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CONTENT

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Relative Gol'dberg Order and Type of Multiple Entire Dirichlet Series in Terms of Coefficients and Exponents

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Abstract — This paper aims to define and characterize the relative Gol'dberg order and type of a multiple entire Dirichlet series with respect to another multiple entire Dirichlet series in terms of their coefficients and exponents. By using the definition, we study the growth properties of the Hadamard product between two such series.

Keywords — *Multiple entire Dirichlet series, Gol'dberg order and type of multiple entire Dirichlet series, Relative Gol'dberg order and type of multiple entire Dirichlet series*

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1. Introduction

Relative Gol'dberg order and type of a multiple entire Dirichlet series with respect to another multiple entire Dirichlet series in terms of their maximum modulus function, has been defined in [1] and [2] respectively. Those definitions have been used to study about growth properties of sum functions and asymptotically equivalent multiple entire Dirichlet series. Hadamard product between two such series involves coefficients and exponents of them. Therefore, use of the definition which involves maximum modulus function, may not be useful to study about growth property of Hadamard product. So, it is necessary to find an expression of relative Gol'dberg order and type in terms of coefficients and exponents of the two multiple entire Dirichlet series.

We now briefly discuss about entire Dirichlet series in one complex variable and the reasons due to which, the series satisfies the condition to be an entire function.

A Series of the form

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n s} \tag{1}$$

where $s = \sigma + it, a_n \in \mathcal{C}$ and $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ is called a Dirichlet series in one complex variable. If the Dirichlet series is convergent at $s_0 = \sigma_0 + it_0 \in \mathcal{C}$, then it is convergent at any point in the set $D = \{s \in \mathcal{C} : Re(s) < Re(s_0)\}$ and uniformly convergent in the domain $D_1 = \{s \in \mathcal{C} : |arg(s - s_0)| \leq \theta < \frac{\pi}{2}\}$. The abscissa of convergence of (1) is $\sigma_c = \sup\{\sigma \in \mathcal{R} :$ series(1) converges for all $s \in \mathcal{C}$ where $Re(s) < \sigma\}$.

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The series (1) is absolutely convergent if the series

$$\sum_{n=1}^{\infty} |a_n e^{\lambda_n s}| = \sum_{n=1}^{\infty} |a_n| e^{\lambda_n \sigma} \tag{2}$$

is convergent. Let σ_A be the abscissa of absolute convergence of the series (1).

The following theorem gives a general relationship between the abscissa of convergence and the abscissa of absolute convergence.

Theorem 1.1. ([3], p-31) If the exponents λ_n in (1) satisfy the condition $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty$, then $0 \leq \sigma_c - \sigma_A \leq L$.

If $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0$, then $\sigma_A = \sigma_c = -\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n}$.
Therefore, the series (1) will represent an entire function $g(s)$ if and only if [3]

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty \tag{3}$$

In this paper, we have studied about multiple entire Dirichlet series. Next, we write the notations which have been used throughout this paper and then briefly discuss about the series in multiple complex variables.

1.1. Notations

For $s = (s_1, s_2, \ldots, s_n), w = (w_1, w_2, \ldots, w_n) \in \mathcal{C}^n$, and $\alpha \in \mathcal{C}$, we define $s = w$ if and only if $s_i = w_i$, $s + w = (s_1 + w_1, s_2 + w_2, \ldots, s_n + w_n)$, $\alpha s = (\alpha s_1, \alpha s_2, \ldots, \alpha s_n)$, $s.w = s_1 w_1 + s_2 w_2 + \cdots + s_n w_n$, $| \ s \ | = (|s_1|^2 + |s_2|^2 + \cdots + |s_n|^2)^{\frac{1}{2}}$. $s + R = (s_1 + R, s_2 + R, \ldots, s_n + R)$, for $R \in \mathcal{R}$. $\lambda_{n,m} = (\lambda_{1m_1}, \lambda_{2m_2}, \ldots, \lambda_{nm_n}) \in \mathcal{R}^{+^n}$ where $\mathcal{R}^{+^n} = \{x : x \in \mathcal{R}^n, x_i \geq 0\}$. $s.\lambda_{n,m} = s_1\lambda_{1m_1} + s_2\lambda_{2m_2} + \cdots + s_n\lambda_{nm_n}$. $\lambda^k_{n,m} = (\lambda^k_{1m_1}, \lambda^k_{2m_2}, \ldots, \lambda^k_{nm_n}) \in \mathcal{R}^{+^n}$, for $k = 0, 1, 2, \ldots$ $\|\lambda_{n,m}\| = \lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}$ For $r, t \in \mathcal{R}^{+^n}$, we define $r \leq t$ if and only if $r_i \leq t_i$ and $r < t$ if and only if $r_i < t_i$ for $i = 1, 2, \ldots, n$.

Definition 1.2. A multiple entire Dirichlet Series is of the form

$$f(s) = \sum_{\|m\|=1}^{\infty} a_{m_1,\ldots,m_n} e^{s.\lambda_{n,m}} \tag{4}$$

where $a_{m_1,\ldots,m_n} \in \mathcal{C}, s = (s_1, s_2, \ldots, s_n) \in \mathcal{C}^n$, $s_j = \sigma_j + it_j, j = 1, 2, \ldots, n$, and $\{\lambda_{j,m_j}\}_{m_j=1}^{\infty}$, $j = 1, \ldots, n$ are n sequences of exponents satisfying the conditions $0 \leq \lambda_{jm_1} < \lambda_{jm_2} < \cdots < \lambda_{jm_k} \to \infty \ \ as \ \ k \to \infty$, $j = 1, \ldots, n$, and $\lim_{m_j\to\infty} \frac{\log m_j}{\lambda_{jm_j}} = 0, j = 1, 2, \ldots, n$.

As described in Equation (3), series (4) must satisfy the following condition in order to represent an entire function

$$\limsup_{\|m\|\to\infty} \frac{\log |a_{m_1\cdots n}|}{\|\lambda_{n,m}\|} = -\infty \tag{5}$$

Let $D \subset \mathcal{C}^n$ be an arbitrary complete n-half-plane defined by $D = \{s : s \in \mathcal{C}^n, Re(s_i) \leq r_i\}$ where $r = (r_1, r_2, \ldots, r_n) \in \mathcal{R}^n$. Consider a parameter $R \in \mathcal{R}$, define $R + D = D + R = \{s + R : s \in D\}$. Then, for the multiple Dirichlet entire function f, the maximum modulus function $M_{f,D}(R)$ with respect to the region D and $R \in \mathcal{R}$ is defined as

$$M_{f,D}(R) = \sup\{| \ f(s) \ | : s \in D + R\} \tag{6}$$

$M_{f,D}(R)$ is strictly increasing, increases to ∞ and continuous functions of R. The inverse function is $M^{-1}_{f,D} : (L, \infty) \to (-\infty, \infty)$ where $0 \leq L = \lim_{R\to-\infty} M_{f,D}(R)$.

Throughout this paper, we have considered a class of multiple entire Dirichlet series with same sequence of exponents.

For $k = 0, 1, 2, 3, \ldots$, we define

$$f^k(s) = \sum_{\|m\|=1}^{\infty} \|\lambda_{n,m}^k\| a_m e^{s.\lambda_{n,m}} \tag{7}$$

Therefore, for two multiple entire Dirichlet series $f(s)$ and $g(s)$, we have $(f+g)^k = f^k + g^k, k = 0, 1, 2, \ldots$

2. Preliminary

In this section, we write a few preliminary definitions which have been used to prove the theorems and results in the next section.

Definition 2.1. [4] The Gol'dberg order of a multiple entire Dirichlet Series f with respect to the domain D is defined by

$$\rho_f(D) = \limsup_{R\to\infty} \frac{\log\log M_{f,D}(R)}{R} \tag{8}$$

Definition 2.2. [4] The lower Gol'dberg order of a multiple entire Dirichlet Series f with respect to the domain D is defined by

$$\lambda_f(D) = \liminf_{R\to\infty} \frac{\log\log M_{f,D}(R)}{R} \tag{9}$$

f is said to be of regular growth if $\rho_f(D) = \lambda_f(D)$.

Definition 2.3. [4] The Gol'dberg type of a multiple entire Dirichlet Series f with Gol'dberg order $\rho_f(D)$, $(0 < \rho_f(D) < \infty)$ with respect to the domain D, is defined by

$$\sigma_f(D) = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_f(D)}} \tag{10}$$

Definition 2.4. [4] The lower Gol'dberg type of a multiple entire Dirichlet Series f with Gol'dberg order $\rho_f(D)$, $(0 < \rho_f(D) < \infty)$ with respect to the domain D, is defined by

$$\tau_f(D) = \liminf_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_f(D)}} \tag{11}$$

f is said to be of perfectly regular growth if $\rho_f(D) = \lambda_f(D)$ and $\sigma_f(D) = \tau_f(D)$.

Definition 2.5. [1] Let f and g be two multiple entire Dirichlet series. The relative Gol'debrg order of f with respect to g, denoted by $\rho_{g,D}(f)$, is defined as

$$\rho_{g,D}(f) = \limsup_{R\to\infty} \frac{M_{g,D}^{-1}(M_{f,D}(R))}{R} \tag{12}$$

Definition 2.6. [2] The relative Gol'dberg type of a multiple entire Dirichlet series f with respect to another multiple entire Dirichlet series g, with $0 < \rho_{g,D}(f) < \infty$, denoted by $\sigma_{g,D}(f)$, is defined as

$$\sigma_{g,D}(f) = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{\log M_{g,D}(\rho_{g,D}(f)R)} \tag{13}$$

From the definition(2.5) and (2.6) it follows that for $k = 0, 1, 2, \ldots$

$$\rho_{g,D}(f^k) = \limsup_{R\to\infty} \frac{M_{g,D}^{-1}(M_{f^k,D}(R))}{R} \tag{14}$$

and

$$\sigma_{g,D}(f^k) = \limsup_{R\to\infty} \frac{\log M_{f^k,D}(R)}{\log M_{g,D}(\rho_g(f^k)R)} \tag{15}$$

where f^k is defined in Equation (7).

Definition 2.7. [5] P. K. Sarkar defined the Gol'dberg order $\rho_f(D)$ of a multiple entire Dirichlet series f in terms of coefficients and exponents as

$$\rho_f(D) = \limsup_{\|m\|\to\infty} \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{-\log |a_m|} \tag{16}$$

Definition 2.8. [5] P. K. Sarkar also defined the Gol'dberg type $\sigma_f(D)$ of a multiple entire Dirichlet series f in terms of coefficients and exponents as

$$\sigma_f(D) = \frac{1}{e\rho_f(D)} \limsup_{\|m\|\to\infty} \|\lambda_{n,m}\| \left\{ |a_m| \phi_D(m) \right\}^{\frac{\rho_f(D)}{\|\lambda_{n,m}\|}} \tag{17}$$

where $\phi_D(m) = \sup_{s\in D} |\exp\{s.\lambda_{n,m}\}|$

Definition 2.9. For $f(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$, Hadamard product $f * g$ is defined as

$$(f * g)(s) = f(s) * g(s) = \sum_{\|m\|=1}^{\infty} a_m b_m e^{s.\lambda_{n,m}} \tag{18}$$

Then, for $k = 0, 1, 2, 3, \ldots,$

$$(f(s) * g(s))^k = \sum_{\|m\|=1}^{\infty} \|\lambda_{n,m}^k\| a_m b_m e^{s.\lambda_{n,m}} \tag{19}$$

$$f^k(s) * g^k(s) = \sum_{\|m\|=1}^{\infty} \|\lambda_{n,m}^{2k}\| a_m b_m e^{s.\lambda_{n,m}} \tag{20}$$

where f^k is defined in Equation (7).

We know that Gol'dberg order $\rho_f(D)$ and relative Gol'dberg order $\rho_{g,D}(f)$ does not depend on the choice of domain D while Gol'dberg type $\sigma_f(D)$ and relative Gol'dberg type $\sigma_{g,D}(f)$ does ([5]). Henceforth we may write ρ_f and $\rho_g(f)$ instead of writing $\rho_f(D)$ and $\rho_{g,D}(f)$.

3. Theorems and Results

In this section, we have proved the theorems which establish relative Gol'dberg order and type in terms of coefficients and exponents of a multiple entire Dirichlet series. Before proving the theorem, we write the statement of Theorem (4.2.2) ([6], Chapter 4) and Theorem (5.2.5) ([6], Chapter 5) which has been used to prove the main results.

Lemma 3.1. Let f and g be two multiple entire Dirichlet series of finite Gol'dberg orders ρ_f and ρ_g such that $\rho_g \neq 0$. Then the relative Gol'dberg order of f with respect to g satisfies the inequality $\rho_g(f) \geq \frac{\rho_f}{\rho_g}$. If g is of regular growth then $\rho_g(f) = \frac{\rho_f}{\rho_g}$.

Lemma 3.2. Let f and g be two multiple entire Dirichlet series of finite Gol'dberg orders ρ_f , ρ_g and Gol'dberg types $\sigma_f(D)$, $\sigma_g(D)$ respectively such that $\sigma_g(D) \neq 0$ and g is of regular growth. Then the relative Gol'dberg type of f with respect to g satisfies the inequality $\sigma_{g,D}(f) \geq \frac{\sigma_f(D)}{\sigma_g(D)}$. Moreover, if g is of perfectly regular growth then $\sigma_{g,D}(f) = \frac{\sigma_f(D)}{\sigma_g(D)}$.

Theorem 3.3. Let $f(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two multiple entire Dirichlet series of finite Gol'dberg orders ρ_f and ρ_g such that $\rho_g \neq 0$ and g is of regular growth. Then, the relative Gol'dberg order of f with respect to g is given by $\rho_g(f) = \limsup_{\|m\|\to\infty} \frac{\log |b_m|}{\log |a_m|}$.

Proof. Since g is of regular growth, by Lemma (3.1),

$$\begin{aligned}\rho_g(f) &= \frac{\rho_f}{\rho_g}\\ &= \limsup_{\|m\|\to\infty} \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{\log |\frac{1}{a_m}|} \liminf_{\|m\|\to\infty} \frac{\log |\frac{1}{b_m}|}{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}, \quad \text{by Equation (16)}\\ &\le \limsup_{\|m\|\to\infty} \frac{\log |b_m|}{\log |a_m|} = \mu \ \text{(say)} \qquad (21)\end{aligned}$$

Then, for any $\varepsilon > 0$ there is an increasing sequence $\{m_k\}$ of positive integers, increasing to ∞, such that $\frac{\log |\frac{1}{b_{m_k}}|}{\log |\frac{1}{a_{m_k}}|} > \mu - \varepsilon$. Hence,

$$\frac{\log |\frac{1}{b_{m_k}}|}{\|\lambda_{n,m_k}\| \log \|\lambda_{n,m_k}\|} . \frac{\|\lambda_{n,m_k}\| \log \|\lambda_{n,m_k}\|}{\log |\frac{1}{a_{m_k}}|} > \mu - \varepsilon \qquad (22)$$

This implies

$$\begin{aligned}\rho_f &= \limsup_{\|m\|\to\infty} \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{\log |\frac{1}{a_m}|}\\ &\ge \limsup_{\|m_k\|\to\infty} \frac{\|\lambda_{n,m_k}\| \log \|\lambda_{n,m_k}\|}{\log |\frac{1}{a_{m_k}}|}\\ &\ge (\mu-\varepsilon) \limsup_{\|m_k\|\to\infty} \frac{\|\lambda_{n,m_k}\| \log \|\lambda_{n,m_k}\|}{\log |\frac{1}{b_{m_k}}|}, \quad \text{by Equation (22)}\\ &\ge (\mu-\varepsilon) \liminf_{\|m\|\to\infty} \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{\log |\frac{1}{b_m}|}\\ &= (\mu-\varepsilon)\rho_g, \quad [\text{Since } g \text{ is of regular growth}]\end{aligned}$$

Therefore,

$$\frac{\rho_f}{\rho_g} > \mu - \varepsilon \qquad (23)$$

Since $\varepsilon > 0$ is arbitrary, combining (21) and (23)

$$\rho_g(f) = \limsup_{\|m\|\to\infty} \frac{\log |b_m|}{\log |a_m|}$$

□

In the next theorem, we have established relative Gol'dberg type in terms of coefficients and exponents of a multiple entire Dirichlet series.

Theorem 3.4. Let $f(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two non constant multiple entire Dirichlet series of finite Gol'dberg orders ρ_f , ρ_g and Gol'dberg types $\sigma_f(D)$, $\sigma_g(D)$ respectively such that $\sigma_g(D) \neq 0$ and g is of perfectly regular growth. Then, the relative Gol'dberg type of f with respect to g is given by

$$\begin{aligned}\sigma_{g,D}(f) &= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \left[\frac{\{|a_m|\phi_D(m)\}^{\rho_f}}{\{|b_m|\phi_D(m)\}^{\rho_g}}\right]^{\frac{1}{\|\lambda_{n,m}\|}}\\ &= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \left[\frac{\{|a_m|\phi_D(m)\}^{\rho_g(f)}}{|b_m|\phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}}\end{aligned}$$

where $\phi_D(m) = \sup_{s\in D} |\exp\{s.\lambda_{n,m}\}|$.

PROOF. Since $\sigma_g(D) \neq 0$ and g is of perfectly regular growth, then by Lemma (3.2) ,

$$\begin{aligned}
\sigma_{g,D}(f) &= \frac{\sigma_f(D)}{\sigma_g(D)} \\
&= \frac{\frac{1}{e\rho_f} \limsup_{\|m\|\to\infty} \| \lambda_{n,m} \| \{|a_m|\phi_D(m)\}^{\frac{\rho_f}{\|\lambda_{n,m}\|}}}{\frac{1}{e\rho_g} \limsup_{\|m\|\to\infty} \| \lambda_{n,m} \| \{|b_m|\phi_D(m)\}^{\frac{\rho_g}{\|\lambda_{n,m}\|}}}, \quad \text{by Definition (2.7)} \\
&\leq \frac{\rho_g}{\rho_f} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_f}}{\{|b_m|\phi_D(m)\}^{\rho_g}}\Big]^{\frac{1}{\|\lambda_{n,m}\|}} \qquad (24) \\
&= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_g(f)}}{|b_m|\phi_D(m)}\Big]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} = \mu(\text{say}) \qquad (25)
\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there is an increasing sequence $\{m_k\}$ of positive integers, increasing to infinity such that

$$\frac{1}{\rho_g(f)} \Big[\frac{\{|a_{m_k}|\phi_D(m_k)\}^{\rho_f}}{\{|b_{m_k}|\phi_D(m_k)\}^{\rho_g}}\Big]^{\frac{1}{\|\lambda_{n,m_k}\|}} > \mu - \varepsilon$$

Hence,

$$\frac{\{|a_{m_k}|\phi_D(m_k)\}^{\frac{\rho_f}{\|\lambda_{n,m_k}\|}}}{e.\rho_f} \geq (\mu - \varepsilon) \frac{\{|b_{m_k}|\phi_D(m_k)\}^{\frac{\rho_g}{\|\lambda_{n,m_k}\|}}}{e.\rho_g} \qquad (26)$$

Therefore,

$$\begin{aligned}
\sigma_f(D) &= \frac{1}{e\rho_f} \limsup_{\|m\|\to\infty} \| \lambda_{n,m} \| \{|a_m|\phi_D(m)\}^{\frac{\rho_f}{\|\lambda_{n,m}\|}} \\
&\geq \limsup_{\|m_k\|\to\infty} (\mu - \varepsilon) \frac{\| \lambda_{n,m_k} \| \{|b_{m_k}|\phi_D(m_k)\}^{\frac{\rho_g}{\|\lambda_{n,m_k}\|}}}{e.\rho_g}, \quad \text{by Equation (26)} \\
&\geq \liminf_{\|m_k\|\to\infty} (\mu - \varepsilon) \frac{\| \lambda_{n,m_k} \| \{|b_{m_k}|\phi_D(m_k)\}^{\frac{\rho_g}{\|\lambda_{n,m_k}\|}}}{e.\rho_g} \\
&\geq \liminf_{\|m\|\to\infty} (\mu - \varepsilon) \frac{\|\lambda_{n,m}\|\{|b_m|\phi_D(m)\}^{\frac{\rho_g}{\|\lambda_{n,m}\|}}}{e.\rho_g} \\
&= (\mu - \varepsilon)\sigma_g(D), \quad [\text{Since } g \text{ is of perfectly regular growth.}]
\end{aligned}$$

Hence,

$$\sigma_{g,D}(f) = \frac{\sigma_f(D)}{\sigma_g(D)} \geq (\mu - \varepsilon) \qquad (27)$$

Since $\varepsilon > 0$ is arbitrarily small, combining (25) and (27) we get

$$\begin{aligned}
\sigma_{g,D}(f) &= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_f}}{\{|b_m|\phi_D(m)\}^{\rho_g}}\Big]^{\frac{1}{\|\lambda_{n,m}\|}} \\
&= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_g(f)}}{|b_m|\phi_D(m)}\Big]^{\frac{\rho_g}{\|\lambda_{n,m}\|}}
\end{aligned}$$

□

Now, by using the above definitions we discuss about relative growth of Hadamard product of two multiple entire Dirichlet series.

Theorem 3.5. Let f_1, f_2, and g be three multiple entire Dirichlet series of finite order such that g is of regular growth. If $\rho_g(f)$ denotes the relative Gol'dberg order of f with respect to g, then $\rho_g((f_1 * f_2)^k) = \rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)$, for $k = 0, 1, 2, \ldots$

Proof. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$, $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$, and $g(s) = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$. Then, by Theorem (3.3)

$$\begin{aligned}
\frac{1}{\rho_g((f_1 * f_2)^k)} &= \liminf_{\|m\|\to\infty} \frac{\log |(\|\lambda_{n,m}^k\|) a_m b_m|}{\log |c_m|} \\
&= \lim_{\|m\|\to\infty} \frac{\log \| \lambda_{n,m}^k \|}{\log |c_m|} + \liminf_{\|m\|\to\infty} \frac{\log |a_m b_m|}{\log |c_m|} \\
&= \liminf_{\|m\|\to\infty} \frac{\log |a_m b_m|}{\log |c_m|}, \quad \left[\text{Since } g \text{ is entire, by Equation (5), } \limsup_{\|m\|\to\infty} \frac{\log |c_m|}{\|\lambda_{n,m}\|} = -\infty.\right] \\
&= \frac{1}{\limsup_{\|m\|\to\infty} \frac{\log |c_m|}{\log |a_m b_m|}} = \frac{1}{\rho_g(f_1 * f_2)}
\end{aligned}$$

Hence, $\rho_g((f_1 * f_2)^k) = \rho_g(f_1 * f_2)$. Similarly it can be proved that $\rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)$. Therefore, $\rho_g((f_1 * f_2)^k) = \rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)$, for $k = 0, 1, 2, \ldots$ □

Theorem 3.6. Let f_1,f_2 and g be three multiple entire Dirichlet series, then

$$\sigma_{g,D}((f_1 * f_2)^k) = \sigma_{g,D}(f_1^k * f_2^k) = \sigma_{g,D}(f_1 * f_2)$$

Proof. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$, $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$. Then, by Theorem (3.4),

$$\begin{aligned}
\sigma_{g,D}(f_1 * f_2)^k &= \frac{1}{\rho_g(f_1 * f_2)^k} \limsup_{\|m\|\to\infty} \left[\frac{\{|(\|\lambda_{n,m}^k\|) a_m b_m| \phi_D(m)\}^{\rho_g(f_1 * f_2)^k}}{|c_m| \phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} \\
&\text{where } \phi_D(m) = \sup_{s\in D} |\exp\{s.\lambda_{n,m}\}| \\
&= \frac{1}{\rho_g(f_1 * f_2)} \limsup_{\|m\|\to\infty} \left[\frac{\{|(\|\lambda_{n,m}^k\|) a_m b_m| \phi_D(m)\}^{\rho_g(f_1 * f_2)}}{|c_m| \phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} \\
&[\text{Since } \rho_g((f_1 * f_2)^k) = \rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)] \\
&= \frac{1}{\rho_g(f_1 * f_2)} \limsup_{\|m\|\to\infty} \left[\frac{\{|a_m b_m| \phi_D(m)\}^{\rho_g(f_1 * f_2)}}{|c_m| \phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} \\
&\left[\text{Since } \limsup_{\|m\|\to\infty} \left[\lambda_{1m_1}^{k_1} + \cdots + \lambda_{nm_n}^{k_n}\right]^{\frac{1}{\|\lambda_{n,m}\|}} = 1\right] \\
&= \sigma_{g,D}(f_1 * f_2)
\end{aligned}$$

Similarly, it can be proved that $\sigma_{g,D}(f_1^k * f_2^k) = \sigma_{g,D}(f_1 * f_2)$. Therefore,

$$\sigma_{g,D}((f_1 * f_2)^k) = \sigma_{g,D}(f_1^k * f_2^k) = \sigma_{g,D}(f_1 * f_2)$$

□

Theorem 3.7. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two multiple entire Dirichlet series of relative Gol'dberg order $\rho_g(f_1)$ and $\rho_g(f_2)$ with respect to another multiple entire Dirichlet series $g = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$ respectively. Then, for $k = 0, 1, 2, \ldots$

$$\frac{1}{\rho_g(f_1^k * f_2^k)} = \frac{1}{\rho_g(f_1 * f_2)^k} \geq \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}$$

.

Proof. Using Theorem (3.3) and Equation (20), we have

$$\begin{aligned}\frac{1}{\rho_g(f_1 * f_2)^k} &= \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)a_m b_m|}{\log|c_m|}\\ &\geq \liminf_{\|m\|\to\infty} \frac{\log\|\lambda_{n,m}^k\|}{\log|c_m|} + \liminf_{\|m\|\to\infty} \frac{\log|a_m|}{\log|c_m|} + \liminf_{\|m\|\to\infty} \frac{\log|b_m|}{\log|c_m|}\\ &= \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}\end{aligned}$$

Therefore,

$$\frac{1}{\rho_g(f_1 * f_2)^k} \geq \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}$$

By Theorem (3.5), $\rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)^k$. Therefore,

$$\frac{1}{\rho_g(f_1^k * f_2^k)} = \frac{1}{\rho_g(f_1 * f_2)^k} \geq \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}$$

□

Theorem 3.8. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two multiple entire Dirichlet series of relative Gol'dberg order $\rho_g(f_1)$ and $\rho_g(f_2)$ with respect to another multiple entire Dirichlet series $g = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$ respectively. Then, for $k = 0, 1, 2, \ldots,$

$$\rho_g(f_1^k * f_2^k) \leq \left[\rho_g(f_1^k).\rho_g(f_2^k)\right]^{\frac{1}{2}}$$

provided

$$\log|(\|\lambda_{n,m}^{2k}\|)a_m b_m| \sim \left\{\log|(\|\lambda_{n,m}^k\|)a_m| \log|(\|\lambda_{n,m}^k\|)b_m|\right\}^{\frac{1}{2}}$$

Proof. We have

$$\frac{1}{\rho_g(f_1^k)} = \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)a_m|}{\log|c_m|}$$

and

$$\frac{1}{\rho_g(f_2^k)} = \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)b_m|}{\log|c_m|}$$

Therefore, for any arbitrary $\varepsilon > 0$, and for all sufficiently large $\| m \|$

$$\frac{1}{\rho_g(f_1^k)} - \frac{\varepsilon}{2} < \frac{\log|(\|\lambda_{n,m}^k\|)a_m|}{\log|c_m|}$$

and

$$\frac{1}{\rho_g(f_2^k)} - \frac{\varepsilon}{2} < \frac{\log|(\|\lambda_{n,m}^k\|)b_m|}{\log|c_m|}$$

Thus,

$$\Big(\frac{1}{\rho_g(f_1^k)}-\frac{\varepsilon}{2}\Big)\Big(\frac{1}{\rho_g(f_2^k)}-\frac{\varepsilon}{2}\Big)<\frac{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|}{(\log|c_m|)^2}$$

or

$$\Big\{\Big(\frac{1}{\rho_g(f_1^k)}-\frac{\varepsilon}{2}\Big)\Big(\frac{1}{\rho_g(f_2^k)}-\frac{\varepsilon}{2}\Big)\Big\}^{\frac{1}{2}}<\frac{\Big\{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|\Big\}^{\frac{1}{2}}}{\log|c_m|}$$

for all sufficiently large $\| m \|$. Since $\log|(\|\lambda_{n,m}^{2k}\|)a_mb_m| \sim \big\{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|\big\}^{\frac{1}{2}}$, for all sufficiently large $\| m \|$, we have

$$\Big\{\Big(\frac{1}{\rho_g(f_1^k)}-\frac{\varepsilon}{2}\Big)\Big(\frac{1}{\rho_g(f_2^k)}-\frac{\varepsilon}{2}\Big)\Big\}^{\frac{1}{2}}<\frac{\log|(\|\lambda_{n,m}^{2k}\|)a_mb_m|}{\log|c_m|}$$

Therefore,

$$\Big(\frac{1}{\rho_g(f_1^k)\rho_g(f_2^k)}\Big)^{\frac{1}{2}}\leq\liminf_{\|m\|\to\infty}\frac{\log|(\|\lambda_{n,m}^{2k}\|)a_mb_m|}{\log|c_m|}$$

or

$$\Big(\frac{1}{\rho_g(f_1^k)\rho_g(f_2^k)}\Big)^{\frac{1}{2}}\leq\frac{1}{\rho_g(f_1^k*f_2^k)}$$

Hence,

$$\rho_g(f_1^k*f_2^k)\leq\big[\rho_g(f_1^k).\rho_g(f_2^k)\big]^{\frac{1}{2}}$$

□

4. Conclusion

Thus, we understand that, in the study of growth properties of Hadamard product between two multiple entire Dirichlet series, the method of using coefficients and exponents is easy and useful. However, not only in the case of Hadamard product, but also for any study of growth property which involves exponents and coefficients of the series, our result will be useful and an easy method to prove them.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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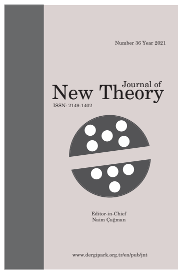

Analytical Approximation for Cahn-Hillard Phase-Field Model for Spinodal Decomposition of a Binary System

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Abstract — Phase transformations which lead to dramatical property change are very important for engineering materials. Phase-field methods are one of the most successful and practical methods for modelling phase transformations in materials. The Cahn-Hillard phase-field model is among the most promising phase-field models. The most successful aspect of the model is that it can predict spinodal decomposition (which is essential to determining the microstructure of an alloy) in a binary system. It is used in both materials science and many other fields, such as polymer science, astrophysics, and computer science. In this study, the Cahn-Hillard phase-field model is evaluated by an analytical approach using the $(1/G')$-expansion method. The solutions obtained are tested for certain thermodynamic conditions, and their accuracy of predicting the spidonal decomposition of a binary system is confirmed.

Keywords — *Cahn-Hillard phase field model, spidonal decomposition,* $(1/G')$*-expansion method*

Mathematics Subject Classification (2020) — 35R11, 34A08

1. Introduction

Foreseeing phase transformations is very important for tailoring the properties of engineering materials. For instance, occurrence of domains may determine magnetic properties of a metallic material [1]. On the other hand, mechanical properties of a material can be altered by a phase transformation [2]. Because of having a significant impact on the physical and chemical properties of the material, phase transformations, which are changes in the microstructure at equilibrium or non- equilibrium states, are the primary and most important concern of a materials engineering scientist. There are certain difficulties that this issue brings along with its importance. A method developed to model phase transformations to predict the properties of the material needs to take into account many thermal, electromagnetic, elastic and chemical effects. Hence, the issue is getting very difficult mathematically.

Phase-field methods are one of the most successful and practical methods for modeling phase transformations in materials. A phase field model can be described as spatiotemporal treating of the microstructure of a material as a scalar field (concentration, strain temperature, etc.). Phase field models have attracted much attention in recent years due to their versatility [3–5]. In these models, the free energy equation based on the thermodynamic state of the system is given in terms of phase fields. Nonlinear partial differential equations play an important role in the phase-field model due to dynamical governing of one or more order parameters as well as heat or mass transfer. These

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dynamical interactions of the parameters are based on the tendency of free energy to reach to its possible minimum value.

Cahn-Hillard phase field model is the one of the most promising phase field models [6,7]. The most successful aspect of the model is that it can predict spinodal decomposition (which is very essential for determining the microstructure of an alloy) in a binary system [8]. Due to the success of the model in this regard, it is being used both in materials science [9, 10] and in many fields such as polymer science [11], astrophysics [12], computer science [13] and astrophysics [12]. The simple structure of the model is mainly due to the fact that it is based on the diffusion laws in the Ginzburg-Landau theorem [14, 15]. Total energy of a binary alloy is illustrated by the sum of two parts which are the bulk part depending on local energy difference and the interfacial part depending on concentration gradient. In summary, the total energy expression can be written as follows:

$$E = \int_{\Omega} f(\varphi) d\Omega + \int_{\Omega} \frac{\epsilon^2}{2} |\nabla \varphi|^2 d\Omega \tag{1}$$

where the first term is bulk and second term is interfacial energy ($\epsilon > 0$ represents the interfacial parameter). In order to evaluate the thermodynamic conditions, on the $f(\varphi)$ which is the free energy function is considered as follows according to Flory-Huggins model [16];

$$f(\varphi) = \frac{1}{2}((1+\varphi)\ln(1+\varphi) + (1-\varphi)\ln(1-\varphi) - \theta\varphi^2)$$

where $\theta = T/T_{critical}$ and φ is the phase field.

In the literature, different numerical techniques such as radial basis functions differential quadrature [17], finite element [18] and reduced differential transform [19] methods have been employed for the solution of Cahn-Hillard equation. Unfortunately, numerical methods give approximate results. Analytical methods can give precise predictions to phase separation processes. Thus, Cahn-Hillard phase field model is evaluated by an analytical approach using the $(1/G')$-expansion method in this study. The obtained analytical solutions are firstly appeared in the literature.

2. Governing Equation

Authors considered the general form of the Cahn-Hillard [20];

$$\varphi_t + \varphi_{xxxx} = (A(\varphi))_{xx} + l\varphi_x . l > 0 \tag{2}$$

In this equation $A(\varphi(x,t))$ denotes an inherent chemical potential arising from driven forces (such as thermodynamic) that has the form of $A(\varphi(x,t)) = \varphi^3(x,t) - \varphi(x,t)$ and $\varphi(x,t)$ states phase field of a binary system that the components are having separated and $l\varphi_x$ indicates the diffusion induced kinetics of the mixture(such as a binary alloy).

3. $(1/G')$-Expansion Method

Many authors used The $(1/G')$-expansion method as a tool to get the exact solutions of various differential equations [21–23]. These studies represent that considered method is a powerful and effective while obtaining the analytical solutions of partial differential equations (PDEs). Suppose that nonlinear PDE for $\varphi(x,t)$ is given in the form

$$H\left(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial^2 \varphi}{\partial t^2}, \frac{\partial^2 \varphi}{\partial x^2}, \ldots\right) = 0 \tag{3}$$

that the unknown function $\varphi(x,t)$ is the function of two independent the variables x, t and also H is the polynomial of the function $\varphi(x,t)$ and its higher order partial derivatives.

Regarding the wave variable as

$$\varphi(x,t) = \varphi(\xi), \xi = kx + ct \tag{4}$$

where k and c can be described as free constants. By using Equation (4) and Equation (3) changes into an nonlinear ordinary differential equation (ODE) with respect to $\varphi = \varphi(\xi)$

$$F\left(\varphi, \varphi', \varphi'', \varphi''', \ldots\right) = 0 \tag{5}$$

where prime indicates Newtonian concept derivative due to ξ. According to considered method the traveling wave solutions of Equation (5) can be expressed as a polynomial of $(1/G')$ as follows

$$\varphi(\xi) = \sum_{i=0}^{n} a_i \left(\frac{1}{G'}\right)^i, \quad a_n \neq 0 \tag{6}$$

where $G = G(\xi)$ satisfies the following differential equation

$$G'' + \lambda G' + \mu = 0 \tag{7}$$

and $a_i(i = 0, \ldots, n), \lambda, \mu$ are random constants to be examined later. To represent the solution of Equation (7) with $G = G(\xi)$, the Equation (6) will include the following equation

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + A\tanh(\lambda\xi) - A\sinh(\lambda\xi)} \tag{8}$$

where A is integral constant.

Step 1. By using the homogeneous balance principle between the highest nonlinear terms and the highest order derivatives of $\varphi(\xi)$ in Equation (5), the positive integer n in Equation (6) can be determined.

Step 2. Subrogating (6) with Equation (7) into Equation (5) and collecting together all the same powered terms of $(1/G')$ together, the left hand side of Equation (5) is turns into a polynomial with respect to $(1/G')$. After equating each coefficient of this polynomial to zero, we handle an algebraic equation system with respect to $a_i(i = 0, \ldots, n), \lambda, \mu, c, k$.

Step 3. In this step symbolic computer software is used to solve the algebraic equations system with respect to arbitrary constants $a_i(i = 0, \ldots, n), \lambda, \mu, c, k$, then subrogating the results with the solutions of Equation (7) into Equation (6) led to traveling wave solutions of Equation (5).

4. The Analytical Solution of Cahn-Hillard Equation

The transformation

$$\varphi(x, t) = \varphi(\xi), \xi = kx + ct \tag{9}$$

converts Equation (2) into nonlinear differential equation as

$$(c - l)\varphi + \varphi^{'''} - (\varphi^3 - \varphi)^{'} = 0 \tag{10}$$

Now, considering the homogeneous balance principle between $\varphi^{'''}$ and $\varphi^2\varphi^{'}$ appearing in Equation (10), we get $n = 1$. Consequently, we can write the equation:

$$\varphi(\xi) = a_1 \left(\frac{1}{G'(\xi)}\right) + a_0, \quad a_1 \neq 0 \tag{11}$$

and therefore

$$\varphi'(\xi) = a_1\mu \left(\frac{1}{G'(\xi)}\right)^2 + a_1\lambda \left(\frac{1}{G'(\xi)}\right) \tag{12}$$

$$\varphi'''(\xi) = 6a_1\mu^3 \left(\frac{1}{G'(\xi)}\right)^4 + 12a_1\mu^2\lambda \left(\frac{1}{G'(\xi)}\right)^3 + 7a_1\lambda^2\mu \left(\frac{1}{G'(\xi)}\right)^2 + a_1\lambda^3 \left(\frac{1}{G'(\xi)}\right) \tag{13}$$

By substituting Equations (11)-(13) into Equation (10) and bringing all terms with the same power of $(1/G')$ together, the left-hand side of Equation (10) is changes into another polynomial in $(1/G')$.

Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for a_0, a_1, k, c, λ and μ follows:

$$\begin{aligned}
\left(\frac{1}{G'(\xi)}\right)^0 &: \ ca_0 - lka_0 = 0 \\
\left(\frac{1}{G'(\xi)}\right)^1 &: \ ca_1 - lka_1 - 3k^2a_1a_0^2\lambda + k^2a_1\lambda + k^4a_1\lambda^3 = 0 \\
\left(\frac{1}{G'(\xi)}\right)^2 &: \ -3k^2a_1a_0^2\mu + 7k^4a_1\lambda^2\mu + k^2a_1\mu - 6k^2a_1^2a_0\lambda = 0 \\
\left(\frac{1}{G'(\xi)}\right)^3 &: \ -3k^2a_1^3\lambda + 12k^4a_1\mu^2\lambda - 6k^2a_1^2a_0\mu = 0 \\
\left(\frac{1}{G'(\xi)}\right)^4 &: \ 6k^4a_1\mu^3 - 3k^2a_1^3\mu = 0
\end{aligned}$$

Solving the algebraic equations above, yields

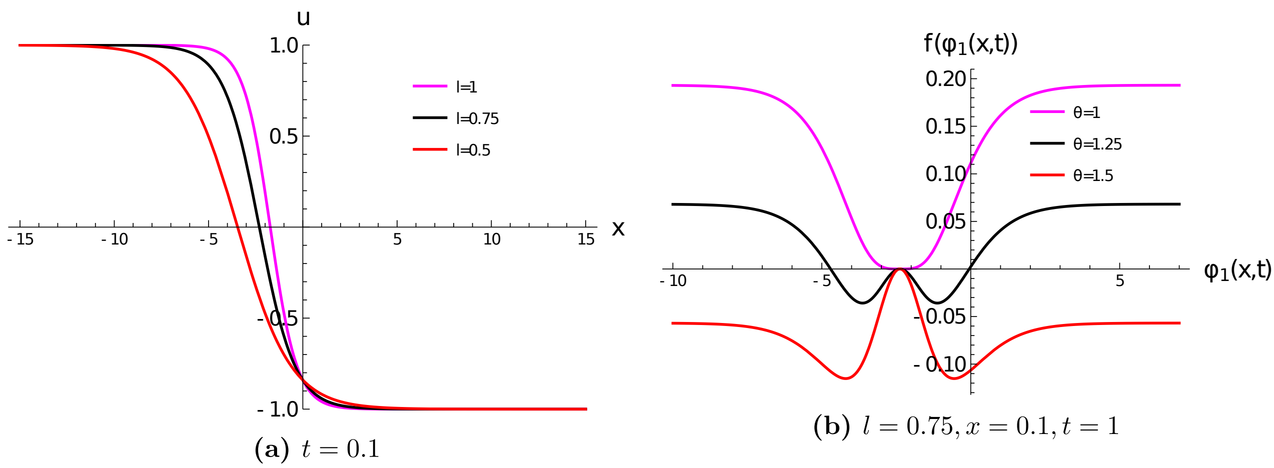
$$c = \pm\frac{l\sqrt{2}}{\lambda}, \quad k = \pm\frac{\sqrt{2}}{\lambda}, \quad a_0 = \pm 1, \quad a_1 = \pm\frac{2\mu}{\lambda} \tag{14}$$

and λ, μ are arbitrary constants. By using (14), expression (11) and (8), we have of travelling wave solutions of Equation (2) as follows:

$$\varphi_{1,2}(x,t) = \pm 1 \pm \frac{2\mu}{-\mu + A\lambda\cosh\left(\sqrt{2}lt + \sqrt{2}x\right) \mp A\lambda\sinh\left(\sqrt{2}lt + \sqrt{2}x\right)} \tag{15}$$

5. Graphical Representation of $\varphi_1(x,t)$ for Some Thermodynamic Conditions

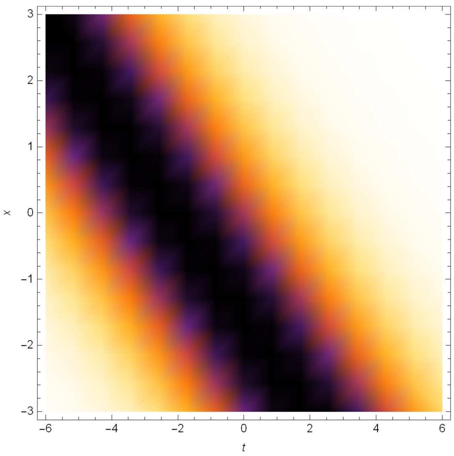
Interfacial parameter ϵ thickening with the decrease in the l parameter can be seen in Figure 1(a). It can be said that in a binary system interfacial region where the metastable phase exist narrows with the increasing l values. As it is expected from the model that the double-well structure for the θ values bigger than one can be seen in the Figure 1(b). The peak between the troughs of the wells corresponds to the metastable phase in the interfacial region. This issue can be more clearly in Figure 2(a-c) which show the energy density distribution for the corresponding θ values of $1, 1.25$, and 1.5.



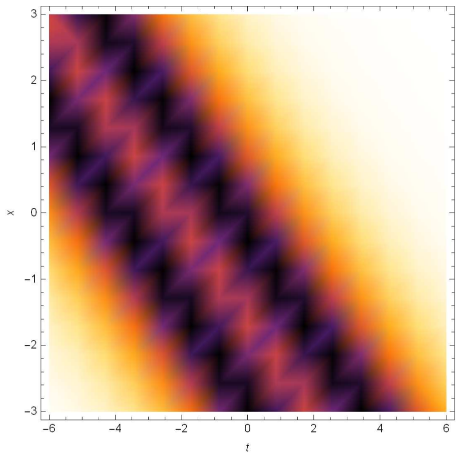
(a) $t = 0.1$

(b) $l = 0.75, x = 0.1, t = 1$

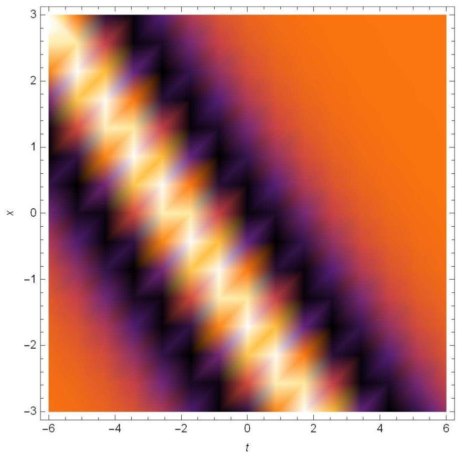
Fig. 1. Phase field (a), free energy (b) for the solution $\varphi_1(x,t)$ for $\mu = -0.1, \lambda = 0.1, A = 0.1$.



(a) $l = 0.75, \theta = 1$



(b) $l = 0.75, \theta = 1.25$



(c) $l = 0.75, \theta = 1.5$

Fig. 2. Spatiotemporal energy distribution (a-c) for the solution $\varphi_1(x,t)$ for $\mu = -0.1, \lambda = 0.1, A = 0.1$.

6. Conclusion

The exact solutions of Cahn-Hillard model have been achieved by the $(1/G')$-expansion method. An energy equation based on Flory-Huggins model was employed for evaluation of the solutions in certain thermodynamic conditions. As it is expected from the model double-well structure of the energy density distribution which corresponds to spinodal decomposition and occurrence of a metastable phase was seen. The results obtained in this study were shown to be accurate in predicting spinodal decomposition phenomenon. Also, some 2D graphical representations and contour plots of the obtained solutions are given for different values of l and θ. These results may be used by material engineers for tailoring the materials properties.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Theory of Generalized Sets in Generalized Topological Spaces

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Abstract — Several specific types of generalized sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) in generalized topological spaces (briefly, $\mathcal{T}_{\mathfrak{g}}$-*spaces*) have been defined and investigated for various purposes from time to time in the literature of $\mathcal{T}_{\mathfrak{g}}$-spaces. Our recent research in the field of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces is reported herein as a starting point for more generalized classes. It is shown that the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets is a superclass of those whose elements are called open, closed, semi-open, semi-closed, pre-open, pre-closed, semi-pre-open, and semi-pre-closed sets in a $\mathcal{T}_{\mathfrak{g}}$-space. A subclass of the $\mathcal{T}_{\mathfrak{g}}$-subspace corresponds to the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathcal{T}_{\mathfrak{g}}$-space. A class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of the Cartesian product of these $\mathcal{T}_{\mathfrak{g}}$-spaces corresponds to the Cartesian product of a finite number of classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, each of which belongs to a $\mathcal{T}_{\mathfrak{g}}$-space. Diagrams establish the various relationships amongst the classes presented here and in the literature, and an ad hoc application supports the overall theory.

Keywords — *Generalized topology, generalized topological space, generalized operations, generalized open sets, generalized closed sets*

Mathematics Subject Classification (2020) — 54A05, 54B05

1. Introduction

Just as the notion of $\mathcal{T}$-set (open or closed set relative to ordinary topology) is fundamental and indispensable in the study of $\mathfrak{T}$-sets in $\mathcal{T}$-spaces (arbitrary sets in ordinary topological spaces) and in the formulation of the concept of $\mathfrak{g}$-$\mathcal{T}$-set (generalized $\mathcal{T}$-open or $\mathcal{T}$-closed set relative to ordinary topology) in the study of $\mathfrak{g}$-$\mathfrak{T}$-sets in $\mathcal{T}$-spaces (generalized sets in ordinary topological spaces) [1–6], so is the notion of $\mathcal{T}_{\mathfrak{g}}$-set (open or closed set relative to generalized topology) in the study of $\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces (arbitrary sets in generalized topological spaces) and in the formulation of the concept of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-set (generalized $\mathcal{T}_{\mathfrak{g}}$-open or $\mathcal{T}_{\mathfrak{g}}$-closed set relative to generalized topology) in the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces (generalized sets in generalized topological spaces) [7]. Thus, the $\mathfrak{g}$-topology maps $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ from the power set $\mathcal{P}(\Omega)$ of Ω into itself, thereby inducing $\mathfrak{g}$-topologies on the underlying set Ω, are classes of distinguished open subsets of a $\mathcal{T}$-space which are not $\mathcal{T}$-open sets but are $\mathcal{T}_{\mathfrak{g}}$-open sets which are related to the families of $\mathfrak{g}$-$\mathcal{T}$-open sets [8, 9]. Examples of $\mathfrak{g}$-$\mathfrak{T}$-sets in $\mathcal{T}$-spaces are α-open and α-closed sets [10], β-open sets [11], and γ-open sets [12]. Examples of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces are Δ_μ-sets and ∇_μ-sets [13], ω-open sets [2], and θ-sets [14]. From these α, β, γ-sets and, Δ_μ, ∇_μ, ω, θ-sets, the theories of $\mathfrak{g}$-$\mathfrak{T}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets then appear to be subjects of primary interest.

To the best of our knowledge, the theory of $\mathfrak{g}$-$\mathfrak{T}$-sets is well-known and that of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets less-known. The earliest works on the theory of $\mathfrak{g}$-$\mathfrak{T}$-sets are those of Levine [15, 16], Njåstad [10], and

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Császár [14,17–20], and the latest works on the theory of $\mathfrak{g}$-$\mathfrak{T}$-sets are those of Rajeshwari et al. [21], Jeyanthi et al. [3,13], Ghour et al. [2], and Tyagi et al. [6], among others. Levine [16] introduced and investigated the weaker forms of open sets, Njåstad [10] introduced and investigated the structures of some classes of more or less nearly open sets, and Császár [20] introduced the notion of $\mathfrak{g}$-topologies; [21] introduced the weaker forms of closed sets and studied some of their characterizations, Jeyanthi et al. [3] gave a unified framework for the study of several types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, Ghour et al. [2] extended the notion of a type of $\mathfrak{g}$-$\mathfrak{T}$-sets in a $\mathcal{T}$-space to its analogue in a $\mathcal{T}_{\mathfrak{g}}$-space, and Tyagi et al. [6] introduced and investigated several types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces.

Several other specific classes of $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets have been defined and investigated by other authors for various purposes from time to time in the literature of $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$-spaces [9,22–38]. The fruitfulness of all these references have made significant contributions to the theory of $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$-spaces, among others.

In this paper, we will show how further contributions can be added to the field in a unified way. The rest of this paper is structured in this manner: In Section 2, preliminary notions are described in Subsection 2.1 and the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces are reported in Section 3. In Section 4, the establishment of the various relationships between the classes of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and illustrated through diagrams in Subsection 4.1. To support the work, a nice application, concentrating on fundamental concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets is presented in Subsection 4.2. Finally, Subsection 5 provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces.

2. Theory

2.1. Preliminaries

Our discussion starts by recalling a carefully chosen set of terms used in this study [39]. Throughout this manuscript, the structures $\mathfrak{T} = (\Omega, \mathcal{T})$ and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, respectively, are called ordinary and generalized topological spaces (briefly, $\mathcal{T}$-space and $\mathcal{T}_{\mathfrak{g}}$-space). The symbols $\mathcal{T}$ and $\mathcal{T}_{\mathfrak{g}}$, respectively, are called ordinary topology and generalized topology (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$-sets and $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathcal{T}$ and $\mathcal{T}_{\mathfrak{g}}$, respectively, are called $\mathcal{T}$-open and $\mathcal{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathcal{T}$-closed and $\mathcal{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathcal{T}$-open and $\mathcal{T}$-closed sets in $\mathcal{T}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathcal{T}$-open and $\mathfrak{g}$-$\mathcal{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets in $\mathcal{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets; $\mathfrak{U}$ stands for the universe of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and containing as elements all sets (Ω, Γ-sets; $\mathcal{T}$, $\mathfrak{g}$-$\mathcal{T}$, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets; $\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets) considered in this theory, and $I_n^0 := \{\nu \in \mathbb{N}^0 : \ \nu \leq n\}$; index sets I_∞^0, I_n^*, I_∞^* are defined similarly. A set $\Gamma \subset \mathfrak{U}$ is a subset of the set $\Omega \subset \mathfrak{U}$ and, for some $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T} \cup \mathfrak{g}\text{-}\mathcal{T} \cup \mathcal{T}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}$, these implications hold:

$$\mathcal{O}_{\mathfrak{g}} \in \mathcal{T} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathcal{T} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \subset \Omega \subset \mathfrak{U} \tag{1}$$

In a natural way, a monotonic map $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ from the power set $\mathcal{P}(\Omega)$ of Ω into itself can be associated to a given mapping $\pi_{\mathfrak{g}} : \Omega \longrightarrow \Omega$, thereby inducing a $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} \subset \mathcal{P}(\Omega)$ on the underlying set Ω [9]. Therefore, the definition of a $\mathcal{T}_{\mathfrak{g}}$-space can be presented in a nice way. Thus, retaining the axioms to be satisfied by its $\mathfrak{g}$-topology [33], and assuming no separation axioms, unless otherwise stated, the following definition is suggestive:

Definition 2.1 ($\mathcal{T}_{\mathfrak{g}}$-Space [39])**.** Let $\Omega \subset \mathfrak{U}$ be a given set and let $\mathcal{P}(\Omega) := \{\mathcal{O}_{\mathfrak{g},\nu} : \ \mathcal{O}_{\mathfrak{g},\nu} \subseteq \Omega\}$ be the family of all subsets $\mathcal{O}_{\mathfrak{g},1}$, $\mathcal{O}_{\mathfrak{g},2}, \ldots$, of Ω. Then, every one-valued map of the type $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ satisfying the following axioms:

i. $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$

ii. $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathcal{O}_{\mathfrak{g}}$

iii. $\mathcal{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I^*_\infty}\mathcal{O}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I^*_\infty}\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$

is called a "$\mathfrak{g}$-topology on Ω," and the structure $\mathfrak{T}_{\mathfrak{g}} := (\Omega,\mathcal{T}_{\mathfrak{g}})$ is called a "$\mathcal{T}_{\mathfrak{g}}$-space."

