □

Corollary 4.9. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the total space M is a locally product manifold of the leaves of $\ker F_*^\perp$ and $\ker F_*$ if and only if

$$v\widetilde{\nabla}_X BY + A_X CY \in \Gamma(D_2),$$

$$g_B(F_*(h\widetilde{\nabla}_X CY), F_*(wZ)) = -g_N(A_X BY, JZ)$$

and

$$v\widetilde{\nabla}_K L + T_K wL + \eta(wL)E \in \Gamma(D_1),$$

$$\begin{aligned} g_B(F_*(T_K\phi L), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_K wL), F_*(CX)) &= -g_N(v\widetilde{\nabla}_K \phi L, BX) - g_N(T_K wL, BX) \\ &\quad -\eta(\phi L) g_N(K, BX) \end{aligned}$$

for all $X, Y \in \Gamma(\ker F_*^\perp), K, L \in \Gamma(\ker F_*)$ and $Z \in \Gamma(D_2)$.

5. Conclusion

Riemannian submersions and SSNMC have an important application for many sciences such as physics and mathematical physics. Researchers have increased studies on this field from different areas in recent years. In this paper, the idea of examining Riemann submersion with different connections is emphasized. We defined and studied Riemannian submersions with SSNMC for the first time. We investigated geometry of foliatons with SSNMC. The works on this subject will be useful tools for the applications of Riemannian submersion with different connections.

Conflicts of Interest

The author declares no conflict of interest.

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Journal of New Theory
ISSN: 2149-1402

35 (2021) 72-90
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Bimultiplications and Annihilators of Crossed Modules in Associative Algebras

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Article History
Received: 29 Mar 2021
Accepted: 19 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract — In this paper, we present a generalization of the concept of the bimultiplication algebra by defining the bimultiplication of crossed modules in associative algebras. Using this structure, we construct the actor of a crossed module in associative algebras, and we obtain annihilators of crossed modules as the kernel of a crossed module morphism from a crossed module to its actor. Moreover, we construct a link between this morphism and a crossed square, two-dimensional crossed module.

Keywords — *Bimultiplication crossed module, actor crossed module, annihilator of crossed module, crossed square*

Mathematics Subject Classification (2020) — 16Y99, 18A05

1. Introduction

In the case of groups, the description of an action is provided by the automorphism group. The action of any group A on itself is given by a group homomorphism $A \longrightarrow Aut(A)$. In some other algebraic contexts, automorphism structure is not enough to give an action. The set of automorphisms of an associative algebra is not usually an algebra. For associative algebra case, the bimultiplication algebra $Bim(G)$ of an associative algebra G which is defined by Lane [1] will accomplish the role of an automorphism group.

The concept of a crossed module for groups is defined in [2, 3] by Whitehead. The commutative algebra version of crossed modules is mentioned in a different name, by Lichtenbaum and Schlessinger [4] also, the paper of Gerstenhaber [5] contains the notion of crossed modules in commutative algebras. The notion of crossed modules in associative algebras is defined by Dedecker and Lue in [6].

In [7], Norrie improved Lue's work [8], and she defined the concept of an actor of crossed modules for groups as analogue of an automorphism group. For commutative algebras, Arvasi and the first author explained that this notion is related to the multiplication algebra in [9]. In this paper based on the second author's PhD dissertation [10], our starting point is the generalization of the bimultiplication algebra. For this, we define the bimultiplication of crossed modules for associative algebras and give details of the construction of actor crossed module for associative algebras via this context. The notion of actor crossed module is also studied simultaneously by Boyacı et al. in [11] as the split extension classifier of a crossed module. By constructing a morphism under certain conditions from a crossed module and its actor, we get the action of any crossed modules in associative algebras on itself. Since the annihilator of an algebra A is given by the kernel of algebra homomorphism $A \longrightarrow Bim(A)$,

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we have a similar notion for a crossed modules in associative algebras via the kernel of the above-mentioned morphism between a crossed module and its actor. Also, while we get $A \longrightarrow Bim(A)$ crossed module as a two-dimensional associative algebra via an algebra A, we can have a crossed square as a two-dimensional crossed module via an appropriate crossed module morphism.

2. Preliminary

Conventions: Throughout this paper, it is supposed that K is a fixed commutative ring with identity $1 \neq 0$, K-algebra G is a unitary K-bimodule G equipped with a K-bilinear associative multiplication and also if G is a K-algebra, then a G -algebra C is a G-bimodule equipped with a G-bilinear associative multiplication. C will not necessarily have identities.

We recall the definition of a crossed module in associative algebras [12].

Definition 2.1. *A crossed module* in associative K-algebras is a morphism from a G-algebra C to a K-algebra G, $\delta : C \longrightarrow G$, with two-sided actions of G on C satisfying

CM1) $\delta(g \cdot c) = g\delta(c)$ and $\delta(c \cdot g) = \delta(c)g$

CM2) $\delta c \cdot c' = cc'$ and $c \cdot \delta c' = cc'$

for all $c \in C, g \in G$. It is denoted by (C, G, δ).

The following are some standard examples of crossed modules in associative algebras:

i. An inclusion map $I \hookrightarrow G$ is a crossed module, where I is any two-sided ideal in G. On the other hand, let $\delta : C \longrightarrow G$ be any crossed module. Then, we get that $\delta(C)$ is a two sided ideal in G by CM1.

ii. Any G-bimodule M has a G-algebra structure with zero multiplication. Thus, we get the crossed module $\mathbf{0} : M \to G, \mathbf{0}(m) = 0$. Conversely, the kernel of crossed module $\delta : N \longrightarrow G$ is an $G/\delta(N)$-bimodule.

Thus, the concept of crossed modules in associative algebras is a generalisation the concepts both of a two-sided ideal and that of a bimodule over an algebra. Also, any associative algebra is considered as a crossed modules by identity map $Id : G \longrightarrow G$.

Definition 2.2. A pair (f, ϕ) of G-algebra morphisms $f : C \longrightarrow C', \phi : G \longrightarrow G'$ such that

$$f(g \cdot c) = \phi(g) \cdot f(c) \quad \text{and} \quad f(c \cdot g) = f(c) \cdot \phi(g)$$

for $c \in C, g \in G$ is called a morphism of crossed modules from (C, G, δ) to (C', G', δ').

Thus, we get the category AssXMod of crossed modules in associative algebras.

The *kernel* of (f, ϕ) and the image $\mathrm{Im}(f, \phi)$ are defined by $(\ker f, \ker \phi, \delta)$ and $(\mathrm{Imf}, \mathrm{Im}\phi, \delta')$, respectively.

Definition 2.3. A crossed module (C', G', δ') is called a *subcrossed module* of a crossed module (C, G, δ) if the following conditions are satisfied:

i. C' and G' are the subalgebra of C and G, respectively

ii. $\delta' = \delta \mid_{C'}$

iii. The action of G' on C' inherits from the action of G on C

The image $\mathrm{Im}(f, \phi)$ of (f, ϕ) is the subcrossed module $(\mathrm{Imf}, \mathrm{Im}\phi, \delta')$ of (C, G, δ).

Definition 2.4. A subcrossed module (C', G', δ') of (C, G, δ) is called *an ideal* if

i. G' is an two sided ideal of G

ii. $g \cdot c' \in C'$

iii. $g' \cdot c \in C'$

for all $g \in G, g' \in G'\ c \in C, c' \in C'$.

If $(f, \phi) : (C, G, \delta) \longrightarrow (C', G', \delta')$ is any crossed module morphism, $(\ker f, \ker \phi, \delta)$ is an ideal of (C, R, δ) and $(\mathrm{Imf}, \mathrm{Im}\phi, \delta')$ is a subcrossed module of (C', G', δ').

Definition 2.5. Let (R, S, δ') be an ideal of (C, G, δ). Then, $\bar{\delta} : C/R \longrightarrow G/S$, $\bar{\delta}(c + R) = \delta(c) + S$ is a crossed module with $(g + S) \cdot (c + R) = (g \cdot c) + R$ and $(c + R) \cdot (g + S) = (c \cdot g) + R$. It is called the quotient crossed module of (C, G, δ) by (R, S, δ') and denoted by $\frac{(C,G,\delta)}{(R,S,\delta')}$.

3. Actor Crossed Modules in Associative Algebras

3.1. A Bimultiplier of a Crossed Module

We recall the structure of the R-algebra of bimultipliers of C, $\mathrm{Bim}(C)$ [1, 13]. $\mathrm{Bim}(C)$ consists of all pairs (γ, δ) of R-linear mappings $\gamma, \delta : C \to C$ such that

$$\gamma(cc') = \gamma(c) \cdot c',$$

$$\delta(cc') = c \cdot \delta\left(c'\right),$$

and

$$c \cdot \gamma\left(c'\right) = \delta(c) \cdot c'$$

It has an R-module structure and a product

$$(\gamma, \delta) \cdot (\gamma', \delta') = (\gamma \circ \gamma', \delta' \circ \delta)$$

Suppose that $\mathrm{Ann}(C) = 0$ or $C^2 = C$. Then, $\mathrm{Bim}(C)$ acts on C by

$$\begin{array}{rcll} \mathrm{Bim}(C) \times C & \to & C; & ((\gamma, \delta), c) \mapsto \gamma(c), \\ C \times \mathrm{Bim}(C) & \to & C; & (c, (\gamma, \delta)) \mapsto \delta(c) \end{array}$$

and there is a

$$\begin{array}{rccc} \mu : & C & \longrightarrow & \mathrm{Bim}(C) \\ & c & \longmapsto & (\gamma_c, \delta_c) \end{array}$$

with

$$\gamma_c(x) = c \cdot x \qquad \text{and} \qquad \delta_c(x) = x \cdot c$$

We give the following result from their work of Lavendhomme and Lucas [14].

Proposition 3.1. Let G be a K-algebra such that Ann $(G) = 0$ or $G^2 = G$. Then, $(G, Bim(G), (\gamma, \sigma))$ is a crossed module.

Proof. $Bim(G)$ acts on G by

$$\begin{array}{rcl} Bim(G) \times G & \longrightarrow & G \\ ((\gamma', \sigma'), g) & \longmapsto & \gamma'(g) \end{array} \quad \text{and} \quad \begin{array}{rcl} G \times Bim(G) & \longrightarrow & G \\ (g, (\gamma', \sigma')) & \longmapsto & \sigma'(g) \end{array}$$

and there is a $(\gamma, \sigma) : G \longrightarrow Bim(G)$ defined by $(\gamma, \sigma)(g) = (\gamma'_g, \sigma'_g)$ with $\gamma'_g(g') = gg'$ and $\sigma'_g(g') = g'g$ for all $g, g' \in G$.

CM1)

$$\begin{aligned}(\gamma,\sigma)((\gamma',\sigma')\cdot g) &= (\gamma,\sigma)(\gamma'(g))\\ &= (\gamma'_{\gamma'(g)},\sigma'_{\gamma'(g)})\\ &\overset{(*)}{=} (\gamma'\gamma'(g),\sigma'(g)\sigma')\\ &= (\gamma',\sigma')(\gamma'(g),\sigma'(g))\\ &= (\gamma',\sigma')((\gamma,\sigma)(g))\end{aligned}$$

$$(*)\qquad \begin{aligned}\left(\gamma'_{\gamma'(g)}\right)(g') &= \gamma'(g)g'\\ &= \gamma'(gg')\\ &= \gamma'(\gamma'_g(g'))\\ &= (\gamma'\gamma'(g))\,(g')\end{aligned}\qquad\qquad \begin{aligned}\left(\sigma'_{\gamma'(g)}\right)(g') &= g'\gamma'(g)\\ &= \sigma'(g')g\\ &= \sigma'_g(\sigma'(g'))\\ &= (\sigma'(g)\sigma')\,(g')\end{aligned}$$

$$\begin{aligned}(\gamma,\sigma)(g\cdot(\gamma',\sigma')) &= (\gamma,\sigma)(\sigma'(g))\\ &= (\gamma'_{\sigma'(g)},\sigma'_{\sigma'(g)})\\ &\overset{(*)}{=} (\gamma'(g)\gamma',\sigma'\sigma'(g))\\ &= (\gamma'(g),\sigma'(g))(\gamma',\sigma')\\ &= ((\gamma,\sigma)(g))(\gamma',\sigma')\end{aligned}$$

$$(*)\qquad \begin{aligned}\left(\gamma'_{\sigma'(g)}\right)(g') &= \sigma'(g)g'\\ &= g\gamma'(g')\\ &= \gamma'_g(\gamma'(g'))\\ &= (\gamma'(g)\gamma')\,(g')\end{aligned}\qquad\qquad \begin{aligned}\left(\sigma'_{\sigma'(g)}\right)(g') &= g'\sigma'(g)\\ &= \sigma'(g'g)\\ &= \sigma'(\sigma'_g(g'))\\ &= (\sigma'\sigma'(g))\,(g')\end{aligned}$$

CM2)

$$\begin{aligned}((\gamma,\sigma)(g))\cdot g' &= (\gamma'_g,\sigma'_g)\cdot g'\\ &= \gamma'_g(g')\\ &= gg'\end{aligned}\qquad\qquad \begin{aligned}g\cdot((\gamma,\sigma)(g')) &= g\cdot(\gamma'_{g'},\sigma'_{g'})\\ &= \sigma'_{g'}(g)\\ &= gg'\end{aligned}$$

□

Definition 3.2. A bimultiplier of the crossed module $\mu : T \longrightarrow L$ is a pair of

$$(\lambda,\chi) = ((\Theta,\Xi),(\Theta',\Xi')) : (T,L,\mu) \longrightarrow (T,L,\mu)$$

satisfying the following conditions:

i. $(\lambda = (\Theta,\Xi) : T \longrightarrow T) \in Bim(T\ )$ and $(\chi = (\Theta',\Xi') : L \longrightarrow L) \in Bim(L\ )$

ii.

$$\begin{array}{ccc} T & \xrightarrow{\ \mu\ } & L\\ {\scriptstyle\lambda=(\Theta,\Xi)}\big\downarrow & & \big\downarrow{\scriptstyle\chi=(\Theta',\Xi')}\\ T & \xrightarrow[\ \mu\ ]{} & L\end{array}$$

$\chi\mu = \mu\lambda$ $(\Theta'\mu = \mu\Theta,\ \Xi'\mu = \mu\Xi)$
and

iii. For $l \in L$, $t \in T$,

$$\begin{aligned} \Theta(l \cdot t) &= \Theta'(l) \cdot t \\ \Theta(t \cdot l) &= \Theta(t) \cdot l \\ \Xi(l \cdot t) &= l \cdot \Xi(t) \\ \Xi(t \cdot l) &= t \cdot \Xi'(l) \\ l \cdot \Theta(t) &= \Xi'(l) \cdot t \\ \Xi(t) \cdot l &= t \cdot \Theta'(l) \end{aligned}$$

The set of bimultipliers of the crossed module $\mu : T \longrightarrow L$ will be denoted by $Bim(T, L, \mu)$.

