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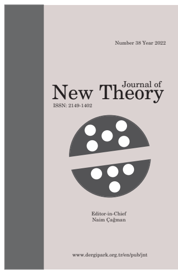

Another View on Picture Fuzzy Soft Sets and Their Product Operations with Soft Decision-Making

Samet Memiş[1]

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Abstract — Cuong [Picture Fuzzy Sets, Journal of Computer Science and Cybernetics 30 (4) (2014) 409–420] has introduced the concept of picture fuzzy soft sets (*pfs*-sets) relying on his definition and operations of picture fuzzy sets (*pf*-sets), in which there exist some inconsistencies. Yang et al. [Adjustable Soft Discernibility Matrix Based on Picture Fuzzy Soft Sets and Its Applications in Decision Making, Journal of Intelligent & Fuzzy Systems 29 (4) (2015) 1711–1722] have claimed that they have introduced the concept of *pfs*-sets with the inconsistencies in Cuong's definition of *pf*-sets. Therefore, this study redefines the concept of *pfs*-sets to deal with the inconsistencies therein. Moreover, it investigates some of the properties of *pfs*-sets and their product operations and proposes a soft decision-making method via *pfs*-sets. Finally, *pfs*-sets, their product operations, and the proposed method are discussed for further research.

Keywords — *Fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, soft sets, picture fuzzy soft sets*

Mathematics Subject Classification (2020) — 03E72, 03E99

1. Introduction

Various uncertainties may occur in real-world problems. Classical mathematical tools are inadequate in modelling such uncertainties. To overcome this problem, introducing of new mathematical tools are needed. One of the well-known mathematical tool to model uncertainty is fuzzy sets [1]. In a short time, it has been applied to pure mathematics such as algebra, topology, and mathematical analysis and computer science such as machine learning, image processing, and artificial intelligence [2]. Shortly after the introducing of fuzzy sets, intuitionistic fuzzy sets [3] have been proposed as an extension of fuzzy sets to model further uncertainty than fuzzy uncertainty. An element of a considered fuzzy set has a membership degree denoted by $\mu(x)$ while those of a considered intuitionistic fuzzy set has the membership and non-membership degrees denoted by $\mu(x)$ and $\nu(x)$ such that $\mu(x) + \nu(x) \leq 1$, respectively. A intuitionistic fuzzy set represents as a fuzzy set if $\mu(x)+\nu(x) = 1$, whose the membership and non-membership degrees are equal to $\mu(x)$ and $1 - \mu(x)$, respectively. Moreover, the indeterminacy degrees of fuzzy sets and intuitionistic fuzzy sets are equal to 0 and $1 - (\mu(x) + \nu(x))$, respectively.

One of the other state-of-the-art mathematical tools is soft sets defined by Molodstov [4] in 1999 to parameterise the alternative set for the considered problems without employing the specific membership functions. Due to its ease of implementation, it has been applied to a great variety of fields such as algebra [5–7], topology [8–10], decision-making [11–15], and machine learning [16–18]. After that, the hybrid structures of fuzzy sets and soft sets are studied, and fuzzy soft sets [19,20], fuzzy parameterized

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soft sets [21], and fuzzy parameterized fuzzy soft sets [22] are introduced to model problems containing fuzzy parameters or alternatives.

In the real world, many more problems and uncertainties are encountered that fuzzy sets and intuitionistic fuzzy sets can not model. For example, let us consider a voting process for an election. The electorate's decisions in the process may separate into three types: yes, no, and abstain. To deal with this problem, Cuong [23] has introduced the concept of picture fuzzy sets (*pf*-sets). The membership, neutral membership, and non-membership degrees are denoted by $\mu(x)$, $\eta(x)$, and $\nu(x)$, respectively, for a *pf*-set such that $\mu(x)+\eta(x)+\nu(x) \leq 1$. In the Cuong's definition, the indeterminacy degree is denoted by $1-(\mu(x)+\eta(x)+\nu(x))$ for a *pf*-set. In the same study [23], Cuong has put forward the concept of picture fuzzy soft sets (*pfs*-sets) to model problems containing picture fuzzy alternatives and investigate some of their properties. However, the investigation is so limited, and Cuong's definitions and operations of *pf*-sets and *pfs*-sets have theoretical inconsistencies.

Recently, *pfs*-sets have been redefined [24] relying on definition of Cuong's *pf*-sets without mentioning the definition of Cuong's *pfs*-sets. Therefore, the concepts of *pfs*-sets in [24] inherit from the inconsistencies [23]. To overcome the problem therein, Memiş [25], has been redefined the concept of *pf*-sets, in which $\mu(x)+\nu(x) \leq 1$ and $\mu(x)+\eta(x)+\nu(x) \leq 2$, improved their operations, and investigated their properties extensively. In this study, the main goal is that *pfs*-sets are redefined relying on the definition of *pf*-sets in [25] to deal with the inconsistencies of definition and operations in *pfs*-sets [24] and to ensure their consistency.

In Section 2 of the present study, we present concepts of fuzzy sets, intuitionistic fuzzy sets, *pf*-sets, and basic operations of *pf*-sets. In Section 3, we present the counter-examples provided in [25] related to Cuong's definitions and operations and motivation of the redefining of *pfs*-sets. In Section 4, we redefine the concept of *pfs*-sets, investigate and revise some of its basic operations, and define the product operations of *pfs*-sets. In Section 5, we propose a soft decision-making method rely on the concept of *pfs*-sets and compare its ranking orders with those in [24]. Finally, we discuss *pfs*-sets, their product operations, and the proposed soft decision-making method and provide conclusive remarks for further research.

2. Preliminaries

This section provides the concepts of fuzzy sets [1], intuitionistic fuzzy sets [3], and picture fuzzy sets (*pf*-sets) [23, 25] and some of *pf*-sets' operations and properties provided in [25] by considering the notations used throughout this paper.

In the present paper, let E be a parameter set, $F(E)$ be the set of all fuzzy sets over E, and $\mu \in F(E)$. Here, a fuzzy set is denoted by $\{^{\mu(x)}x : x \in E\}$ instead of $\{(x,\mu(x)) : x \in E\}$.

Definition 2.1. [3] Let κ be a function from E to $[0,1]\times[0,1]$. Then, the set $\{(x,f(x)) : x \in E\}$, being the graphic of κ is called an intuitionistic fuzzy set (*if*-set) over E.

Here, for all $x \in E$, $\kappa(x) = (\mu(x),\nu(x))$ such that $\mu(x)+\nu(x) \leq 1$. Moreover, μ and ν are called the membership function and non-membership function, respectively, and $\pi(x) = 1-(\mu(x)+\nu(x))$ is called the degree of indeterminacy of the element $x \in E$. For brevity, we represent an intuitionistic fuzzy set over E with $\kappa = \left\{^{\mu(x)}_{\nu(x)}x : x \in E\right\}$ instead of $\kappa = \{(x,\mu(x),\nu(x)) : x \in E\}$. Obviously, each ordinary fuzzy set can be written as $\left\{^{\mu(x)}_{1-\mu(x)}x : x \in E\right\}$.

Definition 2.2. [25] Let κ be a function from E to $[0,1]\times[0,1]\times[0,1]$. Then, the set $\{(x,f(x)) : x \in E\}$, being the graphic of κ is called a picture fuzzy set (*pf*-set) over E.

Here, for all $x \in E$, $\kappa(x) = (\mu(x),\eta(x),\nu(x))$ such that $0 \leq \mu(x)+\nu(x) \leq 1$ and $0 \leq \mu(x)+\eta(x)+\nu(x) \leq 2$. We denote a *pf*-set over E by $\kappa = \left\{\left\langle\begin{smallmatrix}\mu(x)\\ \eta(x)\\ \nu(x)\end{smallmatrix}\right\rangle x : x \in E\right\}$ instead of $\kappa = \{(x,\mu(x),\eta(x),\nu(x)) : x \in E\}$ for brevity.

Moreover, μ, η, and ν are called the membership function, neutral membership function, and non-membership function, respectively,

Note 2.3. Indeterminacy-membership of the element $x \in E$ in a pf-set over E must be defined by $\pi(x) = 1 - (\mu(x) + \nu(x))$ in order to that a pf-set can model a real-world problem and has theoretical consistency.

Manifestly, each ordinary fuzzy set can be written as $\left\{ \left\langle \begin{array}{c} \mu(x) \\ 1 \\ 1-\mu(x) \end{array} \right\rangle x : x \in E \right\}$ and each intuitionistic fuzzy set can be written as $\left\{ \left\langle \begin{array}{c} \mu(x) \\ 1 \\ \nu(x) \end{array} \right\rangle x : x \in E \right\}$.

In the present paper, the set of all the pf-sets over E is denoted by $PF(E)$ and $\kappa \in PF(E)$. In $PF(E)$, since the graph(κ) and κ have generated each other uniquely, the notations are interchangeable. Therefore, we represent a pf-set graph(κ) with κ as long as it causes no confusion.

Example 2.4. Let $E = \{x_1, x_2, x_3, x_4\}$. Then,

$$\kappa_1 = \left\{ \left\langle \begin{array}{c} 0.6 \\ 0.4 \\ 0.2 \end{array} \right\rangle x_1, \left\langle \begin{array}{c} 0.3 \\ 0 \\ 0.4 \end{array} \right\rangle x_2, \left\langle \begin{array}{c} 0.7 \\ 1 \\ 0.2 \end{array} \right\rangle x_3, \left\langle \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right\rangle x_4 \right\}$$

and

$$\kappa_2 = \left\{ \left\langle \begin{array}{c} 0.2 \\ 0.7 \\ 0.1 \end{array} \right\rangle x_1, \left\langle \begin{array}{c} 0.1 \\ 0 \\ 0.9 \end{array} \right\rangle x_2, \left\langle \begin{array}{c} 0.2 \\ 0.8 \\ 0.3 \end{array} \right\rangle x_3, \left\langle \begin{array}{c} 0.8 \\ 0 \\ 1 \end{array} \right\rangle x_4 \right\}$$

are two pf-sets over E.

Definition 2.5. [25] Let $\kappa \in PF(E)$. For all $x \in E$, if $\mu(x) = \lambda$, $\eta(x) = \varepsilon$, and $\nu(x) = \omega$, then κ is called $(\lambda, \varepsilon, \omega)$-$pf$-set and is denoted by $\left\langle \begin{array}{c} \lambda \\ \varepsilon \\ \omega \end{array} \right\rangle E$.

Definition 2.6. [25] Let $\kappa \in PF(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 1$, and $\nu(x) = 1$, then κ is called empty pf-set and is denoted by $\left\langle \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\rangle E$ or 0_E.

Definition 2.7. [25]Let $\kappa \in PF(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 0$, and $\nu(x) = 0$, then κ is called universal pf-set and is denoted by $\left\langle \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\rangle E$ or 1_E.

Definition 2.8. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \geq \eta_2(x)$, and $\nu_1(x) \geq \nu_2(x)$, then κ_1 is called a subset of κ_2 and is denoted by $\kappa_1 \tilde{\subseteq} \kappa_2$.

Definition 2.9. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) = \mu_2(x)$, $\eta_1(x) = \eta_2(x)$, and $\nu_1(x) = \nu_2(x)$, then κ_1 and κ_2 are called equal pf-sets and is denoted by $\kappa_1 = \kappa_2$.

Definition 2.10. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. If $\kappa_1 \tilde{\subseteq} \kappa_2$ and $\kappa_1 \neq \kappa_2$, then κ_1 is called a proper subset of κ_2 and is denoted by $\kappa_1 \tilde{\subsetneq} \kappa_2$.

Definition 2.11. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called union of κ_1 and κ_2 and is denoted by $\kappa_3 = \kappa_1 \tilde{\cup} \kappa_2$.

Definition 2.12. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \max\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called intersection of κ_1 and κ_2 and is denoted by $\kappa_3 = \kappa_1 \tilde{\cap} \kappa_2$.

Definition 2.13. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = 1 - \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then κ_2 is called complement of κ_1 and is denoted by $\kappa_2 = \kappa_1^{\tilde{c}}$.

Definition 2.14. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \nu_2(x)\}$, $\eta_3(x) = \max\{\eta_1(x), 1 - \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \mu_2(x)\}$, then κ_3 is called difference between κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\setminus} \kappa_2$.

Definition 2.15. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\min\{\mu_1(x), \nu_2(x)\}, \min\{\mu_2(x), \nu_1(x)\}\}$, $\eta_3(x) = \min\{\max\{\eta_1(x), 1 - \eta_2(x)\}, \max\{\eta_2(x), 1 - \eta_1(x)\}\}$, and $\nu_3(x) = \min\{\max\{\nu_1(x), \mu_2(x)\}, \max\{\nu_2(x), \mu_1(x)\}\}$, then κ_3 is called symmetric difference between κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\triangle} \kappa_2$.

3. Motivations of the Redefining of Picture Fuzzy Soft Sets

This section presents the definition and basic operations of picture fuzzy sets and the counter examples for the Cuong's definition provided in [23] and [25], respectively, considering the notations used across the present paper.

Definition 3.1. [23] Let κ be a function from E to $[0,1]\times[0,1]\times[0,1]$. Then, the set $\{(x, f(x)) : x \in E\}$, being the graphic of κ is called a picture fuzzy set (*pf*-set) over E.

In this section, the set of all the *pf*-sets over E according to Cuong's definition is denoted by $PF_C(E)$ and $\kappa \in PF_C(E)$.

Definition 3.2. [23] Let $\kappa_1, \kappa_2 \in PF_C(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \leq \eta_2(x)$, and $\nu_1(x) \geq \nu_2(x)$, then κ_1 is called a subset of κ_2 and is denoted by $\kappa_1 \tilde{\subseteq} \kappa_2$.

Definition 3.3. [23] Let $\kappa_1, \kappa_2 \in PF_C(E)$. If $\kappa_1 \tilde{\subseteq} \kappa_2$ and $\kappa_2 \tilde{\subseteq} \kappa_1$, then κ_1 and κ_2 are called equal *pf*-sets and is denoted by $\kappa_1 = \kappa_2$.

Definition 3.4. [23] Let $\kappa_1, \kappa_2, \kappa_3 \in PF_C(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called union of κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\cup} \kappa_2$.

Definition 3.5. [23] Let $\kappa_1, \kappa_2, \kappa_3 \in PF_C(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called intersection of κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\cap} \kappa_2$.

Definition 3.6. [23] Let $\kappa_1, \kappa_2 \in PF_C(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then κ_2 is called complement of κ_1 and is denoted by $\kappa_2 = \kappa_1^{\tilde{c}}$.

Memiş [25] have provided the following several counter-examples related to definition and operations of *pf*-sets in [23]. According to Definition 3.2, the definitions of empty and universal *pf*-sets should be as in Definition 3.7 and Definition 3.8, respectively, to be held the following conditions [25]:

- Empty *pf*-set over E is a subset of all the *pf*-set over E.
- All *pf*-sets over E are the subset of universal *pf*-set over E.

Definition 3.7. [25] Let $\kappa \in PF_C(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 0$, and $\nu(x) = 1$, then κ is called empty *pf*-set and is denoted by $\left\langle\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right\rangle E_C$ or 0_{E_C}.

Definition 3.8. [25] Let $\kappa \in PF_C(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 1$, and $\nu(x) = 0$, then κ is called empty *pf*-set and is denoted by $\left\langle\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right\rangle E_C$ or 1_{E_C}.

Example 3.9. [25] There is a contradiction in Definition 3.8 since $1 + 1 + 0 \nleq 1$, i.e., $1_{E_C} \notin PF_C(E)$. On the other hand, even if $1_{E_C} \in PF_C(E)$, $(1_{E_C})^{\tilde{c}} \neq 0_{E_C}$.

Example 3.10. [25] Let $\kappa \in PF_C(E)$ such that $\kappa = \left\{\left\langle\begin{smallmatrix}0.1\\0.2\\0.3\end{smallmatrix}\right\rangle x\right\}$. Then, $\kappa \tilde{\cup} 0_E \neq \kappa$ and $\kappa \tilde{\cup} 1_{E_C} \neq 1_{E_C}$.

To deal with the aforesaid inconsistencies in Example 3.9 and 3.10, the concept of *pf*-sets and their operations have been redefined by Memiş [25].

Secondly, the definitions of picture fuzzy soft sets (*pfs*-sets) provided in [23, 24] considering the notations used across the present paper.

Definition 3.11. [23] Let E be the set of parameters and $A \subseteq E$ set. A pair (F, A) is called *pfs*-set over U, where F is a mapping given by $F : A \to PF_C(U)$.

Definition 3.12. [24] Let U be the initial universe set and E a set of parameters. By *pfs*-set over U we mean a pair $\langle F, A\rangle$, where $A \subseteq E$ and F is a mapping given by $F : A \to PF_C(U)$.

Cuong [23] has defined the concept of *pfs*-sets relying on his own definition and operations of *pf*-sets. Therefore, the aforementioned inconsistencies have transferred to his concept of *pfs*-sets. Moreover, Yang et. al. [24] have claimed that they have introduced the concept of *pfs*-sets while Cuong has defined the concept of *pfs*-sets in [23]. Although the *pfs*-sets have been redefined in [24], the inconsistencies mentioned above has also transferred to the concept of *pfs*-sets due to it based on the definition and operations of *pf*-sets in [23].

Therefore, the concept of *pfs*-sets should be redefined to overcome the inconsistencies in the concept of *pfs*-sets and their operations.

4. Picture Fuzzy Soft Sets, Some of Their Properties, and Their Product Operations

In this section, we redefine the concepts of *pfs*-sets and investigate some of their properties according to new definition herein by considering the notations used throughout the present paper.

Definition 4.1. Let U be a universal set, E be a parameter set, and f is a function from E to $PF(U)$. Then the set $\{(x, f_A(x)) : x \in E\}$,being the graphic of f, is called a picture fuzzy soft set (*pfs*-set) parameterized via E over U (or briefly over U).

Example 4.2. Let $E = \{x_1, x_2, x_3, x_4\}$ be a parameter set and $U = \{u_1, u_2, u_3, u_4\}$ be a universal set. Then,

$$f = \left\{ \left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.4 \\ 0.1 \\ 0.9 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0 \\ 0.7 \\ 0.3 \end{smallmatrix} \right\rangle u_4 \right\} \right), \left(x_2, \left\{ \left\langle \begin{smallmatrix} 1 \\ 0.2 \\ 0 \end{smallmatrix} \right\rangle u_2 \right\} \right), (x_3, 0_U), \left(\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle x_4, 1_U \right) \right\}$$

is a *pfs*-set over U.

Note 4.3. We do not display the element $(x, 0_U)$ in a *pfs*-set where 0_U is empty *pf*-set over U.

Henceforth, the set of all the *pfs*-sets over U is denoted by $PFS(U)$. In $PFS(U)$, the notations graph(f) and f are interchangeable since they have generated each other uniquely. Thus, a *pfs*-set graph(f) is denoted by f as long as it leads no confusion.

Definition 4.4. Let $f \in PFS(U)$. If for all $x \in E$, $f(x) = \left\langle \begin{smallmatrix} \lambda \\ \varepsilon \\ \omega \end{smallmatrix} \right\rangle U$, then f is called $(\lambda, \varepsilon, \omega)$-*pfs*-set and is denoted by $\left(E, \left\langle \begin{smallmatrix} \lambda \\ \varepsilon \\ \omega \end{smallmatrix} \right\rangle U \right)$.

Definition 4.5. Let $f \in PFS(U)$ and f be $(\lambda, \varepsilon, \omega)$-*pfs*-set. If $\lambda = 0$, $\varepsilon = 1$, and $\omega = 1$, then f is called empty *pfs*-set and is denoted by $\left(E, \left\langle \begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right\rangle U \right)$ or briefly $\tilde{0}$.

Definition 4.6. Let $f \in PFS(U)$ and f be $(\lambda, \varepsilon, \omega)$-*pfs*-set. If $\lambda = 1$, $\varepsilon = 0$, and $\omega = 0$, then f is called universal *pfs*-set and is denoted by $\left(E, \left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle U \right)$ or briefly $\tilde{1}$.

Definition 4.7. Let $f, f_1 \in PFS(U)$ and $A \subseteq E$. Then, A_{f_1}-restriction of f, denoted by f_{Af_1}, is defined by

$$f_{Af_1}(x) := \begin{cases} f(x), & x \in A \\ f_1(x), & x \in E \setminus A \end{cases}$$

Briefly, if $f_1 = \tilde{0}$, then f_A can be employed instead of f_{Af_1}. It is clear that

$$f_A(x) := \begin{cases} f(x), & x \in A \\ \tilde{0}, & x \in E \setminus A \end{cases}$$

Example 4.8. Let us consider the *pfs*-set f provided in Example 4.2, $A = \{x_1, x_3\}$, and $f_1 \in PFS(U)$ such that

$$f_1 = \left\{ (x_1, 1_U), \left(x_4, \left\{ \left\langle \begin{smallmatrix} 0.2 \\ 0.5 \\ 0.4 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.6 \\ 0.3 \\ 0.2 \end{smallmatrix} \right\rangle u_4 \right\} \right) \right\}$$

Then,

$$f_{Af_1} = \left\{ \left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.4 \\ 0.1 \\ 0.9 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0 \\ 0.7 \\ 0.3 \end{smallmatrix} \right\rangle u_4 \right\} \right), \left(x_4, \left\{ \left\langle \begin{smallmatrix} 0.2 \\ 0.5 \\ 0.4 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.6 \\ 0.3 \\ 0.2 \end{smallmatrix} \right\rangle u_4 \right\} \right) \right\}$$

Definition 4.9. 4.10 Let $f_1, f_2 \in PFS(U)$. If for all $x \in E$, $f_1(x) \tilde{\subseteq} f_2(x)$, then f_1 is called a subset of f_2 and is denoted by $f_1 \tilde{\subseteq} f_2$.

Proposition 4.10. Let $f, f_1, f_2, f_3 \in PFS(U)$. Then,

i. $f \tilde{\subseteq} \tilde{1}$

ii. $\tilde{0} \tilde{\subseteq} f$

iii. $f \tilde{\subseteq} f$

iv. $[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_3] \Rightarrow f_1 \tilde{\subseteq} f_3$

Remark 4.11. $f_1 \tilde{\subseteq} f_2$ does not mean that every element of f_1 is an element of f_2. For instance, let $E = \{x_1, x_2\}$ be parameter set, $U = \{u_1, u_2\}$ be a universal set,

$$f_1 = \left\{ \left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.3 \\ 0.8 \\ 0.2 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.8 \\ 0.6 \\ 0.1 \end{smallmatrix} \right\rangle u_2 \right\} \right), \left(x_2, \left\{ \left\langle \begin{smallmatrix} 0.3 \\ 0.6 \\ 0.7 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.2 \\ 0.6 \\ 0.8 \end{smallmatrix} \right\rangle u_2 \right\} \right) \right\}$$

and

$$f_2 = \left\{ \left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.8 \\ 0.6 \\ 0.1 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.9 \\ 0.3 \\ 0.1 \end{smallmatrix} \right\rangle u_2 \right\} \right), \left(x_2, \left\{ \left\langle \begin{smallmatrix} 0.5 \\ 0.3 \\ 0.1 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.3 \\ 0.3 \\ 0.2 \end{smallmatrix} \right\rangle u_2 \right\} \right) \right\}$$

Thus, $f_1 \tilde{\subseteq} f_2$ because $f_1(x) \tilde{\subseteq} f_2(x)$ for all $x \in E$. However, $f_1 \not\subseteq f_2$ since $\left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.3 \\ 0.8 \\ 0.2 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.8 \\ 0.6 \\ 0.1 \end{smallmatrix} \right\rangle u_2 \right\} \right) \notin f_2$ while $\left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.3 \\ 0.8 \\ 0.2 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.8 \\ 0.6 \\ 0.1 \end{smallmatrix} \right\rangle u_2 \right\} \right) \in f_1$, where the notation $\subseteq$ indicates classic inclusion relation.

Definition 4.12. Let $f_1, f_2 \in PFS(U)$. If for all $x \in E$, $f_1(x) = f_2(x)$, then f_1 and f_2 are called equal *pfs*-sets and is denoted by $f_1 = f_2$.

Proposition 4.13. Let $f_1, f_2, f_3 \in PF(E)$. Then,

i. $[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_1] \Leftrightarrow f_1 = f_2$

ii. $[f_1 = f_2 \wedge f_2 = f_3] \Rightarrow f_1 = f_3$

Definition 4.14. Let $f_1, f_2 \in PFS(U)$. If $f_1 \tilde{\subseteq} f_2$ and $f_1 \neq f_2$, then f_1 is called a proper subset of f_2 and is denoted by $f_1 \tilde{\subsetneq} f_2$

Definition 4.15. Let $f_1, f_2, f_3 \in PFS(U)$. If for all $x \in E$, $f_3(x) = f_1(x) \tilde{\cup} f_2(x)$, then f_3 is called union of f_1 and f_2 and is denoted by $f_3 = f_1 \tilde{\cup} f_2$.

Proposition 4.16. Let $f, f_1, f_2, f_3 \in PFS(U)$. Then,

i. $f \tilde{\cup} f = f$

ii. $f \tilde{\cup} \tilde{1} = \tilde{1}$

iii. $f \tilde{\cup} \tilde{0} = f$

iv. $f_1 \tilde{\cup} f_2 = f_2 \tilde{\cup} f_1$

v. $f_1\tilde{\cup}\left(f_2\tilde{\cup}f_3\right)=\left(f_1\tilde{\cup}f_2\right)\tilde{\cup}f_3$

vi. $f_1\tilde{\subseteq}f_2\Rightarrow f_1\tilde{\cup}f_2=f_2$

Definition 4.17. Let $f_1,f_2,f_3\in PFS(U)$. If for all $x\in E$, $f_3\left(x\right)=f_1\left(x\right)\tilde{\cap}f_2\left(x\right)$, then f_3 is called intersection of f_1 and f_2 and is denoted by $f_3=f_1\tilde{\cap}f_2$.

Proposition 4.18. Let $f,f_1,f_2,f_3\in PFS(U)$. Then,

i. $f\tilde{\cap}f=f$

ii. $f\tilde{\cap}\tilde{1}=f$

iii. $f\tilde{\cap}\tilde{0}=\tilde{0}$

iv. $f_1\tilde{\cap}f_2=f_2\tilde{\cap}f_1$

v. $f_1\tilde{\cap}\left(f_2\tilde{\cap}f_3\right)=\left(f_1\tilde{\cap}f_2\right)\tilde{\cap}f_3$

vi. $f_1\tilde{\subseteq}f_2\Rightarrow f_1\tilde{\cap}f_2=f_1$

Proposition 4.19. Let $f_1,f_2,f_3\in PFS(U)$. Then,

i. $f_1\tilde{\cup}\left(f_2\tilde{\cap}f_3\right)=\left(f_1\tilde{\cup}f_2\right)\tilde{\cap}\left(f_1\tilde{\cup}f_3\right)$

ii. $f_1\tilde{\cap}\left(f_2\tilde{\cup}f_3\right)=\left(f_1\tilde{\cap}f_2\right)\tilde{\cup}\left(f_1\tilde{\cup}f_3\right)$

Proof. *i.* Let $f_1,f_2,f_3\in PFS(U)$. Then,

$$\begin{aligned}
f_1\tilde{\cup}(f_2\tilde{\cap}f_3) &= \{(x,f_1(x)):x\in E\}\,\tilde{\cup}\,\{(x,f_2(x)\tilde{\cap}f_3(x)):x\in E\}\\
&= \{(x,f_1(x)\tilde{\cup}(f_2(x)\tilde{\cap}f_3(x))):x\in E\}\\
&= \{(x,(f_1(x)\tilde{\cup}f_2(x))\tilde{\cap}(f_1(x)\tilde{\cup}f_3(x))):x\in E\}\\
&= \{(x,(f_1(x)\tilde{\cup}f_2(x))):x\in E\}\,\tilde{\cap}\,\{(x,(f_1(x)\tilde{\cup}f_3(x))):x\in E\}\\
&= (f_1\tilde{\cup}f_2)\tilde{\cap}(f_1\tilde{\cup}f_3)
\end{aligned}$$

□

Definition 4.20. Let $f_1,f_2\in PFS(U)$. If $f_1\tilde{\cap}f_2=\tilde{0}$, then f_1 and f_2 are called disjoint *pfs*-sets.

Definition 4.21. Let $f_1,f_2\in PFS(U)$. If for all $x\in E$, $f_2\left(x\right)=f_1^{\tilde{c}}\left(x\right)$, then f_2 is called complement of f_1 and is denoted by $f_2=f_1^{\tilde{c}}$.

Proposition 4.22. Let $f,f_1,f_2\in PFS(U)$. Then,

i. $\left(f^{\tilde{c}}\right)^{\tilde{c}}=f$

ii. $\tilde{0}^{\tilde{c}}=\tilde{1}$

iii. $f_1\tilde{\subseteq}f_2\Rightarrow f_2^{\tilde{c}}\tilde{\subseteq}f_1^{\tilde{c}}$

Definition 4.23. Let $f_1,f_2,f_3\in PFS(U)$. If for all $x\in E$, $f_3\left(x\right)=f_1\left(x\right)\tilde{\setminus}f_2\left(x\right)$, then f_3 is called difference between f_1 and f_2 and is denoted by $f_3=f_1\tilde{\setminus}f_2$.

Proposition 4.24. Let $f,f_1,f_2\in PFS(U)$. Then,

i. $f\tilde{\setminus}\tilde{0}=f$

ii. $f\tilde{\setminus}\tilde{1}=\tilde{0}$

iii. $f_1\tilde{\setminus}f_2=f_1\tilde{\cap}f_2^{\tilde{c}}$

Remark 4.25. It must be noted that the difference is non-commutative and non-associative. For example, Let $f_1 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.2\\0\\0.3\end{matrix}\right\rangle u\right\}\right)\right\}$, $f_2 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\0\\0.1\end{matrix}\right\rangle u\right\}\right)\right\}$, and $f_3 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.4\\0.1\\0.6\end{matrix}\right\rangle u\right\}\right)\right\}$. Then,

i. $\left[f_1 \tilde{\setminus} f_2 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.1\\1\\0.3\end{matrix}\right\rangle u\right\}\right)\right\} \wedge f_2 \tilde{\setminus} f_1 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\1\\0.2\end{matrix}\right\rangle u\right\}\right)\right\}\right] \Rightarrow f_1 \tilde{\setminus} f_2 \neq f_2 \tilde{\setminus} f_1$

ii. $\left[f_1 \tilde{\setminus} \left(f_2 \tilde{\setminus} f_3\right) = \left\{\left(x, \left\{\left\langle\begin{matrix}0.2\\0.1\\0.3\end{matrix}\right\rangle u\right\}\right)\right\} \wedge \left(f_1 \tilde{\setminus} f_2\right) \tilde{\setminus} f_3 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.1\\1\\0.4\end{matrix}\right\rangle u\right\}\right)\right\}\right] \Rightarrow f_1 \tilde{\setminus} \left(f_2 \tilde{\setminus} f_3\right) \neq \left(f_1 \tilde{\setminus} f_2\right) \tilde{\setminus} f_3$

Proposition 4.26. Let $f_1, f_2 \in PF(E)$. Then, the following De Morgan's Laws are valid.

i. $(f_1 \tilde{\cup} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cap} f_2^{\tilde{c}}$

ii. $(f_1 \tilde{\cap} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cup} f_2^{\tilde{c}}$

PROOF. *i.* Let $f_1, f_2 \in PFS(U)$. Then,

$$\begin{aligned}
(f_1 \tilde{\cup} f_2)^{\tilde{c}} &= \left(\left\{\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x, f_1\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \tilde{\cup} \left\{\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x, f_2\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \\
&= \left(\left\{\left(\left\langle\begin{matrix}\max\{\mu_1(x), \mu_2(x)\}\\\min\{\eta_1(x), \eta_2(x)\}\\\min\{\nu_1(x), \nu_2(x)\}\end{matrix}\right\rangle x, f_1\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right) \tilde{\cup} f_2\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \\
&= \left\{\left(\left\langle\begin{matrix}\min\{\nu_1(x), \nu_2(x)\}\\1-\min\{\eta_1(x), \eta_2(x)\}\\\max\{\mu_1(x), \mu_2(x)\}\end{matrix}\right\rangle x, f_1^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right) \tilde{\cap} f_2^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \\
&= \left\{\left(\left\langle\begin{matrix}\min\{\nu_1(x), \nu_2(x)\}\\\max\{1-\eta_1(x), 1-\eta_2(x)\}\\\max\{\mu_1(x), \mu_2(x)\}\end{matrix}\right\rangle x, f_1^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right) \tilde{\cap} f_2^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \\
&= \left\{\left\langle\begin{matrix}\nu_1(x)\\1-\eta_1(x)\\\mu_1(x)\end{matrix}\right\rangle x, f_1^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right) : x \in E\right\} \tilde{\cap} \left\{\left(\left\langle\begin{matrix}\nu_2(x)\\1-\eta_2(x)\\\mu_2(x)\end{matrix}\right\rangle x, f_2^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \\
&= \left(\left\{\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x, f_1\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \tilde{\cap} \left(\left\{\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x, f_2\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \\
&= f_1^{\tilde{c}} \tilde{\cap} f_2^{\tilde{c}}
\end{aligned}$$

□

Definition 4.27. Let $f_1, f_2, f_3 \in PFS(U)$. If for all $x \in E$, $f_3(x) = f_1(x) \tilde{\triangle} f_2(x)$, then f_3 is called symmetric difference between f_1 and f_2 and is denoted by $f_3 = f_1 \tilde{\triangle} f_2$.

Proposition 4.28. Let $f, f_1, f_2 \in PFS(U)$. Then,

i. $f \tilde{\triangle} \tilde{0} = f$

ii. $f \tilde{\triangle} \tilde{1} = f^{\tilde{c}}$

iii. $f_1 \tilde{\triangle} f_2 = f_2 \tilde{\triangle} f_1$

iv. $f_1 \tilde{\triangle} f_2 = (f_1 \tilde{\setminus} f_2) \tilde{\cup} (f_2 \tilde{\setminus} f_1)$

Remark 4.29. It must be noted that the symmetric difference is non-associative. Let us consider the *pfs*-sets f_1, f_2, and f_3 provided in Remark 4.25.

Since $f_1 \tilde{\triangle} \left(f_2 \tilde{\triangle} f_3\right) = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\0.1\\0.3\end{matrix}\right\rangle u\right\}\right)\right\}$ and $\left(f_1 \tilde{\triangle} f_2\right) \tilde{\triangle} f_3 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\0.1\\0.4\end{matrix}\right\rangle u\right\}\right)\right\}$, then $f_1 \tilde{\triangle} \left(f_2 \tilde{\triangle} f_3\right) \neq \left(f_1 \tilde{\triangle} f_2\right) \tilde{\triangle} f_3$.

We secondly present the AND, OR, ANDNOT, and ORNOT-products of *pfs*-sets and their examples.

Definition 4.30. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1\times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x)\tilde{\cap} f_2(y)$$

then f_3 is called AND-product of f_1 and f_2 and is denoted by $f_1 \wedge f_2$.

Definition 4.31. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1\times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x)\tilde{\cup} f_2(y)$$

then f_3 is called OR-product of f_1 and f_2 and is denoted by $f_1 \vee f_2$.