In Definition 2.1, by Ax. *i.*, Ax. *ii.*, and Ax. *iii.*, respectively, are meant that the unary operation $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ preserves nullary union, is contracting and preserves binary union. Any element $\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}}(\Omega)$ of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathcal{T}_{\mathfrak{g}}$-open set and its complement element $\complement(\mathcal{O}_{\mathfrak{g}})=\mathcal{K}_{\mathfrak{g}}\notin\mathcal{T}_{\mathfrak{g}}(\Omega)$ is called a $\mathcal{T}_{\mathfrak{g}}$-closed set. If there exists a $\nu\in I^*_\infty$ such that $\mathcal{O}_{\mathfrak{g},\nu}=\Omega$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a strong $\mathcal{T}_{\mathfrak{g}}$-space [9,19]. Moreover, if the relation $\mathcal{T}_{\mathfrak{g}}\big(\bigcap_{\nu\in I^*_n}\mathcal{O}_{\mathfrak{g},\nu}\big)=\bigcap_{\nu\in I^*_n}\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ holds for any index set $I^*_n\subset I^*_\infty$ such that $n<\infty$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a quasi $\mathcal{T}_{\mathfrak{g}}$-space [17].

Definition 2.2 ($\mathfrak{g}$-Closure, $\mathfrak{g}$-Interior Operators [39])**.** Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space on the set $\Omega\subset\mathfrak{U}$ with a $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$. Then,

i. The operator $\mathrm{cl}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ into its closure $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})=\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\subset\mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{g}$-closure operator."

ii. The operator $\mathrm{int}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ into its interior $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})=\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\subset\mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{g}$-interior operator."

By convention, we let $\mathcal{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathcal{T}_{\mathfrak{g}}(\Omega)$, respectively, stand for the classes of all $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$. Their proper definitions are contained below.

Definition 2.3 (Classes: $\mathcal{T}_{\mathfrak{g}}$-Open, $\mathcal{T}_{\mathfrak{g}}$-Closed Sets [39])**.** Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space, let $\complement:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ denotes the absolute complement with respect to the underlying set $\Omega\subset\mathfrak{U}$, and let $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set. The classes

$$\mathcal{T}_{\mathfrak{g}}(\Omega):=\big\{\mathcal{O}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\ \mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}}\big\}\quad\text{and}\quad\neg\mathcal{T}_{\mathfrak{g}}(\Omega):=\big\{\mathcal{K}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\ \complement(\mathcal{K}_{\mathfrak{g}})\in\mathcal{T}_{\mathfrak{g}}\big\}\tag{2}$$

respectively, denote the classes of all $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$, and the classes

$$\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]:=\big\{\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}}:\ \mathcal{O}_{\mathfrak{g}}\subseteq\mathcal{S}_{\mathfrak{g}}\big\}\quad\text{and}\quad\mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]:=\big\{\mathcal{K}_{\mathfrak{g}}\in\neg\mathcal{T}_{\mathfrak{g}}:\ \mathcal{K}_{\mathfrak{g}}\supseteq\mathcal{S}_{\mathfrak{g}}\big\}\tag{3}$$

respectively, denote the classes of $\mathcal{T}_{\mathfrak{g}}$-open subsets and $\mathcal{T}_{\mathfrak{g}}$-closed supersets (complements of the $\mathcal{T}_{\mathfrak{g}}$-open subsets) of the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$.

That $\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]\subseteq\mathcal{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathcal{T}_{\mathfrak{g}}(\Omega)\supseteq\mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]$ are true for the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ in question are clear from the context. To this end, the $\mathfrak{g}$-closure and the $\mathfrak{g}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space define themselves as

$$\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}):=\bigcup_{\mathcal{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]}\mathcal{O}_{\mathfrak{g}}\quad\text{and}\quad\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}):=\bigcap_{\mathcal{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]}\mathcal{K}_{\mathfrak{g}}\tag{4}$$

We note in passing that, $\mathrm{cl}_{\mathfrak{g}}(\cdot)\neq\mathrm{cl}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot)\neq\mathrm{int}(\cdot)$, because the resulting sets obtained from the intersection of all $\mathcal{T}_{\mathfrak{g}}$-closed supersets and the union of all $\mathcal{T}_{\mathfrak{g}}$-open subsets, respectively, relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$ are not necessarily equal to those which would be obtained from the intersection of all $\mathcal{T}$-closed supersets and the union of all $\mathcal{T}$-open subsets relative to the topology $\mathcal{T}$ [23]. Throughout this work, by $\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot)$, $\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot)$, and $\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot)$, respectively, are meant $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\cdot))$, $\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot))$, and $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined in a similar way. Also, the backslash $\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}}$ refers to the set-theoretic relative complement of $\mathcal{S}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$. Finally, for convenience of notation, let $\mathcal{P}^*(\Omega)=\mathcal{P}(\Omega)\setminus\{\emptyset\}$, $\mathcal{T}^*_{\mathfrak{g}}=\mathcal{T}_{\mathfrak{g}}\setminus\{\emptyset\}$, and $\neg\mathcal{T}^*_{\mathfrak{g}}=\neg\mathcal{T}_{\mathfrak{g}}\setminus\{\emptyset\}$.

Definition 2.4 ($\mathfrak{g}$-Operation [39])**.** Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space. Then, a mapping $\mathrm{op}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ on $\mathcal{P}(\Omega)$ ranging in $\mathcal{P}(\Omega)$ is called a "$\mathfrak{g}$-operation" if and only if the following statements hold:

$$(\forall\mathcal{S}_{\mathfrak{g}}\in\mathcal{P}^*(\Omega))(\exists(\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}})\in\mathcal{T}^*_{\mathfrak{g}}\times\neg\mathcal{T}^*_{\mathfrak{g}})\big[(\mathrm{op}_{\mathfrak{g}}(\emptyset)=\emptyset)\vee(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset)=\emptyset)\vee(\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}))\vee(\mathcal{S}_{\mathfrak{g}}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))\big]\tag{5}$$

where $\neg\,\mathrm{op}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathcal{P}(\Omega)$ ranging in $\mathcal{P}(\Omega)$ and, for all $(\mathcal{S}_{\mathfrak{g}},\mathcal{U}_{\mathfrak{g},\mu},\mathcal{V}_{\mathfrak{g},\nu})\in\bigotimes_{\alpha\in I^*_3}\mathcal{P}^*(\Omega)$ such that $\mathcal{W}_{\mathfrak{g}}=\mathcal{U}_{\mathfrak{g},\mu}\cup\mathcal{V}_{\mathfrak{g},\nu}$ and $\big(\hat{\mathcal{W}}_{\mathfrak{g}},\neg\hat{\mathcal{W}}_{\mathfrak{g}}\big)=\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{W}_{\mathfrak{g}}),\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{W}_{\mathfrak{g}})\big)$, the following axioms are satisfied:

i. $\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g}}\right)\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g}}\right)\right)$

ii. $\left(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{S}_{\mathfrak{g}}\right) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g}}\right)\right) \vee \left(\neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{S}_{\mathfrak{g}}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}} \circ \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g}}\right)\right)$

iii. $\left(\hat{\mathcal{W}}_{\mathfrak{g}} \subseteq \bigcup_{\sigma=\mu,\nu} \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\right) \bigvee \left(\neg \hat{\mathcal{W}}_{\mathfrak{g}} \supseteq \bigcup_{\sigma=\mu,\nu} \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\right)$

iv. $\left(\mathcal{U}_{\mathfrak{g},\mu} \subseteq \mathcal{V}_{\mathfrak{g},\nu} \longrightarrow \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\mu}\right) \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\mathcal{U}_{\mathfrak{g},\mu} \supseteq \mathcal{V}_{\mathfrak{g},\nu} \longleftarrow \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mu}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)$

for some $\left(\mathcal{O}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g},\mu}, \mathcal{O}_{\mathfrak{g},\nu}\right) \in \bigotimes_{\alpha \in I_3^*} \mathcal{T}_{\mathfrak{g}}^*$ and $\left(\mathcal{K}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g},\mu}, \mathcal{K}_{\mathfrak{g},\nu}\right) \in \bigotimes_{\alpha \in I_3^*} \neg \mathcal{T}_{\mathfrak{g}}^*$.

The formulation of Definition 2.5 is based on the axioms of the Čech closure operator [25] and the various axioms used by many mathematicians to define closure operators [36]. The class $\mathcal{L}_{\mathfrak{g}}\left[\Omega\right]$ stands for the class of all possible $\mathfrak{g}$-operators and their complementary $\mathfrak{g}$-operators in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

Definition 2.5 ($\mathbf{op}_{\mathfrak{g}}(\cdot)$-Elements [39])**.** Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. The elements of the class $\mathcal{L}_{\mathfrak{g}}\left[\Omega\right] = \mathcal{L}_{\mathfrak{g}}^{\omega}\left[\Omega\right] \times \mathcal{L}_{\mathfrak{g}}^{\kappa}\left[\Omega\right]$, where

$$\mathcal{L}_{\mathfrak{g}}\left[\Omega\right] := \left\{\mathbf{op}_{\mathfrak{g},\nu\mu}\left(\cdot\right) = \left(\mathrm{op}_{\mathfrak{g},\nu}\left(\cdot\right), \neg\, \mathrm{op}_{\mathfrak{g},\mu}\left(\cdot\right)\right) : \ (\nu,\mu) \in I_3^0 \times I_3^0\right\} \tag{6}$$

in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are defined as:

$$\begin{aligned} \mathrm{op}_{\mathfrak{g}}\left(\cdot\right) &\in \mathcal{L}_{\mathfrak{g}}^{\omega}\left[\Omega\right] := \left\{\mathrm{op}_{\mathfrak{g},0}\left(\cdot\right),\ \mathrm{op}_{\mathfrak{g},1}\left(\cdot\right),\ \mathrm{op}_{\mathfrak{g},2}\left(\cdot\right),\ \mathrm{op}_{\mathfrak{g},3}\left(\cdot\right)\right\} \\ &= \left\{\mathrm{int}_{\mathfrak{g}}\left(\cdot\right),\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\left(\cdot\right),\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\left(\cdot\right),\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\left(\cdot\right)\right\} \\ \neg\, \mathrm{op}_{\mathfrak{g}}\left(\cdot\right) &\in \mathcal{L}_{\mathfrak{g}}^{\kappa}\left[\Omega\right] := \left\{\neg\, \mathrm{op}_{\mathfrak{g},0}\left(\cdot\right),\ \neg\, \mathrm{op}_{\mathfrak{g},1}\left(\cdot\right),\ \neg\, \mathrm{op}_{\mathfrak{g},2}\left(\cdot\right),\ \neg\, \mathrm{op}_{\mathfrak{g},3}\left(\cdot\right)\right\} \\ &= \left\{\mathrm{cl}_{\mathfrak{g}}\left(\cdot\right),\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\left(\cdot\right),\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\left(\cdot\right),\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\left(\cdot\right)\right\} \end{aligned} \tag{7}$$

We remark in passing that, $\mathbf{op}_{\mathfrak{g},11}\left(\cdot\right) = \neg\, \mathbf{op}_{\mathfrak{g},22}\left(\cdot\right)$, and the use of $\mathbf{op}_{\mathfrak{g}}\left(\cdot\right) = \left(\mathrm{op}_{\mathfrak{g}}\left(\cdot\right), \neg\, \mathrm{op}_{\mathfrak{g}}\left(\cdot\right)\right) \in \mathcal{L}_{\mathfrak{g}}\left[\Omega\right]$ on a class of $\mathfrak{T}_{\mathfrak{g}}$-sets will construct a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, just as the use of $\mathcal{L}\left[\Omega\right] := \left\{\mathbf{op}_{\nu}\left(\cdot\right) = \left(\mathrm{op}_{\nu}\left(\cdot\right), \neg\, \mathrm{op}_{\nu}\left(\cdot\right)\right) :\ \nu \in I_3^0\right\}$ on the class of $\mathfrak{T}$-sets have constructed the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets. But since $\mathrm{cl}_{\mathfrak{g}}\left(\cdot\right) \neq \mathrm{cl}\left(\cdot\right)$ and $\mathrm{int}_{\mathfrak{g}}\left(\cdot\right) \neq \mathrm{int}\left(\cdot\right)$, in general, it follows that $\mathbf{op}_{\mathfrak{g}}\left(\cdot\right) \neq \mathbf{op}\left(\cdot\right)$ and, therefore, the new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets that will be obtained from the first construction will, in general, differ from the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets that had been obtained from the second construction.

Definition 2.6 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Set [39])**.** A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set" if and only if there exist a pair $\left(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}\right) \in \mathcal{T}_{\mathfrak{g}} \times \neg \mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets, and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g}}\left(\cdot\right) \in \mathcal{L}_{\mathfrak{g}}\left[\Omega\right]$ such that the following statement holds:

$$(\exists \xi)\left[\left(\xi \in \mathcal{S}_{\mathfrak{g}}\right) \wedge \left(\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g}}\right)\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g}}\right)\right)\right)\right] \tag{8}$$

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is said to be of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-sets:

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right] := \left\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \left(\exists \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right)\left[\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g}}\right)\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g}}\right)\right)\right]\right\} \tag{9}$$

It is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open set if it satisfies the first property in $\mathfrak{g}$-ν-$\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed set if it satisfies the second property in $\mathfrak{g}$-ν-$\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The classes of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] &:= \left\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \left(\exists \mathcal{O}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right)\left[\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g}}\right)\right]\right\} \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] &:= \left\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \left(\exists \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right)\left[\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g}}\right)\right]\right\} \end{aligned} \tag{10}$$

From the class $\mathfrak{g}$-ν-$\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, consisting of the classes $\mathfrak{g}$-ν-$\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively, of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν, where $\nu \in I_3^0$, there results in the following definition.

Definition 2.7 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Set [39])**.** Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. If, for each $\nu \in I_3^0$, $\mathfrak{g}$-ν-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathfrak{g}$-ν-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, respectively, denote the classes of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] = \textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] &= \textstyle\bigcup_{\nu\in I_3^0}\big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \\
&= \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \cup \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \\
&= \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cup \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \qquad (11)
\end{aligned}$$

In the sequel, it is interesting to view the concepts of open, semi-open, pre-open, semi-pre-open sets as $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2, and 3; likewise, to view the concepts of closed, semi-closed, pre-closed, semi-pre-closed sets as $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2, and 3. These can be realised by omitting the subscript "$\mathfrak{g}$" in all symbols of the above definitions.

Definition 2.8 ($\mathfrak{g}$-ν-$\mathfrak{T}$-Set [39])**.** A $\mathfrak{T}$-set $\mathcal{S} \subset \mathfrak{T}$ in a $\mathcal{T}$-space is called a "$\mathfrak{g}$-$\mathfrak{T}$-set" if and only if there exists a pair $(\mathcal{O},\mathcal{K}) \in \mathcal{T} \times \neg\mathcal{T}$ of $\mathcal{T}$-open and $\mathcal{T}$-closed sets, and an operator $\mathbf{op}(\cdot) \in \mathcal{L}\big[\Omega\big]$ such that the following statement holds:

$$(\exists\xi)\,\big[(\xi \in \mathcal{S}) \wedge \big((\mathcal{S} \subseteq \mathrm{op}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}(\mathcal{K}))\big)\big] \qquad (12)$$

The $\mathfrak{g}$-$\mathfrak{T}$-set $\mathcal{S} \subset \mathfrak{T}$ is said to be of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathcal{T}$-sets:

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}\big] := \big\{\mathcal{S} \subset \mathfrak{T} :\ (\exists\mathcal{O},\mathcal{K},\mathbf{op}_\nu(\cdot))\,\big[(\mathcal{S} \subseteq \mathrm{op}_\nu(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}_\nu(\mathcal{K}))\big]\big\} \qquad (13)$$

It is called a $\mathfrak{g}$-ν-$\mathfrak{T}$-open set if it satisfies the first property in $\mathfrak{g}$-ν-$\mathrm{S}\big[\mathfrak{T}\big]$ and a $\mathfrak{g}$-ν-$\mathfrak{T}$-closed set if it satisfies the second property in $\mathfrak{g}$-ν-$\mathrm{S}\big[\mathfrak{T}\big]$. The classes of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets, respectively, are defined by

$$\begin{aligned}
\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big] &:= \big\{\mathcal{S} \subset \mathfrak{T} :\ (\exists\mathcal{O},\mathbf{op}_\nu(\cdot))\,\big[\mathcal{S} \subseteq \mathrm{op}_\nu(\mathcal{O})\big]\big\} \\
\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big] &:= \big\{\mathcal{S} \subset \mathfrak{T} :\ (\exists\mathcal{K},\mathbf{op}_\nu(\cdot))\,\big[\mathcal{S} \supseteq \neg\,\mathrm{op}_\nu(\mathcal{K})\big]\big\} \qquad (14)
\end{aligned}$$

As in the previous definitions, from the class $\mathfrak{g}$-ν-$\mathrm{S}\big[\mathfrak{T}\big]$, consisting of the classes $\mathfrak{g}$-ν-$\mathrm{O}\big[\mathfrak{T}\big]$ and $\mathfrak{g}$-ν-$\mathrm{K}\big[\mathfrak{T}\big]$, respectively, of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets of category ν, where $\nu \in I_3^0$, there results in the following definition.

Definition 2.9 (Class: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Sets [39])**.** Let $\mathfrak{T}$ be a $\mathcal{T}$-space. If, for each $\nu \in I_3^0$, $\mathfrak{g}$-ν-$\mathrm{O}\big[\mathfrak{T}\big]$ and $\mathfrak{g}$-ν-$\mathrm{K}\big[\mathfrak{T}\big]$, respectively, denote the classes of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets of category ν then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}\big] = \textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}\big] &= \textstyle\bigcup_{\nu\in I_3^0}\big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big]\big) \\
&= \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big]\big) \cup \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big]\big) \\
&= \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cup \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}\big] \qquad (15)
\end{aligned}$$

The classes of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ as well as the classes of $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets in a $\mathcal{T}$-space $\mathfrak{T}$ are defined as thus:

Definition 2.10 (Families: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Open Sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closed Sets [39])**.** Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space and let $\mathfrak{T} = (\Omega, \mathcal{T})$ be a $\mathcal{T}$-space.

i. The classes $\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ denote the families of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, with $\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cup \mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

ii. The classes $\mathrm{O}\left[\mathfrak{T}\right]$ and $\mathrm{K}\left[\mathfrak{T}\right]$ denote the families of $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in $\mathfrak{T}$, with $\mathrm{S}\left[\mathfrak{T}\right] = \mathrm{O}\left[\mathfrak{T}\right] \cup \mathrm{K}\left[\mathfrak{T}\right]$.

In the following sections, the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets are presented.

3. Main Results

Theorem 3.1. Let $\mathrm{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ and $\mathrm{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, respectively, be $\mathfrak{g}$-closure and $\mathfrak{g}$-interior operators in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Then,

i. $\mathrm{cl}_{\mathfrak{g}}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot)$ are enhancing and contracting, respectively.

ii. $\mathrm{cl}_{\mathfrak{g}}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot)$ are idempotent.

iii. $\mathrm{cl}_{\mathfrak{g}}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot)$ are monotone.

Proof.

i. Since the following logical statement

$$\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ (\forall \xi) \left[(\xi \in \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \longleftarrow \xi \in \mathcal{S}_{\mathfrak{g}}) \vee (\xi \in \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \longrightarrow \xi \in \mathcal{S}_{\mathfrak{g}}) \right]$$

holds, it follows that $\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ or $\mathcal{S}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$.

ii. If $\mathcal{S}_{\mathfrak{g}}$ is open, then $\mathcal{S}_{\mathfrak{g}} = \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$; if it is closed, $\mathcal{S}_{\mathfrak{g}} = \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. Consequently, the substitutions $\mathcal{S}_{\mathfrak{g}} \longmapsto \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ and $\mathcal{S}_{\mathfrak{g}} \longmapsto \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, respectively, give $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathrm{int}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$.

iii. Let $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that $\mathcal{R}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$. Then, $\mathcal{R}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}})$, $\mathcal{R}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}})$, $\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, and $\mathcal{S}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ by *i.* Consequently, $\mathrm{int}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}}) \subseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$.

□

Lemma 3.2. Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space. Then,

i. $(\mathcal{S}_{\mathfrak{g}} = \emptyset) \wedge (\Omega \in \mathcal{T}_{\mathfrak{g}}) \ \Rightarrow \ (\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \emptyset) \wedge (\mathrm{cl}_{\mathfrak{g}}(\emptyset) = \emptyset)$

ii. $(\mathcal{S}_{\mathfrak{g}} = \emptyset) \wedge (\Omega \notin \mathcal{T}_{\mathfrak{g}}) \ \Rightarrow \ (\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \emptyset) \wedge (\mathrm{cl}_{\mathfrak{g}}(\emptyset) \neq \emptyset)$

Proof.

i. If $\mathcal{S}_{\mathfrak{g}} = \emptyset$ and $\Omega \in \mathcal{T}_{\mathfrak{g}}$, then $\left(\emptyset \in \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\right) \wedge \left(\emptyset \in \mathrm{C}^{\mathrm{sup}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\right)$. Consequently, $\mathrm{int}_{\mathfrak{g}}(\emptyset) = \emptyset$ and $\mathrm{cl}_{\mathfrak{g}}(\emptyset) = \emptyset$.

ii. If $\mathcal{S}_{\mathfrak{g}} = \emptyset$ and $\Omega \notin \mathcal{T}_{\mathfrak{g}}$, then $\left(\emptyset \in \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\right) \wedge \left(\emptyset \notin \mathrm{C}^{\mathrm{sup}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\right)$. Consequently, $\mathrm{int}_{\mathfrak{g}}(\emptyset) = \emptyset$ and $\mathrm{int}_{\mathfrak{g}}(\emptyset) \neq \emptyset$.

□

According to Sarsak [40] and Noiri [41], the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ may be called a μ-space when $\mathrm{cl}_{\mathfrak{g}}(\emptyset) = \emptyset$.

Theorem 3.3. If $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$ are $n \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathcal{T}_{\mathfrak{g}}$-space, then,

i. $\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_n^*} \mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

ii. $\mathrm{int}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_n^*} \mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

Proof. Expressed in set-builder notation, the $\mathfrak{g}$-closure and the $\mathfrak{g}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space can also be defined as thus:

$$\begin{aligned} \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) &:= \left\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ (\mathcal{S}_{\mathfrak{g}} \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\right\} \\ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) &:= \left\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ (\mathcal{S}_{\mathfrak{g}} \cap \mathrm{int}(\mathcal{O}_{\mathfrak{g}}) = \mathrm{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\right\} \end{aligned}$$

respectively, from which it is easily seen that,

$$\begin{aligned}\mathrm{cl}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \bigcup_{\nu\in I_n^*}\big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ (\mathcal{S}_{\mathfrak{g},\nu}\cap\mathrm{cl}\,(\mathcal{O}_{\mathfrak{g}})\neq\emptyset)\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big\}\\ &= \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \big((\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu})\cap\mathrm{cl}\,(\mathcal{O}_{\mathfrak{g}})\neq\emptyset\big)\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big\}\\ &= \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \big(\textstyle\bigcup_{\nu\in I_n^*}(\mathcal{S}_{\mathfrak{g},\nu}\cap\mathrm{cl}\,(\mathcal{O}_{\mathfrak{g}}))\neq\emptyset\big)\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big\}\\ &= \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \textstyle\bigvee_{\nu\in I_n^*}\big((\mathcal{S}_{\mathfrak{g},\nu}\cap\mathrm{cl}\,(\mathcal{O}_{\mathfrak{g}})\neq\emptyset)\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big)\big\}\\ &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\end{aligned}$$

Likewise, it is also easily seen that,

$$\begin{aligned}\mathrm{int}_{\mathfrak{g}}\big(\textstyle\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \bigcap_{\nu\in I_n^*}\big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ (\mathcal{S}_{\mathfrak{g},\nu}\cap\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}})=\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}}))\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big\}\\ &= \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \big((\textstyle\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu})\cap\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}})=\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}})\big)\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big\}\\ &= \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \big(\textstyle\bigcap_{\nu\in I_n^*}(\mathcal{S}_{\mathfrak{g},\nu}\cap\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}}))=\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}})\big)\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big\}\\ &= \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \textstyle\bigwedge_{\nu\in I_n^*}\big((\mathcal{S}_{\mathfrak{g},\nu}\cap\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}})=\mathrm{int}\,(\mathcal{O}_{\mathfrak{g}}))\wedge(\xi\in\mathcal{O}_{\mathfrak{g}}\in\mathcal{T}_{\mathfrak{g}})\big)\big\}\\ &= \textstyle\bigcap_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\end{aligned}$$

□

Clearly, $\mathcal{S}_{\mathfrak{g},\mu}\subseteq\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}$ and $\mathcal{S}_{\mathfrak{g},\mu}\supseteq\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}$ hold true for any $\mu\in I_n^*$. The following corollary, then, is an immediate consequence of the above theorem.

Corollary 3.4. If $\mathcal{S}_{\mathfrak{g},1}$, $\mathcal{S}_{\mathfrak{g},2}$, …, $\mathcal{S}_{\mathfrak{g},n}\subset\mathfrak{T}_{\mathfrak{g}}$ are $n\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathcal{T}_{\mathfrak{g}}$-space, then,

i. $\mathrm{cl}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\subseteq\bigcap_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

ii. $\mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\supseteq\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

Proposition 3.5. For any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the following statement holds:

$$\mathfrak{T}_{\mathfrak{g}}\setminus\big(\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\cup\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\big)=\emptyset \tag{16}$$

Proof. Let $\xi\in\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})$. Then, $\xi\in\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}}$ since, $\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})$. But, $\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}}\subseteq\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})$ and, consequently, $\xi\in\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})$. Hence, there follows that, $\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\subseteq\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})$. Conversely, let $\xi\in\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})$. Then, $\xi\in\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}}))$, since $\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}}))$. But, since $\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}}))$, and, consequently, $\xi\in\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})$. Hence, $\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})$. Since $\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})=\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})$ is equivalent to

$$\big(\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\subseteq\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\big)\wedge\big(\mathrm{cl}_{\mathfrak{g}}\,(\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}})\supseteq\mathfrak{T}_{\mathfrak{g}}\setminus\mathrm{int}_{\mathfrak{g}}\,(\mathcal{S}_{\mathfrak{g}})\big)$$

the proof of the proposition at once follows. □

Proposition 3.6. Let $\mathrm{cl}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ and $\mathrm{int}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$, respectively, be $\mathfrak{g}$-closure and $\mathfrak{g}$-interior operators in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. If $\mathcal{S}_{\mathfrak{g},1}$, $\mathcal{S}_{\mathfrak{g},2}$, …, $\mathcal{S}_{\mathfrak{g},n}\subset\mathfrak{T}_{\mathfrak{g}}$ are $n\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then,

i. $\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\supseteq\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

ii. $\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\supseteq\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

iii. $\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\supseteq\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

PROOF. Since the relations

$$\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big),\quad \mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\supseteq\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$$

hold, it follows that

$$\begin{aligned}
\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &\supseteq \mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\big)\\
&= \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\\
\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &\supseteq \mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\big)\\
&= \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\\
\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\big)\\
&\supseteq \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)
\end{aligned}$$

□

From the above proposition, it is obvious that their duals are

$$\begin{aligned}
\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &\subseteq \bigcap_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\\
\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &\subseteq \bigcap_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)\\
\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &\subseteq \bigcap_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big) \qquad (17)
\end{aligned}$$

respectively. On this basis, we have the following corollary:

Corollary 3.7. Let $\mathbf{op}_{\mathfrak{g}}(\cdot)\in\mathcal{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. If $\mathcal{S}_{\mathfrak{g},1},\mathcal{S}_{\mathfrak{g},2},\ldots,\mathcal{S}_{\mathfrak{g},n}\subset\mathfrak{T}_{\mathfrak{g}}$ are $n\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then,

i. $\mathrm{op}_{\mathfrak{g}}\circ\neg\,\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\supseteq\bigcup_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g}}\circ\neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

ii. $\neg\,\mathrm{op}_{\mathfrak{g}}\circ\mathrm{op}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)\subseteq\bigcap_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g}}\circ\mathrm{op}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}\big)$

Theorem 3.8. If $\mathcal{S}_{\mathfrak{g},1},\mathcal{S}_{\mathfrak{g},2},\ldots,\mathcal{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ are $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of a class $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then $\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$.

PROOF. The statement $\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\nu\in I_n^*$ is identical to the logical statement:

$$\exists\left(\mathcal{O}_{\mathfrak{g},\nu},\mathcal{K}_{\mathfrak{g},\nu}\right)\in\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}:\ \left(\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right)\vee\left(\mathcal{S}_{\mathfrak{g},\nu}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)$$

On the other hand, if $\mathbf{op}_{\mathfrak{g}}(\cdot)\in\mathcal{L}_{\mathfrak{g}}[\Omega]$ is a $\mathfrak{g}$-operator in the $\mathcal{T}_{\mathfrak{g}}$-space, then

$$\begin{aligned}
\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}\big) &= \bigcup_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\\
\neg\,\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}\big) &= \bigcup_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\bigvee_{\nu\in I_n^*}\left(\left(\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right)\vee\left(\mathcal{S}_{\mathfrak{g},\nu}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)\right)\\
\Rightarrow\ &\left(\left(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\subseteq\bigcup_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right)\vee\left(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\supseteq\bigcup_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)\right)\\
\Rightarrow\ &\left(\left(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}\big)\right)\vee\left(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}\big)\right)\right)
\end{aligned}$$

But, $\bigcup_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}\in\mathcal{T}_{\mathfrak{g}}$ and $\bigcup_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}\in\neg\mathcal{T}_{\mathfrak{g}}$. Hence, $\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. □

Theorem 3.9. If $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ are $n \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of a class $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then

$$\Big(\bigcap_{\nu\in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\Big) \vee \Big(\bigcap_{\nu\in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\Big) \tag{18}$$

Proof. Because, $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ by hypothesis, the trueness of $\bigcap_{\nu\in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ and $\bigcap_{\nu\in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ evidently depend on the following property:

$$\bigwedge_{\nu\in I_n^*} \Big(\big(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})\big) \vee \big(\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})\big)\Big)$$

where $(\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ for every $\nu \in I_n^*$. Furthermore, because the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-theoretic operations concern finite intersections, it suffices to prove the theorem for $n = 2$. Set the first property preceding $\vee$ to $\mathrm{P}(\nu)$ and that following $\vee$ to $\mathrm{Q}(\nu)$. Then, its decomposition gives

$$\begin{aligned}\textstyle\bigwedge_{\nu\in I_2^*}\big(\mathrm{P}(\nu) \vee \mathrm{Q}(\nu)\big) &= \big(\textstyle\bigwedge_{\nu\in I_2^*} \mathrm{P}(\nu)\big) \vee \big(\textstyle\bigwedge_{\nu\in I_2^*} \mathrm{Q}(\nu)\big)\\ &= \big(\mathrm{P}(1) \wedge \mathrm{Q}(2)\big) \vee \big(\mathrm{P}(2) \wedge \mathrm{Q}(1)\big)\end{aligned}$$

If $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets then $\bigwedge_{\nu\in I_2^*} \mathrm{P}(\nu)$ is true, and if they are both $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets then $\bigwedge_{\nu\in I_2^*} \mathrm{Q}(\nu)$ is true. In these two cases, $\bigcap_{\nu\in I_2^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Because, in general, there does not necessarily exists $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set which is simultaneously $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed, both $\mathrm{P}(1) \wedge \mathrm{Q}(2)$ and $\mathrm{P}(2) \wedge \mathrm{Q}(1)$ are untrue; thus, $\bigcap_{\nu\in I_2^*} \mathcal{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. □

Theorem 3.10. Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and let $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, then

$$\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\big) \tag{19}$$

Proof. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Then, $\big(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big) \vee \big(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)$ for some pair $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to $\mathcal{T}_{\mathfrak{g}}$. Consequently, $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. But, $\mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ and $\neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. Thus, there follows that $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. Hence, $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. □

Proposition 3.11. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ and suppose the logical statement

$$(\exists \mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}})\left[\big(\mathcal{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big) \vee \big(\mathcal{R}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big)\right] \tag{20}$$

holds, then $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. Let there exists a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that $\mathcal{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ or $\mathcal{R}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. But $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Thus, $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. □

Corollary 3.12. Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ denotes a class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, and $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ denotes a class of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, then

$$\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \tag{21}$$

An important remark should be pointed out at this stage.

Remark 3.13. The converse of the statement "if $\mathcal{S}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$" is obviously untrue. Because, the negation of this statement gives

$$\big(\mathcal{S}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\big) \wedge \big(\neg\,(\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}])\big)$$

which is an untrue statements.

Theorem 3.14. Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, then

$$\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \Leftrightarrow \big(\mathcal{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}} \circ \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big) \vee \big(\mathcal{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}} \circ \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big) \tag{22}$$

Proof.
$(\Leftarrow)$: Let

$$\big(\mathcal{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}} \circ \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big) \vee \big(\mathcal{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}} \circ \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big)$$

Then, the substitution of $\neg \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathcal{O}_{\mathfrak{g}}$ in the logical statement preceding $\vee$ and $\operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathcal{K}_{\mathfrak{g}}$ in that following $\vee$ gives $\big(\mathcal{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big) \vee \big(\mathcal{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)$.
$(\Rightarrow)$: Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Then, $\big(\mathcal{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big) \vee \big(\mathcal{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)$. Consequently, substituting $\mathcal{O}_{\mathfrak{g}} = \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ in the logical statement preceding $\vee$ and $\mathcal{K}_{\mathfrak{g}} = \operatorname{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ in that following $\vee$, the required logical statement at once follows, which proves the theorem. □

The class $\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ forms a $\mathfrak{g}$-topology on Ω, which will be denoted by $\mathcal{T}_{\mathfrak{g}\text{-S}}$.

Theorem 3.15. Let $\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a given $\mathfrak{g}$-class in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Then, the one-valued map $\mathcal{T}_{\mathfrak{g}\text{-S}} : \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \longrightarrow \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ forms a $\mathfrak{g}$-topology on Ω in the $\mathcal{T}_{\mathfrak{g}}$-space.

Proof. By definition, $\big(\emptyset = \operatorname{op}_{\mathfrak{g}}(\emptyset)\big) \vee \big(\emptyset = \neg \operatorname{op}_{\mathfrak{g}}(\emptyset)\big)$. Since, either $\operatorname{op}_{\mathfrak{g}}(\emptyset) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ or $\neg \operatorname{op}_{\mathfrak{g}}(\emptyset) \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$ holds, where $\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, respectively, are some $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets in $\mathfrak{T}_{\mathfrak{g}}$, it follows that $\emptyset \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and, hence, $\mathcal{T}_{\mathfrak{g}\text{-S}}(\emptyset) = \emptyset$. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Then, since $\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \subseteq \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it follows that $\mathcal{S}_{\mathfrak{g}}$ is a superset of $\mathcal{T}_{\mathfrak{g}\text{-S}}(\mathcal{S}_{\mathfrak{g}})$. Hence, $\mathcal{T}_{\mathfrak{g}\text{-S}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathcal{S}_{\mathfrak{g}}$. Let $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots$ be $\mathfrak{T}_{\mathfrak{g}}$-sets satisfying, for every $\nu \in I_{\infty}^{*}$, $\mathcal{S}_{\mathfrak{g},\nu}$. Then, there exist classes $\big\{\mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}} : \ \nu \in I_{\infty}^{*}\big\}$ and $\big\{\mathcal{K}_{\mathfrak{g},\nu} \in \neg\mathcal{T}_{\mathfrak{g}} : \ \nu \in I_{\infty}^{*}\big\}$, respectively, of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets such that

$$\big(\textstyle\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu}\big)\big) \vee \big(\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \operatorname{op}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{K}_{\mathfrak{g},\nu}\big)\big)$$

a relation established on the following expressions:

$$\begin{aligned}\textstyle\bigcup_{\nu\in I_{\infty}^{*}} \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) &= \operatorname{op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu}\big)\\ \textstyle\bigcup_{\nu\in I_{\infty}^{*}} \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}) &= \neg \operatorname{op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{K}_{\mathfrak{g},\nu}\big)\end{aligned}$$

Consequently, $\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, since $\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}}$ is a $\mathcal{T}_{\mathfrak{g}}$-open set and $\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu} \in \neg\mathcal{T}_{\mathfrak{g}}$ is a $\mathcal{T}_{\mathfrak{g}}$-closed set. Hence,

$$\mathcal{T}_{\mathfrak{g}\text{-S}}\big(\textstyle\bigcup_{\nu\in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_{\infty}^{*}} \mathcal{T}_{\mathfrak{g}\text{-S}}(\mathcal{S}_{\mathfrak{g},\nu})$$

□

An immediate consequence of the above theorem is the following corollary.

Corollary 3.16. Let a $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. Then, the structure $(\Omega, \mathcal{T}_{\mathfrak{g}\text{-S}})$, where $\mathcal{T}_{\mathfrak{g}\text{-S}} : \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \longrightarrow \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, is a $\mathcal{T}_{\mathfrak{g}}$-space.

To condense the set-builder notation describing the classes $\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and then classify it into subclasses, predicates must be introduced, and the choice made is to consider the so-called *Boolean-valued functions* on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$, the definition of which are given below.

Definition 3.17. Let $(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ and let $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}\big[\Omega\big]$ be a $\mathfrak{g}$-operator in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The first two predicates

$$\begin{aligned}\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big) &:= \big(\exists \mathcal{O}_{\mathfrak{g}}, \operatorname{op}_{\mathfrak{g}}(\cdot)\big)\big(\mathcal{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big)\\ \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big) &:= \big(\exists \mathcal{K}_{\mathfrak{g}}, \neg \operatorname{op}_{\mathfrak{g}}(\cdot)\big)\big(\mathcal{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)\\ \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big) &:= \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big)\\ &\quad \vee \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big)\end{aligned} \tag{23}$$

are called a Boolean-valued functions on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$.

In this respect, $\mathfrak{g}\text{-S}[\mathfrak{T}_\mathfrak{g}] := \{\mathcal{S}_\mathfrak{g} \subset \mathfrak{T}_\mathfrak{g} :\ \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq, \supseteq)\}$. Moreover, employing the set-builder notations, the class of $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-open and $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-closed sets, denoted by $\mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}]$ and $\mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}]$, respectively, may then be defined as thus:

Definition 3.18. Let $\mathfrak{T}_\mathfrak{g}$ be a $\mathcal{T}_\mathfrak{g}$-space. The classes

$$\begin{aligned} \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] &:= \{\mathcal{S}_\mathfrak{g} \subset \mathfrak{T}_\mathfrak{g} :\ \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq)\} \\ \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}] &:= \{\mathcal{S}_\mathfrak{g} \subset \mathfrak{T}_\mathfrak{g} :\ \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq)\} \end{aligned} \tag{24}$$

respectively, such that $\mathfrak{g}\text{-S}[\mathfrak{T}_\mathfrak{g}] = \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] \bigcup \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}]$, denote the families of all $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-open and $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-closed sets in $\mathfrak{T}_\mathfrak{g}$.

It is interesting to demonstrate their usefulness. In this direction, let us prove in a different way that $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-set-theoretic operations is closed under arbitrary unions.

$$\begin{aligned} \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq) &:= (\exists \mathcal{O}_\mathfrak{g}, \mathrm{op}_\mathfrak{g}(\cdot))\,(\mathcal{S}_\mathfrak{g} \subseteq \mathrm{op}_\mathfrak{g}(\mathcal{O}_\mathfrak{g})) \\ \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq) &:= (\exists \mathcal{K}_\mathfrak{g}, \neg\,\mathrm{op}_\mathfrak{g}(\cdot))\,(\mathcal{S}_\mathfrak{g} \supseteq \neg\,\mathrm{op}_\mathfrak{g}(\mathcal{K}_\mathfrak{g})) \\ \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq, \supseteq) &:= \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq) \\ &\quad \vee \mathrm{P}_\mathfrak{g}(\mathcal{S}_\mathfrak{g}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq) \end{aligned} \tag{25}$$

Theorem 3.19. If $\{\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] :\ \nu \in I_n^*\}$ and $\{\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}] :\ \nu \in I_n^*\}$, respectively, are finite collections of $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-open and $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-closed sets in a $\mathcal{T}_\mathfrak{g}$-space $\mathfrak{T}_\mathfrak{g}$, then

$$\begin{aligned} &\bigcup_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_\mathfrak{g} :\ (\exists \nu \in I_\mu^*)\,(\xi \in \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}])\} \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] \\ &\bigcap_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_\mathfrak{g} :\ (\forall \nu \in I_\mu^*)\,(\xi \in \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}])\} \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}] \end{aligned} \tag{26}$$

Proof. Let $\{\mathcal{R}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] :\ \nu \in I_n^*\}$ and $\{\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}] :\ \nu \in I_n^*\}$, respectively, be finite collections of $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-open and $\mathfrak{g}$-$\mathfrak{T}_\mathfrak{g}$-closed sets in a $\mathcal{T}_\mathfrak{g}$-space $\mathfrak{T}_\mathfrak{g}$. Then, since $(\mathcal{R}_{\mathfrak{g},\nu}, \mathcal{S}_{\mathfrak{g},\nu}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}]$, there exists $(\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathcal{T}_\mathfrak{g} \times \neg\mathcal{T}_\mathfrak{g}$ such that the propositional formulas $\mathrm{P}_\mathfrak{g}(\mathcal{R}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\nu}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq)$ and $\mathrm{P}_\mathfrak{g}(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq)$ hold true for every index $\nu \in I_n^*$. Consequently, the propositional formulas $\bigvee_{\nu \in I_n^*} \mathrm{P}_\mathfrak{g}(\mathcal{R}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\nu}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq)$ and $\bigwedge_{\nu \in I_n^*} \mathrm{P}_\mathfrak{g}(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq)$ also hold true. Since

$$\begin{aligned} \bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu} &\longleftrightarrow \bigcup_{\mu \in I_n^*} \{\mathcal{R}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}] :\ \nu \in I_\mu^*\} \\ &\longleftrightarrow \bigcup_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_\mathfrak{g} :\ (\exists \nu \in I_\mu^*)\,(\xi \in \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}])\} \subseteq \bigcup_{\nu \in I_n^*} \mathrm{op}_\mathfrak{g}(\mathcal{O}_{\mathfrak{g},\nu}) \\ \bigcup_{\nu \in I_n^*} \mathrm{op}_\mathfrak{g}(\mathcal{O}_{\mathfrak{g},\nu}) &\longleftrightarrow \bigcap_{\mu \in I_n^*} \{\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}] :\ \nu \in I_\mu^*\} \\ &\longleftrightarrow \bigcap_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_\mathfrak{g} :\ (\forall \nu \in I_\mu^*)\,(\xi \in \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}])\} \supseteq \bigcap_{\nu \in I_n^*} \neg\,\mathrm{op}_\mathfrak{g}(\mathcal{K}_{\mathfrak{g},\nu}) \end{aligned}$$

it results that $\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu} \subseteq \bigcup_{\nu \in I_n^*} \mathrm{op}_\mathfrak{g}(\mathcal{O}_{\mathfrak{g},\nu})$ and $\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \bigcap_{\nu \in I_n^*} \neg\,\mathrm{op}_\mathfrak{g}(\mathcal{K}_{\mathfrak{g},\nu})$. But it holds that $\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_\mathfrak{g}(\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}]$ and $\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_\mathfrak{g}(\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}]$. Consequently, there exists $(\mathcal{O}_\mathfrak{g}, \mathcal{K}_\mathfrak{g}) \in \mathcal{T}_\mathfrak{g} \times \neg\mathcal{T}_\mathfrak{g}$ such that $\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_\mathfrak{g}(\mathcal{O}_\mathfrak{g}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_\mathfrak{g}]$ and $\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_\mathfrak{g}(\mathcal{K}_\mathfrak{g}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_\mathfrak{g}]$, implying that both $\mathrm{P}_\mathfrak{g}(\bigcup_{\nu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu}, \bigcup_{\nu \in I_\mu^*} \mathcal{O}_{\mathfrak{g},\nu}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq)$ and $\mathrm{P}_\mathfrak{g}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}, \bigcup_{\nu \in I_\mu^*} \mathcal{K}_{\mathfrak{g},\nu}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq)$, respectively, hold true. But,

$$\begin{aligned} \mathrm{P}_\mathfrak{g}(\textstyle\bigcup_{\nu \in I_n^*} \mathcal{R}_{\mathfrak{g},\nu}, \mathcal{O}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq) &= \bigvee_{\nu \in I_n^*} \mathrm{P}_\mathfrak{g}(\mathcal{R}_{\mathfrak{g},\nu}, \mathcal{O}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \subseteq) \\ \mathrm{P}_\mathfrak{g}(\textstyle\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq) &= \bigwedge_{\nu \in I_n^*} \mathrm{P}_\mathfrak{g}(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{K}_\mathfrak{g}; \mathbf{op}_\mathfrak{g}(\cdot); \supseteq) \end{aligned}$$

Hence, it suffices to set

$$\mathrm{P}_{\mathfrak{g}}\big(\mathcal{R}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big) = \bigvee_{\nu \in I_n^*} \mathrm{P}_{\mathfrak{g}}\big(\mathcal{R}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big)$$

$$\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big) = \bigvee_{\nu \in I_n^*} \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big)$$

and the theorem is proved. □

If in $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big)$ it be assumed that $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, we have the following theorem:

Theorem 3.20. Let $(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. If $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then

$$\big\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big)\big\} \subseteq \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \tag{27}$$

Proof. It is clear that

$$\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big) = \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big) \vee \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big)$$

and the Boolean-valued functions surrounding $\vee$ hold on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$. Consequently, the following two cases must be considered in proving the theorem:

Case i. Let $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big)$ hold on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$. Then, $\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$. But, $\mathcal{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and consequently, it follows that $\mathcal{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ and $\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ for some $\mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}}$, by the properties of the $\mathfrak{g}$-operator. Hence, $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g},\nu}; \mathrm{op}_{\mathfrak{g}}(\cdot); \subseteq\big)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$.

Case ii. Let $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big)$ hold on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$. Then, $\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. But, $\mathcal{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and consequently, it follows that $\mathcal{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})$ and $\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})$ for some $\mathcal{K}_{\mathfrak{g},\nu} \in \neg\mathcal{T}_{\mathfrak{g}}$, by the properties of the $\mathfrak{g}$-operator. Hence, $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g},\nu}; \mathrm{op}_{\mathfrak{g}}(\cdot); \supseteq\big)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$.

From Case i. and Case ii., it follows that

$$\big\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq\big)\big\} \subseteq \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$$
$$\big\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \supseteq\big)\big\} \subseteq \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$$

But, since $\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] = \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \bigcup \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, the proof of the theorem at once follows. □

The following theorem shows that the class $\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, upon satisfaction of two conditions, is the smallest class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

Theorem 3.21. Let $\mathfrak{g}\text{-}\mathrm{S}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] = \mathfrak{g}\text{-}\mathrm{O}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \cup \mathfrak{g}\text{-}\mathrm{K}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ such that the following two conditions are satisfied:

i. If $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-}\mathrm{O}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$, then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

ii. The relation $\mathcal{S}_{\mathfrak{g}} \in \mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ implies $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

Then, $\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \subseteq \mathfrak{g}\text{-}\mathrm{S}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

Proof. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Then, $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$ for some pair $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. But, $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ implies $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$ by (*i.*), and the latter together with the trueness of $\mathrm{P}_{\mathfrak{g}}\big(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq\big)$ on $\mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-}\mathrm{O}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}_0\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathcal{L}_{\mathfrak{g}}\big[\Omega\big] \times \big\{\subseteq, \supseteq\big\}$ implies $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$ by (*ii.*). Thus, $\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \subseteq \mathfrak{g}\text{-}\mathrm{S}_0\big[\mathfrak{T}_{\mathfrak{g}}\big]$, which completes the proof. □

In the earlier discussion, the set $\Omega \subset \mathfrak{U}$ carried the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}(\Omega)$. A $\mathfrak{g}$-topology of this kind will be termed an *absolute* $\mathfrak{g}$*-topology*. To this end, if $\Gamma \subseteq \Omega$ is any subset of Ω then, obviously, we would expect Γ to carry the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}(\Gamma)$. But, since $\mathcal{T}_{\mathfrak{g}}(\Gamma) \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega)$, as a consequence of the fact that $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Gamma) \longrightarrow \mathcal{P}(\Gamma)$ is the one-valued restriction map of $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, which follows from the statement, $\Gamma \subseteq \Omega$ implies $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Omega)$, it does make sense to term $\mathcal{T}_{\mathfrak{g}}(\Gamma)$ a *relative* $\mathfrak{g}$*-topology*. In order to determine what any $\mathfrak{g}$-set-theoretic concepts for the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$ becomes when discussion is restricted to $\Gamma \subseteq \Omega$, it merely suffices to regard Γ as the set which carries the relative $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}(\Gamma)$ and carry over the discussion verbatim.

Definition 3.22 ($\mathcal{T}_{\mathfrak{g}}$-Subspace)**.** Let $\mathfrak{T}_{\mathfrak{g}}(\Omega) := (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$ be a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, where $\Omega \subset \mathfrak{U}$ carries the absolute $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, and let $\mathcal{P}(\Gamma) := \{\mathcal{O}_{\mathfrak{g},\nu} : \ \mathcal{O}_{\mathfrak{g},\nu} \subset \Gamma\}$ be the family of all subsets $\mathcal{O}_{\mathfrak{g},1}$, $\mathcal{O}_{\mathfrak{g},2}$, ..., of any subset $\Gamma \subseteq \Omega$ of Ω, then every one-valued restriction map of the type

$$\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Gamma) \longmapsto \mathcal{T}_{\mathfrak{g}}(\Gamma) := \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Omega)\} \tag{28}$$

defines a "relative $\mathfrak{g}$-topology on Γ," and the structure $\mathfrak{T}_{\mathfrak{g}}(\Gamma) := (\Gamma, \mathcal{T}_{\mathfrak{g}}(\Gamma))$ is called a "$\mathcal{T}_{\mathfrak{g}}$-subspace."

Theorem 3.23. Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}(\Gamma) \subseteq \mathfrak{T}_{\mathfrak{g}}(\Omega)$, where $\mathfrak{T}_{\mathfrak{g}}(\Gamma) = (\Gamma, \mathcal{T}_{\mathfrak{g}}(\Gamma))$ is the $\mathcal{T}_{\mathfrak{g}}$-subspace of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}(\Omega) = (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$. If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$, then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$.

Proof. If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$, then $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}}(\Omega) \times \mathcal{T}_{\mathfrak{g}}(\Omega) \cup \neg\mathcal{T}_{\mathfrak{g}}(\Omega) \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$. Therefore, if $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$, then $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}} \cap \Gamma, \mathcal{O}_{\mathfrak{g}} \cap \Gamma, \mathcal{K}_{\mathfrak{g}} \cap \Gamma; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}}(\Gamma) \times \mathcal{T}_{\mathfrak{g}}(\Gamma) \cup \neg\mathcal{T}_{\mathfrak{g}}(\Gamma) \times \mathcal{L}_{\mathfrak{g}}[\Gamma] \times \{\subseteq, \supseteq\}$. But, since $\mathcal{S}_{\mathfrak{g}} \cap \Gamma = \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$, $\mathcal{O}_{\mathfrak{g}} \cap \Gamma = \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Gamma)$, and $\mathcal{K}_{\mathfrak{g}} \cap \Gamma = \mathcal{K}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Gamma)$, it follows that $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$ whenever $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$, and the theorem is proved. □

Definition 3.24 (Cartesian Product)**.** The Cartesian product of an arbitrary family $\{\Omega_\nu \subset \mathfrak{U} : \ \nu \in I_n^*\}$ of sets is the set of functions $\phi : I_n^* \longrightarrow \bigcup_{\nu \in I_n^*} \Omega_\nu$ such that $\phi : \nu \longmapsto \Omega_\nu$ for every $\nu \in I_n^*$. It is denoted by $\bigotimes_{\nu \in I_n^*} \Omega_\nu$ and satisfies the following properties:

i. $\bigotimes_{\nu=\mu} \Omega_\nu = \Omega_\mu \quad \forall \mu \in I_n^*$

ii. $\bigotimes_{\nu \in I_{\mu+1}^*} \Omega_\nu = \left(\bigotimes_{\nu \in I_\mu^*} \Omega_\nu\right) \times \Omega_{\mu+1} \quad \forall \mu \in I_{n-1}^*$

The projection map which gives the projection of the Cartesian product set $\bigotimes_{\nu \in I_n^*} \Omega_\nu$ onto the μ^{th} factor of $\bigotimes_{\nu \in I_n^*} \Omega_\nu$ is defined as thus.