Proposition 3.3. The set $Bim(T, L, \mu)$ has an associative K-algebra structure with following operations:

$$\begin{aligned} +) \quad & ((\Theta, \Xi), (\Theta', \Xi')) + ((\overline{\Theta}, \overline{\Xi}), (\overline{\Theta}', \overline{\Xi}')) = ((\Theta + \overline{\Theta}, \Xi + \overline{\Xi}), (\Theta' + \overline{\Theta}', \Xi' + \overline{\Xi}') \\ \cdot) \quad & k((\Theta, \Xi), (\Theta', \Xi')) = (k\Theta, k\Xi), (k\Theta', k\Xi') \\ \circ) \quad & ((\Theta, \Xi), (\Theta', \Xi')) \circ ((\overline{\Theta}, \overline{\Xi}), (\overline{\Theta}', \overline{\Xi}')) = ((\Theta \circ \overline{\Theta}, \overline{\Xi} \circ \Xi), (\Theta' \circ \overline{\Theta}', \overline{\Xi}' \circ \Xi)) \end{aligned}$$

PROOF. See for details [10]. □

Definition 3.4. Let (T, L, μ) be a crossed module. $\mathcal{U}(L, T)$ is the set of pair maps (β_1, β_2) such that

$$\begin{aligned} \beta_1(l_1 l_2) &= \beta_1(l_1) \cdot l_2 \\ \beta_2(l_1 l_2) &= l_1 \cdot \beta_2(l_2) \\ l_1 \cdot \beta_1(l_2) &= \beta_2(l_1) \cdot l_2 \end{aligned}$$

for all $l_1, l_2 \in L$, where $\beta_1, \beta_2 : L \to T$, are K-linear maps.

$\mathcal{U}(L, T)$ has an associative K-algebra structure with the following operations:

$$\begin{aligned} +) \quad & ((\beta_1, \beta_2) + (\theta_1, \theta_2)) = (\beta_1 + \theta_1, \beta_2 + \theta_2) \\ \cdot) \quad & k(\beta_1, \beta_2) = (k\beta_1, k\beta_2) \\ \circ) \quad & ((\beta_1, \beta_2) \circ (\theta_1, \theta_2)) = (\beta_1 \mu \theta_1, \theta_2 \mu \beta_2) \end{aligned}$$

Proposition 3.5. Let (T, L, μ) be a crossed module and $(\beta_1, \beta_2) \in \mathcal{U}(L, T)$. Each such K-linear maps $\beta_1, \beta_2 : L \longrightarrow T$ defines bimultipliers $(\Theta_\beta, \Xi_\beta), (\Theta'_\beta, \Xi'_\beta)$ of T and L respectively, given by $\Theta_\beta = \beta_1 \mu$, $\Xi_\beta = \beta_2 \mu$, $\Theta'_\beta = \mu \beta_1$, $\Xi'_\beta = \mu \beta_2$.

PROOF.

$$\begin{aligned} \Theta_\beta(tt') &= \beta_1(\mu(tt')) \\ &= \beta_1(\mu(t)\mu(t')) \\ &= \beta_1(\mu(t)) \cdot \mu(t') \\ &= \Theta_\beta(t) \cdot \mu(t') \\ &= \Theta_\beta(t) \cdot t' \end{aligned}$$

$$\begin{aligned} \Xi_\beta(tt') &= \beta_2(\mu(tt')) \\ &= \beta_2(\mu(t)\mu(t')) \\ &= \mu(t) \cdot \beta_2(\mu(t')) \\ &= \mu(t) \cdot \Xi_\beta(t') \\ &= t \cdot \Xi_\beta(t') \end{aligned}$$

$$\begin{aligned} t\Theta_\beta(t') &= t\beta_1(\mu(t')) \\ &= \mu(t)\cdot\beta_1(\mu(t')) \\ &= \beta_2(\mu(t))\cdot\mu(t') \\ &= \beta_2(\mu(t))t' \\ &= \Xi_\beta(t)t' \end{aligned}$$

for $t, t' \in T$. Thus, $(\Theta_\beta, \Xi_\beta) \in Bim(T)$. Similarly, $(\Theta'_\beta, \Xi'_\beta) \in Bim(L)$. □

Proposition 3.6. The bimultipliers $\lambda_\beta = (\Theta_\beta, \Xi_\beta)$ of T and $\chi_\beta = (\Theta'_\beta, \Xi'_\beta)$ of L satisfy the following conditions:

i.

$$\begin{aligned} \Theta_\beta\beta_1 = \beta_1\mu\beta_1 = \beta_1\Theta'_\beta \quad &\text{and} \quad \Xi_\beta\beta_2 = \beta_2\mu\beta_2 = \beta_2\Xi'_\beta \\ \Theta_\beta\beta_2 = \beta_1\mu\beta_2 = \beta_1\Xi'_\beta \quad &\text{and} \quad \Xi_\beta\beta_1 = \beta_2\mu\beta_1 = \beta_2\Theta'_\beta \end{aligned}$$

ii. The following diagram

$$\begin{array}{ccc} T & \overset{\mu}{\underset{\beta_1,\ \beta_2}{\rightleftarrows}} & L \\ {\scriptstyle \lambda_\beta=(\Theta_\beta,\Xi_\beta)}\downarrow & & \downarrow{\scriptstyle \chi_\beta=(\Theta'_\beta,\Xi'_\beta)} \\ T & \overset{\mu}{\underset{\beta_1,\ \beta_2}{\rightleftarrows}} & L \end{array}$$

is commutative, that is

$$\begin{aligned} \mu\Theta_\beta = \mu(\beta_1\mu) = (\mu\beta_1)\mu = \Theta'_\beta\mu \\ \mu\Xi_\beta = \mu(\beta_2\mu) = (\mu\beta_2)\mu = \Xi'_\beta\mu \end{aligned}$$

iii.

$$(\lambda_\beta,\ \chi_\beta) = ((\Theta_\beta, \Xi_\beta), (\Theta'_\beta, \Xi'_\beta)) \in Bim(T, L, \mu)$$

Proposition 3.7. If $\mu : T \longrightarrow L$ is a crossed module,then

$$\begin{aligned} \Gamma : \ U(L,T) &\longrightarrow Bim(T) \\ (\beta_1, \beta_2) &\longmapsto (\Theta_\beta, \Xi_\beta) = (\beta_1\mu, \beta_2\mu) \end{aligned}$$

$$\begin{aligned} \Phi : \ U(L,T) &\longrightarrow Bim(L) \\ (\beta_1, \beta_2) &\longmapsto (\Theta'_\beta, \Xi'_\beta) = (\mu\beta_1, \mu\beta_2) \end{aligned}$$

are algebra morphisms.

Lemma 3.8. Let $\mu : T \longrightarrow L$ be a crossed module. Then, the left and right actions of $Bim(T, L, \mu)$ on U(L,T) is given by

$$\begin{aligned} Bim(T, L, \mu) \times U(L, T) &\longrightarrow U(L, T) \\ ((\lambda, \chi), (\theta_1, \theta_2)) &\longmapsto (\Theta\theta_1, \theta_2\Xi') \end{aligned}$$

$$\begin{aligned} U(L, T) \times Bim(T, L, \mu) &\longrightarrow U(L, T) \\ ((\theta_1, \theta_2), (\lambda, \chi)) &\longmapsto (\theta_1\Theta', \Xi\theta_2) \end{aligned}$$

where $(\lambda, \chi) = ((\Theta, \Xi), (\Theta', \Xi'))$

Proof. Since

$$\begin{aligned} (\Theta\theta_1)(ll') &= \Theta(\theta_1(ll')) \\ &= \Theta(\theta_1(l)\cdot l') \\ &= \Theta(\theta_1(l))\cdot l' \\ &= ((\Theta\theta_1)(l))\cdot l' \end{aligned}$$

$$\begin{aligned} (\theta_2\Xi')(ll') &= \theta_2(\Xi'(ll')) \\ &= \theta_2(l\Xi'(l')) \\ &= l\cdot\theta_2(\Xi'(l')) \\ &= l\cdot((\theta_2\Xi')(l')) \end{aligned}$$

$$\begin{aligned} l\cdot((\Theta\theta_1)(l')) &= l\cdot(\Theta(\theta_1(l'))) \\ &= \Xi'(l)\cdot\theta_1(l') \\ &= \theta_2(\Xi'(l))\cdot l' \\ &= ((\theta_2\Xi')(l))\cdot l' \end{aligned}$$

the left action of $Bim(T,L,\mu)$ on U(L,T) is well defined. Since

$$\begin{aligned} (\theta_1\Theta')(ll') &= \theta_1(\Theta'(ll')) \\ &= \theta_1(\Theta'(l)l') \\ &= (\theta_1(\Theta'(l)))\cdot l' \\ &= ((\theta_1\Theta')(l))\cdot l' \end{aligned} \qquad \begin{aligned} (\Xi\theta_2)(ll') &= \Xi(\theta_2(ll')) \\ &= \Xi(l\cdot\theta_2(l')) \\ &= l\cdot\Xi(\theta_2(l')) \\ &= l\cdot((\Xi\theta_2)(l')) \end{aligned}$$

$$\begin{aligned} l\cdot((\theta_1\Theta')(l')) &= l\cdot(\theta_1(\Theta'(l'))) \\ &= \theta_2(l)\cdot(\Theta'(l') \\ &= \Xi(\theta_2(l))\cdot l' \\ &= ((\Xi\theta_2)(l))\cdot l' \end{aligned}$$

the right action of $Bim(T,L,\mu)$ on U(L,T) is well defined.

Moreover, one sees that associative algebra action conditions are satisfied by routine calculations. (See for details [10]) □

The next theorem begins the analysis of the associative algebra version of the actor structure.

Theorem 3.9. If $\mu : T \longrightarrow L$ is a crossed module, then the morphism

$$\triangle : U(L,T) \to Bim(T,L,\mu)$$

is a crossed module given by

$$\triangle(\beta_1,\beta_2) = ((\Theta_{\beta_1},\Xi_{\beta_2}),(\Theta'_{\beta_1},\Xi'_{\beta_2})) = ((\beta_1\mu,\beta_2\mu),(\mu\beta_1,\mu\beta_2)).$$

Proof. The right and left actions of $Bim(T,L,\mu)$ on $\mathcal{U}(L,T)$ are given by

$$((\Theta,\Xi),(\Theta',\Xi'))\cdot(\theta_1,\theta_2) = (\Theta\theta_1,\theta_2\Xi')$$

and

$$(\theta_1,\theta_2)\cdot((\Theta,\Xi),(\Theta',\Xi')) = (\theta_1\Theta',\Xi\theta_2)$$

as mentioned before. We need to check the crossed module axioms:

For $((\Theta,\Xi),(\Theta',\Xi')) \in Bim(T,L,\mu), (\theta_1,\theta_2) \in U(L,T)$,

CM1)

$$\begin{aligned} \triangle\left[((\Theta,\Xi),(\Theta',\Xi'))\cdot(\theta_1,\theta_2)\right] &= \triangle(\Theta\theta_1,\theta_2\Xi') \\ &= (\Theta\theta_1\mu,\theta_2\Xi'\mu),(\mu\Theta\theta_1,\mu\theta_2\Xi') \\ &= (\Theta\theta_1\mu,\theta_2\mu\Xi),(\Theta'\mu\theta_1,\mu\theta_2\Xi') \\ &= ((\Theta\circ\Theta_\theta,\Xi_\theta\circ\Xi),(\Theta'\circ\Theta'_\theta,\Xi'_\theta\circ\Xi')) \\ &= [((\Theta,\Xi)\circ(\Theta_\theta,\Xi_\theta)),((\Theta',\Xi')\circ(\Theta'_\theta,\Xi'_\theta))] \\ &= ((\Theta,\Xi),(\Theta',\Xi'))\circ((\Theta'_\theta,\Xi'_\theta),(\Theta_\theta,\Xi_\theta)) \\ &= ((\Theta,\Xi),(\Theta',\Xi'))\circ\triangle(\theta_1,\theta_2) \end{aligned}$$

$$\begin{aligned}
\triangle\left[(\theta_1,\theta_2)\cdot((\Theta,\Xi),(\Theta',\Xi'))\right] &= \triangle(\theta_1\Theta',\Xi\theta_2)\\
&= (\theta_1\Theta'\mu,\Xi\theta_2\mu),(\mu\theta_1\Theta',\mu\Xi\theta_2)\\
&= (\theta_1\mu\Theta,\Xi\theta_2\mu),(\mu\theta_1\Theta',\Xi'\mu\theta_2)\\
&= ((\Theta_\theta\circ\Theta,\Xi\circ\Xi_\theta),(\Theta'_\theta\circ\Theta',\Xi'\circ\Xi'_\theta))\\
&= [((\Theta_\theta,\Xi_\theta)\circ(\Theta,\Xi)),((\Theta'_\theta,\Xi'_\theta)\circ(\Theta',\Xi'))]\\
&= ((\Theta'_\theta,\Xi'_\theta),(\Theta_\theta,\Xi_\theta))\circ((\Theta,\Xi),(\Theta',\Xi'))\\
&= \triangle(\theta_1,\theta_2)\circ((\Theta,\Xi),(\Theta',\Xi'))
\end{aligned}$$

CM2)

$$\begin{aligned}
\triangle(\beta_1,\beta_2)\cdot(\theta_1,\theta_2) &= ((\Theta_\beta,\Xi_\beta),(\Theta'_\beta,\Xi'_\beta))\cdot(\theta_1,\theta_2)\\
&= (\Theta_\beta\theta_1,\theta_2\Xi'_\beta)\\
&= (\beta_1\mu\theta_1,\theta_2\mu\beta_2)\\
&= (\beta_1,\beta_2)\circ(\theta_1,\theta_2)
\end{aligned}$$

$$\begin{aligned}
(\beta_1,\beta_2)\cdot\triangle(\theta_1,\theta_2) &= (\beta_1,\beta_2)\cdot((\Theta_\theta,\Xi_\theta),(\Theta'_\theta,\Xi'_\theta))\\
&= (\beta_1\Theta'_\theta,\Xi_\theta\beta_2)\\
&= (\beta_1\mu\theta_1,\theta_2\mu\beta_2)\\
&= (\beta_1,\beta_2)\circ(\theta_1,\theta_2)
\end{aligned}$$

□

The crossed module $(U(L,T),Bim(T,L,\mu),\triangle)$ will be called the "*the actor crossed module*" of (T,L,μ) and it will be denoted by $\mathcal{A}(T,L,\mu)$.

Lemma 3.10. If $\mu : T \longrightarrow L$ is a crossed module, then $\eta : T \to U(L,T)$ given by $\eta(t) = (\eta_{1t},\eta_{2t})$ is an algebra morphism where $\eta_{1t}(l) = t\cdot l, \eta_{2t}(l) = l\cdot t$ and $\eta(T)$ is an ideal of $U(L,T)$.

Lemma 3.11.

$$\begin{array}{rccl}
\alpha: & L & \to & Bim(T,L,\mu)\\
 & l & \mapsto & ((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))
\end{array}$$

is an algebra morphism where $\Theta_l(t) = l\cdot t, \Theta'_l(l') = ll', \Xi_l(t) = t\cdot l, \Xi'_l(l') = l'l$, and $\alpha(L)$ is an ideal of $Bim(T,L,\mu)$.

Theorem 3.12. If $\mu : T \longrightarrow L$ is a crossed module, then the morphism

$$(\eta,\alpha) : (T,L,\mu) \to \mathcal{A}(T,L,\mu)$$

is a morphism of crossed modules where $(\eta(t))(l) = \eta_t(l) = (\eta_{1t}(l),\eta_{2t}(l))$ and $\alpha(l) = ((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))$.