Definition 4.32. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1\times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x)\tilde{\cap} f_2^{\tilde{c}}(y)$$

then f_3 is called ANDNOT-product of f_1 and f_2 and is denoted by $f_1 \mathbin{\overline{\wedge}} f_2$.

Definition 4.33. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1\times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x)\tilde{\cup} f_2^{\tilde{c}}(y)$$

then f_3 is called ORNOT-product of f_1 and f_2 and is denoted by $f_1 \underline{\vee} f_2$.

Example 4.34. Let us consider the *pfs*-sets f_1 and f_2 provided in Remark 4.11. Then,

$$f_1 \wedge f_2 = \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{matrix}0.3\\0.8\\0.2\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.8\\0.6\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \left((x_1,x_2), \left\{ \left\langle\begin{matrix}0.3\\0.8\\0.2\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.3\\0.6\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \right.$$

$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.2\\0.6\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \left((x_2,x_2), \left\{ \left\langle\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.2\\0.6\\0.2\end{matrix}\right\rangle u_2 \right\}\right) \right\}$$

$$f_1 \vee f_2 = \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{matrix}0.8\\0.6\\0.1\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.9\\0.3\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \left((x_1,x_2), \left\{ \left\langle\begin{matrix}0.5\\0.3\\0.1\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.8\\0.3\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \right.$$

$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{matrix}0.8\\0.6\\0.1\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.9\\0.3\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \left((x_2,x_2), \left\{ \left\langle\begin{matrix}0.5\\0.3\\0.1\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.3\\0.3\\0.2\end{matrix}\right\rangle u_2 \right\}\right) \right\}$$

$$f_1 \mathbin{\overline{\wedge}} f_2 \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{matrix}0.1\\0.8\\0.8\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.1\\0.6\\0.9\end{matrix}\right\rangle u_2 \right\}\right), \left((x_1,x_2), \left\{ \left\langle\begin{matrix}0.1\\0.8\\0.5\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.1\\0.7\\0.3\end{matrix}\right\rangle u_2 \right\}\right), \right.$$

$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{matrix}0.1\\0.6\\0.8\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.1\\0.8\\0.9\end{matrix}\right\rangle u_2 \right\}\right), \left((x_2,x_2), \left\{ \left\langle\begin{matrix}0.1\\0.7\\0.7\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.1\\0.7\\0.8\end{matrix}\right\rangle u_2 \right\}\right) \right\}$$

$$f_1 \underline{\vee} f_2 \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{matrix}0.3\\0.4\\0.2\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.8\\0.4\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \left((x_1,x_2), \left\{ \left\langle\begin{matrix}0.3\\0.7\\0.2\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.8\\0.6\\0.1\end{matrix}\right\rangle u_2 \right\}\right), \right.$$

$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{matrix}0.3\\0.4\\0.7\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle u_2 \right\}\right), \left((x_2,x_2), \left\{ \left\langle\begin{matrix}0.3\\0.6\\0.5\end{matrix}\right\rangle u_1, \left\langle\begin{matrix}0.2\\0.6\\0.3\end{matrix}\right\rangle u_2 \right\}\right) \right\}$$

5. A Soft Decision-Making Method Based on *pfs*-Sets and Its Comparison

This section proposes a soft decision-making method via *pfs*-sets. Its algorithm steps are as follows:

<u>*Proposed Method*</u>

Step 1. Construct a *pfs*-set $f = \left\{ \left(x, \left\{ \left\langle \begin{smallmatrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{smallmatrix} \right\rangle u \right\} \right) : x \in E \right\}$ over U.

Step 2. Compute the score values

$$s(u) = \frac{1}{n} \sum_{x \in E} [\mu_u(x) - \eta_u(x)\nu_u(x)], \text{ for all } u \in U$$

such that $\mu_u(x)$, $\eta_u(x)$, and $\nu_u(x)$ denotes the membership, neutral membership, and non-membership degrees of the alternative u according to the parameter x.

Step 3. Obtain the decision set $\{^{\hat{s}(u_k)}u_k \mid u_k \in U\}$ such that

$$\hat{s}(u_k) := \begin{cases} \frac{s(u_k) - \min\limits_i\{s(u_i)\}}{\max\limits_i\{s(u_i)\} - \min\limits_i\{s(u_i)\}}, & \max\limits_i\{s(u_i)\} \neq \min\limits_i\{s(u_i)\} \\ 1, & \max\limits_i\{s(u_i)\} = \min\limits_i\{s(u_i)\} \end{cases}$$

Secondly, the section provides the illustrative example in [24] to compare fairly the proposed method with those in [24].

Example 5.1. [24] Suppose that there is an investment firm that wishes to put money into the best option (adapted from [26]). Let us consider the *pfs*-set f, which describes the "attractiveness of projects" being considered for investment by the firm. Assume that there are six alternative projects, i.e., $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ such that $u_1 =$"Project-1", $u_2 =$"Project-2", $u_3 =$"Project-3", $u_4 =$"Project-4" ,$u_5 =$"Project-5", and $u_6 =$"Project-6", and four parameters, i.e., $E = \{x_1, x_2, x_3, x_4\}$ such that $x_1 =$"Risk Analysis", $x_2 =$"Growth Analysis", $x_3 =$"Social-Political Impact Analysis", and $x_4 =$"Environment Analysis", under consideration. The firm evaluates the alternatives according to the parameters and constructs a *pfs*-set f_1 as follows:

$$\begin{aligned} f_1 = \Bigg\{ &\left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.31 \\ 0.22 \\ 0.41 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.12 \\ 0.41 \\ 0.33 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.23 \\ 0.52 \\ 0.21 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.45 \\ 0.09 \\ 0.36 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.57 \\ 0.30 \\ 0.05 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.44 \\ 0.40 \\ 0.13 \end{smallmatrix} \right\rangle u_6 \right\} \right), \\ &\left(x_2, \left\{ \left\langle \begin{smallmatrix} 0.54 \\ 0.21 \\ 0.15 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.81 \\ 0.11 \\ 0.02 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.13 \\ 0.48 \\ 0.37 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.23 \\ 0.59 \\ 0.18 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.60 \\ 0.23 \\ 0.14 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.42 \\ 0.36 \\ 0.22 \end{smallmatrix} \right\rangle u_6 \right\} \right), \\ &\left(x_3, \left\{ \left\langle \begin{smallmatrix} 0.60 \\ 0.14 \\ 0.26 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.26 \\ 0.51 \\ 0.20 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.72 \\ 0.15 \\ 0.03 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.32 \\ 0.49 \\ 0.15 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.81 \\ 0.11 \\ 0.06 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.43 \\ 0.27 \\ 0.13 \end{smallmatrix} \right\rangle u_6 \right\} \right), \\ &\left(x_4, \left\{ \left\langle \begin{smallmatrix} 0.38 \\ 0.21 \\ 0.40 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.65 \\ 0.15 \\ 0.18 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.29 \\ 0.58 \\ 0.12 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.14 \\ 0.32 \\ 0.45 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.43 \\ 0.18 \\ 0.35 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.35 \\ 0.29 \\ 0.34 \end{smallmatrix} \right\rangle u_6 \right\} \right) \Bigg\} \end{aligned}$$

Thirdly, the proposed soft decision-making method is applied to the *pfs*-set f_1 and the decision set is as follows:

$$\left\{ {}^{0.3980}\text{Project-1}, {}^{0.4062}\text{Project-2}, {}^{0.2636}\text{Project-3}, {}^{0.1788}\text{Project-4}, {}^{0.5718}\text{Project-5}, {}^{0.3424}\text{Project-6} \right\}$$

Fourthly, the ranking orders of proposed method and the decision-making method provided in [24] present in Table 1.

Table 1. The ranking orders of the proposed method and literature

Methods	**Structures**	**Ranking Orders**
Literature [24]	*pfs*-sets	Project-4 = Project-6 ≺ Project-1 = Project-3 ≺ Project-2 ≺ Project-5
Proposed Method	*pfs*-sets	Project-4 ≺ Project-3 ≺ Project-6 ≺ Project-1 ≺ Project-2 ≺ Project-5

According to the ranking orders in Table 1, proposed method and the literature is tend to producing the same ranking except for the alternatives Project-1, Project-3, and Project-6. Moreover, they confirm that Project-5 is the most suitable project and Project-4 is not suitable for the firm among the projects.

6. Conclusion

In this paper, we redefined the concept of *pfs*-sets to ensure their theoretical consistency. We then investigated their properties extensively and revised some of their operations. Afterwards, we defined their product operations such as AND, OR, ANDNOT, and ORNOT-products. We then proposed a soft decision-making method based on *pfs*-sets and compared it with the decision-making method provided in [24]. The results manifested that proposed method generate the stable ranking order compared to literature.

The concept of *pfs*-sets is a new mathematical tool for modelling the uncertainties. It has not been applied to real-world problems such as image processing and machine learning. To carry out these implementations, the matrix representation of the concept is required. The algebraic operations of picture fuzzy soft matrices (*pfs*-matrices) [27] have been studied, but the concept therein has not been explored substantially. In addition, it has the consistency resulting from definitions provided in [23, 24]. Hence, redefining of *pfs*-matrices is worth studying. On the other hand, applications of *pfs*-matrices to image processing and machine learning are crucial research topics since fuzzy parameterized fuzzy soft matrices, which is a substructure of *pfs*-matrices, are successfully applied to machine learning [28–32].

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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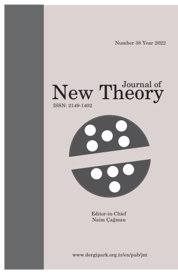

Timelike V-Bertrand Curves in Minkowski 3-Space E_1^3

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Abstract — In this paper, the timelike V-Bertrand curve, a new type Bertrand curve in Minkowski 3-space E_1^3, is characterized. Based on the timelike V-Bertrand curve, the properties of the timelike T, N, and B Bertrand curves are obtained. From the timelike V-Bertrand curve, f-Bertrand curves and Bertrand surfaces are defined. We support the existence of these new curves and surfaces with examples. Finally, we discuss the results for further research.

Keywords — *Bertrand curves, V-Bertrand curves, timelike V-Bertrand curves, Minkowski 3-space E_1^3*

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1. Introduction

The theory of curves has been a popular topic and many studies have been done on them. The Euclidean case (or more generally the Riemann case) of regular curves, a special type of curve, has been explored by many mathematicians. Characterization of a regular curve is one of the important problems in Euclidean space. Also, determining the Serret-Frenet vectors and the curvatures of regular curves is a common way to characterize a space curve in 3-dimensional space.

Minkowski space is one of the mathematical structures in which Einstein's relativity theory is best expressed. Since the inner product in Minkowski 3-space has an index, a vector has three different casual character. Therefore, while there exists only one Serret-Frenet formula in Euclidean 3-space, there exist five different Serret-Frenet formulas in Minkowski 3-space.

Bertrand curves are one of the most studied topics in the theory of curves. These curves have been firstly defined by Bertrand [1]. In this study, he has given an answer to the Saint Venant's open problem in which whether a curve exists on the surface produced by its principal normal vector and whether there exists another curve linearly dependent with principal normal vector of this curve [2]. The necessary and sufficient condition for existence of such a second curve is it satisfies the equation $a\kappa + b\tau = 1$ such that $a, b \in \mathbb{R}$, $a \neq 0$, and κ and τ are curvatures [3]. Moreover, Izumiya and Takeuchi have shown that all Bertrand curves can be obtained from a sphere, and they have given a method in [4] to obtain a Bertrand curve from a sphere. Recently, Camcı et al. [5] have studied Bertrand curves with a novel approach. İlarslan et al. have defined null Cartan and pseudo null Bertrand curves in Minkowski 3-space E_1^3 [6]. Further, (1,3)-Bertrand curves in a timelike (1,3)-normal plane in Minkowski space-time E_1^4 have been examined [7]. Also, Matsuda and Yorozu have shown that there is no Bertrand curve in Euclidean n-space E^n such that $n \geq 4$ and have defined (1,3)-Bertrand curves in Euclidean 4-space E^4 [8]. Lucas and Ortega-Yagües have characterized helices in $\mathbb{S}^3$ as the only

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twisted curves in $\mathbb{S}^3$ having infinite Bertrand conjugate curves [9]. Dede et al. have defined directional Bertrand curves [10]. Additionally, a new type Bertrand curve, called V-Bertrand curve, has been firstly defined and investigated by Camcı in [11].

In Section 2, we present some of definitions and properties to be used in the next sections. In Section 3, we describe timelike V-Bertrand curves in Minkowski 3-space E_1^3 and give a characterization of a timelike V-Bertrand curve. In Section 4, we define f-Bertrand curves using timelike curves. In Section 5, we give a method to obtain another Bertrand curve from a Bertrand curve. In Section 6, we define Bertrand surfaces by timelike curves. Finally, we discuss the need for further research. This study is a part of the first author's master's thesis [12].

2. Preliminaries

We start with recalling the definitions and theorems given by Camcı in [11]. Let $\gamma : I \to \mathbb{R}^3$ be a unit-speed curve with arc-length parameter "s". If Serret-Frenet apparatus are denoted with $\{T, N, B, \kappa, \tau\}$, then we can define a curve $\beta : I \to \mathbb{R}^3$ as

$$\beta(s) = \int V(s)ds + \lambda(s)N(s) \tag{1}$$

where $\lambda : I \to \mathbb{R}^3$ is a differentiable function and V is a unit vector field with

$$V : I \to T(\mathbb{R}^3), V(s) = u(s)T(s) + v(s)N(s) + \omega(s)B(s), \quad u, v, \omega \in C^\infty(I, \mathbb{R})$$

Definition 2.1. [11] Let $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$ be Serret-Frenet apparatus of the curve β defined in (1). If $\{N, \bar{N}\}$ is linearly dependent (e.g. $N = \varepsilon\bar{N}$, $\varepsilon = \pm 1$), then (γ, β) is V-Bertrand curve mate and γ is called V-Bertrand curve. If $V = T$, then $(\gamma,\ \beta)$ is a classical Bertrand mate.

Theorem 2.2. [11] Let γ be a unit-speed curve with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$. The curve γ is a V-Bertrand curve if and only if the following equation holds:

$$\lambda(\kappa \tan\theta + \tau) = u \tan\theta - \omega \tag{2}$$

where

$$\lambda(s) = -\int v(s)ds$$

and θ is a constant angle between T and $\bar{T}$.

Definition 2.3. [11] Let γ be a unit-speed and non-planar curve ($\tau \neq 0$) with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$. If there exist $\lambda \neq 0$ and $\theta \in \mathbb{R}$ satisfying the equation

$$\lambda\kappa + \lambda \cot\theta\tau = 1 \tag{3}$$

then we say that the curve γ is a Bertrand curve (or T-Bertrand curve). In addition, if the equation

$$\lambda\kappa \tan\theta + \lambda\tau = -1 \tag{4}$$

holds, then we say that the curve γ is a B-Bertrand curve.

Remark 2.4. [11] If $u(s) = 1$ and $v(s) = \omega(s) = 0$, then the pair (γ, β) is a T-Bertrand curve mate. Also, if $\omega(s) = 1$ and $u(s) = v(s) = 0$, then the pair (γ, β) is a B-Bertrand curve mate. Furthermore, if $v(s) = 1$ and $u(s) = \omega(s) = 0$, then we say that the pair (γ, β) is an N-Bertrand curve mate.

Next, recall that Minkowski 3-space E_1^3 is Euclidean 3-space E^3 equipped with the metric

$$g := -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3 [13]. In this space, a vector can has one of three casual characters according to this metric. If $g(u, u) > 0$ or $u = 0$, then u is a spacelike vector, if $g(u,u) < 0$, then u is a timelike vector, and if $g(u,u) = 0$ and $u \neq 0$, then u is a null (lightlike) vector. Moreover, an arbitrary curve $\alpha = \alpha(s)$ in Minkowski 3-space E_1^3 can be called according to its the velocity vector $\alpha'(s)$. A curve α is called spacelike, timelike, or null, if $\alpha'(s)$ is spacelike, timelike, or null, respectively. For a timelike curve with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$, the following formulas hold:

$$T' = \kappa N, \ N' = \kappa T + \tau B, \text{ and } B' = -\tau N \tag{5}$$

where

$$g(T,T) = -1, \quad g(N,N) = 1, \quad g(B,B) = 1 \tag{6}$$

$$g(N,B) = 0, \quad g(T,N) = 0, \quad g(T,B) = 0 \tag{7}$$

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N \tag{8}$$

3. Timelike V-Bertrand Curves in Minkowski 3-Space E_1^3

In this section, we define timelike V-Bertrand curves in Minkowski 3-space E_1^3 and investigate some of their basic properties. In addition, we give a characterization for this type curves.

Definition 3.1. Let $\gamma : I \to E_1^3$, $\gamma = \gamma(s)$ be a unit-speed timelike curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\beta : I \to E_1^3$, $\beta = \beta(s)$ be a regular curve with Frenet apparatus $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$. We can define a curve β by

$$\beta(s) = \int V(s)ds + \lambda(s)N(s) \tag{9}$$

where $\lambda : I \to \mathbb{R}^3$ is a differentiable function and V is a unit vector field with

$$V : I \to T(\mathbb{R}^3), V(s) = u(s)T(s) + v(s)N(s) + \omega(s)B(s), \quad u, v, \omega \in C^\infty(I, \mathbb{R}).$$

If $\{N, \bar{N}\}$ is linearly dependent (e.g. $N = \varepsilon\bar{N}$, $\varepsilon = \pm 1$), then the pair (γ, β) is called a timelike V-Bertrand curve mate and γ is called a timelike V-Bertrand curve. Moreover, especially, if $V = T$ (N or B), then (γ, β) is a timelike T (N or B)-Bertrand curve mate.

Theorem 3.2. Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of this curve. The curve γ is a timelike V-Bertrand curve if and only if it satisfies the following condition:

$$\lambda(\tau - \kappa \tanh\theta) = u \tanh\theta - \omega \tag{10}$$

such that

$$\lambda = -\int v(s)ds \tag{11}$$

and θ is a constant angle between T and $\overline{T}$.

Proof. Let $\gamma : I \to E_1^3$, $\gamma = \gamma(s)$ be a unit-speed timelike V-Bertrand curve and $\beta : I \to E_1^3$, $\beta = \beta(\bar{s})$ be V-Bertrand curve mate of γ. Also, let Frenet apparatus of these curves be $\{T, N, B, \kappa, \tau\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$, respectively.

($\Rightarrow$) Derivating β with respect to s, we have the following equation

$$\begin{aligned}\frac{d\bar{s}}{ds}\bar{T} &= uT + vN + \omega B + \lambda' N + \lambda N' \\ &= (u + \lambda\kappa)T + (\lambda' + v)N + (\omega + \lambda\tau)B\end{aligned} \tag{12}$$

Since $\{N, \bar{N}\}$ is linearly dependent, we have

$$\lambda = -\int v(s)ds \tag{13}$$

After, it follows that equation (12), we have

$$\bar{T}=\frac{ds}{d\bar{s}}(u+\lambda\kappa)T+\frac{ds}{d\bar{s}}(\omega+\lambda\tau)B \tag{14}$$

From the equation (14), we get

$$\cosh\theta=\frac{ds}{d\bar{s}}(u+\lambda\kappa) \tag{15}$$

$$\sinh\theta=\frac{ds}{d\bar{s}}(\omega+\lambda\tau) \tag{16}$$

From the equations (15) and (16), we get

$$\lambda(\tau-\kappa\tanh\theta)=u\tanh\theta-\omega$$

Thus, the equation (14) is rewritten as

$$\bar{T}=\cosh\theta T+\sinh\theta B \tag{17}$$

Also, if the derivative of equation (17) according to the arc-parameter s is taken, then we get

$$\frac{d\bar{s}}{ds}\overline{\kappa}\overline{N}=\theta'\sinh\theta T+(\kappa\cosh\theta-\tau\sinh\theta)N+\theta'\cosh\theta B \tag{18}$$

As $\{N,\bar{N}\}$ is linearly dependent, the angle θ is a constant.

($\Leftarrow$) Let the equation (10) be valid for the constant θ. Derivating the equation (9), we have the equation (12). From the equations (11) and (12), we get

$$\bar{T}=\frac{ds}{d\bar{s}}(u+\lambda\kappa)T+\frac{ds}{d\bar{s}}(\omega+\lambda\tau)B=\cosh(w(s))T+\sinh(w(s))B \tag{19}$$

From the equations (10) and (19), we obtain

$$\tanh(w(s))=\frac{u+\lambda\kappa}{\omega+\lambda\tau}=\tanh\theta \tag{20}$$

From the equation (20), $w(s)=\theta$. Since θ is a constant, if the derivative of the equation (19) is taken, then it is seen that $\{N,\bar{N}\}$ is linearly dependent. Therefore, the curve γ is a V-Bertrand curve.

□

Corollary 3.3. Let γ be a unit-speed and non-planar timelike curve and $\{T,N,B,\kappa,\tau\}$ be Frenet apparatus of the curves in Minkowski 3-space E_1^3. If $\overline{\lambda}=\lambda\tanh\theta$ and $\overline{\mu}=-\lambda$ such that λ and θ are non-zero constant numbers, then

1. γ is a timelike T-Bertrand curve if and only if $\overline{\lambda}\kappa+\overline{\mu}\tau=-\tanh\theta$. Further, if $u(s)=1$ and $v(s)=\omega(s)=0$ in the equation $V(s)=u(s)T(s)+v(s)N(s)+\omega(s)B(s)$, then (γ,β) is a timelike T-Bertrand curve mate. From the equation (9), we have

$$\beta(s)=\int T(s)ds+\lambda(s)N(s)$$

If the integral constant is assumed as zero in this equation, then (γ,β) is a classical timelike Bertrand curve mate.

2. γ is a timelike N-Bertrand curve if and only if $\frac{\tau}{\kappa}=\tanh\theta$. Also, if $u(s)=w(s)=0$ and $v(s)=1$ in the equation $V(s)=u(s)T(s)+v(s)N(s)+\omega(s)B(s)$, then (γ,β) is a timelike N-Bertrand curve mate. From Theorem 3.2, $\lambda=-s+c$ and the timelike N-Bertrand curve γ is a general helix such that θ is a constant.

3. γ is a timelike B-Bertrand curve if and only if $\overline{\lambda}\kappa+\overline{\mu}\tau=1$. Morever, let γ be a timelike anti-Salkowski curve, i.e., τ is a constant. If $\lambda=\frac{1}{\tau}$, then

$$(\lambda\tanh\theta)\kappa-\lambda\kappa=1$$

In this case, any timelike anti-Salkowski curve is a timelike B-Bertrand curve.

Example 3.4. Let us consider the curve $\gamma(s) = (\sqrt{2}\sinh s, \sqrt{2}\cosh s, s)$ in Minkowski 3-space E_1^3 provided in [14]. It is clear that γ is a timelike curve. The Frenet vectors and curvatures of γ are as follows:

$$\begin{aligned} T &= (\sqrt{2}\cosh s, \sqrt{2}\sinh s, 1) \\ N &= (\sinh s, \cosh s, 0) \\ B &= \left(\cosh s, \sinh s, \sqrt{2}\right) \\ \kappa &= \sqrt{2} \\ \tau &= -1 \end{aligned} \tag{21}$$

If $V = B$ ($u = v = 0$ and $w = 1$) is taken, then (γ, β) timelike B-Bertrand curve mate is obtained in Definition 3.1. To find the curve β, if timelike B-Bertrand curve characterization is used, then we have

$$\lambda = \frac{\sqrt{2}}{\sqrt{2} + 2\tanh\theta}$$

If the vectors N and B in the equation (21) and λ are written in the Definition 3.1, then we obtain

$$\beta(s) = \left((1+\lambda)\sinh s, (1+\lambda)\cosh s, \sqrt{2}s\right)$$

The tangent vector of the curve β is as follows:

$$\overline{T} = \frac{1}{\sqrt{2-(1+\lambda)^2}}\left((1+\lambda)\cosh s, (1+\lambda)\sinh s, \sqrt{2}s\right)$$

If $1 + \lambda = \frac{1}{\sqrt{2}}$, then the curve β is obtained as

$$\beta(s) = \left(\frac{1}{\sqrt{2}}\sinh s, \frac{1}{\sqrt{2}}\cosh s, \sqrt{2}s\right)$$

Hence, the graph of the timelike B-Bertrand curve mate (γ, β) is as follows:

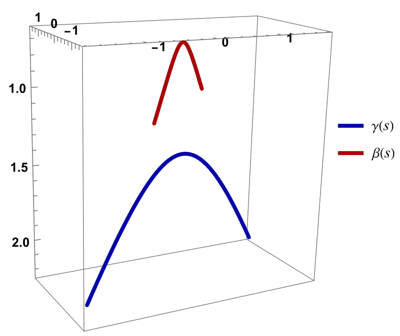


Fig. 1. The timelike B-Bertrand curve mate (γ, β)

4. f-Bertrand Curves Obtained from Timelike Curves

In this section, we propose f-Bertrand curves by using timelike curves. Morever, we provide three examples for f-Bertrand curves.

Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curve in Minkowski 3-space E_1^3. Let V be a timelike unit vector field defined in the Definition 3.1. If $v = 0$, then $-u^2+w^2=-1$. For $\epsilon=\pm1$, then $w=\epsilon\sqrt{u^2-1}$. Applying transformation in the equation (10), we have

$$u\tanh\theta-\epsilon\sqrt{u^2-1}=f \tag{22}$$

If this quadratic equation is solved according to the variable u, then we have

$$u^{\pm}=\frac{f\tanh\theta\pm\sqrt{f^2+1-(\tanh\theta)^2}}{(\tanh\theta)^2-1} \tag{23}$$

From (23), $w_{1,2}^{\pm}=\epsilon\sqrt{(u^{\pm})^2-1}$. Therefore, there are four different situations for timelike unit vector field:

$$V_1^{\pm}=u^+T+w_1^{\pm}B \quad V_2^{\pm}=u^-T+w_2^{\pm}B$$

Thus, $\beta_1^{\pm}$ and $\beta_2^{\pm}$ can be defined as

$$\begin{aligned}\beta_1^{\pm}(s)&=\int V_1^{\pm}ds+\lambda N\\ \beta_2^{\pm}(s)&=\int V_2^{\pm}ds+\lambda N\end{aligned} \tag{24}$$

Then, the curve γ is a timelike V_1^+, V_1^-,V_2^+, and V_2^--curve. Thus, the following definition can be given.

Definition 4.1. Each of the curves $\beta_1^+(s)$, $\beta_1^-(s)$, $\beta_2^+(s)$, and $\beta_2^-(s)$ defined in (23) is called an f-Bertrand curve mate of a timelike curve γ and the timelike curve γ is called an f-Bertrand curve.

Example 4.2. Let us consider the timelike curve γ provided in Example 3.4. To find $\tanh\theta-$Bertrand mates of the timelike curve γ, we suppose that $f=\tanh\theta$ in the equation (22). From the equations (10) and (22),

$$\tanh\theta=-\frac{\lambda}{1+\lambda\sqrt{2}}$$

Morever, $u^+=1-2(\cosh\theta)^2$ and $u^-=-1$ from the equation (23). Therefore, we have $w_1^+=\sinh2\theta$, $w_1^-=-\sinh2\theta$, and $w_2^{\pm}=0$. Hence, the f-Bertrand curve mates of the timelike curve γ are as follows:

$$\beta_1^{\pm}(s)=\begin{pmatrix}\left(\left(1-2(\cosh\theta)^2\right)\sqrt{2}\pm(\sinh2\theta+\lambda)\right)\sinh s,\\ \left(\left(1-2(\cosh\theta)^2\right)\sqrt{2}\pm(\sinh2\theta+\lambda)\right)\cosh s,\\ \left(\left(1-2(\cosh\theta)^2\right)\pm\left(\sqrt{2}\sinh2\theta\right)\right)s\end{pmatrix}$$

$$\beta_2^{\pm}(s)=\beta_2(s)=\left(\left(\sqrt{2}+\lambda\right)\sinh s,\left(\sqrt{2}+\lambda\right)\cosh s,s\right)$$

For $\lambda=\sqrt{2}$, the curve pairs (γ,β_1^+), (γ,β_1^-), and (γ,β_2) are presented in the Fig. 2.

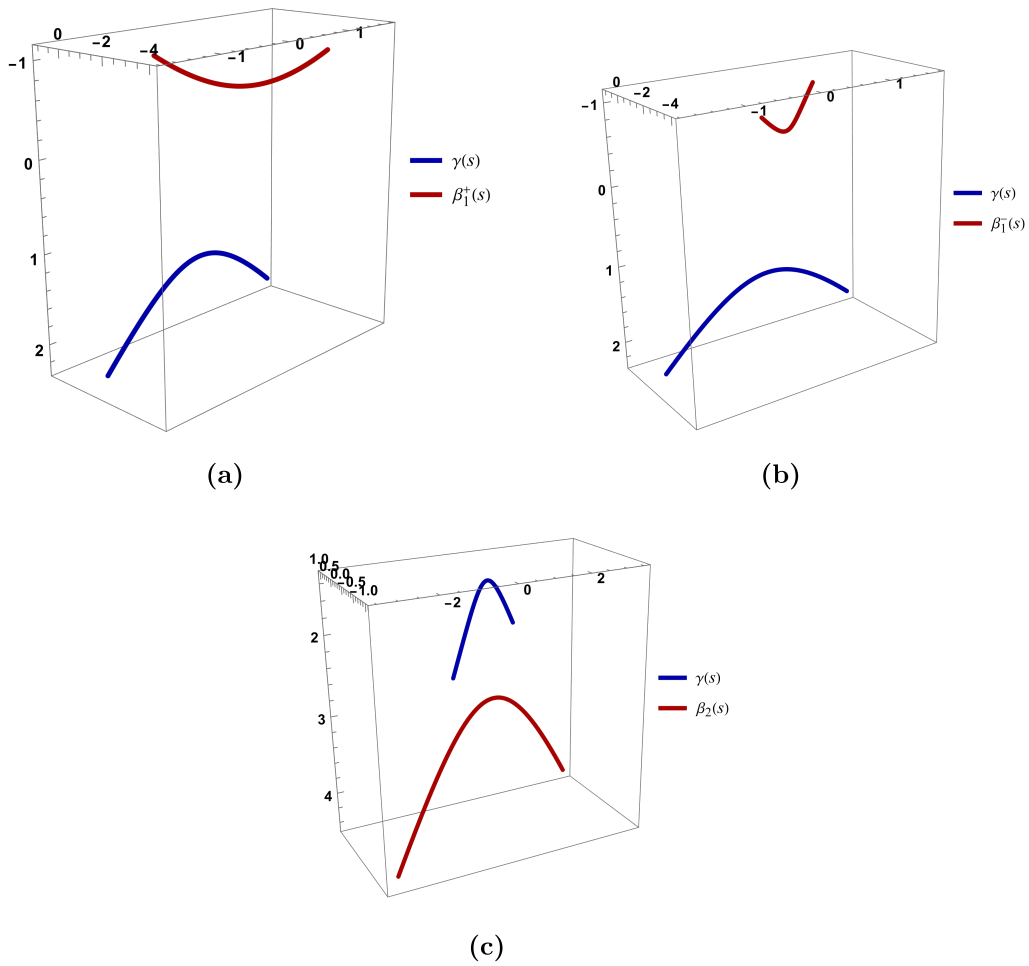


Fig. 2. (a) The curve pair (γ, β_1^+) for $\lambda = \sqrt{2}$ (b) The curve pair (γ, β_1^-) for $\lambda = \sqrt{2}$, and (c) The curve pair (γ, β_2) for $\lambda = \sqrt{2}$

5. Timelike and Spacelike Bertrand Curve Obtained From Timelike Bertrand Curve

In this section, we obtain new timelike and spacelike Bertrand curves using a timelike curve.

Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curve in Minkowski 3-space E_1^3. Considering u and w are constants and $v = 0$ in the unit vector field V in Definition 3.1, V can be rewritten as $V(s) = uT(s) + wB(s)$. Let $\gamma_V = \int V(s)ds$ and its Frenet vectors and curvatures is $\{T_V, N_V, B_V, \kappa_V, \tau_V\}$. In this section, the conditions for a curve γ_V to be a Bertrand curve are investigated.

Lemma 5.1. Let V be a timelike unit vector field. In this case, curvatures of γ are written as follows by curvatures of γ_V:

$$\begin{aligned} \kappa &= w\kappa_V + u\tau_V \\ \tau &= u\kappa_V + w\tau_V \end{aligned}$$

Proof. If V is a timelike unit vector field, we have $-u^2 + w^2 = -1$. Since the tangent vector of curve γ_V is the vector V, the curve γ_V is a timelike curve. Therefore,

$$T_V = uT + wB \tag{25}$$

If the derivative of this equation is taken and $N_V = N$, then

$$\kappa_V = u\kappa - w\tau \tag{26}$$

Applying the cross product to the equation (25) by N_V from the right, we get

$$B_V = uB + wT$$

If we derivative this equation, we have

$$\tau_V = -w\kappa + u\tau \tag{27}$$

From equations (26) and (27), the curvatures of the curve γ are obtained as follows:

$$\begin{aligned} \kappa &= w\kappa_V + u\tau_V \\ \tau &= u\kappa_V + w\tau_V \end{aligned} \tag{28}$$

□

The following theorem is given from the Lemma 5.1.

Theorem 5.2. Let V be a timelike unit vector field. γ is a timelike Bertrand curve if and only if γ_V is a timelike Bertrand curve.

Lemma 5.3. Let V be a spacelike unit vector field. In this case, curvatures of γ are written as follows by curvatures of γ_V:

$$\begin{aligned} \kappa &= -u\kappa_V + w\tau_V \\ \tau &= -w\kappa_V + u\tau_V \end{aligned}$$

PROOF. Let V be a spacelike unit vector field. Thus, $-u^2 + w^2 = 1$. Because the tangent vector of curve γ_V is the vector V, the curve γ_V is a spacelike curve. Hereby,

$$T_V = uT + wB \tag{29}$$

If the equation (29) is differentiated and $N_V = N$, thereby

$$\kappa_V = u\kappa - w\tau \tag{30}$$

Applying the cross product to the equation (29) by N_V from the right, the following equation is obtained:

$$B_V = uB + wT$$

If we derivative this equation, we have

$$\tau_V = w\kappa - u\tau \tag{31}$$

From equations (30) and (31), the curvatures of the curve γ are obtained as follows:

$$\begin{aligned} \kappa &= -u\kappa_V + w\tau_V \\ \tau &= -w\kappa_V + u\tau_V \end{aligned} \tag{32}$$

□

The following theorem is given from the Lemma 5.3.

Theorem 5.4. Let V be a spacelike unit vector field. γ is a timelike Bertrand curve if and only if γ_V is a spacelike Bertrand curve whose binormal is a timelike curve.

6. Bertrand Surface Obtained From Timelike Bertrand Curve

In this section, we suggest the concept of Bertrand surfaces and provide an example for Bertrand surfaces.

Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curves in Minkowski 3-space E_1^3. Because of timelike Bertrand (timelike T-Bertrand) characterization, we have the equation

$$\lambda \tanh\theta\kappa - \lambda\tau = -\tanh\theta$$

If both sides of this equation are multiplied by a real number t, the following equation is obtained

$$\lambda t \tanh\theta\kappa - \lambda t\tau = -t\tanh\theta$$

Putting $-t\tanh\theta$ instead of f in the equation (23), we find

$$u^{\pm}(t) = \frac{-t(\tanh\theta)^2 \pm \sqrt{t^2(\tanh\theta)^2 + 1 - (\tanh\theta)^2}}{(\tanh\theta)^2 - 1} \tag{33}$$

Also,

$$w_1^{\pm} = \epsilon\sqrt{(u^+(t))^2 - 1} \text{ and } w_2^{\pm} = \epsilon\sqrt{(u^-(t))^2 - 1} \tag{34}$$

Thus, the following definition can be given.

Definition 6.1. Let γ be a timelike Bertrand curve. Each of the following surfaces ψ_1^+, ψ_1^-, ψ_2^+, and ψ_2^- is called a Bertrand surface of γ.

$$\begin{aligned} \psi_1^{\pm}(t,s) &= \int V_1^{\pm} ds + \lambda N \\ \psi_2^{\pm}(t,s) &= \int V_2^{\pm} ds + \lambda N \end{aligned} \tag{35}$$

such that $V_1^{\pm}(t,s) = u^+(t)T(s) + w_1^{\pm}(t)B(s)$ and $V_2^{\pm}(t,s) = u^-(t)T(s) + w_2^{\pm}(t)B(s)$ by $u^{\pm}$, $w_1^{\pm}$, and $w_2^{\pm}$ in the equations (33) and (34).

Example 6.2. Let γ be a timelike curve provided in Example 3.4. To find a Bertrand surface of the curve γ, if the curvatures of the curve γ are written by using timelike T-Bertrand characterization, we get

$$\lambda = -\frac{\tanh\theta}{1 + \sqrt{2}\tanh\theta}$$

If $\tanh\theta = -\frac{\sqrt{2}}{3}$, then

$$\begin{aligned} u^+(t) &= \frac{2}{7}t - \frac{3}{7}\sqrt{2t^2+7} \\ w_1^+(t) &= \sqrt{\left(\frac{2}{7}t - \frac{3}{7}\sqrt{2t^2+7}\right)^2 - 1} \end{aligned} \tag{36}$$

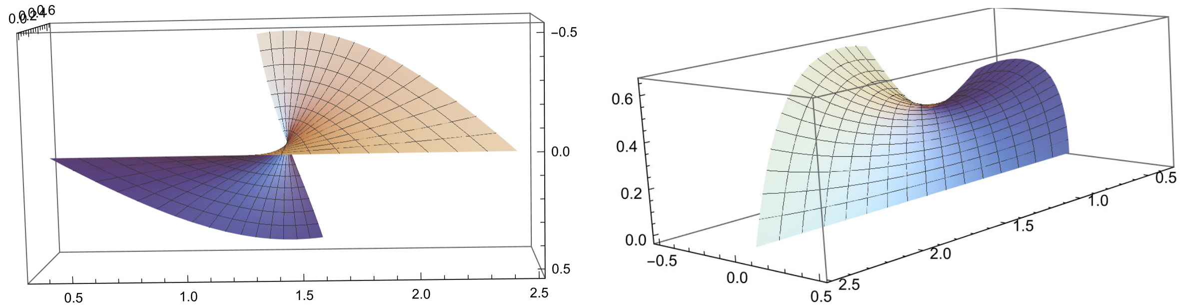
The surface ψ_1^+ in the equation (35) is as follows:

$$\psi_1^+(t,s) = u^+(t)\int T(s)ds + w_1^+(t)\int B(s)ds + \lambda N(s) \tag{37}$$

From the equation (36), the equation (37) is rearranged as follows:

$$\psi_1^+(t,s) = \begin{pmatrix} \left(\frac{2}{7}\sqrt{2}t - \frac{3}{7}\sqrt{2}\sqrt{2t^2+7} + \frac{1}{7}\sqrt{22t^2 + 14 - 12t\sqrt{2t^2+7}} + \sqrt{2}\right)\sinh s, \\ \left(\frac{2}{7}\sqrt{2}t - \frac{3}{7}\sqrt{2}\sqrt{2t^2+7} + \frac{1}{7}\sqrt{22t^2 + 14 - 12t\sqrt{2t^2+7}} + \sqrt{2}\right)\cosh s, \\ \left(\frac{2}{7}st - \frac{3}{7}s\sqrt{2t^2+7} + \frac{1}{7}s\sqrt{22t^2 + 14 - 12t\sqrt{2t^2+7}}\right)\sqrt{2} + \sqrt{2} \end{pmatrix}$$

The graph of the surface ψ_1^+ is provided in Fig. 3.



(a) (b)

Fig. 3. The Bertrand surface ψ_1^+ of the curve γ

7. Conclusion

In this study, we characterized V-Bertrand curves in Minkowski 3-space by V-Bertrand curves in Euclidean 3-space, a new type of Bertrand curve defined by Camcı [11]. Firstly, the characterization of timelike V-Bertrand curves was given by a timelike curve. Afterwards, we defined T-Bertrand, N-Bertrand, and B-Bertrand curves by the timelike V-Bertrand curve and their characterization. Some of the obtained important results are the following: a timelike T-Bertrand curve is a timelike Bertrand curve and a timelike N-Bertrand curve is a timelike circular helix. Furthermore, in the timelike V-Bertrand curve characterization, four f-Bertrand curves were obtained from a timelike V-Bertrand curve and a mapping f. Additionaly, using these f-Bertrand curve characterizations, four Bertrand surfaces were defined by timelike Bertrand curves. Finally, a method was given to obtain a spacelike curve whose binormal vector is a timelike vector and another timelike Bertrand curve from a timelike Bertrand curve. Thus, timelike V-Bertrand curves in Minkowski 3-space, a new curve, has been brought to the literature. With the idea used in this study, the researchers can develop this study for other Frenet frames.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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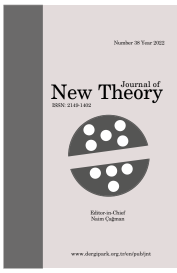

Solutions of Fractional Kinetic Equations using the $(p, q; \ell)$-Extended τ-Gauss Hypergeometric Function

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Abstract − The main objective of this paper is to use the newly proposed $(p, q; \ell)$-extended beta function to introduce the $(p, q; \ell)$-extended τ-Gauss hypergeometric and the $(p, q; \ell)$-extended τ-confluent hypergeometric functions with some of their properties, such as the Laplace-type and the Euler-type integral formulas. Another is to apply them to fractional kinetic equations that appear in astrophysics and physics using the Laplace transform method.

Keywords− *Beta function, hypergeometric function, fractional calculus, pochhammer symbol, integral transforms*

Mathematics Subject Classification (2020) – 33E99, 44A20

1. Introduction

Recently, the applications of the special functions of mathematics have developed significantly in such fields as fractional calculus, approximation theory, mathematical physics, engineering, science and technology [1-3]. One very interesting application area of special functions of mathematics is the extension of the standard kinetic equations by its integration [4]

$$\Lambda(t) - \Lambda_0 = -c^{\partial}\ {}_0D_t^{-\partial}\{\Lambda(t)\} \tag{1}$$

for any positive constant c, $\Lambda(t)$ represents the reaction rate, Λ_0 represents $\Lambda(t)$ at $t = 0$, and ${}_0D_t^{-\partial}$ is the Riemann-Liouville fractional integral operator defined by

$${}_0D_t^{-\partial}\{\Lambda(t)\} = \frac{1}{\Gamma(\partial)}\int_0^t (t-u)^{\partial-1}\Lambda(u)du,\ (Re(\partial) > 0, t > 0)$$

They [4] also give the following solution to equation (1):

$$\Lambda(t) = \Lambda_0 E_{\partial}\left(-c^{\partial}t^{\partial}\right), (\partial \in \mathbb{R}^+)$$

Extensions, generalizations and different forms of equation (1) have been studied by Saxena et al., [5, 6] using functions of Wiman and Prabhakar [7-9], Khan et al., [10] studied the following fractional kinetic equations:

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$$\Lambda(t) - \Lambda_0 t^{\aleph} R^{\tau}_{p,q}\big(a, b; c; \psi t^{\partial}\big) = -\sigma^{\partial} \; {}_0D_t^{-\partial}\{\Lambda(t)\} \tag{2}$$

and

$$\Lambda(t) - \Lambda_0 t^{\aleph} R^{\tau}_{p,q}\big(a, b; c; \psi t^{\partial}\big) = -\left\{\sum_{\omega\geq 1}\binom{\kappa}{\omega}\sigma^{\partial\omega} \; {}_0D_t^{-\partial\omega}\right\}\Lambda(t) \tag{3}$$

where $R^{\tau}_{p,q}(;)$ is the (p,q)-extended τ-Gauss hypergeometric function [11]

$$R^{\tau}_{p,q}\big(a, b; c; \psi t^{\partial}\big) = \sum_{\Bbbk\geq 0}(a)_{\Bbbk}\frac{B_{p,q}(b+\Bbbk\tau, c-b)}{B(b, c-b)}\frac{z^{\Bbbk}}{\Bbbk!}$$

for all $\min\{Re(p), Re(q)\} > 0, \tau \geq 0, \ell \in \mathbb{R}^{+}\backslash\{1\}, Re(a) > Re(b) > 0,$ and $B_{p,q}(\wp, \Im)$ is the extended beta function defined by [12]

$$B_{p,q}(\wp, \Im) = \int_0^1 t^{\wp-1}(1-t)^{\Im-1}\exp\Big(-\frac{p}{t}-\frac{q}{1-t}\Big)\,dt$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\wp), Re(\Im)\} > 0.$

Readers can refer to [13-20] for more generalizations and extensions of extended fractional kinetic equations.

The main objective of this paper is to introduce the new the $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions with some properties and their applications to fractional kinetic equations via the Laplace transforms methods. Furthermore, the resulting functions and equations can be reduced to well-known and perhaps new results. This paper is presented as follows: Section one is compressed with some preliminaries. In section 3, the $(p,q;\ell)$-extended τ-hypergeometric functions and some of their properties have been discussed. In section 4, the solution of the fractional kinetic equations contains the $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions. In section 5, include a conclusion.

2. Preliminaries

In this paper, the extended fractional kinetic equations will be studied by using the following $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions:

Definition 2.1. The new $(p,q;\ell)$-extended τ-Gauss hypergeometric function is

$$R^{\tau;\phi,\varphi}_{p,q}(a, b; c; z; \ell) = \sum_{\Bbbk\geq 0}(a)_{\Bbbk}\frac{B^{\phi,\varphi}_{p,q}(b+\Bbbk\tau, c-b;\ell)}{B(b, c-b)}\frac{z^{\Bbbk}}{\Bbbk!} \tag{4}$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\phi), Re(\varphi)\} > 0, \tau \geq 0, \ell \in \mathbb{R}^{+}\backslash\{1\}, Re(a) > Re(b) > 0.$

Definition 2.2. The new $(p,q;\ell)$-extended $(p,q;\ell)$-confluent hypergeometric function is

$$\Phi^{\tau;\phi,\varphi}_{p,q}(b; c; z; \ell) = \sum_{\Bbbk\geq 0}\frac{B^{\phi,\varphi}_{p,q}(b+\Bbbk\tau, c-b;\ell)}{B(b, c-b)}\frac{z^{\Bbbk}}{\Bbbk!} \tag{5}$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\phi), Re(\varphi)\} > 0, \tau \geq 0, \ell \in \mathbb{R}^{+}\backslash\{1\}, Re(a) > Re(b) > 0,$ and $B^{\phi,\varphi}_{p,q}(\wp, \Im; \ell)$ is the extended beta function proposed in [21]

$$B^{\phi,\varphi}_{p,q}(\wp, \Im; \ell) = \int_0^1 t^{\wp-1}(1-t)^{\Im-1}\ell^{\left(-\frac{p}{t^{\phi}}-\frac{q}{(1-t)^{\varphi}}\right)}dt \tag{6}$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\wp), Re(\Im)\} > 0, \tau \geq 0, \ell \in \mathbb{R}^{+}\backslash\{1\},\ \min\{Re(\wp), Re(\Im)\} > 0.$

3. The $(p, q; \ell)$-Extended τ- Hypergeometric Functions

In this section, the integral representation of the $(p, q; \ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions are established in the following theorem:

Theorem 3.1. The following Laplace-type integral formula holds:

$$R_{p,q}^{\tau;\phi,\varphi}(a, b; c; z; \ell) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp(-t)\, \Phi_{p,q}^{\tau;\phi,\varphi}(b; c; z; \ell) dt$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\phi), Re(\varphi)\} > 0, \tau \geq 0, \ell \in \mathbb{R}^+ \backslash \{1\}, Re(a) > 0, Re(z) < 1.$

Proof. Consider equation (4) and expansion of the pochhammer notation in [22]

$$(a)_{\Bbbk} = \frac{1}{\Gamma(a)} \int_0^\infty t^{a+\Bbbk-1} \exp(-t)\, dt$$

gives

$$R_{p,q}^{\tau;\phi,\varphi}(a, b; c; z; \ell) = \sum_{\Bbbk \geq 0} \left\{ \frac{1}{\Gamma(a)} \int_0^\infty t^{a+\Bbbk-1} \exp(-t)\, dt \right\} \frac{B_{p,q}^{\phi,\varphi}(b + \Bbbk\tau, c - b; \ell)}{B(b, c - b)} \frac{z^{\Bbbk}}{\Bbbk!}$$

As a result of changing the order of integration and summation,

$$\begin{aligned} R_{p,q}^{\tau;\phi,\varphi}(a, b; c; z; \ell) &= \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp(-t) \left\{ \sum_{\Bbbk \geq 0} \frac{B_{p,q}^{\phi,\varphi}(b + \Bbbk\tau, c - b; \ell)}{B(b, c - b)} \frac{(tz)^{\Bbbk}}{\Bbbk!} \right\} dt \\ &= \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp(-t)\, \Phi_{p,q}^{\tau;\phi,\varphi}(b; c; z; \ell) dt \end{aligned}$$

Theorem 3.2. The following Euler-type equality holds:

$$R_{p,q}^{\tau;\phi,\varphi}(a, b; c; z; \ell) = \frac{1}{B(b, c - b)} \int_0^1 t^{a-1}(1 - t)^{c-b-1}(1 - t^\tau z)^{-a} \ell^{\left(-\frac{p}{t^\phi} - \frac{q}{(1-t)^\varphi}\right)} dt$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\phi), Re(\varphi)\} > 0,\ \tau \geq 0,\ \ell \in \mathbb{R}^+ \backslash \{1\},\ Re(b) > Re(b) > 0,$ and $|\arg(1 - z)| < \pi.$

Proof. Rewritten equation (4) in term of $(p, q; \ell)$-extended beta function in (6), yields

$$R_{p,q}^{\tau;\phi,\varphi}(a, b; c; z; \ell) = \sum_{\Bbbk \geq 0} \frac{(a)_{\Bbbk}}{B(b, c - b)} \left\{ \int_0^1 t^{b+\Bbbk\tau-1}(1 - t)^{c-b-1}(1 - t^\tau z)^{-a} \ell^{\left(-\frac{p}{t^\phi} - \frac{q}{(1-t)^\varphi}\right)} dt \right\} \frac{z^{\Bbbk}}{\Bbbk!}$$

Changing the order of integration and summation will result in

$$\begin{aligned} R_{p,q}^{\tau;\phi,\varphi}(a, b; c; z; \ell) &= \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1} \ell^{\left(-\frac{p}{t^\phi} - \frac{q}{(1-t)^\varphi}\right)} \left\{ \sum_{\Bbbk \geq 0} (a)_{\Bbbk} \frac{(tz)^{\Bbbk}}{\Bbbk!} \right\} dt \\ &= \frac{1}{B(b, c - b)} \int_0^1 t^{a-1}(1 - t)^{c-b-1}(1 - t^\tau z)^{-a} \ell^{\left(-\frac{p}{t^\phi} - \frac{q}{(1-t)^\varphi}\right)} dt \end{aligned}$$

Considering equation (5), the following corollary can be obtained:

Corollary 3.1. The following result is also holds true:

$$\Phi_{p,q}^{\tau;\phi,\varphi}(b; c; z; \ell) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1} \exp(t^\tau z)\, \ell^{\left(-\frac{p}{t^\phi} - \frac{q}{(1-t)^\varphi}\right)} dt$$

for all $\min\{Re(p), Re(q)\} > 0,\ \min\{Re(\phi), Re(\varphi)\} > 0,\ \tau \geq 0,\ \ell \in \mathbb{R}^+ \backslash \{1\},\ Re(c) > Re(b) > 0.$

4. Extended Fractional Kinetic Equations Solutions

In this section, the applications of $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions to extended fractional kinetic equations are established using the Laplace transform method in the following theorem:

Theorem 4.1. The extended fractional kinetic equation

$$\Lambda(t)-\Lambda_0 t^{\aleph} R_{p,q}^{\tau;\phi,\varphi}\left(a,b;c;\psi t^{\partial};\ell\right)=-\sigma^{\partial}\ {}_0D_t^{-\partial}\{\Lambda(t)\} \tag{7}$$

for all $\aleph,\partial,\sigma\in\mathbb{R}^+,\psi\in\mathbb{C}$ with $\delta\neq\psi,\tau\in\mathbb{R}_0^+;\min\{Re(p),Re(q)\}>0,\ \min\{Re(\phi),Re(\varphi)\}>0,\ \tau\geq 0$, $\ell\in\mathbb{R}^+\backslash\{1\}$, $Re(c)>Re(b)>0$.

$$\Lambda(t)=\Lambda_0 t^{\aleph-1}\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\left(\psi t^{\partial}\right)^{\mathbb{k}}}{\mathbb{k}!}\Gamma(\partial\mathbb{k}+\aleph)\mathrm{E}_{\partial,\partial\mathbb{k}+\aleph}\left(-\sigma^{\partial}t^{\partial}\right)$$

is the solution.

Proof. Applying the Laplace transform [23] to equation (7), gives

$$\mathcal{L}\{\Lambda(t);s\}-\Lambda_0\mathcal{L}\left\{t^{\aleph}R_{p,q}^{\tau;\phi,\varphi}\left(a,b;c;\psi t^{\partial};\ell\right);s\right\}=-\sigma^{\partial}\mathcal{L}\left\{\ {}_0D_t^{-\partial}\{\Lambda(t)\};s\right\}$$

Consider equation (4) and the Laplace transform of the Riemann-Liouville fractional integral [24]

$$\mathcal{L}\left\{\ {}_0D_t^{-\partial}\{\Lambda(t)\};s\right\}=-s^{\partial}\mathcal{L}\{\Lambda(t)\}$$

yields

$$\mathcal{L}\{\Lambda(t);s\}-\Lambda_0\left[\int_0^{\infty}\exp(-st)\left\{\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\left(\psi t^{\partial}\right)^{\mathbb{k}}}{\mathbb{k}!}\right\}dt\right]=-\sigma^{\partial}s^{\partial}\mathcal{L}\{\Lambda(t)\}$$

When integration and summation are changed, it leads to

$$\mathcal{L}\{\Lambda(t);s\}=\Lambda_0\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\psi^{\mathbb{k}}}{\mathbb{k}!}\left\{\int_0^{\infty}\exp(-st)t^{\partial\mathbb{k}+\aleph-1}dt\right\}\left\{\frac{1}{1+(\sigma s^{-1})^{\partial}}\right\}$$

Using result [25]

$$\int_0^{\infty}\exp(-st)t^{\aleph}dt=\frac{\Gamma(\aleph+1)}{s^{\aleph+1}}\ ,(Re(\aleph)>-1)$$

gives

$$\mathcal{L}\{\Lambda(t);s\}=\Lambda_0\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\psi^{\mathbb{k}}}{\mathbb{k}!}\frac{\Gamma(\partial\mathbb{k}+\aleph)}{s^{\partial\mathbb{k}+\aleph}}\left\{\frac{1}{1+(\sigma s^{-1})^{\partial}}\right\}$$

Applying the geometric series expansion [26]

$$\frac{1}{1+(\sigma s^{-1})^{\partial}}=\sum_{\xi\geq 0}(-1)^{\xi}\sigma^{\sigma\xi}\,s^{-\sigma\xi}$$

leads to

$$\mathcal{L}\{\Lambda(t);s\}\ =\ \Lambda_0\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\psi^{\mathbb{k}}}{\mathbb{k}!}\frac{\Gamma(\partial\mathbb{k}+\aleph)}{s^{\partial\mathbb{k}+\aleph}}\sum_{\xi\geq 0}(-1)^{\xi}\sigma^{\sigma\xi}\,s^{-\sigma\xi}$$

$$= \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \sum_{\xi\geq 0} (-1)^{\xi} \sigma^{\sigma\xi} s^{-(\sigma\xi+\partial\Bbbk+\aleph)}$$

Using the inverse Laplace transform and the result in [25]

$$\mathcal{L}^{-1}\{s^{-\partial}\} = \frac{t^{\partial-1}}{\Gamma(\partial)}$$

one may obtain

$$\Lambda(t) = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \sum_{\xi\geq 0} \frac{(-1)^{\xi}\sigma^{\partial\xi}}{\Gamma(\sigma\xi+\partial\Bbbk+\aleph)} t^{\partial\xi+\partial\Bbbk+\aleph-1}$$

$$= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{(\psi t^{\partial})^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \sum_{\xi\geq 0} \frac{(-\sigma^{\partial}t^{\partial})^{\xi}\sigma^{\sigma\xi}}{\Gamma(\sigma\xi+\partial\Bbbk+\aleph)}$$

$$= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{(\psi t^{\partial})^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}(-\sigma^{\partial}t^{\partial})$$

Theorem 4.2. The extended fractional kinetic equation

$$\Lambda(t) - \Lambda_0 t^{\aleph-1} R_{p,q}^{\tau;\phi,\varphi}(a,b;c;\psi t^{\partial};\ell) = -\left\{\sum_{\omega\geq 1} \binom{\kappa}{\omega} \sigma^{\partial\omega}\, {}_0D_t^{-\partial\omega}\right\} \Lambda(t) \tag{8}$$

for all $\aleph, \partial, \sigma \in \mathbb{R}^+, \psi, \kappa \in \mathbb{C}$ with $\delta \neq \psi, \tau \in \mathbb{R}_0^+$; $\min\{Re(p), Re(q)\} > 0$, $\min\{Re(\phi), Re(\varphi)\} > 0$, $\tau \geq 0$, $\ell \in \mathbb{R}^+\backslash\{1\}$, $Re(c) > Re(b) > 0$.

$$\Lambda(t) = \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{(\psi t^{\partial})^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}^{\kappa}(-\sigma^{\partial}t^{\partial})$$

is the solution.

Proof. Applying the Laplace transform [23] to equation (8), gives

$$\mathcal{L}\{\Lambda(t);s\} - \Lambda_0 \mathcal{L}\left\{t^{\aleph-1} R_{p,q}^{\tau;\phi,\varphi}(a,b;c;\psi t^{\partial};\ell);s\right\} = -\sum_{\omega\geq 1} \binom{\kappa}{\omega} \sigma^{\partial\omega} \mathcal{L}\{{}_0D_t^{-\partial\omega}\Lambda(t);s\}$$

Consider equation (4) and the Laplace transform of the Riemann-Liouville fractional integral [24]

$$\mathcal{L}\{{}_0D_t^{-\partial}\{\Lambda(t)\};s\} = -s^{\partial}\mathcal{L}\{\Lambda(t);s\},$$

yields

$$\mathcal{L}\{\Lambda(t);s\} - \Lambda_0 \left[\int_0^{\infty} \exp(-st) \left\{t^{\aleph} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{(\psi t^{\partial})^{\Bbbk}}{\Bbbk!}\right\} dt\right] = \sum_{\omega\geq 1} \binom{\kappa}{\omega} \sigma^{\partial\omega} s^{\partial} \mathcal{L}\{\Lambda(t);s\}$$

By reordering integral and summation, we get

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \left\{\int_0^{\infty} \exp(-st) t^{\partial\Bbbk+\aleph-1} dt\right\} \left\{\frac{1}{\sum_{\omega\geq 1}\binom{\kappa}{\omega}(\sigma s^{-1})^{\partial\omega}}\right\}$$

Using result [25]

$$\int_0^{\infty} \exp(-st) t^{\aleph} dt = \frac{\Gamma(\aleph+1)}{s^{\aleph+1}}, \ (Re(\aleph) > -1)$$

gives

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \frac{\Gamma(\partial\Bbbk+\aleph)}{s^{\partial\Bbbk+\aleph}} \left\{\frac{1}{\sum_{\omega\geq 1}\binom{\kappa}{\omega}(\sigma s^{-1})^{\partial\omega}}\right\}$$

Applying the geometric series expansion in [27]

$$\sum_{\omega\geq 1}\binom{\kappa}{\omega}\sigma^{\partial} = (1+z)^{\kappa}, (\kappa\in\mathbb{C}, |z|<1)$$

leads to

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \frac{\Gamma(\partial\Bbbk+\aleph)}{s^{\partial\Bbbk+\aleph}} \left(1+\sigma^{\partial}s^{\partial}\right)^{\kappa}$$

Can be rewritten using [27]

$$(1-z)^{\kappa} = \sum_{\omega\geq 0}\frac{(\kappa)_{\omega}}{\omega!}z^{\omega}, (\kappa\in\mathbb{C}, |z|<1)$$

so that

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \frac{\Gamma(\partial\Bbbk+\aleph)}{s^{\partial\Bbbk+\aleph}} \left\{\sum_{\xi\geq 0}\frac{(-1)^{\xi}(\kappa)_{\xi}}{\xi!}\sigma^{\partial\xi}s^{-(\sigma\xi+\partial\Bbbk+\aleph)}\right\}$$

Using the inverse Laplace transform and the result in [25]

$$\mathcal{L}^{-1}\{s^{-\partial}\} = \frac{t^{\partial-1}}{\Gamma(\partial)}$$

The following can be obtained:

$$\begin{aligned}\Lambda(t) &= \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \left\{\sum_{\xi\geq 0}\frac{(-1)^{\xi}\sigma^{\partial\xi}(\kappa)_{\xi}}{\Gamma(\partial\xi+\partial\Bbbk+\aleph)}\frac{t^{\partial\xi+\partial\Bbbk+\aleph-1}}{\xi!}\right\} \\ &= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \left\{\sum_{\xi\geq 0}\frac{(-1)^{\xi}\sigma^{\partial\xi}(\kappa)_{\xi}}{\Gamma(\partial\xi+\partial\Bbbk+\aleph)}\frac{\left(-\sigma^{\partial}t^{\partial}\right)^{\xi}}{\xi!}\right\} \\ &= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}^{\kappa}\left(-\sigma^{\partial}t^{\partial}\right)\end{aligned}$$

Considering equations (5), (7), and (8), the following corollaries can be obtained:

Corollary 4.1. The extended fractional kinetic equation

$$\Lambda(t) - \Lambda_0 t^{\aleph}\Phi_{p,q}^{\tau;\phi,\varphi}\left(b;c;\psi t^{\partial};\ell\right) = -\sigma^{\partial}\ {}_0D_t^{-\partial}\{\Lambda(t)\}$$

for all $\aleph, \partial, \sigma \in \mathbb{R}^+, \psi \in \mathbb{C}$ with $\delta \neq \psi, \tau \in \mathbb{R}_0^+; \min\{Re(p), Re(q)\} > 0$, $\min\{Re(\phi), Re(\varphi)\} > 0$, $\tau \geq 0$, $\ell \in \mathbb{R}^+\backslash\{1\}$, $Re(c) > Re(b) > 0$.

$$\Lambda(t) = \Lambda_0 \sum_{\Bbbk\geq 0} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}\left(-\sigma^{\partial}t^{\partial}\right)$$

is the solution.

Corollary 4.2. The extended fractional kinetic equation

$$\Lambda(t)-\Lambda_0 t^{\aleph-1}\Phi_{p,q}^{\tau;\phi,\varphi}\left(b;c;\psi t^{\partial};\ell\right)=-\left\{\sum_{\omega\geq 1}\binom{\kappa}{\omega}\sigma^{\partial\omega}\ {}_0D_t^{-\partial\omega}\right\}\Lambda(t)$$

for all $\aleph,\partial,\sigma\in\mathbb{R}^+,\psi,\kappa\in\mathbb{C}$ with $\delta\neq\psi,\tau\in\mathbb{R}_0^+$; $\min\{Re(p),Re(q)\}>0$, $\min\{Re(\phi),Re(\varphi)\}>0$, $\tau\geq 0$, $\ell\in\mathbb{R}^+\backslash\{1\}$, $Re(c)>Re(b)>0$.

$$\Lambda(t)=\Lambda_0 t^{\aleph-1}\sum_{\Bbbk\geq 0}\frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau,c-b;\ell)}{B(b,c-b)}\frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!}\Gamma(\partial\Bbbk+\aleph)\mathrm{E}_{\partial,\partial\Bbbk+\aleph}^{\kappa}\left(-\sigma^{\partial}t^{\partial}\right)$$

is the solution.

5. Conclusion

The new $(p,q;\ell)$-extended τ-Gauss hypergeometric and $(p,q;\ell)$-extended τ-confluent hypergeometric functions are defined by using the $(p,q;\ell)$-extended beta function in [21] with some of their properties such as integral formulas and their application to the solutions of extended fractional kinetic equations. If the parameters of these newly established functions and equations are appropriately substituted, a number of works already established in the literature are obtained, for example: if $\ell=e$ and $\phi=\varphi=1$, then the results of Khan et al., [10] and Parmar et al., [11]; by setting $\ell=e,\ \phi=\varphi=1$,and $\tau=1$, the extended Gauss hypergeometric and confluent hypergeometric functions presented by Choi et al., [12] will be obtained; by setting $\ell=e,\ \phi=\varphi=1$ and $p=q=0$, the proposed results will be returned to Virchenko et al., [28] and Virchenko [29]; the substituting $\ell=e,\ \phi=\varphi=1,\tau=1$and $p=q$ leads to the results of Chaudhry et al., [30, 31]; finally, by taking $\ell=e,\ \phi=\varphi=1,\tau=1$, and $p=q=1$, the results under discussion will naturally return to the classical results. The extended kinetic equations are expected to have potential applications in nuclear energy, nuclear physics, astrophysics and other related fields. Furthermore, the functions under discussion can be used to study fractional integrals and derivatives such as the Riemann-Liouville, Caputo, Eydilyi-kober, Saigo, Merichev-Saigo-Maide and the Caputo-type Merichev-Saigo-Maide.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

Reference

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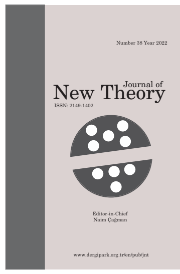

$\mathcal{I}$-Statistical Rough Convergence of Order α

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Abstract — The aim of this paper is to define the concept of $\mathcal{I}$-statistical ($\mathcal{I}$-st) rough convergence of order α $(0 < \alpha \leq 1)$. It proposes the concept of $\mathcal{I}$-st bounded of order α. Moreover, the necessary and sufficient condition for a sequence (x_k) to be $\mathcal{I}$-st bounded of order α is studied. In addition, the necessary and sufficient condition for a sequence (x_k) to be $\mathcal{I}$-st convergent of order α is examined. Finally, the need for further research studies are discussed.

Keywords — *Ideal rough convergence, Ideal statistical rough convergence, Ideal statistical rough convergence of order α.*

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1. Introduction

Statistical convergence is a generalization of the concept of convergence based on the concept of natural density of a subset of $\mathbb{N}$, the set of all natural numbers. This concept has been defined independently by Fast [1] and Steinhaus [2] in 1951. Further, Schoenberg [3] has defined statistical convergence as a summability method. Many mathematicians particularly Salat [4], Freedman and Sember [5], Fridy [6], Connor [7], Kolk [8], and Fridy and Orhan [9, 10], have contributed to the development of statistical convergence.

Let $K \subseteq \mathbb{N}$ and $K(n) = \{k \in K : k \leq n\}$, for all $n \in \mathbb{N}$. In this case, the natural density of the set K is defined as follows:

$$\delta(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$$

such that $|K(n)|$ signifies the number of elements in $K(n)$ [5]. If K is a finite set, its natural density is zero. Let $x = (x_k)$ be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. For every $\varepsilon > 0$, if the natural density of the set $\{k \in \mathbb{N} : ||x_k - x_0|| \geq \varepsilon\}$ is zero, that is

$$\lim_{n\to\infty} \frac{|\{k \leq n : ||x_k - x_0|| \geq \varepsilon\}|}{n}$$

is zero, then the sequence $x = (x_k)$ is said to be statistical convergent to x_0 and is denoted by $st - \lim x = x_0$. Since the natural density of finite sets is zero, every convergence sequence is statistical convergence. Kostryko et al. [11] has introduced the concept of ideal convergence (or briefly $\mathcal{I}$-convergence), a generalization of the concept of statistical convergence. They have defined the concept of $\mathcal{I}$-convergence by the concept of an ideal. They have investigated many properties of $\mathcal{I}$-convergence. Statistical convergence by degree has been first provided by Gadjiev and Orhan [12] as the relationship

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to the statistical convergence of a set of positive linear operators. In 2010, Çolak [13] has generalized the concept of statistical convergence by defining density of order α of a set and defined the concept of the statistical convergence of order α, for sequence of real numbers. After, Savaş and Das [14] has put forward $\mathcal{I}$-statistical ($\mathcal{I}$-st) convergence of order α ($\alpha \in (0, 1]$).

Phu [15] has suggested the concept of rough convergence in finite-dimensional normed spaces and examined between the relation rough convergence and other convergence types. He has also proved that the set of all rough limit points is bounded, closed, and convex. Later on, the concept of rough statistical convergence has been studied by Aytar [16]. Moreover, Aytar [17, 18] has also studied sets of rough statistical limit points and rough statistical cluster points. Pal et al [19] have studied rough ideal convergence and properties of the set of rough $\mathcal{I}$-limit points. Moreover, Dündar and Çakan [20] have also investigated rough ideal convergence. Savaş et al. [21] have defined the concept of $\mathcal{I}$-st rough convergence. Furthermore, the concept of rough statistical convergence of order α has been propounded and studied some properties of the set of all rough statistical limit points of order α by Maity [22].

In the second part of the present study, some basic definitions and properties to be required for the next section are provided. Section 3 proposes the concept of $\mathcal{I}$-st rough convergence of order α such that $0 < \alpha \leq 1$. Additionally, the necessary and sufficient conditions for a sequence (x_k) to be $\mathcal{I}$-st convergent of order α and $\mathcal{I}$-st bounded of order α are proved. Finally, the need for further research studies is discussed.

2. Preliminaries

Throughout this study, a normed space $(X, ||.||)$ will be denoted by X.

Definition 2.1. [15] Let $x = (x_k)$ be a sequence in X and $r \geq 0$. $x = (x_k)$ is said to be rough convergent (r-convergent) to $x_0 \in X$, if for every $\varepsilon > 0$ there exists a $k_\varepsilon \in \mathbb{N}$ such that $k \geq k_\varepsilon$ implies

$$||x_k - x_0|| < r + \varepsilon$$

or equivalently

$$\limsup ||x_k - x_0|| < r$$

and denoted by $x_k \xrightarrow{r} x_0$. Here, r is called the roughness degree of the sequence (x_k). If $r = 0$, then the concept of the r-convergence is equivalent to the concept of the classical convergence. Here, x_0 is called an r-limit point of (x_k) and the set of all r-limit points is denoted by

$$\text{LIM}_r x = \left\{ x_0 \in X : x_k \xrightarrow{r} x_0 \right\}$$

If the set $\text{LIM}_r x$ is non-empty, then the sequence $x = (x_k)$ is r-convergent.