Definition 3.25 (Projection)**.** Let $\{\Omega_\nu \subset \mathfrak{U} : \ \nu \in I_n^*\}$ be any class of sets and let $\bigotimes_{\nu \in I_n^*} \Omega_\nu$ denotes the Cartesian product of these sets. The map

$$\mathrm{proj}_\mu : \bigotimes_{\nu \in I_n^*} \Omega_\nu \longrightarrow \Omega_\mu \quad \left(\mathrm{proj}_\mu\left(\bigotimes_{\nu \in I_n^*} \Omega_\nu\right) = \Omega_\mu\right) \tag{29}$$

is called the projection of the Cartesian product set $\bigotimes_{\nu \in I_n^*} \Omega_\nu$ onto the μ^{th} factor of $\bigotimes_{\nu \in I_n^*} \Omega_\nu$.

To generate all $\mathcal{T}_{\mathfrak{g}}$-open sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, a basis $\mathcal{B}[\mathcal{T}_{\mathfrak{g}}]$ for $\mathfrak{T}_{\mathfrak{g}}$ must be supplied, and the following definition is worth considering.

Definition 3.26 ($\mathcal{T}_{\mathfrak{g}}$-Basis)**.** A subclass $\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_\mu)] \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega_\mu)$ consisting of $\mathcal{T}_{\mathfrak{g}}$-open sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}(\Omega_\mu) := (\Omega_\mu, \mathcal{T}_{\mathfrak{g}}(\Omega_\mu))$, defined by

$$\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_\mu)] := \{\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} : \ (\nu, \mu, \sigma(\nu,\mu)) \in I_\infty^* \times \{\mu\} \times I_\infty^*\} \tag{30}$$

is said to be a base for $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega_\mu) \longrightarrow \mathcal{P}(\Omega_\mu)$ if and only if

$$\forall\left(\mu, \sigma(\mu), \mathcal{O}_{\mathfrak{g},\sigma(\mu)}\right) \in \{\mu\} \times I_\infty^* \times \mathcal{T}_{\mathfrak{g}}(\Omega_\mu), \ \exists I_{\sigma(\mu)} \subseteq I_\infty^* : \mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \tag{31}$$

With regards to the terminology employed, $\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]$ is called a $\mathcal{T}_{\mathfrak{g}}$-basis and its elements, $\mathcal{B}_{\mathcal{T}_{\mathfrak{g}}}$-open sets, because they are $\mathcal{T}_{\mathfrak{g}}$-open sets of $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}\left(\Omega_{\mu}\right)\longrightarrow\mathcal{P}\left(\Omega_{\mu}\right)$. With regards to the definition itself, an immediate consequence follows. By the relation $\mathscr{O}_{\mathfrak{g},\sigma(\mu)}=\bigcup_{\nu\in I^{*}_{\sigma(\mu)}}\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}$, is meant, for every $\left(\nu,\mu,\sigma\left(\mu\right),\sigma\left(\nu,\mu\right)\right)\in I^{*}_{\sigma(\mu)}\times I^{*}_{n}\times I^{*}_{\infty}\times I^{*}_{\infty}$, that $\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\in\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]$ and $\mathscr{O}_{\mathfrak{g},\sigma(\mu)}\in\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)$ in the relation $\mathscr{O}_{\mathfrak{g},\sigma(\mu)}=\bigcup_{\nu\in I^{*}_{\sigma(\mu)}}\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}$, where $\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]$ and $\mathscr{O}_{\mathfrak{g},\sigma(\mu)}\in\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)$ are given by

$$\begin{aligned}&\mathrm{proj}_{\alpha}:\bigotimes\nolimits_{\mu\in I^{*}_{n}}\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]\longrightarrow\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\alpha}\right)\right]\\&\mathrm{proj}_{\alpha}:\bigotimes\nolimits_{\mu\in I^{*}_{n}}\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\longrightarrow\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\alpha}\right)\quad\forall\alpha\in I^{*}_{n}\end{aligned}\tag{32}$$

respectively. To this end, a Cartesian product topology (Cartesian $\mathcal{T}_{\mathfrak{g}}$-product) is one that having for $\mathcal{T}_{\mathfrak{g}}$-basis all $\mathcal{B}_{\mathcal{T}_{\mathfrak{g}}}$-open sets of the form $\mathrm{proj}^{-1}_{\mu}\left(\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\right)$, where $\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\in\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]$ for every $\left(\nu,\mu,\sigma\left(\nu,\mu\right)\right)\in I^{*}_{\sigma(\mu)}\times I^{*}_{n}\times I^{*}_{\infty}$. Therefore, in order to define a Cartesian product $\mathcal{T}_{\mathfrak{g}}$-space, it suffices to take the above descriptions into account and postulate a proper definition on this ground. The following definition presents itself.

Definition 3.27. Let $\left\{\mathfrak{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right):=\left(\Omega_{\mu},\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right):\ \mu\in I^{*}_{n}\right\}$ be a class of $n\geq 1$ $\mathcal{T}_{\mathfrak{g}}$-spaces and, for every $\mu\in I^{*}_{n}$, let $\mathcal{T}_{\mathfrak{g},\Omega_{\mu}}:\mathcal{P}\left(\Omega_{\mu}\right)\longrightarrow\mathcal{P}\left(\Omega_{\mu}\right)$ be the $\mathfrak{g}$-topology for $\mathfrak{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)$. The Cartesian $\mathcal{T}_{\mathfrak{g}}$-product $:=\bigotimes_{\mu\in I^{*}_{n}}\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)$ on the Cartesian product set $\Omega:=\bigotimes_{\mu\in I^{*}_{n}}\Omega_{\mu}$ is that having for $\mathcal{T}_{\mathfrak{g}}$-basis all $\mathcal{B}_{\mathcal{T}_{\mathfrak{g}}}$-open sets belonging to the following class:

$$\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega\right)\right]:=\left\{\mathrm{proj}^{-1}_{\mu}\left(\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\right):\ \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\in\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]\forall\left(\nu,\mu,\sigma\left(\nu,\mu\right)\right)\in I^{*}_{\sigma(\mu)}\times I^{*}_{n}\times I^{*}_{\infty}\right\}\tag{33}$$

The structure $\mathfrak{T}_{\mathfrak{g}}\left(\Omega\right):=\left(\Omega,\mathcal{T}_{\mathfrak{g}}\left(\Omega\right)\right)$ is called a "Cartesian product $\mathcal{T}_{\mathfrak{g}}$-space."

The fact that $\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\in\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]$ and $\mathscr{O}_{\mathfrak{g},\sigma(\mu)}\in\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)$ hold for every $\left(\nu,\mu,\sigma\left(\mu\right),\sigma\left(\nu,\mu\right)\right)\in I^{*}_{\sigma(\mu)}\times I^{*}_{n}\times I^{*}_{\infty}\times I^{*}_{\infty}$ makes it reasonable to write

$$\begin{aligned}\bigotimes\nolimits_{\mu\in I^{*}_{n}}\mathscr{O}_{\mathfrak{g},\sigma(\mu)}&\in\bigotimes\nolimits_{\mu\in I^{*}_{n}}\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\\\bigotimes\nolimits_{\mu\in I^{*}_{n}}\left(\bigcup\nolimits_{\nu\in I^{*}_{\sigma(\mu)}}\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\right)&=\bigcup_{\vec{\nu}\in\bigotimes_{\alpha\in I^{*}_{n}}I^{*}_{\sigma(\alpha)}}\left(\bigotimes\nolimits_{\alpha\in I^{*}_{n}}\mathscr{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}\right)\\&\in\bigotimes\nolimits_{\mu\in I^{*}_{n}}\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]\end{aligned}\tag{34}$$

where $\vec{\nu}:=\left(\nu_{1},\nu_{2},\ldots,\nu_{n}\right)$ and, for every $\alpha\in I^{*}_{n}$, $\nu_{\alpha}\in I^{*}_{\sigma(\alpha)}$. An immediate consequence of such relation is contained in the following lemma.

Lemma 3.28. If $\mathcal{T}_{\mathfrak{g}}:\mathscr{Q}\big(\Omega\big)\longrightarrow\mathscr{Q}\big(\Omega\big)$ is a one-valued map on the Cartesian product set $\Omega=\bigotimes_{\mu\in I^{*}_{n}}\Omega_{\mu}$, where

$$\mathscr{Q}\left(\Omega\right):=\left\{\mathscr{O}_{\mathfrak{g},\sigma}=\bigcup_{\vec{\nu}\in\bigotimes_{\alpha\in I^{*}_{n}}I^{*}_{\sigma(\alpha)}}\left(\bigotimes\nolimits_{\alpha\in I^{*}_{n}}\mathscr{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}\right):\quad\mathscr{O}_{\mathfrak{g},\sigma}\in\bigotimes\nolimits_{\mu\in I^{*}_{n}}\mathcal{B}\left[\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)\right]\right\}\tag{35}$$

then $\mathcal{T}_{\mathfrak{g}}:\mathscr{Q}\left(\Omega\right)\longrightarrow\mathscr{Q}\left(\Omega\right)$ is a $\mathfrak{g}$-topology on the Cartesian product set $\bigotimes_{\mu\in I^{*}_{n}}\Omega_{\mu}$.

Proof. Let $\mathscr{O}_{\mathfrak{g},\sigma}=\bigcup_{\vec{\nu}\in\bigotimes_{\alpha\in I^{*}_{n}}I^{*}_{\sigma(\alpha)}}\left(\bigotimes_{\alpha\in I^{*}_{n}}\mathscr{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}\right)$. Since $\mathscr{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}\in\mathcal{T}_{\mathfrak{g}}\left(\Omega_{\mu}\right)$ holds true for every $\left(\nu_{\alpha},\alpha,\sigma\left(\nu_{\alpha},\alpha\right)\right)\in I^{*}_{\sigma(\alpha)}\times I^{*}_{n}\times I^{*}_{\infty}$, it is evident that $\mathscr{O}_{\mathfrak{g},\sigma}=\emptyset$ only if, for every $\left(\nu_{\alpha},\alpha,\sigma\left(\nu_{\alpha},\alpha\right)\right)\in I^{*}_{\sigma(\alpha)}\times I^{*}_{n}\times I^{*}_{\infty}$, $\mathscr{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}=\emptyset$. Thus, $\mathcal{T}_{\mathfrak{g}}\left(\emptyset\right)=\emptyset$.

Let $\mathscr{O}_{\mathfrak{g},\sigma}=\bigcup_{\vec{\nu}\in\bigotimes_{\alpha\in I^{*}_{n}}I^{*}_{\sigma(\alpha)}}\left(\bigotimes_{\alpha\in I^{*}_{n}}\mathscr{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}\right)$. Then, since $\mathscr{Q}\left(\Omega\right)\subseteq\mathscr{Q}\left(\Omega\right)$, it follows that $\mathscr{O}_{\mathfrak{g},\sigma}$ is a superset of $\mathcal{T}_{\mathfrak{g}}\left(\Omega\right)$. Thus, $\mathcal{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\subseteq\mathscr{O}_{\mathfrak{g},\sigma}$.

Let $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ and $\vec{\kappa} = (\kappa_1, \ldots, \kappa_n)$, and consider

$$\begin{aligned} \mathcal{O}_{\mathfrak{g},\sigma} &= \bigcup_{\vec{\nu} \in \bigotimes_{\alpha \in I_n^*} I^*_{\sigma(\alpha)}} \left(\bigotimes_{\alpha \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\right) \\ \mathcal{O}_{\mathfrak{g},\tau} &= \bigcup_{\vec{\kappa} \in \bigotimes_{\beta \in I_n^*} I^*_{\tau(\beta)}} \left(\bigotimes_{\beta \in I_n^*} \mathcal{O}_{\mathfrak{g},\tau\left(\kappa_\beta,\beta\right)}\right) \end{aligned}$$

Further, let us assume that $\vec{\vec{\eta}} = (\nu_1, \ldots, \nu_n, \kappa_1, \ldots, \kappa_n)$, $\mathbb{I}^*_{\sigma(\alpha)} := \bigotimes_{\alpha \in I_n^*} I^*_{\sigma(\alpha)}$, and $\mathbb{I}^*_{\sigma(\beta)} := \bigotimes_{\beta \in I_n^*} I^*_{\tau(\beta)}$. Then,

$$\mathcal{O}_{\mathfrak{g},\sigma} \cup \mathcal{O}_{\mathfrak{g},\tau} = \bigcup_{\vec{\vec{\eta}} \in \mathbb{I}^*_{\sigma(\alpha)} \times \mathbb{I}^*_{\sigma(\beta)}} \left(\bigotimes_{\mu \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\right) \cup \left(\bigotimes_{\beta \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma\left(\kappa_\beta,\beta\right)}\right)$$

Thus, $\mathcal{T}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\sigma} \cup \mathcal{O}_{\mathfrak{g},\tau}\right) \subseteq \mathcal{T}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right) \cup \mathcal{T}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\tau}\right)$. □

Theorem 3.29. Let $\mathfrak{T}_{\mathfrak{g},1}(\Omega), \mathfrak{T}_{\mathfrak{g},2}(\Omega), \ldots, \mathfrak{T}_{\mathfrak{g},n}(\Omega)$ be $n \geq 1$ $\mathcal{T}_{\mathfrak{g}}$-spaces and let $\mathfrak{T}_{\mathfrak{g}}(\Omega) := \bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)$ be the $\mathcal{T}_{\mathfrak{g}}$-space product. If the relation $(\mathcal{S}_{\mathfrak{g},1}, \ldots, \mathcal{S}_{\mathfrak{g},n}) \in \bigotimes_{\nu \in I_n^*} \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\nu}\right]$ holds, then $\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}\left[\bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)\right]$.

Proof. For every $\sigma \in I_n^*$, let

$$\mathbf{op}_{\mathfrak{g},12\cdots\sigma}(\cdot) = \left(\mathrm{op}_{\mathfrak{g},12\cdots\sigma}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g},12\cdots\sigma}(\cdot)\right) \in \mathcal{L}_{\mathfrak{g},12\cdots\sigma}\left[\Omega\right]$$

denotes the $\mathfrak{g}$-operator in $\bigotimes_{\nu \in I_\sigma^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)$ and, for every $\nu \in I_n^*$, let $(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\nu}\right] \times \mathcal{T}_{\mathfrak{g},\nu} \times \neg\mathcal{T}_{\mathfrak{g},\nu}$. Then,

$$\begin{aligned} \mathrm{op}_{\mathfrak{g},12\cdots n}\left(\bigotimes_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}\right) &= \bigotimes_{\nu \in I_n^*} \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\nu}\right) \\ \neg\,\mathrm{op}_{\mathfrak{g},12\cdots n}\left(\bigotimes_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}\right) &= \bigotimes_{\nu \in I_n^*} \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\nu}\right) \end{aligned}$$

On the other hand, for every $\nu \in I_n^*$, the logical statement

$$\left(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)$$

holds in $\mathfrak{T}_{\mathfrak{g},\nu}$. Consequently,

$$\begin{aligned} & \bigotimes_{\nu \in I_n^*}\left(\left(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right)\right) \\ \Rightarrow \quad & \left(\left(\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \bigotimes_{\nu \in I_n^*} \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \bigotimes_{\nu \in I_n^*} \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)\right) \\ \Rightarrow \quad & \left(\left(\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g},12\cdots n}\left(\bigotimes_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},12\cdots n}\left(\bigotimes_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}\right)\right)\right) \end{aligned}$$

Therefore, the Boolean-valued functions

$$\mathrm{P}_{\mathfrak{g}}\left(\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}, \bigotimes_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}, \bigotimes_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}; \mathbf{op}_{\mathfrak{g},12\cdots n}(\cdot); \subseteq, \supseteq\right)$$

holds on $\mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g},12\cdots n}\left[\Omega\right] \times \left\{\subseteq, \supseteq\right\}$ and, hence, it follows that

$$\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-G}\left[\bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)\right]$$

□

The categorical classifications of $\mathfrak{T}$-sets and $\mathfrak{g}$-$\mathfrak{T}$-sets in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and, $\mathfrak{T}_{\mathfrak{g}}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and diagrammed on this ground in the next sections.

4. Discussion

4.1. Categorical Classifications

Having adopted a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the twofold purposes here are to establish the various relationships between the classes of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and to illustrate them through diagrams.

We have seen that, $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. But, $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}], \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$; $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. In Figure 1, we present the relationships between the class $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets and the class $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

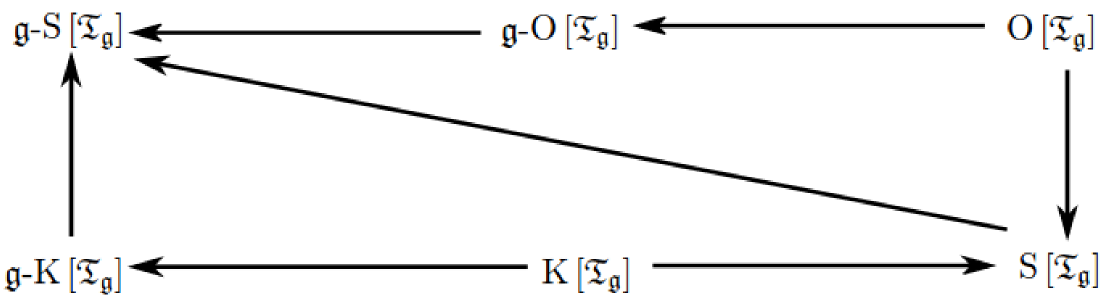

Fig. 1. Relationships: classes of $\mathfrak{T}_{\mathfrak{g}}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets

It is plain that $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}]$ and $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\nu \in I_3^0$. Moreover, it is also clear that, $\mathfrak{g}\text{-}2\text{-}\mathrm{O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}3\text{-}\mathrm{O}[\mathfrak{T}]$ and $\mathfrak{g}\text{-}0\text{-}\mathrm{O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}1\text{-}\mathrm{O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}3\text{-}\mathrm{O}[\mathfrak{T}]$, and $\mathfrak{g}\text{-}2\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}3\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}0\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}1\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}3\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$. In fact, for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ holds. Consequently,

$$\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \quad \forall \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} \tag{36}$$

In Figure 2, we present the relationships between the class $\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of categories 0, 1, 2 and 3 in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and the class $\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}]$ of $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2 and 3 in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$.

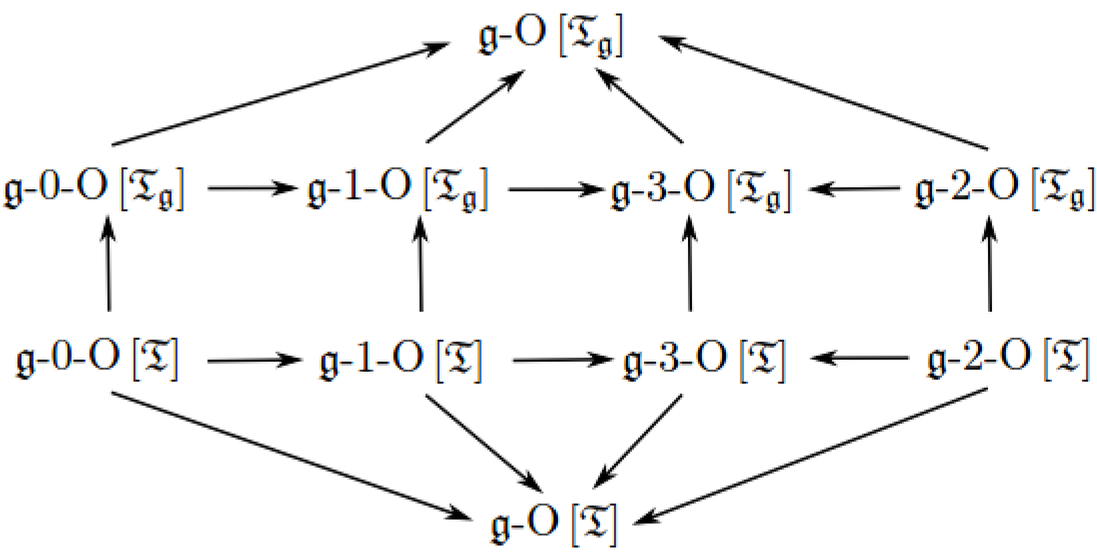

Fig. 2. Relationships: classes of $\mathfrak{g}$-$\mathfrak{T}$-open sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets

It is plain that, $\mathfrak{g}$-ν-K$[\mathfrak{T}] \subseteq \mathfrak{g}$-K$[\mathfrak{T}]$ and $\mathfrak{g}$-ν-K$[\mathfrak{T}] \subseteq \mathfrak{g}$-$\nu$-K$[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}]$ for every $\nu \in I_3^0$. Moreover, it is also clear that, $\mathfrak{g}$-2-K$[\mathfrak{T}] \subseteq \mathfrak{g}$-3-K$[\mathfrak{T}]$ and $\mathfrak{g}$-0-K$[\mathfrak{T}] \subseteq \mathfrak{g}$-1-K$[\mathfrak{T}] \subseteq \mathfrak{g}$-3-K$[\mathfrak{T}]$, and $\mathfrak{g}$-2-K$[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}$-3-K$[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-0-K$[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}$-1-K$[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}$-3-K$[\mathfrak{T}_{\mathfrak{g}}]$. Because, for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relations $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ holds. Consequently,

$$\neg\,\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \subseteq \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \;\; \forall \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} \tag{37}$$

In Figure 3, we present the relations between the class $\mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}$-$\nu$-K$[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories 0, 1, 2 and 3 in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and the class $\mathfrak{g}$-K$[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}$-$\nu$-K$[\mathfrak{T}]$ of $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2 and 3 in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$.

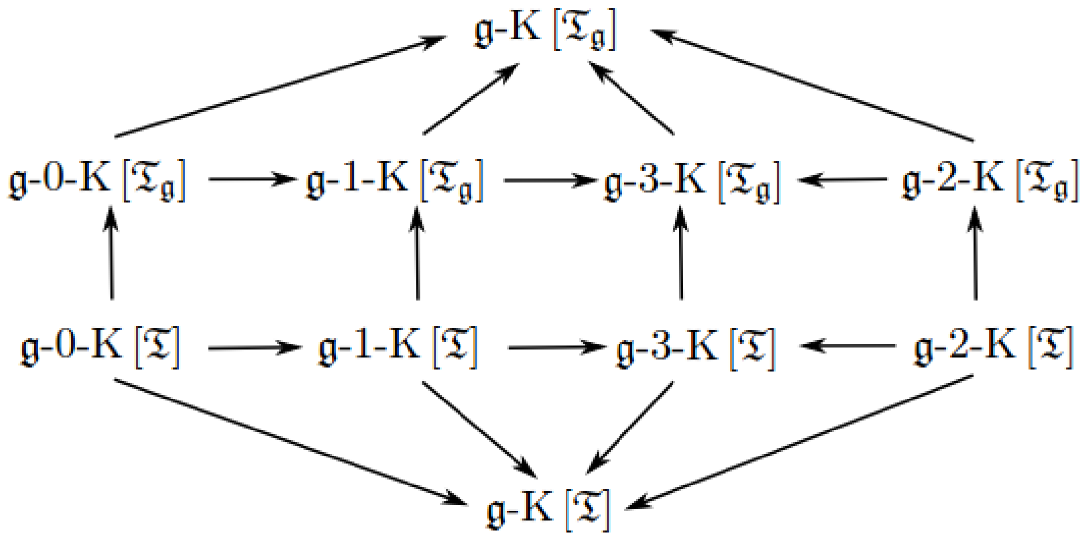

Fig. 3. Relationships: classes of $\mathfrak{g}$-$\mathfrak{T}$-closed sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets

As in the papers of Caldas et al. [42], Dontchev [43], Jun et al. [8], and Tyagi et al. [6], among others, the manner we have positioned the arrows is solely to stress that, in general, none of the implications in Figs 1, 2 and 3 is reversible.

At this stage, a nice application is worth considering, and is presented in the following section.

4.2. A Nice Application

Concentrating on fundamental concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, we shall now present a nice application. Let $\Omega = \{\xi_\nu : \nu \in I_5^*\}$ denotes the underlying set and consider the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, where

$$\begin{aligned} \mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset, \{\xi_1\}, \{\xi_3, \xi_4\}, \{\xi_1, \xi_3, \xi_4\}\} \\ &= \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \mathcal{O}_{\mathfrak{g},3}, \mathcal{O}_{\mathfrak{g},4}\} \qquad (38) \\ \neg\mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\Omega, \{\xi_2, \xi_3, \xi_4, \xi_5\}, \{\xi_1, \xi_2, \xi_5\}, \{\xi_2, \xi_5\}\} \\ &= \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},4}\} \qquad (39) \end{aligned}$$

respectively, stand for the classes of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets. Since conditions $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathcal{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, and $\mathcal{T}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_4^*} \mathcal{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu \in I_4^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ are satisfied, it is clear that the one-valued map $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\{\xi_\nu : \nu \in I_5^*\})$ is a $\mathfrak{g}$-topology. Furthermore, it is easily checked that, $\mathcal{O}_{\mathfrak{g},\mu} \in \mathfrak{g}$-$\nu$-O$[\mathfrak{T}]$ for every $(\nu, \mu) \in I_3^0 \times I_4^*$. Hence, the $\mathcal{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ are $\mathfrak{g}$-$\mathfrak{T}$-open sets relative to the $\mathcal{T}$-space $\mathfrak{T} = (\Omega, \mathcal{T})$.

After calculations, the classes $\mathfrak{g}$-ν-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathfrak{g}$-ν-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, respectively, of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories $\nu \in \{0, 2\}$ then take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] &= \mathcal{T}_{\mathfrak{g}} \cup \big\{\{\xi_3\}, \{\xi_4\}, \{\xi_1, \xi_3\}, \{\xi_1, \xi_4\}\big\} \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] &= \neg\mathcal{T}_{\mathfrak{g}} \cup \big\{\{\xi_2, \xi_4, \xi_5\}, \{\xi_1, \xi_2, \xi_3, \xi_5\} \\ &\qquad \{\xi_1, \xi_2, \xi_4, \xi_5\}, \{\xi_2, \xi_3, \xi_5\}\big\}, \quad \forall \nu \in \{0, 2\} \end{aligned} \tag{40}$$

On the other hand, those of categories $\nu \in \{1, 3\}$ take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] &= \mathcal{T}_{\mathfrak{g}} \cup \big\{\mathcal{O}_{\mathfrak{g}} :\ \mathcal{O}_{\mathfrak{g}} \in \mathcal{P}(\Omega) \setminus \mathcal{T}_{\mathfrak{g}}\big\} \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] &= \neg\mathcal{T}_{\mathfrak{g}} \cup \big\{\mathcal{K}_{\mathfrak{g}} :\ \mathcal{K}_{\mathfrak{g}} \in \mathcal{P}(\Omega) \setminus \neg\mathcal{T}_{\mathfrak{g}}\big\}, \quad \forall \nu \in \{1, 3\} \end{aligned} \tag{41}$$

The discussions carried out in the preceding sections can be easily verified from this nice application. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets discussed in the preceding sections.

5. Conclusion

In this paper, we developed a new theory, called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Sets*. In its own rights, the proposed theory has several advantages. The very first advantage is that the theory holds equally well when $(\Omega, \mathcal{T}_{\mathfrak{g}}) = (\Omega, \mathcal{T})$ and other features adapted on this basis, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$-*Sets*. Hence, in a $\mathcal{T}_{\mathfrak{g}}$-space the theoretical framework categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-closed sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-pre-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-pre-closed sets, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-pre-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-pre-closed sets as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a $\mathcal{T}$-space it categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets, $\mathfrak{g}$-$\mathfrak{T}$-semi-open and $\mathfrak{g}$-$\mathfrak{T}$-semi-closed sets, $\mathfrak{g}$-$\mathfrak{T}$-pre-open and $\mathfrak{g}$-$\mathfrak{T}$-pre-closed sets, and $\mathfrak{g}$-$\mathfrak{T}$-semi-pre-open and $\mathfrak{g}$-$\mathfrak{T}$-semi-pre-closed sets as $\mathfrak{g}$-$\mathfrak{T}$-sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of mixed categories. More precisely, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\big\{\mathcal{O}_{\mathfrak{g}} = \mathcal{O}_{\mathfrak{g},\nu} \cup \mathcal{O}_{\mathfrak{g},\mu} :\ (\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ and the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets belonging to the class $\big\{\mathcal{K}_{\mathfrak{g}} = \mathcal{K}_{\mathfrak{g},\nu} \cup \mathcal{K}_{\mathfrak{g},\mu} :\ (\mathcal{K}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, as Andrijević [22] and Caldas et al. [44] developed the theory of b-open and b-closed sets in a $\mathcal{T}$-space $\mathfrak{T}$. Such two theories are what we thought would certainly be worth considering, and the discussion of this work ends here.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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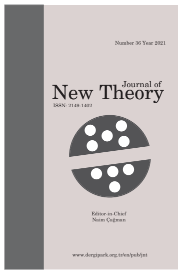

Roots of Second Order Polynomials with Real Coefficients in Elliptic Scator Algebra

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Abstract — The roots of second order polynomials with real coefficients are obtained in the $\mathbb{S}^{1+2}$ scator set. Explicit formulae are computed in terms of the polynomial coefficients. Although the scator product does not distribute over addition, the lack of distributivity is surmountable in order to find the zeros of the polynomial. The structure of the solutions and their distribution in 1+2 dimensional scator space are illustrated and discussed. There exist six, two, or eight solutions, depending on the value of polynomial coefficients. Four of these roots only exist in the hypercomplex $\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$ set.

Keywords — *Quadratic polynomial solutions, non-distributive algebras, hypercomplex numbers*

Mathematics Subject Classification (2020) — 30G35, 20M14

1. Introduction

The quadratic equation $ax^2 + bx + c = 0$, $x \in \mathbb{H}$ in the quaternion set with real coefficients a, b, c: i) if $4ac \leq b^2$, has real solutions $x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$ and ii) if $4ac > b^2$, has an infinite number of solutions $x = -b + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$, with $\beta^2 + \gamma^2 + \delta^2 = 4c - b^2$ and $\beta, \gamma, \delta \in \mathbb{R}$ [1, 2]. In contrast, as we shall presently see, the quadratic equation with real coefficients in elliptic scator algebra has a finite number of roots.

Scator numbers are compound numbers that have one scalar component and n director components in $\mathbb{R}^{1+n}$, $\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j$, where $f_0, f_j \in \mathbb{R}$ for j from 1 to n in $\mathbb{N}$ and $\check{\mathbf{e}}_j \notin \mathbb{R}$. Scator addition, performed component-wise, satisfies commutative group properties. Multiplication in elliptic scator algebra is commutative, possesses an identity element and all elements are invertible if zero is excluded. Multiplication is not associative if the additive scalar component of any two products vanish [3]. However, the non associative products can be isolated by imposing the appropriate conditions (i.e. $a_j b_j \neq a_0 b_0$ in (3a)), so that the additive scalar component of the products does not vanish. In general, scator multiplication is not distributive over addition [4]. Scator algebra has been successfully applied to several problems: time-space description in a deformed Lorentz metric [5], a wave-function evolution and collapse unified description in quantum mechanics [3] and three dimensional fractal structures [6]. Explicit formulae for scator holomorphic functions recently published, will very likely expand applications to other areas [7].

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1.1. Elliptic Scators

Scator multiplication is defined in the scator set,

$$\mathbb{S}^{1+n} = \left\{ \overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j, (f_0; \ldots f_j, \ldots f_n) \in \mathbb{R}^{1+n} : f_0 \neq 0, \text{ if there exists } f_j f_l \neq 0 \text{ for } j \neq l \in n \right\} \tag{1}$$

where the scalar component must be non-zero if two or more director components are non-zero. In particular, in $1+2$ dimensions,

$$\mathbb{S}^{1+2} = \left\{ \overset{o}{\varphi} = f_0 + f_1 \check{\mathbf{e}}_1 + f_2 \check{\mathbf{e}}_2, (f_0; f_1, f_2) \in \mathbb{R}^{1+2} : f_0 \neq 0, \text{ if } f_1 f_2 \neq 0 \right\} \tag{2}$$

The $\mathbb{S}_j^{1+1}$ scator set is isomorphic to the complex set $\mathbb{C}$ for the real component and any one of the j director components. There are n copies of the complex set embedded in $\mathbb{S}^{1+n}$ sharing the real part (named the scalar component in scator algebra) and having n different hyper-imaginary parts. In this communication, we shall restrict to scators in $1+2$ dimensions.

Definition 1.1. The product of two scators $\overset{o}{\alpha} = a_0 + a_1 \check{\mathbf{e}}_1 + a_2 \check{\mathbf{e}}_2 \in \mathbb{S}^{1+2}$ and $\overset{o}{\beta} = b_0 + b_1 \check{\mathbf{e}}_1 + b_2 \check{\mathbf{e}}_2 \in \mathbb{S}^{1+2}$ is defined by:

If $a_0 b_0 \neq 0$,

$$\overset{o}{\alpha}\overset{o}{\beta} = a_0 b_0 \left(1 - \frac{a_1 b_1}{a_0 b_0}\right)\left(1 - \frac{a_2 b_2}{a_0 b_0}\right) + (a_0 b_0 - a_2 b_2)\left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right)\check{\mathbf{e}}_1 + (a_0 b_0 - a_1 b_1)\left(\frac{a_2}{a_0} + \frac{b_2}{b_0}\right)\check{\mathbf{e}}_2 \tag{3a}$$

If $a_0 = 0$ and $b_0 \neq 0$,

$$(a_1 \check{\mathbf{e}}_1)\overset{o}{\beta} = -a_1 b_1 + b_0 a_1 \check{\mathbf{e}}_1 - \left(\frac{a_1 b_1 b_2}{b_0}\right)\check{\mathbf{e}}_2 \tag{3b}$$

$$(a_2 \check{\mathbf{e}}_2)\overset{o}{\beta} = -a_2 b_2 + b_0 a_2 \check{\mathbf{e}}_2 - \left(\frac{a_2 b_2 b_1}{b_0}\right)\check{\mathbf{e}}_1 \tag{3c}$$

If $a_0 = b_0 = 0$,

$$a_1 \check{\mathbf{e}}_1 b_2 \check{\mathbf{e}}_2 = 0, \qquad a_1 \check{\mathbf{e}}_1 b_1 \check{\mathbf{e}}_1 = -a_1 b_1, \qquad a_2 \check{\mathbf{e}}_2 b_2 \check{\mathbf{e}}_2 = -a_2 b_2 \tag{3d}$$

The conjugate of a scator $\overset{o}{\varphi} = f_0 + f_1 \check{\mathbf{e}}_1 + f_2 \check{\mathbf{e}}_2 \in \mathbb{S}^{1+2}$ is obtained by reversing the sign of all the director components while leaving the scalar component unaltered, $\overset{o}{\varphi}^* = f_0 - f_1 \check{\mathbf{e}}_1 - f_2 \check{\mathbf{e}}_2$. The magnitude of a scator $\|\overset{o}{\varphi}\| \in \mathbb{R}$, is given by the positive square root of the scator times its conjugate

$$\|\overset{o}{\varphi}\|^2 = \overset{o}{\varphi}\overset{o}{\varphi}^* = f_0^2 \left(1 + \frac{f_1^2}{f_0^2}\right)\left(1 + \frac{f_2^2}{f_0^2}\right) \tag{4}$$

if $f_0 \neq 0$, and $\|\overset{o}{\varphi}\|^2 = f_j^2$ if $f_0 = 0$.

2. Roots of the Scator Quadratic Polynomial with Real Coefficients

Consider the second order polynomial

$$a \overset{o}{\varphi}^2 + b \overset{o}{\varphi} + c = 0 \tag{5}$$

where $\overset{o}{\varphi}$ is a scator element and $a, b, c \neq 0$ are real numbers. This polynomial cannot be factorized into $(\overset{o}{\varphi} - \overset{o}{r}_1)(\overset{o}{\varphi} - \overset{o}{r}_2) = 0$, since the scator product does not distribute over addition if $\overset{o}{\varphi} \in \mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$. Furthermore, the product $\overset{o}{\alpha}\overset{o}{\beta} = 0$ does not imply that $\overset{o}{\alpha} = 0$ or $\overset{o}{\beta} = 0$. The product of two scators $\overset{o}{\alpha}, \overset{o}{\beta}$ is zero if their components satisfy the conditions

$$a_0 b_0 = a_1 b_1 = a_2 b_2 \tag{6}$$

as may be seen from direct computation from the product definition (3a). For a given scator $\overset{o}{\alpha} \in \mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$, there always exists $\overset{o}{\beta}$ such that (6) is satisfied. Thus, all elements are zero divisors in the Bourbaki sense [8, p.98]. Nonetheless, it is possible to solve the polynomial equation in the scator domain without performing a factorization.

Theorem 2.1. The second order polynomial $a\overset{o}{\varphi}^2 + b\overset{o}{\varphi} + c = 0$, where $\overset{o}{\varphi} \in \mathbb{S}^{1+2}$ is an elliptic scator and $a, b, c \neq 0$ are real coefficients, has the following roots:

If $|4ac| > b^2$, then

$$\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} = -\frac{4ac + b^2}{4ab} \pm \sqrt{\frac{(4ac)^2 - (b^2)^2}{16a^2b^2}}\check{\mathbf{e}}_1 \pm \sqrt{\frac{(4ac)^2 - (b^2)^2}{16a^2b^2}}\check{\mathbf{e}}_2 \tag{7a}$$

If $4ac \leq b^2$, then

$$\overset{o}{\varphi}_{\mathbb{S}^{1+0}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{7b}$$

If $4ac > b^2$, then

$$\overset{o}{\varphi}_{\mathbb{S}_1^{1+1}} = -\frac{b}{2a} \pm \frac{\sqrt{-b^2 + 4ac}}{2a}\check{\mathbf{e}}_1, \quad \overset{o}{\varphi}_{\mathbb{S}_2^{1+1}} = -\frac{b}{2a} \pm \frac{\sqrt{-b^2 + 4ac}}{2a}\check{\mathbf{e}}_2 \tag{7c}$$

Proof. Consider the scator $\overset{o}{\varphi} = f_0 + f_1\check{\mathbf{e}}_1 + f_2\check{\mathbf{e}}_2 \in \mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$, with non-vanishing components $f_0, f_1, f_2 \neq 0$. From the product definition (3a), the square of this scator is,

$$\overset{o}{\varphi}^2 = f_0^2\left(1 - \frac{f_1^2}{f_0^2}\right)\left(1 - \frac{f_2^2}{f_0^2}\right) + 2f_0f_1\left(1 - \frac{f_2^2}{f_0^2}\right)\check{\mathbf{e}}_1 + 2f_0f_2\left(1 - \frac{f_1^2}{f_0^2}\right)\check{\mathbf{e}}_2$$

The polynomial (5), is then

$$a\left[f_0^2\left(1 - \frac{f_1^2}{f_0^2}\right)\left(1 - \frac{f_2^2}{f_0^2}\right) + 2f_0f_1\left(1 - \frac{f_2^2}{f_0^2}\right)\check{\mathbf{e}}_1 + 2f_0f_2\left(1 - \frac{f_1^2}{f_0^2}\right)\check{\mathbf{e}}_2\right] + b\left(f_0 + f_1\check{\mathbf{e}}_1 + f_2\check{\mathbf{e}}_2\right) + c = 0 \tag{8}$$

Grouping components,

$$af_0^2\left(1 - \frac{f_1^2}{f_0^2}\right)\left(1 - \frac{f_2^2}{f_0^2}\right) + bf_0 + c + \left(2af_0f_1\left(1 - \frac{f_2^2}{f_0^2}\right) + bf_1\right)\check{\mathbf{e}}_1 + \left(2af_0f_2\left(1 - \frac{f_1^2}{f_0^2}\right) + bf_2\right)\check{\mathbf{e}}_2 = 0$$

Two scators are equal if and only if all its additive components are equal. In particular, since the zero scator is $0 = \overset{o}{0} = 0 + 0\,\check{\mathbf{e}}_1 + 0\,\check{\mathbf{e}}_2$, the scalar component of the polynomial equation is then

$$af_0^2\left(1 - \frac{f_1^2}{f_0^2}\right)\left(1 - \frac{f_2^2}{f_0^2}\right) + bf_0 + c = 0 \tag{9a}$$

whereas the director components equations are

$$2af_0f_1\left(1 - \frac{f_2^2}{f_0^2}\right) + bf_1 = 0 \tag{9b}$$

$$2af_0f_2\left(1 - \frac{f_1^2}{f_0^2}\right) + bf_2 = 0 \tag{9c}$$

Since f_1 and f_2 are both different from zero, Eq. (9b) can be multiplied by $\frac{1}{f_1}$ and (9c) by $\frac{1}{f_2}$, to obtain

$$2af_0\left(1 - \frac{f_2^2}{f_0^2}\right) + b = 0 \tag{10a}$$

$$2af_0\left(1 - \frac{f_1^2}{f_0^2}\right) + b = 0 \tag{10b}$$

The square of the two director components must then be equal, $f_1^2 = f_2^2$. The term $1 - \frac{f_2^2}{f_0^2}$ can be written in terms of f_0 from (10a) or (10b) , $1 - \frac{f_2^2}{f_0^2} = -\frac{b}{2af_0}$. The scalar component is then

$$f_0 = -\left(\frac{c}{b} + \frac{b}{4a}\right)$$

and the director components are

$$f_2 = f_1 = \pm\sqrt{\frac{c^2}{b^2} - \frac{b^2}{16a^2}}$$

The four hyper-complex roots with non-vanishing director components are then

$$\overset{o}{\varphi}_{\mathbb{S}^{1+2}\backslash\mathbb{S}^{1+1}} = -\left(\frac{c}{b} + \frac{b}{4a}\right) \pm \sqrt{\frac{c^2}{b^2} - \frac{b^2}{16a^2}}\breve{\mathbf{e}}_1 \pm \sqrt{\frac{c^2}{b^2} - \frac{b^2}{16a^2}}\breve{\mathbf{e}}_2$$

These roots can be written in a similar form to the complex solutions as in equation (7a). To establish the interval where these roots exist, recall that the director coefficients were assumed to be non zero, thus $(4ac)^2 \neq \left(b^2\right)^2$. However, non-vanishing director coefficients also imply the inequality $(4ac)^2 > \left(b^2\right)^2$: Assume $(4ac)^2 < \left(b^2\right)^2$, since the radicand is negative, $\sqrt{(4ac)^2 - (b^2)^2} = \sqrt{(b^2)^2 - (4ac)^2}\sqrt{-1}$, where $\sqrt{(b^2)^2 - (4ac)^2} \in \mathbb{R}$. The root of minus one is any of the imaginary director units in scator algebra, let $\sqrt{-1} = \sqrt{\breve{\mathbf{e}}_1\breve{\mathbf{e}}_1} = \pm\breve{\mathbf{e}}_1$. The first director term in (7a) is then $\sqrt{\frac{\left((b^2)^2 - (4ac)^2\right)}{16a^2b^2}}\breve{\mathbf{e}}_1\breve{\mathbf{e}}_1 = -\sqrt{\frac{\left((b^2)^2 - (4ac)^2\right)}{16a^2b^2}} \in \mathbb{R}$, thus this director term is zero. For the second director term in (7a), $\sqrt{\frac{\left((b^2)^2 - (4ac)^2\right)}{16a^2b^2}}\breve{\mathbf{e}}_1\breve{\mathbf{e}}_2 = 0$, the director term is again zero. Therefore the two director coefficients vanish for $(4ac)^2 < \left(b^2\right)^2$.
If one of the director components is zero, then $\overset{o}{\varphi} = f_0 + f_1\breve{\mathbf{e}}_1 \in \mathbb{S}_1^{1+1}$ or $\overset{o}{\varphi} = f_0 + f_2\breve{\mathbf{e}}_2 \in \mathbb{S}_2^{1+1}$.
If $f_2 = 0$, the polynomial (8) is

$$a\left[\left(f_0^2 - f_1^2\right) + 2f_0f_1\breve{\mathbf{e}}_1\right] + b\left(f_0 + f_1\breve{\mathbf{e}}_1\right) + c = 0$$

and the real and $\breve{\mathbf{e}}_1$ equations are $a\left(f_0^2 - f_1^2\right) + bf_0 + c = 0$, and $2f_0f_1 + bf_1 = 0$, respectively.
If $f_1 \neq 0$, $f_0 = -\frac{b}{2a}$, the solutions if $4ac > b^2$ are then,

$$\overset{o}{\varphi}_{\mathbb{S}_1^{1+1}} = f_0 + f_1\breve{\mathbf{e}}_1 = -\frac{b}{2a} \pm \frac{\sqrt{-b^2 + 4ac}}{2a}\breve{\mathbf{e}}_1$$

If $4ac \leq b^2$, $\sqrt{-b^2 + 4ac} = \sqrt{b^2 - 4ac}\,\breve{\mathbf{e}}_1$, and thus

$$\overset{o}{\varphi}_{\mathbb{S}_1^{1+1}} = -\frac{b}{2a} \mp \frac{\sqrt{b^2 - 4ac}}{2a} \in \mathbb{S}^{1+0}$$

Similarly, if $f_1 = 0$ and $4ac > b^2$, then

$$\overset{o}{\varphi}_{\mathbb{S}_2^{1+1}} = f_0 + f_2\breve{\mathbf{e}}_2 = -\frac{b}{2a} \pm \frac{\sqrt{-b^2 + 4ac}}{2a}\breve{\mathbf{e}}_2$$

whereas if $4ac \leq b^2$, $\sqrt{-b^2 + 4ac} = \sqrt{b^2 - 4ac}\,\breve{\mathbf{e}}_2$, thus

$$\overset{o}{\varphi}_{\mathbb{S}_2^{1+1}} = -\frac{b}{2a} \mp \frac{\sqrt{b^2 - 4ac}}{2a} \in \mathbb{S}^{1+0}.$$

If the solutions are real, $\mathbb{S}^{1+0} \in \left(\mathbb{S}_1^{1+1} \cap \mathbb{S}_2^{1+1}\right) = \mathbb{S}^{1+0}$ This exhausts all possibilities stated in the proposition. □

We refer to solutions of the form (7a), as the *hypercomplex roots*. Notice that the scalar component of the solution is zero for the hypercomplex roots (7a) if $4ac = -b^2$, but then the two director components are also zero. Thus, $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} \in \mathbb{S}^{1+2}$. However, the hypercomplex roots are not a solution if $4ac = -b^2$, the only solution is then given by the $\overset{o}{\varphi}_{\mathbb{S}^{1+0}}$ real root. $\overset{o}{\varphi}$ is always in the $\mathbb{S}^{1+2}$ set, where the scator product is defined. Note that two of the conditions $|4ac| > b^2$, $4ac \leq b^2$ and $4ac > b^2$ can be met simultaneously depending on the values of a, b and c, so that the hypercomplex roots can coexist with the real or complex roots.

In order to establish the interval where the hypercomplex roots (7a) exist, $\sqrt{-1}$ was written as $\check{\mathbf{e}}_1$ in the Theorem's proof. It could have equally been written as $\check{\mathbf{e}}_2$, then the product with $\check{\mathbf{e}}_1$ vanishes. As far as the proof is concerned, in both cases, the director component is zero, so no problems arise. However, the two results are not equal, i.e. $-1 \cdot \check{\mathbf{e}}_1 = (\check{\mathbf{e}}_1\check{\mathbf{e}}_1)\,\check{\mathbf{e}}_1 = -\check{\mathbf{e}}_1$, whereas $-1 \cdot \check{\mathbf{e}}_1 = (\check{\mathbf{e}}_2\check{\mathbf{e}}_2)\,\check{\mathbf{e}}_1 \neq \check{\mathbf{e}}_2\,(\check{\mathbf{e}}_2\check{\mathbf{e}}_1) = 0$. The reason being that associativity does not hold when the product of two factors have zero scalar component, as is the case for $(\check{\mathbf{e}}_2\check{\mathbf{e}}_1) = 0$.

It is reassuring to confirm that the solutions satisfy the polynomial equation. The real and complex like expressions are the familiar solutions. Let us evaluate the entirely novel hypercomplex solution (7a). The squared term for equal director components, from (3a) is

$$\overset{o}{\varphi}^2 = f_0^2\left(1 - \frac{f_1^2}{f_0^2}\right)^2 + 2f_0f_1\left(1 - \frac{f_1^2}{f_0^2}\right)\check{\mathbf{e}}_1 + 2f_0f_1\left(1 - \frac{f_1^2}{f_0^2}\right)\check{\mathbf{e}}_2 \tag{11}$$

Evaluation of f_0, f_1 and f_2 from (7a), noting that $f_0\left(1 - \frac{f_1^2}{f_0^2}\right) = -\frac{b}{2a}$, after some algebra gives,

$$\overset{o}{\varphi}^2_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} = \frac{b^2}{4a^2} \mp \frac{b}{a}\sqrt{\frac{(4ac)^2 - (b^2)^2}{16a^2b^2}}\check{\mathbf{e}}_1 \mp \frac{b}{a}\sqrt{\frac{(4ac)^2 - (b^2)^2}{16a^2b^2}}\check{\mathbf{e}}_2$$

The sum of a times this expression plus $b\,\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} + c$, confirms that the quadratic polynomial equation (5) is satisfied.

3. Geometric Representation of the Scator Roots in 1+2 Space

If $4ac \leq b^2$ and $|4ac| > b^2$, there exist six roots: i) two real roots and ii) four hypercomplex roots. The initial conditions imply that $4ac$ is negative, $4ac + b^2 < 0$. Recalling that the hypercomplex scalar component is $-\frac{4ac+b^2}{4ab}$, these hypercomplex roots always lie in the real positive semispace if the product $a\,b$ is positive and in the real negative semispace if $a\,b$ is negative. These roots are depicted in Figure 1, for $a = 1, b = 1, c = -1$. The two real roots (drawn in blue) are $\overset{o}{\varphi}_{\mathbb{S}^{1+0}} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, whereas the four hypercomplex roots (drawn in orange) are $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} = \frac{3}{4} \pm \frac{\sqrt{15}}{4}\check{\mathbf{e}}_1 \pm \frac{\sqrt{15}}{4}\check{\mathbf{e}}_2$. It is interesting to notice that the hypercomplex roots coexist with real roots in this region.

If $4ac > b^2$, there are eight roots because conditions $|4ac| > b^2$ and $4ac > b^2$ in Theorem 2.1 are met: i) two in the $s, \check{\mathbf{e}}_1$ plane (the scalar axis is hereafter labeled s) and ii) two in the $s, \check{\mathbf{e}}_2$ plane, corresponding to the complex roots in $\mathbb{C}$, but the imaginary parts are now the orthogonal $\check{\mathbf{e}}_1$, $\check{\mathbf{e}}_2$ imaginary units. iii) four hypercomplex roots that lie in the negative s semispace if $a\,b$ is positive and in the real positive s semispace if $a\,b$ is negative. These roots are shown in Figure 1, for $a = 1, b = 1, c = 1$. The complex akin roots are $\overset{o}{\varphi}_{\mathbb{S}_1^{1+1}} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\check{\mathbf{e}}_1$ (green) and $\overset{o}{\varphi}_{\mathbb{S}_2^{1+1}} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\check{\mathbf{e}}_2$ (brown). The hypercomplex roots are $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} = -\frac{5}{4} \pm \frac{\sqrt{15}}{4}\check{\mathbf{e}}_1 \pm \frac{\sqrt{15}}{4}\check{\mathbf{e}}_2$ (drawn in red).

In the interval $-b^2 < 4ac \leq b^2$, only the two real roots exist; these roots collapse to the same $-\frac{b}{2a}$ value when $4ac = b^2$. The regions where the different roots coexist are illustrated in Figure 2. Since the hypercomplex roots have directors with equal square components $f_1^2 = f_2^2$, the hypercomplex roots always lie in planes at 45° with respect to the $\check{\mathbf{e}}_1, \check{\mathbf{e}}_2$ axes. Let the function $\overset{o}{f}(a, b, c) = a\,\overset{o}{\varphi}^2 + b\,\overset{o}{\varphi} + c$. The values of $\overset{o}{\varphi}$ where $\overset{o}{f}(a, b, c)$ is zero as a function of c, for constant a, b are shown in Figure 3. The possible values for the real, complex like and hypercomplex roots are shown in different colors. The

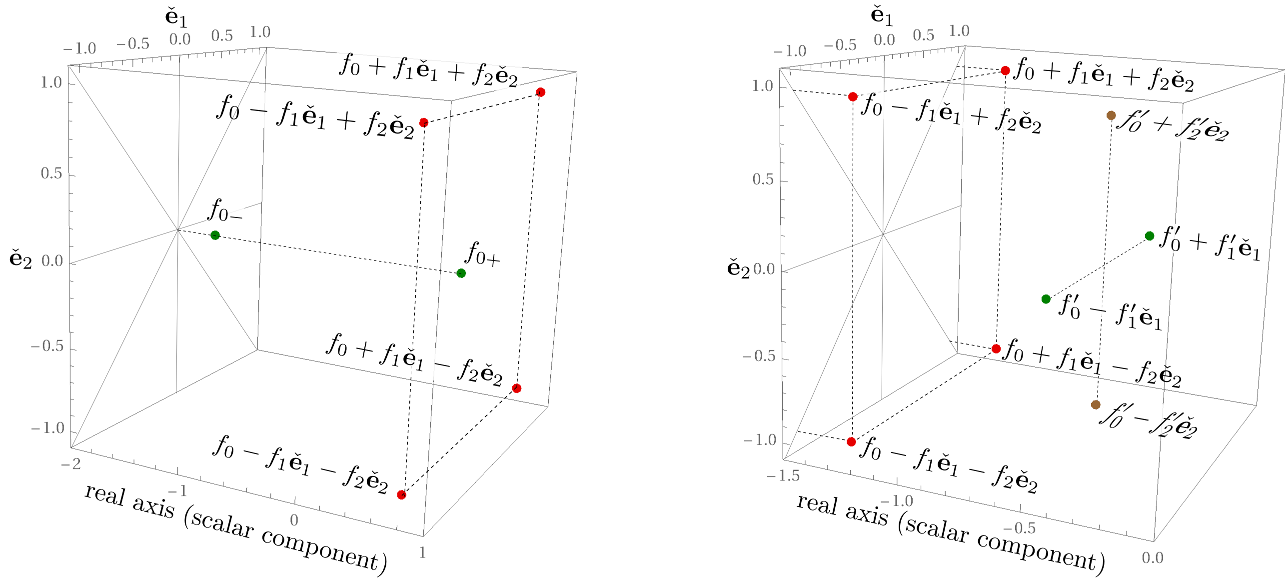

Fig. 1. Roots of the scator quadratic polynomial. Left: $4ac \leq b^2$ and $|4ac| > b^2$ with $a = 1, b = 1, c = -1$. Hypercomplex roots in orange; real roots in blue. Right: $4ac > b^2$ with $a = 1, b = 1, c = 1$. Hypercomplex roots in red; complex like roots in the $s, \check{\mathbf{e}}_1$ plane in green and in the $s, \check{\mathbf{e}}_2$ plane in brown.