PROOF.

$$\begin{array}{ccc}
T & \xrightarrow{\ \eta\ } & U(L,T)\\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\Delta}\\
L & \xrightarrow[\ \alpha\ ]{} & Bim(T,L,\mu)
\end{array}$$

i.

$$\begin{aligned}
\triangle\eta(t) &= \triangle\eta_t(l)\\
&= \triangle(\eta_{1t}(l),\eta_{2t}(l))\\
&= \left[(\Theta_{\eta_{1t}}(t'),\Xi_{\eta_{2t}}(t')),(\Theta'_{\eta_{1t}}(l),\Xi'_{\eta_{2t}}(l))\right]\\
&= [(\eta_{1t}\mu(t'),\eta_{2t}\mu(t')),(\mu\eta_{1t}(l),\mu\eta_{2t}(l))]\\
&= [(t\cdot\mu(t'),\mu(t')\cdot t),(\mu(t\cdot l),\mu(l\cdot t))]\\
&= [(\mu(t)\cdot t',t'\cdot\mu(t)),(\mu(t)l,l\mu(t))]\\
&= \left[(\Theta_{\mu(t)},\Xi_{\mu(t)}),(\Theta'_{\mu(t)},\Xi'_{\mu(t)})\right]\\
&= \alpha(\mu(t))\\
&= \alpha\mu(t)
\end{aligned}$$

ii.

$$\begin{aligned}
(\alpha(l)\circ\eta(t))(l') &= [((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\cdot(\eta_{1t},\eta_{2t})]\,(l')\\
&= (\Theta_l\eta_{1t},\eta_{2t}\Xi'_l)(l')\\
&= [\Theta_l\eta_{1t}(l'),\eta_{2t}\Xi'_l(l')]\\
&= (\Theta_l(t\cdot l'),\eta_{2t}(l'l))\\
&= (l\cdot(t\cdot l'),(l'l)\cdot t)\\
&= ((l\cdot t)\cdot l',l'\cdot(l\cdot t))\\
&= (\eta_{1l\cdot t}(l'),\eta_{2l\cdot t}(l'))\\
&= \eta(l\cdot t)(l')
\end{aligned}$$

and

$$\begin{aligned}
(\eta(t)\circ\alpha(l))(l') &= [(\eta_{1t},\eta_{2t})\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))]\,(l')\\
&= (\eta_{1t}\Theta'_l,\Xi_l\eta_{2t})(l')\\
&= [\eta_{1t}\Theta'_l(l'),\Xi_l\eta_{2t}(l')]\\
&= (\eta_{1t}(l'l),\Xi_l(l'\cdot t))\\
&= (\eta_{1t}(l')\cdot l,(l'\cdot t)\cdot l)\\
&= ((t\cdot l)\cdot l',(l'\cdot t)\cdot l)\\
&= ((t\cdot l)\cdot l',l'\cdot(t\cdot l))\\
&= (\eta_{1t\cdot l}(l'),\eta_{2t\cdot l}(l'))\\
&= \eta(t\cdot l)(l')
\end{aligned}$$

□

Proposition 3.13. $\mathrm{Im}(\eta,\alpha)=(\mathrm{Im}(\eta)\,,Im\,(\alpha))$ is an ideal of $(\mathcal{U}(L,T),Bim(T,L,\mu),\Delta)$.

PROOF.

i. For $((\Theta,\Xi),(\Theta',\Xi'))\in Bim(T,L,\mu),((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\in \mathrm{Im}\ (\alpha)$,

$$\begin{aligned}
((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\circ((\Theta,\Xi),(\Theta',\Xi')) &= ((\Theta_l\circ\Theta,\Xi\circ\Xi_l),(\Theta'_l\circ\Theta',\Xi'\circ\Xi'_l))\\
&= ((\Theta_{\Xi'(l)},\Xi_{\Xi'(l)}),(\Theta'_{\Xi'(l)},\Xi'_{\Xi'(l)}))\in Im(\alpha)
\end{aligned}$$

and

$$((\Theta,\Xi),(\Theta',\Xi'))\circ((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))=((\Theta_{\Theta'(l)},\Xi_{\Theta'(l)}),(\Theta'_{\Theta'(l)},\Xi'_{\Theta'(l)}))\in Im(\alpha)$$

ii. For $((\Theta,\Xi),(\Theta',\Xi'))\in Bim(T,L,\mu)$, $(\eta_{1t},\eta_{2t})\in \mathrm{Im}(\eta)$, since

$$\left[((\Theta,\Xi),(\Theta',\Xi'))\cdot(\eta_{1t},\eta_{2t})\right](l)=(\eta_{1\Theta(t)},\eta_{2\Theta(t)})(l)$$

and

$$\left[(\eta_{1t},\eta_{2t})\cdot((\Theta,\Xi),(\Theta',\Xi'))\right](l)=(\eta_{1\Xi(t)},\eta_{2\Xi(t)})(l)$$

We get $(\eta_{1\Theta(t)},\eta_{2\Theta(t)})$ and $(\eta_{1\Xi(t)},\eta_{2\Xi(t)})$ belong to $\mathrm{Im}(\eta)$.

iii. For $((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\in \mathrm{Im}\ (\alpha)$, $(\beta_1,\beta_2)\in U(L,T), l'\in L$, since

$$\begin{aligned}
\left[((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\cdot\ (\beta_1,\beta_2)\right](l') &= (\Theta_l\beta_1,\beta_2\Xi'_l)(l')\\
&= (\Theta_l\beta_1(l'),\beta_2\Xi'_l(l'))\\
&= (l\cdot\beta_1(l'),\beta_2(l'l))\\
&= (l\cdot\beta_1(l'),l'\cdot\beta_2(l))\\
&= (\beta_2(l)\cdot l',l'\cdot\beta_2(l))\\
&= (\eta_{1\beta_2(l)}(l'),\eta_{2\beta_2(l)}(l'))\\
\left[\ (\beta_1,\beta_2)\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\right](l') &= (\beta_1\Theta'_l,\Xi_l\beta_2)(l')\\
&= (\beta_1(\Theta'_l(l')),\Xi_l(\beta_2(l')))\\
&= (\beta_1(ll'),\beta_2(l')\cdot l)\\
&= (\beta_1(l)\cdot l',l'\cdot\beta_1(l))\\
&= (\eta_{1\beta_1(l)}(l'),\eta_{2\beta_1(l)}(l')),
\end{aligned}$$

then

$$\left[((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\cdot\ (\beta_1,\beta_2)\right]=(\eta_{1\beta_2(l)},\eta_{2\beta_2(l)})$$

and

$$(\beta_1,\beta_2)\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))=(\eta_{1\beta_1(l)},\eta_{2\beta_1(l)})\in Im(\alpha)$$

□

4. The Annihilator of a Crossed Module

In this section, our object is to generalise the notion of annihilator in associative algebras by using actor crossed module.

Definition 4.1. Let (T,L,μ) be a crossed module. An annihilator of the crossed module in associative algebras is the kernel of the homomorphism

$$(\eta,\alpha):(T,L,\mu)\to\mathcal{A}(T,L,\mu)$$

and it is denoted by $\mathrm{Ann}(T,L,\mu)$.

$$Ann(T,L,\mu)=\mathrm{Ker}(\eta,\alpha)=(Ann_T(L),Ann_L(T)\cap Ann_L(L),\mu)$$

where

$$\mathrm{Ker}\eta=\{t\in T\mid \eta_1(t)=0,\eta_2(t)\}=\{t\in T\mid t\cdot l=0,l\cdot t=0,l\in L\}=Ann_T(L)$$

and

$$\begin{aligned}
\mathrm{Ker}\alpha &= \{l\in L\mid \alpha(l)=((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))=(0,0)\}\\
&= \{l\in L\mid \Theta_l(t)=l\cdot t=0,\Xi_l(t)=t\cdot l=0,\Theta'_l(l')=ll'=0,\Xi'_l(l')=l'l=0,t\in T,l'\in L\}\\
&= Ann_L(T)\cap Ann_L(L)
\end{aligned}$$

$\mathrm{Ann}(T,L,\mu)$ is a two sided ideal of the crossed module $\mu:T\longrightarrow L$, because it is the kernel of a crossed module morphism under certain conditions.

5. An Application for Crossed Squares

It is considered that a crossed n-cubes is a higher dimensional analogous of crossed modules. If we take the case $n = 2$, a crossed square is obtained. As an associative algebra G gives rise to a crossed module $\mu : G \to Bim(G)$, that idea allows us to get an generalisation for higher dimension.

The definition of a crossed square in associative algebras is given by Ellis in [12]. Some related studies are [15], [16], and [17].

5.1. Crossed Squares

If a commutative diagram of associative algebras

$$\begin{array}{ccc} T & \xrightarrow{\lambda} & N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ L & \xrightarrow[\lambda']{} & P \end{array}$$

satisfies the following axioms together with functions $h : L \times N \to T,\ h' : N \times L \to T$ and actions of P on T, L and N, then it is called as a crossed square:

i. The maps $\mu, \lambda, \lambda', \nu$ and $\nu\lambda$ are crossed modules.

ii. The maps λ, μ preserve the action of P.

iii. $kh(l, n) = h(kl, n) = h(l, kn)$

$kh'(n, l) = h'(kn, l) = h'(n, kl)$

iv. $p \cdot h(l, n) = h(p \cdot l, n)$

$h(l, n) \cdot p = h(l, n \cdot p)$

$h(l \cdot p, n) = h(l, n \cdot p)$

$p \cdot h'(n, l) = h'(p \cdot n, l)$

$h'(n, l) \cdot p = h'(n, l \cdot p)$

$h'(n \cdot p, l) = h'(n, p \cdot l)$

v. $h(l_1 + l_2, n) = h(l_1, n) + h(l_2, n)$

$h(l, n_1 + n_2) = h(l, n_1) + h(l, n_2)$

$h'(n_1 + n_2, l) = h'(n_1, l) + h'(n_2, l)$

$h'(n, l_1 + l_2) = h'(n, l_1) + h'(n, l_2)$

vi. $\lambda h(l, n) = \lambda' l \cdot n,\ \lambda h'(n, l) = n \cdot \lambda' l$

$\lambda' h(l, n) = l \cdot \nu n,\ \lambda' h'(n, l) = \nu n \cdot l$

vii. $h(l, \lambda t) = \lambda' l \cdot t,\ h'(\lambda t, l) = t \cdot \lambda' l$

$h(\mu t, n) = t \cdot \nu n,\ h'(n, \mu t) = \nu n \cdot t$

viii. $n_2 \cdot h(l, n_1) = h'(n_2, l) \cdot n_2$

$l_2 \cdot h'(n, l_1) = h(l_2, n) \cdot l_1$

for all $l, l_1, l_2 \in L$, $n, n_1, n_2 \in N, p \in P, t \in T$, $k \in K$, where $l \cdot t$ means $\lambda'(l) \cdot t$. There are actions of L on T and N, via λ', and of N on T and L via ν.

As a crossed module $\mu : G \to Bim(G)$ arises from an associative algebra G, we have been able to deduce the result shown below:

Theorem 5.1. If $\mu : T \longrightarrow L$ is a crossed module, then the morphism

$$(\eta, \alpha) : (T, L, \mu) \to \mathcal{A}(T, L, \mu)$$

gives rise to the crossed square

$$\begin{array}{ccc} T & \xrightarrow{\eta=(\eta_{1t},\eta_{2t})} & U(L,T) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \Delta} \\ L & \xrightarrow[\alpha=((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))]{} & Bim(T,L,\mu) \end{array}$$

with functions

$$\begin{array}{rccl} h: & U(L,T) \times L & \to & T \\ & ((\beta_1, \beta_2), l) & \mapsto & h((\beta_1, \beta_2), l) = \beta_1(l) \end{array}$$

and

$$\begin{array}{rccl} h': & L \times U(L,T) & \to & T \\ & (l, (\beta_1, \beta_2)) & \mapsto & h'(l, (\beta_1, \beta_2)) = \beta_2(l) \end{array}$$

where $Bim(T, L, \mu)$ acts on L and T via the appropriate projections.

PROOF.

i. $\triangle : U(L, T) \longrightarrow Bim(T, L, \mu)$ is a crossed module as mentioned before. Since

CM1)

$$\begin{array}{rcl} \alpha(((\Theta, \Xi), (\Theta', \Xi')) \cdot l) & = & \alpha(\Theta'_l) \\ & = & ((\Theta_{\Theta'_l}, \Xi_{\Theta'_l}), (\Theta'_{\Theta'_l}, \Xi'_{\Theta'_l})) \\ & = & ((\Theta, \Xi), (\Theta', \Xi')) \circ ((\Theta_l, \Xi_l), (\Theta'_l, \Xi'_l)) \\ & = & ((\Theta, \Xi), (\Theta', \Xi')) \circ \alpha(l) \end{array}$$

and

$$\begin{array}{rcl} \alpha(l \cdot ((\Theta, \Xi), (\Theta', \Xi'))) & = & \alpha(\Xi'_l) \\ & = & ((\Theta_{\Xi'_l}, \Xi_{\Xi'_l}), (\Theta'_{\Xi'_l}, \Xi'_{\Xi'_l})) \\ & = & ((\Theta_l, \Xi_l), (\Theta'_l, \Xi'_l)) \circ ((\Theta, \Xi), (\Theta', \Xi')) \\ & = & \alpha(l) \circ ((\Theta, \Xi), (\Theta', \Xi')) \end{array}$$

CM2)

$$\begin{array}{rcl} \alpha(l) \cdot l' & = & ((\Theta_l, \Xi_l), (\Theta l', \Xi'_l)) \cdot l' \\ & = & \Theta'_l(l') = ll' \end{array}$$

and

$$\begin{array}{rcl} l \cdot \alpha(l') & = & l \cdot ((\Theta_{l'}, \Xi l'), (\Theta'_{l'}, \Xi'_{l'})) \\ & = & \Xi_{l'}(l) = ll', \end{array}$$

for $((\Theta, \Xi), (\Theta', \Xi')) \in Bim(T, L, \mu)$, $l \in L$, α is a crossed module.