Definition 2.2. [16] Let $x = (x_k)$ be a sequence in X and $r \geq 0$. $x = (x_k)$ is said to be statistical rough convergent to $x_0 \in X$, if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{|\{k \leq n : ||x_k - x_0|| \geq r + \varepsilon\}|}{n} = 0$$

or equivalently

$$st - \limsup ||x_k - x_0|| \leq r$$

and denoted by $x_k \xrightarrow{st-r} x_0$. Thus,

$$st - \text{LIM}_r x = \left\{ x_0 \in X : x_k \xrightarrow{st-r} x_0 \right\}$$

If $r = 0$, then the concept of statistical rough convergence is equivalent to the concept of the statistical convergence. If the set $st - \text{LIM}_r x$ is non-empty, then the sequence $x = (x_k)$ is statistical rough convergent.

Definition 2.3. [11] Let $X \neq \emptyset$ and $\mathcal{I} \subseteq P(X)$. If

i. $\emptyset \in \mathcal{I}$,

ii. $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,

iii. $(A \in \mathcal{I} \wedge B \subseteq A) \Rightarrow B \in \mathcal{I}$,

then $\mathcal{I}$ is called an ideal of X.

Definition 2.4. [11] Let $\mathcal{I}$ be an ideal of X. Then,

i. $\mathcal{I}$ is called non-trivial ideal, if $\mathcal{I} \neq \emptyset$ and $X \notin I$.

ii. A non-trivial ideal $\mathcal{I} \subseteq P(X)$ is called admissible if and only if $\{\{x\} : x \in X\} \subseteq \mathcal{I}$.

From now on, let $\mathcal{I}$ be a non-trivial admissible ideal of X.

Definition 2.5. [11] Let $X \neq \emptyset$ and $\emptyset \neq F \subseteq P(X)$. If

i. $\emptyset \notin F$,

ii. $A, B \in F \Rightarrow A \cap B \in F$,

iii. $(A \in F \wedge B \subseteq A) \Rightarrow B \in F$,

then F is called a filter on X.

Remark 2.6. [11] Let $\mathcal{I} \subseteq P(X)$ be a non-trivial ideal. Then, the family

$$F(\mathcal{I}) = \{M \subset X : M = X \setminus A, \text{ for some } A \in \mathcal{I}\}$$

is a filter on X, is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.7. [19] Let $x = (x_k)$ be a sequence in X and $r \geq 0$. $x = (x_k)$ is said to be ideal rough convergent to $x_0 \in X$ if, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : ||x_k - x_0|| \geq r + \varepsilon\} \in \mathcal{I}$$

or equivalently

$$\mathcal{I} - \limsup ||x_k - x_0|| \leq r$$

and denoted by $x_k \xrightarrow{\mathcal{I}-r} x_0$. Thus,

$$\mathcal{I} - \text{LIM}_r x = \left\{x_0 \in X : x_k \xrightarrow{\mathcal{I}-r} x_0\right\}$$

If $r = 0$, then the concepts of the ideal rough convergence and the $\mathcal{I}$-convergence are equivalent. If the set $\mathcal{I} - \text{LIM}_r x$ is non-empty, then the sequence $x = (x_k)$ is $\mathcal{I} - r$ convergent.

Definition 2.8. [23] Let $x = (x_k)$ be a sequence in X. $x = (x_k)$ is said to be $\mathcal{I}$-statistical convergent to $x_0 \in X$, if, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : ||x_k - x_0|| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$$

and denoted by $\mathcal{I} - st - \text{LIM}x = x_0$ or $x_k \xrightarrow{\mathcal{I}-st} x_0$. If $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbb{N} : A \text{ is a finite set}\}$, then $\mathcal{I}$-statistical convergence is the same as the classical convergence. If $\mathcal{I} = \mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, then $\mathcal{I}$-statistical convergence is the same as the statistical convergence.

Definition 2.9. [17] Let $x = (x_k)$ be a sequence in X. $x = (x_k)$ is said to be $\mathcal{I}$-statistical convergent of order $\alpha \in (0, 1]$ to $x_0 \in X$, if, for any $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - x_0|| \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}$ and denoted by $\mathcal{I} - st - \text{LIM}^\alpha x = x_0$ or $x_k \xrightarrow{\mathcal{I}-st\alpha} x_0$.

Definition 2.10. [19] Let $x = (x_k)$ be a sequence in X and $r \geq 0$. $x = (x_k)$ is said to be $\mathcal{I}$ statistical rough convergent ($\mathcal{I}$-st rough convergent) to $x_0 \in X$ if, for every $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - x_0\| \geq r + \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$$

and denoted by $\mathcal{I} - st - \text{LIM}_r x = x_0$ or $x_k \xrightarrow{\mathcal{I}-st-r} x_0$.

Definition 2.11. [24] Let $x = (x_k)$ be a sequence in X. If there exist a real number M such that $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : ||x_k|| > M\}| > \delta\} \in \mathcal{I}$, then (x_k) is called $\mathcal{I}$-st bounded.

Definition 2.12. [21] Let $x = (x_k)$ be a sequence in X and $c \in X$. If, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : ||x_k - c|| \geq \varepsilon\}| < \delta\right\} \notin \mathcal{I}$$

then c is called an $\mathcal{I}$-statistical cluster point of $x = (x_k)$.

The set of all $\mathcal{I}$-statistical ($\mathcal{I} - st$) cluster points denoted by $\mathcal{I} - S(\Gamma_x)$.

Theorem 2.13. [21] Let $x = (x_k)$ be an $\mathcal{I} - st$ bounded sequence. If the sequence $x = (x_k)$ has one cluster point, then it is $\mathcal{I} - st$ convergent.

3. Main Results

This section defines the concept of ideal statistical rough convergence of order α and studies some of its basic properties.

Definition 3.1. [22] Let $x = (x_k)$ be a sequence in X, $r \geq 0$, and $0 < \alpha \leq 1$. $x = (x_k)$. For all $\varepsilon > 0$, if

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n : ||x_k - x_0|| \geq r + \varepsilon\}| = 0$$

then the sequence $x = (x_k)$ is said to be statistical rough convergence of order α to x_0 and denoted by $x_k \xrightarrow{st-r\alpha} x_0$. Then, the set of all statistical rough of order α limit points of a sequence (x_k) is denoted by $st - \text{LIM}_r^\alpha x$.

Definition 3.2. Let $x = (x_k)$ be a sequence in X. For any $\varepsilon > 0$ and $\delta > 0$, if

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - x_0|| \geq r + \varepsilon\}| \geq \delta\right\} \in \mathcal{I}, \quad \alpha \in (0, 1]$$

then $x = (x_k)$ is called to be ideal statistical rough convergent of order α to x_0 and denoted by $x_k \xrightarrow{\mathcal{I}-st-r\alpha} x_0$. Then, the set of all ideal statistical rough of order α limit points is denoted by $\mathcal{I} - st - \text{LIM}_r^\alpha x$.

Definition 3.3. Let $x = (x_k)$ be a sequence in X. Then, $x = (x_k)$ is called $\mathcal{I}$-statistical bounded of order α, if there exists a real number H such that

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k|| > H\}| > \delta\right\} \in \mathcal{I}$$

Limit of a convergent sequence is unique, however limit of a rough convergent sequence is not need to be unique, for the degree of roughness $r > 0$.

Theorem 3.4. Let $x = (x_k)$ be a sequence in X. x is $\mathcal{I}$-st bounded of order α if and only if there exists $r \geq 0$ such that $\mathcal{I} - st - \text{LIM}_r^\alpha x \neq \emptyset$.

PROOF. Let X be a normed space and $x = (x_k)$ be a sequence in X.
($\Rightarrow$) Assume that $x = (x_k)$ be $\mathcal{I}$-st bounded sequence. Then, there exists a real number H such that

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : ||x_k|| > H\}| > \delta \right\} \in \mathcal{I}$$

Let $\bar{r} = \sup_{\substack{t \leq m \\ m \in \mathbb{N} \setminus A}} \{||x_t||\}$. $\mathcal{I} - st - \text{LIM}_{\bar{r}}^\alpha x$ includes the origin of X $\mathcal{I} - st - \text{LIM}_{\bar{r}}^\alpha x \neq \emptyset$. Here, since the normed space X is a vector space, then the origin of X is θ_X.
($\Leftarrow$) Suppose that for an arbitrary $r \geq 0$, $\mathcal{I} - st - \text{LIM}_r^\alpha x \neq \emptyset$. Then, for any $\varepsilon > 0$ and $\delta > 0$, there exists $x_0 \in \mathcal{I} - st - \text{LIM}_r^\alpha x$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : ||x_k - x_0|| \geq r + \varepsilon\}| \geq \delta \right\} \in \mathcal{I}$$

Choose $\varepsilon = \|x_0\|$. Then, for each $\delta > 0$ and $H = r + \|x_0\|$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : \|x_k - x_0\| \geq H\}| \geq \delta \right\} \in \mathcal{I}$$

Therefore, $x = (x_k)$ is an $\mathcal{I}-$ st bounded sequence of order α. □

Theorem 3.5. Let $x = (x_k)$ be a sequence in X. The set $\mathcal{I} - st - \text{LIM}_r^\alpha x$ is closed.

PROOF. If $\mathcal{I} - st - \text{LIM}_r^\alpha x = \emptyset$, then the proof is clear. Let $\mathcal{I} - st - \text{LIM}_r^\alpha x \neq \emptyset$ and there exists $(y_n) \subseteq \mathcal{I} - st - \text{LIM}_r^\alpha x$ such that $y_n \to x_0$ as $n \to \infty$. Then, for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $n > n_\varepsilon \Rightarrow ||y_n - x_0|| < \varepsilon$. Choose $n_0 \in \mathbb{N}$, $y_{n_0} \in (y_n) \subseteq \mathcal{I} - st - \text{LIM}_r^\alpha x$, then

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : ||x_k - y_{n_0}|| \geq r + \frac{\varepsilon}{2}\}| < \delta \right\} \in F(\mathcal{I})$$

For $t \in A$, $\frac{1}{t^\alpha} |\{k \leq t : ||x_k - y_{n_0}|| \geq r + \frac{\varepsilon}{2}\}| < \delta$. For maximum $k \leq t$, $||x_k - y_{n_0}|| < r + \frac{\varepsilon}{2}$ and for an arbitrary $n_0 > n_\varepsilon$

$$||x_k - x_0|| \leq ||x_k - y_{n_0}|| + ||y_{n_0} - x_0|| < r + \varepsilon$$

and for maximum $k \leq t \in A$

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : ||x_k - x_0|| \geq r + \varepsilon\}| < \delta \right\} \in \mathcal{I}$$

Hence, $x_0 \in \mathcal{I} - st - \text{LIM}_r^\alpha x$. Consequently, $\mathcal{I} - st - \text{LIM}_r^\alpha x$ is a closed set. □

Theorem 3.6. Let $x = (x_k)$ be a sequence in X. The set $\mathcal{I} - st - \text{LIM}_r^\alpha x$ is convex.

PROOF. Let $y_0, y_1 \in \mathcal{I} - st - \text{LIM}_r^\alpha x$. For any $\varepsilon > 0$ and $\delta > 0$,

$$A_1 = \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : ||x_k - y_0|| \geq r + \varepsilon\}| \geq \delta \right\} \in \mathcal{I}$$

$$A_2 = \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : ||x_k - y_1|| \geq r + \varepsilon\}| \geq \delta \right\} \in \mathcal{I}$$

As $S = \mathbb{N} \setminus (A_1 \cup A_2) \in F(\mathcal{I})$, then S is a finite set. For $s \in S$, let $B_1 = \{k \leq s : ||x_k - y_0|| \geq r + \varepsilon\}$ and $B_2 = \{k \leq s : ||x_k - y_1|| \geq r + \varepsilon\}$. Thus, $\lim_{n \to \infty} \frac{1}{n^\alpha} |B_1| = \lim_{n \to \infty} \frac{1}{n^\alpha} |B_2| = 0$. For all $k \in B_1^c \cap B_2^c$ and $\lambda \in [0, 1]$,

$$||x_k - ((1 - \lambda)y_0 + \lambda y_1)|| = ||(1 - \lambda)(x_k - y_0) + \lambda(x_k - y_1)|| < r + \varepsilon$$

and so $\lim_{k\to\infty} \frac{1}{k^\alpha}|B_1^c \cap B_2^c| = 1$. Thereby,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - ((1-\lambda)y_0 + \lambda y_1)|| \geq r + \varepsilon\}| < \delta\right\} \supseteq S \in F(\mathcal{I})$$

Consequently, $\mathcal{I} - st - \text{LIM}_r^\alpha x$ is convex. □

Theorem 3.7. For an arbitrary $b \in \mathcal{I} - S(\Gamma_x)$. Then, for all $x_* \in \mathcal{I} - st - \text{LIM}_r^\alpha x$, $||x_* - b|| \leq r$.

Proof. Assume that $b \in \mathcal{I} - S(\Gamma_x)$ and $x_* \in \mathcal{I} - st - \text{LIM}_r^\alpha x$ such that $||x_* - b|| > r$. Let $\varepsilon = \frac{||x_* - b|| - r}{2}$. Therefore,

$$A = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - b|| \geq \varepsilon\}| < \delta\right\} \notin \mathcal{I}$$

Let

$$B = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - x_*|| \geq r + \varepsilon\}| \geq \delta\right\}$$

For $t \in A$, $\frac{1}{m^\alpha}|\{k \leq t : ||x_k - b|| \geq \varepsilon\}| < \delta$. So, for maximum $k \leq t, ||x_k - b|| < \varepsilon$. Hence, for all $k \leq t \in A$,

$$||x_k - x_*|| \geq ||x_* - b|| - ||x_k - b|| > r + \varepsilon$$

Therefore, $B \supseteq A$ implies that B is not an element of $\mathcal{I}$ and this is a contradiction. Thus, $||x_* - b|| \leq r$, for all $x_* \in \mathcal{I} - st - \text{LIM}_r^\alpha x$ and $b \in \mathcal{I} - S(\Gamma_x)$. □

Theorem 3.8. Let $r \geq 0$. $x = (x_k)$ is $\mathcal{I}$-statistical rough convergent of order α to x_* if and only if there exists a sequence $y = (y_k)$ such that $\mathcal{I} - st - \text{LIM}^\alpha y = x_*$ and for all $k \in \mathbb{N}$, $||x_k - y_k|| \leq r$.

Proof. ($\Leftarrow$) Assume that $\mathcal{I} - st - \text{LIM}^\alpha y = x_*$ and for all $k \in \mathbb{N}$, $||x_k - y_k|| \leq r$. For any $\varepsilon > 0$ and $\delta > 0$

$$A = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||y_k - x_*|| \geq \varepsilon\}| < \delta\right\} \in \mathcal{I}$$

Moreover,

$$||x_k - x_*|| \leq ||x_k - y_k|| + ||y_k - x_*|| < r + \varepsilon, \quad k \leq s \in A^c$$

Therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - x_*|| \geq r + \varepsilon\}| < \delta\right\} \supset A^c \in F(\mathcal{I})$$

Then, $x = (x_k)$ is $\mathcal{I}$-statistical rough convergent of order α to x_*
($\Rightarrow$) Suppose that $\mathcal{I} - st - \text{LIM}^\alpha x_k = x_*$. For any $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||x_k - x_*|| \geq \varepsilon\}| > \delta\right\} \in \mathcal{I}$$

For the sequence $y = (y_k)$ defined by

$$y_k = \begin{cases} x_*, & \text{if } ||x_k - x_*|| \leq r \\ x_k + r\frac{x_* - x_k}{||x_k - x_*||}, & \text{otherwise} \end{cases}$$

the following inequalities, for $k \in \mathbb{N}$,

$$||y_k - x_*|| \leq \begin{cases} 0, & \text{if } ||x_k - x_*|| \leq r \\ ||x_k - x_*|| + r, & \text{if otherwise} \end{cases}$$

and

$$||x_k - y_k|| \leq r$$

are hold. Thereby,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : ||y_k - x_*|| \geq r + \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$$

Consequently, $\mathcal{I} - st - \text{LIM}^\alpha y = x_*$ and for all $k \in \mathbb{N}$, $||x_k - y_k|| \leq r$. □

4. Conclusion

In this study, ideal statistical rough convergence of order α was defined. Moreover, some important properties of the set of all ideal statistical rough of order α limit points were studied. In addition, this study proved two theorems that "a sequence (x_k) in a normed space is $\mathcal{I}$-st bounded of order α if and only if there exists $r \geq 0$ such that $\mathcal{I} - st - \text{LIM}_r^\alpha x \neq \emptyset$" and "a sequence (x_k) in a normed space is $\mathcal{I}$-statistical rough convergent of order α to x_* if and only if there exists a sequence $y = (y_k)$ such that $\mathcal{I} - st - \text{LIM}^\alpha y = x_*$ and for all $k \in \mathbb{N}$, $||x_k - y_k|| \leq r$".

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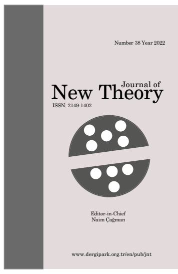

Fixed Soft Points on Parametric Soft Metric Spaces

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Abstract − This manuscript is devoted to investigating the existence of fixed soft points under conditions in the parametric soft metric spaces. Since the parametric soft metric spaces are the parametric expansions of the parametric metric and soft metric spaces, the observations of the fixed-point results are meaningful to consider in such spaces.

Keywords *– Soft set, parametric soft metric, self-soft mapping, fixed-point theory.*

Mathematics Subject Classification (2020) – 54E35, 54H25

1. Introduction

Investigations on indicating the existence and uniqueness of fixed points of self-mappings have several applications in mathematics, economics, engineering, and statistics. In the mathematical aspect, fixed point theory is worth investigating by its applicability in various problems that consist of differential and integral equations, approximations, games, and so on. For these reasons, to determine the existence and uniqueness of (common) fixed points and coincident fixed points in different types of metric spaces, the researchers working in the different branches of mathematics pay attention.

The notion of a parametric metric space is defined by Hussain et al. [1]. According to this definition, the distance between the points of the space takes values according to the parameters. Since the measurement tool depends on the parameters in these spaces, it can be thought as the parameterized extension of the classical metric. This idea has taken attention by several authors and applied to the weak and strong forms of the metric spaces. Rao et al. [2] presented parametric S-metric spaces and proved common fixed-point theorems in parametric S-metric spaces. Later, Çetkin [3] introduced the concept of parametric 2-metric spaces and investigated some of their characteristics and fixed-point results. Different versions of parametric metric spaces and investigations on fixed points of the proposed spaces have been considered by several authors [4-8]. Besides, the notion of soft metric spaces introduced by Das and Samanta [9] is one of the generalizations of metric spaces based on the parameterization tool. In fact, soft metric is a crisp distance function which measures the distance between two soft points in any soft universe. That is, the parameterization tool is used just for the points of the soft universe. Nowadays, research on soft metric spaces and their fixed-point theorems is prevalent. By expanding the role of the parameterization tool in the parametric metric spaces and soft metric spaces, Tunçay and Çetkin [10] defined the concept of a parametric soft metric space and observed the basic

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features of these spaces. Hence, by using the parametric soft metric, one measures the distance between two soft points according to some parameters chosen in the soft universe.

This study aims to show the existence and uniqueness of fixed soft points and common fixed soft points of self-soft mappings described on a (complete) parametric soft metric space. Hence, by providing the parametrized distance between two soft points, where the parameters are soft real numbers, we observe extensions of some fixed-point results in the proposed spaces. This study is arranged in the following manner. In section 2, we recall some basic notions and notations necessary for the main sections. In Section 3, we discuss the existence and uniqueness of fixed soft points in parametric soft metric spaces. In section 4, we show under what conditions two self-soft mappings have common fixed soft points.

2. Preliminaries

This section mentions the concepts and the findings that we need to understand in this manuscript.

Definition 2.1. [11] Let X denote the nonempty universal set, and E represent the set of parameters. Then, the mapping $F: E \to \mathcal{P}(X)$ is called a soft set over the universe X, denoted by the pair (F, E). The collection of all soft sets over the universe X, with the set of parameters E, is represented by $SS(X, E)$.

Definition 2.2.[12] Let (F, E) and (G, E) be two soft sets over X. Then, the set operations for soft sets are defined as follows:

(1) (F, E) is a soft subset of (G, E) and write $(F, E) \sqsubseteq (G, E)$ if $F(e) \subseteq G(e)$, for each $e \in E$.

(2) the union of (F, E) and (G, E) is a soft set $(H, E) = (F, E) \sqcup (G, E)$, where $H(e) = F(e) \cup G(e)$, for all $e \in E$.

(3) the intersection of (F, E) and (G, E) is a soft set $(K, E) = (F, E) \sqcap (G, E)$, where $H(e) = F(e) \cap G(e)$, for all $e \in E$.

(4) (F, E) is called an absolute soft set denoted by $\tilde{X}$, if $F(e) = X$, for all $e \in E$.

(5) (F, E) is called a null soft set denoted by $\tilde{\emptyset}$, if $F(e) = \emptyset$, for all $e \in E$.

Definition 2.3. [13] A soft set (F, E) is called a soft real number if the mapping F is a parameterized family of nonempty bounded subsets of the real line, i.e., $F: E \to \mathcal{B}(\mathbb{R})$. For simplicity, soft real numbers are denoted by the symbols $\tilde{r}, \tilde{s}, \tilde{t}$ and the constant soft real numbers are denoted by $\bar{r}, \bar{s}, \bar{t}$. For instance, $\bar{0}$ represents the zero soft real number which means that $\bar{0}(e) = 0$, for all $e \in E$. Moreover, $\mathbb{R}(E)^*$ denotes the collection of non-negative soft real numbers.

Definition 2.4. [13] The pair $(\mathbb{R}(E)^*, \leq)$ is a partially ordered set. Here, the order " $\leq$ " is the natural order of reals over the parameters.

Definition 2.5. [9]

(1) A soft point over X is a soft set (P, E) if there is exactly one $\lambda \in E$ such that

$$P: E \to \mathcal{P}(X);\ P(e) = \begin{cases} \{x\}, & if\ e = \lambda \\ \emptyset, & if\ e \neq \lambda \end{cases}$$

Therefore, it is indicated by the symbol P_λ^x.

(2) If $P(\lambda) = \{x\} \subseteq F(\lambda)$, then $P_\lambda^x \tilde{\in} (F, E)$.

(3) $P_\lambda^x = P_\mu^y$ if and only if $\lambda = \mu$ and $x = y$. Thus, $P_\lambda^x \neq P_\mu^y$ if and only if $\lambda \neq \mu$ or $x \neq y$.

The notation $SP(\tilde{X})$ indicates the family of all soft points of the universe $\tilde{X}$.

Definition 2.6. [13] Let $\mathcal{S}$ denote a family of soft points. Then, the induced soft set by taking off all collection elements is symbolized by $SS(\mathcal{S})$. Besides, the notation $SP(F,E)$ represents the family of all soft points of the soft set (F,E).

Definition 2.7. [14] Let $(F,E_1) \in SS(X,E_1)$ and $(G,E_2) \in SS(Y,E_2)$. Then, the pair $(\varphi,\psi) \coloneqq \varphi_\psi: SS(X,E_1) \to SS(Y,E_2)$ is called a soft mapping. Here, $\varphi: X \to Y$ and $\psi: E_1 \to E_2$ are the crisp functions. In this case, the image and the preimage of the soft sets (F,E_1) and (G,E_2) under the soft mapping φ_ψ are also soft sets which are defined as follows, respectively.

$$\varphi_\psi\big((\mathrm{F},\mathrm{E}_1)\big)(\mathrm{k}) = \bigcup_{e\in\psi^{-1}(k)} \varphi(F(e))\,, \forall\, k \in E_2$$

and

$$\varphi_\psi^{-1}\big((\mathrm{G},\mathrm{E}_2)\big)(\mathrm{e}) = \varphi^{-1}\Big(G\big(\psi(e)\big)\Big)\,, \forall\, e \in E_1$$

If φ and ψ are both injective (surjective), then the soft mapping φ_ψ is said to be injective (surjective).

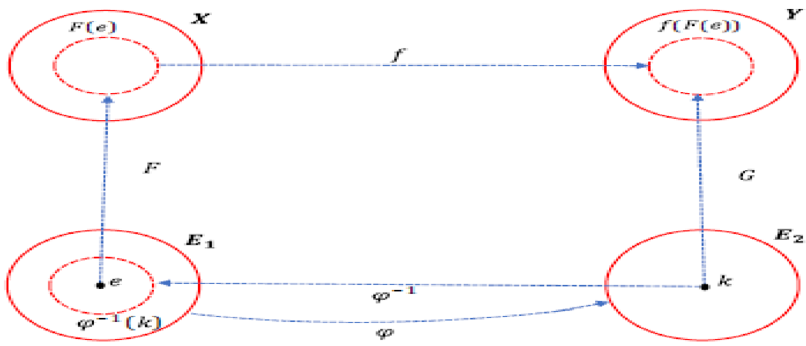

Figure 1. Graphical representation of a soft mapping

Definition 2.8. [10] A parametric soft metric on $\tilde{X}$ is a mapping $d: SP(\tilde{X}) \times SP(\tilde{X}) \times \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ which satisfies the following axioms.

(P1) $d(P_\lambda^x, P_\mu^y, \bar{t}) = \bar{0}$, for all $\bar{t} \tilde{>} \bar{0}$ if and only if $P_\lambda^x = P_\mu^y$.

(P2) $d(P_\lambda^x, P_\mu^y, \bar{t}) = d(P_\mu^y, P_\lambda^x, \bar{t})$, for all $\bar{t} \tilde{>} \bar{0}$.

(P3) $d(P_\lambda^x, P_\mu^y, \bar{t}) \tilde{\leq} d(P_\lambda^x, P_\gamma^z, \bar{t}) + d(P_\gamma^z, P_\mu^y, \bar{t})$, for all $P_\lambda^x, P_\mu^y, P_\gamma^z \tilde{\in} SP(\tilde{X})$ and all $\bar{t} \tilde{>} \bar{0}$.

In this case, $\tilde{X}$ is said to be a parametric soft metric space and represented by the pair $(\tilde{X}, d)$.

If the parameter set is one-pointed, then this definition turns to the original parametric metric definition of Hussain et al. [1]. Besides, in a case when one considers the interval $(0,\infty)$ instead of $\mathbb{R}(E)^*$, then Definition 2.8 coincides with the definition of Bhardwaj et al. [8]. If one also considers the parameter set is one-pointed for the parameters only, then Definition 2.8 coincides with the soft metric definition of Das and Samanta [9]. Hence, our definition can be thought as the parametric extension of the parametric metric, soft metric and soft parametric metric, since both of the points and the parameters of the distance function are all have softness.

Example 2.9.[10] Define a mapping $d: SP(\tilde{X}) \times SP(\tilde{X}) \times \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ by follows: for all $\bar{t} \tilde{>} \bar{0}$

$$d(P_\lambda^x, P_\mu^y, \bar{t}) = \begin{cases} \bar{1}, & P_\lambda^x \neq P_\mu^y \\ \bar{0}, & P_\lambda^x = P_\mu^y \end{cases}$$

Therefore, d is a parametric soft metric over $\tilde{X}$.

Example 2.10. [10] Let $\mathbb{R}$ be the set of all reals and define a mapping $d: SP(\widetilde{\mathbb{R}}) \times SP(\widetilde{\mathbb{R}}) \times \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ by follows: for each $\bar{t} > \bar{0}$,

$$d\,(P_\lambda^x, P_\mu^y, \bar{t}) = \bar{t}\big[|\bar{x} - \bar{y}| + |\bar{\lambda} - \bar{\mu}|\big]$$

Then, d is a parametric soft metric over $\widetilde{\mathbb{R}}$.

Definition 2.11.[10] Let $(\tilde{X}, d)$ be a parametric soft metric space.

(1) If $\lim_{n\to\infty} d(P_{\lambda_n}^{x_n}, P_\lambda^x, \bar{t}) = \bar{0}$, for all $\bar{t} \gtrsim \bar{0}$, then the sequence of soft points $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ is convergent to a soft point $P_\lambda^x \tilde{\in} SP(\tilde{X})$. This is denoted by $P_{\lambda_n}^{x_n} \to P_\lambda^x$ or $\lim_{n\to\infty} P_{\lambda_n}^{x_n} = P_\lambda^x$.

(2) If $\lim_{n,m\to\infty} d(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}) = \bar{0}$, for each $\bar{t} \gtrsim \bar{0}$, then the sequence of soft points $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ satisfies the Cauchy property.

(3) If each sequence of soft points that satisfies the Cauchy property is convergent to some point in the given space, then the space is called complete.

Definition 2.12. [10] Let $\varphi_\psi: (\tilde{X}, d_1) \to (\tilde{Y}, d_2)$ be a soft mapping between parametric soft metric spaces. Then, the continuity of φ_ψ at P_λ^x in $\tilde{X}$, is described sequentially in the following manner:

if for any sequence $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ such that $\lim_{n\to\infty} P_{\lambda_n}^{x_n} = P_\lambda^x$, then $\lim_{n\to\infty} \varphi_\psi\,(P_{\lambda_n}^{x_n}) = \varphi_\psi(P_\lambda^x)$.

3. Main Results

This section is devoted to investigating the existence and the uniqueness of the (common) fixed soft points of self-soft mappings in the parametric soft metric spaces.

Lemma 3.1. Let $(\tilde{X}, d)$ be a parametric soft metric space. Then, the sequence of soft points $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ satisfies the Cauchy property if the following equality is satisfied for all $\bar{k} \tilde{\in} [\bar{0}, \bar{1})$ and $n \in \mathbb{N}$.

$$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) \lesseqqgtr \bar{k}.\, d\left(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}\right) \tag{1}$$

Proof. Let $m > n \geq 1$ be chosen. Then, it implies that the following for all $\bar{t} \gtrsim \bar{0}$,

$$\begin{aligned} d\left(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}\right) &\lesseqqgtr d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) + d\left(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_{n+2}}^{x_{n+2}}, \bar{t}\right) + \cdots + d\left(P_{\lambda_{m-1}}^{x_{m-1}}, P_{\lambda_m}^{x_m}, \bar{t}\right) \\ &\lesseqqgtr \left(\bar{k}^n + \bar{k}^{n+1} + \cdots + \bar{k}^{m-1}\right) d\left(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}\right) \\ &\lesseqqgtr \frac{\bar{k}^n}{\bar{1} - \bar{k}}\, d\left(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}\right) \end{aligned} \tag{2}$$

Assume that $d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}) \gtrsim \bar{0}$. Since $\bar{k} \lesssim \bar{1}$, if one taken limit as $m, n \to +\infty$ in the previous inequality, then the following is gained

$$\lim_{n,m\to\infty} d(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}) = \bar{0} \tag{3}$$

As a result, $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ satisfies the Cauchy property. Also, in case $d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}) = \bar{0}$, we have $d\left(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}\right) = \bar{0}$, for all $m > n$ and hence the result is clear.

Theorem 3.2. Let $(\tilde{X}, d)$ be a complete parametric soft metric space and $\varphi_\psi : (\tilde{X}, d) \to (\tilde{X}, d)$ be a surjective soft self-mapping. If there exist non-negative soft real numbers such that $\bar{\alpha} + \bar{\beta} + \bar{\gamma} \tilde{>} \bar{1}$ satisfying the following for all $P_\lambda^x, P_\mu^y \tilde{\in} SP(\tilde{X})$ and for all $\bar{t} \tilde{>} \bar{0}$.

$$d\left(\varphi_\psi (P_\lambda^x), \varphi_\psi \left(P_\mu^y\right), \bar{t}\right) \tilde{\geq} \bar{\alpha}\, d\left(P_\lambda^x, P_\mu^y, \bar{t}\right) + \bar{\beta}\, d\left(P_\lambda^x, \varphi_\psi (P_\lambda^x), \bar{t}\right) + \bar{\gamma}\, d\left(P_\mu^y, \varphi_\psi \left(P_\mu^y\right), \bar{t}\right) \quad (4)$$

then there exists a fixed soft point of φ_ψ .

Proof. By the assumptions, it is evident that φ_ψ is an injective soft mapping. Let δ_ρ denote the inverse mapping of φ_ψ , for simplicity. Choose $P_{\lambda_0}^{x_o} \tilde{\in} SP(\tilde{X})$, and set a sequence as follows:

$$P_{\lambda_1}^{x_1} = \delta_\rho \left(P_{\lambda_0}^{x_0}\right),\ P_{\lambda_2}^{x_2} = \delta_\rho \left(P_{\lambda_1}^{x_1}\right) = \delta_\rho^2 \left(P_{\lambda_0}^{x_0}\right), \cdots, P_{\lambda_{n+1}}^{x_{n+1}} = \delta_\rho \left(P_{\lambda_n}^{x_n}\right) = \delta_\rho{}^{n+1} \left(P_{\lambda_0}^{x_0}\right) \cdots$$

Let us choose $P_{\lambda_{n-1}}^{x_{n-1}} \neq P_{\lambda_n}^{x_n}$ for all positive integers (otherwise, if there exists some $P_{\lambda_{n_0}}^{x_{n_0}}$ such that $P_{\lambda_{n_0-1}}^{x_{n_0-1}} = P_{\lambda_{n_0}}^{x_{n_0}}$, then $P_{\lambda_{n_0}}^{x_{n_0}}$ is a fixed point of φ_ψ . By the condition (4), we gain the following inequalities:

$$\begin{aligned} d\left(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}\right) &= d\left(\varphi_\psi \left(\varphi_\psi{}^{-1} \left(P_{\lambda_{n-1}}^{x_{n-1}}\right)\right), \varphi_\psi \left(\varphi_\psi{}^{-1} \left(P_{\lambda_n}^{x_n}\right)\right), \bar{t}\right) \\ &\tilde{\geq} \bar{\alpha} d\left(\varphi_\psi{}^{-1} \left(P_{\lambda_{n-1}}^{x_{n-1}}\right), \varphi_\psi^{-1} \left(P_{\lambda_n}^{x_n}\right), \bar{t}\right) + \bar{\beta}\, d\left(\varphi_\psi{}^{-1} \left(P_{\lambda_{n-1}}^{x_{n-1}}\right), \varphi_\psi \left(\varphi_\psi{}^{-1} \left(P_{\lambda_{n-1}}^{x_{n-1}}\right)\right), \bar{t}\right) \\ &+ \gamma d\left(\varphi_\psi^{-1}(P_{\lambda_n}^{x_n}), \varphi_\psi \left(\varphi_\psi{}^{-1}(P_{\lambda_n}^{x_n})\right), \bar{t}\right) \\ &= \bar{\alpha} d(\delta_\rho(P_{\lambda_{n-1}}^{x_{n-1}}), \delta_\rho(P_{\lambda_n}^{x_n}), \bar{t}) + \bar{\beta} d(\delta_\rho(P_{\lambda_{n-1}}^{x_{n-1}}), P_{\lambda_{n-1}}^{x_{n-1}}, \bar{t}) + \bar{\gamma} d(\delta_\rho(P_{\lambda_n}^{x_n}), P_{\lambda_n}^{x_n}, \bar{t}) \\ &= \bar{\alpha} d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) + \bar{\beta} d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n-1}}^{x_{n-1}}, \bar{t}\right) + \bar{\gamma} d\left(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_n}^{x_n}, \bar{t}\right) \end{aligned} \quad (5)$$

By arranging the right side of the previous inequality, it is obtained that

$$(\bar{1} - \bar{\beta})\, d(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}) \ \tilde{\geq}\ (\bar{\alpha} + \bar{\gamma})\, d(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_n}^{x_n}, \bar{t}) \quad (6)$$

If $\bar{\alpha} + \bar{\gamma} = \bar{0}$, then $\bar{\beta} \tilde{>} \bar{1}$. This fact contradicts with the inequality (6). Thus, $\bar{\alpha} + \bar{\gamma}$ must be non-negative and $(\bar{1} - \bar{\beta}) \tilde{>} \bar{0}$.