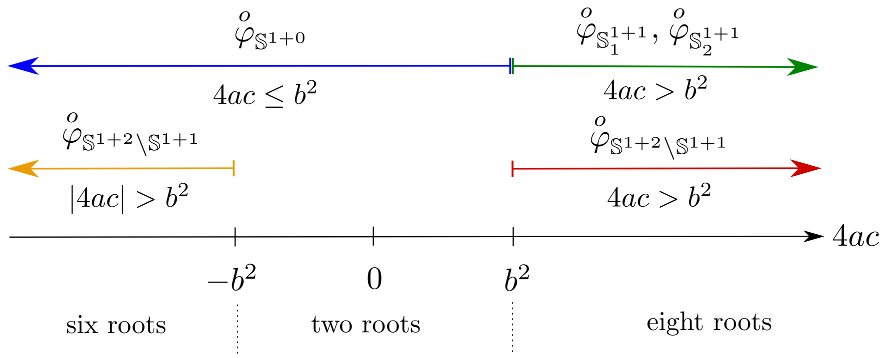

Fig. 2. The scator quadratic equation with real coefficients has six, two or eight roots, depending on the value of $4ac$.

complex like roots (in green) have constant scalar component since the real (or scalar) part does not depend on c. There are however, two perpendicular branches, one for each hypercomplex direction $\check{\mathbf{e}}_1$ or $\check{\mathbf{e}}_2$. The hypercomplex roots with two nonvanishing director components, have been drawn in orange when they coexist with the real (blue) solutions and in red, when they coexist with the complex (green) solutions.

Lemma 3.1. Given a hypercomplex solution $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\backslash\mathbb{S}^{1+1}} = f_0 + f_1\check{\mathbf{e}}_1 + f_2\check{\mathbf{e}}_2$ to the quadratic polynomial equation with real coefficients $a\overset{o}{\varphi}^2 + b\overset{o}{\varphi} + c = 0$, the scators $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\backslash\mathbb{S}^{1+1}} = f_0 \pm f_1\check{\mathbf{e}}_1 \pm f_2\check{\mathbf{e}}_2$ are also solutions to this equation.

Proof. The solution $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\backslash\mathbb{S}^{1+1}} = f_0 - f_1\check{\mathbf{e}}_1 - f_2\check{\mathbf{e}}_2$ can be obtained following the usual complex proof of the conjugate polynomial equation. To prove that the sign of any one director can be changed, consider one such component. The $\check{\mathbf{e}}_1$ component of the squared scator can only cancel out with the $\check{\mathbf{e}}_1$ component of the linear scator term since the polynomial coefficients are real, thus

$$a \cdot 2f_0 f_1 \left(1 - \frac{f_1^2}{f_0^2}\right)\check{\mathbf{e}}_1 = b f_1 \check{\mathbf{e}}_1 \quad \Rightarrow \quad 2f_0\left(1 - \frac{f_1^2}{f_0^2}\right) = \frac{b}{a}$$

If this result is satisfied for f_1 it also holds for $-f_1$. A similar argument is true for the $\check{\mathbf{e}}_2$ component. □

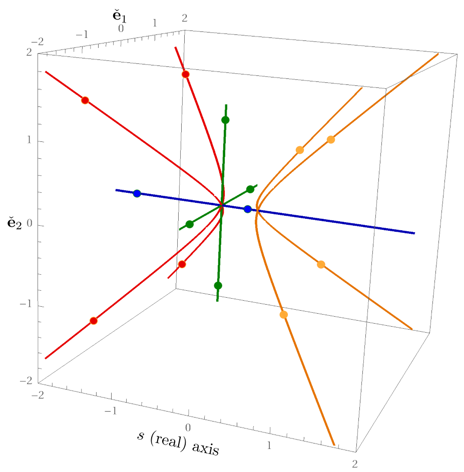

Fig. 3. Image of the polynomial roots for $a = b = 1$ with c as a parameter in the ± 2 interval in each axis, $\overset{o}{f}(1, 1, c) = \overset{o}{\varphi}{}^2 + \overset{o}{\varphi} + c$. Real solutions in blue, complex like solutions in green and hypercomplex solutions for $4ac > 0$ in red and $4ac < 0$ in orange. Roots for $c = 1$ (red and green dots) and $c = -1$ (blue and orange dots)

The hypercomplex roots clearly have the same magnitude (4), since $\|\overset{o}{\varphi}\|$ depends only on the square of the scator coefficients. The magnitude of the hypercomplex roots (7a), noticing that the squared ratio of director over scalar components is $\frac{f_1^2}{f_0^2} = \frac{4ac-b^2}{4ac+b^2}$, is given by

$$\left\| \overset{o}{\varphi}_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} \right\| = 2 \left| \frac{c}{b} \right| \tag{12}$$

The solutions as a function of a, depicted in figure 4, lie in an isometric surface since the above equation is independent of a. The elliptic scators isometric surface, named a cusphere, is illustrated in Figs. 5 and 6 in [9]. The teal curves in figure 4, are scator isometric ellipses lying on one of the two planes at $\pm 45°$ with respect to the $\check{\mathbf{e}}_1, \check{\mathbf{e}}_2$ director axes. The projection of these ellipses in the $s, \check{\mathbf{e}}_1$ and $s, \check{\mathbf{e}}_2$ planes are actually circles with unit radius centered at -1, as we shall now see.

Consider the change of variable $u = \frac{1}{4a}$, the hypercomplex solutions (7a), can then be written as

$$\overset{o}{\varphi}_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} = -\frac{c}{b} - bu \pm \sqrt{\frac{c^2}{b^2} - b^2u^2}\, \check{\mathbf{e}}_1 \pm \sqrt{\frac{c^2}{b^2} - b^2u^2}\, \check{\mathbf{e}}_2 \tag{13}$$

Let c and b be constants. The above expression is then recognized as the parametric representation of the circle equation $x^2 + y^2 = \frac{c^2}{b^2}$, shifted by $\frac{c}{b}$ in the negative scalar axis, where $x = -bu$ and $y = \sqrt{\frac{c^2}{b^2} - b^2u^2}$ is the director coefficient in the $\check{\mathbf{e}}_1$ or the $\check{\mathbf{e}}_2$ axes. These circle projections in the $s, \check{\mathbf{e}}_1$ and $s, \check{\mathbf{e}}_2$ planes, exclude the points when the circles cross the scalar axis, i.e. $c^2 = b^4u^2$. If b and u are considered constant, a parametric representation of the hyperbolic equation $x^2 - y^2 = b^2u^2$, is obtained, this time shifted by $-bu$ in the scalar axis, where $x = -\frac{c}{b}$ and y is the director coefficient in either axis. These curves are plotted in red in figure 4, the asymptotes lie at $\pm 45°$ in the $s, \check{\mathbf{e}}_1$ or $s, \check{\mathbf{e}}_2$ planes. The vertices are located at 0 and $-2bu = -\frac{b}{2a}$, but these two points are not solutions to $\overset{o}{\varphi}_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}}$, since $\left|\frac{c}{u}\right| = |4ac|$ must be greater than b^2. If c and u are constant, the parametric equation (13) as a function of b, is no longer a conic curve.

The limit of the hypercomplex, real and complex like roots when $4ac \to b^2$ is the same,

$$\lim_{4ac \to b^2} \left(\overset{o}{\varphi}_{\text{sol } \mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} \right) = \lim_{4ac \to b^2} \left(\overset{o}{\varphi}_{\text{sol } \mathbb{S}^{1+0}} \right) = \lim_{4ac \to b^2} \overset{o}{\varphi}_{\text{sol } \mathbb{S}^{1+1}_{1,2}} = -\frac{b}{2a}$$

Hypercomplex (red), real (blue) and complex like (green) curves approach $-\frac{b}{2a} = -\frac{1}{2}$ in the Figure 3.

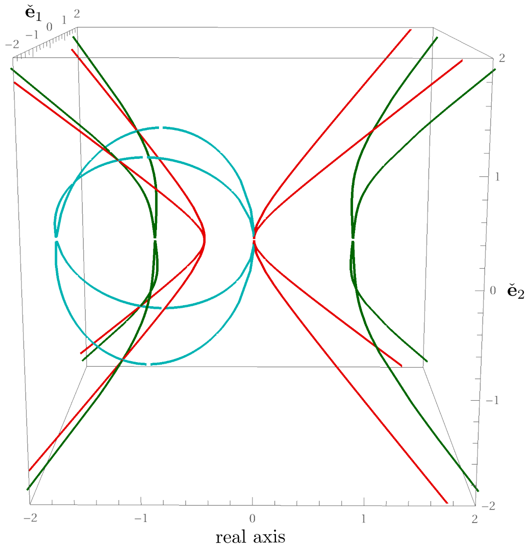

Fig. 4. Hypercomplex roots of the scator function, $\overset{o}{f}(a,b,c) = a\,\overset{o}{\varphi}{}^2 + b\,\overset{o}{\varphi} + c$ for $|4ac| > b^2$ in the ± 2 interval in each direction. $a = b = 1$ with c as a parameter in red (red and orange in figure 3); $a = c = 1$ with b as a parameter in green; $b = c = 1$ with a as a parameter in teal. All curves lie in planes that are at $\pm 45°$ with respect to the $\check{\mathbf{e}}_1, \check{\mathbf{e}}_2$ axes.

The limit of the hypercomplex and real roots when $4ac \to -b^2$ are

$$\lim_{4ac \to -b^2} \left(\overset{o}{\varphi}_{\mathrm{sol}\ \mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} \right) = 0 \qquad \text{and} \qquad \lim_{4ac \to -b^2} \left(\overset{o}{\varphi}_{\mathrm{sol}\ \mathbb{S}^{1+0}} \right) = -\frac{b}{2a} \left(1 \pm \sqrt{2} \right)$$

However, it should be noted that the hypercomplex root function domain excludes $4ac = -b^2$, so that 0 is not a root of the polynomial. In Figure 3, the hypercomplex curve (orange) does not intersect the real curve (blue) at zero. However, as $4ac$ approaches $-b^2$ from the negative real axis, all three hypercomplex scator coefficients become infinitesimal. Nonetheless, from (12), the magnitude of the scator in this limit is $\lim\limits_{4ac \to -b^2} \left\| \overset{o}{\varphi}_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} \right\| = 2 \left| \frac{c}{b} \right| = \left| \frac{b}{2a} \right|$.

3.1. Roots of Minus One

The square roots of minus one have been studied in quaternions [10,11], complexified quaternions [12], split quaternions [13] and more generally in Clifford algebras [14]. In scator algebra, the hypercomplex roots tend to infinity as b tends to zero, the green curves in Figure 4 are then asymptotic to $\pm 45°$ lines. If $b^2 \ll |4ac|$, the binomial expansion of the roots to first non-vanishing order is

$$\overset{o}{\varphi}_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} \approx -\frac{c}{b}(1+\delta) \pm \frac{c}{b}\left(1 - \frac{1}{2}\delta^2\right) \check{\mathbf{e}}_1 \pm \frac{c}{b}\left(1 - \frac{1}{2}\delta^2\right) \check{\mathbf{e}}_2$$

where $\delta = \frac{b^2}{4ac}$. In the limit when $b \to 0$, each component diverges so that the solution diverges. The hypercomplex solutions in this limit become $\overset{o}{\varphi}_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} = -\frac{c}{b} \pm \frac{c}{b}\check{\mathbf{e}}_1 \pm \frac{c}{b}\check{\mathbf{e}}_2$, but recall that scators whose three components have equal absolute value are square nilpotent [6, Lemma 1]. This result can also be readily seen from (11), for $f_0 = f_1 = f_2$. The quadratic polynomial leading term is thus zero $a \cdot \overset{o}{\varphi}{}^2_{\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}} = 0$, and the polynomial equation becomes $b\,\overset{o}{\varphi} + c = 0$. However, no scator in $\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$ satisfies this equation since b, c are real. Therefore, there are no hypercomplex square roots in $\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$ of a real number, be it positive or negative. If $b = 0$, then the only possible polynomial roots are

$$\overset{o}{\varphi}_{\mathbb{S}_1^{1+1}} = \pm\sqrt{\frac{c}{a}}\check{\mathbf{e}}_1, \quad \overset{o}{\varphi}_{\mathbb{S}_2^{1+1}} \pm \sqrt{\frac{c}{a}}\check{\mathbf{e}}_2, \text{ if } ac > 0$$

For $a = c$, the roots of -1 in $\mathbb{S}^{1+2}$ are thus $\pm\check{\mathbf{e}}_1$ and $\pm\check{\mathbf{e}}_2$.

4. Conclusions

The quadratic polynomial $ax^2 + bx + c = 0$, $x = \overset{o}{\varphi} \in \mathbb{S}^{1+2}$ in the imaginary scator set with real coefficients a, b, c, has a finite number of roots. In sharp contrast, an infinite number of roots to the quadratic equation are encountered in quaternions and more generally in Clifford algebras. In the $\mathbb{S}^{1+2}$ scator set:

- If $4ac > b^2$, there exist eight roots in three sets, $\overset{o}{\varphi}_{\mathbb{S}_1^{1+1}} = -\frac{b}{2a} \pm \frac{\sqrt{-b^2+4ac}}{2a}\check{\mathbf{e}}_1$ and $\overset{o}{\varphi}_{\mathbb{S}_2^{1+1}} = -\frac{b}{2a} \pm \frac{\sqrt{-b^2+4ac}}{2a}\check{\mathbf{e}}_2$ give two sets of two roots that are akin to the complex roots but the hyperimaginary units are now $\check{\mathbf{e}}_1$ and $\check{\mathbf{e}}_2$ instead of i. Four hypercomplex roots $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}} = -\frac{4ac+b^2}{4ab} \pm \sqrt{\frac{(4ac)^2-(b^2)^2}{16a^2b^2}}\check{\mathbf{e}}_1 \pm \sqrt{\frac{(4ac)^2-(b^2)^2}{16a^2b^2}}\check{\mathbf{e}}_2$, that involve non zero components in both hypercomplex axes.
- If $|4ac| \leq b^2$ there exist two real roots, $\overset{o}{\varphi}_{\mathbb{S}^{1+0}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ identical to the roots in the real set.
- If $4ac < b^2$ and $|4ac| > b^2$, there exist six roots in two sets, two real roots $\overset{o}{\varphi}_{\mathbb{S}^{1+0}}$ and four hypercomplex roots $\overset{o}{\varphi}_{\mathbb{S}^{1+2}\setminus\mathbb{S}^{1+1}}$.

Hypercomplex roots always come in sets of four in $\mathbb{S}^{1+2}$, in as much as complex roots come in sets of two in $\mathbb{C}$. Hypercomplex roots coexist with the real or complex like roots, in contrast with roots in the complex set, where the roots are either real xor complex. If b, the polynomial linear coefficient vanishes, the hypercomplex roots become square nilpotent so that the only roots of minus one are $\pm\check{\mathbf{e}}_1$ and $\pm\check{\mathbf{e}}_2$. Arbitrary integer powers of scators and nilpotent elements are discussed in [15]. The $\mathbb{S}^{1+2}$ scator roots can be visualized geometrically in a three dimensional space, where the scalar (real) axis and the two hypercomplex axes are drawn in orthogonal directions.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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Robust Logistic Modelling for Datasets with Unusual Points

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Abstract − Unusual Points (UPs) occur for different reasons, such as an observational error or the presence of a phenomenon with unknown cause. Influential Points (IPs), one of the UPs, have a negative effect on parameter estimation in the Logistic Regression model. Many researchers in fisheries sciences face this problem and have recourse to some manipulations to overcome this problem. The limitations of these manipulations have prompted researchers to use more suitable and innovative estimation techniques to deal with the problem. In this study, we examine the classification accuracies and parameter estimation performances of the Maximum Likelihood (ML) estimator and robust estimators through modified real datasets and simulation experiments. Besides, we discuss the potential applicability of the assessed robust estimators to the estimation models when the IPs are kept in the dataset. The obtained results show that the Weighted Maximum Likelihood (WML) and Weighted Bianco-Yohai (WBY) estimators of robust estimators outperform the others.

Keywords *– Influential point, robust estimators, unusual point, logistic regression*

Mathematics Subject Classification (2020) – 62G32, 65C60

1. Introduction

The most frequently adopted statistical method to obtain parameter estimates of the explanatory variables relationship with the binary outcome (0 and 1) is Logistic Regression. Binary Logistic Regression (BLR) models the functional relationship between the binary response variable and one/more explanatory variable [1-4]. Maximum Likelihood Estimator (MLE), which has the optimal properties under proper circumstances, is utilized to estimate the parameters in BLR; however, it is considerably affected by the presence of an unusual data point(s) in the dataset and may cause misleading inferences and misinterpretations in parameter estimates [5-11].

The unusual data point(s) (UP(s)) is generally defined as point(s) that are relatively far from the central tendency compared to all values [12-13]. These types of point(s) may derive from errors existing during the recording of observations, sampling errors, and experimental errors or may originate from an unknown phenomenon in a study area (e.g., economy, applied science, health, engineering).

The UP(s) are differently named as an outlier(s), influential point(s), or leverage point(s) according to their locations in the two-dimensional space. Among these definitions, influential point(s) (IPs) can be described as the product of dangerous outliers and bad leverage points and significantly affect the fit of the

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model or the estimation of the parameters compared to the others [14-17]. If the variable on the x-axis is continuous and the one on the y-axis is binary, unusual points can only occur as a transposition $0 \rightarrow 1$ or $1 \rightarrow 0$ in the y-axis direction [6]. This type of UP(s) is also recognized as a residual outlier or misclassification-type error [16].

These point(s) can be observed in datasets of numerous studies conducted in applied areas, and most researchers have been confused about what to do with them and how to manage them. To manage IP(s), researchers generally have had to decide among such strategies as keeping them, removing them, or recoding them [12]. [18] reported as a result of their research on the frequency of these points in different scientific disciplines that there is no overarching explanation and the frequency varies according to the study area and sample size; and they have claimed that if these outliers occur in about 1-10% of the dataset, it is normal. Although this decision differs according to the scientific field studied, it is recommended to estimate parameters using a more robust estimator instead of MLE by [18] if these points are to be kept in the data. Robust estimator instead of MLE has become the focus of many research fields in statistics [19].

In the BLR model, [20] is the first to indicate the problem of parameter estimation in the presence of IPs. After that, several robust alternative parameter estimation methods much less influenced by these points are suggested in the literature (i.e., [5,6,18,21-31]). Besides, many researchers have also studied to compare the performances of the estimators to examine the robustness of these proposed estimators on simulation experiments [11,16,27,32,33]. These studies have shown that MLE can be influenced even by the presence of 1% IPs in the dataset, and therefore robust estimators were recommended [34]. However, there are very few studies examining these points in terms of their effects on parameter estimates as they move away from the centre. Our hypothesis in the present study is to display that IP(s) occurs in different levels of percentage amounts in the dataset, three standard deviations away from the centre influence parameter estimations and to illustrate to researchers the possibility of being anomaly as research questions. The present paper was built on two purposes within the framework of our hypothesis: (a) to examine the performance of MLE and some robust estimators in parameter estimation in extreme situations, such as different sample size data have different percentages of influential points (i.e., contamination rate) in simulation experiments, and to contribute to the literature by providing information on what kind of results researchers may obtain if they encounter such data points.

2. Material

In this study, we carried out comprehensive simulation experiments to examine commonly cited or recently proposed robust estimators for BLR.

In the simulation study, to examine the performance of the estimators in different situations, we generated specific datasets created in combinations that vary according to different percentages of IPs occurring farther from the centre of the dataset in three different sample sizes (100, 250, and 500). We generated datasets with IPs, which we called a contaminated dataset, by adding IPs that fall 1.5, 3, and 5 whiskers away from the centre of the dataset that constituted 1%, 5%, 10%, and 15% of the dataset in each sample size and IP-free datasets (0% contaminated), which we called a clean dataset in each sample size for control purposes. The simulated datasets contain a response and two explanatory variables. We first generated a design matrix of explanatory variables of size $n \times p$ by drawing each observation from a bivariate normal distribution $\left(x_i \sim N(\mu, \Sigma)\right)$. Where μ is a mean vector of length $p = 2$ and Σ is a 2×2 non-singular covariance matrix. The considered true values of the BLR model parameters are set to be $\beta = (\beta_0, \beta_1, \beta_2)' = (0, 2, 2)'$. Then, we produced the binary response variable according to the BLR model as follows:

$$y_i = \begin{cases} 0 & \text{if} \quad \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon_i < 0 \\ 1 & \text{if} \quad \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon_i \geq 0 \end{cases} \tag{1}$$

where the error terms were generated according to a logistic distribution, $\varepsilon_i \sim logistic(0,1)$. We added the contaminants (IPs) to the dataset by inflating the covariance matrix and deriving it in the % amounts denoted in the simulation scenario. In the simulation study, we obtained the design matrix of the contaminated and uncontaminated explanatory variables with the configuration denoted below by [35].

$$(1-\gamma)N_{ss}(\mu, \Sigma) + \gamma N_{ss}\,(\mu, k \times \Sigma)$$

where N_{ss} is the sample size (100, 250 and 500), γ represents the percentage of contaminants (γ = 1%, 5%, 10%, and 15% contamination rate) in a dataset, and k represents a scalar which determines the separation of the contaminants from the rest of the data (k = 1.5, 3, and 5 whiskers), for any amount of contamination.

The aforesaid processes were applied to all the estimators used in this study. Each simulation study was replicated 1000 times by using the Monte Carlo simulation.

3. Method

The logistic regression model is a special case of GLMs, especially for a binary response variable y_i, with the assigned values 1 (success) and 0 (failure). The explanatory variables ($x_i \in R, i = 1, 2, \dots, n$) and the probability of response variable $p(Y_i = 1|X_i = x_i)$ are linked to explanatory variables by the mean of a link function $g(\pi) = X\beta$, such that $g^{-1}(X\beta)$ is the logit link function, which transforms the covariate values in the internal (0,1). The BLR model can be defined by:

$$p(Y_i = 1|X_i = x_i) = F(x_i'\beta) = \frac{\exp(x_i'\beta)}{1+\exp\left(x_i'\beta\right)}\,, \quad i = 1, 2, \dots, n \tag{2}$$

where $X = \left(1, x_1, \dots, x_p\right)$ is an $n \times k$ matrix of explanatory variables with $k = p + 1$ and $\beta' = \left(\beta_0, \beta_1, \dots, \beta_p\right)$ is the vector of the unknown regression coefficient. The BLR model can be defined by:

$$\eta_i = x_i'\beta \tag{3}$$

where η_i is a linear predictor known as transformation function and $\eta_i = \text{logit}(\pi_i) = \log\left(\frac{\pi_i}{1-\pi_i}\right)$. Suppose that the response variable y_i has Bernoulli distribution and the joint probability density function for the i^{th} observation is,

$$f(y_i) = \pi(x_i)^{y_i}[1-\pi(x_i)]^{1-y_i}, \;\; i = 1, 2, \dots, n \tag{4}$$

and each y_i observation takes the value 1 or 0. The likelihood function is given by:

$$l(\beta; y_i) = \prod_{i=1}^{n} f_i(y_i) = \prod_{i=1}^{n} \pi(x_i)^{y_i}[1-\pi(x_i)]^{1-y_i} \tag{5}$$

Then, we take a logarithm of the likelihood function (log-likelihood), which can be written as:

$$\begin{aligned} l(\beta; y_i) &= ln \prod_{i=1}^{n} f_i(y_i) = \sum_{i=1}^{n} l\,(y_i, \beta) \\ &= \sum_{i=1}^{n}\left[y_i ln\left(\frac{\pi(x_i)}{1-\pi(x_i)}\right)\right] + \sum_{i=1}^{n} ln\left(1-\pi(x_i)\right) \end{aligned} \tag{6}$$

To estimate the parameters in BLR, Maximum Likelihood Estimator (MLE) is used. The likelihood function is produced by maximizing the logarithm of and is defined as:

$$\hat{\beta}_{MLE} = \underset{\beta}{\operatorname{argmax}} \sum_{i=1}^{n} l\,(y_i, \beta) \tag{7}$$

As an alternative, MLE deviation statistics are minimized according to β [16], and it is defined as:

$$d_i = \left[-y_i \ln\left(\frac{\hat{\pi}_i}{y_i}\right) - (1-y_i)\ln\left(\frac{1-\hat{\pi}_i}{1-y_i}\right)\right]$$
$$\hat{\beta}_{MLE} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{n} d_i \tag{8}$$

It is known that MLE is the most efficient estimator, but it may behave very inadequately in the presence of outlying observations in terms of their location and impact. Many robust estimators have been proposed in the literature to replace MLE in order to solve this problem, but in this study, we aspired to evaluate the performances of the most cited and most recommended estimators. These robust estimators are briefly discussed in the subsequent sections.

3.1. The Mallows Type Leverage Dependent Weights Estimator (MALLOWS)

MALLOWS type estimator, introduced by [22] and intensively examined by [26], was obtained by minimizing log-likelihood function using weights dependent on explanatory variables. A robust estimate of β can be obtained by the solution of the following function [23]:

$$\sum_{i=1}^{n} w_i\{y_i \ln[\pi(x_i)] + (1-y_i)\ln[1-\pi(x_i)]\} \tag{9}$$

where $w_i = W\big(h_n(x_i)\big)$ are the weights function. W is bounded by depending on $W(u)$ and a non-increasing function. $W(u)$ is dependent on a parameter $c > 0$, and $W(u) = \left(1 - \frac{u^2}{c^2}\right)^3 I(|u| \le c)$. If $w_i \equiv 1$ and $c(x_i, \beta) \equiv 0$, then Eq. (8) supplies the usual BLR model parameter estimate. If $w_i = w(x_i, x_i'\beta)$, $c(x_i, x_i'\beta) \equiv 0$, and the weights depend only on the design, this estimate is called Weighted Maximum Likelihood (MALLOWS type estimator).

3.2. Weighted Maximum Likelihood Estimator (WMLE)

This estimator is obtained in a similar way to the strategy used in constructing the MALLOWS type estimator. That is, it detects unusual values and makes the parameter estimation by equalizing the weights of these values to zero. WMLEs for BLR can be obtained with a solution in (Eq. 8). However, in this study, parameters were estimated by equalizing the weights obtained by the weighting function introduced by [36] and proposed by [33]. First, the square of the Mahalanobis distances of the explanatory variables is calculated according to the computed $\hat{\mu}^{(0)}$ and $\hat{\Sigma}^{(0)}$ values. The square of the Mahalanobis distances (m^2) is calculated by:

$$m^2 = \left(x_i - \hat{\mu}^{(0)}\right)'\left(\hat{\Sigma}^{(0)}\right)^{-1}\left(x_i - \hat{\mu}^{(0)}\right)$$

The weight function proposed by [33] is defined as:

$$w_i = (0.8 * m^2 + 0.2)$$

Then, WMLEs for BLR can be obtained by the solution of the following:

$$\sum_{i=1}^{n} w_i\{y_i \ln[\pi(x_i)] + (1-y_i)\ln[1-\pi(x_i)]\} \tag{10}$$

3.3. The Conditionally Unbiased Bounded Influence Function (CUBIF) Estimator

In CUBIF estimator, introduced by [22], the weights depend on the response variables besides the explanatory variables. This method minimises a measure of efficiency based on the asymptotic (co)variance matrix to bound the measures of infinitesimal sensitivity. The M-estimators are the solution of the form of $\sum \psi(y_i, x_i, \beta) = 0$, where ψ is a known function. Its optimal function is written by:

$$\psi(y, x, \beta, B) = W(\beta, y, x, b, B)\left[y - g(\beta' x) - c\left(\beta' x, \frac{b}{h(x, B)}\right) x\right] \tag{11}$$

where B is a (co)variance matrix, b is bounded infinitesimal sensitively and $h(x, B) = (x' B^{-1} x)^{1/2}$ is a leverage measure. $c\left(\beta' x, \frac{b}{h(x,B)}\right)$ is a bias correction with corrected residual

$$\left(r(y, x, \beta, b, B) = y_i - g(\beta' x) - c\left(\beta' x, \frac{b}{h(x, B)}\right)\right)$$

The weight function $W(\beta, y, x, b, B) = W_b r\big((y, x, \beta) h(x, B)\big)$ downweighs observations with high leverage points and largely corrected residuals making M-estimator have bonded influence.

3.4. Consistent Misclassification Estimator (CME)

It is a known fact that unusual points in the dataset cause misclassification, and this issue has been studied by many researchers under different assumptions [37]. Misclassification is a stand-alone issue, and there are estimators developed for parameter estimation in case of misclassification. In this study, we used the Consistent Misclassification estimator (CME), proposed by [6], since we consider the parameter estimation in contaminated datasets. If $P(Y = 1|x_i) = F(x_i' | \beta_L)$ considered robust estimation in the BLR model, a misclassification model in which each response is misclassified with probability γ, so that [23]:

$$P(Y = 1|x_i) = F\big(x_i{}' \beta_{Mc}\big) + \gamma\{1 - 2F\big(x_i{}' \beta_{Mc}\big)\} = G\big(x_i{}' \beta_{Mc}, \gamma\big) \tag{12}$$

where β_L is the true regression parameter for the conventional BLR model and β_{Mc} is the true regression parameter under the misclassification model. [6] has investigated small values of γ and the use of (Eq. 11) in generating robust estimators and diagnostics and suggested a bias-corrected version that is suitable for small γ.

3.5. Robust Quasi-Likelihood Estimator (RQL)

The quasi-likelihood estimator, proposed by [38], is defined as solutions of the following equation:

$$\sum_{i=1}^{n} \frac{y_i - \mu(\beta' x_i)}{V(\beta' x_i)} \mu'^{(\beta' x_i)} x_i = 0$$

Then, the quasi-likelihood approach to parameter estimation was robustified by [26] by bounding and centring the quasi-likelihood score function [39].

$$\psi(y, \beta) = \frac{y_i - \mu(\beta' x)}{V(\beta' x)} \mu'(\beta' x) x,$$

To deal with high leverage points, they suggest putting weight on each point [39].

3.6. Bianco Yohai Estimator (BYE) and Weighted Bianco Yohai Estimator (WBYE)

[25] have found that Pregibon's estimator based on deviation statistics (Eq. 7) does not reduce the weight of high leverage points and is inconsistent. They have improved the consistent and more robust Bianco and Yohai Estimator (BYE) by shrinking Pregibon's estimator as follows:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{n} \left[\rho\left(\left[d(x_i'\beta; y_i) + G\left(F(x_i'\beta)\right) + G\left(1 - F(x_i'\beta)\right)\right]\right)\right] \tag{13}$$

where $\rho(x) = \left(x - x^2/(2c)\right)I_{(-\infty,c)}(x) + (c/2)I_{(c,\infty)}(x)$ is Huber's loss function and c is a tuning parameter,

$$G(x) = \int_0^x \rho'\left(-ln(u)\right)du$$

and I_A stands for the usual indicator function. $G\left(F(x_i'\beta)\right) + G\left(1 - F(x_i'\beta)\right)$ is a bias correction term [40].

[25] have also stressed that other choices of the bounded function ρ are possible. To reduce the effect of unusual points in the covariate space, [27] have proposed to include an extra weight to downweigh the high leverage points in (Eq. 10). Weighted Bianco and Yohai (WBY) estimator can be defined as follows [27-40]:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{n} w(x_i)\left[\rho\left(\left[d(x_i'\beta; y_i) + G\left(F(x_i'\beta)\right) + G\left(1 - F(x_i'\beta)\right)\right]\right)\right] \tag{14}$$

where

$$w(x_i) = \begin{cases} 1 & if\ (RMD_i)^2 \leq \chi^2_{m,0.975} \\ 0 & otherwise \end{cases}$$

are the weights for a decreasing function of Robust Mahalanobis Distances, and distances are computed by using the Minimum Covariance Determinant (MCD) estimator [41].

WBYE remains consistent because the weighting is merely applied to the explanatory variables. Unfortunately, the above weighting procedure also decreases the weights of the good leverage points, which is not required, and can lead to a loss of efficiency [11-16].

To test the performance of the estimators, we conducted computational experiments on Monte Carlo simulation and modified real datasets. The evaluations focused on the magnitude and severity of the IPs and the number of observations by adding outliers to the uncontaminated data. In the study, we used R 3.0.2. [42-44] to set up the Monte Carlo simulation and to examine the performance of the estimators via BLR analysis procedure.

The performances of the estimators are evaluated in view of each predicted beta parameter based on the bias and MSE (mean-squared errors):

$$\text{Bias} = \left\| \frac{1}{m}\sum_{i=1}^{m} \hat{\beta}_i - \beta_i \right\|,$$

$$\text{MSE} = \frac{1}{m}\sum_{i=1}^{m} \left\|\hat{\beta}_i - \beta_i\right\|^2$$

where $\|.\|$ indicates the Euclidean norm.

4. Results

The values for the bias and the MSE of the MLE, and the seven robust estimators are given in this section in Table 1-4 for the simulation study. A "good estimator" is one that has the values of the bias, and MSE is relatively small or close to zero. The bias and MSE of the eight estimators are shown in Table 1. In the uncontaminated dataset, it can be seen that the biases and MSEs of all the estimators are considerably close to each other and also will reduce when the number of observations is increased.

Table 1. Bias, variance, and MSE values of ML, WMLE, and robust estimators for uncontaminated dataset

Sample Size	Output	MLE	WMLE	CUBIF	CME	MALLOWS	RQL	BYE	WBYE
$n=100$	Bias	0.256	0.261	0.255	0.294	0.253	0.280	0.286	0.292
	MSE	0.746	0.772	0.747	0.960	0.744	0.839	0.849	0.882
$n=250$	Bias	0.107	0.106	0.106	0.110	0.105	0.130	0.112	0.111
	MSE	0.224	0.230	0.224	0.234	0.224	0.274	0.240	0.246
$n=500$	Bias	0.038	0.037	0.038	0.038	0.037	0.046	0.039	0.038
	MSE	0.099	0.102	0.100	0.103	0.099	0.121	0.107	0.110

MLE: Maximum Likelihood Estimator, **WMLE:** Weighted Maximum likelihood estimator, **CUBIF:** The Conditionally Unbiased Bounded Influence Function, **CME**: Consistent Misclassification Estimator, **MALLOWS:** The Mallows Type Leverage Dependent Weights Estimator**, RQL**: Robust Quasi-Likelihood Estimator, **BYE**: Bianco Yohai Estimator, **WBYE**: Weighted Bianco Yohai Estimator, **MSE**: Mean square error

In Table 2-4, the Bias and MSE outputs of the simulation derived from examining the estimator's behaviour under different conditions are given. As seen in the tables, the MLE method was quickly affected by the 1% degradation rate (percentage of IPs) that occurred, and outputs are the same in other studies. The presence of moderate and extreme IPs (5%, 10%, 15%) changes the results dramatically. Whereas the WMLE performs best in terms of Bias and MSE as the percentage of IP (degradation rate) increases, MLE appears to behave very poorly. The closest values to WMLE in terms of MSE and bias were observed in WBY and MALLOWS, respectively. The weighting process in the WML and WBY estimators becomes more advantageous in extreme contamination. It can be observed that the CUBIF, CME and RQL estimators do not perform well even at 5% contamination. The robustness performance of MLE dramatically decreases as the rate of contamination increases as IPs move away from the centre. At a distance of 3 whiskers, WMLE, WBY, and MALLOWS show the best performance at medium and high fouling rates, respectively, while MALLOWS, WBY, and WMLE estimators at a distance of 5 whiskers have the best performance, respectively, in terms of biases and MSEs. Meanwhile, it can be observed that bias and MSE decrease when the sample size is increased. WMLE, WBY, and Mallows have the overall best performance among all the compared estimators for different sample sizes. CUBIF, CME, and RQL estimators did not perform as well as WMLE, WBY and Mallows, even as the sample size was increased. This situation is thought to be due to the location of the unusual points. Finally, the WMLE, WBY, and Mallows estimators exhibited reasonable perform in the contaminated dataset.

Table 2. Bias, Variance, and MSE values of MLE and robust estimators over m = 1000 replication for 100 sample sizes in all cases

γ	k	Output	MLE	WMLE	CUBIF	CME	MALLOWS	RQL	BYE	WBYE
1%	1.5	Bias	1.460	0.229	0.798	0.287	1.024	0.644	0.129	0.259
		MSE	2.258	0.846	0.851	2.016	1.229	1.884	0.957	0.851
	3	Bias	2.132	0.246	0.814	0.251	0.099	0.334	0.094	0.266
		MSE	4.644	0.745	0.875	0.954	0.565	1.122	0.768	0.862
	5	Bias	2.569	0.204	0.803	0.543	0.234	0.390	0.184	0.253
		MSE	6.686	0.749	0.855	2.953	0.742	3.340	0.841	0.896
5%	1.5	Bias	2.358	0.250	1.748	2.253	2.024	0.229	1.073	0.253
		MSE	5.650	0.768	3.164	5.528	4.197	1.867	1.681	0.921
	3	Bias	2.692	0.249	1.745	2.692	0.664	2.616	2.692	0.280
		MSE	7.336	0.755	3.148	7.338	0.937	7.448	7.614	0.923
	5	Bias	2.753	0.251	1.742	2.753	0.228	2.751	2.807	0.274
		MSE	7.672	0.758	3.136	7.671	0.749	7.669	7.934	0.891
10%	1.5	Bias	2.624	0.239	2.414	2.625	2.455	2.621	2.623	0.039
		MSE	6.983	0.733	5.924	6.988	6.119	7.015	6.983	1.607
	3	Bias	2.785	0.281	2.411	2.785	1.352	2.780	2.780	0.258
		MSE	7.862	0.802	5.912	7.859	2.333	7.841	7.840	0.876
	5	Bias	2.802	0.253	2.407	2.799	0.236	2.785	2.788	0.056
		MSE	7.949	0.778	5.887	7.940	0.727	7.863	7.881	1.569
15%	1.5	Bias	2.740	0.284	2.710	2.741	2.623	2.746	2.746	0.262
		MSE	7.607	0.898	7.439	7.612	6.977	7.652	7.647	0.931
	3	Bias	2.835	0.209	2.725	2.833	1.932	2.814	2.819	0.243
		MSE	8.139	0.719	7.524	8.128	4.136	8.033	8.061	0.813
	5	Bias	2.830	0.223	2.725	2.826	0.216	2.804	2.806	0.243
		MSE	8.113	0.721	7.525	8.095	0.712	7.978	7.987	0.814

$\boldsymbol{\gamma}$: contamination rate, $\boldsymbol{k}$: whiskers distance from the centre of the data

Table 3. Bias, Variance, and MSE values of MLE and robust estimators over m = 1000 replication for 250 sample sizes in all cases

γ	k	Output	MLE	WMLE	CUBIF	CME	MALLOWS	RQL	BYE	WBYE
1%	1.5	Bias	1.071	0.081	0.549	0.088	0.720	0.101	0.142	0.073
		MSE	1.212	0.221	0.408	0.240	0.609	0.300	0.224	0.237
	3	Bias	1.643	0.084	0.558	0.068	0.077	0.089	0.022	0.084
		MSE	2.743	0.225	0.419	0.235	0.200	0.276	0.228	0.243
	5	Bias	2.127	0.082	0.551	0.086	0.078	0.106	0.055	0.054
		MSE	4.560	0.241	0.414	0.249	0.233	0.299	0.260	0.260
5%	1.5	Bias	2.267	0.101	1.604	1.920	1.898	0.096	0.934	0.088
		MSE	5.175	0.240	2.617	4.515	3.640	0.318	1.016	0.241
	3	Bias	2.658	0.085	1.609	2.658	0.635	2.221	1.435	0.086
		MSE	7.100	0.236	2.632	7.100	0.593	6.163	3.610	0.251
	5	Bias	2.742	0.087	1.609	2.742	0.086	2.742	2.741	0.076
		MSE	7.556	0.230	2.634	7.556	0.247	7.561	7.555	0.282
10%	1.5	Bias	2.603	0.104	2.335	2.604	2.431	2.602	2.598	0.101
		MSE	6.811	0.244	5.487	6.814	5.946	6.838	6.790	0.250
	3	Bias	2.777	0.087	2.334	2.777	1.422	2.775	2.773	0.099
		MSE	7.750	0.240	5.482	7.748	2.183	7.738	7.732	0.262
	5	Bias	2.799	0.093	2.333	2.797	0.078	2.783	2.783	0.107
		MSE	7.872	0.241	5.477	7.863	0.219	7.788	7.786	0.271
15%	1.5	Bias	2.725	0.077	2.700	2.725	2.607	2.731	2.732	0.088
		MSE	7.460	0.227	7.325	7.463	6.829	7.496	7.503	0.253
	3	Bias	2.826	0.090	2.719	2.825	2.033	2.808	2.813	0.104
		MSE	8.029	0.235	7.430	8.019	4.260	7.930	7.953	0.260
	5	Bias	2.826	0.067	2.721	2.823	0.069	2.801	2.802	0.083
		MSE	8.022	0.215	7.441	8.006	0.224	7.888	7.895	0.242

$\boldsymbol{\gamma}$: contamination rate, $\boldsymbol{k}$: whiskers distance from the centre of the data

Table 4. Bias, Variance, and MSE values of MLE and robust estimators over m = 1000 replication for 500 sample sizes in all cases

γ	k	Output	MLE	WMLE	CUBIF	CME	MALLOWS	RQL	BYE	WBYE
1%	1.5	Bias	1.08	0.053	0.569	0.038	0.738	0.035	0.154	0.056
		MSE	1.201	0.109	0.378	0.111	0.591	0.129	0.128	0.122
	3	Bias	1.642	0.051	0.567	0.047	0.091	0.064	0.044	0.054
		MSE	2.717	0.108	0.374	0.111	0.102	0.135	0.113	0.119
	5	Bias	2.126	0.049	0.567	0.046	0.042	0.058	0.020	0.055
		MSE	4.538	0.111	0.376	0.114	0.111	0.133	0.115	0.120
5%	1.5	Bias	2.268	0.039	1.614	2.131	1.909	0.005	0.955	0.043
		MSE	5.160	0.110	2.627	4.771	3.663	0.131	0.986	0.121
	3	Bias	2.658	0.038	1.615	2.658	0.67	2.408	0.546	0.039
		MSE	7.08	0.102	2.629	7.08	0.534	6.509	0.655	0.110
	5	Bias	2.742	0.04	1.613	2.742	0.043	2.743	2.741	0.044
		MSE	7.537	0.112	2.62	7.537	0.100	7.541	7.532	0.122
10%	1.5	Bias	2.596	0.044	2.312	2.597	2.422	2.605	2.585	0.051
		MSE	6.756	0.100	5.363	6.76	5.883	6.804	6.703	0.112
	3	Bias	2.774	0.046	2.31	2.774	0.413	2.772	2.771	0.048
		MSE	7.715	0.110	5.353	7.713	0.250	7.706	7.701	0.118
	5	Bias	2.797	0.033	2.307	2.795	0.052	2.782	2.784	0.037
		MSE	7.841	0.111	5.337	7.832	0.109	7.758	7.768	0.119
15%	1.5	Bias	2.724	0.036	2.701	2.724	2.608	2.729	2.732	0.040
		MSE	7.438	0.104	7.315	7.442	6.821	7.469	7.483	0.112
	3	Bias	2.795	0.792	2.626	2.794	0.632	2.783	2.785	0.791
		MSE	7.831	0.767	6.913	7.825	0.522	7.766	7.778	0.782
	5	Bias	2.826	0.046	2.722	2.823	0.038	2.802	2.802	0.048
		MSE	8.007	0.107	7.427	7.991	0.103	7.872	7.873	0.119

$\boldsymbol{\gamma}$: contamination rate, $\boldsymbol{k}$: whiskers distance from the centre of the data

Figure 2 shows the changes in the performance of the eight estimators concerning MSE in their respective datasets have IP(s) under different conditions. As clear from the plots, the WMLE, WBY, and MALLOWS estimators had reasonable perform in the contaminated dataset.

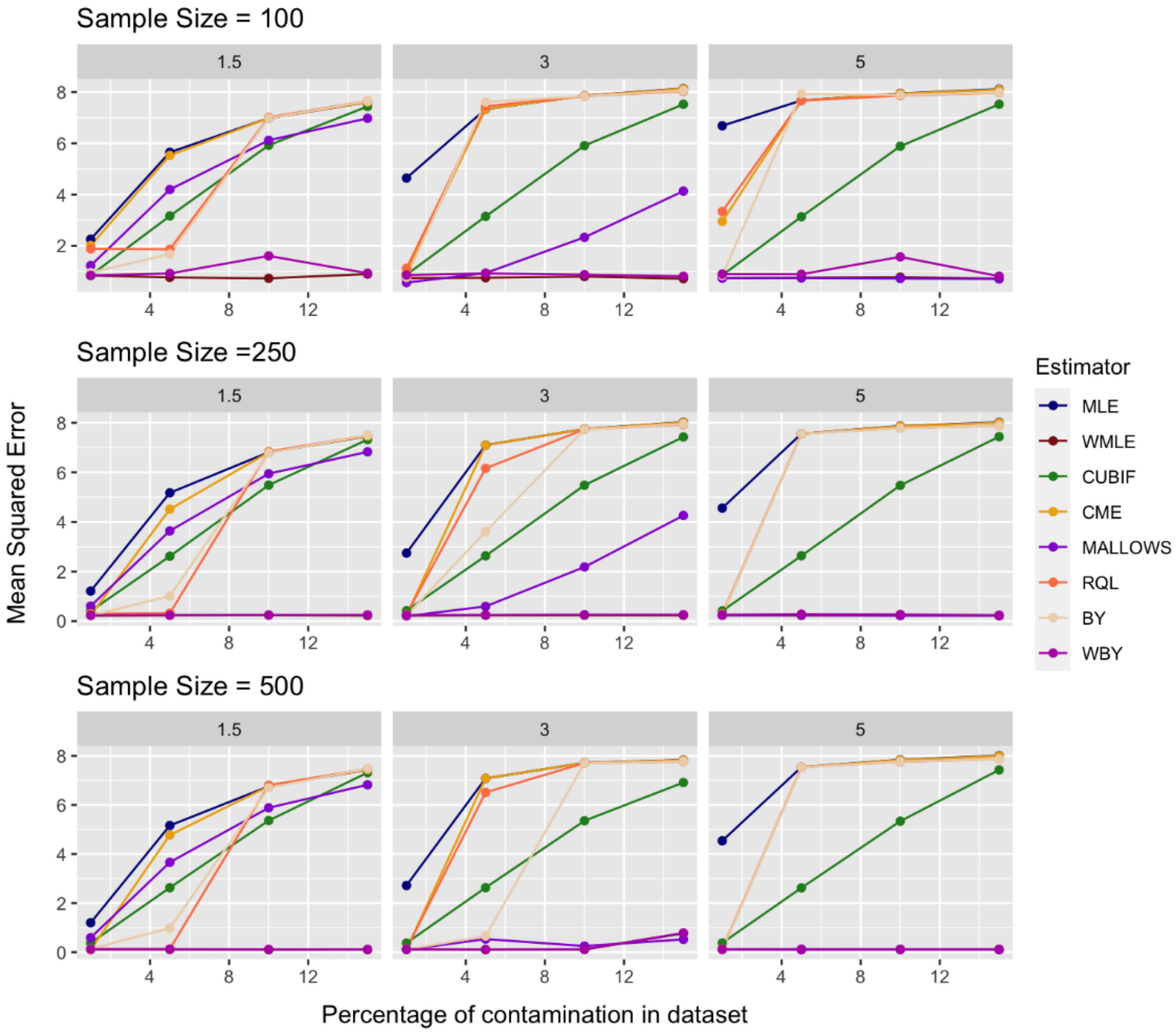

Fig. 2. Plots of change of estimator performance in terms of MSE under different conditions

MLE: Maximum Likelihood Estimator, **WMLE:** Weighted Maximum likelihood estimator, **CUBIF:** The Conditionally Unbiased Bounded Influence Function, **CME**: Consistent Misclassification Estimator, **MALLOWS:** The Mallows Type Leverage Dependent Weights Estimator**,** **RQL**: Robust Quasi-Likelihood Estimator, **BYE**: Bianco Yohai Estimator, **WBYE**: Weighted Bianco Yohai Estimator

5. Conclusion

Since datasets containing Influential Points (IP(s)), one of the Unusual Points (UP(s)), are possible to be encountered in every field, it becomes essential to use estimators that give more robust results than the MLE estimator to make parameter estimation. Since the size of these kinds of points is as important as their distances to the centre, the estimators designed are approaches developed with lessening the weight depending on both the location of the points and their sample size. Therefore, the simulation scenario in this study is developed considering the modelling principles of robust estimators focused on weighting.

The first of these approaches is the WML estimator, which was developed for weighting the likelihood function, and then [23] expanded the estimators by adding weights to reduce the effect of unusual points and developed new estimators. Parameter estimates (CUBIF, CME and MALLOWS) are made by using IP(s) the effect of which is reduced by this extended method, according to their position to the centre and their values.

Another approach is extended estimators (RQL), with less weight given to IP(s) and minimizing deviations from the estimated parameters. [25] developed another robust method, the Bianco and Yohai (BY) estimator, adding a function limited, differentiable, and decreasing. However, since this approach was also ineffective in reducing the weight of IP(s) with an increasing amount, [27] extended the estimator by adding a different weight to the Bianco and Yohai (BY) estimator, they defined Bianco and Yohai (WBY) estimator obtaining more consistent results.

In this study, we evaluated the MLE and seven robust estimators from the contaminated dataset with IP(s) whether it is feasible to obtain consistent parameter estimates. We conducted simulation experiments under different scenarios to examine the performance of MLE and robust estimators under contaminated and uncontaminated datasets. According to the simulation results, turned out that the uncontaminated dataset MLE and robust estimators exhibited performances similar to each other, the classical ML estimates lacked robustness and could be biased when IPs were present, while robust estimators gave better results. Among the robust estimators, the WML WBY and MALLOWS estimators, respectively, produced the smallest BIAS and MSE in the contaminated data. With the increase of the contamination at five whiskers, the MALLOWS, WBY, and WML estimators, respectively, produced the smallest bias and MSE. The results demonstrated that there might be frequent and significant differences in the case of IPs in the dataset and, therefore, should be taken into account as an example of how results can differ in different research areas. It can, thus, be concluded that the WML, WBY, and MALLOWS estimators outperformed the ML estimator and the rest of the robust estimators in the presence of IP(s).

In addition to examining the performances of robust estimators, we evaluated the problem regarding what percentage of UP(s) should be kept in the dataset. The results of our study showed that, like other studies comparing predictors, the traditional ML estimator deteriorated even at 1% contamination; for this reason, if the datasets contain approximately 1-10% or more unusual points, we recommend that they should be examined carefully. A robust method is needed, especially when there is an UP(s) at 1.5 or more whisker distance from the centre. These data should be treated from an objective perspective, and they should then be examined specifically. After being examined in detail with as different analytical methods as possible, it should be kept in the dataset, or one of the other strategies (removed or transformed) should be opted for. If the distance and amount of contamination are high, these points, determined by analytical and graphical methods, may be the possibility of anomaly, depending on the field of study. Anomaly detection methods are different from the detection of IP(s), and in such cases, these points should be considered a separate research subject without treating them as IP(s) or outliers. More studies are needed to develop and research more suitable robust methods that can be used to detect unusual points and anomalies in BLR and for parameter estimation in these types of datasets.

The first point to be considered on which estimator should be used for performance in further studies is the location of the IP(s) and the amount of the IP(s) in the dataset.

As the distance of IP(s) to the centre increases, it can be said that WML and MALLOWS estimators, in which weighting is performed according to the location to lessen the effect of the points, are better. On the other hand, the WBY estimator is a better alternative in case the number of IP(s) is high (1-10% and/or more).

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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Enumeration of Involutions of Finite Rings

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Abstract − In this paper, we study a special class of elements in the finite commutative rings called involutions. An involution of a ring R is an element with the property that $x^2 - 1 = 0$ for some x in R. This study describes both the implementation and enumeration of the involutions of various rings, such as cyclic rings, non-cyclic rings, zero-rings, finite fields, and especially rings of Gaussian integers. The paper begins with simple well-known results of an equation $x^2 - 1 = 0$ over the finite commutative ring R. It provides a concrete setting to enumerate the involutions of the finite cyclic and non-cyclic rings R, along with the isomorphic relation $I(R) \cong Z_2^k$.