Also $\chi = \alpha\mu = \triangle\eta$ is a crossed module, because

CM1)

$$\begin{aligned}
\chi(((\Theta,\Xi),(\Theta',\Xi'))\cdot t) &= \chi(\Theta_t)\\
&= \alpha\mu(\Theta_t)\\
&= ((\Theta_{\mu\Theta_t},\Xi_{\mu\Theta_t}),(\Theta'_{\mu\Theta_t},\Xi'_{\mu\Theta_t}))\\
&= ((\Theta_{\Theta'\mu_t},\Xi_{\Theta'\mu_t}),(\Theta'_{\Theta'\mu_t},\Xi'_{\Theta'\mu_t}))\\
&= ((\Theta,\Xi),(\Theta',\Xi'))\circ((\Theta_{\mu_t},\Xi_{\mu_t}),(\Theta'_{\mu_t},\Xi'_{\mu_t}))\\
&= ((\Theta,\Xi),(\Theta',\Xi'))\circ\alpha\mu(t)\\
&= ((\Theta,\Xi),(\Theta',\Xi'))\circ\chi(t)
\end{aligned}$$

and

$$\begin{aligned}
\chi(t\cdot(\Theta,\Xi),(\Theta',\Xi'))) &= \chi(\Xi_t)\\
&= \alpha\mu(\Xi_t)\\
&= ((\Theta_{\mu\Xi_t},\Xi_{\mu\Xi_t}),(\Theta'_{\mu\Xi_t},\Xi'_{\mu\Xi_t}))\\
&= ((\Theta_{\Xi'\mu_t},\Xi_{\Xi'\mu_t}),(\Theta'_{\Xi'\mu_t},\Xi'_{\Xi'\mu_t}))\\
&= ((\Theta_{\mu_t},\Xi_{\mu_t}),(\Theta'_{\mu_t},\Xi'_{\mu_t}))\circ((\Theta,\Xi),(\Theta',\Xi'))\\
&= \alpha\mu(t)\circ((\Theta,\Xi),(\Theta',\Xi'))\\
&= \chi(t)\circ((\Theta,\Xi),(\Theta',\Xi'))
\end{aligned}$$

CM2)

$$\begin{aligned}
\chi(t)\cdot t' &= \alpha\mu(t)\cdot t'\\
&= ((\Theta_{\mu_t},\Xi_{\mu_t}),(\Theta'_{\mu_t},\Xi'_{\mu_t}))\cdot t'\\
&= \Theta_{\mu_t}(t')\\
&= \mu(t)\cdot t'\\
&= tt'
\end{aligned}$$

and

$$\begin{aligned}
t\cdot\chi(t') &= t\cdot\alpha\mu(t')\\
&= t\cdot((\Theta_{\mu_{t'}},\Xi_{\mu_{t'}}),(\Theta'_{\mu_{t'}},\Xi'_{\mu_{t'}}))\\
&= \Xi_{\mu_{t'}}(t)\\
&= t\cdot\mu(t')\\
&= tt'
\end{aligned}$$

for $((\Theta,\Xi),(\Theta',\Xi'))\in Bim(T,L,\mu)$, $t\in T$.

ii. Since

$$\begin{aligned}
\alpha(((\Theta,\Xi),(\Theta',\Xi'))\cdot t) &= \mu(\Theta_t)\\
&= \Theta'\mu(t)\\
&= ((\Theta,\Xi),(\Theta',\Xi'))\circ\mu(t)
\end{aligned}$$

$$\begin{aligned}
\alpha(t\cdot((\Theta,\Xi),(\Theta',\Xi'))) &= \mu(\Xi_t)\\
&= \Xi'\mu(t)\\
&= \mu(t)\circ((\Theta,\Xi),(\Theta',\Xi'))
\end{aligned}$$

and

$$\begin{aligned}
\eta(((\Theta,\Xi),(\Theta',\Xi'))\cdot t) &= \eta(\Theta_t)\\
&= (\eta_{1\Theta_t},\eta_{2\Theta_t})\\
&= ((\Theta,\Xi),(\Theta',\Xi'))\cdot(\eta_{1t},\eta_{2t})\\
&= ((\Theta,\Xi),(\Theta',\Xi'))\cdot\eta(t)
\end{aligned}$$

$$\begin{aligned}
\eta(t\cdot((\Theta,\Xi),(\Theta',\Xi'))) &= \eta(\Xi_c)\\
&= (\eta_{1\Xi_t},\eta_{2\Xi_t})\\
&= (\eta_{1t},\eta_{2t})\cdot((\Theta,\Xi),(\Theta',\Xi'))\\
&= \eta(t)\cdot((\Theta,\Xi),(\Theta',\Xi'))
\end{aligned}$$

for $((\Theta,\Xi),(\Theta',\Xi'))\in Bim(T,L,\mu)$, $t\in T$, μ and η preserve the action of $Bim(T,L,\mu)$.

iii.

$$\begin{aligned}
kh((\beta_1,\beta_2),l) &= k(\beta_1(l))\\
&= k\beta_1(l)\\
&= \beta_1(kl)\\
&= h((\beta_1,\beta_2)(kl))\\
&= k(\beta_1(l))\\
&= (k(\beta_1))l\\
&= h(k(\beta_1,\beta_2),l)
\end{aligned}$$

$$\begin{aligned}
kh'(l,(\beta_1,\beta_2)) &= k(\beta_2(l))\\
&= k\beta_2(l)\\
&= \beta_2(kl)\\
&= h'((kg),(\beta_1,\beta_2))\\
&= k(\beta_2(l))\\
&= (k(\beta_2))l\\
&= h'(l,k(\beta_1,\beta_2))
\end{aligned}$$

for $k\in K,(\beta_1,\beta_2)\in U(L,T)$, $l\in L$.

iv. For $((\Theta,\Xi),(\Theta',\Xi'))\in Bim(T,L,\mu)$, $l\in L$,

$$\begin{aligned}
((\Theta,\Xi),(\Theta',\Xi'))\cdot(h((\beta_1,\beta_2),l)) &= ((\Theta,\Xi),(\Theta',\Xi'))\cdot\beta_1(l)\\
&= \Theta_{\beta_1(l)}\\
&= \Theta\beta_1(l)\\
&= h((\Theta\beta_1,\beta_2\Xi'),l)\\
&= h((((\Theta,\Xi),(\Theta',\Xi'))\cdot(\beta_1,\beta_2)),l)
\end{aligned}$$

$$\begin{aligned}
h((\beta_1,\beta_2),l))\cdot((\Theta,\Xi),(\Theta',\Xi')) &= \beta_1(l)\cdot((\Theta,\Xi),(\Theta',\Xi'))\\
&= \Xi_{\beta_1(l)}\\
&= \Xi\beta_1(l)\\
&= \beta_1\Xi'_l\\
&= h((\beta_1,\beta_2),\Xi'_l)\\
&= h((\beta_1,\beta_2),l\cdot((\Theta,\Xi),(\Theta',\Xi')))
\end{aligned}$$

$$
\begin{aligned}
h(((\Theta,\Xi),(\Theta',\Xi'))\cdot(\beta_1,\beta_2),l)) &= h((\Theta\beta_1,\beta_2\Xi'),l)) \\
&= \Theta_{\beta_1(l)} \\
&= \Theta\beta_1(l) \\
&= \beta_1\Theta'(l) \\
&= h((\beta_1\Theta',\Xi\beta_2),l) \\
&= h(((\beta_1,\beta_2)\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l)),l)
\end{aligned}
$$

$$
\begin{aligned}
((\Theta,\Xi),(\Theta',\Xi'))\cdot(h'(l,(\beta_1,\beta_2))) &= ((\Theta,\Xi),(\Theta',\Xi'))\cdot\beta_2(l) \\
&= \Theta_{\beta_2(l)} \\
&= \Theta\beta_2(l) \\
&= \beta_2\Theta'(l) \\
&= h'(\Theta'_l,(\beta_1,\beta_2)) \\
&= h'(((\Theta,\Xi),(\Theta',\Xi'))\cdot l,(\beta_1,\beta_2))
\end{aligned}
$$

$$
\begin{aligned}
h'((\beta_1,\beta_2),l))\cdot((\Theta,\Xi),(\Theta',\Xi')) &= \beta_2(l)\cdot((\Theta,\Xi),(\Theta',\Xi')) \\
&= \Xi_{\beta_2(l)} \\
&= \Xi\beta_2(l) \\
&= \beta_2\Xi'_l \\
&= h'(l,(\Theta\beta_1,\beta_2\Xi')) \\
&= h'(l,(\beta_1,\beta_2)\cdot((\Theta,\Xi),(\Theta',\Xi')))
\end{aligned}
$$

$$
\begin{aligned}
h'((\beta_1,\beta_2)\cdot((\Theta_\beta,\Xi_\beta),(\Theta'_\beta,\Xi'_\beta)),\beta)) &= h'(\beta;(\beta_1\Theta',\Xi\beta_2)) \\
&= \Xi_{\beta_2(l)} \\
&= \Xi\beta_2(l) \\
&= \beta_2\Xi'(l) \\
&= h'(l,(\Theta\beta_1,\beta_2\Xi')) \\
&= h'(l,(\beta_1,\beta_2)\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l)))
\end{aligned}
$$

v.

$$
\begin{aligned}
h((\beta_1,\beta_2)+(\theta_1,\theta_2),l) &= h((\beta_1+\theta_1,\beta_2+\theta_2),l) \\
&= (\beta_1+\theta_1)l \\
&= \beta_1(l)+\theta_1(l) \\
&= h((\beta_1,\beta_2),l)+h((\theta_1,\theta_2),l)
\end{aligned}
$$

$$
\begin{aligned}
h((\beta_1,\beta_2),l_1+l_2) &= \beta_1(l_1+l_2) \\
&= \beta_1(l_1)+\beta_1(l_2) \\
&= h((\beta_1,\beta_2),l_1)+h((\beta_1,\beta_2)),l_2)
\end{aligned}
$$

$$
\begin{aligned}
h'(l_1+l_2,(\beta_1,\beta_2)) &= \beta_2(l_1+l_2) \\
&= \beta_2(l_1)+\beta_2(l_2) \\
&= h'(l_1,(\beta_1,\beta_2))+h'(l_2,(\beta_1,\beta_2))
\end{aligned}
$$

$$
\begin{aligned}
h'(l,(\beta_1,\beta_2)+(\theta_1,\theta_2)) &= h'(l,(\beta_1+\theta_1,\beta_2+\theta_2)) \\
&= (\beta_2+\theta_2)(l) \\
&= \beta_2(g)+\theta_2(l) \\
&= h'(l,(\beta_1,\beta_2))+h'(l,(\theta_1,\theta_2))
\end{aligned}
$$

vi.

$$\begin{aligned}
\mu h((\beta_1,\beta_2),l) &= \mu(\beta_1(l)) \\
&= \mu\beta_1(l) \\
&= ((\beta_1\mu,\beta_2\mu),(\mu\beta_1,\mu\beta_2))\cdot l \\
&= ((\Theta_\beta,\Xi_\beta),(\Theta'_\beta,\Xi'_\beta))\cdot l \\
&= (\triangle(\beta_1,\beta_2))\cdot l
\end{aligned}$$

$$\begin{aligned}
\mu h'(l,(\beta_1,\beta_2)) &= \mu(\beta_2(l)) \\
&= \mu\beta_2(l) \\
&= l\cdot((\beta_1\mu,\beta_2\mu),(\mu\beta_1,\mu\beta_2)) \\
&= l\cdot((\Theta_\beta,\Xi_\beta),(\Theta'_\beta,\Xi'_\beta)) \\
&= l\cdot(\triangle(\beta_1,\beta_2))
\end{aligned}$$

$$\begin{aligned}
\eta h((\beta_1,\beta_2),l) &= \eta(\beta_1(l)) \\
&= (\eta_{1\beta_1(l)},\eta_{2\beta_1(l)}) \\
&\overset{(*)}{=} ((\beta_1\Theta'_l,\Xi_l\beta_2) \\
&= (\beta_1,\beta_2)\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l)) \\
&= (\beta_1,\beta_2)\cdot\alpha(l)
\end{aligned}$$

$$(*)\quad \begin{aligned}
\eta_{1\beta_1(l)}(l') &= \beta_1(l)\cdot l' \\
&= \beta_1(ll') \\
&= \beta_1\Theta'_l(l')
\end{aligned} \quad \text{and} \quad \begin{aligned}
\eta_{2\beta_1(l)}(l') &= l'\cdot\beta_1(l) \\
&= \beta_2(ll') \\
&= \Xi_l\beta_2(l')
\end{aligned}$$

$$\begin{aligned}
\eta h'((\beta_1,\beta_2),l) &= \eta(\beta_2(l)) \\
&= (\eta_{1\beta_2(l)},\eta_{2\beta_2(l)}) \\
&\overset{(*)}{=} ((\Theta_l\beta_1,\beta_2\Xi'_l) \\
&= ((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\cdot(\beta_1,\beta_2) \\
&= \alpha(l)\cdot(\beta_1,\beta_2)
\end{aligned}$$

$$(*)\quad \begin{aligned}
\eta_{1\beta_2(l)}(l') &= \beta_2(l)\cdot l' \\
&= l\cdot\beta_1(l') \\
&= \Theta_l\beta_1(l')
\end{aligned} \quad \text{and} \quad \begin{aligned}
\eta_{2\beta_2(l)}(l') &= l'\cdot\beta_2(l) \\
&= \beta_2(ll') \\
&= \beta_2\Xi'_l(l')
\end{aligned}$$

vii.

$$\begin{aligned}
h((\beta_1,\beta_2),\mu(t)) &= \beta_1(\mu(t)) \\
&= \beta_1\mu(t) \\
&= ((\beta_1\mu,\beta_2\mu),(\mu\beta_1,\mu\beta_2))\cdot t \\
&= ((\Theta_\beta,\Xi_\beta),(\Theta'_\beta,\Xi'_\beta))\cdot t \\
&= (\triangle(\beta_1,\beta_2))\cdot t
\end{aligned}$$

$$\begin{aligned} h'(\mu(t),(\beta_1,\beta_2)) &= \beta_2(\mu(t)) \\ &= \beta_2\mu(t) \\ &= t\cdot((\beta_1\mu,\beta_2\mu),(\mu\beta_1,\mu\beta_2)) \\ &= t\cdot((\Theta_\beta,\Xi_\beta),(\Theta'_\beta,\Xi'_\beta)) \\ &= t\cdot(\triangle(\beta_1,\beta_2)) \end{aligned}$$

$$\begin{aligned} h(\eta(t),l) &= h((\eta_{1t},\eta_{2t}),l) \\ &= \eta_{1t}(l) \\ &= t\cdot l \\ &= \Xi_l(t) \\ &= t\cdot((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l)) \\ &= t\cdot\alpha(l) \end{aligned}$$

$$\begin{aligned} h'(l,\eta(t)) &= h'(l,(\eta_{1t},\eta_{2t})) \\ &= \eta_{2t}(l) \\ &= l\cdot t \\ &= \Theta_l(t) \\ &= ((\Theta_l,\Xi_l),(\Theta'_l,\Xi'_l))\cdot t \\ &= \alpha(l)\cdot t \end{aligned}$$

viii.

$$\begin{aligned} l'\cdot h((\beta_1,\beta_2),l) &= l'\cdot\beta_1(l) \\ &= \beta_2(l')\cdot l \\ &= h'(l',(\beta_1,\beta_2))\cdot l \end{aligned}$$

$$\begin{aligned} (\theta_1,\theta_2)\cdot h'(l,(\beta_1,\beta_2)) &= \triangle(\theta_1,\theta_2)\cdot\beta_2(l) \\ &= \theta_1\mu\beta_2(l) \\ &\overset{(*)}{=} \beta_2\mu\theta_1(l) \\ &= \theta_1(l)\cdot\triangle((\beta_1,\beta_2) \\ &= h((\theta_1,\theta_2),l)\cdot(\beta_1,\beta_2) \end{aligned}$$

$(*)$ Because of $L^2=L$, for every $l\in L$, we can take $l=l_1l_2$.

$$\begin{aligned} \theta_1\mu\beta_2(l_1l_2) &= \theta_1\mu(l_1\cdot\beta_2(l_2)) \\ &= \Theta_\theta(l_1\cdot\beta_2(l_2)) \\ &= \Theta'_\theta(l_1)\cdot\beta_2(l_2) \\ &= \mu\theta_1(l_1)\cdot\beta_2(l_2) \\ &= \beta_2((\mu\theta_1(l_1))l_2) \\ &= \beta_2(\mu(\theta_1(l_1)l_2)) \\ &= \beta_2(\mu(\theta_1(l_1)\cdot l_2)) \\ &= \beta_2(\mu(\theta_1(l_1l_2))) \\ &= \beta_2\mu\theta_1(l_1l_2) \end{aligned}$$

□

6. Conclusion

It is well known that the action of any group on itself is given by a group homomorphism from any group to its automorphism group. In the case of associative algebra, the role of automorphism groups replaces with bimultiplication algebra. It is considered that the concept of a crossed module in associative algebra is a generalisation of the concept of an associative algebra. Thus we can generalise the notion of bimultiplication algebra for a crossed module in associative algebras. Thus, we conclude that in this context the notion of action is given by the actor crossed module which is obtained via bimultiplication crossed module. Also since the annihilator of an associative algebra A is given by the kernel of algebra morphism $A \to Bim(A)$, we get the annihilator crossed module as the kernel of the crossed module morphism $(\eta, \alpha) : (T, L, \mu) \to \mathcal{A}(T, L, \mu)$. Furthermore, we see that this morphism gives rise to a crossed square which is a two-dimensional analogous to crossed module.