This implies the following result in the case $\bar{k} = \frac{\bar{1} - \bar{\beta}}{\bar{\alpha} + \bar{\gamma}} \tilde{<} \bar{1}$ and for all $n \in \mathbb{N} \cup \{0\}$

$$d(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_n}^{x_n}, \bar{t}) \ \tilde{\leq}\ \bar{k} d(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}) \quad (7)$$

By repeating (7) n-times, we obtain the following inequality for all $\bar{t} \tilde{>} \bar{0}$,

$$d(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_n}^{x_n}, \bar{t}) \ \tilde{\leq}\ \bar{k}^n d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}) \quad (8)$$

By Lemma 3.1, $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ satisfies the Cauchy property.

So, by the hypothesis, $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ converges to some soft point $P_\rho^\varpi \tilde{\in} SP(\tilde{X})$. Now since φ_ψ is surjective, we may write $P_\rho^\varpi = \varphi_\psi (P_\mu^y)$ for some $P_\mu^y \in SP(\tilde{X})$. Taking into consideration, we obtain that

$$
\begin{aligned}
d\left(P_{\lambda_n}^{x_n}, P_\rho^{\varpi}, \bar{t}\right) &= d\left(\varphi_\psi\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \varphi_\psi\left(P_\mu^y\right), \bar{t}\right) \\
&\widetilde{\leq} \bar{\alpha}\, d(P_{\lambda_{n+1}}^{x_{n+1}}, P_\mu^y, \bar{t}) + \bar{\beta}\, d(P_{\lambda_{n+1}}^{x_{n+1}}, \varphi_\psi(P_{\lambda_{n+1}}^{x_{n+1}}), \bar{t}) + \bar{\gamma}\, d(P_\mu^y, \varphi_\psi(P_\mu^y), \bar{t}) \qquad (9) \\
&= \bar{\alpha} d(P_{\lambda_{n+1}}^{x_{n+1}}, P_\mu^y, \bar{t}) + \bar{\beta}\, d(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_n}^{x_n}, \bar{t}) + \bar{\gamma}\, d(P_\mu^y, P_\rho^{\varpi}, \bar{t})
\end{aligned}
$$

This inequality witnesses the following result in case n tends to infinity

$$\bar{0} \geq (\bar{\alpha} + \bar{\gamma})\, \mathrm{d}(P_\mu^y, P_\rho^{\varpi}, \bar{t}) \qquad (10)$$

As a result, $P_\mu^y = P_\rho^{\varpi}$ is obtained as claimed.

Corollary 3.3. Let $(\tilde{X}, d)$ be a complete parametric soft metric space and $\varphi_\psi : (\tilde{X}, \mathrm{d}) \to (\tilde{X}, d)$ be a surjective self-soft mapping. If there exists a constant $\bar{k} \widetilde{\geq} \bar{1}$ satisfying for all $P_\lambda^x, P_\mu^y \widetilde{\in} SP(\tilde{X}), P_\lambda^x \neq P_\mu^y$, and for all $\bar{t} \widetilde{>} \bar{0}$,

$$d(\varphi_\psi(P_\lambda^x), \varphi_\psi(P_\mu^y), \bar{t}) \widetilde{\geq} \bar{k}\, d(P_\lambda^x, P_\mu^y, \bar{t}) \qquad (11)$$

Then, there exists a unique fixed soft point of φ_ψ.

Proof. By the previous theorem, in the case $\bar{\beta} = \bar{\gamma} = \bar{0}$ and $\bar{\alpha} = \bar{k}$, the existence of the fixed soft point is clear. So, it is sufficient only to prove that the uniqueness. To do this, let us suppose the converse. That is, let $P_\rho^{\varpi}, P_\gamma^z$ be two fixed soft points of φ_ψ, then from condition (11), we obtain the following

$$d(P_\rho^{\varpi}, P_\gamma^z, \bar{t}) = d(\varphi_\psi(P_\rho^{\varpi}), \varphi_\psi(P_\gamma^z), \bar{t}) \widetilde{\geq} \bar{k}\, d(P_\rho^{\varpi}, P_\gamma^z, \bar{t}) \qquad (12)$$

which implies $d(P_\rho^{\varpi}, P_\gamma^z, \bar{t}) = \bar{0}$, that is $P_\rho^{\varpi} = P_\gamma^z$ as desired.

Corollary 3.4. Let $(\tilde{X}, d)$ be a complete parametric soft metric space and φ_ψ be a surjective self soft mapping in this space. If the following is satisfied for some positive integer n and a soft constant $\bar{k} \widetilde{\geq} \bar{1}$,

$$d(\varphi_\psi^n(P_\lambda^x), \varphi_\psi^n(P_\mu^y), \bar{t}) \widetilde{\geq} \bar{k}\, \mathrm{d}(P_\lambda^x, P_\mu^y, \bar{t}) \qquad (13)$$

for all $P_\lambda^x, P_\mu^y \widetilde{\in} SP(\tilde{X})$, $P_\lambda^x \neq P_\mu^y$, and for all $\bar{t} \widetilde{>} \bar{0}$, then there exists a unique fixed soft point of φ_ψ.

Proof. By the previous corollary, φ_ψ^n has a fixed soft point, such as P_ρ^{ϖ}. However, $\varphi_\psi^n(\varphi_\psi(P_\rho^{\varpi})) = \varphi_\psi(\varphi_\psi^n(P_\rho^{\varpi})) = \varphi_\psi(P_\rho^{\varpi})$. So $\varphi_\psi(P_\rho^{\varpi})$ is also a fixed soft point of the soft mapping φ_ψ^n. Hence $\varphi_\psi\left(P_\rho^{\varpi}\right) = P_\rho^{\varpi}$. Since the mappings φ_ψ and φ_ψ^n have the same fixed soft points. The result is obtained.

Definition 3.5. Let δ_ρ and φ_ψ be two self soft mappings of the soft universe $\tilde{X}$. Then, δ_ρ and φ_ψ are said to be weakly compatible if $\delta_\rho(P_\lambda^x) = \varphi_\psi(P_\lambda^x)$, for some $P_\lambda^x \widetilde{\in} SP(\tilde{X})$ and $\delta_\rho(\varphi_\psi(P_\lambda^x)) = \varphi_\psi(\delta_\rho(P_\lambda^x))$.

Theorem 3.6. Let $(\tilde{X}, d)$ be a complete parametric soft metric space and $\delta_\rho, \varphi_\psi : (\tilde{X}, d) \to (\tilde{X}, d)$ be the weakly compatible mappings such that $\varphi_\psi(\tilde{X}) \subseteq \delta_\rho(\tilde{X})$. If the following inequality holds for some $\bar{k} \widetilde{\geq} \bar{1}$ and for all $P_\lambda^x, P_\mu^y \widetilde{\in} \tilde{X}$,

$$d(\delta_\rho(P_\lambda^x), \delta_\rho(P_\mu^y), \bar{t}) \widetilde{\geq} \bar{k}\, d(\varphi_\psi(P_\lambda^x), \varphi_\psi(P_\mu^y), \bar{t}) \qquad (14)$$

and if besides one of the images $\varphi_\psi(\tilde{X})$ or $\delta_\rho(\tilde{X})$ is complete, then these mappings have a unique common fixed soft point in $\tilde{X}$.

Proof. Let $P_{\lambda_0}^{x_0} \tilde{\in} SP(\tilde{X})$ be taken arbitrarily. Since $\varphi_\psi(\tilde{X}) \subseteq \delta_\rho(\tilde{X})$, choose $P_{\lambda_1}^{x_1}$ such that $P_{\mu_1}^{y_1} = \delta_\rho(P_{\lambda_1}^{x_1}) = \varphi_\psi(P_{\lambda_0}^{x_0})$. In general, choose $P_{\lambda_{n+1}}^{x_{n+1}}$ such that

$P_{\mu_{n+1}}^{y_{n+1}} = \delta_\rho(P_{\lambda_{n+1}}^{x_{n+1}}) = \varphi_\psi(P_{\lambda_n}^{x_n})$, then from the condition (14) we gain the following

$$d\left(P_{\mu_{n+1}}^{y_{n+1}}, P_{\mu_{n+2}}^{y_{n+2}}, \bar{t}\right) = \mathrm{d}\left(\varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \varphi_\psi\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \bar{t}\right) \tilde{\leq} \frac{\bar{1}}{k} d\left(P_{\mu_n}^{y_n}, P_{\mu_{n+1}}^{y_{n+1}}, \bar{t}\right) \tag{15}$$

By repeating (15) $(n+1)$ –times, we obtain the following

$$d\left(P_{\mu_{n+1}}^{y_{n+1}}, P_{\mu_{n+2}}^{y_{n+2}}, \bar{t}\right) \tilde{\leq} \bar{\ell}^{n+1} d\left(P_{\mu_0}^{y_0}, P_{\mu_1}^{y_1}, \bar{t}\right) \tag{16}$$

where $\ell = \frac{1}{k}$. Hence for $n > m$, we have for all $\bar{t} \tilde{>} \bar{0}$,

$$\begin{aligned} d\left(P_{\mu_n}^{y_n}, P_{\mu_m}^{y_m}, \bar{t}\right) &\tilde{\leq} d\left(P_{\mu_n}^{y_n}, P_{\mu_{n+1}}^{y_{n+1}}, \bar{t}\right) + d\left(P_{\mu_n}^{y_n}, P_{\mu_{n+1}}^{y_{n+1}}, \bar{t}\right) + \ldots \ldots + d\left(P_{\mu_{m-1}}^{y_{m-1}}, P_{\mu_m}^{y_m}, \mathrm{t}\right) \\ &\tilde{\leq} \left(\bar{\ell}^n + \bar{\ell}^{n+1} + \cdots + \bar{\ell}^{m-1}\right) d\left(P_{\mu_0}^{y_0}, P_{\mu_1}^{y_1}, \bar{t}\right) \end{aligned} \tag{17}$$

The previous inequality witnesses the fact that if n and m tend to infinity,

$\lim\limits_{n,m\to\infty} d(P_{\lambda_n}^{x_n}, P_{\mu_m}^{y_m}, \bar{t}) = \bar{0}$. Therefore, $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ satisfies the Cauchy property and by the hypothesis $P_{\mu_n}^{y_n} \to P_\rho^\varpi$ for some $P_\rho^\varpi \tilde{\in} SP(\tilde{X})$. Hence, we get

$$\lim_{n\to\infty} P_{\mu_n}^{y_n} = \lim_{n\to\infty} \varphi_\psi\left(P_{\lambda_n}^{x_n}\right) = \lim_{n\to\infty} \delta_\rho\left(P_{\lambda_n}^{x_n}\right) = P_\rho^\varpi \tag{18}$$

Since one of the soft images $\varphi_\psi(\tilde{X})$ or $\delta_\rho(\tilde{X})$ is complete and $\varphi_\psi(\tilde{X}) \subseteq \delta_\rho(\tilde{X})$, $\delta_\rho(P_u^v) = P_\rho^\varpi$ for some $P_u^v \tilde{\in} \tilde{X}$. Now from (14), we have for all $\bar{t} \tilde{>} \bar{0}$,

$$d\left(\varphi_\psi(P_u^v), \varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \bar{t}\right) \tilde{\leq} \frac{\bar{1}}{\bar{k}} d\left(\delta_\rho(P_u^v), \delta_\rho\left(P_{\mu_n}^{y_n}\right), \bar{t}\right) \tag{19}$$

This implies the following for all $\bar{t} \tilde{>} \bar{0}$

$$d\left(\varphi_\psi(P_u^v), P_\rho^\varpi, \bar{t}\right) \tilde{\leq} \frac{\bar{1}}{\bar{k}} d\left(\delta_\rho(P_u^v), P_\rho^\varpi, \bar{t}\right) \tag{20}$$

The last inequality witnesses the fact that $\varphi_\psi(P_u^v) = P_\rho^\varpi$. Therefore $\varphi_\psi(P_u^v) = \delta_\rho(P_u^v) = P_\rho^\varpi$.

Since φ_ψ and δ_ρ are weakly compatible self-soft mappings, we have $\delta_\rho(\varphi_\psi(P_u^v)) = \varphi_\psi(\delta_\rho(P_u^v))$, that is $\delta_\rho(P_\rho^\varpi) = \varphi_\psi(P_\rho^\varpi)$. Now we show that P_ρ^ϖ is a fixed point of δ_ρ and φ_ψ . From (14), we have

$$d(\delta_\rho(P_\rho^\varpi), \delta_\rho(P_{\lambda_n}^{x_n}), \bar{t})) \tilde{\geq} \bar{k}\, d(\varphi_\psi(P_\rho^\varpi), \varphi_\psi(P_{\lambda_n}^{x_n}), \bar{t}) \tag{21}$$

If one takes limit as $n \to \infty$ in (21), then the following inequality

$$d(\delta_\rho(P_\rho^\varpi), P_\rho^\varpi, \bar{t})) \tilde{\geq} \bar{k} d(\varphi_\psi(P_\rho^\varpi), P_\rho^\varpi, \bar{t})) \tag{22}$$

implies the fact that $\delta_\rho(P_\rho^\varpi) = P_\rho^\varpi$. Hence, we have $\delta_\rho(P_\rho^\varpi) = \varphi_\psi(P_\rho^\varpi) = P_\rho^\varpi$.

Uniqueness: Let us suppose the converse, that is, let $P_{\rho}^{\varpi} \neq P_{\gamma}^{z}$ be two common fixed points of the given self-soft mappings. Then, we have $d(\delta_\rho(P_{\rho}^{\varpi}), \delta_\rho(P_{\gamma}^{z}), \bar{t})) \cong \bar{k} d(\varphi_\psi(P_{\rho}^{\varpi}), \varphi_\psi(P_{\gamma}^{z}), \bar{t}))$, for all $\bar{t} \tilde{>} \bar{0}$, which witnesses the fact that $P_{\rho}^{\varpi} = P_{\gamma}^{z}$. Hence, we get uniqueness.

Theorem 3.7. Let $(\tilde{X}, d)$ be a complete parametric soft metric space and $\varphi_\psi, \delta_\rho: \tilde{X} \to \tilde{X}$ be two surjective self soft mappings which satisfy the following conditions for some soft real numbers $\bar{\alpha}, \bar{\beta}$ and $\bar{k}$ such that $\bar{\alpha} > \bar{1} + 2\bar{k}$ and $\bar{\beta} > \bar{1} + 2\bar{k}$.

$$d(\varphi_\psi\ \delta_\rho(P_\lambda^x), \delta_\rho(P_\lambda^x), \bar{t}) + \bar{k} d(\varphi_\psi\, \delta_\rho(P_\lambda^x), P_\lambda^x, \bar{t}) \cong \bar{\alpha}\, d(\delta_\rho(P_\lambda^x), P_\lambda^x, \bar{t}) \tag{23}$$

and

$$d(\delta_\rho\ \varphi_\psi(P_\lambda^x), \varphi_\psi(P_\lambda^x), \bar{t}) + \bar{k} d\big(\delta_\rho \varphi_\psi(P_\lambda^x), P_\lambda^x, \tilde{t}\big) \cong \bar{\beta} d\big(\varphi_\psi(P_\lambda^x), P_\lambda^x, \bar{t}\big) \tag{24}$$

for all $P_\lambda^x \tilde{\in} SP(\tilde{X})$, all $\bar{t} \tilde{>} \bar{0}$. If one of the soft mappings is continuous, then they have a common fixed soft point.

Proof. Choose a soft point $P_{\lambda_0}^{x_0} \tilde{\in} SP(\tilde{X})$. Since φ_ψ is surjective, $P_{\lambda_0}^{x_0} = \varphi_\psi(P_{\lambda_1}^{x_1})$ for some $P_{\lambda_1}^{x_1} \tilde{\in} SP(\tilde{X})$. Since δ_ρ is surjective, too, $P_{\lambda_2}^{x_2} = \delta_\rho(P_{\lambda_1}^{x_1})$ for some $P_{\lambda_2}^{x_2} \tilde{\in} SP(\tilde{X})$. Continuing this process, we may set a sequence of soft points $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ in such a way that

$$P_{\lambda_{2n}}^{x_{2n}} = \varphi_\psi\left(P_{\lambda_{2n+1}}^{x_{2n+1}}\right) \quad \text{and} \quad P_{\lambda_{2n+1}}^{x_{2n+1}} = \delta_\rho\left(P_{\lambda_{2n+2}}^{x_{2n+2}}\right), \forall\, n \in \mathbb{N} \cup \{0\} \tag{25}$$

Now for $n \in \mathbb{N} \cup \{0\}$, we have

$$d(\varphi_\psi\ \delta_\rho(P_{\lambda_{2n+2}}^{x_{2n+2}}), \delta_\rho(P_{\lambda_{2n+2}}^{x_{2n+2}}), \bar{t}) + \bar{k}\, d(\varphi_\psi\ \delta_\rho(P_{\lambda_{2n+2}}^{x_{2n+2}}), P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}) \cong \bar{\alpha} d\left(\delta_\rho\left(P_{\lambda_{2n+2}}^{x_{2n+2}}\right), P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \tag{26}$$

Thus, we get

$$d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) + \bar{k}\, d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \cong \bar{\alpha} d\left(P_{\lambda_{2n+1}}^{x_{2n+1}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \tag{27}$$

Since

$$d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \mathrm{t}\right) \cong d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) + d\left(P_{\lambda_{2n+1}}^{x_{2n+1}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right)$$

Hence from (27),

$$d\left(P_{\lambda_{2n+1}}^{x_{2n+1}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \cong \frac{\bar{1} + \bar{k}}{\bar{\alpha} - \bar{k}} d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \tag{28}$$

On the other hand, we have

$$d(\delta_\rho\ \varphi_\psi(P_{\lambda_{2n+1}}^{x_{2n+1}}), \varphi_\psi(P_{\lambda_{2n+1}}^{x_{2n+1}}), \bar{t}) + \bar{k}\, d(\delta_\rho\ \varphi_\psi(P_{\lambda_{2n+1}}^{x_{2n+1}}), P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}) \cong \bar{\beta} d(\varphi_\psi(P_{\lambda_{2n+1}}^{x_{2n+1}}), P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}) \tag{29}$$

Thus, we have

$$d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n}}^{x_{2n}}, \bar{t}\right) + \bar{k}\, d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \cong \bar{\beta} d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \tag{30}$$

Since

$$d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \lesseqqgtr d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n}}^{x_{2n}}, \bar{t}\right) + d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right)$$

And from (30), we have

$$d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \lesseqqgtr \frac{\bar{1} + \bar{k}}{\bar{\beta} - \bar{k}} d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n}}^{x_{2n}}, \bar{t}\right) \tag{31}$$

Let $\bar{\ell} = \max\{\frac{\bar{1}+\bar{k}}{\bar{\beta}-\bar{k}}, \frac{\bar{1}+\bar{k}}{\bar{\alpha}-\bar{k}}\}$. Then, by combining (28) and (31), we have for each $n \in \mathrm{N} \cup \{0\}$ and $\bar{t} \gtrsim \bar{0}$

$$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) \lesseqqgtr \bar{\ell}\, d\left(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}\right) \tag{32}$$

By repeating (31) n-times, we get for all $n \in \mathrm{N} \cup \{0\}$ and all $\bar{t} \gtrsim \bar{0}$,

$$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) \lesseqqgtr \bar{\ell}^{n}\, d\left(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}\right) \tag{33}$$

By Lemma 3.1, $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ satisfies the Cauchy property. Then by the hypothesis, the sequence converges to some soft point as $P_{\lambda_n}^{x_n} \to P_{\rho}^{\varpi}$. So, $P_{\lambda_{2n+1}}^{x_{2n+1}} \to P_{\rho}^{\varpi}$ and $P_{\lambda_{2n+2}}^{x_{2n+2}} \to P_{\rho}^{\varpi}$. If φ_{ψ} is continuous, then $\varphi_{\psi}(P_{\lambda_{2n+1}}^{x_{2n+1}}) \to \varphi_{\psi}(P_{\rho}^{\varpi})$. But $\varphi_{\psi}(P_{\lambda_{2n+1}}^{x_{2n+1}}) = P_{\lambda_{2n}}^{x_{2n}} \to P_{\rho}^{\varpi}$. As a result, $\varphi_{\psi}(P_{\rho}^{\varpi}) = P_{\rho}^{\varpi}$. By the surjectivity of δ_{ρ}, we have $\delta_{\rho}(P_{\rho}^{\varpi}) = P_{\rho}^{\varpi}$ for some soft point P_{σ}^{ϑ}. Now

$$d(\varphi_{\psi}(\delta_{\rho}(P_{\sigma}^{\vartheta}), \delta_{\rho}P_{\sigma}^{\vartheta}), \bar{t}) + \bar{k}d(\varphi_{\psi}(\delta_{\rho}(P_{\sigma}^{\vartheta}), P_{\sigma}^{\vartheta}, \bar{t}) \gtreqqless \bar{\alpha}d(\delta_{\rho}(P_{\sigma}^{\vartheta}), P_{\sigma}^{\vartheta}, \bar{t})$$

implies that

$$\bar{k}d(P_{\rho}^{\varpi}, P_{\sigma}^{\vartheta}, \bar{t}) \gtreqqless \bar{\alpha}d(P_{\rho}^{\varpi}, P_{\sigma}^{\vartheta}, \bar{t}).$$

Thus, we gain the following

$$d(P_{\rho}^{\varpi}, P_{\sigma}^{\vartheta}, \mathrm{t}) \lesseqqgtr \frac{\bar{k}}{\bar{\alpha}} d(P_{\rho}^{\varpi}, P_{\sigma}^{\vartheta}, \bar{t})$$

Since $\bar{\alpha} \gtrsim \bar{k}$, we conclude that $d(P_{\rho}^{\varpi}, P_{\sigma}^{\vartheta}, \bar{t}) = \bar{0}$. So $P_{\rho}^{\varpi} = P_{\sigma}^{\vartheta}$.

Hence, $\varphi_{\psi}(P_{\rho}^{\varpi}) = \delta_{\rho}(P_{\rho}^{\varpi}) = P_{\rho}^{\varpi}$. This completes the proof as claimed.

4. Conclusion

Fixed point theory is essential in the surveys given in metric and topological spaces. Several authors have applied/embedded this theory in different metric spaces and applied sciences. Since the solutions of integral and differential equations are based on the fixed-point theory constructed in normed spaces, in this merit, we decide to investigate the existence and the uniqueness of fixed soft points of self-soft mappings in parametric soft metric spaces, which spaces the parameterization tool plays the key role. Moreover, the studies on fixed-circle results have gained attention in metric and metric-like spaces [15,16,17], nowadays. For further research, we hope to investigate some different kinds of fixed soft point theorems, some fixed-circle theorems and also, we plan to give some applications in such spaces.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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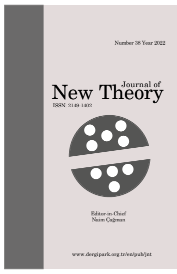

Numerical Treatment of Uniformly Convergent Method for Convection Diffusion Problem

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Abstract − In this paper, we will study the convergence properties of the method designed for the convection-diffusion problem. We will prove that the analytical and numerical methods give the same result. Merging the ideas in previous research, we introduce a numerical algorithm on a uniform mesh that requires no exact solution to the local convection-diffusion problem. We display how to obtain the numerical solution of the local Boundary Value Problem (BVP) in a suitable way to ensure that the resulting numerical algorithm recaptures the same convergence properties when using the exact solution of the local BVP. We prove that the proposed algorithm nodally converges to the exact solution.

Keywords − *Trapezoidal rule, convection-diffusion problem, boundary value problem, singular points, Green's function, Lagrange interpolation*

Mathematics Subject Classification (2020) − 34B27, 65L10

1. Introduction

It is well-known that the piecewise- uniform fitted meshes studied by Shishkin [1] and the corresponding numerical algorithms were developed and shown to be ε −uniform in various studies including the book by Shishkin [2]. The numerical results using a fitted mesh method were firstly presented in [3]. We refer the readers to Bakhvalov [4], Gartland [5] and Vulanovic [6] for other approaches to adapting the mesh, involving complicated redistribution of the mesh points [7, 8]. We note that none has the simplicity of the piecewise uniform fitted meshes.

Motivating by ~~this~~ these considerations, we remark that both fitted operators and fitted meshes need to be studied. Since the methods using fitted meshes are usually easier to implement than the methods using fitted operators in practice, they recommended to be applied whenever possible. We also note that the methods using fitted meshes are easier to generalize to the problems in more than one dimension and to the nonlinear problems.

In this paper, the following convection−diffusion problem with a concentrated source is considered and we prove that ε-uniformly convergent methods can be designed for the problem (1). In other words, in this article, to investigate the numerical solution of equation (1) and to obtain a suitable method, we will focus on the following boundary value problem [9]

$$Lu = -\varepsilon u'' + bu' + c\,u = f(x), \;\; u(0) = 0, \;\; u(1) = 0 \tag{1}$$

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It is worth mentioning that the modelling of real-world problems including physical, chemical, and biological phenomena contain interactions of convection and diffusion processes, which can be described by the convection-diffusion- problem [10].

We remark that, we have studied the following

$$-\varepsilon g_i'(x_{i-1})\, u_{i-1} + u_i + \varepsilon g_i'(x_{i+1})\, u_{i+1} = (f - cu) \int_{x_{i-1}}^{x_{i+1}} g_i \, dx$$

and we obtained the analytical solution

$$-\frac{e^{\rho_i}}{e^{\rho_i}+1} U_{i-1} + U_i - \frac{1}{e^{\rho_i}+1} U_{i+1} = (f_i - c_i\, U_i)\, \frac{h}{b} \left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right) \tag{2}$$

see [10] and [11] for details. In this article, we will use the equation (1), and after applying various numerical treatments, we will get the same solution given by the equation (2) which was studied before in [12]. In this study, we have,

$$g_i'(x_{i-1}) \cong D^+ G_0 = \frac{G_1 - G_0}{h_1^*} \text{ and } g_i'(x_{i+1}) \cong D^- G_M = \frac{G_M - G_{M-1}}{h_2^*}$$

$$-\varepsilon\, D^+ G_0 U_{i-1} + U_i + \varepsilon\, D^- G_M U_{i+1} = (f_i - c\, U_i) \int_{x_{i-1}}^{x_{i+1}} G^i \, dx$$

$$T_1(\varepsilon, b_i, c_i, h, M) = \varepsilon D^+ G_0$$

$$T_2(\varepsilon, b_i, c_i, h, M) = -\varepsilon D^- G_M$$

$$T_3(\varepsilon, b_i, c_i, M) = \int_{x_{i-1}}^{x_{i+1}} G^i \, dx$$

At the end of this paper, we will show that

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = \frac{e^{\rho_i}}{e^{\rho_i}+1}$$

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = -\frac{1}{\varepsilon}\left(\frac{1}{e^{\rho_i}+1}\right)$$

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i \, dx = \lim_{M\to\infty} T_3(\varepsilon, b_i, c_i, M) = \left(\frac{h}{b_i}\right)\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

Now, consider

$$-\varepsilon\, D^+ D^- G_j - b D^+ G_j = \Delta X_{i,j}, \qquad j = 1, 2, 3, \dots, M-1$$

$$-\varepsilon\left(\frac{G_{j+1} - G_j}{h_{j+1}} - \frac{G_j - G_{j-1}}{h_j}\right)\frac{1}{h_j} - b\left(\frac{G_{j+1} - G_j}{h_{j+1}}\right) = \Delta X_{i,j}$$

where

$$\Delta X_{i,j} = \begin{cases} \dfrac{1}{h_{j+1}}, & x_i \in (x_j, x_{j+1}) \\ 0, & \text{otherwise} \end{cases}$$

If $j = 0$ or $j = M$, then $G_0 = 0$ or $G_M = 0$.

$$h_j = \begin{cases} h_1 \;, 1 \le j \le \frac{M}{4} - 1 \\ h_2, \; \frac{M}{4} \le j \le \frac{2M}{4} - 1 \\ h_1, \; \frac{2M}{4} \le j \le \frac{3M}{4} - 1 \\ h_2, \; \frac{3M}{4} \le j \le \frac{4M}{4} - 1 \end{cases}$$

$$-G_{j+1}\left(1 + \frac{b\,h_1}{\varepsilon}\right) + G_j\left(2 + \frac{b\,h_1}{\varepsilon}\right) + G_{j-1}(-1) = 0$$

In the previous equation, if we take $G_{j+1} = r^2$, $G_j = r$, $G_{j-1} = r^0 = 1$ and $\lambda_1 = 1 + \frac{b\,h_1}{\varepsilon}$ then we will get

$$-r^2\left(1 + \frac{b\,h_1}{\varepsilon}\right) + r\left(2 + \frac{b\,h_1}{\varepsilon}\right) - 1 = 0$$

$$-r^2\lambda_1 + r(1 + \lambda_1) - 1 = 0 \;\Rightarrow (1 - r\lambda_1)(r - 1) = 0$$

Then, the roots of the quadratic equation are given by: $r_1 = 1$ and $r_2 = \frac{1}{\lambda_1}$. Similarly, we get

$$-G_{j+1}\left(1 + \frac{b\,h_2}{\varepsilon}\right) + G_j\left(2 + \frac{b\,h_2}{\varepsilon}\right) + G_{j-1}(-1) = 0$$

In the previous equation, if we take $G_{j+1} = r^2$, $G_j = r$, $G_{j-1} = r^0 = 1$ and $\lambda_2 = 1 + \frac{b\,h_2}{\varepsilon}$, then we will get

$$-r^2\left(1 + \frac{b\,h_2}{\varepsilon}\right) + r\left(2 + \frac{b\,h_2}{\varepsilon}\right) - 1 = 0$$

$$-r^2\lambda_2 + r(1 + \lambda_2) - 1 = 0 \;\Rightarrow (1 - r\lambda_2)(r - 1) = 0$$

Then, the roots of the quadratic equation are given by: $r_1 = 1$ and $r_2 = \frac{1}{\lambda_2}$.

2. Derivation of Trapezoidal Rule

We can derive the trapezoidal rule by using polynomial interpolants of $f(x)$ function. The usage of a Lagrange interpolant for each sub-interval $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$ leads to the trapezoidal rule in [13], that is,

$$\int_{x_{i-1}}^{x_i} f(x)\,dx \approx \int_{x_{i-1}}^{x_i} P(x)\,dx$$

where

$$P(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i)$$

$$\begin{aligned}\int_{x_{i-1}}^{x_i} f(x)\,dx \approx \int_{x_{i-1}}^{x_i} P(x)dx &= \int_{x_{i-1}}^{x_i} \left(\frac{(x - x_i)}{x_{i-1} - x_i} f(x_{i-1}) + \frac{(x - x_{i-1})}{x_i - x_{i-1}} f(x_i)\right) dx \\ &= \frac{f(x_{i-1})}{x_{i-1} - x_i} \int_{x_{i-1}}^{x_i} (x - x_i)dx + \frac{f(x_i)}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (x - x_{i-1})dx \\ &= \frac{f(x_{i-1})}{-(x_i - x_{i-1})} \frac{(x - x_i)^2}{2} \Bigg|_{x = x_{i-1}}^{x_i} + \frac{f(x_i)}{x_i - x_{i-1}} \frac{(x - x_{i-1})^2}{2} \Bigg|_{x = x_{i-1}}^{x_i}\end{aligned}$$

$$= -\frac{f(x_{i-1})}{(x_i - x_{i-1})}\left[\frac{(x_i - x_i)^2}{2} - \frac{(x_{i-1} - x_i)^2}{2}\right] + \frac{f(x_i)}{(x_i - x_{i-1})}\left[\frac{(x_i - x_{i-1})^2}{2} - \frac{(x_{i-1} - x_{i-1})^2}{2}\right]$$

$$= -\frac{f(x_{i-1})}{(x_i - x_{i-1})} \cdot \left[0 - \frac{(x_{i-1} - x_i)^2}{2}\right] + \frac{f(x_i)}{(x_i - x_{i-1})}\left[\frac{(x_i - x_{i-1})^2}{2} - 0\right]$$

$$= \frac{f(x_{i-1})}{(x_i - x_{i-1})}\left[\frac{(x_{i-1} - x_i)^2}{2}\right] + \frac{f(x_i)}{(x_i - x_{i-1})}\left[\frac{(x_i - x_{i-1})^2}{2}\right]$$

$$\int_{x_{i-1}}^{x_i} P(x)\,dx = \frac{(x_i - x_{i-1})}{2}\left[\frac{f(x_{i-1})}{2} + \frac{f(x_i)}{2}\right]$$

For the composite trapezoidal rule, we have,

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \quad = \sum_{i=1}^{n} (x_i - x_{i-1})\left[\frac{f(x_{i-1})}{2} + \frac{f(x_i)}{2}\right]$$

$$\int_a^b P(x)\,dx = \frac{h}{2}\sum_{i=1}^{n}[\,f(x_{i-1}) + f(x_i)\,] = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n} f(x_i) + (x_n)\right]$$

We note that, this is known as the composite trapezoidal rule in [13].

Lemma 2.1: If

$$T_1(\varepsilon, b_i, c_i, h, M) = \varepsilon D^+ G_0$$

then

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = \frac{e^{\rho_i}}{e^{\rho_i} + 1}$$

Proof. Consider the uniform case, that is $\tau = \frac{h}{2}$. Then, the mesh parameters can be written as $h_1^* = h_2^* = \frac{2h}{M}$ and $\lambda_1 = \lambda_2 = 1 + \frac{2bh}{\varepsilon M}$.