Keywords − *Cyclic rings, noncyclic rings, zero rings, finite fields, involutions*

Mathematics Subject Classification (2020) − 16W10, 11K65

1. Introduction

In this paper, R denotes a commutative finite ring with unity. We call that a nonzero element u in R is a unit if there is some $x \in R$ such that $ux = 1$. When such an element x exists, it is called the multiplicative inverse of u and denoted by $x = u^{-1}$. The collection of units of the ring R is denoted by $U(R)$. However, $U(R)$ is a multiplicative group concerning the multiplication defined on the ring R. If R is a finite field, then $U(R)$ is a cyclic group. If the unit group $U(R)$ of R is cyclic, then $U(R)$ is finite. The order of R and the order of its group of units will be denoted by $|R|$ and $|U(R)|$, respectively. In the case when $R = Z_n$, $|U(R)| = \varphi(n)$, where $\varphi(n)$ is Euler's phi-function, the number of positive integers less than n and relatively prime to n. If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ is the decomposition of n into product of distinct prime powers, then $\varphi(n) = n \prod_{p|n}(1 - 1/p)$. It is well known that if a finite commutative ring with unity R decomposes as a direct product $R = R_1 \times R_2 \times \dots \times R_k$, then its group of units decomposes naturally as a direct product of groups. That is, $U(R)$ is isomorphic to $U(R_1) \times U(R_2) \times \dots \times U(R_k)$. The symbol $\cong$ will be used for both ring and group isomorphism. Note that if two rings R and $R'$are isomorphic, $R \cong R'$, then their group of units is isomorphic, $U(R) \cong U(R')$. Since the number of units of Z_n is $|U(Z_n)| = \varphi(n)$ and the number of units in the ring $Z_m \times Z_n$ is $\varphi(m)\varphi(n)$, but in general $\varphi(mn) \neq \varphi(m)\varphi(n)$ for some $m, n \geq 1$. If R is a finite field, then $U(R)$ is a cyclic group. Otherwise, $U(R)$ is an abelian group but not cyclic. If the unit group $U(R)$ of R is cyclic, then $U(R)$ is finite and $|U(R)|$ must be an even number.

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Up to isomorphism, there is a unique cyclic group $C_n = \{1, a, a^2, \dots, a^{n-1} : a^n = 1\} = \langle a \rangle$ of order n. But the fundamental theorem of finite abelian groups states that any finite non cyclic abelian group G is isomorphic to a direct product of cyclic groups $C_{n_1}, C_{n_2}, \dots, C_{n_k}$. That is, $G \cong C_{n_1} \times C_{n_2} \times \dots \times C_{n_k}$. Hence, the group of units of a finite commutative ring with unity is isomorphic to a direct product of cyclic groups. For instance, $U(Z_m \times Z_n) \cong U(Z_{mn})$ if and only if $(m,n) = 1$ if and only if $\varphi(mn) = \varphi(m)\varphi(n)$. The problem of classifying the group of units of an arbitrary finite commutative ring with identity is an open problem. However, the problem is solved for certain classes. In the case when $R = Z_n$, its group of units $U(Z_n)$ is characterized by using the following, see [1].

The group of units of the ring Z_n when n is a prime power integer is given by

(1) $U(Z_2) \cong C_1$,
(2) $U(Z_4) \cong C_2$,
(3) $U(Z_{2^k}) \cong U(Z_2) \times U(Z_{2^{k-1}})$, for every $k > 1$. For instance, $U(Z_8) \cong U(Z_2) \times U(Z_4)$.
For every prime p, we have $U(Z_{p^\alpha}) \cong C_p \times C_{p^{\alpha-1}}$.

Cross [2] gave a characterization of the group of units of $Z[i]/\langle\alpha\rangle$, where $Z[i]$ is the ring of Gaussian integers and α is an element in $Z[i]$. Smith and Gallian [3] solved the problem when $R = F[i]/\langle f(x)\rangle$ where F is a finite field and $f(x)$ is an irreducible polynomial over F. The related problem of determining the cyclic groups of units for each of the above classes of rings is completely solved. It is well known that $U(Z_n)$ is a cyclic group if and only if $n = 2,4, p^\alpha$ or $2p^\alpha$, where p is an odd prime integer. In [2], Cross showed that the group of units of $Z[i]/\langle\alpha\rangle$ is a cyclic group if and only if (1) $\alpha = (1+i)^k$, where $k = 1,2,3$ and (2) $\alpha = p, (1+i)p$, where p is a prime integer of the form $4k+3$ and α is a Gaussian prime such that $\alpha\bar{\alpha}$ is a prime integer of the form $4k+1$. The problem of determining all quotient rings of polynomials over a finite field with a cyclic group of units was solved by El-Kassar et al., see [4]. For more details about the unit groups and their corresponding properties, we refer to the work [5-6].

A ring R is called cyclic if $(R, +)$ is a cyclic group. In [7], the author Buck proved that every cyclic ring is a commutative and commutative finite cyclic ring with unity is isomorphic to the ring Z_n. Further, a ring $(R^0, +, \cdot)$ is a zero ring [8], if $ab = 0$ for every, $a, b \in R^0$, where '0' is the additive identity in R^0. For any finite commutative cyclic ring R without unity, we have $R \cong R^0$ and hence $U(R^0) = \phi$. Let B be a finite Boolean ring with unity, then $b^2 = b$ for every $b \in B$. If $B \cong Z_2$, then B is a Boolean ring with two elements 0,1 and $B^n = B \times B \times \dots \times B$ is a Boolean ring with 2^nelements, and clearly $|U(B^n)| = 1$.

The purpose of this paper is to enumerate the involutions in the group of units of a finite commutative ring with unity and to examine the properties of the involutions in a group of units. For this first, we shall define involutions in various fields of mathematics and their other related fields. Generally, in mathematics and other related fields, involution is a function f and it is equal to its inverse. This means that $f(f(x)) = x$ for all x in the domain of f. So, the involution is a bijection. For this reason, many fields in modern mathematics contain the term involution such as Group theory, Ring theory, and Vector spaces. Moreover, in the Euclidean and the Projective geometry, the involution is a reflection through the origin, and an involution is a projectivity of period 2, respectively. In mathematical logic, the operation of complement in Boolean algebra is called involution, and in classical logic, the negation that satisfies the law of double negation is called involution. Finally, in Computer science, the XOR bitwise operation with a given value for one parameter is also an involution, and RC4 cryptographic cipher is involution, as encryption and decryption operations use the same function. Recently in [9], the authors Fakieh and Nauman studied involutions and their minimalities of Reversible Rings. For further representations of involutions of various rings, the reader refers [10-13].

2. Properties of Involutions of Rings

Throughout this section, we are interested in involutions that have a special property in the elements of rings. Also, this section provides a useful theory that can be used to help to find solutions of equations of the form $x^2 = 1$, where 1 is the multiplicative unity of R.

Definition 2.1. An element u in a finite ring R with unity 1 is called an involution of R if $u^2 = 1$ where 1 is the unity of R. We denote it with $I(R)$, the set of all involutions of R. In particular, $I(R) \subseteq U(R) \subset R$.

For instance, 4 and 6 are the involutions of the ring $R = \{0,2,4,6,8\}$ with unity 6 modulo 10. When the cyclic ring $R = Z_n$, for a given positive integer n, we will use the symbol I_n to denote the set of all involutions of the ring Z_n and we will call it the set of involutions modulo n. For instance, $I_3 = \{1,2\}$, $I_8 = \{1,3,5,7\}$ and $I_{10} = \{1,9\}$. For any finite cyclic ring R with unity and finite zero rings R^0, we have $I(R) \neq \phi$ and $I(R^0) = \phi$. But we can simply verify that I_n is a subgroup of $U(Z_n)$. This is a basic property for the ring R with an abelian unit group $U(R)$. Now we show that $I(R)$ is a subgroup of $U(R)$.

Theorem 2.2. Let R be a commutative ring with unity. Then, $I(R)$ is a subgroup of $U(R)$.

Proof. Since $I(R)$ is a nonempty subset of $U(R)$. It is sufficient to prove that if $u, v \in I(R)$, then $uv^{-1} \in I(R)$. Indeed, if $u^2 = 1$ and $v^2 = 1$, then clearly $(uv^{-1})^2 = u^2(v^{-1})^2 = u^2(v^2)^{-1} = 1$. □

Example 2.3. Let us take the ring $R = Z_5$. Then, $I(R) = \{1,4\}$ and $U(R) = \{1,2,3,4\}$. This clearly shows that $I(R)$ is a subgroup of $U(R)$.

Here, we recall that the Cartesian product of two rings and the results about these rings. For a complete treatment of these rings, see [1]. Let R and S be any two rings. Then, $(R \times S, +, \cdot)$ is again a ring concerning the component-wise addition and component-wise multiplication: $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, bd)$, for every $(a, b),(c, d) \in R \times S$. It is well known that $(1_R, 1_S) \in R \times S$ if and only if $1_R \in R$ and $1_S \in S$. Also, Z_{mn} is not isomorphic to $Z_m \times Z_n$ if and only if $gcd(m, n) \neq 1$. In general, the following result is well known in the literature for $U(R)$ and $U(S)$.

Theorem 2.4. If R and S are commutative rings with unity, then $U(R \times S) = U(R) \times U(S)$.

Proof. Since $(1_R, 1_S) \in R \times S$. For $(u, v) \in U(R \times S)$, there exists $(u^{-1}, v^{-1}) \in U(R \times S)$ such that

$$\begin{aligned}(u, v)(u^{-1}, v^{-1}) = (1_R, 1_S) \quad &\Leftrightarrow (uu^{-1}, vv^{-1}) = (1_R, 1_S)\\ &\Leftrightarrow uu^{-1} = 1_R\end{aligned}$$

for some $u^{-1} \in R$ and $vv^{-1} = 1_S$ for some $v^{-1} \in S \Leftrightarrow u \in U(R)$ and $v \in U(S) \Leftrightarrow (u, v) \in U(R) \times U(S)$. Therefore, $U(R \times S) = U(R) \times U(S)$. □

Example 2.5. Let $R = Z_2$ and $S = Z_3$. Then, $R \times S = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$, $U(R) = 1$, and $U(S) = \{1,2\}$. Also, $U(R \times S) = \{1\} \times \{1,2\} = \{(1,1), (1,2)\}$ and $U(R) \times U(S) = \{(1,1), (1,2)\}$.

By Theorem 2.4, the following remark is obvious.

Remark 2.6. For any ring R, we have $I(R) = I(U(R))$.

The strategy is then to express $I(R \times S)$ in terms of $I(R)$ and $I(S)$. It is essential for finding the number of involutions in a finite commutative ring with unity.

Theorem 2.7. For any finite cyclic rings R and S with unity, then we $I(R \times S) = I(R) \times I(S)$.

Proof. Let R be a commutative ring with unity 1_R and S be a commutative ring with unity 1_S. Then by the Theorem 2.2 and Theorem 2.4, $I(R) \subseteq U(R)$, $I(S) \subseteq U(S)$ and $I(R \times S) \subseteq U(R \times S)$. This implies that $I(R) \times I(S)$ is a non-empty subset of $U(R) \times U(S)$.

First, we have to prove that $I(R \times S) \subseteq I(R) \times I(S)$. For any $(r, s) \in R \times S$, if $(r, s) \in I(R \times S)$ then $(r, s)^2 = (1,1)$, or $(r^2, s^2) = (1,1)$. This is the same as $r^2 = 1$ and $s^2 = 1$. Consequently, $r \in I(R)$ and $s \in I(S)$. Therefore, $(r, s) \in I(R) \times I(S)$. Thus $I(R \times S) \subseteq I(R) \times I(S)$. Similarly, we can show that $I(R) \times I(S) \subseteq I(R \times S)$. Hence, by the set inclusions, $I(R \times S) = I(R) \times I(S)$. □

Example 2.8. Let $R = Z_2$ and $S = Z_3$. Then, $R \times S = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$, $I(R) = \{1\}$, and $I(S) = \{1,2\}$. Therefore, $I(R \times S) = \{(1,1), (1,2)\} = I(R) \times I(S)$.

We will denote with $|I(R)|$, the number of involutions of R. Particularly, if the ring $R = Z_n$, the number $|I_n|$ will represent the number of involutions modulo n. We now state and prove the basic theorem for the involutions of R that shows that the number $|I(R)| > 1$ is even.

Theorem 2.9. For any finite commutative ring R with $|I(R)| > 1$, then $|I(R)|$ is even.

Proof. Let $u \in I(R)$ and $|I(R)| > 1$. Then, $u^2 = 1$, and $|u|$ divides 2. This implies that $|u| \in \{1,2\}$. By the consequence of Lagrange's theorem [1] for finite groups, $|u|\big||I(R)|$. Therefore, for some positive integer q, $|I(R)| = |u|q$. Suppose $|u| = 1$. Then, clearly $u = 1$, because $u^2 = 1$. So, our assumption $|u| = 1$ is not true. Thus, for every unit $u \neq 1$ in $I(R)$, we have $|u| = 2$. Hence, $|I(R)| = 2q$. This concludes that $|I(R)|$ must be even. □

We observe that $|I(R)|$ is even except $R \cong B^n$, as the following remark illustrates how Theorem 2.9 is applicable.

Remark 2.10. If R is a finite cyclic ring with unity and $|I(R)|$ is an odd number, then it must be equal to one, that is $I(R) = \{1\}$. If $|R| > 2$ and $R \ncong R^0, B^n$ then either $|I(R)| = 1$, or $|I(R)|$ must be even. For instance, $R = \frac{Z_2[x]}{(x^3+1)}$ and $R' = \frac{Z_2[x]}{(x^3+x)}$ are both commutative rings with unity 1, so $I(R) = \{1\}$ and $I(R') = \{1, 1 + x + x^2\}$.

Before we proceed, we need to solve the equation $x^2 - 1 = 0$ over the ring R with unity. Note that if $Char(R) = 2$, and then the set of solutions of $x^2 - 1 = 0$ is the same as the set of solutions of $x^2 + 1 = 0$ and vice versa. If $Char(R) \neq 2$, then $x^2 + 1 = 0$ contains either finite or infinite number of solutions over R. In [14], the authors Khanna and Bhambri proved that the equation $x^2 + 1 = 0$ has an infinite number of solutions over the ring of Quaternions. Recently, Suzanne discussed and described the solution of $x^2 + 1 = 0$ in [15]. For finite fields, the following result is well known.

Theorem 2.11. Let F be a finite field with unity 1 and $x^2 = 1$ for some $x \in F$. Then, $x = \pm 1$, in particular, $|I(F)| = 2$.

Proof. Assume F is a finite field with unity 1 and $x^2 = 1$ over F. Then, algebraically $x^2 - 1 = 0$ implies that $(x - 1)(x + 1) = 0$. If both $(x - 1) \neq 0$ and $(x + 1) \neq 0$, then they are both zero-divisors of F. But F has no zero-divisors because every field is an integral domain. So, either $x - 1 = 0$, or $x + 1 = 0$ for some $x \in F$, so that either $x = 1$, or $x = -1$. Hence, $|I(F)| = 2$. □

Example 2.12. Let $F = \{0,2,4,6,8\}$. Then, $(F, +_{10}, \times_{10})$ is a field with unity 6 and the set of involutions $I(F) = \{4,6\}$.

Now we consider the solutions of the equation $x^2 - 1 = 0$ over the finite commutative ring R. For this, we need to consider two cases, i.e., (i) $U(R)$ is a cyclic group and (ii) $U(R)$ is a non-cyclic group.

Before getting started for the enumeration of involutions, we need to recall two familiar theorems from finite group theory.

Theorem 2.13 (Fundamental theorem of cyclic groups) [1]**.** Every subgroup of a cyclic group is cyclic.

Theorem 2.14 (Fundamental theorem of finite abelian groups) [1]**.** Every finite abelian group is isomorphic to a direct product of cyclic groups of prime power order.

Theorem 2.15. Let R be a finite cyclic ring with unity 1. Then, $U(R)$ is a cyclic group if and only if $|I(R)| = 2$.

Proof. Let x be a generator of a finite cyclic group $U(R)$. Then, $U(R) = \langle x \rangle$. Because of $I(R) \subseteq U(R)$, every involution u in $I(R)$ can be written as $u = x^m$ for some positive integer m, $1 \leq m \leq |U(R)|$. Therefore,

$$\begin{aligned} u^2 = 1 \quad &\Leftrightarrow (x^m)^2 = 1 \\ &\Leftrightarrow x^{2m} = 1 \\ &\Leftrightarrow 2m \equiv 0(mod|U(R)|) \end{aligned}$$

Because of $gcd(2, |U(R)|) = 2$, this linear congruence has exactly two solutions. Hence, $|I(R)| = 2$ if and only if $U(R)$ is cyclic. □

Example 2.16. Let us take the ring $R = Z_5$. Then, $I(R) = \{1,4\}$ and $U(R) = \{1,2,3,4\}$. Clearly, $U(R) =< 2 >=< 3 >$ is a cyclic group, and $|I(R)| = 2$.

Theorem 2.17. Let $U(R)$ be the unit group of a finite cyclic ring R with unity 1. For some $k > 1$, $U(R)$ is a non-cyclic group if and only if $|I(R)| = 2^k$.

Proof. By Theorem 2.14, the finite abelian non-cyclic group $U(R)$ is isomorphic to the direct product of cyclic groups of prime power order. Suppose that the prime factorization of $|U(R)|$ is $p_1^{a_1}p_2^{a_2}\dots p_k^{a_k}$, where each p_i is a distinct prime and $k \geq 2$. Then, clearly there exist cyclic groups $U\left(Z_{p_1^{a_1}}\right), U\left(Z_{p_2^{a_2}}\right), \dots, U\left(Z_{p_k^{a_k}}\right)$ of prime power orders such that

$$U(R) \cong U\left(Z_{p_1^{a_1}}\right) \times U\left(Z_{p_2^{a_2}}\right) \times \dots \times U\left(Z_{p_k^{a_k}}\right) \quad \Rightarrow I(U(R)) \cong I\left(U\left(Z_{p_1^{a_1}}\right) \times U\left(Z_{p_2^{a_2}}\right) \times \dots \times U\left(Z_{p_k^{a_k}}\right)\right)$$
$$\Rightarrow I(U(R)) \cong I\left(U\left(Z_{p_1^{a_1}}\right)\right) \times I\left(U\left(Z_{p_2^{a_2}}\right)\right) \times \dots \times I\left(U\left(Z_{p_k^{a_k}}\right)\right)$$

In view of the Remark 2.6 and the Theorem 2.7, we have $I(R) \cong I\left(Z_{p_1^{a_1}}\right) \times I\left(Z_{p_2^{a_2}}\right) \times \dots \times I\left(Z_{p_k^{a_k}}\right)$. From the Theorem 2.15, $U\left(Z_{p_i^{a_i}}\right)$ is a cyclic group and hence $I\left(Z_{p_i^{a_i}}\right) = \left|I\left(U\left(Z_{p_i^{a_i}}\right)\right)\right| = 2$. Therefore, the number of Involutions of a finite cyclic ring R is equal to $|I(R)|$. Clearly, $|U(R)| = p_1^{a_1}p_2^{a_2}\dots p_k^{a_k}$, we have $|I(R)| = \left|I\left(Z_{p_1^{a_1}}\right)\right| \left|I\left(Z_{p_2^{a_2}}\right)\right| \dots \left|I\left(Z_{p_k^{a_k}}\right)\right| = 2.2\dots 2$ (k times) $= 2^k$. □

Example 2.18. Let the ring $R = Z_8$. Then $U(R) = I(R) = \{1,3,5,7\}$ and therefore $U(R)$ is a non-cyclic and $|I(R)| = 4$.

3. Properties of Involutions of Rings

In the previous section, we studied the properties of the set of involutions of finite commutative rings, particularly, finite cyclic rings. A specifically appealing of elementary number theory is that many fundamental properties of the positive integers relating to their primality, divisibility, and factorization can be carried over to the other sets and algebraic structures of numbers. In this section, we study the set of involutions of Gaussian integers modulo n, complex numbers of the form $a + ib$, where a and b are integers modulo n and $i^2 = -1$. We introduce the concept of Gaussian involution and establish the basic properties of Gaussian involutions over addition and multiplication of complex integers over modulo n.

For any positive integer $n \geq 1$, $< n >$ be the proper principal ideal generated by n in the infinite ring of Gaussian integers $Z_n[i]$. So there exists a quotient ring $Z_n[i]/\langle n\rangle$. In [16], the authors Dresden and Dymacek proved that $Z_n[i]/\langle n\rangle$ is isomorphic to $Z_n[i]$, the ring of Gaussian integers modulo n with unity $1 = 1 + i0$ where $n > 1$. If $n = 1$, then $Z_n[i] = \{0 + i\,0\}$. When $R = Z_n[i]$, for given positive integer $n > 1$, we will use the symbols $U_n[i]$, $I_n[i]$ to denote the set of units and involutions of the ring $Z_n[i]$, and call it the set of all Gaussian units and Gaussian involutions modulo n, respectively. It is well known that $|Z_n[i]| = n^2$ and $Z_n[i]$ is a field if and only if $n \equiv 3(mod\,4)$ and also for more information about $Z_n[i]$, see [1]. First, we prove that the basic property of the ring $Z_n[i]$, it indicated that $Z_n[i]$ is not a cyclic ring. First, we notice that $Z_n[i] = \{0\}$ if and only if $n = 1$. Consequently, the following theorem is true for $n > 1$.

Theorem 3.1. The ring $Z_n[i]$ of Gaussian integers modulo n is not a cyclic ring.

Proof. We use proof by contradiction. Suppose $Z_n[i]$ is a cyclic ring for some values of n. Then there exists an element $\alpha = a + bi \in Z_n[i]$ such that $Z_n[i] =< \alpha >$ with respect to the addition of Gaussian integers modulo n. Now we have reached a contradiction. Note that $c + di \in Z_n[i]$ implies there exists a positive integer m such that

$$\begin{aligned} c + di = m(a + bi)(mod\, n) \quad &\Rightarrow ma \equiv c(mod\, n) \text{and } mb \equiv d(mod\, n) \\ &\Rightarrow Z_n = \langle a \rangle \text{ and } Z_n = \langle b \rangle \\ &\Rightarrow Z_n \times Z_n = \langle a \rangle \times \langle b \rangle \\ &\Rightarrow Z_n \times Z_n = \langle (a, b) \rangle \end{aligned}$$

This implies that the ring $Z_n \times Z_n$ is generated by the element (a, b) and thus $Z_n \times Z_n$ is a cyclic group with a generator (a, b) under addition modulo n, which is a contradiction to the fact that $Z_n \times Z_n$ is not a cyclic group for addition modulo n. This completes the proof. □

It is well known that a Diophantine equation is a polynomial equation for which you seek integer solutions, see [17]. For example, the Pythagorean triples (a, b, c) are positive integer solutions to the equation $a^2 + b^2 = c^2$. Here is another Diophantine equation $a^2 - b^2 = 1$ over the infinite ring of integers $\mathbb{Z}$ to the usual addition and multiplication of integers. According to the literature survey of algebraic equations, there are no positive integer solutions to the Diophantine equation $a^2 - b^2 = 1$ over the ring Z. But we observe that there exist integer solutions over the finite ring Z_n. For instance, the pair $(3, 4)$ satisfies the equation $a^2 - b^2 = 1$ over the ring Z_8. The identity $(a + bi)^2 = 1$ is true over the ring $Z_n[i]$ if and only if $a^2 - b^2 = 1$ and $2ab = 0$ over modulo n.

Now we are going to study basic properties of Gaussian involutions $I_n[i]$ and next investigate the cardinality of $I_n[i]$ for various values of n.

Definition 3.2. A Gaussian integer $a + ib$ in $Z_n[i]$ is called a Gaussian unit if $a^2 + b^2 \in U_n$ and the set of Gaussian units $Z_n[i]$ is $U_n[i]$. For example, $U_2[i] = \{1, i\}$.

Properties 3.3. The set $U_n[i]$, the collection of Gaussian units in $Z_n[i]$ has the following basic properties.

- *i.* $U_n \subset U_n[i]$ for every $n > 1$.
- *ii.* If $a + ib$ is a Gaussian unit in, then $Z_n[i]$ then $b + ia$ is also a Gaussian unit in $Z_n[i]$.
- *iii.* If $u, v \in U_n$, then $u + iv$ may not be in $U_n[i]$.
- *iv.* For any odd prime p, $p \not\equiv 3(mod\, 4)$, the unit group U_p is cyclic but $U_p[i]$ may not be cyclic.

Example 3.4.

- *i.* For the rings Z_2 and $Z_2[i]$, the corresponding sets of units are $U_2 = \{1\}$ and $U_2[i] = \{1, i\}$. So that clearly $U_2 \subset U_2[i]$.
- *ii.* In the ring $Z_3[i]$, $1 + 2i$ and $2 + i$ are both Gaussian units.
- *iii.* 1 is a unit in U_4, but $1 + i$ is not a unit in $U_4[i]$.
- *iv.* For the prime $p = 5$, the unit group U_5 is cyclic but $U_5[i]$ may not be cyclic.

Definition 3.5. A Gaussian unit $\alpha = a + ib$ is called a Gaussian involution modulo n if $\alpha^2 = 1$. The set of all Gaussian involutions modulo n is denoted by $I_n[i]$, with cardinality $|I_n[i]|$. For example, $|I_2[i]| = |\{i, 1\}| = 2$, $|I_3[i]| = |\{1,2\}| = 2$, and $|I_4[i]| = |\{1, 1 + 2i, 3, 3 + 2i\}| = 4$.

To determine the structure of the group $I_n[i]$, we must first derive a relation for determining when an element of $I_n[i]$ is a Gaussian involution. Recall that in a finite commutative ring R, a nonzero element is a unit if and only if it is not a zero divisor. Particularly, this is true for the rings Z_n, $Z_n \times Z_n$, $Z_n[i]$, and $Z_n[i] \times Z_n[i]$. Since, $I_n \subseteq U_n$ and $I_n[i] \subseteq U_n[i]$. It is clear that $I_n \subseteq I_n[i]$, it is not surprising that there is an interrelationship between the elements in the groups I_n and $I_n[i]$.

Theorem 3.6. Let $\alpha = a + ib$ be a nonzero element in the ring $Z_n[i]$. Then $a + bi \in I_n[i]$ if and only if $a^2 - b^2 = 1$ and $2ab = 0$ over modulo n.

Proof. Suppose that $\alpha = a + ib \in Z_n[i]$ and $\alpha \neq 0$. By the definition of involution,

$$\begin{aligned} \alpha \in I_n[i] \quad &\Leftrightarrow \ \alpha^2 = 1 \text{ under modulo } n \\ &\Leftrightarrow (a + bi)(a + bi) = 1 \\ &\Leftrightarrow a^2 - b^2 + i2ab = 1 + i0 \\ &\Leftrightarrow a^2 - b^2 = 1 \text{ and } 2ab = 0 \end{aligned}$$

□

Remark 3.7.

i. Every Gaussian involution is a Gaussian unit, but the converse is not true. For instance, $2 + 3i$ is a Gaussian unit in $Z_4[i]$ but not a Gaussian involution, since $2^2 - 3^2 = 3 \neq 1$.

ii. If $a + bi$ is a Gaussian involution, then $b + ai$ may not be a Gaussian involution. For example, $3 + 2i$ is a Gaussian involution in $Z_4[i]$, but $2 + 3i$ is not a Gaussian involution.

In general, it is not clear to satisfy finite groups and their subgroups by resolving the orders of each of its members. According to the Chinese remainder's theorem [18] of numbers, a standard method is to resolve the finite groups to its orders like primes and prime powers as recommended in the following theorems.

Theorem 3.8. [17] If l and m are both relatively prime, then

i. $Z_{lm} \cong Z_l \times Z_m$ and $Z_{lm}[i] \cong Z_l[i] \times Z_m[i]$

ii. $U_{lm} \cong U_l \times U_m$ and $U_{lm}[i] \cong U_l[i] \times U_m[i]$

Theorem 3.9. [17] If $n > 1$ is a positive integer with the canonical form $n = {p_1}^{a_1}{p_2}^{a_2} \dots {p_r}^{a_r}$. Then,

i. $U_n \cong U_{p_1^{a_1}} \times U_{{p_2}^{a_2}} \times \dots \times U_{{p_r}^{a_r}}$

ii. $U_n[i] \cong U_{{p_1}^{a_1}}[i] \times U_{{p_2}^{a_2}}[i] \times \dots \times U_{{p_r}^{a_r}}[i]$

We observe the previous results do hold good for the collection of Gaussian involutions modulo n. We know that the collection of positive integers is partitioned into the sets of positive integers n such that $n \equiv 3(mod\, 4)$, $n \equiv 2(mod\, 4)$, $n \equiv 1(mod\, 4)$, and $n \equiv 0(mod\, 4)$. Also, every odd prime can be written as $n \equiv 3(mod\, 4)$ and $n \equiv 1(mod\, 4)$. We observe that, for the even prime 2, $I_2[i] = \{1, i\}$ and thus $|I_2[i]| = 2$. But, for the collection of Gaussian involutions, we accomplish many results.

Theorem 3.10. If p is a prime of the form $p \equiv 3(mod\, 4)$, then $\left|I_p[i]\right| = 2$.

Proof. Because of the prime p of the form $p \equiv 3(mod\, 4)$, the ring $Z_p[i]$ is a field, and this $U_p[i]$ is a cyclic group. Hence, by the Theorem [2.11], it is well known that every finite field contains exactly two involutions, so $\left|I_p[i]\right| = 2$. □

Example 3.11.

i. For $p = 3$, $I_3[i] = \{1,3\}$ and $|I_3[i]| = 2$.

ii. For $p = 7$, $I_7[i] = \{1,6\}$ and $|I_7[i]| = 2$.

Theorem 3.12. For every prime$p$, $p \equiv 3(mod\ 4)$ and $k \geq 1$ then $|I_{p^k}[i]| = |I_{p^k}| = 2$.

Proof. By the definition of Gaussian involutions,

$$I_{p^k}[i] = \{a + ib \in Z_{p^k}[i] : (a + ib)^2 = 1\} = \{a + ib \in Z_{p^k}[i] : a^2 - b^2 \equiv 1(mod\ p^k), 2ab \equiv 0(mod\ p^k)\}$$

For the condition $2ab \equiv 0(mod\ p^k)$, there are the following possibilities exist. First suppose $a = 0$ and $b = 0$, then $a^2 - b^2 = 0$. This is a contradiction to the fact that $a^2 - b^2 \equiv 1(mod\ p^k)$. So at least one of a and b must be not equal to zero. Suppose the elements a and b are both not equal to 0. Without loss of generality we may assume that $a = p^q$ and $b = p^{k-q}$ $(q > 0)$, $a^2 - b^2 = (p^q)^2 - (p^{k-q})^2 = p^{2q} - p^{2(k-q)} \not\equiv 1(mod\ p^k)$, a contradiction. Hence, we conclude that the condition $b = 0$ holds good because Gaussian involution is not purely imaginary over modulo p^k. This clears that $I_{p^k}[i] = I_{p^k}$.

Now enumerate the total number of Gaussian involutions in $I_{p^k}[i]$. For this let $x \in I_{p^k}[i]$, we have $\alpha = a + bi = a + 0i = a$ and $a^2 = 1$. This implies that

$$\begin{aligned} a^2 - 1 \equiv 0(mod\ p^k) &\Rightarrow ((a-1)+1)((a-1)+1) - 1 \equiv 0(mod\ p^k) \\ &\Rightarrow ((a-1)+1)^2 - 1 \equiv 0(mod\ p^k) \\ &\Rightarrow (a-1)^2 + 2(a-1) \equiv 0(mod\ p^k) \\ &\Rightarrow (a-1)(a+1) \equiv 0(mod\ p^k) \\ &\Rightarrow p^k|(a-1)(a+1) \end{aligned}$$

This shows that $p^k|(a-1)$, or $p^k|(a+1)$. Now suppose $p^k|(a-1)$, then $a - 1 \equiv 0(mod\ p^k)$. Therefore, $a \equiv 1(mod\ p^k)$ implies that $\alpha = 1$. Again suppose $p^k|(a+1)$, then there exists a positive integer r such that $a + 1 = p^k r$. Now we claim that $r = 1$. Suppose $r > 1$. Then, $a = p^k r - 1$ and $a^2 = 1$. This implies that $(p^k r - 1)^2 = 1$. It follows that, either $r = 0$, or $r = 2(p^{-k})$, this is again a contradiction. So, our assumption that $r > 1$ is not true, and hence $r = 1$. Therefore, $a + 1 = p^k$, and thus $a = \alpha = p^k - 1$. This shows that $\alpha = 1$ and $\alpha = p^k - 1$ are the only two elements in $I_{p^k}[i]$. So, for every prime $p \equiv 3(mod\ 4)$ there is a cyclic subgroup $\langle 1, p^k - 1 : (p^k - 1)^2 \equiv 1(mod\ p^k)\rangle$ in the group $U_{p^k}[i]$ such that $I_{p^k}[i] \cong \langle 1, p^k - 1 : (p^k - 1)^2 \equiv 1(mod\ p^k)\rangle \cong I_{p^k}$. Hence, $|I_{p^k}[i]| = |I_{p^k}| = 2$. □

Example 3.13.

i. For $p = 3$ and $k = 2$, $I_{3^2}[i] = I_9[i] = \{1,8\}$ and $|I_{3^2}[i]| = 2$.

ii. For $p = 7$ and $k = 2$, $I_{7^2}[i] = I_{49}[i] = \{1,48\}$ and $|I_{7^2}[i]| = 2$.

Theorem 3.14. If p is a prime of the form $p \equiv 1(mod\ 4)$ and $k \geq 1$, then $|I_{p^k}[i]| = 4$.

Proof. For the prime p of the form $p \equiv 1(mod\ 4)$, the set of Gaussian involutions of the ring $Z_{p^k}[i]$ is $I_{p^k}[i] = \{a + ib \in Z_{p^k}[i] : (a + ib)^2 \equiv 1(mod\ p^k)\}$. Let $a + ib \in I_{p^k}[i]$, then

$$(a + ib)^2 \equiv 1(mod\ p^k) \Rightarrow a^2 - b^2 \equiv 1(mod\ p^k) \text{ and } 2ab \equiv 0(mod\ p^k)$$

First, $2ab \equiv 0(mod\ p^k)$ means $a = 0$ or $b = 0$. From this condition, the group $I_{p^k}[i]$ reduces to $I_{p^k}[i] = \{a, ib \in Z_{p^k}[i] : a^2 \equiv 1(mod\ p^k)\}, (ib)^2 \equiv 1(mod\ p^k)\}$. This shows that for $a, ib \in I_{p^k}[i]$, we have $p^k|(a^2 - 1)$ and $p^k|(b^2 + 1) \Rightarrow a^2 - 1 \equiv 0(mod\ p^k)$ and $b^2 + 1 \equiv 0(mod\ p^k)$.

These two quadratic congruences give two distinct values for a and two distinct values for b over modulo p^k. Consequently, for α and β in $U_{p^k}[i]$, there is a non-cyclic subgroup $I_{p^k}[i]$ of the group $U_{p^k}[i]$ such that $I_{p^k}[i] = \langle \alpha, \beta : \alpha^2 - 1 \equiv 0(mod\ p^k),\ \beta^2 + 1 \equiv 0(mod\ p^k)\rangle$ whenever the prime $p \equiv 1(mod\ 4)$. Therefore, $|I_{p^k}[i]| = 4$. □

Example 3.15.

1. Let $p = 5$.
   i. If $\alpha = 1$, then $I_5[i] = \{1,4,2i,3i\}$ and $|I_5[i]| = 4$.
   ii. If $\alpha = 2$, then $I_{5^2}[i] = I_{25}[i] = \{1,24,7i,18i\}$ and $|I_{5^2}[i]| = 4$.
2. Let $p = 13$.
   i. If $\alpha = 1$, then $I_{13}[i] = \{1,12,5i,8i\}$ and $|I_{13}[i]| = 4$.
   ii. If $\alpha = 2$, then $I_{13^2}[i] = I_{169}[i] = \{1,168,70i,99i\}$ and $|I_{13^2}[i]| = 4$.

Theorem 3.16. For even prime 2 and $k > 1$ then $I_{2^k}[i] \cong I_2[i] \times I_2[i] \times \dots \times I_2[i](k\text{times})$ and $|I_{2^k}[i]| = 2^k$.

Proof. Since $I_2[i]$ is a cyclic group of order 2, and thus $I_{2^k}[i]$ is a finite abelian but not cyclic. Accordingly, by the fundamental theorem of finite abelian groups, the group $I_{2^k}[i]$ can be written as $I_{2^k}[i] \cong I_2[i] \times I_{2^{k-1}}[i] \cong I_2[i] \times I_2[i] \times I_{2^{k-2}}[i] \cong \cdots \cong I_2[i] \times I_2[i] \times \dots \times I_2[i](k \text{ times})$ and hence

$$\begin{aligned} |I_{2^k}[i]| &= |I_2[i] \times I_2[i] \times \dots \times I_2[i](k\text{times})| \\ &= |I_2[i]| \cdot |I_2[i]| \cdot \dots \cdot |I_2[i]| \ (k \text{ times}) \\ &= 2 \cdot 2 \cdot \dots \cdot 2(k \text{ times}) \\ &= 2^k \end{aligned}$$

□

Example 3.17. For $k = 2$, $I_{2^2}[i] = I_4[i] = \{1,3,1+2i,3+2i\}$ and $|I_{2^2}[i]| = 4 = 2^2$.

If the prime $p > 2$ then Theorem 3.16 is not true, that is $|I_{p^k}[i]| \neq p^k$ because $I_{p^k}[i] \ncong I_p[i] \times I_{p^{k-1}}[i]$. For example, $I_{5^2}[i] \ncong I_5[i] \times I_5[i]$. In particular, the following results are well cleared. For any $k > 1$,

i. $Z_{2^k} \ncong Z_2 \times Z_{2^{k-1}}$ and $Z_{2^k}[i] \ncong Z_2[i] \times Z_{2^{k-1}}[i]$

ii. $U_{2^k} \ncong U_2 \times U_{2^{k-1}}$ and $U_{2^k}[i] \ncong U_2[i] \times U_{2^{k-1}}[i]$

iii. $I_{2^k} \cong I_2 \times I_{2^{k-1}}$ and $I_{2^k}[i] \cong I_2[i] \times I_{2^{k-1}}[i]$

Theorem 3.18. If p and q are relatively prime, then $I_{pq}[i] \cong I_p[i] \times I_q[i]$.

Proof. Without loss of generality, assume that $p \equiv 2(mod\, 4)$ and $q \equiv 3(mod\, 4)$. Now we define a map $f: I_p[i] \times I_q[i] \to I_{pq}[i]$ by the relation $f((a,b)) = iqa + pb$ for every $(a,b) \in I_p[i] \times I_q[i]$ and the element $iqa + pb \in I_{pq}[i]$ for all a and b. One can easily verify that$f$ is a well-defined group homomorphism. Now to show that f is an injection. For $(a,b),(c,d) \in I_p[i] \times I_q[i]$, we have $f((a,b)) = f((c,d))$. This implies that

$$\begin{aligned} iqa + pb = iqc + pd \quad &\Rightarrow a = c \text{ and } b = d \\ &\Rightarrow (a,b) = (c,d) \end{aligned}$$

Thus f is injective. Since the finite groups $I_p[i] \times I_q[i]$ and $I_{pq}[i]$ have the same cardinality, so that f is surjective and hence f is a group isomorphism. □

For example, take $p \equiv 2$ and $q \equiv 3$, $I_6[i] \cong I_2[i] \times I_3[i]$. We have $I_2[i] = \{1,i\}$, $I_3[i] = \{1,2\}$ and $I_6[i] = \{1,5,2+3i,4+3i\}$. Clearly, $(1,1) \to 2+3i$, $(1,2) \to 4+3i$, $(i,1) \to 5$, and $(i,2) \to 1$.

Theorem 3.19. Let $n > 1$ be a positive integer with the canonical form $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$. Then,

$$I_n[i] \cong I_{p_1^{a_1}}[i] \times I_{p_2^{a_2}}[i] \times \dots \times I_{p_r^{a_r}}[i] \text{ and } |I_n[i]| \cong |I_{p_1^{a_1}}[i]| \times |I_{p_2^{a_2}}[i]| \times \dots \times |I_{p_r^{a_r}}[i]|$$

Proof. It is clear from the Chinese remainder theorem [18]. □

Generally, now establish a formula for enumerating the total number of Gaussian involutions in the Gaussian ring for various values of n. Remember that the cardinality of the Gaussian involutions of the non-cyclic ring $Z_n[i]$ is $|I_n[i]|$ and $I\ (Z_n[i]) = I\ \ (U_n[\ i])$, and the representation theory of the finite cyclic group is a critical base case for the representation theory of more general finite groups. For any integer $n \geq 1$, there exists a finite cyclic group C_n with representation $C_n = \langle a : a^n = 1\rangle$ for multiplication. For instance, a group $C_2 = \{1, a : a^2 = 1\}$ is a cyclic group of order 2, and it is also isomorphic to the cyclic group $Z_2 = \{0,1\}$ for addition modulo 2.

Theorem 3.20. If n is a positive integer, then $|I_n[i]| = 2^k$ for some positive integer k.

Proof. The result is clear if $n = 2$. If $n = 2$ so that $|Z_n[i]| = 4$, then there is only one subgroup, namely $\{1, i\}$ in $Z_n[i]$ with the property that $a^2 = 1$, and so $|I_n[i]| = 2 = 2^1$. Assume that $n > 2$. We now prove this by the two cases, namely, $I_n[i]$ is either cyclic or not. First, suppose $I_n[i]$ is cyclic. Then, there is nothing to prove. Now suppose $I_n[i]$ is a non-cyclic abelian group, then we have to prove that $|I_n[i]| = 2^k$ for some positive integer k. For this, we define a map $f : Z_2 \times Z_2 \times \dots \times Z_2 \to I_n[i]$ by the relation $f(a_1, a_2, \dots, a_k) = \alpha_1^{a_1}\alpha_2^{a_2} \dots \alpha_k^{a_k}$ for every element $a_1, a_2, \dots, a_k$ in the non-cyclic group $Z_2 \times Z_2 \times \dots \times Z_2 \cong Z_2^k$, where $\alpha_1^{a_1}, \alpha_2^{a_2}, \dots, \alpha_k^{a_k}$ are distinct k involutions of $I_n[i]$. By Theorem 3.18, $I_n[i] \cong Z_2^k$, and hence $|I_n[i]| = \left|Z_2^k\right| = 2^k$. □

For verification of the above results, we obtain the following set of Gaussian involutions of the Gaussian ring $Z_n[i]$ with fixed values of $n = 2,3,4, \dots ,13$, respectively.

$I_2[i] = \{1, i\} \cong C_2,$

$I_3[i] = \{1,2\} \cong C_2,$

$I_4[i] = \{1,3,1 + 2\,i, 3 + 2i\} \cong C_2 \times C_2,$

$I_5[i] = \{1,4,2\,i, 3\,i\} \cong C_2 \times C_2,$

$I_6[i] = \{1,5,2 + 3i, 4 + 3i\} \cong C_2 \times C_2,$

$I_7[i] = \{1,6\} \cong C_2,$

$I_8[i] = \{1,3,5,7,1 + 4i, 3 + 4i, 5 + 4\,i, 7 + 4i\} \cong C_2 \times C_2 \times C_2,$

$I_9[i] = \{1,8\} \cong C_2,$

$I_{10}[i] = \{1,9,3i, 7i, 4 + 5i, 5 + 2i, 6 + 5i, 5 + 8i\}\} \cong C_2 \times C_2 \times C_2,$

$I_{11}[i] = \{1,10\} \cong C_2,$

$I_{12}[i] = \{1,5,7,11,1 + 6i, 5 + 6i, 7 + 6i, 11 + 6i\} \cong C_2 \times C_2 \times C_2,$

$I_{13}[i] = \{1,12,5i, 8i\} \cong C_2 \times C_2$

4. Conclusion

Owing to the involution theory, involutions over finite commutative rings have been widely used in applications such as algebraic cryptography, network security, and coding theory. Further, quadratic polynomials like $x^2 - 1 = 0$ over finite rings and fields have been extensively studied due to their wide applications in block cipher designs, algebraic coding theory, and combinatorial design theory. Following these applications of involutions to characterize the involutory behaviour of the digital control systems, digital logic systems, modern algebraic systems, and generalized cyclotomic systems and this paper gives more concise criterion analytical methods for enumerating Involutions over the finite cyclic and non-cyclic rings.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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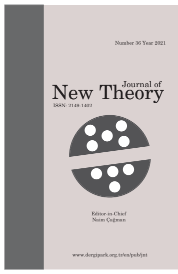

Construction of Developable Surface with Geodesic or Line of Curvature Coordinates

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Abstract — In this paper, a developable surface with geodesic or line of curvature coordinates is constructed in the Euclidean 3-space. A developable surface is coordinated by two families of parametric curves, base curves (directrices) and lines (rulings). Since any part of a straight line on a developable surface is geodesic and line of curvature, we only need to show that the directrices curves are geodesics or lines of curvature to ensure that the developable surface is parameterized by geodesic or line of curvature coordinates. The necessary and sufficient conditions for the directrices curves to be geodesics or lines of curvature are studied. The main results of this paper show that the developable surface with geodesic coordinates is a generalized cylinder, and the developable surface with line of curvature coordinates is a tangent surface.

Keywords — *Developable surface, geodesic, line of curvature, parametric curves, coordinates*

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1. Introduction

A ruled surface is constructed by the continuous motion of a straight line called the ruling or generator through a given curve called the base or directrix curve. Developable surfaces are a special class of the ruled surfaces that can be mapped isometrically into the plane, therefore the developable surface has zero Gaussian curvature as the plane. Cylinders (including planes), cones, and tangent surfaces are three basic types of developable surface. The developable surface has been used in geometric modeling, architecture, and many applications in manufacturing industries [1–3].

Geodesic is a curve that travel in directions of zero geodesic curvature on a surface. The shortest path between two given points on a curved space is given by geodesic. Any (part of) a straight line on a surface is a geodesic as are the rulings of any ruled surface. Line of curvature is a curve that travel in directions for which the surface curvature takes its maximum or minimum values, in other words, the directions in which a shape bends extremely. The lines of curvatures exists as orthogonal curves on non-umbilical points of the surface. Rulings of any developable surface are lines of curvature.

A coordinate on a surface is said to be geodesic or line of curvature if the two families of coordinate curves are geodesics or lines of curvature. Extracting the geometric information from the surface depends on the parametrization that is using as a coordinate system on the surface. Using geodesic or line of curvature coordinates are suitable for many tasks not only in differential geometry but involves other areas of applications, such as integrable systems [4], surfaces motions [5, 6], and architectural shape design [7, 8].

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In this paper, we discuss how a developable surface can be parameterized by geodesic or line of curvature coordinates. For our purpose, we parameterize the ruled surface by its directrix curve and director vector that is expressed by a linear combination of Frenet frame with angular functions as coefficients. The two families of parametric curves are rulings and directrices. It is well known that the rulings are geodesics and lines of curvature on a developable surface. Consequently, throughout this paper, our focus lies on the family of directrices. Many constraints are assumed on the angular functions through many steps to generate parametrization whose directrices are geodesics or lines of curvature. The main results show that the developable surface with geodesic coordinates is a generalized cylinder, and the developable surface with line of curvature coordinates is a tangent surface.

There are many articles for constructing a developable surface family that possesses the directrix curve as a common characteristic curve, see for example [9–11]. For our approach, [11] is a relevant paper, but it is restricted and limited. More explicitly, it focused on the sufficient and necessary conditions for only one directrix curve to be a line of curvature, whereas our paper deals with conditions on the family of directrices to be geodesics or lines of curvature in the constructed developable surface.

The rest of this paper is organized as follows: In section 2, some basic notations, facts and definitions of the space curves and ruled surfaces are reviewed. The main results are studied in section 3, where the developable surface with geodesic or line of curvature coordinates is constructed, the necessary and sufficient conditions for the directrices curves to be geodesics or lines of curvature are derived, some geometric properties of such coordinates are analyzed. Examples is presented in section 4. Finally, the conclusion is given in section 5.

2. Preliminary

This section introduces some basic concepts on the classical differential geometry of space curves and ruled surfaces in Euclidean 3-space, as well as some basic definitions and notions that are required subsequently. More details can be found in such standard references as [12, 13].

2.1. Curves in Euclidean 3-space

A smooth space curve in Euclidean 3-space is parameterized by a map $\gamma : I \subseteq \mathbb{R} \to E^3$, γ is called a regular curve if $\gamma' \neq 0$ for every point of an interval $I \subseteq \mathbb{R}$, and if $|\gamma'(s)| = 1$ where $|\gamma'(s)| = \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}$, then γ is said to be of unit speed (or parameterized by arc-length s). For a unit speed regular curve $\gamma(s)$ in E^3, the unit tangent vector $t(s)$ of γ at $\gamma(s)$ is given by $t(s) = \gamma'(s)$. If $\gamma''(s) \neq 0$, the unit principal normal vector n(s) of the curve at $\gamma(s)$ is given by $n(s) = \frac{\gamma''(s)}{\|\gamma''\|}$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at $\gamma(s)$. Physically, the vectors $\gamma'(s)$ and $\gamma''(s)$ are called the velocity and acceleration vectors respectively. For each point of $\gamma(s)$ where $\gamma''(s) \neq 0$, we associate the Serret-Frenet frame $\{t, n, b\}$ along the curve γ. As the parameter s traces out the curve, the Serret-Frenet frame moves along γ and satisfies the following formula.

$$\begin{cases} t'(s) &= \kappa(s)n(s) \\ n'(s) &= -\kappa(s)t(s) + \tau b(s) \\ b'(s) &= -\tau(s)n(s) \end{cases} \tag{1}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and torsion functions. The planes spanned by $\{t(s), n(s)\}$, $\{t(s), b(s)\}$ and $\{n(s), b(s)\}$ are respectively called the osculating plane, the rectifying plane and the normal plane. Kinematically, when the point moves along the unit speed curve with non vanishing curvature and torision, the Serret-Frenet frame $\{t, n, b\}$ is drawn to the curve at each position of the moving point, this motion consists of translation with rotation and described by the following Darboux vector

$$\omega = \tau t + \kappa b \tag{2}$$

The direction of Darboux vector is the direction of rotational axis and its magnitude gives the angular velocity of rotation. The unit Darboux vector field is defined by

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}t + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}b \tag{3}$$

A necessary and sufficient condition that a curve be of constant slope (or general helix) is that the ratio of curvature to torsion is constant ($\frac{\tau}{\kappa} = c$). The general helix lies on a general cylinder and also known as a cylindrical helix. The circular helix (a helix on a circular cylinder) is a special helix with both of $\kappa(s) \neq 0$ and $\tau(s)$ are constants. The Darboux vector is constant for circular helix. For the cylindrical helix, the unit Darboux vector is constant, where (3) can be written as

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}t + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}b = \frac{\tau/\kappa}{\sqrt{(\tau/\kappa)^2+1}}t + \frac{1}{\sqrt{(\tau/\kappa)^2+1}}b = \frac{c}{\sqrt{c^2+1}}t + \frac{1}{\sqrt{c^2+1}}b \tag{4}$$

For a regular curve on a surface, there exists another frame which is called Darboux frame and denoted by $\{t(s), g(s), N(s)\}$. In this frame $t(s)$ is the unit tangent of the curve, $N(s)$ is the unit normal of the surface and g is a unit vector given by $g = N \times t$. Derivative of the Darboux frame according to arc-length parameter is governed by the following relations

$$\begin{cases} t'(s) &= \kappa_g g(s) + \kappa_n N(s) \\ g'(s) &= -\kappa_g t(s) + \tau_g N(s) \\ N'(s) &= -\kappa_n t(s) - \tau_g g(s) \end{cases} \tag{5}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion at each point of the curve $\gamma(s)$ which are given by the following

$$\kappa_g =< \gamma''(s), g >, \qquad \kappa_n =< \gamma''(s), N >, \qquad \text{and} \qquad \tau_g =< N', g > \tag{6}$$

The unit tangent t is common in both Frenet frame and Darboux frame, then the vectors N,g,n and b lie on the same plane and the relations between these frames can be given as follows:

$$\begin{pmatrix} t \\ g \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix} \tag{7}$$

where

$$\begin{cases} g(s) &= \cos\phi(s)n(s) + \sin\phi(s)b(s) \\ N(s) &= -\sin\phi(s)n(s) + \cos\phi(s)b(s) \end{cases} \tag{8}$$

Differentiating (8), by using (5) and (1) we get the relation between geodesic curvature, normal curvature, geodesic torsion with curvature, torsion as follows:

$$\kappa_g = \kappa\cos\phi, \qquad \kappa_n = \kappa\sin\phi, \qquad \text{and} \qquad \tau_g = \tau + \frac{d\phi}{ds} \tag{9}$$

A unit-speed curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere ($\kappa_g = 0$) [13]. From (9) and (8), this condition is equivalent to the following

$$N = \pm n \tag{10}$$

Hence a curve is a geodesic on the surface if and only if the principal normal n to the curve and the surface normal N are parallel to each other at any point on the curve. Equivalently, a curve $\gamma(s)$ on the surface is a geodesic provided its acceleration vector $\gamma''(s)$ is always normal to the surface, i.e.

$$\gamma''(s) \times N = 0 \tag{11}$$

A unit-speed curve on a surface is a line of curvature if and only if its geodesic torsion is zero everywhere ($\tau_g = 0$) [13]. From (9), this condition is equivalent to the following

$$\tau + \frac{d\phi}{ds} = 0 \tag{12}$$

2.2. Ruled Surfaces

Let $X(u, v)$ be a regular parameterization of a smooth surface in Euclidean 3-space, and defined by

$$X(u, v) : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

The variables (u, v) are called the (curvilinear) coordinates on the surface, the two families of u-curves ($v = const$), and v-curves ($u = const$), are called the parametric curves (or coordinate curves). Their directions are defined by the tangents vectors X_u and X_v respectively. The surface $X(u, v)$ is regular if the condition $X_u \times X_v \neq 0$ is satisfied for all points, that means the vectors X_u and X_v do not vanish and have different directions. Consequently, the surface normal is defined at every point on the regular surface as a unit vector on the tangent plane and given by

$$N(u, v) = \frac{X_u \times X_v}{|X_u \times X_v|} \tag{13}$$

The first and second fundamental form of the parameterized regular surface are given by

$$I = Eds^2 + 2Fdsdv + Gdv^2, \quad II = eds^2 + 2fdsdv + gdv^2 \tag{14}$$

where their coefficients can be calculated respectively as

$$E =< X_s, X_s >, \quad F =< X_s, X_v >, \quad G =< X_v, X_v > \tag{15}$$

$$e =< N, X_{ss} >, \quad f =< N, X_{vs} >, \quad g =< N, X_{vv} > \tag{16}$$

The fundamental quantities I and II are important tools to describe the intrinsic and extrinsic geometry of surface. In particular, type of the parametric curves and their characteristics properties are described by the coefficients of the fundamental quantities I and II. For example, the coordinate curves are orthogonal if $F = 0$, conjugate if $f = 0$, and lines of curvature if satisfy both conditions.