Conflicts of Interest

The authors declare no conflict of interest.

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Journal of
New Theory
ISSN: 2149-1402

35 (2021) 91-102
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
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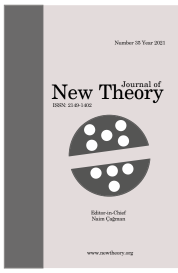

Analysis and Simulation of a Two-Stage Blocked Tandem Queueing System

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Article History
Received: 17 May 2021
Accepted: 08 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract − In this paper, a new blocked tandem queueing model is given and analysed. The arrival process to this queueing model is Poisson with parameter λ. There is one service unit at the first stage of the system, and the service time of this unit is exponentially distributed with μ_1 parameter. There are two parallel service units at the second stage, and the service time of these service units are exponentially distributed with parameters μ_2 and μ_3. No queue is allowed at the first stage of the system. Upon completing service at the first stage, a customer proceeds to the second stage if at least one of the service units at the second stage is available. If both service units at the second stage are busy, the customer blocks the service unit at the first stage, which results in loss. The most important measure of performance of this queueing system is the loss probability π_{loss}. First of all, the state probabilities of the system are obtained and then using these probabilities, the steady-state distribution of the system is obtained. Transition probabilities of the system are calculated by using steady-state probabilities, and finally an equation is obtained for π_{loss} in terms of transition probabilities. Furthermore, another measure of performance, the mean number of customers, is obtained in terms of transition probabilities. Since the Equation for π_{loss} is very complex, a numerical method is used to calculate the minimum π_{loss} probabilities. After numerical optimal π_{loss} calculations, a simulation of the queueing system is done, and it is seen that the obtained numerical π_{loss} values tend to simulation results.

Keywords – *Stochastic processes, Markov chains, queueing theory, optimization, queue performance*

Mathematics Subject Classification (2020) − 60K25, 60K30

1. Introduction

Stochastic queueing models are widely used in production lines, telecommunication technologies and computer sciences. In recent years, the studies mostly focused on queueing networks. A queueing network is simply a combination of several queueing systems. Bertsimas studied on performance analyse of queueing networks via Robust optimization [1]. A semi-open queueing network with a Markovian arrival process having a finite number of nodes is considered in a study [2]. A study on evaluating the performance of general queueing networks in manufacturing systems is given in [3]. Dudina et al. considered a multi-service retrial queueing system with Markovian arrival flow to model a call centre [4]. For the first time, Hunt [5] defined the customer's blocking effects in a queue sequence. Various performance measures, namely the average number of customers in the queueing system, the proportion of customers entering the queueing system, average waiting time, and a blocked series queue, have been obtained in the study [6]. Basharin et al. show

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how properties of Markovian Arrival Processes can be derived from the general theory of Markov processes with a homogeneous second component [7]. In another study [8], a two-station tandem queue with blocking is considered, and an accurate solution with correct stationary distribution is given. A stochastic queueing model, consisting of two heterogeneous service channels and having no waiting room, is considered [9]. In that study, Sağlam et al. calculated the expected number of customers and loss probability, an optimal ordering of service channels is given and minimizing parameters of the queueing system are found. In another related study [10], a queueing model with two sequential stations is constructed. In this model, there is a single service at each station, and no queue is allowed at the second station. The state probabilities and loss probability of this model are obtained. Furthermore, the model is simulated. In a recent study [2], a semi-open queueing network having a finite number of nodes is considered, and the stationary behaviour of queueing states is analysed.

In this paper, a new blocked tandem queueing system is constructed and analysed. The model we have analysed is a modified version of the model studied in [9]. The arrival process to this new queueing system is Poisson. There is one service unit at the first stage of the system, and the service time of this unit is exponentially distributed. There are two parallel service units at the second stage, and the service time of these service units are exponentially distributed. No queue is allowed at the first stage of the system. Upon completing service at the first stage, a customer proceeds to the second stage if at least one of the service units at the second stage is available. If both service units at the second stage are busy, the customer blocks the service unit at the first stage; hence, loss occurs. The most important measure of performance of this queueing system is the loss probability.

2. The Stochastic Queueing Model

The queueing model we considered in this study has Poisson arrival flow with parameter λ. At the first station, there is a single service unit that has exponentially distributed service time with parameter μ_1 and no queue is allowed at this phase. The second station of the system consists of two parallel service units, and they also have exponentially distributed service times with parameters μ_2 and μ_3 respectively. As well as the first station, no queue is allowed at the second station. Upon receiving service at the first station, a customer proceeds to the second station of the queueing system. If both service units at the second station are empty, the customer enters the first service unit with probability α_1 or second service unit with probability $\alpha_2 = 1 - \alpha_1$. If only one of the service units at the second station is available, the customer proceeds to this server. On the other hand, if both servers at the second station are busy, the customer waits at the first station until any of the service units of the second station is available; hence the customer blocks the first station. This queueing model is mathematically stated as follows: At any given time t, let $\xi_1(t)$ random variable be the number of customers in service unit of the first station, $\xi_2(t)$ and $\xi_3(t)$ random variables be the number of customers in the services of the second station. Then the 3-dimensional continuous-time Markov chain of the model is stated as $\{\xi_1(t), \xi_2(t), \xi_3(t); t \geq 0\}$ and the state probabilities of the Markov chain is p_{n_1,n_2,n_3} where $n_1 \in \{0,1\}$, $n_2 \in \{0,1\}$, $n_3 \in \{0,1\}$. Finally, the state space of the defined Markov chain is $\mathfrak{I} = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1), (b,1,1)\}$.

2.1. The Transition Probabilities of the Queueing System

First, we need to find the state probabilities of the system to obtain transition probabilities. Then Kolmogorov differential equations are acquired using state probabilities, and by using Kolmogorov equations, the stationary distribution of the chain is obtained. Finally, the transition probabilities are found with the help of stationary distribution. The probability at any given time t, in which there are n_1 customers in the first service unit, n_2 customers in the second, and n_3 customers in the third, is defined as

$$P(\xi_1(t) = n_1, \xi_2(t) = n_2, \xi_3(t) = n_3) = p_{n_1 n_2 n_3}(t)$$

For $\Delta t \to 0$, the state probabilities of the Markov chain $\{\xi_1(t), \xi_2(t), \xi_3(t); t \geq 0\}$ are obtained as below:

$$p_{000}(t+h) = p_{000}(t)\big(1-\lambda h + o(h)\big) + p_{010}(t)\big(\mu_2 h + o(h)\big) + p_{001}(t)\big(\mu_3 h + o(h)\big) + o(h) \tag{2.1}$$

$$p_{010}(t+h) = p_{010}(t)\big(1-\lambda h + o(h)\big)\big(1-\mu_2 h + o(h)\big) + p_{011}(t)\big(\mu_3 h + o(h)\big) + p_{100}(t)\alpha_1\big(\mu_1 h + o(h)\big) + o(h) \tag{2.2}$$

$$p_{001}(t+h) = p_{001}(t)\big(1-\lambda h + o(h)\big)\big(1-\mu_3 h + o(h)\big) + p_{011}(t)\big(\mu_2 h + o(h)\big) + p_{100}(t)\alpha_2\big(\mu_1 h + o(h)\big) + o(h) \tag{2.3}$$

$$p_{011}(t+h) = p_{011}(t)\big(1-\lambda h + o(h)\big)\big(1-\mu_2 h + o(h)\big)\big(1-\mu_3 h + o(h)\big) + p_{110}(t)\big(\mu_1 h + o(h)\big) + p_{101}(t)\big(\mu_1 h + o(h)\big) + p_{b11}(t)\big(\mu_2 h + o(h)\big) + p_{b11}(t)\big(\mu_3 h + o(h)\big) + o(h) \tag{2.4}$$

$$p_{100}(t+h) = p_{100}(t)\big(1-\mu_1 h + o(h)\big) + p_{000}(t)\big(\lambda h + o(h)\big) + p_{110}(t)\big(\mu_2 h + o(h)\big) + p_{101}(t)\big(\mu_3 h + o(h)\big) + o(h) \tag{2.5}$$

$$p_{110}(t+h) = p_{110}(t)\big(1-\mu_1 h + o(h)\big)\big(1-\mu_2 h + o(h)\big) + p_{111}(t)\big(\mu_3 + o(h)\big) + p_{010}(t)\big(\lambda h + o(h)\big) + o(h) \tag{2.6}$$

$$p_{101}(t+h) = p_{101}(t)\big(1-\mu_1 h + o(h)\big)\big(1-\mu_3 h + o(h)\big) + p_{111}(t)\big(\mu_2 + o(h)\big) + p_{001}(t)\big(\lambda h + o(h)\big) + o(h) \tag{2.7}$$

$$p_{111}(t+h) = p_{111}(t)\big(1-\mu_1 h + o(h)\big)\big(1-\mu_2 h + o(h)\big)\big(1-\mu_3 h + o(h)\big) + p_{011}(t)\big(\lambda h + o(h)\big) + o(h) \tag{2.8}$$

$$p_{b11}(t+h) = p_{b11}(t)\big(1-\mu_2 h + o(h)\big)\big(1-\mu_3 h + o(h)\big) + p_{111}(t)\big(\mu_1 + o(h)\big) + o(h) \tag{2.9}$$

under the assumption of limit distribution by using state probabilities, we have the stationary state probabilities as following:

$$0 = -\lambda p_{000} + \mu_2 p_{010} + \mu_3 p_{001} \tag{2.10}$$

$$0 = -(\lambda + \mu_2)p_{010} + \mu_3 p_{011} + \alpha_1 \mu_1 p_{100} \tag{2.11}$$

$$0 = -(\lambda + \mu_3)p_{001} + \mu_2 p_{011} + \alpha_2 \mu_1 p_{100} \tag{2.12}$$

$$0 = -(\lambda + \mu_2 + \mu_3)p_{011} + \mu_1 p_{110} + \mu_1 p_{101} + \mu_2 p_{b11} + \mu_3 p_{b11} \tag{2.13}$$

$$0 = -\mu_1 p_{100} + \lambda p_{000} + p_{110}\mu_2 + p_{101}\mu_3 \tag{2.14}$$

$$0 = -(\mu_1 + \mu_2)p_{110} + \mu_3 p_{111} + \lambda p_{010} \tag{2.15}$$

$$0 = -(\mu_1 + \mu_3)p_{101} + \mu_2 p_{111} + \lambda p_{001} \tag{2.16}$$

$$0 = -(\mu_1 + \mu_2 + \mu_3)p_{111} + \lambda p_{011} \tag{2.17}$$

$$0 = -(\mu_2 + \mu_3)p_{b11} + \mu_1 p_{111} \tag{2.18}$$

Now, we calculate the transition probabilities of the queueing system by using stationary state probabilities. Using Equation (2.17), we have

$$p_{111} = \left(\frac{\lambda}{\mu_1 + \mu_2 + \mu_3}\right) p_{011} \tag{2.19}$$

Equations (2.18) and (2.19) lead us to:

$$p_{b11} = \left(\frac{\mu_1}{\mu_2 + \mu_3}\right)\left(\frac{\lambda}{\mu_1 + \mu_2 + \mu_3}\right) p_{011} \tag{2.20}$$

With the same solution manner, all transition probabilities are obtained in terms of p_{011}:

$$p_{001} = \left[\frac{\mu_2}{\lambda + \mu_3} + \alpha_2 \frac{\mu_1}{\lambda + \mu_3} \Delta_4\right] p_{011} \tag{2.21}$$

$$p_{010} = \left[\frac{\mu_3}{\lambda + \mu_2} + \alpha_1 \frac{\mu_1}{\lambda + \mu_2} \Delta_4\right] p_{011} \tag{2.22}$$

$$p_{000} = \left[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda + \mu_2} + \alpha_1 \frac{\mu_1}{\lambda + \mu_2} \Delta_4\right) + \frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda + \mu_3} + \alpha_2 \frac{\mu_1}{\lambda + \mu_3} \Delta_4\right)\right] p_{011} \tag{2.23}$$

$$p_{101} = \left[\frac{\mu_2}{\mu_1 + \mu_3} \cdot \frac{\lambda}{\mu_1 + \mu_2 + \mu_3} + \frac{\lambda}{\mu_1 + \mu_3} \cdot \frac{\mu_2}{\lambda + \mu_3} + \alpha_2 \frac{\lambda}{\mu_1 + \mu_3} \cdot \frac{\mu_1}{\lambda + \mu_3} \Delta_4\right] p_{011} \tag{2.24}$$

$$p_{100} = \Delta_4 . p_{011} \tag{2.25}$$

$$p_{110} = \left[\frac{\lambda \mu_3}{(\mu_1 + \mu_2)(\mu_1 + \mu_2 + \mu_3)} + \frac{\lambda}{\mu_1 + \mu_2}\left(\frac{\mu_3}{\lambda + \mu_2} + \alpha_1 \frac{\mu_1}{\lambda + \mu_2} \Delta_4\right)\right] p_{011} \tag{2.26}$$

where,

$$\Delta_1 = 1 - \alpha_1 \frac{\mu_2}{\lambda + \mu_2}\left(1 + \frac{\lambda}{\mu_1 + \mu_2}\right) - \alpha_2 \frac{\mu_3}{\lambda + \mu_3}(1 + \frac{\lambda}{\mu_1 + \mu_3})$$

$$\Delta_2 = \frac{\mu_3}{\lambda + \mu_2} + (\frac{\mu_3}{\mu_1 + \mu_2} \cdot \frac{\lambda}{\mu_1 + \mu_2 + \mu_3}) + (\frac{\lambda}{\mu_1 + \mu_2} \cdot \frac{\mu_3}{\lambda + \mu_2})$$

$$\Delta_3 = \frac{\mu_2}{\lambda + \mu_3} + (\frac{\mu_2}{\mu_1 + \mu_3} \cdot \frac{\lambda}{\mu_1 + \mu_2 + \mu_3}) + (\frac{\lambda}{\mu_1 + \mu_3} \cdot \frac{\mu_2}{\lambda + \mu_3})$$

and

$$\Delta_4 = \frac{\mu_2 \Delta_2 + \mu_3 \Delta_3}{\mu_1 \Delta_1}$$

Hence, the sum of all these probabilities is 1, i.e.,

$$p_{000} + p_{001} + p_{010} + p_{100} + p_{011} + p_{101} + p_{110} + p_{111} + p_{b11} = 1 \tag{2.27}$$