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = \lim_{M\to\infty} \frac{G_1 - G_0}{h_1^*} = \frac{e^{\rho_i}}{e^{\rho_i} + 1}$$

Lemma 2.2: If

$$T_2(\varepsilon, b_i, c_i, h, M) = -\varepsilon D^- G_M$$

then

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = -\frac{1}{\varepsilon}\left(\frac{1}{e^{\rho_i} + 1}\right)$$

Proof. We follow the same steps in the proof of Lemma 2.1. For the uniform case when $\tau = \frac{h}{2}$, we use the difference solution G^i and the fact that $h_1^* = h_2^* = \frac{2h}{M}$;

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = \lim_{M\to\infty} \frac{G_M - G_{M-1}}{h_2^*} = -\frac{1}{\varepsilon}\left(\frac{1}{e^{\rho_i} + 1}\right)$$

Lemma 2.3: If

$$T_3(\varepsilon, b_i, c_i, M) = \int_{x_{i-1}}^{x_{i+1}} G^i\,dx$$

then

$$\lim_{M\to\infty} T_3(\varepsilon, b_i, c_i, M) = \left(\frac{h}{b_i}\right)\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

PROOF. In order to calculate the following integral

$$\int_{x_{i-1}}^{x_{i+1}} G^i\,dx$$

the trapezoidal rule is used for the exact solution of

$$G_j^i = \begin{cases} a_1 r_1^j + a_2 r_2^j \quad 0 \le j \le \dfrac{M}{4} \\ a_3 r_3^j + a_4 r_4^j, \quad \dfrac{M}{4} \le j \le \dfrac{2M}{4} \\ a_5 r_1^j + a_6 r_2^j, \quad \dfrac{2M}{4} \le j \le \dfrac{3M}{4} \\ a_7 r_3^j + a_8 r_4^j, \quad \dfrac{3M}{4} \le j \le \dfrac{4M}{4} \end{cases}$$

$$G_j^i = \begin{cases} a_1 + a_2 \lambda_1^{-j}, \quad 0 \le j \le \dfrac{M}{4} \\ a_3 + a_4 \lambda_2^{-j}, \quad \dfrac{M}{4} \le j \le \dfrac{2M}{4} \\ a_5 + a_6 \lambda_1^{-j}, \quad \dfrac{2M}{4} \le j \le \dfrac{3M}{4} \\ a_7 + a_8 \lambda_2^{-j}, \quad \dfrac{3M}{4} \le j \le \dfrac{4M}{4} \end{cases}$$

Using the properties of Green's function in [14–18], we get,

$$a_1 + a_2\lambda_1^{-j} = a_3 + a_4\lambda_2^{-j}$$

$$a_3 + a_4\lambda_2^{-j} = a_5 + a_6\lambda_1^{-j}$$

$$a_5 + a_6\lambda_1^{-j} = a_7 + a_8\lambda_2^{-j}$$

For $G_0 = G_M = 0$, we have,

$$a_1 + a_2\lambda_1^{-j} = 0$$

$$a_7 + a_8\lambda_2^{-j} = 0$$

For j = M/4, we have, $a_1\lambda_1 + a_2(k_1^{-1}(1-\lambda_2+\lambda_1)) - a_3\lambda_1 + a_4 k_3 k_2^{-1} = 0$.

For $\frac{h_2}{\varepsilon}$, we have, $a_3\lambda_2 + a_4(k_2^{-2}(1-\lambda_2+\lambda_1)) - a_5\lambda_2 - a_6 k_1^{-2} k_3^{-1} = \frac{h_2}{\varepsilon}$.

For j=3M/4, we have, ⇒ $a_5\lambda_1 + a_6(k_1^{-3}(1-\lambda_2+\lambda_1)) - a_7\lambda_1 - a_8 k_3 k_2^{-3} = 0$.

In order to get the difference solution exactly, we need to determine the eight unknown coefficients. Two equations can be obtained by using the boundary conditions: $G_0 = G_M = 0$; the difference equations related to the nodes $x_{M/4}$, $x_{2M/4}$ and $x_{3M/4}$ give us other three equations; and finally, the continuity conditions can be applied to obtain the other three equations. Next, the corresponding numerical algorithm can be obtained by using the fitted finite difference operator in order to get a system of finite difference equations on a standard mesh. We remark that the mesh is often a uniform mesh in practice. Finally, the obtained system can be solved in a practical way to get the numerical solutions. We refer the readers to [16] for other approaches in constructing fitted finite difference operators.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & k_1^{-1} & -1 & k_2^{-1} & 0 & 0 & 0 & 0 \\ \lambda_1 & \xi_1 & -\lambda_1 & -k_3k_2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & k_2^{-2} & -1 & -k_1^{-2} & 0 & 0 \\ 0 & 0 & \lambda_2 & \xi_2 & -\lambda_2 & -k_1^{-2}k_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & k_1^{-3} & -1 & -k_2^{-3} \\ 0 & 0 & 0 & 0 & \lambda_1 & \xi_3 & -\lambda_1 & -k_3k_2^{-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & k_2^{-4} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{h_2}{\varepsilon} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $k_1 = \lambda_1^{\frac{M}{4}}$, $k_2 = \lambda_2^{\frac{M}{4}}$, $k_3 = \lambda_1\lambda_2^{-1}$, $\xi_1 = k_1^{-1}(1-\lambda_2+\lambda_1)$, $\xi_2 = k_2^{-2}(1-\lambda_1+\lambda_2)$, $\xi_3 = k_1^{-3}(1-\lambda_2+\lambda_1)$, and $\eta = \varepsilon(\lambda_1-1)(1+\lambda_1^{\frac{M}{4}}\lambda_2^{\frac{M}{4}})$.

Using the symbolic programming MATHEMATICA, one can solve $AX = B$ linear system and obtain the following results:

$$a_1 = \frac{h_2}{\eta} k_1\ k_2\ k_3$$

$$a_2 = -\frac{h_2}{\eta} k_1\ k_2\ k_3$$

$$a_3 = \frac{h_2}{\eta}(-k_2\ k_3 + k_1\ k_2\ k_3 + k_2)$$

$$a_4 = -\frac{h_2}{\eta} k_2^2$$

$$a_5 = -\frac{h_2}{\eta}(\lambda_2 - \lambda_2 k_2 + \lambda_1 k_2)\,\lambda_2^{-1}$$

$$a_6 = \frac{h_2}{\eta} k_1^3 k_2\ k_3$$

$$a_7 = \frac{h_2}{\eta}$$

$$a_8 = \frac{h_2}{\eta} k_2^4$$

$$\int_{x_{i-1}}^{x_{i+1}} G^i\,dx = \int_{x_{i-1}}^{x_{i-1}+\tau} G^i\,dx + \int_{x_{i-1}+\tau}^{x_i} G^i\,dx + \int_{x_i}^{x_i+\tau} G^i\,dx + \int_{x_i+\tau}^{x_{i+1}} G^i\,dx$$

where $\tau = \frac{h}{2}$.

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i\,dx = \lim_{M\to\infty}(I_1 + I_2 + I_3 + I_4)$$

Unless otherwise indicated, we will apply the trapezoidal rule for numerical integration until the end of this work.

$$I_1 = \int_{x_{i-1}}^{x_{i-1}+\tau} G^i\,dx = h_1\left[\frac{G_0}{2} + G_1 + G_2 + \cdots + G_{\frac{M}{4}-1} + \frac{G_{\frac{M}{4}}}{2}\right]$$

$$I_1 = \int_{x_{i-1}}^{x_{i-1}+\tau} G^i\,dx = h_1\left[\frac{0}{2} + G_1 + G_2 + \cdots + G_{\frac{M}{4}-1} + \frac{G_{\frac{M}{4}}}{2}\right]$$

$$I_1 = h_1\left[\sum_{j=1}^{\frac{M}{4}-1} G^j\right] + \frac{h_1}{2} G_{\frac{M}{4}}$$

$$I_1 = h_1 \sum_{j=1}^{\frac{M}{4}-1}\left(a_1 r_1^j + a_2 r_2^j\right) + \frac{h_1}{2}\left(a_1 r_1^{M/4} + a_2 r_2^{M/4}\right)$$

$$I_1 = h_1 \sum_{j=1}^{\frac{M}{4}-1}\left(a_1 r_1^j\right) + h_1 \sum_{j=1}^{\frac{M}{4}-1}\left(a_2 r_2^j\right) + \frac{h_1}{2}\left(a_1 r_1^{M/4} + a_2 r_2^{M/4}\right)$$

$$I_2 = \int_{x_{i-1}+\tau}^{x_i} G^i\,dx = h_2\left[\frac{G_{\frac{M}{4}}}{2} + G_{\frac{M}{4}+1} + G_{\frac{M}{4}+2} + \cdots + G_{\frac{2M}{4}-1} + \frac{G_{\frac{2M}{4}}}{2}\right]$$

$$I_2 = \int_{x_{i-1}+\tau}^{x_i} G^i\,dx = h_2\left[\frac{G_{\frac{M}{4}}}{2} + \sum_{j=\frac{M}{4}+1}^{\frac{2M}{4}-1} G^j + \frac{G_{\frac{2M}{4}}}{2}\right]$$

$$I_2 = \frac{h_2}{2}\left(a_3 r_3^{M/4} + a_4 r_4^{M/4}\right) + h_2 \sum_{j=\frac{M}{4}+1}^{\frac{2M}{4}-1}\left(a_3 r_3^j + a_4 r_4^j\right) + \frac{h_2}{2}\left(a_3 r_3^{M/2} + a_4 r_4^{M/2}\right)$$

$$I_3 = \int_{x_i}^{x_i+\tau} G^i\,dx = h_1\left[\frac{G_{\frac{M}{2}}}{2} + G_{\frac{M}{2}+1} + G_{\frac{M}{2}+2} + \cdots + G_{\frac{3M}{4}-1} + \frac{G_{\frac{3M}{4}}}{2}\right]$$

$$I_3 = \int_{x_i}^{x_i+\tau} G^i\,dx = h_1\left[\frac{G_{\frac{M}{2}}}{2} + \sum_{j=\frac{M}{2}+1}^{\frac{3M}{4}-1} G^j + \frac{G_{\frac{3M}{4}}}{2}\right]$$

$$I_3 = \frac{h_1}{2}\left(a_5 r_1^{M/2} + a_6 r_2^{M/2}\right) + h_1 \sum_{j=\frac{M}{2}+1}^{\frac{3M}{4}-1}\left(a_5 r_1^j + a_6 r_2^j\right) + \frac{h_1}{2}\left(a_5 r_1^{3M/4} + a_6 r_2^{3M/4}\right)$$

$$I_4 = \int_{x_i+\tau}^{x_{i+1}} G^i\,dx = h_2\left[\frac{G_{\frac{3M}{4}}}{2} + G_{\frac{3M}{4}+1} + G_{\frac{3M}{4}+2} + \cdots + G_{M-1} + \frac{G_M}{2}\right]$$

$$I_4 = \int_{x_i+\tau}^{x_{i+1}} G^i\,dx = h_2\left[\frac{G_{\frac{3M}{4}}}{2} + G_{\frac{3M}{4}+1} + G_{\frac{3M}{4}+2} + \cdots + G_{M-1} + \frac{0}{2}\right]$$

$$I_4 = \int_{x_i+\tau}^{x_{i+1}} G^i\,dx = h_2\left[\frac{G_{\frac{3M}{4}}}{2} + \sum_{j=\frac{3M}{4}+1}^{M-1} G^j\right]$$

$$I_4 = \frac{h_2}{2}\left(a_7 r_3^{3M/4} + a_8 r_4^{3M/4}\right) + h_2 \sum_{j=\frac{3M}{4}+1}^{M-1} \left(a_7 r_3^j + a_8 r_4^j\right)$$

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i\, dx = \lim_{M\to\infty} (I_1 + I_2 + I_3 + I_4)$$

Since the integral T_3 ~~integral~~ can be written as the sum of the integrals I_1, I_2 I_3 and I_4, we have

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i\, dx = \left(\frac{h}{b_i}\right)\tanh\left(\frac{b_i h}{2\varepsilon}\right) = \left(\frac{h}{b_i}\right) \frac{e^{\frac{b_i h}{\varepsilon}} - 1}{e^{\frac{b_i h}{\varepsilon}} + 1}$$

$$T_3 = \lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i\, dx = \lim_{M\to\infty} T_3(\varepsilon, b_i, c_i, M) = \left(\frac{h}{b_i}\right)\left(\frac{e^{\rho_i} - 1}{e^{\rho_i} + 1}\right)$$

Finally, we proved that the numerical and analytical results converge exactly (see [11]), that is,

$$-\frac{e^{\rho_i}}{e^{\rho_i}+1} U_{i-1} + U_i - \frac{1}{e^{\rho_i}+1} U_{i+1} = (f_i - c_i\, U_i)\ \frac{h}{b}\left(\frac{e^{\rho_i} - 1}{e^{\rho_i} + 1}\right)$$

3. Conclusion

In this paper, we studied different finite difference methods for the convection-diffusion problem. We presented numerical behaviour of the convection-diffusion problem. We applied a uniformly convergent numerical algorithm, called Il'in-Allen-Southwell scheme, with better accuracy throughout the domain for various values of ε. At the end of the study, we showed how to construct such a method. Finally, we have constructed a uniformly convergent numerical method for the convection-diffusion problem.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

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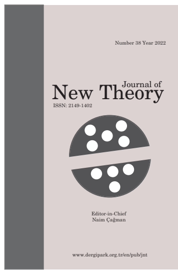

On Smarandache Curves in Affine 3-space

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Abstract — In this paper, we introduce Smarandache curves of an affine C^{∞}-curve in affine 3-space. Besides, we present the relationship between the Frenet frames of the curve couple and the Frenet apparatus of each obtained curve.

Keywords — *Affine curve, affine frame, Smarandache curves, affine* 3*-space*

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1. Introduction

In the theory of curves in differential geometry, one of the interesting problems studied by many mathematicians is to characterize a regular curve and give information about its structure. Using the curvatures κ and τ of a regular curve, its shape and size can be determined, so the curvatures play an important role in the problem's solution. The relationship between the corresponding Frenet vectors of the two curves gives another approach to solving the problem. For example; involute-evolute curve couple, Bertrand mate curves and Mannheim mate curves result from this relationship. Another example is Smarandache curves, which are defined as regular curves with the location vector generated by the Frenet vectors of the regular curve. Smarandache curves have been widely studied in different ambient spaces ([1–17]).

While Euclidean differential geometry is the study of differential invariants regarding the group of rigid motions, affine differential geometry is the study of differential invariants regarding the group of affine transformations $x \longrightarrow Ax + b$, $A \in GL(n, \mathbb{R})$, $b \in \mathbb{R}^n$ acting on $x \in \mathbb{R}^n$, i.e., nonsingular linear transformations together with translations, denoted by the Lie group $A(n, \mathbb{R}) = GL(n, \mathbb{R}) \times \mathbb{R}^n$ with a semi-direct product structure, (see [18, 19]). Moreover, "affine geometry" is also called "equi-affine geometry", where we restrict to the subgroup $SA(n, \mathbb{R}) = SL(n, \mathbb{R}) \times \mathbb{R}^n$ of volume-preserving linear transformations together with translations.

In this paper, we introduce TN, TB, NB and $TNB-$Smarandache curves corresponding to a regular $C^{\infty}-$curve in affine $3-$space A_3. We also establish the relationship between the Frenet frames of the pair of curves and the Frenet apparatus of each curve.

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2. Basic Concepts

In this section, we give the information to understand the main subjects in this paper (see for details [20–22]).

A set of points whose elements correspond to a vector of the vector space V over a field is called the *affine space associated with* V. We refer to A_3 as an affine 3-space associated with $\mathbb{R}^3$.

An arbitrary curve $\alpha : I \subset \mathbb{R} \to A_3$ is called a *regular affine curve* if for all $r \in I$

$$\det\left(\frac{d\alpha}{dr}(r), \frac{d^2\alpha}{dr^2}(r), \frac{d^3\alpha}{dr^3}(r)\right) \neq 0$$

and the *arc-length* of α is defined as

$$s(r) := \int_{r_1}^{r_2} \left|\det\left(\frac{d\alpha}{dr}(r), \frac{d^2\alpha}{dr^2}(r), \frac{d^3\alpha}{dr^3}(r)\right)\right|^{1/6} dr$$

Here, s is called *the parameter of the affine arc-length* if

$$\det\left(\frac{d\alpha}{ds}(s), \frac{d^2\alpha}{ds^2}(s), \frac{d^3\alpha}{ds^3}(s)\right) = 1$$

Remark 2.1. In this paper, the prime denotes differentiation concerning the parameter s, i.e., $\alpha' = \frac{d\alpha}{ds}$ etc., while a dot is reserved for differentiation concerning any arbitrary parameter r, i.e., $\dot{\alpha} = \frac{d\alpha}{dr}$ etc..

For an affine $C^\infty-$curve α in A_3 parameterized by the parameter of the affine arc-length s, κ and τ are called the *affine curvature* and the *affine torsion* of α given by

$$\kappa(s) = \det\left[\alpha'(s), \alpha'''(s), \alpha^{(iv)}(s)\right] \tag{1}$$

and

$$\tau(s) = -\det\left[\alpha''(s), \alpha'''(s), \alpha^{(iv)}(s)\right] \tag{2}$$

From the definition of $\kappa(s)$ and $\tau(s)$, we get

$$\alpha^{(iv)}(s) + \kappa(s)\alpha''(s) + \tau(s)\alpha'(s) = 0$$

that is

$$\frac{d}{ds}\begin{bmatrix} \alpha'(s) \\ \alpha''(s) \\ \alpha'''(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tau & -\kappa & 0 \end{bmatrix}\begin{bmatrix} \alpha'(s) \\ \alpha''(s) \\ \alpha'''(s) \end{bmatrix} \tag{3}$$

Let us set

$$T = \alpha', \quad N = \alpha'', \quad B = \alpha'''$$

Then, we can write the relation (3) as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tau & -\kappa & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{4}$$

Here, T, N and B are called the *tangent vector*, the *normal vector*, and the *binormal vector* of α, respectively. Also, $\{T, N, B\}$ is called an *affine Frenet frame* of α.

Example 2.2. Let α be an affine $C^\infty-$curve in A_3 with parametric equation

$$\alpha(s) = (\cos s, \sin s, s)$$

The affine Frenet frame of α reads

$$
\begin{aligned}
T(s) &= (-\sin s, \cos s, 1) \\
N(s) &= (-\cos s, -\sin s, 0) \\
B(s) &= (\sin s, -\cos s, 0)
\end{aligned}
$$

It follows that the curvatures of α have the form

$$\kappa(s) = \det\left[\alpha'(s), \alpha'''(s), \alpha^{(iv)}(s)\right] = 1$$

and

$$\tau(s) = -\det\left[\alpha''(s), \alpha'''(s), \alpha^{(iv)}(s)\right] = 0$$

Then, we can write

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

3. Smarandache Curves in Affine 3−space

In this section, we consider an affine $C^\infty-$curve α and define its affine Smarandache curves in affine 3−space A_3. Let $\alpha = \alpha(s)$ be a regular affine $C^\infty-$curve with affine Frenet frame $\{T, N, B\}$ in A_3. Denote by $\beta = \beta(u)$ arbitrary affine $C^\infty-$curve, where u is the parameter of the affine arc-length of β.

Definition 3.1. Let α be an affine $C^\infty-$curve in affine 3−space A_3. A curve β defined by

$$\beta(u(s)) = \frac{1}{\sqrt{2}}(T(s) + N(s)) \tag{5}$$

is called the $TN-$affine Smarandache curve of α.

Definition 3.2. Let α be an affine $C^\infty-$curve in affine 3−space A_3. A curve β defined by

$$\beta(u(s)) = \frac{1}{\sqrt{2}}(T(s) + B(s)) \tag{6}$$

is called the $TB-$affine Smarandache curve of α.

Definition 3.3. Let α be an affine $C^\infty-$curve in affine 3−space A_3. A curve β defined by

$$\beta(u(s)) = \frac{1}{\sqrt{2}}(N(s) + B(s)) \tag{7}$$

is called the $NB-$affine Smarandache curve of α.

Definition 3.4. Let α be an affine $C^\infty-$curve in affine 3−space A_3. A curve β defined by

$$\beta(u(s)) = \frac{1}{\sqrt{3}}(T(s) + N(s) + B(s)) \tag{8}$$

is called the $TNB-$affine Smarandache curve of α.

Next, we obtain the affine Frenet frame $\{T_\beta, N_\beta, B_\beta\}$, and the curvatures κ_β and τ_β of affine Smarandache curves of α.

3.1. $TN-$affine Smarandache curve

Taking the derivatives $\beta(u\,(s))$ concerning u, we obtain

$$\frac{d\beta}{du} = \frac{d\beta}{ds}\frac{ds}{du} \tag{9}$$

$$\frac{d^2\beta}{du^2} = \frac{d^2\beta}{ds^2}\left(\frac{ds}{du}\right)^2 + \frac{d\beta}{ds}\frac{d^2s}{du^2} \tag{10}$$

$$\frac{d^3\beta}{du^3} = \frac{d^3\beta}{ds^3}\left(\frac{ds}{du}\right)^3 + 3\frac{d^2\beta}{ds^2}\frac{ds}{du}\frac{d^2s}{du^2} + \frac{d\beta}{ds}\frac{d^3s}{du^3} \tag{11}$$

and since u is the parameter of the affine arc-length of β, i.e., $\det(\frac{d\beta}{du}, \frac{d^2\beta}{du^2}, \frac{d^3\beta}{du^3}) = 1$, we can easily obtain

$$\left(\frac{du}{ds}\right)^6 = \det\left(\frac{d\beta}{ds}, \frac{d^2\beta}{ds^2}, \frac{d^3\beta}{ds^3}\right) \tag{12}$$

Using the relations (4) and (5) we get

$$\frac{d\beta}{ds} = \frac{1}{\sqrt{2}}(N+B)$$

$$\frac{d^2\beta}{ds^2} = \frac{1}{\sqrt{2}}(-\tau T - \kappa N + B)$$

$$\frac{d^3\beta}{ds^3} = \frac{1}{\sqrt{2}}\left[\left(-\tau' - \tau\right)T + \left(-\kappa' - \tau - \kappa\right)N - \kappa B\right]$$

and so, from the relation (12)

$$\frac{du}{ds} = \left(\frac{1}{2\sqrt{2}}\left[\left(\tau' + \tau\right)(\kappa+1) - \tau\left(\kappa' + \tau\right)\right]\right)^{1/6} \tag{13}$$

Without loss of generality, we assume that $du = ds.$ Then the affine Frenet frame's vectors are given by

$$T_\beta = \frac{1}{\sqrt{2}}N + \frac{1}{\sqrt{2}}B \tag{14}$$

$$N_\beta = -\frac{\tau}{\sqrt{2}}T - \frac{\kappa}{\sqrt{2}}N + \frac{1}{\sqrt{2}}B \tag{15}$$

and

$$B_\beta = -\frac{\tau' + \tau}{\sqrt{2}}T - \frac{\kappa' + \tau + \kappa}{\sqrt{2}}N - \frac{\kappa}{\sqrt{2}}B \tag{16}$$

Differentiating the equation (16) concerning s and using the relations (4) we obtain

$$B'_\beta = -\frac{\tau'' + \tau' - \kappa\tau}{\sqrt{2}}T - \frac{\kappa'' + 2\tau' + \kappa' + \tau + \kappa^2}{\sqrt{2}}N - \frac{2\kappa' + \tau + \kappa}{\sqrt{2}}B \tag{17}$$

Since $B'_\beta = -\tau_\beta T_\beta - \kappa_\beta N_\beta$, from the relations (14)-(17) we have

$$\kappa_\beta = -\frac{\tau'' + \tau' - \kappa\tau}{\tau}$$

and

$$\tau_\beta = -\frac{\tau'' - 2\kappa'\tau - \tau' + \tau^2 + 2\kappa\tau}{\tau}$$

Theorem 3.5. Let $\alpha : I \subseteq R \mapsto A_3$ be an affine $C^\infty-$curve in affine 3−space A_3 with the affine Frenet frame $\{T, N, B\}$ and the curvatures κ and τ. If $\beta : I \subseteq R \mapsto A_3$ is $TN-$affine Smarandache curve of α, then its frame $\{T_\beta, N_\beta, B_\beta\}$ is given by

$$\begin{bmatrix} T_\beta \\ N_\beta \\ B_\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\tau}{\sqrt{2}} & -\frac{\kappa}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\tau'+\tau}{\sqrt{2}} & -\frac{\kappa'+\tau+\kappa}{\sqrt{2}} & -\frac{\kappa}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{18}$$

and the corresponding curvature κ_β and τ_β read

$$\kappa_\beta = -\frac{\tau'' + \tau' - \kappa\tau}{\tau}, \quad \tau_\beta = -\frac{\tau'' - 2\kappa'\tau - \tau' + \tau^2 + 2\kappa\tau}{\tau} \tag{19}$$

3.2. $TB-$affine Smarandache curve

Using the relations (4) and (6) we get

$$\begin{aligned} \frac{d\beta}{ds} &= \frac{1}{\sqrt{2}}\left(-\tau T - (\kappa - 1) N\right) \\ \frac{d^2\beta}{ds^2} &= \frac{1}{\sqrt{2}}\Big(-\tau' T - \big(\kappa' + \tau\big) N - (\kappa - 1) B\Big) \\ \frac{d^3\beta}{ds^3} &= \frac{1}{\sqrt{2}}\left[-\big(\tau'' - \kappa\tau + \tau\big) T - \big(\kappa'' + 2\tau' - \kappa^2 + \kappa\big) N - \big(2\kappa' + \tau\big) B\right] \end{aligned}$$

and so, from the relation (12)

$$\frac{du}{ds} = \left(\frac{1}{2\sqrt{2}}\Big(-(\kappa-1)^2\big(\tau''+\tau\big) + (\kappa-1)\big(\tau\kappa'' + 3\tau\tau' + 2\kappa'\tau'\big) - \big(2\kappa'+\tau\big)\big(\kappa'+\tau\big)\tau\Big)\right)^{1/6} \tag{20}$$

Without loss of generality, we assume that $du = ds$. Then the affine Frenet frame's vectors are given by

$$T_\beta = -\frac{\tau}{\sqrt{2}}T - \frac{\kappa - 1}{\sqrt{2}}N \tag{21}$$

$$N_\beta = -\frac{\tau'}{\sqrt{2}}T - \frac{\kappa' + \tau}{\sqrt{2}}N - \frac{\kappa - 1}{\sqrt{2}}B \tag{22}$$

and

$$B_\beta = -\frac{\tau'' - \kappa\tau + \tau}{\sqrt{2}}T - \frac{\kappa'' + 2\tau' - \kappa^2 + \kappa}{\sqrt{2}}N - \frac{2\kappa' + \tau}{\sqrt{2}}B \tag{23}$$

Differentiating the equation (23) concerning s and using the relations (4) we obtain

$$B'_\beta = -\frac{\tau''' - 3\kappa'\tau - \kappa\tau' + \tau' - \tau^2}{\sqrt{2}}T - \frac{\kappa''' + 3\tau'' - 4\kappa\kappa' + \kappa' - 2\kappa\tau + \tau}{\sqrt{2}}N - \frac{3\kappa'' + 3\tau' - \kappa^2 + \kappa}{\sqrt{2}}B \tag{24}$$

Since $B'_\beta = -\tau_\beta T_\beta - \kappa_\beta N_\beta$, from the relations (21)-(24) we have

$$\kappa_\beta = -\frac{3\kappa'' + 3\tau' - \kappa^2 + \kappa}{\kappa - 1}$$

and

$$\tau_\beta = 3\kappa' + \tau - \frac{\tau''' - 2\kappa\tau' + \tau'}{\tau} - \frac{3\kappa''\tau' - 3(\tau')^2}{\kappa - 1}$$

where $\kappa(s) \neq 1$ for all s.

Theorem 3.6. Let $\alpha : I \subseteq R \mapsto A_3$ be an affine $C^\infty-$curve in affine 3$-$space A_3 with the affine Frenet frame $\{T, N, B\}$ and the curvatures κ and τ. If $\beta : I \subseteq R \mapsto A_3$ is $TB-$affine Smarandache curve of α, then its frame $\{T_\beta, N_\beta, B_\beta\}$ is given by

$$\begin{bmatrix} T_\beta \\ N_\beta \\ B_\beta \end{bmatrix} = \begin{bmatrix} -\frac{\tau}{\sqrt{2}} & -\frac{\kappa-1}{\sqrt{2}} & 0 \\ -\frac{\tau'}{\sqrt{2}} & -\frac{\kappa'+\tau}{\sqrt{2}} & -\frac{\kappa-1}{\sqrt{2}} \\ -\frac{\tau''-\kappa\tau+\tau}{\sqrt{2}} & -\frac{\kappa''+2\tau'-\kappa^2+\kappa}{\sqrt{2}} & -\frac{2\kappa'+\tau}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{25}$$

and the corresponding curvature κ_β and τ_β read

$$\kappa_\beta = -\frac{3\kappa'' + 3\tau' - \kappa^2 + \kappa}{\kappa - 1}, \quad \tau_\beta = 3\kappa' + \tau - \frac{\tau''' - 2\kappa\tau' + \tau'}{\tau} - \frac{3\kappa''\tau' - 3(\tau')^2}{\kappa - 1} \tag{26}$$

everywhere $\kappa(s) \neq 1$.

3.3. $NB-$affine Smarandache curve

Using the relations (4) and (7) we get

$$\begin{aligned} \frac{d\beta}{ds} &= \frac{1}{\sqrt{2}}(-\tau T - \kappa N + B) \\ \frac{d^2\beta}{ds^2} &= \frac{1}{\sqrt{2}}\left(-\left(\tau' + \tau\right)T - \left(\kappa' + \tau + \kappa\right)N - \kappa B\right) \\ \frac{d^3\beta}{ds^3} &= \frac{1}{\sqrt{2}}\left[-\left(\tau'' + \tau' - \kappa\tau\right)T - \left(\kappa'' + 2\tau' + \kappa' + \tau - \kappa^2\right)N - \left(2\kappa' + \tau + \kappa\right)B\right] \end{aligned}$$

and so, from the relation (12)

$$\frac{du}{ds} = \begin{pmatrix} \frac{1}{2\sqrt{2}}(-2(\kappa')^2\tau - 3\kappa'\tau^2 - \tau^3 + \kappa''\kappa\tau + 3\kappa\tau'\tau + 2\tau'\kappa'\kappa - \kappa^2\tau'' + \kappa^3\tau + 2(\tau')^2 \\ +\kappa''\tau' - \kappa^2\tau' + \kappa''\tau + 2\tau'\tau + \kappa'\tau + \tau^2 - \kappa'\tau'' + \kappa'\kappa\tau - \tau''\tau + \kappa\tau^2 - \kappa\tau'' - \kappa\tau') \end{pmatrix}^{1/6} \tag{27}$$

Without loss of generality, we assume that $du = ds$. Then the affine Frenet frame's vectors are given by

$$T_\beta = -\frac{\tau}{\sqrt{2}}T - \frac{\kappa}{\sqrt{2}}N + \frac{1}{\sqrt{2}}B \tag{28}$$

$$N_\beta = -\frac{\tau' + \tau}{\sqrt{2}}T - \frac{\kappa' + \tau + \kappa}{\sqrt{2}}N - \frac{\kappa}{\sqrt{2}}B \tag{29}$$

and

$$B_\beta = -\frac{\tau'' + \tau' - \kappa\tau}{\sqrt{2}}T - \frac{\kappa'' + 2\tau' + \kappa' + \tau - \kappa^2}{\sqrt{2}}N - \frac{2\kappa' + \tau + \kappa}{\sqrt{2}}B \tag{30}$$

Differentiating the equation (30) concerning s and using the relations (4) we obtain

$$\begin{aligned} B'_\beta = &-\frac{\tau''' + \tau'' - 3\kappa'\tau - \kappa\tau' - \tau^2 - \kappa\tau}{\sqrt{2}}T - \frac{\kappa''' + \kappa'' + 3\tau'' + 2\tau' - 4\kappa'\kappa - 2\kappa\tau - \kappa^2}{\sqrt{2}}N \\ &-\frac{3\kappa'' + 2\kappa' + 3\tau' + \tau - \kappa^2}{\sqrt{2}}B \end{aligned} \tag{31}$$

Since $B'_\beta = -\tau_\beta T_\beta - \kappa_\beta N_\beta$, from the relations (27)-(31) we have

$$\kappa_\beta = -\frac{\tau''' + \tau'' - \kappa'\tau - \kappa\tau' - \kappa\tau + 3\kappa''\tau + 3\tau'\tau + \kappa^2\tau}{\tau' + \tau + \kappa\tau}$$

and

$$\tau_\beta = -\frac{\kappa\tau''' + \kappa\tau'' - 3\kappa''\tau' - 3(\tau')^2 - 2\kappa'\tau' - 3\kappa''\tau - 4\tau'\tau - 2\kappa'\tau - 3\kappa'\kappa\tau - \kappa\tau^2 - \tau^2}{\tau' + \tau + \kappa\tau}$$

Theorem 3.7. Let $\alpha : I \subseteq R \mapsto A_3$ be an affine $C^\infty-$curve in affine 3−space A_3 with the affine Frenet frame $\{T, N, B\}$ and the curvatures κ and τ. If $\beta : I \subseteq R \mapsto A_3$ is $NB-$affine Smarandache curve of α, then its frame $\{T_\beta, N_\beta, B_\beta\}$ is given by

$$\begin{bmatrix} T_\beta \\ N_\beta \\ B_\beta \end{bmatrix} = \begin{bmatrix} -\frac{\tau}{\sqrt{2}} & -\frac{\kappa}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\tau'+\tau}{\sqrt{2}} & -\frac{\kappa'+\tau+\kappa}{\sqrt{2}} & -\frac{\kappa}{\sqrt{2}} \\ -\frac{\tau''+\tau'-\kappa\tau}{\sqrt{2}} & -\frac{\kappa''+2\tau'+\kappa'+\tau-\kappa^2}{\sqrt{2}} & -\frac{2\kappa'+\tau+\kappa}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{32}$$

and the corresponding curvature κ_β and τ_β read

$$\begin{aligned} \kappa_\beta &= -\frac{\tau''' + \tau'' + 3\kappa''\tau + (3\tau - \kappa)\tau' - \kappa'\tau + \left(\kappa^2 - \kappa\right)\tau}{\tau' + \tau + \kappa\tau} \\ \tau_\beta &= -\frac{\kappa\tau''' + \kappa\tau'' - 3(\tau' + \tau)\kappa'' - 3(\tau')^2 - 2\kappa'\tau' - 4\tau'\tau - (2\tau + 3\kappa\tau)\kappa' - (1 + \kappa)\tau^2}{\tau' + \tau + \kappa\tau} \end{aligned} \tag{33}$$

3.4. $TNB-$affine Smarandache curve

The next theorem can be proved analogously as in the previous three cases.