Theorem 2.1. [14] A necessary and sufficient condition for the coordinate curves of a parametrization to be lines of curvature in a neighborhood of a nonumbilical point is that $F = f = 0$.

The ruled surfaces are defined by moving a straight line on a given curve and parameterized by

$$X(s, v) = \gamma(s) + vD(s), \quad 0 \leq s \leq \ell, \quad v \in \mathbb{R} \tag{17}$$

A unit regular curve $\gamma(s)$ is called a base curve (or directrix), and the line passing through $\gamma(s)$ that is parallel to $D(s)$ is called the ruling (or generator). $D(s)$ is a unit director vector field that gives the direction of the ruling. Different ruled surfaces can be constructed based on different $\gamma(s)$ and $D(s)$. By the above discussion, the ruled surface has two families of parametric curves, the family of lines (or rulings), and the family of base curves (or directrices), which can be defined respectively by

$$X(s_0, v) = \{\gamma(s_0) + vD(s_0),\, 0 \leq s_0 \leq \ell\} \quad \text{and} \quad X(s, v_0) = \{\gamma(s) + v_0 D(s),\, v_0 \in \mathbb{R}\} \tag{18}$$

In this paper, we call $\gamma(s)$ the first directrix curve, where $X(s, 0) = \gamma(s)$. The unit normal vector field (surface normal, for shortly) is defined by

$$N(s, v) = \frac{X_s \times X_v}{|X_s \times X_v|} = \frac{[\alpha'(s) + vD'(s)] \times D(s)}{[\alpha'(s) + vD'(s)] \times D(s)|} \tag{19}$$

A point on a ruled surface that satisfies $X_s \times X_v = 0$ is called a singular point. A ruled surface may have singular points, which are located (if exist) on the striction curve that parameterized by [12]

$$\beta(s) = \gamma(s) - \frac{\langle \gamma'(s), D'(s) \rangle}{\langle D'(s), D'(s) \rangle} D(s), \quad D'(s) \neq 0 \tag{20}$$

A ruled surface is said to be cylindrical if D(s) is constant, i.e. $D'(s) = 0$. Otherwise, it is said to be non-cylindrical. The Gaussian curvature is non-positive for a ruled surface, it vanishes identically for special classes called the developable surfaces. Equivalently, a ruled surface (17) is developable if and only if [12]

$$det\langle \gamma'(s), D(s), D'(s) \rangle = 0 \tag{21}$$

Remark 2.2. For a developable surface, the singular points can be classified using (20) into three cases based on the type of developable surface.

i. Cylindrical surface (all rulings are parallel i.e. $D'(s) = 0$), has no singular points.

ii. Conical surface (all rulings meet at a vertex), has one singular point at its vertex.

iii. Tangent surface has singularities along the curve which is called the cuspidal edge.

A curve is called a twisted curve if has nonzero curvature and torsion. Through this paper, we assume that the first directrix curve $\gamma(s)$ is a unit speed twisted curve. $D(s)$ is a unit director vector field lies in the space formed by moving frame $\{t, n, b\}$ of $\gamma(s)$ and can be written using (7) as

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)g(s), \quad \text{where} \quad g(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s) \tag{22}$$

Therefore, $D(s)$ can be decomposed as following [15]

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)) \tag{23}$$

Definition 2.3. A surface defined by:

$$X(s, v) = \gamma(s) + vD(s), \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{24}$$

is ruled surface where $D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))$.

The singular points on the constructed ruled surface are avoided, therefore the vectors $\gamma'(s)$ and $D(s)$ must be not collinear. It requires that $\sin\theta(s) \neq 0$ which can be used as a regularity condition. By using (8) the surface normal along the first directrix curve is given by

$$N(s, 0) = -\sin\phi(s)n(s) + \cos\phi(s)b(s) \tag{25}$$

Lemma 2.4. [15] A ruled surfaces family (24) is developable if and only if the following condition is satisfied,

$$\sin\theta(s)(\frac{d\phi}{ds} + \tau(s)) - \kappa(s)\sin\phi(s)\cos\theta(s) = 0 \tag{26}$$

The main result of this paper is the following main theorem which is proved in the next section.

Theorem 2.5. Let $X(s, v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$ be a ruled surface, where $\gamma(s)$ is a unit speed twisted curve, and $D(s)$ is a unit director vector defined by (23), then the developable surface with geodesic coordinates is a generalized cylinder. And the developable surface with lines of curvatures coordinates is a tangent surface.

3. Developable Surface with Geodesic and Line of Curvature Coordinates

This section is the main part of this paper, it consists of two subsections which are devoted to parameterizing the developable surface with geodesics and lines of curvature coordinate respectively. As well known, the parametric curves of the developable surface are base curves (directrices) and lines (rulings). Any part of a straight line on a developable surface is geodesic and line of curvature, therefore, this section is devoted to giving the necessary and sufficient condition for the directrices to be geodesics or lines of curvature. We show that the developable surface with geodesic coordinates is a generalized cylinder, and the developable surface with line of curvature coordinates is a tangent surface.

3.1. Developable Surface with Geodesics Coordinates

This subsection is devoted to construct a ruled developable surface parameterized by (24) with geodesic coordinates. In particular, to design a developable surface with geodesic directrices curves. For this purpose, three conditions must be satisfied through three theorems, the first one is a geodesic condition (10) that insures the first directrix curve is a geodesic. The second one is the developability condition (26). Finally, the third condition is that makes the other directrices are geodesics. In the following, we start with the first theorem which gives the condition that makes the first directrix curve is a geodesic.

Theorem 3.1. Let X be a ruled surface parameterized by $X(s,v) = \gamma(s) + vD(s)$, $0 \leq s \leq L$, $v \in \mathbb{R}$, where $\gamma(s)$ is the first directrx curve. $D(s)$ is a unit director vector defined by (23), then $\gamma(s)$ is a geodesic if and only if $D(s)$ is a rectifying vector.

PROOF. From (25), it is clear that $N = \pm n$, if and only if $\cos\phi(s) = 0$, by (10) this is equivalent to $\gamma(s)$ is a geodesic if and only if $D(s)$ is a rectifying vector. □

Definition 3.2. A ruled surface defined by:

$$X(s,v) = \gamma(s) + vD(s), \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{27}$$

possesses the first directrix curve as geodesic where $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$.

The following theorem gives the second condition that makes the ruled surface with geodesic first directrix (27) is a developable

Theorem 3.3. A ruled surface that has a geodesic first directrix (27) is developable if and only if $\tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0$.

PROOF. Since $\cos\phi(s) = 0$ for a ruled surface whose the first directrix is a geodesic (27), then the developability condition (26) can be abbreviated as

$$\tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0 \tag{28}$$

□

Definition 3.4. A ruled surface defined by:

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)b(s)], \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{29}$$

is a developable surface with geodesic first directrix where $\tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0$.

The developability condition (28) for a ruled surface whose first directrix curve is a geodesic has geometric restriction should be imposed on the unit director vector as given in the following proposition

Proposition 3.5. [16] Suppose that $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ is a rectifying vector field defined along a unit speed twisted curve $\gamma(s)$, then $D(s)$ is a unit Darboux vector field if and only if $\kappa\cos\theta - \tau\sin\theta = 0$

PROOF. Let $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ be a unit Darboux vector. From (3),

$$\cos\theta = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \quad \sin\theta(s) = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \quad \text{and} \quad \cot\theta = \frac{\tau}{\kappa}$$

This implies that $\kappa\cos\theta - \tau\sin\theta = 0$, and vice versa. □

Definition 3.6. A ruled surface defined by:

$$X(s,v) = \gamma(s) + vD(s), \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{30}$$

is developable with geodesic first directrix where $D(s) = \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s)$.

The developable surface whose first directrix curve is a geodesic has an equivalent two parameterization (29) and (30). The parameterization (30) has a nice geometric expression, where the constraints are viewed on the director vector, whereas the calculations based on differentiation become easier with parameterization (29). This can be shown as follows, where the first and second derivatives of the developable surface with geodesic first directrix (29) are listed in the following equations

$$\begin{cases} X_s &= (1 - v\sin\theta\theta'(s))t + v\cos\theta\theta'(s)b \\ X_{ss} &= -v(\sin\theta\theta'' + \cos\theta\theta'^2)t + [\kappa - v\theta'(\kappa\sin\theta + \tau\cos\theta)]n + v(\cos\theta\theta'' - \sin\theta\theta'^2)b \\ X_{sv} &= -\sin\theta\theta'(s)t(s) + v\cos\theta\theta'(s)b(s) \\ X_v &= \cos\theta(s)t(s) + \sin\theta(s)b(s) \\ X_{vv} &= 0 \end{cases} \tag{31}$$

The inner and cross products of the tangents vectors X_s and X_v are given by

$$\begin{cases} \langle X_s, X_v \rangle &= \cos\theta(s) \\ X_s \times X_v &= (-\sin\theta(s) + v\theta'(s))n(s) \end{cases} \tag{32}$$

By using the regularity condition $\sin\theta(s) \neq 0$, and from (32) the unit normal of developable surface with geodesic first directrix (29) is defined everywhere and given by the following

$$N(s,v) = \frac{X_s \times X_v}{|X_s \times X_v|} = n \tag{33}$$

Until now, we get a parameterization (30) that is characterized by two properties, the first directrix curve is geodesic and the ruled surface is developable. These are satisfied under two constraints, the director vector must be rectifying, and Darboux vector receptively. The following third theorem gives the third condition that makes the directrices are geodesics as one of the main results of this paper.

Theorem 3.7. Let $X(s,v)$ be a developable surface parameterized by (29). Then, the directrices curves are geodesics if and only if θ is constant.

Proof. The directrices curves on a developable surface (29) are geodesics if and only if its acceleration vector X_{ss} is normal to the surface according to (11), then we have

$$N(s,v) \times X_{ss} = 0 \tag{34}$$

From (31) and (33), it follows that

$$-v(\sin\theta\theta'' + \cos\theta\theta'^2)b + v(\cos\theta\theta'' - \sin\theta\theta'^2)t = 0$$

The above condition is satisfied when $v = 0$, i.e., the first directrix curve is geodesic. Or the coefficients of tangent and binormal vectors must vanish simultaneously as the following

$$\begin{cases} \sin\theta\theta'' + \cos\theta\theta'^2 &= 0 \\ \cos\theta\theta'' - \sin\theta\theta'^2 &= 0 \end{cases} \tag{35}$$

Using the elimination method, we obtain their common solution $\theta' = 0$, which implies that θ is a constant and vice versa. □

Definition 3.8. A ruled surface defined by:

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)b(s)], \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{36}$$

is developable with geodesic directrices where $\tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0$ and $\theta'(s) = 0$.

As we know from poposition (3.5), the second condition $\tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0$ means that the unit director vector is a unit Darboux vector. At the same time, the third condition, $\theta'(s) = 0$, implies that a unit Darboux vector is constant.

Definition 3.9. A ruled surface defined by:

$$X(s,v) = \gamma(s) + vD(s), \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{37}$$

is developable with geodesic directrices where $D(s) = \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s)$ and $D'(s) = 0$.

There is an important hidden property in an equivalent parameterizations (36) and (37) for a developable surface with geodesic directrices, and can be extracted by the following proposition

Proposition 3.10. For a developable surface with geodesic directrices and parametrized by (36) or (37), the first directrix is a helix.

PROOF. For parameterization (36), we have the second condition

$$\tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0, \quad \text{which gives} \quad \cot\theta = \frac{\tau}{\kappa}$$

At the same time, we have the third condition $\theta'(s) = 0$, which implies that $\frac{\tau}{\kappa} = c$, then the first directrix is a helix by definition. For parameterization (37), the unit Darboux vector is a constant, which means that the first directrix is a helix as discussed in (4), which confirms that every helix described by its constant unit Darboux vector. □

The first directrix curve and the director vector are responsible to build the ruled developable surface, therefore the following theorem gives the conditions that can be applied on the first directrix curve and the director vector at the same time to generate a developable parameterization with geodesic directrices.

Theorem 3.11. Let $X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$ be a ruled surface, where $\gamma(s)$ is a unit speed twisted curve, $D(s)$ is a unit director vector defined by (23). Then every ruling is geodesic and the directrices curves are geodesics if and only if $\gamma(s)$ is a helix and $D(s)$ is a unit Darboux vector.

Definition 3.12. A ruled surface defined by:

$$X(s,v) = \gamma(s) + vD(s), \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{38}$$

is developable with geodesic directrices where $D(s) = \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s)$ and $\gamma(s)$ is a helix.

The developable surface with geodesic directrices is a generalized cylinder, where the director vector is constant, i.e., $D'(s) = 0$ (37). Therefore, the generalized cylinder is the only developable surface that can be coordinated by geodesics parametric curves with additional assumptions. This result is given in the following theorem

Theorem 3.13. Among all developable surfaces parameterized by (24), the generalized cylinder can be equipped with geodesic coordinates if and only if the directrix curve is a helix and the director vector is a unit Darboux vector.

For a developable surface with geodesic coordinates (38), the directrices geodesic curves have the same curvature and torsion, and differ only by the rigid motion modeled by a constant unit Darboux vector with fixed direction and fixed angular velocity, this result formulated as following

Proposition 3.14. Let $X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$, be a ruled surface, $D(s)$ is a unit director vector defined by (23), if the directrices are geodesics, then they are congruent.

Corollary 3.15. The geodesic parametric curves of the developable surface (38) are lines and helices.

Corollary 3.16. There is no a skew ruled surface parameterized by (24) whose parametric curves are geodesics.

3.2. Developable Surface with Line of Curvature Coordinates

This subsection investigates how to construct a ruled developable surface parametrized by (24) with lines of curvature coordinates. In particular, to design a developable surface with lines of curvature directrices curves. For this aim, two conditions are required, the first one insures that the first directrix curve is a line of curvature. The second one is a developability condition (26). After that, we show that the other directrices of a ruled developable surface are lines of curvature without further assumptions. The following definition combines the line of curvature condition (12) with parametrization (24) to obtain a ruled surface whose the first directrix curve is a line of curvature.

Definition 3.17. A ruled surface defined by:

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))], \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{39}$$

has line of curvature first directrix where $\tau + \frac{d\phi}{ds} = 0$.

Remark 3.18. If $\cos\phi(s) = 0$ or $\sin\phi(s) = 0$, then $\phi(s)$ has constant value and $\frac{d\phi}{ds} = 0$, this implies that $\tau = 0$, it is contradiction with the assumption that $\gamma(s)$ is a twisted curve. In general, $\cos\phi(s) = const.$ or $\sin\phi(s) = const.$, lead to the same contradiction. Therefore, through this section we suppose that neither $\cos\phi(s) = const.$ nor $\sin\phi(s) = const.$

Theorem 3.19. A ruled surface with line of curvature first directrix (39) is developable if and only if $\cos\theta(s) = 0$.

Proof. A ruled surface whose first directrix is the line of curvature (39) satisfies $\tau + \frac{d\phi}{ds} = 0$. After substitution, the developabity condition (26) becomes

$$\kappa(s)\sin\phi(s)\cos\theta(s) = 0 \tag{40}$$

Since $\gamma(s)$ is a twisted curve, then $\sin\phi(s) \neq 0$ and $\kappa(s) \neq 0$. Therefore, a ruled surface with line of curvature first directrix (39) is developable if and only if $\cos\theta(s) = 0$. □

Definition 3.20. A ruled surface defined by:

$$X(s,v) = \gamma(s) + v[\cos\phi(s)n(s) + \sin\phi(s)b(s)], \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R} \tag{41}$$

with line of curvature first directrix is developable where $\tau + \frac{d\phi}{ds} = 0$.

The first and second derivatives of the developable surface parametrization whose the first directrix is line of curvature (41) are listed in the following equations

$$\begin{cases} X_s &= (1 - v\kappa\cos\phi(s))t(s) \\ X_{ss} &= -v(\dfrac{d\kappa}{ds}\cos\phi(s) - \kappa\sin\phi(s)\frac{d\phi(s)}{ds})t(s) + \kappa(1 - v\kappa\cos\phi(s))n(s) \\ X_{sv} &= -\kappa\cos\phi(s)\,t(s) \\ X_v &= \cos\phi(s)\,n(s) + \sin\phi(s)\,b(s) \\ X_{vv} &= 0 \end{cases} \tag{42}$$

The inner and cross products of the tangents vectors X_s and X_v are given by

$$\begin{cases} \langle X_s, X_v\rangle &= 0 \\ X_s \times X_v &= (1 - v\kappa\cos\phi(s))[-\sin\phi(s)\,t(s) + \cos\phi(s)\,b(s)] \end{cases} \tag{43}$$

By using the regularity condition $1 - v\kappa\cos\phi(s) \neq 0$, and from (43) the unit normal of developable surface with line of curvature first directrix (41) is defined everywhere and given by the following

$$N(s,v) = \frac{X_s \times X_v}{|X_s \times X_v|} = -\sin\phi(s)\,n(s) + \cos\phi(s)\,b(s) \tag{44}$$

Theorem 3.21. Let X be a developable surface parameterized by (41), then the directrices curves are line of curvature.

PROOF. By Theorem (2.1), the directrices curves on a developable surface (41) are line of curvature if and only if $F = f = 0$. First, its clearly that $F = \langle X_s, X_v \rangle = 0$ from equation (43). Second, using equations (42) and (44), we obtain that

$$f =< N, X_{vs} >=< -\sin\phi(s)\, n(s) + \cos\phi(s)\, b(s), -\kappa\cos\phi(s)\, t(s) >= 0 \tag{45}$$

□

Theorem (3.21) proves that for the developable parametrization (41), the condition $\tau + \frac{d\phi}{ds} = 0$ for the first directrix to be a line of curvature is sufficient to make other directrices are lines of curvature.

Theorem 3.22. Let $X(s, v) = \gamma(s) + vD(s), 0 \le s \le L, v \in \mathbb{R}$ be a developable ruled surface, where $\gamma(s)$ is a unit speed twisted curve, and $D(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s)$ is a unit director vector. Then every ruling is line of curvature and the directrices curves are lines of curvature if and only if $\tau + \frac{d\phi}{ds} = 0$.

Previously, the condition $\tau + \frac{d\phi}{ds} = 0$ with this form has been used to simplify the calculations. For the surface construction, it is useful to write the condition in terms of $\phi(s)$ explicitly as $\phi(s) = \phi_0 - \int_{s_0}^{s} \tau ds$. If we choose $s_0 = 0$, hence $\phi_0 = \phi(0)$, then the condition becomes $\phi(s) = \phi(0) - \int_0^s \tau ds$.

Definition 3.23. A ruled surface defined by:

$$X(s, v) = \gamma(s) + v[\cos\phi(s)n(s) + \sin\phi(s)b(s)], \quad 0 \le s \le L, \quad \text{and} \quad v \in \mathbb{R} \tag{46}$$

is developable surface with line of curvature coordinates where $\phi(s) = \phi(0) - \int_0^s \tau ds$.

As we have seen, the developable surface with geodesic coordinates is a generalized cylinder. The type of developable surface with line of curvature coordinates (46) is determined in the following.

Theorem 3.24. A developable surface with line of curvature coordinates (46) is non cylindrical.

PROOF. Assume by contradiction that a developable surface with line of curvature coordinates (46) is a cylinder, then $D'(s) = 0$. Since $D(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s)$, then $D'(s) = -\kappa\cos\phi(s)t(s) = 0$ implies that $\kappa = 0$ or $\cos\phi = 0$, which contradicts to the assumption that $\gamma(s)$ is a twisted curve. Hence $D'(s) \neq 0$, and a developable surface with line of curvature coordinates (41) is non cylindrical. □

Non-cylindrical developable surface is a tangent or a generalized cone surfaces. The singular points or the striction curve $\beta(s)$ (20) can be used to distinguish between them as given in the following.

Definition 3.25. A non-cylindrical developable surface is

i. A generalized cone if and only if the striction curve $\beta(s)$ (20) is a fixed point, i.e., $\beta'(s) = 0$.

ii. A tangent surface if and only if, there exists striction curve $\beta(s)$ (20) such that $\beta'(s) \neq 0$.

Theorem 3.26. For a developable surface with line of curvature coordinates (46). The striction curve and its first derivative are given respectively by

$$\begin{cases} \beta(s) & = \gamma(s) + \frac{1}{\kappa\cos\phi} D(s) \\ \beta'(s) & = \dfrac{\kappa\sin\phi\phi'(s) - \kappa'\cos\phi}{\kappa^2\cos^2\phi} D(s) \end{cases} \tag{47}$$

PROOF. The striction curve for non-cylindrical developable surface is defined by

$$\beta(s) = \gamma(s) - \frac{\langle \gamma'(s), D'(s)\rangle}{\langle D'(s), D'(s)\rangle} D(s), \;\; D'(s) \neq 0$$

For a developable surface with line of curvature coordinates (46), $\langle \gamma'(s), D'(s)\rangle = -\kappa \cos\phi$ and $\langle D'(s), D'(s)\rangle = \kappa^2 \cos^2\phi$. Then, the striction curve (20) is given by

$$\beta(s) = \gamma(s) + \frac{1}{\kappa\cos\phi} D(s)$$

The first derivative of the striction curve is

$$\begin{aligned} \beta'(s) &= t(s) + \frac{\kappa\cos\phi(-\kappa\cos\phi t(s)) - (\kappa'\cos\phi - \kappa\sin\phi\phi'(s))D(s)}{\kappa^2\cos^2\phi} \\ &= \frac{\kappa\sin\phi\phi'(s) - \kappa'\cos\phi}{\kappa^2\cos^2\phi} D(s) \end{aligned}$$

□

Theorem 3.27. A non cylindrical developable surface with line of curvature coordinates (46) is a tangent surface.

PROOF. Suppose that a non cylindrical developable surface with lines of curvature coordinates (46) is a generalized cone, then $\beta'(s) = 0$. By using (47), the following equation is satisfied

$$\kappa\sin\phi\phi'(s) - \kappa'\cos\phi = 0, \text{ which gives } \;\; \frac{\kappa'}{\kappa} = \frac{\sin\phi}{\cos\phi}\phi'(s)$$

By taking the integral of both sides and some calculations, we get

$$\kappa\cos\phi = \kappa_0\cos\phi_0 \tag{48}$$

Since κ_0 and $\cos\phi_0$ are constants, then κ and $\cos\phi$ are both also constants, this yields a contradiction with a twisted curve. Therefore $\beta'(s) \neq 0$, and a non cylindrical developable surface with line of curvature coordinates (46) must be a tangent surface, where $\beta'(s)$ is parallel with director $D(s)$. □

Li et al. [11] derived a similar condition with (48) in order to make the developable surface through a given line of curvature is cone or tangent surface. Finally, the following theorem is more identified.

Theorem 3.28. Among all developable surfaces parametrized by (24), the tangent surface can be equipped with lines of curvature coordinates if and only if the unit director vector is $D(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s)$ where $\phi(s) = \phi(0) - \int_0^s \tau ds$.

4. Examples

In this section we give an example of a developable surface whose coordinates is geodesic (generalized cylinder) or line of curvature (tangent surface) and draw their pictures by using Mathematica.

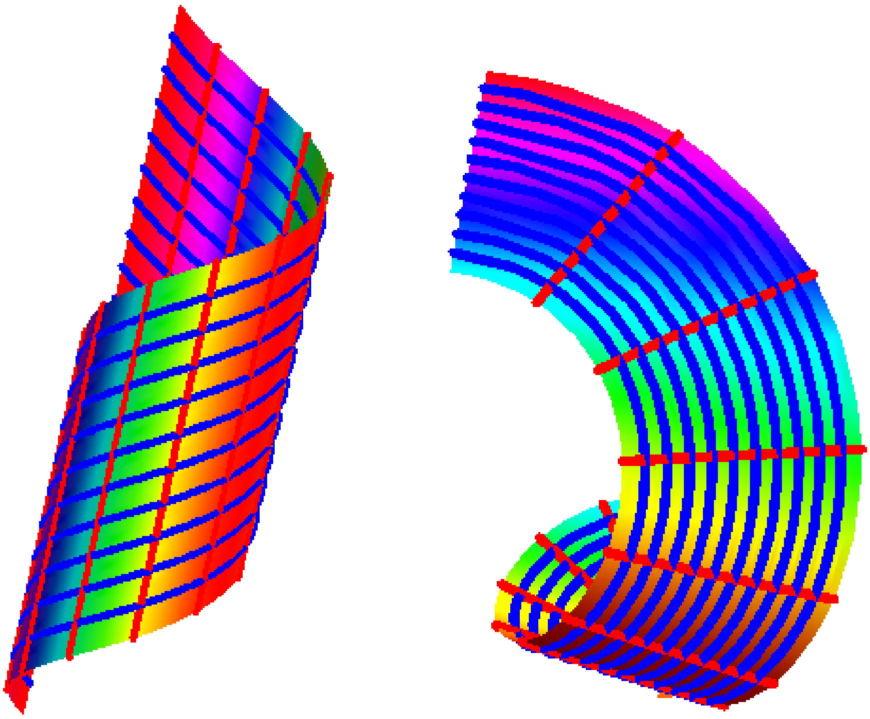
Example 4.1. Let $\gamma(s) = (\frac{\sqrt{3}}{2}\sin(s), \frac{s}{2}, \frac{\sqrt{3}}{2}\cos(s))$ be a unit speed circular helix curve, the unit tangent, the unit normal and the unit binormal are $t = \left(\frac{\sqrt{3}}{2}\cos(s), \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin(s)\right)$, $n = (-\sin(s), 0, -\cos(s))$ and $b = (-\frac{1}{2}\cos(s), \frac{\sqrt{3}}{2}, \frac{1}{2}\sin(s))$. The curvature and torsion of $\alpha(s)$ are $\kappa = \frac{\sqrt{3}}{2}$ and $\tau = \frac{1}{2}$.

(a) According to theorem (3.11) and definition (38), the developable geodesic coordinates is given by

$$X(s, v) = \gamma(s) + vD(s), \quad 0 \leq s \leq L, \quad \text{and} \quad v \in \mathbb{R}$$

where $D(s) = \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}} t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}} b(s)$ and $\gamma(s)$ is a helix.

By substitution $\frac{\tau}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{2}$ and $\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{\sqrt{3}}{2}$, and for $0 \leq s \leq 2\pi$, $0 \leq v \leq \pi$, the constructed developable surface is a generalized cylinder with geodesic coordinates as shown in Figure 1(a).



(a) Cylinder with geodesic coordinates. **(b)** Tangent surface with line of curvature coordinates.

Fig. 1. Developable surface with geodesic or line of curvature coordinates.

(b) By theorem (3.22) and definition (46), the developable line of curvature coordinates is given by

$$X(s,v) = \gamma(s) + v[\cos\phi(s)n(s) + \sin\phi(s)b(s)], \quad 0 \le s \le L, \quad \text{and} \quad v \in \mathbb{R}$$

where $\phi(s) = \phi(0) - \int_0^s \tau ds$.

If we choose $\phi(0) = 0$, then $\phi(s) = -\frac{1}{2}s$, and for $0 \le s \le 2\pi$, $0 \le v \le \pi/2$, the developable constructed surface is a tangent surface with line of curvature coordinates as shown in Figure 1(b).

5. Conclusion

In this paper, using a ruled parametrization (24), we constructed a developable surface whose coordinates curves are geodesics or lines of curvature. The main results asserted that the developable surface with geodesic coordinates is a generalized cylinder, and the developable surface with the line of curvature coordinates is a tangent surface. Also, the first directrix curve must be a helix as a first condition to generate a developable surface with geodesic coordinates, but this condition need not to be satisfied to construct a developable surface with the line of curvature coordinates. Furthermore, the developable line of curvature coordinates is orthogonal, but the developable geodesic coordinates is not. Finally, among the three types of developable surfaces, the generalized cylinder and the tangent surface can be equipped with geodesic coordinates and line of curvature coordinates respectively.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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Generalisations of SDM Methods in *fpfs*-Matrices Space to Render Them Operable in *ifpifs*-Matrices Space and Their Application to Performance Ranking of the Noise-Removal Filters

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Abstract − Recently, the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) has allowed to mathematically model some problems in which the parameters and alternatives exhibit intuitionistic fuzzy uncertainties. To this end, the present study aims to generalise 24 soft decision-making (SDM) methods operating in the fuzzy parameterized fuzzy soft matrices space with a single *fpfs*-matrix to the *ifpifs*-matrices spaces. Afterwards, we propose five test scenarios to analyse whether the SDM methods constructed by *ifpifs*-matrices consistently work. Moreover, we make performance comparisons of the generalised SDM methods successful in the five test scenarios by applying them to the performance-based value assignment (PVA) problem of the well-known noise-removal filters. Finally, we discuss the need for further research.

Keywords − *Fuzzy sets, intuitionistic fuzzy sets, soft sets, ifpifs-matrices, soft decision making*

Mathematics Subject Classification (2020) − 03E72, 15B15

1. Introduction

The concept of intuitionistic fuzzy sets, characterized by membership and non-membership degree of an element's belonging to a set, has been propounded by Atanassov [1] as a generalization of fuzzy sets [2]. Furthermore, many hybrid versions of this concept, together with soft sets [3], have been introduced, such as intuitionistic fuzzy soft sets [4], intuitionistic fuzzy parameterized soft sets [5], intuitionistic fuzzy parameterized fuzzy soft sets [6], fuzzy parameterized intuitionistic fuzzy soft sets [7], and intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets (*ifpifs*-sets) [8]. Afterwards, the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) [9] modelling all the problems that these concepts can and allowing the data in such problems to be processed in a computer environment has been put forward. Thus, especially for the problems containing a large number of data, it has become possible to process these data on the computer. Therefore, a significant advantage has been gained in decision-making process.

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The recently proposed fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) [10] are capable of modelling situations where parameters and alternatives (objects) are fuzzy. Therefore, 136 soft decision-making (SDM) methods [11-22] constructed with the subspaces [3,23-29] of the *fpfs*-sets/matrices space between 1999 and 2019 have been rendered operable in the *fpfs*-matrices space. Moreover, more successful methods with shorter running time have been produced by mathematically simplifying some configured SDM methods [21,22,30-33]. However, these SDM methods are incapable of modelling the problems in which parameters and alternatives have intuitionistic fuzzy uncertainties. To this end, to model such problems and to achieve the same or similar modelling success of the SDM methods configured in the *fpfs*-matrices space within the *ifpifs*-matrices space, the generalisations of these methods have great importance. Hence, the main motivation of the present study is to generalise the SDM methods [10,11,15,16,20] operating with a single *fpfs*-matrix to the *ifpifs*-matrices space.

Section 2 presents several of the basic notions to be required in the next sections. Section 3 generalises the aforesaid SDM methods to the *ifpifs*-matrices space. Section 4 proposes five test cases to test the generalised SDM methods' performance of ranking the alternatives in the presence of decision-making problems and to determine the successful SDM methods. Section 5 applies the methods passing all the tests to a performance-based value assignment (PVA) problem and compares their ranking performances. The final section discusses the need for further research.

2. Preliminaries

This section presents the concepts of fuzzy sets [2], intuitionistic fuzzy sets [1], *ifpifs*-sets [8], and *ifpifs*-matrices [9] to be employed in the next sections. Throughout this study, let U be a universal set and E be a parameter set.

Definition 2.1. [2] Let μ be a function from E to $[0,1]$. Then, the set $\left\{{}^{\mu(x)}x : x \in E\right\}$, being the graphic of μ, is called a fuzzy set over E.

Definition 2.2. [1] Let f be a function from E to $[0,1] \times [0,1]$. Then, the set $\left\{\left(x, f(x)\right) | \ x \in E\right\}$, being the graphic of f, is called an intuitionistic fuzzy set over E.

Here, for all $x \in E$, $f(x) := \left(\mu(x), \nu(x)\right)$ such that $0 \leq \mu(x) + \nu(x) \leq 1$. Moreover, μ and ν are called membership function and non-membership function in an intuitionistic fuzzy set, respectively. Thus, for brevity, we represent an intuitionistic fuzzy set over E with $f := \left\{{}^{\mu(x)}_{\nu(x)}x : x \in E\right\}$ instead of $f = \left\{\left(x, \mu(x), \nu(x)\right) : x \in E\right\}$. Besides, $IF(E)$ denotes the set of all the intuitionistic fuzzy sets over E. For convenience, we do not display the elements ${}^{0}_{1}x$ in an intuitionistic fuzzy set.

Definition 2.3. [8] Let $f \in IF(E)$ and α be a function from f to $IF(U)$. Then, the set $\left\{\left({}^{\mu(x)}_{\nu(x)}x, \alpha\left({}^{\mu(x)}_{\nu(x)}x\right)\right) x \in E\right\}$, being the graphic of α, is called an intuitionistic fuzzy parameterized intuitionistic fuzzy soft set (*ifpifs*-set) parameterized via E over U (or briefly over U).

In the present study, the set of all the *ifpifs*-sets over U is denoted by $IFPIFS_E(U)$. In $IFPIFS_E(U)$, since the $\text{graph}(\alpha)$ and α generate each other uniquely, the notations are interchangeable. Therefore, as long as it causes no confusion, we denote an *ifpifs*-set $\text{graph}(\alpha)$ by α. Moreover, for convenience, we do not display the elements $({}^{0}_{1}x, 0_U)$ in an *ifpifs*-set. Here, 0_U is the empty intuitionistic fuzzy set over U.

Example 2.1. Let $E = \{x_1, x_2, x_3, x_4\}$ and $U = \{u_1, u_2, u_3, u_4\}$. Then,

$$\alpha = \left\{\left({}^{0.3}_{0.6}x_1, \left\{{}^{0.6}_{0.2}u_1, {}^{0.4}_{0.3}u_3, {}^{0.1}_{0.5}u_4\right\}\right), \left({}^{0}_{1}x_2, \left\{{}^{0.6}_{0.3}u_1, {}^{0}_{0.8}u_2, {}^{0.1}_{0.4}u_4\right\}\right), \left({}^{0.4}_{0.4}x_4, \left\{{}^{0.2}_{0.7}u_1, {}^{1}_{0}u_3, {}^{0.5}_{0.5}u_4\right\}\right)\right\}$$

is an *ifpifs*-set over U.

Definition 2.4. [9] Let $\alpha \in IFPIFS_E(U)$. Then, $[a_{ij}]$ is called *ifpifs*-matrix of α and is defined by

$$[a_{ij}] := \begin{bmatrix} a_{01} & a_{02} & a_{03} & \cdots & a_{0n} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{0,1,2,\cdots\}$ and $j \in \{1,2,\cdots\}$,

$$a_{ij} := \begin{cases} {}^{\mu(x_j)}_{\nu(x_j)}, & i = 0 \\ \alpha\left({}^{\mu(x_j)}_{\nu(x_j)}x_j\right)(u_i), & i \neq 0 \end{cases}$$

or briefly $a_{ij} := {}^{\mu_{ij}}_{\nu_{ij}}$. Here, if $|U| = m - 1$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

From now on, as long as it causes no confusion, the membership and non-membership functions of $[a_{ij}]$, i.e. μ_{ij} and ν_{ij}, will be represented by μ_{ij}^{a} and ν_{ij}^{a}, respectively. Moreover, the set of all the *ifpifs*-matrices parameterized via E over U is denoted by $IFPIFS_E[U]$.

Example 2.2. The *ifpifs*-matrix of α provided in Example 2.1 is as follows:

$$[a_{ij}] = \begin{bmatrix} {}^{0.3}_{0.6} & {}^{0}_{1} & {}^{0}_{1} & {}^{0.4}_{0.4} \\ {}^{0.6}_{0.2} & {}^{0.6}_{0.3} & {}^{0}_{1} & {}^{0.2}_{0.7} \\ {}^{0}_{1} & {}^{0}_{0.8} & {}^{0}_{1} & {}^{0}_{1} \\ {}^{0.4}_{0.3} & {}^{0}_{1} & {}^{0}_{1} & {}^{1}_{0} \\ {}^{0.1}_{0.5} & {}^{0.1}_{0.4} & {}^{0}_{1} & {}^{0.5}_{0.5} \end{bmatrix}$$

Proposition 2.1. [34] Let $IFV([0,1])$ be the set of all the intuitionistic fuzzy values and ${}^{\mu_1}_{\nu_1}, {}^{\mu_2}_{\nu_2} \in IFV([0,1])$. Then, the relation $\tilde{\leqq}$ defined by

$${}^{\mu_1}_{\nu_1} \tilde{\leqq} {}^{\mu_2}_{\nu_2} \Leftrightarrow \left[s_1\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix} < s_1\begin{pmatrix}\mu_2\\\nu_2\end{pmatrix} \vee \left(s_1\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix} = s_1\begin{pmatrix}\mu_2\\\nu_2\end{pmatrix} \wedge s_2\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix} \leq s_2\begin{pmatrix}\mu_2\\\nu_2\end{pmatrix}\right)\right]$$

is a linear ordering relation over $IFV([0,1])$. Here, $s_1\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix} := \mu_1 - \nu_1$ and $s_2\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix} := \mu_1 + \nu_1$. Moreover, $s_1\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix}$ and $s_2\begin{pmatrix}\mu_1\\\nu_1\end{pmatrix}$ are called score value and accuracy value of intuitionistic fuzzy value ${}^{\mu_1}_{\nu_1}$, respectively.

3. Generalisations of SDM Methods

This section generalises the SDM methods [10,11,15,16,20] employed a single *fpfs*-matrix and have been constructed with *fpfs*-matrices [10]. Hereinafter, I_n indicates the set of all unsigned integer numbers from 1 to n, i.e., $I_n = \{1,2,\cdots,n\}$. Similarly, I_n^* denotes the set of all nonnegative numbers from 0 to n, i.e., $I_n^* = \{0,1,2,\cdots,n\}$. Moreover, the variables (inputs) R, w, λ, λ_1, and λ_2 are used in algorithms. Here, R is a set of indices, w is an intuitionistic fuzzy valued row matrix, $\lambda \in (0,1]$, and $\lambda_1, \lambda_2 \in [0,1]$. Furthermore, the notation of each algorithm is created by inserting the first letter of the word "intuitionistic" at the beginning of the algorithm notation proposed in [16].

Algorithm 3.1. iMBR01

Step 1. Construct *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b^1_{ik}]_{(m-1)\times(m-1)}$ and $[b^2_{ik}]_{(m-1)\times(m-1)}$ defined by

$$b^1_{ik} := \sum_{j=1}^{n} \mu^a_{0j}\chi(\mu^a_{ij},\mu^a_{kj}) \quad \text{and} \quad b^2_{ik} := \sum_{j=1}^{n} \nu^a_{0j}\psi(\nu^a_{ij},\nu^a_{kj})$$

such that $i,k \in I_{m-1}$,

$$\chi(\mu^a_{ij},\mu^a_{kj}) := \begin{cases} 1, & \mu^a_{ij} \geq \mu^a_{kj} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \psi(\nu^a_{ij},\nu^a_{kj}) := \begin{cases} 0, & \nu^a_{ij} \leq \nu^a_{kj} \\ 1, & \text{otherwise} \end{cases}$$

Step 3. Obtain $[c_{i1}]_{(m-1)\times 1}$

$$c_{i1} := \sum_{k=1}^{m-1} (b^1_{ik} - b^1_{ki}), \quad i \in I_{m-1}$$

Step 4. Obtain $[d_{i1}]_{(m-1)\times 1}$

$$d_{i1} := \sum_{k=1}^{m-1} (b^2_{ik} - b^2_{ki}), \quad i \in I_{m-1}$$

Step 5. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} = \begin{cases} \dfrac{c_{i1} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu^s_{i1} = \begin{cases} 1 - \dfrac{c_{i1} + |d_{i1}| + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right| = 0 \end{cases}$$

Step 6. Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k \middle| u_k \in U\right\}$

Algorithm 3.2. isMBR01

Step 1. Construct *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} := \sum_{k=1}^{m-1}\sum_{j=1}^{n} \mu^a_{0j}\operatorname{sgn}(\mu^a_{ij} - \mu^a_{kj}) \quad \text{and} \quad c_{i1} := \sum_{k=1}^{m-1}\sum_{j=1}^{n} \nu^a_{0j}\operatorname{sgn}(\nu^a_{ij} - \nu^a_{kj})$$

such that $i \in I_{m-1}$

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s}=\begin{cases}1-\dfrac{b_{i1}+|c_{i1}|+\left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|\neq 0\\ 0, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|=0\end{cases}$$

Step 4. Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}}u_k \middle| u_k\in U\right\}$

Algorithm 3.3. iMBR01/2

Step 1. Construct *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ik}^{1}]_{(m-1)\times(m-1)}$ and $[b_{ik}^{2}]_{(m-1)\times(m-1)}$ defined by

$$b_{ik}^{1}:=\sum_{j=1}^{n}\mu_{0j}^{a}\chi\left(\mu_{ij}^{a},\mu_{kj}^{a}\right)\quad\text{and}\quad b_{ik}^{2}:=\sum_{j=1}^{n}\nu_{0j}^{a}\psi\left(\nu_{ij}^{a},\nu_{kj}^{a}\right)$$

such that $i,k\in I_{m-1}$ and

$$\begin{matrix}\chi\left(\mu_{ij}^{a},\mu_{kj}^{a}\right)\\ \psi\left(\nu_{ij}^{a},\nu_{kj}^{a}\right)\end{matrix}:=\begin{cases}{}^{1}_{0}, & \mu_{ij}^{a}\geq\mu_{kj}^{a}\text{ ve }\nu_{ij}^{a}\leq\nu_{kj}^{a}\\ {}^{0}_{1}, & \text{otherwise}\end{cases}$$

Step 3. Obtain $[c_{i1}]_{(m-1)\times 1}$

$$c_{i1}:=\sum_{k=1}^{m-1}\left(b_{ik}^{1}-b_{ki}^{1}\right),\quad i\in I_{m-1}$$

Step 4. Obtain $[d_{i1}]_{(m-1)\times 1}$

$$d_{i1}:=\sum_{k=1}^{m-1}\left(b_{ik}^{2}-b_{ki}^{2}\right),\quad i\in I_{m-1}$$

Step 5. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1}:={}^{\mu_{i1}^{s}}_{\nu_{i1}^{s}}$ such that $i\in I_{m-1}$,

$$\mu_{i1}^{s}=\begin{cases}\dfrac{c_{i1}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{c_{k1}+|d_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{c_{k1}+|d_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|\neq 0\\ 1, & \max\limits_{k\in I_{m-1}}\{c_{k1}+|d_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|=0\end{cases}$$

and

$$\nu_{i1}^{s}=\begin{cases}1-\dfrac{c_{i1}+|d_{i1}|+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{c_{k1}+|d_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{c_{k1}+|d_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|\neq 0\\ 0, & \max\limits_{k\in I_{m-1}}\{c_{k1}+|d_{k1}|\}+\left|\min\limits_{k\in I_{m-1}}\{c_{k1}\}\right|=0\end{cases}$$

Step 6. Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}}u_k \middle| u_k\in U\right\}$

Algorithm 3.4. iMRB02(R)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Determine a set R of indices such that $R\subseteq I_n$

Step 3. Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1}:=\sum_{j\in R}\mu_{0j}^{a}\mu_{ij}^{a}\quad\text{and}\quad c_{i1}:=\sum_{j\in R}\nu_{0j}^{a}\nu_{ij}^{a}$$

such that $i\in I_{m-1}$

Step 4. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \frac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^s = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^s = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

Step 5. Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s}u_k \middle| u_k \in U\right\}$

Algorithm 3.5. iKM11($\boldsymbol{R}$)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Determine a set R of indices such that $R \subseteq I_n$

Step 3. Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} := \prod_{j\in R} \mu_{0j}^a \mu_{ij}^a \quad \text{and} \quad c_{i1} := \prod_{j\in R} \nu_{0j}^a \nu_{ij}^a$$

such that $i \in I_{m-1}$

Step 4. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \frac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^s = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^s = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{b_{k1}+|c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

Step 5. Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s}u_k \middle| u_k \in U\right\}$

Algorithm 3.6. iCCE11

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \frac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$, $K = \{j : \mu_{0j}^a \neq 0 \vee \nu_{0j}^a \neq 1\}$,

$$\mu_{i1}^s := \begin{cases} \dfrac{1}{|K|}\sum_{j=1}^{n} \mu_{0j}^a \mu_{ij}^a, & |K| \neq 0 \\ 0, & |K| = 0 \end{cases} \quad \text{and} \quad \nu_{i1}^s := \begin{cases} \dfrac{1}{|K|}\sum_{j=1}^{n} \nu_{0j}^a \nu_{ij}^a, & |K| \neq 0 \\ 0, & |K| = 0 \end{cases}$$

Here, $|K|$ denotes the cardinality of K.

Step 3. Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

Algorithm 3.7. iYE12

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} := \begin{cases} \frac{1}{\sum_{j=1}^n \mu^a_{0j}} \sum_{j=1}^n \mu^a_{0j}\mu^a_{ij}, & \sum_{j=1}^n \mu^a_{0j} \neq 0 \\ 0, & \sum_{j=1}^n \mu^a_{0j} = 0 \end{cases} \quad \text{and} \quad \nu^s_{i1} := \begin{cases} \frac{1}{\sum_{j=1}^n \nu^a_{0j}} \sum_{j=1}^n \nu^a_{0j}\nu^a_{ij}, & \sum_{j=1}^n \nu^a_{0j} \neq 0 \\ 0, & \sum_{j=1}^n \nu^a_{0j} = 0 \end{cases}$$

Step 3. Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

Algorithm 3.8. iCCE10

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} := \frac{1}{n}\sum_{j=1}^n \mu^a_{0j}\mu^a_{ij} \quad \text{and} \quad \nu^s_{i1} := \frac{1}{n}\sum_{j=1}^n \nu^a_{0j}\nu^a_{ij}$$

Step 3. Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

Algorithm 3.9. iCEC11

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{1j}]_{1\times n}$ defined by $b_{1j} := {}^{\mu^b_{1j}}_{\nu^b_{1j}}$ such that $j \in I_n$,

$$\mu^b_{1j} := \frac{\mu^a_{0j}}{m-1}\sum_{i=1}^{m-1} \mu^a_{ij} \quad \text{and} \quad \nu^b_{1j} := \frac{\nu^a_{0j}}{m-1}\sum_{i=1}^{m-1} \nu^a_{ij}$$

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} := \frac{1}{n}\sum_{j=1}^n \mu^a_{ij}\mu^b_{1j} \quad \text{and} \quad \nu^s_{i1} := \frac{1}{n}\sum_{j=1}^n \nu^a_{ij}\nu^b_{1j}$$

Step 4. Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

Algorithm 3.10. iM11

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b^1_{ik}]_{(m-1)\times(m-1)}$ and $[b^2_{ik}]_{(m-1)\times(m-1)}$ defined by

$$b^1_{ik} := \sum_{j=1}^n \mu^a_{0j}\left(\mu^a_{ij} - \mu^a_{kj}\right) \quad \text{and} \quad b^2_{ik} := \sum_{j=1}^n \nu^a_{0j}\left(\nu^a_{ij} - \nu^a_{kj}\right)$$

such that $i, k \in I_{m-1}$

Step 3. Obtain $[c_{i1}]_{(m-1)\times 1}$ ve $[d_{i1}]_{(m-1)\times 1}$ defined by

$$c_{i1} = \sum_{k=1}^{m-1} b_{ik}^{1} \quad \text{and} \quad d_{i1} \coloneqq \sum_{k=1}^{m-1} b_{ik}^{2}$$

such that $i \in I_{m-1}$

Step 4. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{c_{i1} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{c_{i1} + |d_{i1}| + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

Step 5. Obtain the decision set $\left\{ {}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}} u_k \mid u_k \in U \right\}$

Algorithm 3.11. iKKT13

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{(n+1)\times n}$

Step 2. Obtain $[b_{i1}]_{n\times 1}$ defined by $b_{i1} \coloneqq \frac{\mu_{i1}^{b}}{\nu_{i1}^{b}}$ such that $i \in I_n$,

$$\mu_{i1}^{b} \coloneqq \frac{1}{n}\sum_{j=1}^{n} \mu_{ij}^{a} \quad \text{and} \quad \nu_{i1}^{b} \coloneqq \frac{1}{n}\sum_{j=1}^{n} \nu_{ij}^{b}$$

Step 3. Obtain the score matrix $[s_{i1}]_{n\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_n$,

$$\mu_{i1}^{s} \coloneqq \mu_{0i}^{a}\mu_{i1}^{b} \quad \text{and} \quad \nu_{i1}^{s} \coloneqq \nu_{0i}^{a} + \nu_{i1}^{b} - \nu_{0i}^{a}\nu_{i1}^{b}$$

Step 4. Obtain the decision set $\left\{ {}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}} u_k \mid u_k \in U \right\}$

iKKT13 is used in decision-making problems containing the same number of alternatives and parameters.

Algorithm 3.12. iFJLL10$(\boldsymbol{R}, \boldsymbol{w})$

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} \coloneqq \frac{\mu_{1j}}{\nu_{1j}}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \leq \mu_{1j} + \nu_{1j} \leq 1$, for all $j \in I_n$

Step 3. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\frac{\mu_{ij}^{b}}{\nu_{ij}^{b}} \coloneqq \begin{cases} \frac{1}{0}, & \mu_{ij}^{a} \geq \mu_{1j} \text{ and } \nu_{ij}^{a} \leq \nu_{1j} \\ \frac{0}{1}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

Step 4. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.13. iFJLL10/2($\boldsymbol{R}, \boldsymbol{w}$)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} := \begin{matrix}\mu_{1j} \\ \nu_{1j}\end{matrix}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \leq \mu_{1j} + \nu_{1j} \leq 1$, for all $j \in I_n$

Step 3. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^b \\ \nu_{ij}^b\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^a \\ \nu_{0j}^a\end{matrix}, & i = 0 \\ \begin{matrix}1 \\ 0\end{matrix}, & i \neq 0, \mu_{ij}^a \geq \mu_{1j}, \text{and } \nu_{ij}^a \leq \nu_{1j} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

Step 4. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.14. iFJLL10/3($\boldsymbol{R}, \boldsymbol{w}$)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} := \begin{matrix}\mu_{1j} \\ \nu_{1j}\end{matrix}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \leq \mu_{1j} + \nu_{1j} \leq 1$, for all $j \in I_n$

Step 3. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^b \\ \nu_{ij}^b\end{matrix} := \begin{cases} \begin{matrix}1 \\ 0\end{matrix}, & \mu_{ij}^a \geq \max_{k\in I_n} \mu_{1k} \text{ and } \nu_{ik}^a \leq \min_{k\in I_n} \nu_{1k} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

Step 4. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.15. iFJLL10/4($\boldsymbol{R}, \boldsymbol{w}$)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} := \begin{matrix}\mu_{1j} \\ \nu_{1j}\end{matrix}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \leq \mu_{1j} + \nu_{1j} \leq 1$, for all $j \in I_n$

Step 3. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^b \\ \nu_{ij}^b\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^a \\ \nu_{0j}^a\end{matrix}, & i = 0 \\ \begin{matrix}1 \\ 0\end{matrix}, & i \neq 0, \mu_{ij}^a \geq \max_{k\in I_n} \mu_{1k}, \text{and } \nu_{ij}^a \leq \min_{k\in I_n} \nu_{1k} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

Step 4. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.16. iFJLL10m($\boldsymbol{R}$)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}1 \\ 0\end{matrix}, & \mu_{ij}^{a} \geq \frac{1}{m-1}\sum_{k=1}^{m-1}\mu_{kj}^{a} \text{ and } \nu_{ij}^{a} \leq \frac{1}{m-1}\sum_{k=1}^{m-1}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

Step 3. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.17. iFJLL10/2m$(\boldsymbol{R})$

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^{a} \\ \nu_{0j}^{a}\end{matrix}, & i = 0 \\ \begin{matrix}1 \\ 0\end{matrix}, & i \neq 0, \mu_{ij}^{a} \geq \frac{1}{m-1}\sum_{k=1}^{m-1}\mu_{kj}^{a}, \text{and } \nu_{ij}^{a} \leq \frac{1}{m-1}\sum_{k=1}^{m-1}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

Step 3. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.18. iFJLL10max$(\boldsymbol{R})$

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}1 \\ 0\end{matrix}, & \mu_{ij}^{a} \geq \max_{k\in I_{m-1}}\mu_{kj}^{a} \text{ and } \nu_{ij}^{a} \leq \min_{k\in I_{m-1}}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

Step 3. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.19. iFJLL10/2max$(\boldsymbol{R})$

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^{a} \\ \nu_{0j}^{a}\end{matrix}, & i = 0 \\ \begin{matrix}1 \\ 0\end{matrix}, & i \neq 0, \mu_{ij}^{a} \geq \max_{k\in I_{m-1}}\mu_{kj}^{a}, \text{and } \nu_{ij}^{a} \leq \min_{k\in I_{m-1}}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

Step 3. Apply iMRB02(R) to $[b_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.20. iF10$(\boldsymbol{\lambda})$

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{1j}]_{1\times n}$ defined by $b_{1j} := \begin{matrix}\mu_{1j}^{b} \\ \nu_{1j}^{b}\end{matrix}$ such that $i \in I_{m-1}$, $j \in I_n$, $\delta_i := f\left(\frac{i}{m-1}\right) - f\left(\frac{i-1}{m-1}\right)$,

$$\mu_{1j}^{b} \coloneqq \sum_{i=1}^{m-1} \mu_j^{a_i}\delta_i \quad \text{and} \quad \nu_{1j}^{b} \coloneqq \sum_{i=1}^{m-1} \nu_j^{a_i}\delta_i$$

Here, for $\lambda \in (0,1]$, f is a function defined by $f(x) = x^{\frac{1-\lambda}{\lambda}}$. Moreover, $\mu_j^{a_i}$ denotes i^{th} largest membership degree of the elements with index nonzero in j^{th} column of $[a_{ij}]$. Similarly, $\nu_j^{a_i}$ indicates i^{th} smallest non-membership degree of the elements with index nonzero in j^{th} column of $[a_{ij}]$.