Substituting all obtained transition probabilities in Equation (2.27), we have

$$p_{011}\left[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda + \mu_2} + \alpha_1 \frac{\mu_1}{\lambda + \mu_2} \Delta_4\right) + \frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda + \mu_3} + \alpha_2 \frac{\mu_1}{\lambda + \mu_3} \Delta_4\right)\right.$$

$$+ \left(\frac{\mu_2}{\lambda + \mu_3} + \alpha_2 \frac{\mu_1}{\lambda + \mu_3} \Delta_4\right) + \left(\frac{\mu_3}{\lambda + \mu_2} + \alpha_1 \frac{\mu_1}{\lambda + \mu_2} \Delta_4\right) + \Delta_4 + 1$$

$$+ \left(\frac{\mu_2}{\mu_1 + \mu_3} \cdot \frac{\lambda}{\mu_1 + \mu_2 + \mu_3} + \frac{\lambda}{\mu_1 + \mu_3} \cdot \frac{\mu_2}{\lambda + \mu_3} + \alpha_2 \frac{\lambda}{\mu_1 + \mu_3} \cdot \frac{\mu_1}{\lambda + \mu_3} \Delta_4\right)$$

$$+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)$$

$$+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\Bigg]$$

$$=1$$

Note that the last Equation is only in terms of p_{011}. Hereby, the transition probability p_{011} is obtained in terms of system parameters precisely as:

$$p_{011} = \Bigg[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)$$

$$+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1$$

$$+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)$$

$$+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)$$

$$+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\Bigg]^{-1}$$

The last thing to do is to rewrite all transition probabilities in terms of system parameters:

$$p_{001} = \left[\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right]\cdot\Bigg[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)$$

$$+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1$$

$$+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)$$

$$+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)$$

$$+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\Bigg]^{-1}$$

$$p_{010} = \left[\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right]\cdot\Bigg[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)$$

$$+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1$$

$$+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)$$

$$+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)$$

$$+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\Bigg]^{-1}$$

$$\begin{aligned}
p_{000} &= \left[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\right]\\
&\cdot\left[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\right.\\
&+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1\\
&+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)\\
&\left.+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\right]^{-1}
\end{aligned}$$

$$\begin{aligned}
p_{101} &= \left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&\cdot\left[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\right.\\
&+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1\\
&+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)\\
&\left.+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\right]^{-1}
\end{aligned}$$

$$
\begin{aligned}
p_{100} &= \Delta_4 \cdot \Bigg[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1\\
&+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)\\
&+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\Bigg]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
p_{110} &= \left[\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right]\\
&\cdot\Bigg[\frac{\mu_2}{\lambda}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\frac{\mu_3}{\lambda}\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&+\left(\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)+\Delta_4+1\\
&+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)\\
&+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}\left(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4\right)\right)\\
&+\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)\Bigg]^{-1}
\end{aligned}
$$

2.2. The Loss Probability and The Mean Customer Number of the Queue

The most important performance measure of the defined queueing model is the loss probability. In this queueing model, customer loss occurs only at the first station of the queue, as previously stated. In this context, the loss probability can be simply calculated as:

$$\pi_{loss} = p_{110} + p_{100} + p_{101} + p_{b11} + p_{111} \tag{2.28}$$

Substituting the previously calculated probabilities p_{110}, p_{100}, p_{101}, p_{b11}, and p_{111}, which are the equations (2.26), (2.25), (2.24), (2.20), and (2.19), respectively, in Equation (2.28); we precisely have loss probability in terms of system parameters as below:

$$\pi_{loss} = \Bigg[\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\left(\frac{\mu_1}{\mu_2+\mu_3}\right)\left(\frac{\lambda}{\mu_1+\mu_2+\mu_3}\right)$$

$$+\left(\frac{\mu_2}{\mu_1+\mu_3}\cdot\frac{\lambda}{\mu_1+\mu_2+\mu_3}+\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_2}{\lambda+\mu_3}+\alpha_2\frac{\lambda}{\mu_1+\mu_3}\cdot\frac{\mu_1}{\lambda+\mu_3}\Delta_4\right)+\Delta_4$$

$$+\left(\frac{\lambda\mu_3}{(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)}+\frac{\lambda}{\mu_1+\mu_2}(\frac{\mu_3}{\lambda+\mu_2}+\alpha_1\frac{\mu_1}{\lambda+\mu_2}\Delta_4)\right)\Bigg]p_{011} \tag{2.29}$$

Furthermore, the mean number of customers in the system is obtained as:

$$E(N)=\sum_{n_1\in\mathfrak{I}}\sum_{n_2\in\mathfrak{I}}\sum_{n_3\in\mathfrak{I}}(n_1+n_2+n_3)\,p_{n_1,n_2,n_3} \tag{2.30}$$

where $\mathfrak{I}=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1),(b,1,1)\}$ is the state space of the defined Markov chain. Hence the mean number of customers in the system is calculated as follows:

$$E(N)=0.p_{000}+1.(p_{100}+p_{010}+p_{001})+2.(p_{110}+p_{101}+p_{011})+3.(p_{111}+p_{b11})$$

2.3. The Optimization of the Loss Probability

In a blocked queueing system, one of the most notable performance measures is the loss probability. When a service is busy, a new incoming customer cannot enter the service and has two options: to leave the system without having service or wait until the service is available. In the system we are interested in, when a service is busy, the incoming customer leaves the queue system without being served, so that loss occurs. The probability of this loss is called the loss probability. In this section, we aim to minimize the loss probability π_{loss}, obtained in the previous section, and calculate the mean customer number in the queueing system. When the loss probability, given with the Equation (2.29), is examined; clearly, it is very complex and difficult to reach the minimum value of π_{loss} with the help of algebraic methods. Therefore, under two configurations of the queuing system, the loss probability is numerically calculated, and the minimum loss probability values of both configurations are determined. These two different configurations of the queuing system are based on customer arrival rate and the total service capacity of the system. In order to calculate the loss probability and mean number of customers, the arrival rate λ is chosen as constant. In this context, $c = 2\lambda$ and $c = \lambda$ configurations are established where $c=\mu_1+\mu_2+\mu_3$ is the total service capacity of the queueing system.

First, the optimal service capacity of the $1st$ service unit is searched, and in the next step, the optimal service capacities of the 2nd and 3rd service units are determined for the changing α_1 and α_2 possibilities. The data obtained by applying this method are given in Table 1, 2, 3, and 4. According to the data obtained from the tables, minimum π_{loss} values for $c = 2\lambda$ and $c = \lambda$ configurations are found. The results are also shown in Figure 1 and 2.

Table 1. Optimal service capacity selection of 1st service unit for $c = 2\lambda$ system configuration and the corresponding loss probability and mean customer number values

λ	μ_1	μ_2	μ_3	α_1	α_2	π_{loss}	$E(N)$
2	0.2	1.9	1.9	0.5	0.5	0.9090	1.0047
2	0.8	1.6	1.6	0.5	0.5	0.7150	1.0712
2	1.4	1.3	1.3	0.5	0.5	0.5955	1.2177
2	2	1	1	0.5	0.5	0.5322	1.4677
2	2.2424	0.8889	0.8889	0.5	0.5	0.5233	1.5958
2	3	0.5	0.5	0.5	0.5	0.6013	2.1958
2	3.6	0.2	0.2	0.5	0.5	0.8073	2.7342

Table 2. Selection of optimal service capacities of $2nd$ and $3rd$ service units for $c = 2\lambda$ system configuration and the corresponding loss probability and mean customer number values

λ	μ_1	μ_2	μ_3	α_1	α_2	π_{loss}	$E(N)$
2	2.2424	0.2	1.5575	0.5	0.5	0.5389	1.8454
2	2.2424	0.8889	0.8889	0.5	0.5	0.5233	1.5958
2	2.2424	1.7	0.0575	0.5	0.5	0.5474	1.9877

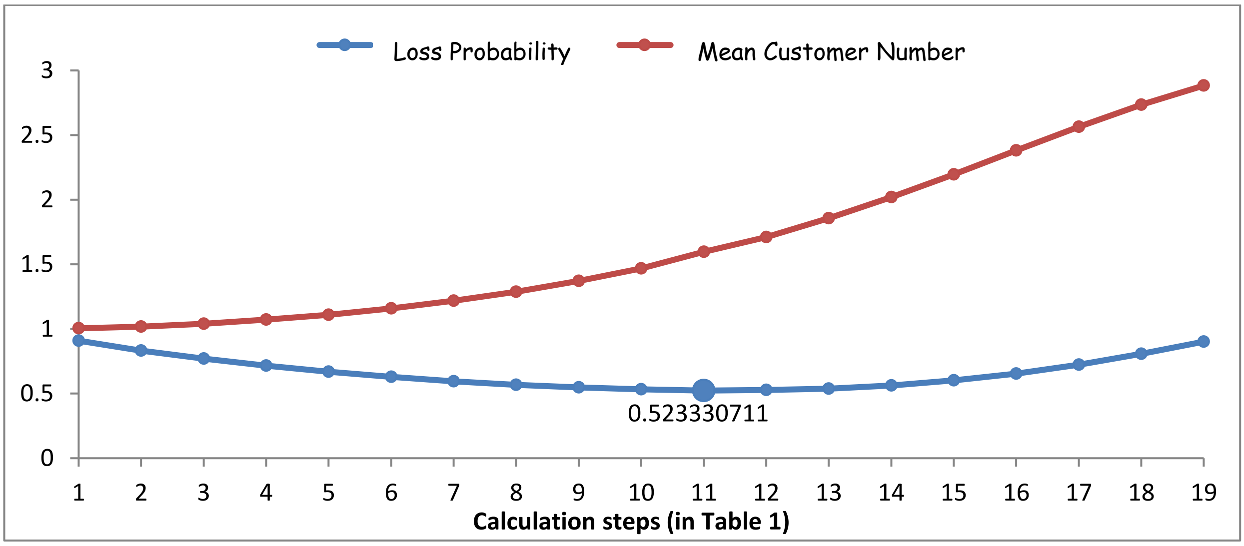

Fig. 1. The obtained π_{loss} and $E(N)$ values for $c = 2\lambda$ configuration

As seen in Figure 1, under the condition $\mu_1 = \mu_2$ and $\alpha_1 = \alpha_2$, minimum $\pi_{loss} = 0.523330711$ is obtained for $c = 2\lambda$ configuration.

Table 3. Optimal service capacity selection of 1st service unit for $c = \lambda$ system configuration and the corresponding loss probability and mean customer number values

λ	μ_1	μ_2	μ_3	α_1	α_2	π_{loss}	$E(N)$
2	0.2	0.9	0.9	0.5	0.5	0.9091	1.1110
2	0.8	0.6	0.6	0.5	0.5	0.7316	1.6262
2	1.07	0.465	0.465	0.5	0.5	0.7079	1.9639
2	1.5	0.25	0.25	0.5	0.5	0.7785	2.5503
2	1.9	0.05	0.05	0.5	0.5	0.9501	2.9422

Table 4. Selection of optimal service capacities of 2nd and 3rd service units for $c = \lambda$ system configuration and the corresponding loss probability and mean customer number values

λ	μ_1	μ_2	μ_3	α_1	α_2	π_{loss}	$E(N)$
2	1.07	0.1	0.83	0.5	0.5	0.7183	2.1628
2	1.07	0.4	0.53	0.5	0.5	0.7082	1.9695
2	1.07	0.465	0.465	0.5	0.5	0.7079	1.9639
2	1.07	0.7	0.23	0.5	0.5	0.7120	2.0406
2	1.07	0.9	0.03	0.5	0.5	0.7236	2.2625

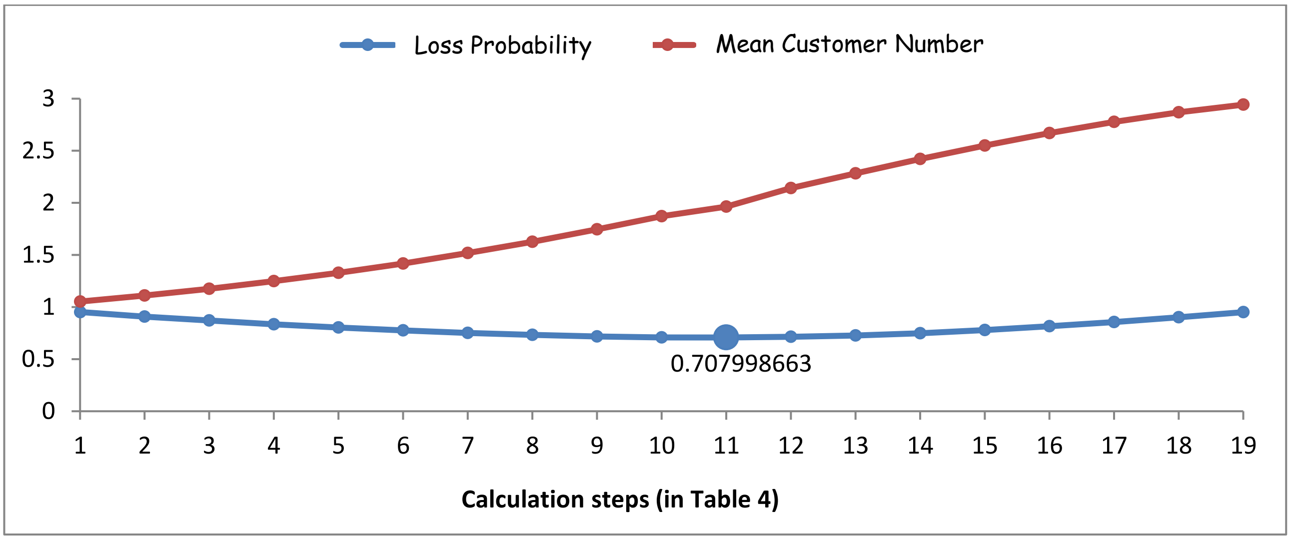

Fig. 2. The obtained π_{loss} and $E(N)$ values of $c = \lambda$ configuration for $\alpha_1 = \alpha_2$

As shown in Figure 2, as in $c = 2\lambda$ configuration, under the condition of $\mu_1 = \mu_2$ and $\alpha_1 = \alpha_2$, the minimum π_{loss} value of $c = \lambda$ configuration is reached, and this value is obtained as $\pi_{loss} = 0.707998663$.

3. The Simulation of the Model

In this section, the previously obtained loss probabilities are simulated for $n = 100$, $n = 1000$, and $n = 10000$ using Matlab R2010a program under the system configuration in which they are obtained. As shown in Table 5 and 6 below, previously obtained loss probability π_{loss}values are found close to the simulation values. This shows that the formula of π_{loss}, which is theoretically found with Equation (2.29), is obtained correctly. In Table 7 and 8, the simulation values are given separately for each of the loss probabilities obtained for $c = \lambda$ and $c = 2\lambda$ configurations, respectively. As seen from these tables, the numerical loss possibilities obtained for each system configuration converge to the simulation results.