Theorem 3.8. Let $\alpha : I \subseteq R \mapsto A_3$ be an affine $C^\infty-$curve in affine 3−space A_3 with the affine Frenet frame $\{T, N, B\}$ and the curvatures κ and τ. If $\beta : I \subseteq R \mapsto A_3$ is T$NB-$affine Smarandache curve of α, then its frame $\{T_\beta, N_\beta, B_\beta\}$ is given by

$$\begin{bmatrix} T_\beta \\ N_\beta \\ B_\beta \end{bmatrix} = \begin{bmatrix} -\frac{\tau}{\sqrt{3}} & -\frac{\kappa-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{\tau'+\tau}{\sqrt{3}} & -\frac{\kappa'+\tau+\kappa}{\sqrt{3}} & -\frac{\kappa-1}{\sqrt{3}} \\ -\frac{\tau''+\tau'-(\kappa-1)\tau}{\sqrt{3}} & -\frac{\kappa''+2\tau'+\kappa'-\kappa^2+\tau+\kappa}{\sqrt{3}} & -\frac{2\kappa'+\tau+\kappa}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{34}$$

and the corresponding curvature κ_β and τ_β read

$$\begin{aligned} \kappa_\beta &= -\frac{\tau''' + \tau'' + 3\kappa''\tau + (3\tau - \kappa + 1)\tau' - \kappa'\tau - \kappa^2\tau}{\tau' + \kappa\tau} \\ \tau_\beta &= -\frac{(\kappa-1)\tau''' + (\kappa-1)\tau'' - 3(\tau' + \tau)\kappa'' - 3(\tau')^2 + (\kappa-1)\tau' - (3\kappa - 1)\kappa'\tau - (4\tau + 2\kappa')\tau' - \kappa\tau^2}{\tau' + \kappa\tau} \end{aligned}$$

4. Conclusion

Recently, many studies have been done on the curve theory in affine 3-space (see [23–26]). However, until now, Smarandache curves in affine 3-space have not been defined and their characteristics have not been examined. Therefore, in this paper, TN, TB, NB and TNB-Smarandache curves whose position vector are made by Frenet frame vectors on another regular affine C^∞-curve α with the affine Frenet frame $\{T, N, B\}$ in affine 3-space A_3 are introduced. The affine curvature κ_β, the affine torsion τ_β, and the expression of the affine frame vectors $\{T_\beta, N_\beta, B_\beta\}$ of Smarandache curves are obtained. Also, the relationship between the Frenet frames of the curve α and Smarandache curves is given.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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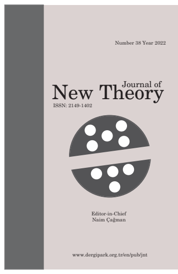

Anisotropic Conformal Model in $f(R, \phi)$ Theory

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Abstract — In this study, we examine conformal spherically symmetric space-time with anisotropic fluid in $f(R, \phi)$ theory. The exact solutions of field equations are obtained for $f(R, \phi) = (1 + \lambda\eta^2\phi^2)R$ model. All the quantities for anisotropic fluid are investigated through equation of state constant, ω. The models for three different selections of ω are represented for the constructed model. Moreover, string gas is the only condition that anisotropic fluid behaves as an isotropic fluid for the constructed model. Furthermore, the anisotropy parameter and causality conditions are examined. Lastly, the results for the solutions are concluded from the physical and geometrical viewpoint.

Keywords — *Conformal symmetry, $f(R, \phi)$ theory, extended theory, anisotropic fluid*

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1. Introduction

Generalization of Einstein-Hilbert action is quite attractive topic in recent years. It is an alternative way to understand dynamical characteristic of universe. Especially, last observations such as supernova type Ia [1–3] and cosmic microwave background radiation [4, 5] lead to expansion universe with acceleration. Although studies indicate that the universe has exhibited different dynamic behaviors in different epochs, current time expansion of universe is correlated with exotic matter components called as "dark energy" on it. Matter form has negative pressure which causes to expansion. Within this framework, many researchers described great numbers of different dark energy models associated with scalar field. Although the existence of dark energy, which has such a repulsive effect cosmologically, is sufficient to explain the movement of late time universe, the origin and dynamic structure of this form of matter cannot be fully explained theoretically. Extended theories or generalizations of Einstein-Hilbert action give researchers to examine universe from beginning to current time as an alternative way. $f(R, \phi)$ theory is one of the most attractive generalization of Einstein-Hilbert action by way of a general function depending on Ricci curvature scalar, scalar field and it's terms. Einstein-Hilbert action for $f(R, \phi)$ theory is firstly studied by Hwang and his collaborators [6–9]. Existence of scalar field in action makes the theory quite interesting in order to study different epoch of universe correlated with scalar field. In this context, Myrzakulov et al. [10] examined possible inflation scenario for some models in $f(R, \phi)$ theory. Mathew et al. [11] obtained a possible exact inflationary model in $f(R, \phi)$ theory. They showed that an inflationary model with an exit is possible in theory. Stabile and Capozzielo [12] studied galaxy rotating curves without needing dark energy in $f(R, \phi)$ theory. They showed that Yukawa-like correction in theory could be explain problem of dark matter in spiral galaxies. In theory, many cosmological issue is studied by researchers [13–17].

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Conformal Killing vectors (CKVs) are quite important symmetrical property which could be used for General Relativity caused to simplification of space-time [18]. It is easier to find exact solutions for models by way of this isometries in field equations. Also, CKVs could be considered to explore conservation laws, as well. CKVs are defined by [19]

$$\mathfrak{L}_\xi g_{ik} = \psi g_{ik} \tag{1}$$

where $\mathfrak{L}_\xi$ represent the Lie derivative operator, ψ is conformal factor and ξ is vector field that generates conformal symmetry [19]. Classification of conformal Killing vectors such as Killing vectors, Homothetic Killing vectors, special conformal Killing vectors and non-special conformal Killing vectors depends on the structure of the conformal factor, ψ [20]. In literature, great number of cosmological models and issues in various theory are examined through conformal symmetry [21–24].

In this study, our main purpose to understand effect of conformal symmetry on anisotropic model in the framework of $f(R,\phi)$ theory. In this context, we investigated conformal spherically symmetric space-time filled with anisotropic fluid in $f(R,\phi)$ theory. Kinematic term of scalar field related with dynamic structure of theory is attained for constructed model.

This study organized as: In section 2, we recaptured field equations of $f(R,\phi)$ theory. After, we obtained field equations conformal spherically symmetric space-time with anisotropic fluid in $f(R,\phi)$ theory. Exact solutions of field equations are obtained. Matter distribution investigated through equation of state constant, ω. Anisotropy parameter and causality conditions are examined. In section 3, Results for solutions have been concluded in the point of view physical and geometrical.

2. Anisotropic Conformal Spherically Symmetric Model in $f(R,\phi)$ Theory

Action function of $f(R,\phi)$ gravity can be written as [25]

$$\mathcal{S} = \int d^4\sqrt{-g}\left[\frac{1}{\kappa^2}(f(R,\phi) + u(\phi)\phi_{;\ell}\phi^{;\ell})\right] + \mathcal{S}_m \tag{2}$$

where $\mathcal{S}_m$ is Lagrange density of matter field. Also, $f(R,\phi)$ is a general function connected with Ricci curvature scalar, R, and scalar field, ϕ [25]. $u(\phi)$ represents kinematic term of scalar field and g is determination of metric tensor $g_{\mu\nu}$ [26]. Variation of Eq.(2) leads to field equations of extended theory as follows:

$$f_R R_{ik} - \frac{1}{2}(f + u(\phi)\phi_{;\ell}\phi^{;\ell})g_{ik} - f_{R;ik} + u(\phi)\phi_{;i}\phi_{;k} + g_{ik}\Box f_R = \kappa T_{ik} \tag{3}$$

where $\Box$ corresponds to d'Alambertian operator ($\Box = \nabla_\ell\nabla^\ell$). Energy-momentum tensor is defined as $T_{ik} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{S}_m)}{\delta g^{ik}}$ [27]. From Eq.(2), the generalization of Klein-Gordon equation for $f(R,\phi)$ theory is attained as [26]

$$2u(\phi)\Box\phi + u_\phi(\phi)\phi_{;\ell}\phi^{;\ell} - f_\phi = 0 \tag{4}$$

Line element of static spherically symmetric metric is described by

$$ds^2 = e^{\mu(r)}dt^2 - e^{\nu(r)}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\Phi^2 \tag{5}$$

where $\mu(r)$ and $\nu(r)$ are metric potentials depending on radial coordinate. Also, Ricci curvature scalar for selected metric is attained as

$$R = e^{-\nu}\left[\frac{1}{2}(2\ddot{\mu} - \dot{\mu}\dot{\nu} + \dot{\mu}) - \frac{2}{r}(\dot{\nu} - \dot{\mu}) + \frac{2}{r^2}\right] - \frac{2}{r^2} \tag{6}$$

where dot signs partial derivative according to radial coordinate. On the other hand, energy-momentum tensor of anisotropic fluid is given by

$$T_{ik} = (\rho + p_t)u_i u_k - p_t g_{ik} + (p_r - p_t)x_i x_k \tag{7}$$

where $\rho(r)$ is energy density, $p_r(r)$ is radial pressure and $p_t(r)$ is tangential pressure of the fluid. u_i is four-velocity in co-moving coordinates and x_i is unit four-vector along the radial direction. By using Eqs.(3),(5) and (7), field equations for spherically symmetric anisotropic fluid in $f(R,\phi)$ theory are attained in the following form:

$$e^{-\nu}\left[\frac{1}{4}f_R\left(\dot{\mu}\dot{\nu}+\frac{4\dot{\nu}}{r}-2\ddot{\mu}-\dot{\mu}^2\right)+\frac{1}{2}e^{\nu}f+\frac{1}{2}u(\phi)\dot{\phi}^2+\partial_r f_R\left(\frac{2}{r}+\frac{\dot{\mu}}{2}\right)\right]=\kappa p_r \tag{8}$$

$$e^{-\nu}\left[\frac{1}{2r}f_R\left(\dot{\nu}-\dot{\mu}+\frac{2e^{\nu}}{r}-\frac{2}{r}\right)+\frac{1}{2}e^{\nu}f-\frac{1}{2}u(\phi)\dot{\phi}^2+\frac{1}{2}\partial_r f_R\left(\frac{2}{r}+\dot{\mu}-\dot{\nu}\right)+\partial_{rr}f_R\right]=\kappa p_t \tag{9}$$

and

$$e^{-\nu}\left[\frac{1}{4}f_R\left(\dot{\mu}^2-\dot{\mu}\dot{\nu}+2\ddot{\mu}+\frac{4\dot{\mu}}{r}\right)-\frac{1}{2}e^{\nu}f+\frac{1}{2}u(\phi)\dot{\phi}^2+\frac{1}{2}\partial_r f_R\left(\dot{\nu}-\frac{4}{r}\right)-\partial_{rr}f_R\right]=\kappa\rho \tag{10}$$

It is clearly seen that constructed field equations have eight unknown components. In consequence, we have considered a viable model in $f(R,\phi)$ theory. The model is [25]

$$f(R,\phi)=(1+\lambda\eta^2\phi^2)R \tag{11}$$

Also, kinematic term of scalar field is considered as power-law form as $u(\phi)=u_0\phi^m$. In this study, we assumed constant as $m=1$. Conformal symmetry gives us to an opportunity to simplify space-time via vector field, ξ. Conformal symmetry for static spherically symmetric metric is studied by Herrera and Ponce de Leon [28]. They obtained homothetic vector fields and metric potentials in the following form:

$$\xi^a=k_1\delta_4^a+(\frac{\psi r}{2}\delta_1^a) \tag{12}$$

$$e^{\mu}=k_2^2r^2 \tag{13}$$

and

$$e^{\nu}=\frac{k_3^2}{\psi^2} \tag{14}$$

Line element of conformal spherically symmetric metric could be defined as

$$ds^2=k_2^2r^2dt^2-\frac{k_3^2}{\psi^2}dr^2-r^2d\theta^2+r^2\sin^2\theta d\Phi^2 \tag{15}$$

Under all consideration, it is possible to rewrite field equations of constructed model in $f(R,\phi)$ theory in the following form:

$$\left(\frac{1}{r^2}+\frac{\lambda\eta^2\phi^2}{r^2}\right)\left(\frac{3\psi}{k_3^2}-1\right)+\frac{1}{2}\frac{\psi^2}{k_3^2}\phi\dot{\phi}\left(\omega\phi+12\frac{\lambda\eta^2}{r}\right)=\kappa p_r \tag{16}$$

$$2\frac{\psi}{rk_3^2}\left(\dot{\psi}+\frac{\psi}{2r}\right)+2\frac{\psi\dot{\psi}}{k_3^2}\lambda\eta^2\phi\left(\frac{\phi}{r}+\dot{\phi}\right)+2\frac{\psi^2}{k_3^2}\lambda\eta^2\dot{\phi}\left(2\frac{\phi}{r}+\dot{\phi}\right)+\frac{\psi^2}{k_3^2}\lambda\eta^2\phi\left(\frac{\phi}{r^2}+2\ddot{\phi}\right)-\frac{1}{2}\frac{\psi^2}{k_3^2}u_0\phi\dot{\phi}^2=\kappa p_t \tag{17}$$

and

$$-2\frac{\psi}{rk_3^2}\left(\dot{\psi}+\frac{\psi}{2r}\right)+\frac{1}{r^2}\left(1+\lambda\eta^2\phi^2\right)-2\frac{\psi\dot{\psi}}{k_3^2}\lambda\eta^2\phi\left(\frac{\phi}{r}+\dot{\phi}\right)-2\frac{\psi^2}{k_3^2}\lambda\eta^2\dot{\phi}\left(2\frac{\phi}{r}+\dot{\phi}\right)$$
$$-2\frac{\psi^2}{k_3^2}\lambda\eta^2\phi\left(\frac{\phi}{2r^2}+\ddot{\phi}\right)+\frac{1}{2}\frac{\psi^2}{k_3^2}u_0\phi\dot{\phi}^2=\kappa\rho \tag{18}$$

In order to simplify our solution, it is considered constants in $f(R,\phi)$ model as $\lambda=-1$ and $\eta=1$. Also, equation of state between radial pressure and density of fluid is given by

$$p_r=\omega\rho \tag{19}$$

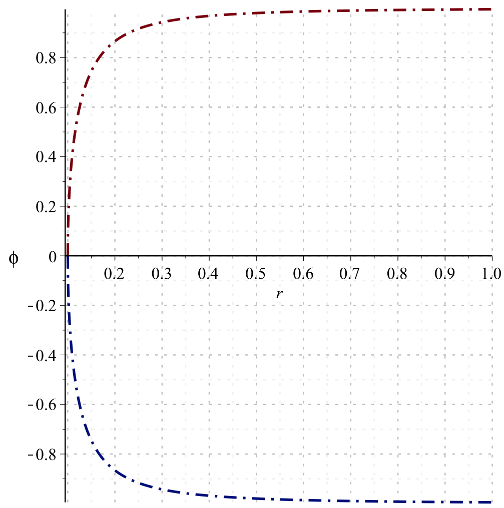

Fig. 1. Evolution of scalar field with r. Positive scalar field (red line) and positive scalar field (red line) are represented for $k_4 = -0.01$.

By using Eqs.(16)-(19), we obtained conformal factor as

$$\psi(r) = \frac{\sqrt{2k_3^2 r(\omega^2-1)\sqrt{r^2+k_4}(6r\sqrt{r^2+k_4}-k_4u_0)}}{6(\omega-1)(\frac{1}{6}k_4u_0 - r\sqrt{r^2+k_4})} \tag{20}$$

Scalar field of theory is obtained as:

$$\phi(r) = \pm\frac{\sqrt{r^2+k_4}}{r} \tag{21}$$

Evolution of scalar field is represented with respect to radial coordinate in Fig. 1. The scalar field is estranged from x-axis for bigger value of radial coordinate. Also, constructed model refers to both complex scalar field and complex conformal factor depending on value of k_4. In order to avoid that condition, it is a way to define critical radius for constructed model. It could be given by $r_{cri}^2 > -k_4$. Also, matter components such as density, radial pressure and tangential pressure are attained as

$$\rho(r) = \frac{2k_4}{\kappa r^4(\omega-1)} \tag{22}$$

$$p_r(r) = \frac{2\omega k_4}{\kappa r^4(\omega-1)} \tag{23}$$

and

$$p_t(r) = -\frac{k_4(\omega+1)}{\kappa r^4(\omega-1)} \tag{24}$$

Line element of conformal spherically symmetric space-time in the presence of anisotropic fluid in $f(R,\phi)$ theory is rewritten as follows:

$$ds^2 = k_2^2 r^2 dt^2 - \frac{(\omega-1)(6r\sqrt{r^2+k_4}-k_4u_0)}{2r(\omega+1)\sqrt{r^2+k_4}}dr^2 - r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2 \tag{25}$$

One can get a physically meaningful density in the case of $\rho > 0$. In Eq.(22), it depends on arbitrary constant, k_4, and ω. k_4 must be negative when equation of state parameter is selected as $\omega < 1$. On the other hand, k_4 can be selected as positive in the case of $\omega > 1$ which isn't expressive because this condition refers to sound speed bigger than the speed of light. In order to get physically meaningful matter distribution, one must select arbitrary constant as negative for constructed model. Also, it is obviously seen that constructed model does not allow to value of $\omega = 1$. In Eq.(24), tangential

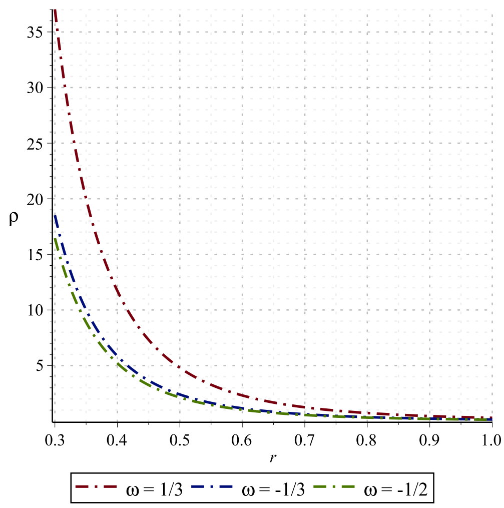

Fig. 2. Evolution of energy density with r for different selection of ω. ($k_4 = -0.1$ and $\kappa = 1$)

pressure and radial pressure could be attractive or repulsive depending on value of ω. In the case of $-1 < \omega < 1$, tangential pressure has repulsive effect. Also, radial pressure has same effect for $\omega < 0$ or $\omega > 1$, as well. At the same time, tangential pressure given by Eq.(24) vanishes for selection of $\omega = -1$. Under this assumption, it could be said that cosmological constant case breaks structure of anisotropy for constructed model in $f(R, \phi)$ theory. In Fig. 2, graphical representation of energy density for different selection of ω is represented with respect to radial coordinate. Energy density for all selections is decreasing for bigger value of radial coordinate. In Fig. 3 and Fig. 4, graphical representations of radial and tangential pressures are represented with respect to radial coordinate. Both figure shows that pressure components of fluid are approaching x-axis with bigger value of radial coordinate. For $\omega = 1/3$, radial pressure has attractive effect, while tangential pressure is repulsive effect for constructed model. In addition to this, cases of $\omega = -1/3$ and $\omega = -1/2$, both pressure components show repulsive behavior for constructed model.

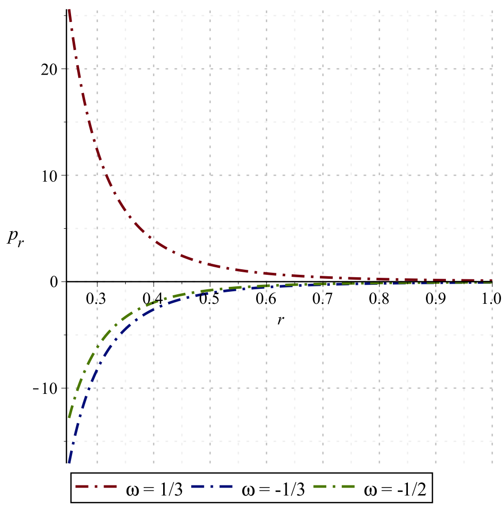

Fig. 3. Evolution of radial pressure with r for different selection of ω. ($k_4 = -0.1$ and $\kappa = 1$)

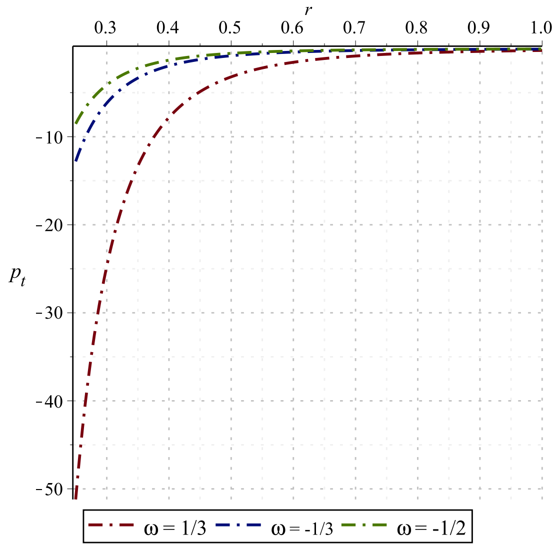

Fig. 4. Evolution of tangential pressure with r for different selection of ω . ($k_4 = -0.1$ and $\kappa = 1$)

Also, spherically symmetric anisotropic models give a opportunity to investigate behavior of radial structures by way of anisotropy measurement indicated by $\triangle$. $\triangle$ is represented as $\triangle = \kappa(p_t - p_r)$. Positive values of $\triangle$ are correlated with outward pressure, otherwise the opposite occurs on it. For constructed model, anisotropy is

$$\triangle = \kappa p_t - p_r = -\frac{k_4(3\omega + 1)}{r^4(\omega - 1)} \tag{26}$$

Considering that arbitrary constant, k_4, is negative, as we studied earlier, constructed model in $f(R, \phi)$ theory indicates outward pressure in the cases of $\omega > 1$ which is physically meaningless and $\omega < -1/3$ which corresponds to dark energy. At the same time, isotropic case ($p_t = p_r = p$) for fluid is possible in the case of $\omega = -1/3$. Under that condition, fluid behaves as string cloud. Also constructed model could be practised for radial objects under some conditions. So, it is good to examine radial and transverse sound velocities for constructed model.

$$\upsilon_{sr}^2 = \frac{dp_r}{d\rho} = \omega \tag{27}$$

and

$$\upsilon_{st}^2 = \frac{dp_t}{d\rho} = -\frac{1}{2}(\omega + 1) \tag{28}$$

Both velocities must be satisfied for condition given by $0 < \upsilon_{sd,st}^2 \leq 1$. From Eqs.(27) and (28), both velocities could not be satisfied because $0 < \omega < 1$ case allows valid radial sound, while $-3 \leq \omega < -1$ case is possible condition in order to get valid transverse sound. Even thought both condition can not be satisfied for same region for ω, one can examine the stabile region in the case of $-1 \leq \upsilon_{st}^2 - \upsilon_{sr}^2 \leq 0$ offered by Abreu *et al.* [29] for constructed model. Stabile region for constructed model is described as $-1/3 \leq \omega \leq 1/3$.

3. Conclusion

In this study, we investigated conformal spherically symmetric space-time in the presence of anisotropic fluid in $f(R, \phi)$ theory. Firstly, we get field equation for field equations of spherically symmetric space-time with anisotropic fluid in $f(R, \phi)$ theory. After that, we considered conformal symmetry for spherically symmetric metric offered by Herrera and Ponce de Leon [28] and field equations of anisotropic conformal symmetric model are examined in $f(R, \phi)$ theory. Exact solution of field equation for constructed model are obtained to take notice of equation of state for $f(R, \phi) = (1 + \lambda\eta^2\phi^2)R$ model.

We defined line element of conformal spherically symmetric metric with anisotropic fluid in $f(R, \phi)$ theory. Also, it is shown that constructed model allow complex scalar field. In order to avoid complex scalar field, critical radius is defined for constructed model. At the same time, effect of scalar field is increasing with bigger value of radial coordinate in theory. All matter components are investigated via arbitrary constants and equation of state parameter. Constructed model allows negative value of arbitrary constant, k_4. All possible cases for repulsive or attractive behaviors of radial and tangential pressure are examined by favour of ω. Anisotropic conformal spherically symmetric model in $f(R, \phi)$ theory for selected model has singularity for $\omega = 1$. Also, cosmological constant case ($\omega = -1$) breaks structure of anisotropy for constructed model in $f(R, \phi)$ theory. For different selection of ω, behaviors of matter components with respect to radial coordinate are investigated via graphical representations of them. All quantities approaches to zero for bigger values of radial coordinate. Anisotropy parameter is investigated for conformal spherically symmetric model. The parameter shows that outward pressure is possible when $\omega < -1/3$. $\omega < -1/3$ corresponds to dark energy. For constructed model, $f(R, \phi)$ theory designates dark energy for outward pressure. Also, $\omega = -1/3$ case which corresponds to string gas is a only condition that anisotropic fluid behaves as isotropic fluid. Lastly, constructed spherical model could be practised for radial objects under some conditions. For this reason, we investigated causality condition and we defined stabile region for constructed model.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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Internal Cat-1 and XMod

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Abstract — In this study, internal categories in the category of cat-1 groups, 1-cat, are determined, and it is investigated whether there is a natural equivalence between the category of these categories and the category of internal categories within the category of crossed modules of groups, XMod.

Keywords — *Crossed module, cat-1 group, internal category, natural equivalence*

Mathematics Subject Classification (2020) — 18D35, 18G45

1. Introduction

The equivalence between the homotopy category of connected CW-complexes X whose homotopy groups $\pi_i(X)$ are trivial for $i > 1$ and the category of groups is well known. In [1] it is given that an analogous equivalence for $i > n+1$ (where n is a constant natural number). Whitehead invented the concept of a crossed module for $n = 1$. This notion replaces that of a group and gives a satisfactory answer. Loday reformulates the concept of crossed module to produce an "n-cat-group", which is a generalization to any n.

The notion of a cat-1 group is merely another method to express the axioms of a strict two-group. Nevertheless the type of characterization used for cat-1 groups, and, modified for cat-n groups is in strictly group theoretic terms and so is frequently better to check than the more categorically defined variant. As an example, getting a cat-1 group structure from a simplicial group, or a cat-n group structure from an n-fold simplicial group is typically easy. The notion of a category can be formulated internal to any other category with enough pullbacks. Since algebraic structures can be defined in a category by giving suitable objects and morphisms, we can sometimes construct categories within a category, $\mathfrak{C}$. In order to form a category (with objects O and morphisms A) inside $\mathfrak{C}$, we need to define a composition $m : A_t \times_s A \to A$ which is associative and respects identities; note in particular that m is also a morphism in $\mathfrak{C}$.

A cat-1 group is essentially another way of expressing an internal category in the category of Groups, Grp, where the kernel commutator condition determines the interchange law. It is well-known that these latter objects are equivalent to crossed modules, and so it's not surprising to see an equivalence between the category of cat-1 group and that of crossed modules in Loday's study.

Alp and Wensley present a share package XMOD consisting functions for computing with finite, permutation crossed modules, cat-1 groups and their morphisms, written using the GAP group theory

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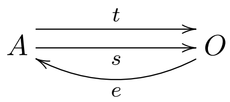
programming language [2]. Also in [3,4], Porter generalized the category of categorical groups to that of categories of groups with operations.

The main object of this paper is to formulate internal categories for the category of cat-1 group in which the objects contain elements we can describe pullbacks in terms of. Also, it is observed that the equivalence of between internal 1-Cat group and internal crossed modules. As we know the category of cat-1 group is equivalent to that of crossed modules, we expect to be able to go between internal category of cat-1 groups and that of crossed modules without hindrance, and we can prove that the equivalence between the category of cat-1 group and that of crossed modules is also preserved for their internal categories.

2. Preliminaries

Definition 2.1. Let $\mathfrak{C}$ be a category with finite products. The internal category $\mathcal{C}$ in $\mathfrak{C}$ consists of the objects A, O with the morphisms $s, t : A \longrightarrow O$, e: O $\longrightarrow A$, $m : A \times A \longrightarrow A$. The diagram of the morphisms



has the following equalities:

$$i.se = te = id_O$$
$$ii.sm = s\pi_2, tm = t\pi_1$$
$$iii.m(1_A \times m) = m(m \times 1_A)$$
$$iv.m(es, 1_A) = m(1_A, et) = 1_A$$

where π_1 and π_2 are the projections. In the internal category s, t, e and m are denoted the source, target, identity and composition morphism, respectively.

(A, O, s, t, e, m) is called the internal category of $\mathcal{C}$ in $\mathfrak{C}$, [5].

Definition 2.2. A cat-1 group consists of a group G with a normal subgroup N and the morphisms s, t from G to N satisfied the following conditions:

- $s|_N = t|_N = id_N$
- $[Kers, Kert] = 1$

A cat-1 group is denoted by (G, N, s, t), [1], [6].

Definition 2.3. Let (G, N, s, t) and (G', N', s', t') be cat-1 groups. A cat-1 group morphism $(G, N, s, t) \to (G', N', s', t')$ is an $\alpha : G \to G'$ group homomorphism satisfied the below equations:

Example 2.4. Let G be a group with a normal subgroup $N = G$. We get a cat-1 group (G, G, id_G, id_G) for $s = t = id_G$.

Example 2.5. Let G be an abelian group with a normal subgroup $N = \{1\}$. $(G, \{1\}, s, t)$ is a cat-1 group with $s(g) = t(g) = 1$ for $g \in G$.

Definition 2.6. Let G and N be two groups, $\partial : N \longrightarrow G$ a group homomorphism and G acts on N on the left. So (G, N, ∂) is a crossed module if and only if

(CM1) $\partial(g \cdot n) = g + \partial(n) - g$
(CM2) $\partial(n) \cdot n_1 = n + n_1 - n$

for $\forall n, n_1 \in N$ and $g \in G$, [7].

Definition 2.7. Let (G_1, N_1, ∂_1) and (G_0, N_0, ∂_0) be crossed modules. The crossed module morphism

$$(\alpha, \beta) : (G_0, N_0, \partial_0) \to (G_1, N_1, \partial_1)$$

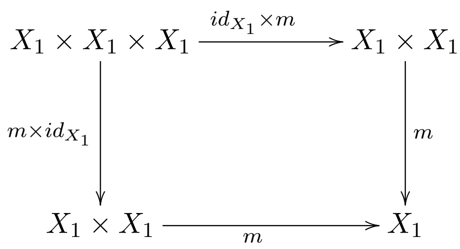
is a pair of homomorphisms $\alpha : G_0 \to G_1$ and $\beta : N_0 \to N_1$ such that

- $\partial_1\beta(n) = \alpha\partial_0(n)$, for all $n \in N_0$,
- $\beta({}^{g_0}n_0) = {}^{\alpha(g_0)}\beta(n_0)$, for all $g_0 \in G_0, n_0 \in N_0$, [8].

Definition 2.8. Let $XMod$ be a category of crossed modules over groups and $\mathfrak{X}$ be an internal category. So $\mathfrak{X}$ includes $X_1 = (A_1, B_1, \partial_1)$ and $X_0 = (A_0, B_0, \partial_0)$ with source $s = (s_A, s_B)$, target $t = (t_A, t_B)$, identity $e = (e_A, e_B)$ and composition $m = (m_A, m_B)$ morphism defined by $m_A(a_1, a_1') = a_1 \circ a_1'$, $m_B(b_1, b_1') = b_1 \circ b_1'$ with $s_1(a_1) = t_1(a_1')$ and $s_0(b_1) = t_0(b_1')$. Then we have the following features:

$$i.s_Ae_A = t_Ae_A = id_{A_0}, s_Be_B = t_Be_B = id_{B_0}$$
$$ii.s_Am_A = s_A\pi_2, t_Am_A = t_A\pi_1, s_Bm_B = s_B\pi_2, t_Bm_B = t_B\pi_1$$
$$iii.m(1_{X_1} \times m) = m(m \times 1_{X_1})$$
$$iv.m(e_As_A, 1_{X_1}) = m(1_{X_1}, e_At_A) = 1_{X_1}, m(e_Bs_B, 1_{X_1}) = m(1_{X_1}, e_Bt_B) = 1_{X_1}$$

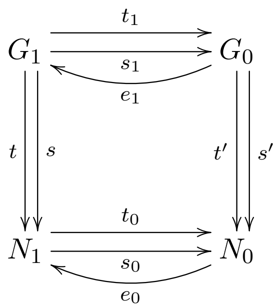
The condition iii can be expressed the following diagram:



$\mathcal{X} = (X_1, X_0, s, t, e, m)$ is called the internal category of crossed modules over groups, [9], [10], [11].

3. Internal of cat-1

Let C be an internal category in the category 1-Cat of cat-1 groups. Then $\mathcal{C}$ consists of two cat-1 groups $X_1 = (G_1, N_1, s, t)$, $X_0 = (G_0, N_0, s', t')$ with $s^* = (s_1, s_0)$, $t^* = (t_1, t_0)$, $e^* = (e_1, e_0)$ illustrated below diagram



and $m = (m_G, m_N) : X_1 \times X_1 \longrightarrow X_1$ morphisms. $(G_1, G_0, s_1, t_1, m_1)$ and $(N_1, N_0, s_0, t_0, m_0)$ are the internal of groups.

$$i.s_1e_1 = t_1e_1 = id_{G_0}, s_0e_0 = t_0e_0 = id_{N_0}$$
$$ii.s_1m_G = s_1\pi_2, t_1m_G = t_1\pi_1, s_0m_N = s_0\pi_2, t_0m_N = t_0\pi_1$$
$$iii.m(1_{G_1} \times m) = m(m \times 1_{G_1})$$
$$iv.m(e_1s_1, 1_{G_1}) = m(1_{G_1}, e_1t_1) = 1_{G_1}, m(e_0s_0, 1_{N_1}) = m(1_{N_1}, e_0t_0) = 1_{N_1}$$

where $m = (m_G, m_N)$ is the composition map for the internal of the category Grp of groups and $m_G(g_1, g_1') = g_1 \circ g_1'$, $m_N(n_1, n_1') = n_1 \circ n_1'$ with $s_1(g_1) = t_1(g_1')$ and $s_0(n_1) = t_0(n_1')$. The category of internal categories within the category of cat-1 groups is denoted by C(Int(cat-1)).

Proposition 3.1. Let (G_1, N_1, s, t) be a cat-1 group. Then $(G_1 \times G_1, N_1 \times N_1, (s, s), (t, t))$ is a cat-1 group and $(X_1, X_0, s^*, t^*, e^*, m^*)$ becomes an internal category in 1-Cat where $X_1 = (G_1 \times G_1, N_1 \times N_1, (s, s), (t, t)), X_0 = (G_1, N_1, s, t), s^* = (s_G, s_N), t^* = (t_G, t_N), e^* = (e_G, e_N), m = (m_G, m_N)$.

PROOF. We show that $(G_1 \times G_1, N_1 \times N_1, (s,s), (t,t))$ is a cat-1 group.

$$\begin{aligned} Ker(s,s) &= \{(g,g') \in G_1 \times G_1 | (s,s)(g,g') = 1_{N_1 \times N_1}\} \\ &= \{(g,g') \in G_1 \times G_1 | (sg, sg') = (1,1)\} \\ &= Kers \times Kers \end{aligned}$$

As the same way, $Ker(t,t) = Kert \times Kert$.
Since for $(n_1, n_1') \in N_1 \times N_1, (g_1, g_2) \in Kers, (g_3, g_4) \in Kert$ the equations

$$(s,s)|_{N_1 \times N_1}(n_1, n_1') = (s(n_1), s(n_1')) = (n_1, n_1') = Id_{N_1 \times N_1}(n_1, n_1'),$$

$$(t,t)|_{N_1 \times N_1}(n_1, n_1') = (t(n_1), t(n_1')) = (n_1, n_1') = Id_{N_1 \times N_1}(n_1, n_1')$$

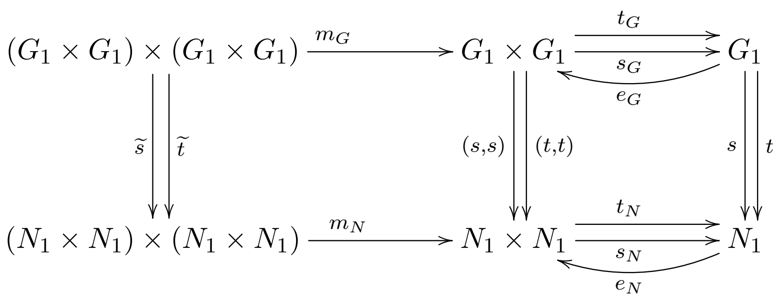
and

$$\begin{aligned} (g_1, g_2)(g_3, g_4)(g_1^{-1}, g_2^{-1})(g_3^{-1}, g_4^{-1}) &= (g_1 g_3 g_1^{-1} g_3^{-1}, g_2 g_4 g_2^{-1} g_4^{-1}) \\ &= (1,1) \end{aligned}$$

are valid, the cat-1 group conditions are satisfied.
Also, we get

$$s_N(s,s)(g_1, g_1') = s_N(sg_1, sg_1') = s(g_1)s(g_1') = s(g_1 g_1') = ss_G(g_1, g_1')$$

$t_N(s,s) = st_G, tt_G = t_N(t,t)$ and $st_G = t_N(s,s)$ for the commutativity of below diagram. Thus, s^* and t^* are cat-1 group morphisms.