Step 3. Obtain $[c_{ij}]_{m\times n}$ defined by $c_{ij} \coloneqq \begin{smallmatrix}\mu_{ij}^{c}\\ \nu_{ij}^{c}\end{smallmatrix}$ such that $i \in I_{m-1}^{*}$, $j \in I_n$, and

$$\begin{matrix}\mu_{ij}^{c}\\ \nu_{ij}^{c}\end{matrix} \coloneqq \begin{cases} \begin{matrix}\mu_{0j}^{a}\\ \nu_{0j}^{a}\end{matrix}, & i = 0 \\ \begin{matrix}1\\ 0\end{matrix}, & i \neq 0, \mu_{ij}^{a} \geq \mu_{1j}^{b}, \text{and } \nu_{ij}^{a} \leq \nu_{1j}^{b} \\ \begin{matrix}0\\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

Step 4. Apply iMRB02(R) to $[c_{ij}]$ such that $R \subseteq I_n$

Algorithm 3.21. iKSM10

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ij}]_{m\times n}$ and $[c_{ij}]_{m\times n}$ defined by

$$b_{ij} \coloneqq \begin{cases} \dfrac{1}{n-1}\left(1 - \dfrac{b_{1j}^{*}}{\sum_{k=1}^{n} b_{1k}^{*}}\right), & i = 0 \text{ and } \sum_{k=1}^{n} b_{1k}^{*} \neq 0 \\ \dfrac{1}{n}, & i = 0 \text{ and } \sum_{k=1}^{n} b_{1k}^{*} = 0 \\ \mu_{ij}^{a}, & i \neq 0 \end{cases}$$

and

$$c_{ij} \coloneqq \begin{cases} \dfrac{1}{n-1}\left(1 - \dfrac{c_{1j}^{*}}{\sum_{k=1}^{n} c_{1k}^{*}}\right), & i = 0 \text{ and } \sum_{k=1}^{n} c_{1k}^{*} \neq 0 \\ \dfrac{1}{n}, & i = 0 \text{ and } \sum_{k=1}^{n} c_{1k}^{*} = 0 \\ \nu_{ij}^{a}, & i \neq 0 \end{cases}$$

$i \in I_{m-1}^{*}$ and $j \in I_n$

Here,

$$b_{1j}^{*} \coloneqq \begin{cases} \dfrac{1}{m-1}\sum_{i=1}^{m-1}\left(\mu_{02}^{a}\mu_{i2}^{a} - \mu_{01}^{a}\mu_{i1}^{a}\right), & j = 1 \\ \dfrac{1}{2(m-1)}\sum_{i=1}^{m-1}\left(\mu_{0(j+1)}^{a}\mu_{i(j+1)}^{a} - \mu_{0(j-1)}^{a}\mu_{i(j-1)}^{a}\right), & j \in \{2,3,\dots,n-1\} \\ \dfrac{1}{m-1}\sum_{i=1}^{m-1}\left(\mu_{0n}^{a}\mu_{in}^{a} - \mu_{0(n-1)}^{a}\mu_{i(n-1)}^{a}\right), & j = n \end{cases}$$

and

$$c_{1j}^* := \begin{cases} \dfrac{1}{m-1}\displaystyle\sum_{i=1}^{m-1}(\nu_{02}^a\nu_{i2}^a - \nu_{01}^a\nu_{i1}^a), & j=1 \\ \dfrac{1}{2(m-1)}\displaystyle\sum_{i=1}^{m-1}\left(\nu_{0(j+1)}^a\nu_{i(j+1)}^a - \nu_{0(j-1)}^a\nu_{i(j-1)}^a\right), & j \in \{2,3,\dots,n-1\} \\ \dfrac{1}{m-1}\displaystyle\sum_{i=1}^{m-1}\left(\nu_{0n}^a\nu_{in}^a - \nu_{0(n-1)}^a\nu_{i(n-1)}^a\right), & j=n \end{cases}$$

such that $j \in I_n$

Step 3. Obtain $[d_{ij}]_{m\times n}$ defined by $d_{ij} := \begin{matrix}\mu_{ij}^d \\ \nu_{ij}^d\end{matrix}$ such that $i \in I_{m-1}^*$, $j \in I_n$,

$$\mu_{ij}^d = \begin{cases} \dfrac{b_{ij} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{kj} + |c_{kj}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{kj} + |c_{kj}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{kj} + |c_{kj}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| = 0 \end{cases}$$

and

$$\nu_{ij}^d = \begin{cases} 1 - \dfrac{b_{ij} + |c_{ij}| + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{kj} + |c_{kj}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{kj} + |c_{kj}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{b_{kj} + |c_{kj}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| = 0 \end{cases}$$

Step 4. Apply iM11 to $[d_{ij}]$

Algorithm 3.22. iKWW11(λ_1, λ_2)

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Apply iMBR01 and iMRB02(R) to $[a_{ij}]$ such that $R \subseteq I_n$ and obtain the score matrices $[s_{i1}]_{(m-1)\times 1}$ and $[\tilde{s}_{i1}]_{(m-1)\times 1}$, respectively

Step 3. Obtain $[b_{i1}^1]_{(m-1)\times 1}$, $[b_{i1}^2]_{(m-1)\times 1}$, $[c_{i1}^1]_{(m-1)\times 1}$, and $[c_{i1}^2]_{(m-1)\times 1}$ defined by

$$b_{i1}^1 := \max_{k\in I_{m-1}} \mu_{k1}^s - \mu_{i1}^s \quad \text{and} \quad b_{i1}^2 := \left|\min_{k\in I_{m-1}} \nu_{k1}^s - \nu_{i1}^s\right|$$

and

$$c_{i1}^1 := \max_{k\in I_{m-1}} \mu_{k1}^{\tilde{s}} - \mu_{i1}^{\tilde{s}} \quad \text{and} \quad c_{i1}^2 := \left|\min_{k\in I_{m-1}} \nu_{k1}^{\tilde{s}} - \nu_{i1}^{\tilde{s}}\right|$$

such that $i \in I_{m-1}$

Step 4. For $\lambda_1 \in [0,1]$, obtain $[d_{i1}^1]_{(m-1)\times 1}$, $[d_{i1}^2]_{(m-1)\times 1}$, $[e_{i1}^1]_{(m-1)\times 1}$, and $[e_{i1}^2]_{(m-1)\times 1}$defined by

$$d_{i1}^1 := \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}}\{\min\{b_{k1}^1, c_{k1}^1\}\} + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^1, c_{k1}^1\}\}}{b_{i1}^1 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^1, c_{k1}^1\}\}}, & b_{i1}^1 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^1, c_{k1}^1\}\} \neq 0 \\ 1, & b_{i1}^1 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^1, c_{k1}^1\}\} = 0 \end{cases}$$

$$d_{i1}^2 := \begin{cases} 1 - \dfrac{\min\limits_{k\in I_{m-1}}\{\min\{b_{k1}^2, c_{k1}^2\}\} + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^2, c_{k1}^2\}\}}{c_{i1}^2 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^2, c_{k1}^2\}\}}, & c_{i1}^2 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^2, c_{k1}^2\}\} \neq 0 \\ 0, & c_{i1}^2 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^2, c_{k1}^2\}\} = 0 \end{cases}$$

$$e_{i1}^1 := \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}}\left\{\min\left\{b_{k1}^1, c_{k1}^1\right\}\right\} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^1, c_{k1}^1\right\}\right\}}{b_{i1}^1 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^1, c_{k1}^1\}\}}, & b_{i1}^1 + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^1, c_{k1}^1\right\}\right\} \neq 0 \\ 1, & b_{i1}^1 + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^1, c_{k1}^1\right\}\right\} = 0 \end{cases}$$

and

$$e_{i1}^2 := \begin{cases} 1 - \dfrac{\min\limits_{k\in I_{m-1}}\left\{\min\left\{b_{k1}^2, c_{k1}^2\right\}\right\} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^2, c_{k1}^2\right\}\right\}}{c_{i1}^2 + \lambda_1 \max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^2, c_{k1}^2\}\}}, & c_{i1}^2 + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^2, c_{k1}^2\right\}\right\} \neq 0 \\ 0, & c_{i1}^2 + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^2, c_{k1}^2\right\}\right\} = 0 \end{cases}$$

such that $i \in I_{m-1}$

Step 5. For $\lambda_2 \in [0,1]$, obtain $\left[f_{i1}^1\right]_{(m-1)\times 1}$ and $\left[f_{i1}^2\right]_{(m-1)\times 1}$ defined by

$$f_{i1}^1 := \lambda_2 d_{i1}^1 + (1-\lambda_2)e_{i1}^1 \quad \text{and} \quad f_{i1}^2 := \lambda_2 d_{i1}^2 + (1-\lambda_2)e_{i1}^2$$

such that $i \in I_{m-1}$

Step 6. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \dfrac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^s = \begin{cases} \dfrac{f_{i1}^1 + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right|}{\max\limits_{k\in I_{m-1}}\{f_{k1}^1 + |f_{k1}^2|\} + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right|}, & \max\limits_{k\in I_{m-1}}\{f_{k1}^1 + |f_{k1}^2|\} + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{f_{k1}^1 + |f_{k1}^2|\} + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^s = \begin{cases} 1 - \dfrac{f_{i1}^1 + |f_{i1}^2| + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right|}{\max\limits_{k\in I_{m-1}}\{f_{k1}^1 + |f_{k1}^2|\} + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right|}, & \max\limits_{k\in I_{m-1}}\{f_{k1}^1 + |f_{k1}^2|\} + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{f_{k1}^1 + |f_{k1}^2|\} + \left|\min\limits_{k\in I_{m-1}}\{f_{k1}^1\}\right| = 0 \end{cases}$$

Step 7. Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s}u_k \middle| u_k \in U\right\}$

Algorithm 3.23. iSM11

Step 1. Construct an *ifpifs*-matrix $\left[a_{ij}\right]_{m\times n}$

Step 2. Obtain $\left[b_{1j}^1\right]_{1\times n}$ and $\left[b_{1j}^2\right]_{1\times n}$ defined by

$$b_{1j}^1 := \max_{i\in I_{m-1}}\left\{\mu_{0j}^a \mu_{ij}^a\right\} \quad \text{and} \quad b_{1j}^2 := \min_{i\in I_{m-1}}\left\{\nu_{0j}^a \nu_{ij}^a\right\}$$

such that $j \in I_n$

Step 3. Obtain $\left[c_{i1}^1\right]_{(m-1)\times 1}$ and $\left[c_{i1}^2\right]_{(m-1)\times 1}$ defined by

$$c_{i1}^1 := \min_{j\in I_n}\left\{\max\left\{1-\mu_{0j}^a \mu_{ij}^a, b_{1j}^1\right\}\right\} \quad \text{and} \quad c_{i1}^2 := \max_{j\in I_n}\left\{\min\left\{1-\nu_{0j}^a \nu_{ij}^a, b_{1j}^2\right\}\right\}$$

such that $i \in I_{m-1}$

Step 4. Obtain $\left[d_{i1}^1\right]_{(m-1)\times 1}$ and $\left[d_{i1}^2\right]_{(m-1)\times 1}$ defined by

$$d_{i1}^1 := \max_{j\in I_n}\left\{\min\left\{\mu_{0j}^a \mu_{ij}^a, b_{1j}^1\right\}\right\} \quad \text{and} \quad d_{i1}^2 := \min_{j\in I_n}\left\{\max\left\{\nu_{0j}^a \nu_{ij}^a, b_{1j}^2\right\}\right\}$$

such that $i \in I_{m-1}$

Step 5. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \dfrac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$

$$\mu_{i1}^s := c_{i1}^1 + d_{i1}^1 - c_{i1}^1 d_{i1}^1 \quad \text{and} \quad \nu_{i1}^s := c_{i1}^2 d_{i1}^2$$

Step 6. Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s}u_k \middle| u_k \in U\right\}$

Algorithm 3.24. iPEM/iEC20

Step 1. Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} \coloneqq \sum_{j=1}^{n}\left[\left(\frac{1}{m-1}\sum_{k=1}^{m-1}\mu_{kj}^{a}\right)\left(\frac{1}{n}\sum_{t=1}^{n}\mu_{it}^{a}\right)\mu_{0j}^{a}\mu_{ij}^{a}\right] \quad \text{and} \quad c_{i1} \coloneqq \sum_{j=1}^{n}\left[\left(\frac{1}{m-1}\sum_{k=1}^{m-1}\nu_{kj}^{a}\right)\left(\frac{1}{n}\sum_{t=1}^{n}\nu_{it}^{a}\right)\nu_{0j}^{a}\nu_{ij}^{a}\right]$$

such that $i \in I_{m-1}$

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

Step 4. Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}}u_k \mid u_k \in U\right\}$

4. Proposed Test Cases for Generalised SDM Methods

This section proposes five new test cases by availing of the test cases provided in [13] to compare the decision-making performances of the generalised SDM methods. Each test case generating the same ranking order of the alternatives without using an SDM method consists of t *ifpifs*-matrices $[a_{ij}^{1}]$, $[a_{ij}^{2}]$, …, $[a_{ij}^{t}]$, with the order of $m \times n$. If an SDM method produces the same ranking order of the alternatives presented in a given test case, it means that the method is successful therein. Moreover, Proposition 2.1 of this study is utilised to rank the alternatives in the proposed test cases. Besides, because the numbers of the alternatives and the parameters are required to be equal in Test Case 3 and Test Case 4, we utilise equal numbers of alternatives and parameters in the remaining test cases as well. Therefore, for all the test cases, let $U = \{u_1, u_2, \dots, u_n\}$ be the set of alternatives and $E = \{x_1, x_2, \dots, x_n\}$ be the set of parameters.

4.1. Test Case 1

Test case 1 constructs *ifpifs*-matrices $[a_{ij}^{1}]_{(n+1)\times n}$, $[a_{ij}^{2}]_{(n+1)\times n}$, …, $[a_{ij}^{t}]_{(n+1)\times n}$ such that, $\mu_{01}^{a^k} = \mu_{02}^{a^k} = \dots = \mu_{0n}^{a^k}$, $\nu_{01}^{a^k} = \nu_{02}^{a^k} = \dots = \nu_{0n}^{a^k}$, $\mu_{1j}^{a^k} < \mu_{2j}^{a^k} < \dots < \mu_{nj}^{a^k}$, and $\nu_{nj}^{a^k} < \dots < \nu_{2j}^{a^k} < \nu_{1j}^{a^k}$, for all $k \in I_t$ and $j \in I_n$. Therefore,

$$\mu_{0j}^{a^k}\mu_{1j}^{a^k} - \nu_{0j}^{a^k}\nu_{1j}^{a^k} < \mu_{0j}^{a^k}\mu_{2j}^{a^k} - \nu_{0j}^{a^k}\nu_{2j}^{a^k} < \dots < \mu_{0j}^{a^k}\mu_{nj}^{a^k} - \nu_{0j}^{a^k}\nu_{nj}^{a^k}$$

for all $k \in I_t$ and $j \in I_n$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_1 \prec u_2 \prec \dots \prec u_n$.

4.2. Test Case 2

Test case 2 constructs *ifpifs*-matrices $[b_{ij}^1]_{(n+1)\times n}, [b_{ij}^2]_{(n+1)\times n}, \dots, [b_{ij}^t]_{(n+1)\times n}$ such that, $\mu_{01}^{b^k} = \mu_{02}^{b^k} = \cdots = \mu_{0n}^{b^k}$, $\nu_{01}^{b^k} = \nu_{02}^{b^k} = \cdots = \nu_{0n}^{b^k}$, $\mu_{nj}^{b^k} < \cdots < \mu_{2j}^{b^k} < \mu_{1j}^{b^k}$, and $\nu_{1j}^{b^k} < \nu_{2j}^{b^k} < \cdots < \nu_{nj}^{b^k}$, for all $k \in I_t$ and $j \in I_n$. Therefore,

$$\mu_{0j}^{b^k}\mu_{nj}^{b^k} - \nu_{0j}^{b^k}\nu_{nj}^{b^k} < \cdots < \mu_{0j}^{b^k}\mu_{2j}^{b^k} - \nu_{0j}^{b^k}\nu_{2j}^{b^k} < \mu_{0j}^{b^k}\mu_{1j}^{b^k} - \nu_{0j}^{b^k}\nu_{1j}^{b^k}$$

for all $k \in I_t$ and $j \in I_n$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_n \prec \cdots \prec u_2 \prec u_1$.

4.3. Test Case 3

Test case 3 constructs *ifpifs*-matrices $[c_{ij}^1]_{(n+1)\times n}, [c_{ij}^2]_{(n+1)\times n}, \dots, [c_{ij}^t]_{(n+1)\times n}$ such that for all $i,j \in I_n$ and $k \in I_t$, $\mu_{01}^{c^k} < \mu_{02}^{c^k} < \cdots < \mu_{0n}^{c^k}$ and $\nu_{0n}^{c^k} < \cdots < \nu_{02}^{c^k} < \nu_{01}^{c^k}$, $\frac{\mu_{ii}^{c^k}}{\nu_{ii}^{c^k}} = \frac{\lambda}{\varepsilon}$ such that $\lambda, \varepsilon \in [0,1]$ and $\lambda + \varepsilon \leq 1$, and if $i \neq j$, then $\frac{\mu_{ij}^{c^k}}{\nu_{ij}^{c^k}} = \frac{0}{1}$. Therefore,

$$\mu_{01}^{c^k}\mu_{11}^{c^k} - \nu_{01}^{c^k}\nu_{11}^{c^k} < \mu_{02}^{c^k}\mu_{22}^{c^k} - \nu_{02}^{c^k}\nu_{22}^{c^k} < \cdots < \mu_{0n}^{c^k}\mu_{nn}^{c^k} - \nu_{0n}^{c^k}\nu_{nn}^{c^k}$$

and if $i \neq j$, then $\mu_{0j}^{c^k}\mu_{ij}^{c^k} - \nu_{0j}^{c^k}\nu_{ij}^{c^k} = 0 - \nu_{0j}^{c^k} = -\nu_{0j}^{c^k}$, for all $i,j \in I_n$ and $k \in I_t$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_1 \prec u_2 \prec \cdots \prec u_n$.

4.4. Test Case 4

Test case 4 constructs *ifpifs*-matrices $[d_{ij}^1]_{(n+1)\times n}, [d_{ij}^2]_{(n+1)\times n}, \dots, [d_{ij}^t]_{(n+1)\times n}$ such that for all $i,j \in I_n$ and $k \in I_t$, $\mu_{0n}^{d^k} < \cdots < \mu_{02}^{d^k} < \mu_{01}^{d^k}$ and $\nu_{01}^{d^k} < \nu_{02}^{d^k} < \cdots < \nu_{0n}^{d^k}$, $\frac{\mu_{ii}^{d^k}}{\nu_{ii}^{d^k}} = \frac{\lambda}{\varepsilon}$ such that $\lambda, \varepsilon \in [0,1]$ and $\lambda + \varepsilon \leq 1$, and if $i \neq j$, then $\frac{\mu_{ij}^{d^k}}{\nu_{ij}^{d^k}} = \frac{0}{1}$. Therefore,

$$\mu_{0n}^{d^k}\mu_{nn}^{d^k} - \nu_{0n}^{d^k}\nu_{nn}^{d^k} < \cdots < \mu_{02}^{d^k}\mu_{22}^{d^k} - \nu_{02}^{d^k}\nu_{22}^{d^k} < \mu_{01}^{d^k}\mu_{11}^{d^k} - \nu_{01}^{d^k}\nu_{11}^{d^k}$$

and if $i \neq j$, then $\mu_{0j}^{d^k}\mu_{ij}^{d^k} - \nu_{0j}^{d^k}\nu_{ij}^{d^k} = 0 - \nu_{0j}^{d^k} = -\nu_{0j}^{d^k}$, for all $i,j \in I_n$ and $k \in I_t$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_n \prec \cdots \prec u_2 \prec u_1$.

4.5. Test Case 5

Test case 5 constructs *ifpifs*-matrices $[e_{ij}^1]_{(n+1)\times n}, [e_{ij}^2]_{(n+1)\times n}, \dots, [e_{ij}^t]_{(n+1)\times n}$ such that for all $i,j \in I_n$ and $k \in I_t$, $\frac{\mu_{ii}^{e^k}}{\nu_{ii}^{e^k}} = \frac{\lambda}{\varepsilon}$, $\lambda, \varepsilon \in [0,1]$, and $\lambda + \varepsilon \leq 1$. Therefore, $\mu_{ij}^{e^k} - \nu_{ij}^{e^k} = \mu_{lj}^{e^k} - \nu_{lj}^{e^k}$ and $\mu_{ij}^{e^k} + \nu_{ij}^{e^k} = \mu_{lj}^{e^k} + \nu_{lj}^{e^k}$, for all $i,l,j \in I_n$ and $k \in I_t$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_1 \approx u_2 \approx \cdots \approx u_n$. Here, $\approx$ denotes the same ranking order.

4.6. Results of Test Cases

This subsection tests the generalised SDM methods using aforesaid test cases. Thus, it determines SDM methods being successful in all the test cases. Since the generalised SDM methods herein employ only a single matrix, we consider $t = 1$, $n = 4$, $U = \{u_1, u_2, u_3, u_4\}$, and $E = \{x_1, x_2, x_3, x_4\}$, for all the test cases. Therefore, we use *ifpifs*-matrices provided in Table 1.

Table 1. *ifpifs*-matrices employed in the test cases

Test Case 1	Test Case 2	Test Case 3	Test Case 4	Test Case 5
$[a_{ij}^1] := \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.6 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.25 & 0.3 \\ 0.8 & 0.7 & 0.6 & 0.5 \\ 0.1 & 0.15 & 0.2 & 0.25 \\ 0.9 & 0.8 & 0.7 & 0.6 \\ 0.05 & 0.1 & 0.15 & 0.2 \\ 1 & 0.9 & 0.8 & 0.7 \\ 0 & 0.05 & 0.1 & 0.15 \end{bmatrix}$	$[b_{ij}^1] := \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.8 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 1 & 0.9 & 0.8 & 0.7 \\ 0 & 0.05 & 0.1 & 0.15 \\ 0.9 & 0.8 & 0.7 & 0.6 \\ 0.05 & 0.1 & 0.15 & 0.2 \\ 0.8 & 0.7 & 0.6 & 0.5 \\ 0.1 & 0.15 & 0.2 & 0.25 \\ 0.7 & 0.6 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.25 & 0.3 \end{bmatrix}$	$[c_{ij}^1] := \begin{bmatrix} 0.6 & 0.7 & 0.8 & 0.9 \\ 0.2 & 0.15 & 0.1 & 0.05 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$[d_{ij}^1] := \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 \\ 0.05 & 0.1 & 0.15 & 0.2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$[e_{ij}^1] := \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$

Moreover, since all the parameters should be considered in all the test cases, we determine the variables (inputs) R, R_1, R_2, and R_3 as I_4. Additionally, the other variables are intently selected so that the methods can pass the largest number of test cases. Table 2 provides the test results of the SDM methods created using MATLAB R2021a. Furthermore, the numbers of the passed tests are presented in the last column of Table 2. 12 of 24 methods are observed to be successful in all the test cases. These methods are iMBR01, isMBR01, iMBR01/2, iMRB02(I_4), iCCE11, iCCE10, iCEC11, iKKT13, iFJLL10/2$\left(I_4, \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}\right)$ (Test 1, 2), iFJLL10/2$\left(I_4, \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}\right)$ (Test 3, 4, 5), iFJLL10/4$\left(I_4, \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}\right)$ (Test 1, 2), iFJLL10/4$\left(I_4, \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}\right)$ (Test 3, 4, 5), iKWW11(I_4, 0.5,0.5), and iPEM.

Table 2. Performance of the generalised SDM methods in the test cases

	Algorithms\Test Cases	**Test Case 1**	**Test Case 2**	**Test Case 3**	**Test Case 4**	**Test Case 5**	**Numbers of Tests Passed**
1.	iMBR01	✓	✓	✓	✓	✓	**5**
2.	isMBR01	✓	✓	✓	✓	✓	**5**
3.	iMBR01/2	✓	✓	✓	✓	✓	**5**
4.	iMRB02(I_4)	✓	✓	✓	✓	✓	**5**
5.	iKM11(I_4)	✓	✓	–	–	✓	3
6.	iCCE11	✓	✓	✓	✓	✓	**5**
7.	iYE12	✓	✓	–	–	✓	3
8.	iCCE10	✓	✓	✓	✓	✓	**5**
9.	iCEC11	✓	✓	✓	✓	✓	**5**
10.	iM11	–	–	–	–	✓	1
11.	iKKT13	✓	✓	✓	✓	✓	**5**

12.	iFJLL10$\left(I_4,\begin{bmatrix}0.7&0.7&0.7&0.7\\0.1&0.1&0.1&0.1\end{bmatrix}\right)$ (Test 1, 2) iFJLL10$\left(I_4,\begin{bmatrix}0.4&0.4&0.4&0.4\\0.4&0.4&0.4&0.4\end{bmatrix}\right)$ (Test 3, 4, 5)	–	✓	–	–	✓	2
13.	iFJLL10/2$\left(I_4,\begin{bmatrix}0.7&0.7&0.7&0.7\\0.1&0.1&0.1&0.1\end{bmatrix}\right)$ (Test 1, 2) iFJLL10/2$\left(I_4,\begin{bmatrix}0.4&0.4&0.4&0.4\\0.4&0.4&0.4&0.4\end{bmatrix}\right)$ (Test 3, 4, 5)	✓	✓	✓	✓	✓	**5**
14.	iFJLL10/3$\left(I_4,\begin{bmatrix}0.7&0.7&0.7&0.7\\0.1&0.1&0.1&0.1\end{bmatrix}\right)$ (Test 1, 2) iFJLL10/3$\left(I_4,\begin{bmatrix}0.4&0.4&0.4&0.4\\0.4&0.4&0.4&0.4\end{bmatrix}\right)$ (Test 3, 4, 5)	–	✓	–	–	✓	2
15.	iFJLL10/4$\left(I_4,\begin{bmatrix}0.7&0.7&0.7&0.7\\0.1&0.1&0.1&0.1\end{bmatrix}\right)$ (Test 1, 2) iFJLL10/4$\left(I_4,\begin{bmatrix}0.4&0.4&0.4&0.4\\0.4&0.4&0.4&0.4\end{bmatrix}\right)$ (Test 3, 4, 5)	✓	✓	✓	✓	✓	**5**
16.	iFJLL10m(I_4)	–	–	–	–	✓	1
17.	iFJLL10/2m(I_4)	–	–	✓	✓	✓	3
18.	iFJLL10max(I_4)	–	–	–	–	✓	1
19.	iFJLL10/2max(I_4)	–	–	✓	✓	✓	3
20.	iF10$(I_4, 0.5)$	–	–	✓	✓	✓	3
21.	iKSM10	–	–	–	–	✓	1
22.	iKWW11$(I_4, 0.5,0.5)$	✓	✓	✓	✓	✓	**5**
23.	iSM11	–	✓	✓	✓	✓	4
24.	iPEM	✓	✓	✓	✓	✓	**5**
	Total	14	17	16	16	24	12

Bold values in the last column indicate the SDM methods passing all the test cases (✓: Successful, –: Unsuccessful)

5. An Application of the Generalised SDM Methods Being Successful in All the Test Cases to a PVA Problem

This section applies the SDM methods generalised in Section 3, which is successful in all test cases to a real problem related to performance-based value assignment (PVA) to seven noise-removal filters, namely "Based on Pixel Density Filter (BPDF)" [35], "Decision-Based Algorithm (DBAIN)" [36], "Modified Decision Based Unsymmetrical Trimmed Median Filter (MDBUTMF)" [37], "Noise Adaptive Fuzzy Switching Median Filter (NAFSMF)" [38], "Different Applied Median Filter (DAMF)" [39], "Adaptive Weighted Mean Filter (AWMF)" [40], and "Adaptive Riesz Mean Filter (ARmF)" [41]. In this PVA problem, we indicate the set of alternatives $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ such that $u_1 =$ "BPDF", $u_2 =$ "DBAIN", $u_3 =$ "MDBUTMF", $u_4 =$ "NAFSMF", $u_5 =$ "DAMF", $u_6 =$ "AWMF", and $u_7 =$ "ARmF". Moreover, we denote the parameters set $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ such that $x_1 =$ "noise density 10%", $x_2 =$ "noise density 20%", $x_3 =$ "noise density 30%", $x_4 =$ "noise density 40%", $x_5 =$ "noise density 50%", $x_6 =$ "noise density 60%", $x_7 =$ "noise density 70%", $x_8 =$ "noise density 80%", and $x_9 =$ "noise density 90%". Therefore, we present the Structural Similarity (SSIM) [42] results of aforesaid filters for 20 traditional images, i.e., "Lena", "Cameraman", "Barbara", "Baboon", "Peppers", "Living Room", "Lake", "Plane", "Hill", "Pirate", "Boat", "House", "Bridge", "Elaine", "Flintstones", "Flower", "Parrot", "Dark-Haired Woman", "Blonde Woman", and "Einstein", at noise density ranging from 10% to 90% in Table 3-8. We obtain these results using MATLAB R2021a.

Table 3. SSIM results of BPDF for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9848	0.9657	0.9411	0.9087	0.8689	0.8120	0.7247	0.5683	0.3063
Cameraman	0.9911	0.9782	0.9608	0.9344	0.8966	0.8453	0.7726	0.6722	0.5105
Barbara	0.9743	0.9427	0.9046	0.8606	0.8024	0.7289	0.6258	0.4597	0.2316
Baboon	0.9795	0.9516	0.9112	0.8556	0.7812	0.6841	0.5622	0.4080	0.1377
Peppers	0.9735	0.9460	0.9158	0.8798	0.8363	0.7780	0.7001	0.5584	0.2194
Living Room	0.9747	0.9432	0.9056	0.8569	0.7962	0.7153	0.6012	0.4372	0.2337
Lake	0.9795	0.9526	0.9218	0.8796	0.8253	0.7468	0.6464	0.4839	0.2226
Plane	0.9885	0.9733	0.9533	0.9220	0.8797	0.8194	0.7309	0.5631	0.1894
Hill	0.9761	0.9480	0.9129	0.8676	0.8062	0.7275	0.6232	0.4954	0.3573
Pirate	0.9801	0.9549	0.9232	0.8817	0.8266	0.7506	0.6494	0.4797	0.2741
Boat	0.9753	0.9456	0.9085	0.8608	0.8010	0.7245	0.6155	0.4697	0.2851
House	0.9938	0.9858	0.9730	0.9550	0.9241	0.8835	0.8113	0.7002	0.4932
Bridge	0.9705	0.9335	0.8856	0.8269	0.7503	0.6452	0.5159	0.3648	0.1815
Elaine	0.9707	0.9405	0.9052	0.8649	0.8149	0.7517	0.6628	0.4927	0.2911
Flintstones	0.9726	0.9417	0.9021	0.8550	0.7912	0.7099	0.5908	0.4125	0.1259
Flower	0.9808	0.9618	0.9346	0.8998	0.8446	0.7718	0.6634	0.4970	0.2249
Parrot	0.9791	0.9663	0.9490	0.9270	0.8992	0.8580	0.7955	0.6816	0.3541
Dark-Haired Woman	0.9909	0.9802	0.9665	0.9471	0.9200	0.8789	0.8100	0.6828	0.4483
Blonde Woman	0.9657	0.9385	0.9055	0.8664	0.8191	0.7561	0.6624	0.5003	0.2184
Einstein	0.9830	0.9614	0.9361	0.9051	0.8640	0.8085	0.7315	0.5892	0.3465

Table 4. SSIM results of DBA for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9885	0.9741	0.9555	0.9291	0.8989	0.8560	0.7942	0.7139	0.5979
Cameraman	0.9948	0.9867	0.9758	0.9586	0.9332	0.8977	0.8452	0.7805	0.6917
Barbara	0.9769	0.9502	0.9174	0.8762	0.8279	0.7662	0.6880	0.5882	0.4589
Baboon	0.9844	0.9644	0.9352	0.8933	0.8373	0.7605	0.6587	0.5422	0.4161
Peppers	0.9742	0.9508	0.9239	0.8909	0.8535	0.8034	0.7387	0.6565	0.5402
Living Room	0.9802	0.9557	0.9251	0.8857	0.8368	0.7693	0.6888	0.5838	0.4565
Lake	0.9768	0.9565	0.9315	0.8988	0.8561	0.7984	0.7228	0.6267	0.5053
Plane	0.9885	0.9781	0.9642	0.9423	0.9124	0.8706	0.8139	0.7343	0.6268
Hill	0.9801	0.9578	0.9287	0.8912	0.8410	0.7784	0.6997	0.6036	0.4833
Pirate	0.9832	0.9637	0.9387	0.9062	0.8605	0.8017	0.7286	0.6247	0.5002
Boat	0.9767	0.9532	0.9239	0.8844	0.8396	0.7785	0.6968	0.5992	0.4825
House	0.9969	0.9920	0.9832	0.9703	0.9522	0.9238	0.8777	0.8142	0.7234
Bridge	0.9728	0.9424	0.9047	0.8552	0.7917	0.7104	0.6060	0.4880	0.3518
Elaine	0.9746	0.9483	0.9173	0.8800	0.8358	0.7832	0.7157	0.6292	0.5121
Flintstones	0.9769	0.9533	0.9210	0.8793	0.8239	0.7487	0.6490	0.5308	0.3807
Flower	0.9854	0.9722	0.9517	0.9259	0.8841	0.8330	0.7579	0.6588	0.5230
Parrot	0.9840	0.9741	0.9607	0.9440	0.9209	0.8900	0.8467	0.7871	0.6951
Dark-Haired Woman	0.9925	0.9850	0.9754	0.9614	0.9414	0.9133	0.8715	0.8065	0.7056
Blonde Woman	0.9666	0.9449	0.9184	0.8856	0.8441	0.7938	0.7259	0.6470	0.5432
Einstein	0.9867	0.9706	0.9500	0.9236	0.8881	0.8449	0.7839	0.7102	0.6142

Table 5. SSIM results of MDBUTMF for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9865	0.9479	0.8498	0.8155	0.8655	0.8898	0.8668	0.7830	0.4010
Cameraman	0.9911	0.9412	0.8160	0.7818	0.8653	0.9174	0.9052	0.8265	0.4667
Barbara	0.9741	0.9228	0.8235	0.7757	0.7962	0.7914	0.7477	0.6573	0.3884
Baboon	0.9727	0.9321	0.8655	0.8228	0.8126	0.7869	0.7317	0.6333	0.3625
Peppers	0.9794	0.9331	0.8263	0.7884	0.8321	0.8484	0.8206	0.7382	0.4131
Living Room	0.9764	0.9338	0.8567	0.8137	0.8251	0.8066	0.7621	0.6682	0.3744
Lake	0.9802	0.9275	0.8097	0.7749	0.8177	0.8374	0.8066	0.7192	0.4084
Plane	0.9884	0.9317	0.7907	0.7539	0.8392	0.8978	0.8833	0.7857	0.3518
Hill	0.9781	0.9340	0.8335	0.7938	0.8193	0.8220	0.7827	0.6976	0.3921
Pirate	0.9813	0.9381	0.8418	0.8072	0.8363	0.8430	0.8096	0.7178	0.4185
Boat	0.9783	0.9353	0.8450	0.8064	0.8268	0.8243	0.7833	0.6906	0.3796
House	0.9950	0.9491	0.8178	0.7831	0.8833	0.9449	0.9425	0.8641	0.4270
Bridge	0.9699	0.9236	0.8433	0.7994	0.7855	0.7572	0.6950	0.6000	0.3651
Elaine	0.9774	0.9324	0.8347	0.7965	0.8224	0.8295	0.7925	0.6973	0.3492
Flintstones	0.9764	0.9304	0.8315	0.7932	0.8169	0.8128	0.7671	0.6735	0.3965
Flower	0.9820	0.9486	0.8681	0.8407	0.8732	0.8832	0.8523	0.7679	0.4292
Parrot	0.9771	0.9334	0.8242	0.7958	0.8655	0.9042	0.8911	0.8123	0.4008
Dark-Haired Woman	0.9923	0.9395	0.7833	0.7576	0.8620	0.9294	0.9272	0.8566	0.4772
Blonde Woman	0.9642	0.9236	0.8294	0.7952	0.8214	0.8258	0.7936	0.7017	0.3539
Einstein	0.9833	0.9418	0.8476	0.8127	0.8528	0.8677	0.8393	0.7561	0.4127

Table 6. SSIM results of NAFSMF for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9839	0.9669	0.9485	0.9279	0.9080	0.8821	0.8511	0.8040	0.6862
Cameraman	0.9798	0.9643	0.9500	0.9340	0.9177	0.8988	0.8727	0.8325	0.7207
Barbara	0.9749	0.9472	0.9174	0.8843	0.8483	0.8039	0.7533	0.6896	0.5729
Baboon	0.9612	0.9216	0.8767	0.8305	0.7800	0.7211	0.6540	0.5777	0.4671
Peppers	0.9772	0.9551	0.9328	0.9068	0.8810	0.8512	0.8154	0.7665	0.6470
Living Room	0.9704	0.9382	0.9047	0.8687	0,8301	0.7839	0.7329	0.6678	0.5472
Lake	0.9754	0.9489	0.9210	0.8925	0.8588	0.8229	0.7805	0.7221	0.6021
Plane	0.9845	0.9685	0.9524	0.9334	0.9136	0.8892	0.8596	0.8175	0.7019
Hill	0.9733	0.9451	0.9148	0.8824	0.8463	0.8064	0.7585	0.7010	0.5843
Pirate	0.9766	0.9511	0.9248	0.8970	0.8635	0.8251	0.7844	0.7227	0.6093
Boat	0.9723	0.9422	0.9115	0.8766	0.8414	0.8005	0.7528	0.6898	0.5778
House	0.9914	0.9831	0.9733	0.9643	0.9535	0.9405	0.9210	0.8918	0.7827
Bridge	0.9631	0.9222	0.8788	0.8337	0.7818	0.7237	0.6544	0.5766	0.4578
Elaine	0.9774	0.9542	0.9295	0.9025	0.8730	0.8404	0.8010	0.7470	0.6310
Flintstones	0.9659	0.9333	0.8983	0.8631	0.8220	0.7743	0.7165	0.6464	0.5215
Flower	0.9763	0.9568	0.9363	0.9143	0.8883	0.8600	0.8218	0.7682	0.6492
Parrot	0.9785	0.9653	0.9519	0.9380	0.9209	0.9030	0.8774	0.8418	0.7331
Dark-Haired Woman	0.9906	0.9815	0.9723	0.9622	0.9513	0.9361	0.9192	0.8891	0.7756
Blonde Woman	0.9606	0.9366	0.9104	0.8833	0.8526	0.8184	0.7805	0.7259	0.6113
Einstein	0.9801	0.9591	0.9364	0.9132	0.8878	0.8591	0.8231	0.7732	0.6698

Table 7. SSIM results of DAMF for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9902	0.9789	0.9653	0.9488	0.9310	0.9085	0.8796	0.8396	0.7657
Cameraman	0.9961	0.9908	0.9844	0.9759	0.9652	0.9512	0.9321	0.9012	0.8347
Barbara	0.9815	0.9588	0.9327	0.9013	0.8675	0.8261	0.7786	0.7176	0.6308
Baboon	0.9884	0.9748	0.9572	0.9356	0.9086	0.8738	0.8237	0.7466	0.6037
Peppers	0.9804	0.9594	0.9372	0.9110	0.8835	0.8515	0.8152	0.7707	0.7018
Living Room	0.9846	0.9654	0.9422	0.9152	0.8824	0.8443	0.7976	0.7325	0.6295
Lake	0.9856	0.9690	0.9499	0.9285	0.9020	0.8689	0.8293	0.7737	0.6842
Plane	0.9938	0.9861	0.9769	0.9648	0.9505	0.9331	0.9086	0.8714	0.7987
Hill	0.9841	0.9656	0.9438	0.9181	0.8875	0.8515	0.8075	0.7495	0.6571
Pirate	0.9875	0.9722	0.9542	0.9332	0.9063	0.8744	0.8362	0.7784	0.6853
Boat	0.9833	0.9634	0.9407	0.9123	0.8829	0.8463	0.8011	0.7419	0.6514
House	0.9982	0.9955	0.9912	0.9861	0.9796	0.9709	0.9577	0.9376	0.8852
Bridge	0.9798	0.9560	0.9276	0.8953	0.8563	0.8072	0.7465	0.6667	0.5415
Elaine	0.9774	0.9534	0.9270	0.8961	0.8620	0.8230	0.7784	0.7248	0.6584
Flintstones	0.9840	0.9658	0.9430	0.9173	0.8865	0.8464	0.7980	0.7268	0.6061
Flower	0.9878	0.9786	0.9662	0.9513	0.9321	0.9089	0.8772	0.8290	0.7404
Parrot	0.9839	0.9763	0.9666	0.9563	0.9423	0.9270	0.9064	0.8775	0.8226
Dark-Haired Woman	0.9950	0.9891	0.9826	0.9743	0.9647	0.9525	0.9362	0.9134	0.8664
Blonde Woman	0.9700	0.9518	0.9301	0.9053	0.8764	0.8424	0.8015	0.7505	0.6753
Einstein	0.9894	0.9765	0.9619	0.9445	0.9244	0.8989	0.8666	0.8208	0.7472

Table 8. SSIM results of AWMF for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9822	0.9740	0.9636	0.9497	0.9349	0.9134	0.8852	0.8447	0.7737
Cameraman	0.9883	0.9849	0.9813	0.9759	0.9681	0.9563	0.9371	0.9059	0.8401
Barbara	0.9718	0.9540	0.9331	0.9065	0.8762	0.8366	0.7879	0.7250	0.6382
Baboon	0.9720	0.9616	0.9487	0.9343	0.9135	0.8824	0.8331	0.7550	0.6108
Peppers	0.9609	0.9560	0.9410	0.9204	0.8952	0.8633	0.8256	0.7789	0.7096
Living Room	0.9693	0.9539	0.9358	0.9144	0.8879	0.8523	0.8062	0.7394	0.6356
Lake	0.9742	0.9620	0.9474	0.9297	0.9067	0.8758	0.8361	0.7799	0.6904
Plane	0.9850	0.9796	0.9733	0.9645	0.9532	0.9376	09133	0.8760	0.8055
Hill	0.9724	0.9576	0.9409	0.9195	0.8929	0.8593	0.8152	0.7562	0.6632
Pirate	0.9753	0.9624	0.9489	0.9322	0.9088	0.8790	0.8417	0.7834	0.6913
Boat	0.9706	0.9555	0.9375	0.9146	0.8887	0.8543	0.8091	0.7483	0.6571
House	0.9933	0.9924	0.9905	0.9878	0.9834	0.9760	0.9630	0.9426	0.8948
Bridge	0.9638	0.9440	0.9209	0.8948	0.8611	0.8148	0.7551	0.6736	0.5469
Elaine	0.9684	0.9514	0.9296	0.9021	0.8696	0.8313	0.7857	0.7310	0.6640
Flintstones	0.9551	0.9502	0.9364	0.9167	0.8908	0.8541	0.8058	0.7334	0.6118
Flower	0.9752	0.9684	0.9594	0.9488	0.9333	0.9126	0.8820	0.8340	0.7459
Parrot	0.9779	0.9727	0.9655	0.9572	0.9457	0.9316	0.9112	0.8828	0.8309
Dark-Haired Woman	0.9910	0.9870	0.9823	0.9761	0.9678	0.9565	0.9404	0.9177	0.8744
Blonde Woman	0.9579	0.9450	0.9273	0.9061	0.8802	0.8476	0.8069	0.7554	0.6814
Einstein	0.9798	0.9701	0.9588	0.9450	0.9280	0.9043	0.8724	0.8259	0.7531

Table 9. SSIM results of ARmF for 20 traditional images

Images/Noise Densities	10%	20%	30%	40%	50%	60%	70%	80%	90%
Lena	0.9910	0.9810	0.9697	0.9554	0.9398	0.9176	0.8885	0.8471	0.7752
Cameraman	0.9970	0.9933	0.9890	0.9828	0.9743	0.9614	0.9416	0.9093	0.8426
Barbara	0.9841	0.9654	0.9438	0.9172	0.8861	0.8450	0.7949	0.7291	0.6394
Baboon	0.9915	0.9818	0.9689	0.9523	0.9294	0.8960	0.8442	0.7630	0.6150
Peppers	0.9826	0.9640	0.9439	0.9205	0.8939	0.8618	0.8241	0.7779	0.7096
Living Room	0.9856	0.9699	0.9514	0.9294	0.9018	0.8653	0.8171	0.7483	0.6418
Lake	0.9867	0.9716	0.9553	0.9361	0.9113	0.8793	0.8391	0.7828	0.6926
Plane	0.9947	0.9887	0.9816	0.9719	0.9599	0.9433	0.9182	0.8795	0.8080
Hill	0.9860	0.9703	0.9526	0.9310	0.9038	0.8690	0.8240	0.7626	0.6672
Pirate	0.9884	0.9750	0.9600	0.9424	0.9181	0.8875	0.8487	0.7886	0.6939
Boat	0.9842	0.9664	0.9467	0.9223	0.8957	0.8606	0.8142	0.7529	0.6610
House	0.9987	0.9970	0.9946	0.9913	0.9863	0.9786	0.9652	0.9446	0.8962
Bridge	0.9823	0.9621	0.9385	0.9113	0.8762	0.8285	0.7663	0.6819	0.5515
Elaine	0.9773	0.9532	0.9272	0.8971	0.8630	0.8239	0.7791	0.7270	0.6631
Flintstones	0.9847	0.9688	0.9491	0.9267	0.8987	0.8608	0.8112	0.7381	0.6154
Flower	0.9877	0.9796	0.9696	0.9577	0.9411	0.9195	0.8876	0.8384	0.7489
Parrot	0.9851	0.9786	0.9706	0.9621	0.9499	0.9351	0.9141	0.8848	0.8320
Dark-Haired Woman	0.9956	0.9909	0.9854	0.9787	0.9701	0.9585	0.9420	0.9189	0.8753
Blonde Woman	0.9718	0.9551	0.9355	0.9132	0.8864	0.8531	0.8114	0.7582	0.6825
Einstein	0.9911	0.9805	0.9687	0.9543	0.9367	0.9121	0.8788	0.8305	0.7551

In this PVA problem, we construct an *ifpifs*-matrix $[a_{ij}]_{8\times 9}$ by using multiple fuzzy values provided in Table 3-9. We calculate the other rows of this *ifpifs*-matrix except for its zero-index row by employing the membership function and non-membership function defined by

$$\mu_{ij}^{a} := \min_{t} S_{ij}^{t} \quad \text{and} \quad \nu_{ij}^{a} := 1 - \max_{t} S_{ij}^{t}$$

such that $i \in I_7$, $j \in I_9$, and $t \in I_{20}$. Here, (S_{ij}^{t}) denotes ordered s-tuples such that S_{ij}^{t} corresponds to SSIM results originating from t^{th} image for i^{th} filter and j^{th} noise density. Moreover, s is the number of images. That is, $s = 20$. For instance,

$$(S_{11}^{t}) = (0.9848,0.9911,0.9743,0.9795,0.9735,0.9747,0.9795,0.9885,0.9761,0.9801,0.9753,0.9938,\\ 0.9705,0.9707,0.9726,0.9808,0.9791,0.9909,0.9657,0.9830)$$

Hence, $\mu_{11}^{a} = 0.9657$ and $\nu_{11}^{a} = 0.0062$. Similarly, the values of the other alternatives can be calculated. Moreover, suppose that the noise removal success of the filters at high noise densities is more significant than at the other densities, it is anticipated that the membership degrees at high noise densities are greater than the non-membership degrees and the former at low noise densities are smaller than the latter. In other words, we consider the first row of $[a_{ij}]_{8\times 9}$ to be

$$\begin{bmatrix} 0.05 & 0.15 & 0.25 & 0.35 & 0.5 & 0.65 & 0.75 & 0.85 & 0.9 \\ 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.3 & 0.2 & 0.1 & 0.05 \end{bmatrix}$$

Thus, *ifpifs*-matrix $[a_{ij}]_{8\times 9}$ is constructed as follows:

$$[a_{ij}] = \begin{bmatrix} 0.05 & 0.15 & 0.25 & 0.35 & 0.5 & 0.65 & 0.75 & 0.85 & 0.9 \\ 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.3 & 0.2 & 0.1 & 0.05 \\[6pt] 0.9657 & 0.9335 & 0.8856 & 0.8269 & 0.7503 & 0.6452 & 0.5159 & 0.3648 & 0.1259 \\ 0.0062 & 0.0142 & 0.0270 & 0.0450 & 0.0759 & 0.1165 & 0.1887 & 0.2998 & 0.4895 \\[6pt] 0.9666 & 0.9424 & 0.9047 & 0.8552 & 0.7917 & 0.7104 & 0.6060 & 0.4880 & 0.3518 \\ 0.0031 & 0.0080 & 0.0168 & 0.0297 & 0.0478 & 0.0762 & 0.1223 & 0.1858 & 0.2766 \\[6pt] 0.9642 & 0.9228 & 0.7833 & 0.7539 & 0.7855 & 0.7572 & 0.6950 & 0.6000 & 0.3492 \\ 0.0050 & 0.0509 & 0.1319 & 0.1593 & 0.1167 & 0.0551 & 0.0575 & 0.1359 & 0.5228 \\[6pt] 0.9606 & 0.9216 & 0.8767 & 0.8305 & 0.7800 & 0.7211 & 0.6540 & 0.5766 & 0.4578 \\ 0.0086 & 0.0169 & 0.0267 & 0.0357 & 0.0465 & 0.0595 & 0.0790 & 0.1082 & 0.2173 \\[6pt] 0.9700 & 0.9518 & 0.9270 & 0.8953 & 0.8563 & 0.8072 & 0.7465 & 0.6667 & 0.5415 \\ 0.0018 & 0.0045 & 0.0088 & 0.0139 & 0.0204 & 0.0291 & 0.0423 & 0.0624 & 0.1148 \\[6pt] 0.9551 & 0.9440 & 0.9209 & 0.8948 & 0.8611 & 0.8148 & 0.7551 & 0.6736 & 0.5469 \\ 0.0067 & 0.0076 & 0.0095 & 0.0122 & 0.0166 & 0.0240 & 0.0370 & 0.0574 & 0.1052 \\[6pt] 0.9718 & 0.9532 & 0.9272 & 0.8971 & 0.8630 & 0.8239 & 0.7663 & 0.6819 & 0.5515 \\ 0.0013 & 0.0030 & 0.0054 & 0.0087 & 0.0137 & 0.0214 & 0.0348 & 0.0554 & 0.1038 \end{bmatrix}$$

In Table 10, $w = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$. Moreover, since the number of alternatives and parameters is not equal in this PVA problem, iKKT13 is not applied to $[a_{ij}]$.