Table 5. Comparison of the optimal π_{loss} value obtained for $c = \lambda$ configuration with simulation results

	Optimal numerical value	Simulation value ($n = 100$)	Simulation value ($n = 1000$)	Simulation value ($n = 10000$)	Simulation value ($n = 1000000$)
π_{loss}	0.707999	0.70	0.704	0.690	0.6899

Table 6. Comparison of the optimal π_{loss} value obtained for $c = \lambda$ configuration with simulation results

	Optimal numerical value	Simulation value ($n = 100$)	Simulation value ($n = 1000$)	Simulation value ($n = 10000$)	Simulation value ($n = 1000000$)
π_{loss}	0.523330711	0.53	0.519	0.521	0.5158

Table 7. π_{loss} values and simulation ($n = 100000$) results obtained for $c = \lambda$ configuration

μ_1	μ_2	μ_3	π_{loss}(Numerical)	π_{loss}(Simulation)
0.2	0.9	0.9	0.9091	0.9084
0.9	0.55	0.55	0.7176	0.7093
1.07	0.465	0.465	0.7079	0.6899
1.3	0.35	0.35	0.7279	0.6930
1.8	0.1	0.1	0.9016	0.8440
1.9	0.05	0.05	0.9501	0.9107

Table 8. π_{loss} values and simulation ($n = 100000$) results obtained for $c = 2\lambda$ configuration

μ_1	μ_2	μ_3	π_{loss}(Numerical)	π_{loss}(Simulation)
0.2	1.9	1.9	0.9090	0.9095
0.8	1.6	1.6	0.7150	0.7192
1	1.5	1.5	0.6685	0.6654
1.6	1.2	1.2	0.5681	0.5667
1.8	1.1	1.1	0.5468	0.5421
2	1	1	0.5322	0.5252
2.2424	0.8889	0.8889	0.5233	0.5158
2.4	0.8	0.8	0.5267	0.5197
2.6	0.7	0.7	0.5388	0.5239
2.8	0.6	0.6	0.5630	0.5387
3	0.5	0.5	0.6013	0.5401

4. Conclusion and Discussion

In this study, a blocked stochastic queueing model consisting of parallel service units is given. In this queueing model, the customer arrivals are Poisson distributed with an average of λ. In the first stage of the system, there is one service unit with an exponential service time with an average of $1/\mu_1$ and in the second stage, there are two service units with exponentially distributed service times, with averages of $1/\mu_2$ and $1/\mu_3$, respectively. This queue model is not allowed to wait in front of the first stage service unit, so there is no queue in this system. An arriving customer is served if the service unit in the first stage is empty. Then, if the service units in the second stage are both empty, the customer continues to the first unit with the probability of α_1, the second unit with a probability of $\alpha_2 = 1 - \alpha_1$ or if only one of the service units in the second stage is empty, completes its service in this unit and leaves the system. A third case is that if both service units in the second stage are full, the customer expects at least one of these service units to be empty by blocking the service unit in the 1st stage. Loss occurs when the customer continues to put into service at the 1^{st} stage service unit or when he/she blocks this service unit. One of the main problems in such a queueing model is to calculate this probability of loss and calculate what its optimal value will be. This given stochastic queueing model is mathematically defined by a three-dimensional continuous parameter Markov chain. Limit probabilities of this model are obtained by Kolmogorov difference and differential equations. Afterwards, transition probabilities and blocking probabilities are obtained by the elimination method. The loss probability of the system is found with the help of transition probabilities and blocking probability. In addition, the average number of customers in the system $E(N)$, which is one of the performance measures of the system, is calculated. Since the optimal value of π_{loss} cannot be calculated algebraically, optimal π_{loss} is numerically analysed under the condition $\mu_1 + \mu_2 + \mu_3 = c$ instead. As a result, the optimal π_{loss} value for $c = \lambda$ and $c = 2\lambda$ is reached when $\mu_2 = \mu_3$ and $\alpha_1 = \alpha_2$. The model is simulated with MATLAB R2010 software. The simulation results are compared with the optimal π_{loss} values and the simulation results are found to be very close to the optimal π_{loss} values achieved in the study.

Conflict of Interest

The authors declare no conflict of interest.

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Journal of New Theory
ISSN: 2149-1402

35 (2021) 103-113
Journal of New Theory
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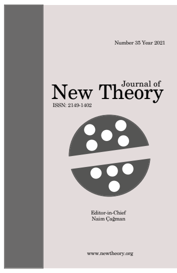

An Application of Soft Multisets to a Decision-Making Problem Concerning Side Effects of COVID-19 Vaccines

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Article History
Received: 06 Jun 2021
Accepted: 30 Jun 2021
Published: 30 Jun 2021
Research Article

Abstract − Soft multi-criteria decision-making, a developing area, is among the most prevalent problems handled by researchers. This study aims to introduce a soft decision-making method and apply it to rank the side effects of COVID-19 vaccines. Based on the literature, the present study features the advantages and disadvantages of previously observed multi-criteria decision-making (MCDM) methods are summarized. This paper achieves to utilize multisets simultaneously with the known soft decision-making methods. The primary concern hereof is to offer an insightful everyday-life example. Finally, the authors discuss the need for further research.

Keywords − *Soft sets, multisets, soft multisets, soft decision-making, MCDM*

Mathematics Subject Classification (2020) − 03E75, 54A05

1. Introduction

Multi-criteria decision-making (MCDM) methods encompass a diverse set of approaches. These methods can be broadly divided into two categories: discrete MCDM or discrete multi-attribute decision-making (MADM) and continuous multi-objective decision-making (MODM) methods [1,2]. A great many publications have recently been released on the development and application of MCDM methods in various fields. This article aims to document the exponentially growing interest in MCDM methods and techniques and reviews the latest literature on MCDM methods and their applications. The foundations of the modern MCDM were established in the 1950s and 1960s. The 1970s marked a critical decade for many pioneering works. The development of MCDM research built momentum in the 1980s and early 1990s and seems to have continued to grow exponentially up to the present time [3]. [4] has formulated the fundamentals of decision-making with multiple objectives. [5] has reviewed the development of MODM methods and their applications in a relatively short period. Later, [6] has analysed the MADM methods: Simple Additive Weighting (SAW) [7], Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [7], Elimination and Choice Expressing Reality (ELECTRE) [8], and LINMAP [9]. [10] has published a detailed study on Analytic Hierarchy Process (AHP). Then, the author has published a study on the further development of Analytic Network Process (ANP) and a book which deals with the problem of the compromise theory. [11] has authored a book that addresses the same theory. [5] has studied MCDM in groups. [8] has summed up the available information on the ELECTRE

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group method. [12] has written several seminal research papers. Yet the development of hybrid and modular methods has recently become more critical. They are based on the well-known methods, such as SAW [7], TOPSIS [5], AHP [10], and ELECTRE [8], and their modifications by applying fuzzy number and grey number theory. It is evident from the foregoing theoretical discussion that two articles [13,14] stand out. Two other relevant papers have featured soft multisets [15] and soft multi-criteria decision-making [16].

In the current study, section 2 analyses MCDM methods. Section 3 presents some of the basic notions needed for the following sections. Section 4 proposes a new algorithm by modifying the algorithm provided in [16], and then propounds its other version to allow for a comparison with the proposed method. The last section discusses the need for new methods further research.

2. Analysis of Multi-Criteria Decision-Making Methods

This section presents a review of 11 methods in the literature. These methods are 1) MAUT, 2) AHP, 3) FST, 4) CBR, 5) DEA, 6) SMART, 7) GP, 8) ELECTRE, 9) PROMETHEE, 10) SAW, and 11) TOPSIS. It is expected that this detailed review will give a deeper insight into these methods.

2.1. Multi-Attribute Utility Theory (MAUT): The most commonly used MCDM approach in this analysis is MAUT [17-19]. This theory has been summarized by Loken as “a more systematic approach to incorporate risk expectations and uncertainty into decision support approaches with multiple parameters” [20]. MAUT’s main benefit is that it takes confusion into account. It potentially has a utility attributed to it, which is not a quality that is accounted for in many MCDM methods.

MAUT has been widely applied to attend to economic, environmental, actuarial, and agricultural issues and water and energy management problems. These issues typically exhibit large quantities of ambiguities and provide ample data to make MAUT a proper decision-making process.

2.2. Analytic Hierarchy Process (AHP): While examining the methods, their relationship with the above or other predefined methods is included as well. The MAUT and AHP approaches are based on various assumptions on value measures, and AHP is developed independently of other decision theories. The use of pair-wise comparisons to evaluate alternatives in terms of several parameters as well as to estimate criteria weights is the main characteristic of the AHP system. AHP’s implementation and its position in the studies on MCDM have followed a similar pattern as with MAUT and experienced increased usage in real-world application examples. AHP is used to compare weighting and ranking. AHP is capable of navigating various indicators.

2.3. Fuzzy Set Theory (FST): Modelling and handling uncertainties have become essential issues in solving complex problems. FST was introduced by [21] to overcome the problems caused by uncertainties in a wide variety of fields. An efficient MCDM technique itself has been proved to be fuzzy logic. The use of cost-benefit analysis as the primary tool for decision analysis to discuss environmental projects has been tackled by [22]. Fuzzy logic “takes into account the insufficient information and the evolution of available knowledge” [23]. Fuzzy systems can also be challenging to build because of drawbacks. In certain instances, before being used in the real world, they can require multiple simulations. FST has been developed and used in such fields as engineering, economics, medicine, and environmental and social sciences.

2.4. Case-Based Reasoning (CBR): There are two popular ways to distinguish between companies in financial distress and those in healthy financial situations: human preference-oriented forecasting and data-driven forecasting. [24] uses CBR to provide a new framework for forecasting financial distress in businesses one year before the real distress. Employing the Manhattan distance, Euclidean distance, and inductive form, CBR compares three different models and their respective results with a ranking-order case-based model of reasoning (ROCBR). One of CBR’s key advantages over most MCDM techniques is that it can improve over time, especially as more instances are added to the database. Through its database of events, it can also respond to environmental changes. Its significant downside is its vulnerability to data inconsistencies.

2.5. Data Envelopment Analysis (DEA) [25]: DEA is used to develop a model that will help policymakers of any country prioritize their actions. The goal thereof is to improve the relevant highways in the most efficient way possible. This method is able to successfully score the productivity of countries by obtaining 21 separate data. In this method, a mutual comparison is made. The comparison method refers to the grading of the efficiency of the most efficient alternatives. With a rating of 1.0, all the other alternatives are a fraction of 1.0. This offers several advantages. The most essential one is that multiple inputs and outputs can be processed.

2.6. Simple Multi-Attribute Rating Technique (SMART) [26]: SMART counts as one of the most accessible categories of MAUT. Its name derives from its convenient use. This approach requires two assumptions, “preferential independence” and “utility independence”. In conjunction with the real numbers, this approach transforms significance weights into real numbers. In addition to those described in MAUT, the key benefits of SMART relative to the MAUT system are that it is easy to use and genuinely facilitates any form of weight assignment technique.

2.7. Goal Programming (GP) [13]: GP is a realistic type of programming that provides an unlimited number of solutions to choose from. All of its strengths are that it can address large-scale concerns. Its most notable value, according to some methods, is the potential to generate limitless alternatives. A significant downside to this strategy is that the coefficients are not weighted. To accurately weight the coefficients, many implementations find it appropriate to use other approaches, such as AHP. This condition is not, however, present in this process. It eliminates one of its drawbacks by doing this while choosing infinite options, which can cause option inconsistencies. This follows a general trend that in applications that avoid many of their drawbacks, MCDM approaches are most frequently used – i.e., that coefficient weight does not care.

2.8. ELECTRE [8]: The areas in which ELECTRE is used are issues with electricity, economy, environment, water management, and transport. It considers ambiguity as other approaches do. ELECTRE is a form of transformation of several iterations dependent on compatibility analysis. Its greatest value is that it takes into account complexity and uncertainty. Its downside is that it may be difficult to describe the mechanism and its consequences concerning its terms and poor comprehensibility.

2.9. PROMETHEE [27]: PROMETHEE is similar to the aforesaid ELECTRE method in that it has multiple iterations and is also a transformation method. Its value is that it is convenient to use. The presumption that the parameters are proportional does not require it. The drawbacks are that it provides no explicit weight distribution method and allows weights to be allocated. Still, it fails to offer a consistent method for assigning these values.

2.10. SAW [7]**:** “SAW is a value function established based on a simple addition of scores representing the goal achievement under each criterion, multiplied by the particular weights” [7]. Its ability to compensate between criteria is among the reasons for its selection in usage. For policymakers, it is intuitive as well. It is easy to use thanks to its ability to render calculations without basic and complicated computer programs.

2.11. TOPSIS [5]: Its main benefits are that it has a clear method and it is accessible and programmable. Regardless of the number of attributes, the number of phases remain the same. It can be inferred from most of the uses in the literature that TOPSIS confirms the responses proposed by other methods of MCDM. The value of its flexibility and the potential to retain the same number of steps regardless of the challenge scale helps it be easily used as a decision-making mechanism for evaluating or retaining other approaches in its own right.

3. Preliminaries

This section provides some of the basic definitions to be needed for the following sections.

Definition 3.1. *Soft sets* [28, as cited in 29] Let U be an initial universe, $P(U)$ be the power set of U, E be the set of parameters, and $A \subseteq E$. Then, a soft set F_A over U is defined as $F_A \coloneqq \{(x, f_A(x)): x \in E\}$ where $f_A: E \to P(U)$ is a mapping and $x \notin A$ for $f_A(x) = \emptyset$.

Definition 3.2. *Multisets* [30] Let U be universal set, $\mathbb{N}$ be a set of unsigned integer numbers, and $X \subseteq U$. Then, a multiset M_X over U is defined as

$$M_X := \left\{\frac{f_X(u)}{u} : u \in U\right\}$$

where $f_X: U \to \mathbb{N}$ be a mapping such that $u \notin X$ for $f_X(u) = 0$ Here, $\frac{f_X(u)}{u}$ means that u occurring $f_X(u)$ times. Moreover, if $f_X(u) = 0$, then $\frac{f_X(u)}{u}$ is not shown in the multiset. Here, if $X = U$, MSs can be denoted by M or M_1, M_2, …

Definition 3.3. [30] Let M_X, M_Y be two multisets over U. If, for all $u \in U$, $f_X(u) \leq f_Y(u)$, then M_X is called multi-subset of M_Y and is denoted by $M_X \tilde{\subseteq} M_Y$.

Example 3.4. Let $U = \{u_1, u_2, u_3\}$, $X = \{u_1, u_3\}$, and $f_X: U \to \mathbb{N}$ is a mapping defined by $f_X(u_1) = 25$, $f_X(u_2) = 33$, and $f_X(u_3) = 40$. Then, the multiset $M_X = \left\{\frac{25}{u_1}, \frac{0}{u_2}, \frac{40}{u_3}\right\}$ briefly $M_X = \left\{\frac{25}{u_1}, \frac{40}{u_3}\right\}$. Similarly, let $Y = U$ and $M_Y = \left\{\frac{32}{u_1}, \frac{10}{u_2}, \frac{50}{u_3}\right\}$. Then, $M_X \tilde{\subseteq} M_Y$ since

$$f_X(u_1) = 25 \leq f_Y(u_1) = 32$$

$$f_X(u_2) = 0 \leq f_Y(u_2) = 10$$

$$f_X(u_3) = 40 \leq f_Y(u_3) = 50$$

Definition 3.5. [30] Let M_U be a multiset over U and $n \in \mathbb{N}$. If, for all $u \in U$, $f_U(u) = n$, then M_U is called n-multi-set over U and is denoted by M^n. If, for all $X \subseteq U$, $\max_{u \in X} f_X(u) = n$, then M^n is referred to as n-universal multiset over U. Here, M^0 is called empty multiset over U. It can be observed that $M_\emptyset$ is empty multiset over U.