Table 10. Decision sets produced by SDM methods*

Algorithms	**Matrix**	**Decision Sets**
iMBR01	$[a_{ij}]$	$\{{}^{0}_{0.8098}\text{BPDF}, {}^{0.1770}_{0.8005}\text{DBAIN}, {}^{0.2008}_{0.5865}\text{MDBUTMF}, {}^{0.1955}_{0.6433}\text{NAFSMF}, {}^{0.4571}_{0.3699}\text{DAMF}, {}^{0.5271}_{0.3884}\text{AWMF}, {}^{0.6711}_{0}\text{ARmF}\}$
isMBR01	$[a_{ij}]$	$\{{}^{0}_{0.8098}\text{BPDF}, {}^{0.1770}_{0.8005}\text{DBAIN}, {}^{0.2008}_{0.5865}\text{MDBUTMF}, {}^{0.1955}_{0.6433}\text{NAFSMF}, {}^{0.4571}_{0.3699}\text{DAMF}, {}^{0.5271}_{0.3884}\text{AWMF}, {}^{0.6711}_{0}\text{ARmF}\}$
iMBR01/2	$[a_{ij}]$	$\{{}^{0}_{0.8210}\text{BPDF}, {}^{0.1582}_{0.8257}\text{DBAIN}, {}^{0.1508}_{0.6475}\text{MDBUTMF}, {}^{0.2131}_{0.6005}\text{NAFSMF}, {}^{0.4444}_{0.3720}\text{DAMF}, {}^{0.5255}_{0.4088}\text{AWMF}, {}^{0.6662}_{0}\text{ARmF}\}$
iMRB02($\boldsymbol{I_9}$)	$[a_{ij}]$	$\{{}^{0.8080}_{0.1520}\text{BPDF}, {}^{0.8878}_{0.0872}\text{DBAIN}, {}^{0.9085}_{0.0285}\text{MDBUTMF}, {}^{0.9209}_{0.0546}\text{NAFSMF}, {}^{0.9833}_{0.0064}\text{DAMF}, {}^{0.9869}_{0.0025}\text{AWMF}, {}^{0.9924}_{0}\text{ARmF}\}$
iCCE11	$[a_{ij}]$	$\{{}^{0.2560}_{0.0253}\text{BPDF}, {}^{0.3065}_{0.0158}\text{DBAIN}, {}^{0.3197}_{0.0399}\text{MDBUTMF}, {}^{0.3275}_{0.0155}\text{NAFSMF}, {}^{0.3670}_{0.0066}\text{DAMF}, {}^{0.3693}_{0.0067}\text{AWMF}, {}^{0.3728}_{0.0048}\text{ARmF}\}$
iCCE10	$[a_{ij}]$	$\{{}^{0.2560}_{0.0253}\text{BPDF}, {}^{0.3065}_{0.0158}\text{DBAIN}, {}^{0.3197}_{0.0399}\text{MDBUTMF}, {}^{0.3275}_{0.0155}\text{NAFSMF}, {}^{0.3670}_{0.0066}\text{DAMF}, {}^{0.3693}_{0.0067}\text{AWMF}, {}^{0.3728}_{0.0048}\text{ARmF}\}$
iCEC11	$[a_{ij}]$	$\{{}^{0.2129}_{0.0035}\text{BPDF}, {}^{0.2469}_{0.0021}\text{DBAIN}, {}^{0.2556}_{0.0055}\text{MDBUTMF}, {}^{0.2598}_{0.0021}\text{NAFSMF}, {}^{0.2895}_{0.0009}\text{DAMF}, {}^{0.2912}_{0.0009}\text{AWMF}, {}^{0.2938}_{0.0007}\text{ARmF}\}$
iFJLL10/2($\boldsymbol{I_9}, \boldsymbol{w}$)	$[a_{ij}]$	$\{{}^{0.7552}_{0.2238}\text{BPDF}, {}^{0.8741}_{0.1189}\text{DBAIN}, {}^{0.8741}_{0.1189}\text{MDBUTMF}, {}^{1}_{0}\text{NAFSMF}, {}^{1}_{0}\text{DAMF}, {}^{1}_{0}\text{AWMF}, {}^{1}_{0}\text{ARmF}\}$
iFJLL10/4($\boldsymbol{I_9}, \boldsymbol{w}$)	$[a_{ij}]$	$\{{}^{0.7552}_{0.2238}\text{BPDF}, {}^{0.8741}_{0.1189}\text{DBAIN}, {}^{0.8741}_{0.1189}\text{MDBUTMF}, {}^{1}_{0}\text{NAFSMF}, {}^{1}_{0}\text{DAMF}, {}^{1}_{0}\text{AWMF}, {}^{1}_{0}\text{ARmF}\}$
iKWW11($\boldsymbol{I_9}, \mathbf{0.5}, \mathbf{0.5}$)	$[a_{ij}]$	$\{{}^{0.6198}_{0.0827}\text{BPDF}, {}^{0.6793}_{0.0542}\text{DBAIN}, {}^{0.6950}_{0.0968}\text{MDBUTMF}, {}^{0.7018}_{0.0662}\text{NAFSMF}, {}^{0.8114}_{0.0325}\text{DAMF}, {}^{0.8429}_{0}\text{AWMF}, {}^{0.9430}_{0.0569}\text{ARmF}\}$
iPEM	$[a_{ij}]$	$\{{}^{0.7485}_{0.2507}\text{BPDF}, {}^{0.8465}_{0.1532}\text{DBAIN}, {}^{0.8602}_{0.1388}\text{MDBUTMF}, {}^{0.8803}_{0.1195}\text{NAFSMF}, {}^{0.9850}_{0.0149}\text{DAMF}, {}^{0.9886}_{0.0113}\text{AWMF}, {}^{1}_{0}\text{ARmF}\}$

*In the event that noise removal performance at high noise densities is more important.

The intuitionistic fuzzy values in the decision sets provided in Table 10 are generated on MATLAB R2021a. Moreover, using the relation in Proposition 2.1, the ranking orders of the alternatives are presented in Table 11. The number of the algorithms producing the same ranking order is signified in the last column of Table 11. According to the table, iMBR01/2, iCEC11, and iPEM have the same ranking orders just as iCCE11, iCCE10, and iKWW11$(I_9, 0.5, 0.5)$ do. Moreover, these six methods produce the same ranking orders with the exception of DBAIN and MDBUTMF's ranks. Besides, iMBR01, isMBR01, and iMRB02(I_9) generate the same ranking orders. However, iFJLL10/2(I_9, w) and iFJLL10/4(I_9, w) have anomalous ranking orders unlike the other SDM methods. Although the decision-making abilities of all the SDM methods herein differ, all signify that BPDF has the lowest noise removal performance. Similarly, all the SDM methods but iFJLL10/2(I_9, w) and iFJLL10/4(I_9, w) yield that ARmF has the highest noise removal performance.

Table 11. Ranking orders of SDM methods*

Algorithms	**Ranking Orders**	**Frequency**
iMBR01	BPDF≺DBAIN≺NAFSMF≺MDBUTMF≺DAMF≺AWMF≺ARmF	3
isMBR01	BPDF≺DBAIN≺NAFSMF≺MDBUTMF≺DAMF≺AWMF≺ARmF	3
iMBR01/2	BPDF≺DBAIN≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	3
iMRB02$(\boldsymbol{I_9})$	BPDF≺DBAIN≺NAFSMF≺MDBUTMF≺DAMF≺AWMF≺ARmF	3
iCCE11	BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF	3
iCCE10	BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF	3
iCEC11	BPDF≺DBAIN≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	3
iFJLL10/2$(\boldsymbol{I_9}, \boldsymbol{w})$	BPDF≺DBAIN≈MDBUTMF≺NAFSMF≈DAMF≈AWMF≈ARmF	2
iFJLL10/4$(\boldsymbol{I_9}, \boldsymbol{w})$	BPDF≺DBAIN≈MDBUTMF≺NAFSMF≈DAMF≈AWMF≈ARmF	2
iKWW11$(\boldsymbol{I_9}, \mathbf{0.5}, \mathbf{0.5})$	BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF	3
iPEM	BPDF≺DBAIN≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF	3

*In the event that noise removal performance at high noise densities is more important.

On the other hand, suppose that the noise removal success of the filters at low noise densities are more significant than at the other densities, it is anticipated that the membership degrees at high noise densities are smaller than the non-membership degrees and the former at low noise densities are greater than the latter. In other words, we consider the first row of $[b_{ij}]_{8\times 9}$ to be

$$\begin{bmatrix} 0.9 & 0.85 & 0.75 & 0.65 & 0.5 & 0.35 & 0.25 & 0.15 & 0.05 \\ 0.05 & 0.1 & 0.2 & 0.3 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \end{bmatrix}$$

Thereby, *ifpifs*-matrix $[b_{ij}]_{8\times 9}$ is constructed as follows:

$$[b_{ij}] = \begin{bmatrix} 0.9 & 0.85 & 0.75 & 0.65 & 0.5 & 0.35 & 0.25 & 0.15 & 0.05 \\ 0.05 & 0.1 & 0.2 & 0.3 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9657 & 0.9335 & 0.8856 & 0.8269 & 0.7503 & 0.6452 & 0.5159 & 0.3648 & 0.1259 \\ 0.0062 & 0.0142 & 0.0270 & 0.0450 & 0.0759 & 0.1165 & 0.1887 & 0.2998 & 0.4895 \\ 0.9666 & 0.9424 & 0.9047 & 0.8552 & 0.7917 & 0.7104 & 0.6060 & 0.4880 & 0.3518 \\ 0.0031 & 0.0080 & 0.0168 & 0.0297 & 0.0478 & 0.0762 & 0.1223 & 0.1858 & 0.2766 \\ 0.9642 & 0.9228 & 0.7833 & 0.7539 & 0.7855 & 0.7572 & 0.6950 & 0.6000 & 0.3492 \\ 0.0050 & 0.0509 & 0.1319 & 0.1593 & 0.1167 & 0.0551 & 0.0575 & 0.1359 & 0.5228 \\ 0.9606 & 0.9216 & 0.8767 & 0.8305 & 0.7800 & 0.7211 & 0.6540 & 0.5766 & 0.4578 \\ 0.0086 & 0.0169 & 0.0267 & 0.0357 & 0.0465 & 0.0595 & 0.0790 & 0.1082 & 0.2173 \\ 0.9700 & 0.9518 & 0.9270 & 0.8953 & 0.8563 & 0.8072 & 0.7465 & 0.6667 & 0.5415 \\ 0.0018 & 0.0045 & 0.0088 & 0.0139 & 0.0204 & 0.0291 & 0.0423 & 0.0624 & 0.1148 \\ 0.9551 & 0.9440 & 0.9209 & 0.8948 & 0.8611 & 0.8148 & 0.7551 & 0.6736 & 0.5469 \\ 0.0067 & 0.0076 & 0.0095 & 0.0122 & 0.0166 & 0.0240 & 0.0370 & 0.0574 & 0.1052 \\ 0.9718 & 0.9532 & 0.9272 & 0.8971 & 0.8630 & 0.8239 & 0.7663 & 0.6819 & 0.5515 \\ 0.0013 & 0.0030 & 0.0054 & 0.0087 & 0.0137 & 0.0214 & 0.0348 & 0.0554 & 0.1038 \end{bmatrix}$$

In Table 12, $w = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$. Moreover, since the numbers of the alternatives and the parameters are not equal in this PVA problem, iKKT13 is not applied to $[b_{ij}]$.

Table 12. Decision sets of SDM methods*

Algorithms	**Matrix**	**Decision Sets**
iMBR01	$[b_{ij}]$	$\{{}^{0.0491}_{0.6250}\text{BPDF}, {}^{0.2217}_{0.6042}\text{DBAIN}, {}^{0.0238}_{0.7589}\text{MDBUTMF}, {}^{0}_{0.9271}\text{NAFSMF}, {}^{0.4449}_{0.4092}\text{DAMF}, {}^{0.2827}_{0.4702}\text{AWMF}, {}^{0.6027}_{0}\text{ARmF}\}$
isMBR01	$[b_{ij}]$	$\{{}^{0.0491}_{0.6250}\text{BPDF}, {}^{0.2217}_{0.6042}\text{DBAIN}, {}^{0.0238}_{0.7589}\text{MDBUTMF}, {}^{0}_{0.9271}\text{NAFSMF}, {}^{0.4449}_{0.4092}\text{DAMF}, {}^{0.2827}_{0.4702}\text{AWMF}, {}^{0.6027}_{0}\text{ARmF}\}$
iMBR01/2	$[b_{ij}]$	$\{{}^{0.0255}_{0.6294}\text{BPDF}, {}^{0.2078}_{0.6242}\text{DBAIN}, {}^{0}_{0.8237}\text{MDBUTMF}, {}^{0.0173}_{0.8762}\text{NAFSMF}, {}^{0.4314}_{0.4164}\text{DAMF}, {}^{0.2993}_{0.4576}\text{AWMF}, {}^{0.5994}_{0}\text{ARmF}\}$
iMRB02($\boldsymbol{I_9}$)	$[b_{ij}]$	$\{{}^{0.8760}_{0}\text{BPDF}, {}^{0.8913}_{0.0340}\text{DBAIN}, {}^{0.8755}_{0.0202}\text{MDBUTMF}, {}^{0.8868}_{0.0585}\text{NAFSMF}, {}^{0.9125}_{0.0589}\text{DAMF}, {}^{0.9104}_{0.0638}\text{AWMF}, {}^{0.9148}_{0.0607}\text{ARmF}\}$
iCCE11	$[b_{ij}]$	$\{{}^{0.3830}_{0.1084}\text{BPDF}, {}^{0.3963}_{0.0653}\text{DBAIN}, {}^{0.3826}_{0.0911}\text{MDBUTMF}, {}^{0.3924}_{0.0479}\text{NAFSMF}, {}^{0.4149}_{0.0250}\text{DAMF}, {}^{0.4130}_{0.0226}\text{AWMF}, {}^{0.4169}_{0.0214}\text{ARmF}\}$
iCCE10	$[b_{ij}]$	$\{{}^{0.3830}_{0.1084}\text{BPDF}, {}^{0.3963}_{0.0653}\text{DBAIN}, {}^{0.3826}_{0.0911}\text{MDBUTMF}, {}^{0.3924}_{0.0479}\text{NAFSMF}, {}^{0.4149}_{0.0250}\text{DAMF}, {}^{0.4130}_{0.0226}\text{AWMF}, {}^{0.4169}_{0.0214}\text{ARmF}\}$
iCEC11	$[b_{ij}]$	$\{{}^{0.3775}_{0.0311}\text{BPDF}, {}^{0.3882}_{0.0184}\text{DBAIN}, {}^{0.3735}_{0.0267}\text{MDBUTMF}, {}^{0.3832}_{0.0134}\text{NAFSMF}, {}^{0.4032}_{0.0071}\text{DAMF}, {}^{0.4010}_{0.0065}\text{AWMF}, {}^{0.4049}_{0.0063}\text{ARmF}\}$
iFJLL10/2($\boldsymbol{I_9}, \boldsymbol{w}$)	$[b_{ij}]$	$\{{}^{0.8205}_{0}\text{BPDF}, {}^{0.8308}_{0.0769}\text{DBAIN}, {}^{0.8308}_{0.0769}\text{MDBUTMF}, {}^{0.8359}_{0.1641}\text{NAFSMF}, {}^{0.8359}_{0.1641}\text{DAMF}, {}^{0.8359}_{0.1641}\text{AWMF}, {}^{0.8359}_{0.1641}\text{ARmF}\}$
iFJLL10/4($\boldsymbol{I_9}, \boldsymbol{w}$)	$[b_{ij}]$	$\{{}^{0.8205}_{0}\text{BPDF}, {}^{0.8308}_{0.0769}\text{DBAIN}, {}^{0.8308}_{0.0769}\text{MDBUTMF}, {}^{0.8359}_{0.1641}\text{NAFSMF}, {}^{0.8359}_{0.1641}\text{DAMF}, {}^{0.8359}_{0.1641}\text{AWMF}, {}^{0.8359}_{0.1641}\text{ARmF}\}$
iKWW11($\boldsymbol{I_9}, \boldsymbol{0.5}, \boldsymbol{0.5}$)	$[b_{ij}]$	$\{{}^{0.7130}_{0.1208}\text{BPDF}, {}^{0.7508}_{0.0657}\text{DBAIN}, {}^{0.7097}_{0.0985}\text{MDBUTMF}, {}^{0.7158}_{0.0588}\text{NAFSMF}, {}^{0.8317}_{0}\text{DAMF}, {}^{0.7801}_{0.0391}\text{AWMF}, {}^{0.9332}_{0.0331}\text{ARmF}\}$
iPEM	$[b_{ij}]$	$\{{}^{0.8585}_{0.1366}\text{BPDF}, {}^{0.9154}_{0.0829}\text{DBAIN}, {}^{0.8968}_{0.0988}\text{MDBUTMF}, {}^{0.9211}_{0.0779}\text{NAFSMF}, {}^{0.9918}_{0.0080}\text{DAMF}, {}^{0.9891}_{0.0107}\text{AWMF}, {}^{0.9998}_{0}\text{ARmF}\}$

*In the event that noise removal performance at low noise densities is more important.

The intuitionistic fuzzy values in the decision sets provided in Table 12 are obtained with MATLAB R2021a. Moreover, using the relation in Proposition 2.1, the ranking orders of the alternatives are presented in Table 13. The number of the algorithms producing the same ranking orders is signified in the last column of Table 13. According to these ranking orders, iMBR01, isMBR01, and iMBR01/2 produce the same ranking orders just as iCEC11 and iKWW11(I_9, 0.5,0.5) do. Furthermore, these methods have the same ranking orders with the exception of BPDF and NAFSMF's ranks. Besides, iCCE11 and iCCE10 generate the same ranking orders. However, iFJLL10/2(I_9, w) and iFJLL10/4(I_9, w) have anomalous ranking order unlike the other SDM methods. Additionally, all the SDM methods except for iMBR01, isMBR01, iMBR01/2, iMRB02(I_9), iFJLL10/2(I_9, w), and iFJLL10/4(I_9, w) confirm that BPDF exhibits the lowest performance. Further, all the SDM methods but iMRB02(I_9), iFJLL10/2(I_9, w), and iFJLL10/4(I_9, w) validate that ARmF performs better than the other filters.

Table 13. Ranking orders of SDM methods*

Algorithms	**Ranking Orders**	**Frequency**
iMBR01	NAFSMF≺MDBUTMF≺BPDF≺DBAIN≺AWMF≺DAMF≺ARmF	3
isMBR01	NAFSMF≺MDBUTMF≺BPDF≺DBAIN≺AWMF≺DAMF≺ARmF	3
iMBR01/2	NAFSMF≺MDBUTMF≺BPDF≺DBAIN≺AWMF≺DAMF≺ARmF	3
iMRB02($\boldsymbol{I_9}$)	NAFSMF≺AWMF≺DAMF≺ARmF≺MDBUTMF≺DBAIN≺BPDF	1
iCCE11	BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF	2
iCCE10	BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF	2
iCEC11	BPDF≺MDBUTMF≺NAFSMF≺DBAIN≺AWMF≺DAMF≺ARmF	2
iFJLL10/2($\boldsymbol{I_9, w}$)	NAFSMF≈DAMF≈AWMF≈ARmF≺DBAIN≈MDBUTMF≺BPDF	2
iFJLL10/4($\boldsymbol{I_9, w}$)	NAFSMF≈DAMF≈AWMF≈ARmF≺DBAIN≈MDBUTMF≺BPDF	2
iKWW11($\boldsymbol{I_9, 0.5, 0.5}$)	BPDF≺MDBUTMF≺NAFSMF≺DBAIN≺AWMF≺DAMF≺ARmF	2
iPEM	BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺AWMF≺DAMF≺ARmF	1

*In the event that noise removal performance at low noise densities is more important.

6. Conclusion

In this study, we generalised 24 SDM methods [10,11,15,16,20], constructed by the concept of *fpfs*-matrices, in the *ifpifs*-matrices space. We then suggested five new test scenarios by inspiring from the scenarios in [13] to examine the performance consistency of the SDM methods in decision-making problems. Thus, we determined the SDM methods which successfully passed all the tests. Afterwards, we applied the successful SDM methods to a PVA problem to rank the state-of-the-art noise removal filters according to their noise removal performance.

The present study encourages researchers to generalised other SDM methods to render them operable in the *ifpifs*-matrices space. Researchers can also focus on SDM methods constructed with intuitionistic fuzzy sets, soft sets, or their hybrid versions [4-8]. Moreover, classification algorithms can be developed using a generalised method (for more on classification methods, see [43-47]). This study ignored the SDM methods suggested by using the superstructures of *ifpifs*-sets/matrices. Thereby, future papers can study the generalisations of SDM methods for such spaces as interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets/matrices space [48,49] and the hybrid versions of picture fuzzy sets [50,51] and soft sets.

Author Contributions

S. Enginoğlu directed the project and supervised the process whereby the findings were obtained. T. Aydın and B. Arslan generalised the SDM methods. S. Memiş and B. Arslan produced the application results of the SDM methods by writing their MATLAB codes. T. Aydın and B. Arslan wrote the manuscript with the support of S. Enginoğlu and S. Memiş. S. Enginoğlu reviewed and edited the manuscript. All the authors discussed the results and contributed to the final manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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A Robust Alternative to Environmental Performance Index

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Abstract − A composite index called the Environmental Performance Index (EPI) was obtained to evaluate countries' environmental performance. This index was calculated for 180 countries concerning 24 environmental indicators. However, it is well known that there are huge differences between countries regarding environmental factors besides social, economic, and cultural factors. This case aggravates the doubt that the data set has outliers. Therefore, the index values should be obtained such that they are unsensitive to outliers. This study aims to generate a composite index, which is a robust alternative to EPI. For this aim, we use the Robust Principal Component Analysis (ROBPCA) and the Technique for Order Preferences by Similarity to an Ideal Solution (TOPSIS), which is a multi-criteria decision-making method.

Keywords − *Environmental performance index (EPI), robust EPI, ROBPCA, TOPSIS, composite index*

Mathematics Subject Classification (2020) − 62P12, 62H25

1. Introduction

Rapid industrialization and population growth cause harmful effects on the environment. In recent years, even ordinary people have realized this fact because of global warming and climate change. Therefore, we need evaluation and comparing countries concerning environmental factors. For this aim, a composite index called EPI was defined to measure the environmental performance of countries. This index was obtained with the collaboration of the Yale Center for Environmental Law and Policy (YCELP), Yale University, Columbia University Center for International Earth Science Information Network (CIESIN), and the World Economic Forum (WEF). The result of this index was released in Davos, Switzerland, at the annual meeting of the World Economic Forum in 2018. According to this report, 180 countries were sorted according to their EPI values, calculated from 24 environmental indicators [1].

In the literature, there are many studies related to environmental factors. In some of these studies, researchers investigated the relationship between the environmental performance of counties and different factors, such as socioeconomic, cultural, financial, ideological, economic growth [2-7]. In other studies, authors focused on obtaining a new composite index, which measures the environmental performance of countries, by using data envelopment analysis and Malmquist approaches [8-13].

Also, the principal components analysis (PCA) is one of the valuable methods to obtain a composite index [14]. Generally, researchers are interested in topics on human development, quality of life, and economic development in the studies that purpose a composite index using PCA [15-19]. Moreover, Bulut and Öner used robust PCA to obtain a composite index that is not sensitive to outliers. Thus, they evaluated the regions

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robustly in Turkey about their socioeconomic development [20]. Also, Alpaykut investigated the well-being of cities in Turkey by using classical PCA, which is sensitive to outliers, and TOPSIS methods [21].

It is well known that the economic and cultural features of countries may affect environmental factors. For example, companies may call in their top model cars because of unsuitable emissions in a developed country, while old vehicles may be on the roads by polluting the environment in an undeveloped country. Because of similar reasons, when countries' environmental performance is evaluated, it should not forget that the data sets consist of countries having different development levels. This case may cause the data set to include outliers. Hence, a robust approach is needed to obtain a composite index from like data.

This study purposes construction of a composite index, which is not sensitive to outliers, to evaluate countries' environmental performance. For this purpose, we use the ROBPCA method, which is a robust principal component analysis algorithm, and the TOPSIS algorithm, which is a multi-criteria decision method. In this way, we have robustly constructed a composite index measuring the environmental performances of countries and sort countries according to these values.

The remainder of the paper is organized as follows. The principal component analysis and TOPSIS methods are introduced in Section 2. In Section 3, a robust alternative to the EPI is constructed called the robust EPI (REPI). The REPI values of countries are obtained, and the countries are ordered according to these values. Finally, we conclude from the obtained results in the last section.

2. Materials and methods

In this section, we introduce the principal component analysis and TOPSIS methods used to construct a composite index called robust environmental performance index.

2.1. Principal Component Analysis

The principal component analysis is one of the most popular multivariate statistical methods. The PCA aims to obtain the new variables, which are the linear combinations of variables that are correlated with each other, and components number is less than the number of the original variables (p). These new variables are called principal components. However, it is well known that classical PCA is sensitive to outliers [20]. A robust principal component analysis method called ROBPCA was developed [22].

The ROBPCA algorithm consists of three stages which are given below.

- **Stage 1:** The data is reduced to space that has maximum $(n-1)$ dimension using the projection pursuit approach.
- **Stage 2:** The initial covariance matrix Σ_0 is obtained, and q, which is the number of important components, is determined.
- **Stage 3:** The data points are projected on this subspace where their location and scatter matrix are robustly estimated, from which its k nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$ are computed. The corresponding eigenvectors are the q robust principal components [20,22]

Principal component scores are obtained from (1):

$$T_{n,q} = \left(X_{n,p} - 1_n\, \widehat{\mu}^T\right) P_{p,q} \tag{1}$$

where $X\colon n \times p$ is data matrix, n is observation number, p is the variable number, $P\colon p \times q$ is eigenvectors matrix, $\widehat{\mu}$ which is called a robust location estimation is a column vector with p-dimension 1_n is the column vector with all n components equal to 1, and $(.)^T$ is the transpose operator. The robust scatter matrix is also calculated using spectral demonstration, as below

$$\Sigma_{p,p} = P_{p,q} L_{q,q} P'_{q,p} \tag{2}$$

where $L_{q,q}$ is eigenvalues matrix [22].

An essential advantage of the ROBPCA algorithm is that it detects outliers by calculating orthogonal and score distances and using critical values for these distances. The critical value of score distance is $\sqrt{\chi^2_{q,0.975}}$ and the critical value of the orthogonal distance is $(\hat{\mu}+\hat{\sigma}Z_{0.975})^2$, where g_1 and g_2 are unknown parameters, $\hat{\mu}=(g_1g_2)^{\frac{1}{3}}\left(1-\frac{2}{9g_2}\right)$ and $\hat{\sigma}^2=\frac{2g_1^{\frac{2}{3}}}{9g_2^{\frac{1}{3}}}$. Score and orthogonal distances are calculated as below, respectively:

$$SD_i=\sqrt{\sum_{j=1}^{q}\frac{t_{ij}^2}{\lambda_j}}\,,(i=1,2,\dots,n) \tag{3}$$

$$OD_i=\left\|x_i-\hat{\mu}-P_{p,q}t_i'\right\|\,,(i=1,2,\dots,n) \tag{4}$$

where t_{ij} is a member in i_{th} row and j_{th} column of $T_{n,q}$ matrix, which is defined in (1). t_i is also i_{th} row vector of $T_{n,q}$ matrix [22].

In this study, “rrcov” package in the R programming language has been used for calculations regarding the ROBPCA algorithm [23].

2.2. The technique for Order Preferences by Similarity to an Ideal Solution (TOPSIS)

Hwang and Yoon [24] suggested the TOPSIS method. In the TOPSIS method, the aim is to select the best solution between different alternatives. The main idea of the TOPSIS method is based on the selection of a solution, which is the nearest to the positive ideal solution and is the farthest to the negative ideal solution. Thus, the TOPSIS method obtains the best sorting [21].

In the TOPSIS method, one needs a decision matrix and a weights vector. Criteria are in rows of the decision matrix, and alternative values are in columns of the decision matrix. Weight vector consists of weights of alternative solutions.

In this study, “topsis” package in the R programming language has been used for calculations regarding the TOPSIS algorithm [25].

3. Construction of Robust Environmental Performance Index

This study uses the data set consisted of the values of 180 countries’ 24 environmental indicators. These indicators are given in Table 1. We have downloaded the data set from web site EPI 2018 [26].

Table 1. The environmental indicators used in this study

Indicator	Code	Indicator	Code
Household Solid Fuels	HAD	Marine Protected Areas	MPA
$PM_{2.5}$ Exposure	PME	Biome Protection (National)	TBN
$PM_{2.5}$ Exceedance	PMW	Biome Protection (Global)	TBG
Drinking-Water	UWD	Species Protection Index	SPI
Sanitation	USD	Representativeness Index	PAR
Lead Exposure	PBD	Species Habitat Index	SHI
Tree Cover Loss	TCL	Methane Emissions	DMT
Fish Stock Status	FSS	N_2O Emissions	DNT
Regional Marine Trophic Index	MTR	Black Carbon Emissions	DBT
CO_2 Emissions – Total	DCT	SO_2 Emissions	DST
CO_2 Emissions – Power	DPT	NO_X Emissions	DXT
Sustainable Nitrogen Management	SNM	Wastewater Treatment	WWT

Firstly, we investigate whether the data set has outliers by using both classical Mahalanobis distances and the ROBPCA algorithm. While we cannot determine outliers in the classical approach, we determine outliers in the data set via the ROBPCA algorithm. Because classical Mahalanobis distances are based on classical mean vector and sample covariance matrix, which are sensitive to outliers, they may fail to determine outliers. In the robust literature, this case is called masking. The results of outlier detection are given in Table A in the Appendix.

Moreover, we investigate the relationship between 24 environmental indicators and give the graph of the obtained correlation matrix in Figure A in the Appendix. The X icon of this graph means that the relationship is statistically unimportant. According to the graphic, we decide to use PCA for dimension reduction because there are many statistically important correlations.

Table 2. The proportion of explained variance

Method	Values	PC_1	PC_2	PC_3	PC_4	PC_5	PC_6	PC_7	PC_8
CPCA	Standard deviation	73.27	55.87	52.17	35.90	33.31	29.13	25.86	24.87
	Proportion of Variance	0.28	0.16	0.14	0.07	0.06	0.04	0.03	0.03
	Cumulative Proportion	0.275	0.436	0.575	0.641	0.698	0.742	0.776	**0.808**
ROBPCA	Standard deviation	79.59	57.43	56.32	35.42	31.37	26.23	24.48	22.92
	Proportion of Variance	0.36	0.19	0.18	0.07	0.06	0.04	0.03	0.03
	Cumulative Proportion	0.358	0.545	0.724	0.795	**0.851**	0.890	0.924	0.953

Also, we decide to use robust principal component analysis because the data set has outliers. Table 2 gives the explained variance's proportions obtained from classical PCA (CPCA) and robust PCA (ROBPCA). According to Table 2, 8 components explain 80.8% of the variance in CPCA, while only 5 components explain 85.1% of the variance in ROBPCA. Therefore, we use the scores obtained from the ROBPCA, which the number of important components is 5.

To obtain only a composite index by basing five principal components, we use the TOPSIS method. In the TOPSIS method, we take countries as criteria and the important components as alternative values. We also take the marginal proportions of explained variance as weights for each alternative. Therefore, the first principal component, which explains the biggest proportion of variance, has the biggest weight on the composite index. In this way, the obtained composite index is called the Robust Environmental Performance Index (REPI) because it is not sensitive to outliers. The EPI and the REPI values and the ranks of countries according to these values are given in Table 3. We show these values of indexes on the world map in Figure 1.

According to Table 3, there are dramatic differences in the results of the REPI and the EPI. The performance rankings of some countries (Armenia, Azerbaijan, Bolivia, Hungary, Czech Republic, Turkmenistan, Makedonia, etc.) decrease, while the performance rankings of other countries (Bahrain, Bangladesh, Chile, China, Malaysia, Maldives, etc.) increase. We detect an essential difference for top countries. Accordingly, the rank of Malta is 1 instead of 4, the rank of Israel is 2 instead of 19, the rank of Sweden is 3 instead of 5, the rank of Finland is 4 instead of 10, the rank of Holland is 5 instead of 18, the rank of South Korea is 6 instead of 60, the rank of Singapore is 7 instead of 49, and the rank of Japan is 8 instea of 20. On the contrary, the rank of Switzerland decreases from 1 to 52, the rank of France decreases from 2 to 10, the rank of Denmark decreases from 3 to 17, the rank of Luxembourg decreases from 7 to 69 and the rank of United Kingdom decreases from 6 to 12.

Table 3. The Index Values and Ranking of Countries According to EPI2018 and REPI

Country	EPI 2018		REPI		Country	EPI 2018		REPI	
	Value	Rank	Value	Ran		Value	Ran	Value	Ran
Afghanistan	37.74	168	31.46	161	Djibouti	40.04	163	40.05	136
Albania	65.46	40	62.83	34	Dominica	59.38	73	49.01	98
Algeria	57.18	88	51.49	88	Dominican Republic	64.71	46	60.23	46
Angola	37.44	170	37.27	146	Ecuador	57.42	87	53.47	75
Antigua and Barbuda	59.18	76	55.08	67	Egypt	61.21	66	55.77	61
Argentina	59.3	74	60.03	48	El Salvador	53.91	106	43.58	118
Armenia	62.07	63	43.07	123	Equatorial Guinea	60.4	71	54.62	70
Australia	74.12	21	67.89	19	Eritrea	39.34	165	35.34	153
Austria	78.97	8	54.89	68	Estonia	64.31	48	61.41	41
Azerbaijan	62.33	59	44.04	115	Ethiopia	44.78	141	21.88	175
Bahamas	54.99	98	52.36	81	Fiji	53.09	107	49.77	93
Bahrain	55.15	96	63.54	29	Finland	78.64	10	71.26	4
Bangladesh	29.56	179	43.44	121	France	83.95	2	70.12	10
Barbados	55.76	93	53.09	78	Gabon	45.05	140	42.36	126
Belarus	64.98	44	48.58	101	The Gambia	42.42	156	37.14	147
Belgium	77.38	15	68.68	15	Georgia	55.69	94	53.22	77
Belize	57.79	81	50.71	90	Germany	78.37	13	68.89	14
Benin	38.17	167	30.50	162	Ghana	49.66	124	46.60	111
Bhutan	47.22	131	30.42	164	Greece	73.6	22	64.21	26
Bolivia	55.98	92	35.66	152	Grenada	50.93	118	48.05	102
Bosnia and Herzegovina	41.84	158	34.23	155	Guatemala	52.33	110	51.51	86
Botswana	51.7	113	32.82	156	Guinea	46.62	134	38.58	140
Brazil	60.7	69	58.08	55	Guinea-Bissau	44.67	143	36.48	149
Brunei Darussalam	63.57	53	61.78	39	Guyana	47.93	128	38.52	141
Bulgaria	67.85	30	59.87	49	Haiti	33.74	174	35.17	154
Burkina Faso	42.83	154	20.97	176	Honduras	51.51	114	48.60	100
Burundi	27.43	180	19.91	178	Hungary	65.01	43	46.84	110
Cabo Verde	56.94	89	47.64	107	Iceland	78.57	11	66.54	22
Côte d'Ivoire	45.25	139	42.52	125	India	30.57	177	43.65	117
Cambodia	43.23	150	43.45	120	Indonesia	46.92	133	49.08	97
Cameroon	40.81	161	31.50	160	Iran	58.16	80	52.62	80
Canada	72.18	25	62.99	32	Iraq	43.2	152	36.11	150
The central African	36.42	171	17.29	180	Ireland	78.77	9	69.63	11
Chad	45.34	137	23.55	171	Israel	75.01	19	73.86	2
Chile	57.49	84	61.92	37	Italy	76.96	16	67.06	21
China	50.74	120	61.79	38	Jamaica	58.58	78	48.98	99
Colombia	65.22	42	63.28	30	Japan	74.69	20	70.14	8
Comoros	44.24	146	38.16	143	Jordan	62.2	62	49.54	95
Costa Rica	67.85	31	57.47	57	Kazakhstan	54.56	101	40.65	132
Croatia	65.45	41	59.02	53	Kenya	47.25	130	40.37	134
Cuba	63.42	55	61.24	42	Kiribati	55.26	95	49.37	96
Cyprus	72.6	24	66.45	24	Kuwait	62.28	61	64.25	25
Czech Republic	67.68	33	47.79	104	Kyrgyzstan	54.86	99	32.61	157
Dem. Rep. Congo	30.41	178	20.35	177	Laos	42.94	153	30.20	166
Denmark	81.6	3	68.32	17	Latvia	66.12	37	59.67	50
Lebanon	61.08	67	60.47	44	São Tomé and Príncipe	54.01	104	44.69	114
Lesotho	33.78	173	29.86	167	Saint Lucia	56.18	91	51.80	85
Liberia	41.62	160	40.06	135	Saint Vincent and the	66.48	36	55.33	63
Libya	49.79	123	46.55	112	Samoa	54.5	102	49.74	94
Lithuania	69.33	29	63.64	28	Saudi Arabia	57.47	86	61.44	40
Luxembourg	79.12	7	54.85	69	Senegal	49.52	126	43.21	122
Macedonia	61.06	68	41.56	129	Serbia	57.49	85	41.36	130
Madagascar	33.73	175	39.23	137	Seychelles	66.02	39	52.21	83
Malawi	49.21	127	22.76	172	Sierra Leone	42.54	155	37.50	145
Malaysia	59.22	75	64.12	27	Singapore	64.23	49	70.22	7
Maldives	52.14	111	55.57	62	Slovakia	70.6	28	51.88	84
Mali	43.71	147	22.22	173	Slovenia	67.57	34	47.03	109

Table 3. (Continued) The Index Values and Ranking of Countries According to EPI2018 and REPI

Country	EPI 2018		REPI		Country	EPI 2018		REPI	
	Values	Rank	Values	Rank		Values	Rank	Values	Rank
Malta	80.9	4	74.24	1	Solomon Islands	43.22	151	41.61	128
Mauritania	39.24	166	40.77	131	South Africa	44.73	142	50.35	91
Mauritius	56.63	90	52.33	82	South Korea	62.3	60	71.05	6
Mexico	59.69	72	56.71	59	Spain	78.39	12	66.52	23
Micronesia	49.8	122	47.49	108	Sri Lanka	60.61	70	52.64	79
Moldova	51.97	112	42.64	124	Sudan	51.49	115	47.77	105
Mongolia	57.51	83	38.09	144	Suriname	54.2	103	50.92	89
Montenegro	61.33	65	54.14	73	Swaziland	40.32	162	30.22	165
Morocco	63.47	54	55.78	60	Sweden	80.51	5	72.45	3
Mozambique	46.37	135	39.16	138	Switzerland	87.42	1	59.11	52
Myanmar	45.32	138	47.66	106	Taiwan	72.84	23	67.81	20
Namibia	58.46	79	49.94	92	Tajikistan	47.85	129	31.70	159
Nepal	31.44	176	22.16	174	Tanzania	50.83	119	43.46	119
Netherlands	75.46	18	71.23	5	Thailand	49.88	121	55.21	65
New Zealand	75.96	17	67.92	18	Timor-Leste	49.54	125	43.84	116
Nicaragua	55.04	97	51.49	87	Togo	41.78	159	31.78	158
Niger	35.74	172	19.09	179	Tonga	62.49	57	54.59	71
Nigeria	54.76	100	45.45	113	Trinidad and Tobago	67.36	35	59.53	51
Norway	77.49	14	68.60	16	Tunisia	62.35	58	61.23	43
Oman	51.32	116	54.41	72	Turkey	52.96	108	53.94	74
Pakistan	37.5	169	38.94	139	Turkmenistan	66.1	38	48.00	103
Panama	62.71	56	58.57	54	Uganda	44.28	145	24.77	170
Papua New Guinea	39.35	164	36.94	148	Ukraine	52.87	109	56.99	58
Paraguay	53.93	105	36.04	151	United Arab Emirates	58.9	77	63.07	31
Peru	61.92	64	60.35	45	United Kingdom	79.89	6	69.37	12
Philippines	57.65	82	55.32	64	United States of America	71.19	27	68.92	13
Poland	64.11	50	60.10	47	Uruguay	64.65	47	62.15	36
Portugal	71.91	26	62.86	33	Uzbekistan	45.88	136	38.42	142
Qatar	67.8	32	70.13	9	Vanuatu	44.55	144	41.69	127
Republic of Congo	42.39	157	40.45	133	Venezuela	63.89	51	55.21	66
Romania	64.78	45	57.82	56	Viet Nam	46.96	132	53.24	76
Russia	63.79	52	62.32	35	Zambia	50.97	117	30.48	163
Rwanda	43.68	148	25.42	168	Zimbabwe	43.41	149	25.04	169

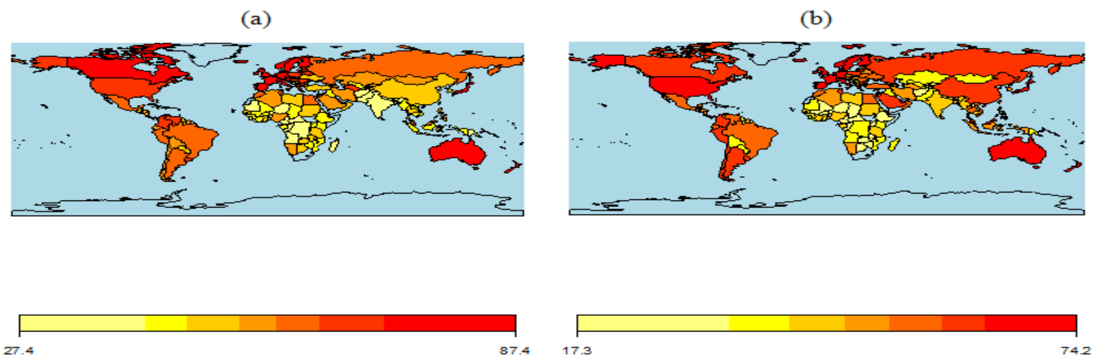

Fig. 1. The world maps according to environmental performance indexes (a) EPI2018 (b) REPI

These results based on the REPI are more confidential than those based on the EPI because the REPI is not sensitive to outliers in the data set. Moreover, it is seen that African countries have poorer environmental performances than American and European countries, according to both the EPI and the REPI, when maps given in Figure 1 are investigated.

4. Conclusion

This study aims to construct a robust composite index, an alternative to the EPI. For this aim, firstly, we have investigated whether the data set has outliers or not and decided the data set has outliers. Therefore, we used the ROBPCA algorithm, a robust principal component analysis, for dimension reduction and obtained five important principal components scores for each country. We have used the TOPSIS method to construct a composite index from five principal components scores. Finally, we have obtained the REPI values, which are not sensitive to outliers in the data set, for each country and have ranked countries according to these index values. When they are compared with the EPI results, the REPI results have dramatic differences. The reason for these differences is the impact of outliers in data sets. Therefore, we suggest using methods that are not sensitive to outliers when constructing a composite index.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

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Appendix

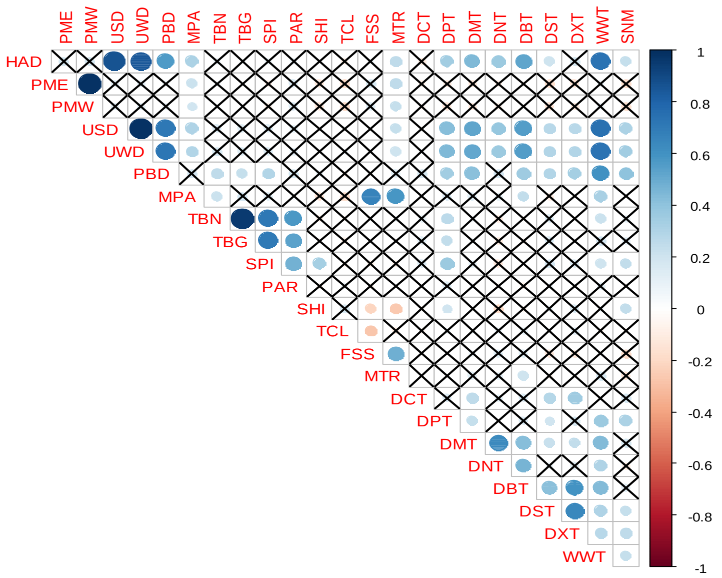

Fig. A. Correlation matrix for environmental indicators

Table A. Outlier Detection

Country	Mah_C	SD	OD	Decision	Country	Mah_C	SD	OD	Decision
Afghanistan	37.14	4.84	82.58	FALSE	Djibouti	14.00	2.54	45.46	TRUE
Albania	31.39	2.56	76.74	TRUE	Dominica	23.77	2.55	111.72	FALSE
Algeria	28.21	3.10	67.56	TRUE	Dominican Republic	15.97	2.19	46.18	TRUE
Angola	16.41	3.00	46.30	TRUE	Ecuador	21.97	2.58	50.36	TRUE
Antigua and Barbuda	34.28	3.03	116.98	FALSE	Egypt	42.66	3.35	66.42	TRUE
Argentina	29.78	2.30	64.39	TRUE	El Salvador	21.67	2.88	61.76	TRUE
Armenia	29.15	2.63	71.40	TRUE	Equatorial Guinea	21.17	3.89	29.69	TRUE
Australia	18.79	3.26	37.74	TRUE	Eritrea	32.78	4.70	52.54	FALSE
Austria	21.72	2.96	47.66	TRUE	Estonia	13.61	2.42	40.73	TRUE
Azerbaijan	38.94	3.88	63.17	TRUE	Ethiopia	22.20	4.49	37.21	TRUE
Bahamas	23.19	3.84	45.33	TRUE	Fiji	19.64	2.14	58.48	TRUE
Bahrain	31.74	4.37	64.33	TRUE	Finland	17.35	2.46	50.94	TRUE
Bangladesh	67.70	5.74	79.49	FALSE	France	17.38	2.71	39.51	TRUE
Barbados	30.74	3.52	93.33	FALSE	Gabon	31.01	3.27	52.13	TRUE
Belarus	13.16	2.18	32.26	TRUE	Gambia	15.03	2.38	40.92	TRUE
Belgium	19.76	1.86	48.80	TRUE	Georgia	34.73	4.27	60.79	TRUE
Belize	16.34	2.55	57.53	TRUE	Germany	16.39	2.48	45.88	TRUE
Benin	16.56	2.63	51.72	TRUE	Ghana	9.53	1.82	37.38	TRUE
Bhutan	22.49	4.07	44.86	TRUE	Greece	23.74	3.08	44.66	TRUE
Bolivia	16.00	2.46	35.63	TRUE	Grenada	29.50	2.51	113.88	FALSE
Bosnia and Herzegovina	31.18	3.24	84.71	FALSE	Guatemala	28.83	2.96	58.94	TRUE
Botswana	20.72	3.43	38.51	TRUE	Guinea	18.98	3.12	39.70	TRUE
Brazil	31.14	2.82	49.21	TRUE	Guinea-Bissau	15.60	2.91	34.90	TRUE
Brunei Darussalam	49.77	3.27	109.83	FALSE	Guyana	26.52	2.39	88.64	FALSE
Bulgaria	22.03	2.10	49.47	TRUE	Haiti	27.84	3.24	52.89	TRUE
Burkina Faso	13.05	2.86	28.12	TRUE	Honduras	17.65	2.50	41.55	TRUE
Burundi	12.82	2.86	31.44	TRUE	Hungary	19.46	2.82	37.60	TRUE
Cabo Verde	18.89	3.70	35.28	TRUE	Iceland	33.92	2.75	81.39	FALSE
Côte d’Ivoire	17.71	2.20	56.14	TRUE	India	28.61	5.67	30.80	FALSE
Cambodia	21.54	2.89	41.71	TRUE	Indonesia	8.35	2.38	22.62	TRUE
Cameroon	25.38	4.04	39.14	TRUE	Iran	36.01	4.35	62.79	TRUE
Canada	14.87	3.27	32.86	TRUE	Iraq	31.83	3.09	46.05	TRUE
Central African Republic	16.02	3.16	26.76	TRUE	Ireland	21.01	2.83	45.92	TRUE
Chad	28.70	3.34	57.87	TRUE	Israel	30.45	2.75	55.63	TRUE
Chile	32.27	5.22	43.99	FALSE	Italy	14.75	2.31	36.99	TRUE
China	12.21	2.84	28.80	TRUE	Jamaica	21.99	1.85	56.78	TRUE
Colombia	10.73	1.53	36.14	TRUE	Japan	23.19	3.54	36.47	TRUE
Comoros	25.72	2.78	49.65	TRUE	Jordan	31.61	3.23	65.28	TRUE
Costa Rica	14.34	2.38	43.09	TRUE	Kazakhstan	33.83	2.96	65.89	TRUE
Croatia	20.06	2.64	45.38	TRUE	Kenya	17.69	2.26	40.71	TRUE
Cuba	15.07	2.41	51.40	TRUE	Kiribati	29.31	3.11	103.39	FALSE
Cyprus	27.11	2.86	57.96	TRUE	Kuwait	34.25	2.94	89.96	FALSE
Czech Republic	19.39	2.85	37.06	TRUE	Kyrgyzstan	30.46	3.88	61.15	TRUE
Dem. Rep. Congo	28.54	5.11	51.21	FALSE	Laos	28.66	5.02	29.13	FALSE
Denmark	16.64	2.21	46.20	TRUE	Latvia	14.64	2.67	40.76	TRUE
Critical Values	39.36	4.53	77.28			39.36	4.53	77.28	

Table A. Outlier Detection (Continue)

Country	Mah_C	SD	OD	Decision	Country	Mah_C	SD	OD	Decision
Lebanon	26.37	3.72	46.77	TRUE	São Tomé and Príncipe	28.93	2.12	117.10	FALSE
Lesotho	17.64	2.98	38.86	TRUE	Saint Lucia	43.98	3.52	135.91	FALSE
Liberia	22.94	2.67	55.91	TRUE	Saint Vincent and the Grenadines	18.44	2.71	40.96	TRUE
Libya	31.52	4.99	41.92	FALSE	Samoa	25.84	2.92	94.54	FALSE
Lithuania	15.54	1.43	49.76	TRUE	Saudi Arabia	15.70	2.83	49.83	TRUE
Luxembourg	26.27	3.52	46.70	TRUE	Senegal	8.55	2.34	26.51	TRUE
Macedonia	18.12	2.10	46.99	TRUE	Serbia	15.64	2.11	46.85	TRUE
Madagascar	33.00	3.16	60.31	TRUE	Seychelles	41.32	3.70	101.34	FALSE
Malawi	13.56	2.65	36.08	TRUE	Sierra Leone	22.62	2.60	39.06	TRUE
Malaysia	28.60	3.35	59.82	TRUE	Singapore	48.81	5.70	79.33	FALSE
Maldives	21.10	3.59	35.72	TRUE	Slovakia	22.53	2.91	43.10	TRUE
Mali	20.18	3.05	34.34	TRUE	Slovenia	15.98	2.45	32.93	TRUE
Malta	32.86	3.80	92.14	FALSE	Solomon Islands	15.89	2.78	38.75	TRUE
Mauritania	23.90	3.17	51.35	TRUE	South Africa	18.17	3.41	43.32	TRUE
Mauritius	20.15	1.88	47.33	TRUE	South Korea	32.02	3.99	61.69	TRUE
Mexico	11.16	1.63	48.37	TRUE	Spain	15.45	2.17	41.29	TRUE
Micronesia	43.49	4.89	105.69	FALSE	Sri Lanka	17.57	2.94	48.24	TRUE
Moldova	9.65	2.61	23.11	TRUE	Sudan	36.42	6.47	41.20	FALSE
Mongolia	26.61	2.46	48.52	TRUE	Suriname	29.89	2.90	88.79	FALSE
Montenegro	31.79	2.01	61.92	TRUE	Swaziland	19.40	2.68	53.63	TRUE
Morocco	17.21	2.48	42.77	TRUE	Sweden	17.03	2.58	48.08	TRUE
Mozambique	16.42	2.87	36.38	TRUE	Switzerland	27.22	4.03	34.05	TRUE
Myanmar	29.86	4.57	31.48	FALSE	Taiwan	21.27	3.21	40.95	TRUE
Namibia	27.95	3.47	57.82	TRUE	Tajikistan	29.09	5.36	37.50	FALSE
Nepal	47.19	5.75	46.73	FALSE	Tanzania	17.30	2.71	33.99	TRUE
Netherlands	17.13	2.65	47.59	TRUE	Thailand	37.07	3.49	40.67	TRUE
New Zealand	11.06	2.10	38.00	TRUE	Timor-Leste	13.35	2.68	37.02	TRUE
Nicaragua	24.34	3.84	40.99	TRUE	Togo	30.06	4.08	72.13	TRUE
Niger	18.44	3.23	47.25	TRUE	Tonga	35.89	3.99	105.62	FALSE
Nigeria	17.93	3.31	28.50	TRUE	Trinidad and Tobago	28.69	2.80	59.20	TRUE
Norway	18.24	3.18	41.76	TRUE	Tunisia	20.85	2.62	46.97	TRUE
Oman	29.00	3.98	58.47	TRUE	Turkey	16.85	3.14	45.37	TRUE
Pakistan	31.09	6.07	37.91	FALSE	Turkmenistan	33.01	3.22	60.17	TRUE
Panama	13.20	2.78	27.65	TRUE	Uganda	10.67	2.67	17.03	TRUE
Papua New Guinea	20.58	2.76	55.59	TRUE	Ukraine	20.87	2.81	47.67	TRUE
Paraguay	31.18	4.66	43.90	FALSE	United Arab Emirates	41.79	2.24	100.27	FALSE
Peru	15.63	2.66	30.39	TRUE	United Kingdom	18.76	3.07	45.88	TRUE
Philippines	19.86	1.95	44.49	TRUE	United States of America	31.40	2.39	40.21	TRUE
Poland	22.87	2.54	57.67	TRUE	Uruguay	38.47	3.59	87.97	FALSE
Portugal	30.64	2.13	60.00	TRUE	Uzbekistan	29.48	4.94	48.61	FALSE
Qatar	20.21	3.10	51.45	TRUE	Vanuatu	18.73	2.99	57.38	TRUE
Republic of Congo	26.25	3.89	45.24	TRUE	Venezuela	23.42	3.75	49.59	TRUE
Romania	12.15	1.82	43.08	TRUE	Viet Nam	18.87	3.33	40.50	TRUE
Russia	13.55	2.17	30.24	TRUE	Zambia	17.62	2.76	50.04	TRUE
Rwanda	19.20	2.59	39.96	TRUE	Zimbabwe	20.56	3.46	38.00	TRUE
	39.36	4.53	77.28			39.36	4.53	77.28	