Definition 3.6. *Soft multisets* (SMSs) [31] Let M_V be a multiset, $M(M_V)$ be the set of all the multiset of U, E be parameters set, and $A \subseteq E$. Then, a soft multiset (SMS) Ω_A over Z_K is defined as

$$\Omega_A := \{(x, \Omega_A(x)) : x \in E\}$$

where $\Omega_A: E \to M(U)$ is a mapping such that $\Omega_A(x) = M^0$ if $x \notin A$. Here, if $A = E$, SMSs can be denoted by Ω or Ω_1, Ω_2, …

Definition 3.7. *Fuzzy set* [21] Let U be an initial universe, [0,1] be unit closed interval, and $\mu: U \to [0,1]$ be a mapping, Then, a fuzzy set μ over U is defined as

$$\mu := \{(x, \mu(x)) : x \in U\} \text{ or briefly } \mu := \left\{ {}^{\mu(x)}x : x \in U\right\}$$

Definition 3.8. *Fuzzy soft set* [32, as cited 33] Let U be an initial universe, $F(U)$ be the set of all the fuzzy sets over U, E be parameters set, and $A \subseteq E$. Then, a fuzzy soft set Γ_A over U is defined as

$$\Gamma_A := \{(x, \Gamma_A(x)) : x \in E\}$$

where $\Gamma_A: E \to F(U)$ is a mapping such that $\Gamma_A(x) = \mathbf{0}$ if $x \notin A$. Here, $\mathbf{0}$ denotes the empty fuzzy set. Moreover, if $A = E$, fuzzy soft sets can be denoted by Γ or Γ_1, Γ_2, …

Definition 3.9. [34, as cited in 35] Let U be an initial universe, E be parameters set, $A \subseteq E$, and Γ_A be a fuzzy soft set over U. Then, $[a_{ij}]$ is called fuzzy soft matrix of Γ_A and is defined by

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i, j \in \{0,1,2,\cdots\}$, $a_{ij} := \Gamma_{\mathrm{A}}(x_j)(u_i)$ where $\Gamma_{\mathrm{A}}(x_j)(u_i)$ refers to the membership degree of u_i in the fuzzy set $\Gamma_{\mathrm{A}}(x_j)$. Here, if $|U| = m$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

Definition 3.10. Let $[s_{i1}]$ be a real matrix has order $m \times 1$. Then, normalisation $[\hat{s}_{i1}]$ of $[s_{i1}]$ is defined by

$$\hat{s}_{i1} := \begin{cases} \dfrac{s_{i1} - \min\limits_k s_{k1}}{\max\limits_k s_{k1} - \min\limits_k s_{k1}}, & \max\limits_k s_{k1} \neq \min\limits_k s_{k1} \\ 1, & \max\limits_k s_{k1} = \min\limits_k s_{k1} \end{cases}$$

4. An Application of Soft Multisets to a Decision-Making Problem Concerning Side Effects of COVID-19 Vaccines

Firstly, this section presents the data on the post-treatment side effects of the COVID-19 vaccine provided in [36-39] and obtained by the feedback received from 791, 530, 80, and 814 people located in Manisa/Turkey, the United States, Konya/Turkey, and the Czech Republic, respectively. The COVID-19 pandemic, identified in 59 suspect cases (In Hubei/Wuhan, China), has spread to affect the whole world [40-41]. Authorities have launched vaccination campaigns to prevent its spread. In this sense, several types of vaccines have been developed (for more about the properties of the COVID-19 vaccines, see [41-42]).

Table 1. By-Symptom Distribution of the frequency of side effects (%) of COVID-19 vaccine

	Symptoms / Sources	n*=791 [36]	n=530 [37]	n=80 [38]	n=814 [39]
s_1	Headache	11.9	13	13.8	45.6
s_2	Muscle/joint pain	9.5	11.6	46.3	64.9
s_3	Sore Throat	3.7	8.5	23.8	-
s_4	Chills/Fever	3.2	2.8	37.5	55.6
s_5	Diarrhoea/Nausea/Vomiting	3.4	5.7	5	13
s_6	Loss/change in taste or smell	1.2	1.6	6.3	10
s_7	Cough	1.6	9.9	61.3	-
s_8	Hypertension	0.9	-	26.2	-
s_9	Shortness of breath	0.4	4.1	20	-
s_{10}	Injection site pain	18	-	-	89.8
s_{11}	Injection site swelling	1.3	-	-	25.6
s_{12}	Injection site itching/redness	1.4	-	-	23
s_{13}	Lymphadenopathy	0.4	-	-	16.2
s_{14}	Fatigue	-	17.9	50	62.2

*number of people

Secondly, this section presents two soft decision-making (SDM) methods provided in [43,44]. Since these SDM methods have been configured to operate in the fuzzy parameterized fuzzy soft matrices space, they are reduced to the fuzzy soft matrices space.

Algorithm Steps of sMBR01 [43]

Step 1: Construct a fuzzy soft matrix $[a_{ij}]$ has order $m \times n$

Step 2: Obtain $[s_{i1}]$ defined by

$$s_{i1} \coloneqq \sum_{k=1}^{m} \sum_{j=1}^{n} \operatorname{sgn}(a_{ij} - a_{kj}), \quad i \in \{1,2,\ldots,m\}$$

Step 3: Obtain the decision set $\left\{{}^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

Algorithm Steps of CCE10 [44]

Step 1: Construct a fuzzy soft matrix $[a_{ij}]$ has order $m \times n$

Step 2: Obtain the score matrix $[s_{i1}]$ defined by

$$s_{i1} \coloneqq \frac{1}{\mathrm{n}} \sum_{j=1}^{n} a_{ij}, \quad i \in \{1,2,\ldots,m\}$$

Step 3: Obtain the decision set $\left\{{}^{\hat{s}_{k1}}u_k | u_k \in U\right\}$

Thirdly, this section proposes a new algorithm, denoted by KPS21, by modifying the algorithm provided in [16]. Then, it propounds its other version, i.e., KPS21/2, to allow for a comparison with KPS21. These methods achieve to utilize multisets simultaneously with the known soft decision-making methods.

Algorithm Steps of KPS21

Step 1: Input a parameter set E, $A \subseteq E$, a universal set U, and $X \subseteq U$

Step 2: Construct a multiset M_X over U

Step 3: Construct an SMS Ω_{A} over U

Step 4: Compute the fuzzy soft set $\Gamma_A = \left\{\left(x, \Gamma_A(x)\right): x \in E\right\}$ defined by

$$\Gamma_A(x) = \left\{\frac{\Omega_A(x)(u) / \sum_v \Omega_A(x)(v)}{u} : x \in E\right\}$$

Step 5: Obtain the fuzzy soft matrices $[a_{ij}]$

Step 6: Apply sMBR01 to the $[a_{ij}]$

Algorithm Steps of KPS21/2

Step 1: Input a parameter set E, $A \subseteq E$, a universal set U, and $X \subseteq U$

Step 2: Construct a multiset M_X over U

Step 3: Construct an SMS Ω_{A} over U

Step 4: Compute the fuzzy soft set $\Gamma_A = \left\{\left(x, \Gamma_A(x)\right): x \in E\right\}$ defined by

$$\Gamma_A(x) = \left\{\frac{\Omega_A(x)(u) / \sum_v \Omega_A(x)(v)}{u} : x \in E\right\}$$

Step 5: Obtain the fuzzy soft matrices $[a_{ij}]$

Step 6: Apply CEC11 to the $[a_{ij}]$

Fourthly, this section applies KPS21 and KPS21/2 to the side-effect data provided in Table 1.

Step 1: $E = \{x_1, x_2, x_3, x_4\}$, $A = E$, $U = \{u_1, u_2, \ldots, u_{14}\}$, and $X = U$ such that

$u_i = s_i$ for all $i \in \{1,2,\ldots,14\}$

x_1 = Data located in Manisa/Turkey

x_2 = Data located in the United States

x_3 = Data located in Konya/Turkey

x_4 = Data located in the Czech Republic

Step 2: The multisets M_1, M_2, M_3 and M_4 over U, which show the distribution of side effects and whose values are obtained by rounding the values provided in Table 1 to the nearest integer number, are as follows:

$$M_1 = \langle \frac{12}{u_1}, \frac{10}{u_2}, \frac{4}{u_3}, \frac{3}{u_4}, \frac{3}{u_5}, \frac{1}{u_6}, \frac{2}{u_7}, \frac{1}{u_8}, \frac{18}{u_{10}}, \frac{1}{u_{11}}, \frac{1}{u_{12}} \rangle$$

$$M_2 = \langle \frac{13}{u_1}, \frac{12}{u_2}, \frac{9}{u_3}, \frac{3}{u_4}, \frac{6}{u_5}, \frac{2}{u_6}, \frac{10}{u_7}, \frac{4}{u_9}, \frac{18}{u_{14}} \rangle$$

$$M_3 = \langle \frac{14}{u_1}, \frac{46}{u_2}, \frac{24}{u_3}, \frac{38}{u_4}, \frac{5}{u_5}, \frac{6}{u_6}, \frac{61}{u_7}, \frac{26}{u_8}, \frac{20}{u_9}, \frac{50}{u_{14}} \rangle$$

$$M_4 = \langle \frac{46}{u_1}, \frac{65}{u_2}, \frac{56}{u_4}, \frac{13}{u_5}, \frac{10}{u_6}, \frac{90}{u_{10}}, \frac{26}{u_{11}}, \frac{23}{u_{12}}, \frac{16}{u_{13}}, \frac{62}{u_{14}} \rangle$$

Step 3: Thus, an SMS Ω over U is as follows:

$$\Omega = \{(x_1, M_1), (x_2, M_2), (x_3, M_3), (x_3, M_4)\}$$

Step 4: Therefore, the fuzzy soft set Γ

$$\Gamma = \left\{ \begin{array}{c} \left(x_1, \left\{ {}^{0.21}u_1, {}^{0.18}u_2, {}^{0.07}u_3, {}^{0.05}u_4, {}^{0.05}u_5, {}^{0.02}u_6, {}^{0.04}u_7, {}^{0.02}u_8, {}^{0.32}u_{10}, {}^{0.02}u_{11}, {}^{0.02}u_{12} \right\}\right), \\ \left(x_2, \left\{ {}^{0.17}u_1, {}^{0.16}u_2, {}^{0.12}u_3, {}^{0.04}u_4, {}^{0.08}u_5, {}^{0.03}u_6, {}^{0.13}u_7, {}^{0.05}u_9, {}^{0.23}u_{14} \right\}\right), \\ \left(x_3, \left\{ {}^{0.05}u_1, {}^{0.16}u_2, {}^{0.08}u_3, {}^{0.13}u_4, {}^{0.02}u_5, {}^{0.02}u_6, {}^{0.21}u_7, {}^{0.09}u_8, {}^{0.07}u_9, {}^{0.17}u_{14} \right\}\right), \\ \left(x_4, \left\{ {}^{0.11}u_1, {}^{0.16}u_2, {}^{0.14}u_3, {}^{0.03}u_4, {}^{0.02}u_5, {}^{0.22}u_6, {}^{0.06}u_7, {}^{0.06}u_8, {}^{0.04}u_9, {}^{0.15}u_{14} \right\}\right), \end{array} \right\}$$

Step 5: The fuzzy soft matrix of Γ is as follows:

$$[a_{ij}] = \begin{bmatrix} 0.21 & 0.17 & 0.05 & 0.11 \\ 0.18 & 0.16 & 0.16 & 0.16 \\ 0.07 & 0.12 & 0.08 & 0 \\ 0.05 & 0.04 & 0.13 & 0.14 \\ 0.05 & 0.08 & 0.02 & 0.03 \\ 0.02 & 0.03 & 0.02 & 0.02 \\ 0.04 & 0.13 & 0.21 & 0 \\ 0.02 & 0 & 0.09 & 0 \\ 0 & 0.05 & 0.07 & 0 \\ 0.32 & 0 & 0 & 0.22 \\ 0.02 & 0 & 0 & 0.06 \\ 0.02 & 0 & 0 & 0.06 \\ 0 & 0 & 0 & 0.04 \\ 0 & 0.23 & 0.17 & 0.15 \end{bmatrix}$$

Step 6: The decision set and the rank of the alternatives obtained by sMBR01 are as follows:

$$\left\{ {}^{0.83}u_1, {}^{1}u_2, {}^{0.52}u_3, {}^{0.7}u_4, {}^{0.45}u_5, {}^{0.22}u_6, {}^{0.61}u_7, {}^{0.19}u_8, {}^{0.03}u_9, {}^{0.7}u_{10}, {}^{0.14}u_{11}, {}^{0.14}u_{12}, {}^{0}u_{13}, {}^{0.77}u_{14} \right\}$$

and

$$u_{13} \prec u_9 \prec u_{11} \approx u_{12} \prec u_8 \prec u_6 \prec u_5 \prec u_3 \prec u_7 \prec u_4 \approx u_{10} \prec u_{14} \prec u_1 \prec u_2$$

Similarly, the decision set and the rank of the alternatives obtained by CEC11 is as follows:

$$\{{}^{0.82}u_1, {}^{1}u_2, {}^{0.41}u_3, {}^{0.54}u_4, {}^{0.27}u_5, {}^{0.14}u_6, {}^{0.58}u_7, {}^{0.17}u_8, {}^{0.11}u_9, {}^{0.89}u_{10}, {}^{0.12}u_{11}, {}^{0.12}u_{12}, {}^{0.06}u_{13}, {}^{0.83}u_{14}\}$$

and

$$u_{13} \prec u_9 \prec u_{11} \approx u_{12} \prec u_8 \prec u_6 \prec u_5 \prec u_3 \prec u_4 \prec u_7 \prec u_1 \prec u_{14} \prec u_{10} \prec u_2$$

Both of the results manifest that the appearance rates of the side effects $u_{13}, u_9, u_{11}, u_{12}, u_8, u_6$, and u_5 account for less than 50%, and the methods produce the same ranking orders for these alternatives. Moreover, the appearance rates of the side effects $u_3, u_7, u_4, u_{10}, u_{14}, u_1$, and u_2 are greater than 50%, and the methods produce the same ranking orders for u_3 and u_2, while those of the others are different. sMBR01 and CEC11 methods are reliable since they pass all the tests provided in [45]. KPS21 and KPS21/2 methods are also reliable as they are based on the aforesaid methods, respectively. However, the uncertainty inherent in the problem causes some of the produced ranking orders to be different.

5. Conclusion

In this study, an example of an algorithm used in alternative decision-making processes was introduced to provide a working example of the MCDM method for selecting alternatives used in everyday life and the decisions that can be reached with the help of the concept of soft sets emerging on the concept of uncertainties. It is believed that the algorithms proposed in this study can be used in different disciplines and will offer guidance for future studies. Moreover, to improve the proposed method, it is worth studying the current soft decision-making methods [46-52].

Conflict of Interest

The authors declare no conflict of interest.

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