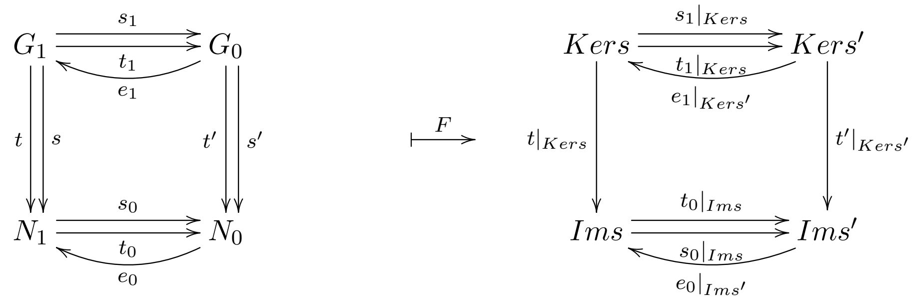
(m_G, m_N) is a morphism of cat-1 groups because of the diagram's commutativity. □

4. Natural Equivalence

Theorem 4.1. The category of internal categories within the category of cat-1 groups is natural equivalent to the category of internal categories within the category of crossed modules over groups.

PROOF. Let F be a functor

$$\mathcal{C}(Int(cat\text{-}1)) \xrightarrow{F} C(Int(XMod))$$



where $s_1|_{Kers}, t_1|_{Kers}, s_0|_{Ims}$ and $t_0|_{Ims}$ are well-defined morphisms with the following relations:

$$s't_1|_{Kers}(x) = t_0 s(x) = t_0(1) = 1$$

and

$$s's_1|_{Kers}(x) = s_0' s(x) = s_0'(1) = 1$$

for $x \in Kers$,

$$s_0|_{Ims}(n) \in Ims'$$

and

$$t_0|_{Ims}(n) \in Ims'$$

with $n = s(g), s_0 s(g) = s's_1(g)$ and $t_0|_{Ims}(n) = t_0|_{Ims} s(g) = s't_1(g)$ for $g \in G$ and $n \in Ims$.

Since $t't_1(g) = t_0 t(g)$ and $s's_1(g) = s_0 s(g)$ for $g \in G_1$, we get

$$t'|_{Kers'} t_1|_{Kers}(x) = t_0|_{Ims} t|_{Kers}(x)$$

for $x \in Kers$. $t|_{Kers} : Kers \to Ims$ is a crossed module with the conjugation action of Ims on $Kers$, [1]. The composition $m_1 : Kers \times Kers \to Kers$

$$m_1(x, y) = m_G|_{Kers}(x, y) = m_N(s, s)(x, y)$$

is well-defined group homomorphism for $x, y \in Kers$, since

$$sm_1(x, y) = sm_G|_{Kers}(x, y) = m_N(s, s)(x, y) = m_N(sx, sy) = m_N(1, 1) = 1.$$

It is clear that $m_0 : Ims \times Ims \to Ims$

$$m_0(a, b) = m_N(a, b)$$

is also well-defined for $a, b \in Ims$.

$$\begin{array}{ccc} Kers \times Kers & \xrightarrow{m_1} & Kers \\ \downarrow{\scriptstyle (t|_{Kers}, t|_{Kers})} & & \downarrow{\scriptstyle t|_{Kers}} \\ Ims \times Ims & \xrightarrow{m_0} & Ims \end{array}$$

The above diagram is commutative:

$$\begin{array}{rcl} t|_{Kers} m_1(x, y) & = & t|_{Kers}(m_G|_{Kers}(x, y)) = t|_{Kers} m_N(s(x), s(y)) \\ m_0(t|_{Kers}, t|_{Kers})(x, y) & = & m_0(t|_{Kers}(x), t|_{Kers}(y)) = m_N(t|_{Kers}(x), t|_{Kers}(y)) = t|_{Kers} m_G(x, y) \end{array}$$

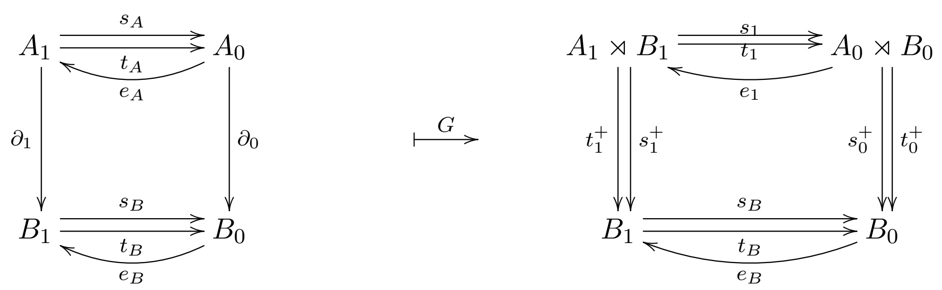
for $x, y \in Kers$. Also,

$$\begin{array}{rcl} {}^{m_0(a,b)} m_1(x, y) & = & m_N(a, b) m_1(x, y) m_N(a, b)^{-1} \\ & = & m_G(a, b) m_G(x, y) m_G(a, b)^{-1} \\ & = & m_G((a, b)(x, y)(a, b)^{-1}) \\ & = & m_G(axa^{-1}, byb^{-1}) \\ & = & m_G({}^a x, {}^b y) \\ & = & m_1({}^{(a,b)}(x, y)) \end{array}$$

for $a, b \in Ims$. Thus (m_0, m_1) is a crossed module morphism.

Let G be a functor as the following

$$C(Int(XMod)) \xrightarrow{G} C(Int(cat{-}1))$$



where

$$\begin{aligned} s_1^+(a_1,b_1) &= b_1 \\ s_0^+(a_0,b_0) &= b_0 \end{aligned}$$

and

$$\begin{aligned} t_1^+(a_1,b_1) &= \partial_1(a_1)b_1 \\ t_0^+(a_0,b_0) &= \partial_0(a_0)b_0 \end{aligned}$$

for $(a_1,b_1) \in A_1 \rtimes B_1$,$(a_0,b_0) \in A_0 \rtimes B_0$. It is clear that the first axiom of the definition of cat-1 group is satisfied.

$$\begin{aligned} Kers_1^+ &= \{(a_1,b_1) \in A_1 \times B_1 | s_1^+(a_1,b_1) = b_1 = 1_{B_1}\} = A_1 \times 1_{B_1} \\ Kert_1^+ &= \{(a_1,b_1) \in A_1 \times B_1 | t_1^+(a_1,b_1) = \partial_1(a_1)b_1 = 1_{B_1}\} \\ &= \{(a_1^{-1}, \partial_1(a_1)) | a_1 \in A_1\} \end{aligned}$$

Since ∂_1 is a crossed module, we have $a_1 {a'_1}^{-1} = {a'_1}^{-1}({}^{\partial_1(a'_1)}a_1)$.

Thus $[Kers_1^+, Kert_1^+] = 1$ because of

$$(a_1,1)({a'_1}^{-1},\partial_1(a'_1))((({a'_1}^{-1},\partial_1(a'_1))(a_1,1))^{-1} = (a_1{a'_1}^{-1},\partial_1(a'_1))({a'_1}^{-1}({}^{\partial_1(a'_1)}a_1),\partial_1(a'_1))^{-1} = 1,$$

[1]. So, $(A_1 \rtimes B_1, B_1, s^*, t^*)$ is a cat-1 group.

The composition $m_A : (A_1 \rtimes B_1) \times (A_1 \rtimes B_1) \to A_1 \rtimes B_1$ given by

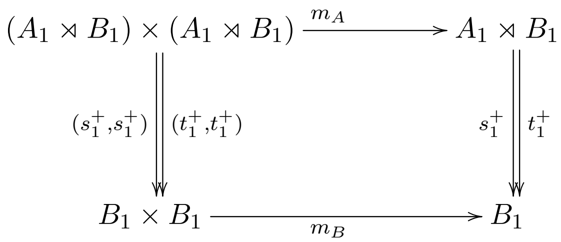
$$m_A(((a_1,b_1),(a'_1,b'_1)) = (m_1((a_1,a'_1)), m_0((b_1,b'_1)))$$

is a group homomorphism with

$$\begin{aligned} m_A(((a_1,b_1),(a'_1,b'_1))((\overline{a_1},\overline{b_1}),(\overline{a'_1},\overline{b'_1}))) &= m_A((a_1,b_1) * (\overline{a_1},\overline{b_1}),(a'_1,b'_1) * (\overline{a'_1},\overline{b'_1})) \\ &= m_A((a_1^{b_1}\overline{a_1}, b_1\overline{b_1}),({a'_1}^{b'_1}\overline{a'_1}, b'_1\overline{b'_1})) \\ &= (m_1(a_1^{b_1}\overline{a_1}, {a'_1}^{b'_1}\overline{a'_1}), m_0(b_1\overline{b_1}, b'_1\overline{b'_1})) \\ &= (m_1((a_1,a'_1)^{(b_1,b'_1)}(\overline{a_1},\overline{a'_1})), m_0((b_1,b'_1)(\overline{b_1},\overline{b'_1}))) \\ &= (m_1(a_1,a'_1)m_1({}^{(b_1,b'_1)}(\overline{a_1},\overline{a'_1})), m_0(b_1,b'_1)m_0(\overline{b_1},\overline{b'_1}))) \\ &= (m_1(a_1,a'_1)^{m_0(b_1,b'_1)}m_1(\overline{a_1},\overline{a'_1}), m_0(b_1,b'_1)m_0(\overline{b_1},\overline{b'_1})) \\ &= (m_1(a_1,a'_1), m_0(b_1,b'_1)) * (m_1(\overline{a_1},\overline{a'_1}), m_0(\overline{b_1},\overline{b'_1})) \\ &= m_A((a_1,b_1),(a'_1,b'_1)) * m_A((\overline{a_1},\overline{b_1}),(\overline{a'_1},\overline{b'_1})). \end{aligned}$$

Also, since $(X_1, X_0, s, t, e, (m_1, m_2))$ in $C(Int(XMod))$ where $X_1 = (A_1, B_1, \partial_1), X_0 = (A_0, B_0, \partial_0), s = (s_A, s_B), t = (t_A, t_B), e = (e_A, e_B)$, we get group morphism

$$m_2 = m_B : B_1 \times B_1 \to B_1.$$

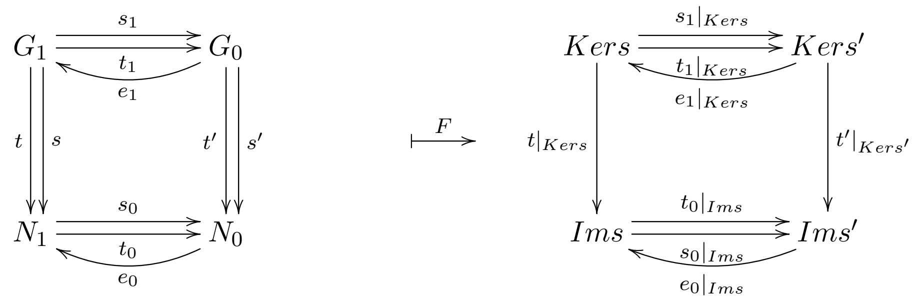


In this diagram,

$$\begin{aligned} s_1^+ m_A &= m_B(s_1^+, s_1^+) \\ t_1^+ m_A &= m_B(t_1^+, t_1^+) \end{aligned}$$

are hold. So, (m_A, m_B) is a cat-1 group morphism.

For any object $(X_1, X_0, s^*, t^*, e^*, m)$ in $C(Int(cat{-}1))$ with $X_1 = (G_1, N_1, s, t), X_0 = (G_0, N_0, s', t'), s^* = (s_1, s_0), t^* = (t_1, t_0), e^* = (e_1, e_0)$ and $m = (m_G, m_N)$, we get the following diagrams



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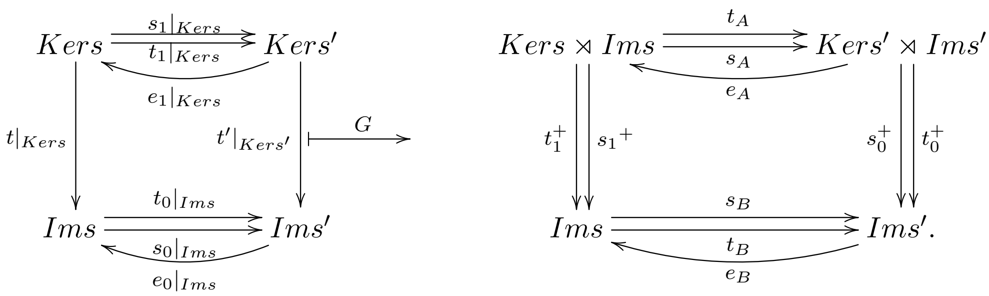
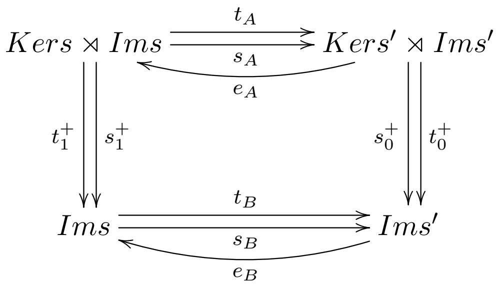


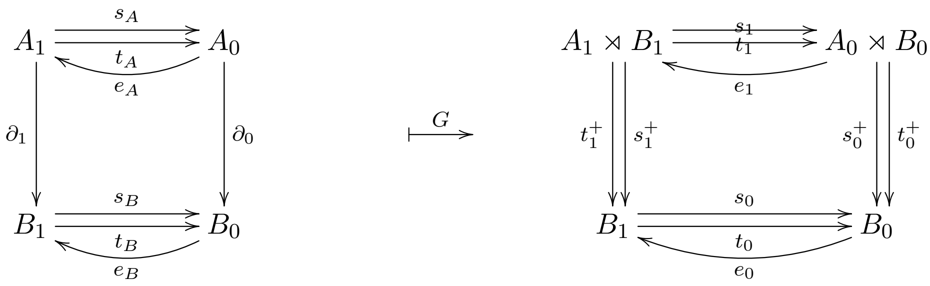
Thus, we have $GF(X_1, X_0, s^*, t^*, e^*, m)$ as an object in $C(Int(cat{-}1))$.



Also, we have $G_1 \cong Kers \rtimes Ims, N_1 \cong Ims, G_0 \cong Kers' \rtimes Ims'$ and $N_0 \cong Ims'$ and the natural transformation

$$\zeta : 1_{C(Int(cat-1))} \Rightarrow GF$$

Conversely, for any object (X_1, X_0, s, t, e, m) in $C(Int(XMod))$ where $X_1 = (A_1, B_1, \partial_1), X_0 = (A_0, B_0, \partial_0)$, we get the following diagrams



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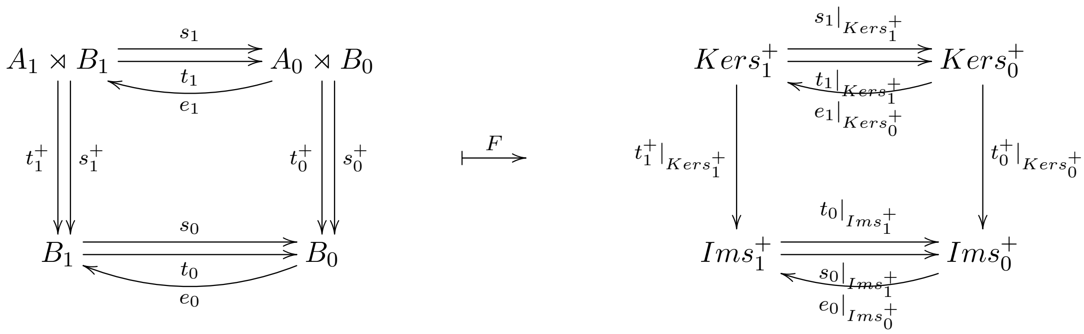
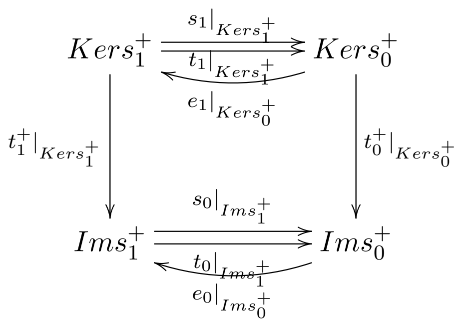


Therefore, we get $F(G(X_1, X_0, s, t, e))$



as an object in $C(Int(XMod))$. We can easily find

$$Kers_1^+ \cong A_1, Kers_0^+ \cong A_0$$
$$Ims_1^+ \cong B_1, Ims_0^+ \cong B_0.$$

So, there is a

$$\xi : 1_{C(Int(XMod))} \Rightarrow FG$$

natural transformation.

Finally, there is a natural equivalence between the category of internal categories in cat-1 and the category of internal categories in XMod. □

5. Conclusion

It is possible that each category containing pullbacks can generate other categories inside that category. By this idea, we construct internal categories in the category of cat-1 groups. Since the category of crossed modules is equivalent to that of cat-1 groups, we conclude that this equivalence is also between their internal categories valid. This idea can be extended to other equivalent categories.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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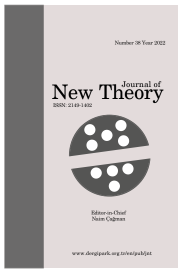

Geometry of Curves with Fractional Derivatives in Lorentz Plane

Meltem Öğrenmiş[1]

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Abstract — In this paper, the geometry of curves is discussed based on the Caputo fractional derivative in the Lorentz plane. Firstly, the tangent vector of a spacelike plane curve is defined in terms of the fractional derivative. Then, by considering a spacelike curve in the Lorentz plane, the arc length and fractional ordered frame of this curve are obtained. Later, the curvature and Frenet-Serret formulas are found for this fractional ordered frame. Finally, the relation between the fractional curvature and classical curvature of a spacelike plane curve is obtained. In the last part of the study, considering the timelike plane curve in the Lorentz plane, new results are obtained with the method in the previous section.

***Keywords* –** *Caputo fractional derivative, tangent vectors, curvature, Frenet-Serret formulas*

Mathematics Subject Classification (2020) – 26A33, 53A04

1. Introduction

The fractional derivative was first established in the 17th century and with an adding number of studies, it has come the focus of attention for many researchers in numerous fields. Fractional analysis has lately become one of the important fields of study in differential geometry. While, in the classical sense, the differential and integral are determined by integer order, in fractional calculus the orders of the differential and integral are not necessarily integers but any real number. That is, fractional calculus is the generalization of ordinary differential and integral to arbitrary order. The difference between the fractional derivative from the integer derivative is that it is given by the integration of a function.

Many studies have been conducted on this subject, and it can be found in detail [1-4]. We can also say that a non-local fractional derivative of a function is related to history or a space-range interaction. Furthermore, fractional calculus has many applications to viscoelastic [5-11], analytical mechanics [12-14], and dynamical systems [15-19]. Fractional analysis has also started to be studied from a differential geometry perspective in recent studies. There are many types of fractional operators, but it is recommended to study the geometry of curves and surfaces mostly based on the Caputo fractional derivative [20]. However, the Caputo fractional derivative is not yet directly used to formulate the differential geometry of curves. Using the Caputo fractional derivative is more appropriate than other fractional derivative operators for formulating a geometric theory since the fractional derivative of the constant function is zero [21-25]. Based on the advantages of the Caputo fractional derivative, it is discussed in [22,24] as a quantification of Lagrangian mechanics and in the theory of gravity [21,23,26].

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In [27], the fractional geometry of curves in Euclidean 3-space is studied using the Caputo fractional derivative. Using the Caputo fractional derivative, the fractional geometry of curves in higher-dimensional Euclidean space is studied [28].

In this study, firstly, by considering a spacelike curve in the Lorentz plane, fractional ordered frame and Frenet-Serret formulas of this curve are obtained. Later, the relation between the fractional curvature and classical curvature of a spacelike plane curve is obtained. In the last part of this paper, considering the timelike plane curve in the Lorentz plane, new results are obtained with the method in the previous section.

2. Preliminaries

In general, the concepts of the Leibnitz rule and derivative of the composite function are needed when studying fractional differential geometry. However, within the scope of fractional analysis, these concepts are obtained with infinite series and are used in impact situations at the initial moment and after a long period [3,4].

Leibnitz's rule and derivative of the composite function can be given as follows for two functions *f(x)* and *g(x)* [29]:

$$(D_x^\alpha fg)(x) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{d^i f}{dx^i} \left(D_x^{\alpha-i} g\right)(x) - \frac{f(0)g(0)}{\Gamma(1-\alpha)} x^{-\alpha}$$

and

$$(D_x^\alpha f)\big(g(x)\big) = \sum_{i=1}^{\infty} \binom{\alpha}{i} \frac{x^{i-\alpha}}{\Gamma(i-\alpha+1)} \frac{d^i f\big(g(x)\big)}{dx^i} + \frac{f\big(g(x)\big) - f\big(g(0)\big)}{\Gamma(1-\alpha)} x^{-\alpha} \tag{1}$$

This different form of the integer derivative presents a challenge for deriving geometric concepts such as the curvature of a curve and the unit tangent vector. So, a certain simplification of the infinite series is used to construct the geometric theory of the derivative. With this simplification, most fundamental terms are removed from the infinite series, which retain the properties of the fractional derivative. Hence, with $t = g(x)$, the following equality is achieved [30]:

$$(D_x^\alpha f)\big(g(x)\big) = \frac{\alpha x^{1-\alpha}}{\Gamma(2-\alpha)} \frac{df}{dt} \frac{dg}{dx} \tag{2}$$

This simplification formula is obtained by taking only the $i = 1$ term of the infinite series in equation (1). This formula gives a partial effect of the fractional derivative and is expressed by the ordinary derivative. After this simplification, the construction of the fractional geometric theory based on the direct Caputo derivative can be expected using the simplified Leibnitz rule and the derivative of the composite function. In other words, using the Caputo derivative researchers have an advantage when studying the differential geometry of curves and surfaces, especially since it is ineffective on a constant function. Throughout the study, the derivative formula given by (2) is discussed.

Now, we will talk about some basic concepts in the Lorentz plane that we will use in the following sections. More detailed information on the following topics can be found in [31].

The Lorentz plane L^2 is the Euclidean plane R^2 with metric given by $g = -dx_1^2 + dx_2^2$ where (x_1, x_2) is a rectangular coordinate system of L^2. It is known that a vector $v \in L^2 \setminus \{0\}$ can be spacelike if $g(v,v) > 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$. The null (lightlike) curves in L^2 are lines, which curvature is identically zero.

Therefore, in this study, we will only deal with spacelike and timelike plane curves. The norm of any vector v in them is given by $\|v\| = \sqrt{g(v,v)|}$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$. An arbitrary curve $\gamma(s)$ in L^2 , can locally be spacelike or timelike if all of its velocity vectors $\dot{\gamma}(s)$ are spacelike, respectively timelike. A spacelike or timelike curve γ is parameterized by the arc-length parameter s if $g(\dot{\gamma}(s), \dot{\gamma}(s)) = \pm 1$.

The curvature κ and the Frenet formulas of the spacelike curve γ can be given as follows:

$$\kappa = -g\,(\dot{t}, n)$$

and

$$\begin{aligned} \dot{t} &= \kappa n \\ \dot{n} &= \kappa t \end{aligned}$$

where $\dot{\gamma} = t$, t and n are the tangent, and unit normal vector of a spacelike curve γ, respectively. If γ is a spacelike curve in Lorentz plane, then $g(t,t) = 1$ and $g(n,n) = -1$. Moreover, the curvature k and the Frenet formulas of the timelike curve β can be given as follows:

$$k = g\,(\dot{v_1}, v_2)$$

and

$$\begin{aligned} \dot{v_1} &= \kappa v_2 \\ \dot{v_2} &= \kappa v_1 \end{aligned}$$

where $\dot{\beta} = v_1$, v_1 and v_2 are the tangent, and unit normal vector of a spacelike curve β, respectively. If β is a timelike curve in Lorentz plane, then $g(v_1, v_1) = -1$ and $g(v_2, v_2) = 1$.

3. Geometry of Spacelike Curves with Fractional Derivative

In this section, the geometry of spacelike curves is discussed based on the Caputo fractional derivative in the Lorentz plane.

Let us consider a smooth spacelike curve γ in the 2-dimensional L^2 space is given by

$$\gamma: I \subset \mathbb{R} \to L^2, \gamma(t) = \big(x(t), y(t)\big)$$

where t is an arbitrary parameter. From the definition of the length σ of a spacelike curve γ, we can write

$$\sigma = \int_0^t \sqrt{|-\dot{x}^2 + \dot{y}^2|}\, dt, t \in I \tag{3}$$

where $\dot{x}$ and $\dot{y}$ denote the ordinary derivatives of x and y concerning t, respectively. The above formula is arclength of the spacelike curve for the tangent vector: $t(\sigma) = \left(\frac{dx}{d\sigma}, \frac{dy}{d\sigma}\right)$.

Let us now investigate the effect of the fractional derivative on the curvature of a spacelike curve. Since curvature is generally related to the change of the tangent vector of a spacelike curve, take a fractional tangent vector:

$$t^{(\alpha)}(\sigma) = \left(\frac{d^\alpha x(\sigma)}{d\sigma^\alpha}, \frac{d^\alpha y(\sigma)}{d\sigma^\alpha}\right) \tag{4}$$

Considering the infinite series given in (1), from the fractional derivative of the composite function, we can write $\left\|t^{(\alpha)}(\sigma)\right\| \neq 1$. This means that the classical arclength given by (3) cannot be used in the geometry of curves with fractional derivatives. To define a fractional unit tangent vector, it is necessary to consider the fractional derivative of the composite function in a simpler form. Therefore, instead of the formula (1), only the first term of the summation is considered in the fractional derivative of the composite function. Thus, both the effect of fractional derivative and first-order derivative are obtained. In this case, we can write

$$\frac{d^{\alpha}\gamma\big(t(s)\big)}{ds^{\alpha}} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\frac{d\gamma}{dt}\frac{dt}{ds} \tag{5}$$

Throughout this study, the theory of curves in the Lorentz plane are examined by considering this simple version of the derivative of the composite function. Using equation (5), we can give the following transformation:

$$s = \left[\frac{\alpha^2}{\Gamma(2-\alpha)}\sigma\right]^{\frac{1}{\alpha}} \tag{6}$$

where α denotes the order of the fractional derivative and $0 < \alpha \leq 1$. For the parameter s given by (6), we write

$$\frac{ds}{dt} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\sqrt{|-\dot{x}^2 + \dot{y}^2|} \tag{7}$$

Since $s > 0$ and $0 < \alpha \leq 1$ in (7), $\frac{ds}{dt}$ is positive. So that parameter t becomes a function dependent on s: $t = t(s)$. In this case, the spacelike curve γ can be written depending on the parameter s and is denoted by $\gamma(s) = \big(x(s), y(s)\big)$.

Now let us define the tangent vector of a given spacelike curve using the parameter s and the Caputo fractional derivative:

$$t^{(\alpha)}(s) \equiv \frac{d^{\alpha}\gamma(s)}{ds^{\alpha}} = \left(\frac{d^{\alpha}x(s)}{ds^{\alpha}}, \frac{d^{\alpha}y(s)}{ds^{\alpha}}\right) \tag{8}$$

Considering Equation (5), the norm of the tangent vector of the spacelike curve is

$$\left\|t^{(\alpha)}(s)\right\| = \sqrt{\left|-\left(\frac{d^{\alpha}x}{ds^{\alpha}}\right)^2 + \left(\frac{d^{\alpha}y}{ds^{\alpha}}\right)^2\right|} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\frac{dt}{ds}\sqrt{|-\dot{x}^2 + \dot{y}^2|} = 1 \tag{9}$$

Then, using (5), a unit vector of the spacelike curve γ orthogonal to *t* can be defined as follows:

$$n^{(\alpha)}(s) \equiv \left(\frac{d^{\alpha}y}{ds^{\alpha}}, \frac{d^{\alpha}x}{ds^{\alpha}}\right) \tag{10}$$

So, we can give the following theorem.

Theorem 3.1. Let γ be a spacelike curve in the Lorentz plane that satisfies the condition (5) and has the parameter *s* given by (6). Then $t^{(\alpha)}(s)$ and $n^{(\alpha)}(s)$ given by (8) and (10) are the unit tangent vector and unit normal vector of the spacelike curve γ, respectively, and s is the arclength.

In the following, we is constructed the geometry of a spacelike curve with the fractional $\left(t^{(\alpha)}(s), n^{(\alpha)}(s)\right)$ Frenet-Serret frame using the Caputo derivative.

Let us take a smooth spacelike curve $\gamma(s) = \big(x(s), y(s)\big)$ given by the arclength parameter (6) in the Lorentz plane. Based on the fractional frenet frame, let us define the frenet-serret formulas and the curvature of the spacelike curve γ. From Theorem 3.1, the tangent vector $t^{(\alpha)}(s)$ of the spacelike curve provides $g(t^{(\alpha)}(s), t^{(\alpha)}(s)) = 1$. If we take the derivative of both sides of this equation concerning s, we get

$$g\left(t^{(\alpha)}(s), \frac{dt^{(\alpha)}(s)}{ds}\right) = 0 \tag{11}$$

which means that $\frac{dt^{(\alpha)}(s)}{ds}$ can be expressed with the normal vector $n^{(\alpha)}(s)$:

$$\frac{dt^{(\alpha)}(s)}{ds}=\kappa^{(\alpha)}(s)n^{(\alpha)}(s) \tag{12}$$

where $\kappa^{(\alpha)}(s)$ is the fractional curvature of the spacelike curve γ. Then the norm of the normal vector $n^{(\alpha)}(s)$ is also equal to one, $\left\|n^{(\alpha)}(s)\right\|=1$. Then we write $g(n^{(\alpha)}(s),n^{(\alpha)}(s))=-1$. If we take the derivative of both sides of this last equation concerning s, we have

$$g\left(n^{(\alpha)}(s),\frac{dn^{(\alpha)}(s)}{ds}\right)=0 \tag{13}$$

From (13), $\frac{dn^{(\alpha)}(s)}{ds}$ can be given using a certain arclength function $\lambda^{(\alpha)}$:

$$\frac{dn^{(\alpha)}(s)}{ds}=\lambda^{(\alpha)}(s)t^{(\alpha)}(s) \tag{14}$$

Considering the orthogonality relation $g\big(t^{(\alpha)}(s),n^{(\alpha)}(s)\big)=0$, taking the derivative of both sides of this relation concerning s, we get the following expression:

$$t^{(\alpha)}(s)\frac{dn^{(\alpha)}}{ds}+\frac{dt^{(\alpha)}}{ds}n^{(\alpha)}(s)=0 \tag{15}$$

If the equations (12) and (14) are substituted in the expression (15), we get $\lambda^{(\alpha)}=\kappa^{(\alpha)}$. Thus, the following theorem is obtained.

Theorem 3.2. Let γ be a spacelike curve in the Lorentz plane that satisfies the condition (5) and has the parameter s given by (6). Let us consider $\left(t^{(\alpha)}(s),n^{(\alpha)}(s)\right)$ as the fractional frame of this spacelike curve γ. Then the Frenet-Serret formulas for γ can be given as

$$\frac{dt^{(\alpha)}(s)}{ds}=\kappa^{(\alpha)}(s)n^{(\alpha)}(s) \tag{16}$$

$$\frac{dn^{(\alpha)}(s)}{ds}=\kappa^{(\alpha)}(s)t^{(\alpha)}(s) \tag{17}$$

Let us now investigate the relationship between the fractional curvature and classical curvature of a given spacelike curve γ. Considering (8) and (16), we can write

$$\frac{d}{ds}\left(\frac{d^{\,\alpha}\gamma(s)}{ds^{\,\alpha}}\right)=\kappa^{(\alpha)}(s)n^{(\alpha)}(s) \tag{18}$$

If the normal vector $n^{(\alpha)}$ is applied to both sides of (18), we can write the fractional curvature as

$$\kappa^{(\alpha)}(s)=-g\left(n^{(\alpha)}(s),\frac{d}{ds}\left(\frac{d^{\,\alpha}\gamma(s)}{ds^{\,\alpha}}\right)\right) \tag{19}$$

Considering the normal vector $n^{(\alpha)}(s)$ in (19), $\kappa^{(\alpha)}(s)$ according to the fractional derivative is written as

$$\kappa^{(\alpha)}(s)=\frac{d^{\,\alpha}y}{ds^{\,\alpha}}\frac{d}{ds}\left(\frac{d^{\,\alpha}x}{ds^{\,\alpha}}\right)-\frac{d^{\,\alpha}x}{ds^{\,\alpha}}\frac{d}{ds}\left(\frac{d^{\,\alpha}y}{ds^{\,\alpha}}\right) \tag{20}$$

If a curve is given by an arbitrary parameter t and not by the arc length s, then we must calculate the fractional curvature according to an arbitrary parameter t. Then let us calculate the fractional curvature according to an arbitrary parameter t. From the expression (5) for the composite function $t=t(s)$, we can write

$$\frac{d}{ds}\left(\frac{d^{\alpha}x}{ds^{\alpha}}\right)=\frac{\alpha s^{-\alpha}\dot{x}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds}+s\frac{d^2t}{ds^2}\right)+\frac{\alpha s^{1-\alpha}\ddot{x}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \quad (21)$$

$$\frac{d}{ds}\left(\frac{d^{\alpha}y}{ds^{\alpha}}\right)=\frac{\alpha s^{-\alpha}\dot{y}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds}+s\frac{d^2t}{ds^2}\right)+\frac{\alpha s^{1-\alpha}\ddot{y}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \quad (22)$$

where $\ddot{x}=\frac{d^2x}{dt^2}$ and $\ddot{y}=\frac{d^2y}{dt^2}$. If the expressions (21) and (22) are written instead in (20), the fractional-order curvature is

$$\kappa^{(\alpha)}(t)=\left\{\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right\}^2(-\dot{x}\ddot{y}+\ddot{x}\dot{y})\left(\frac{dt}{ds}\right)^3 \quad (23)$$

Moreover, from (7), (23) it can be rewritten by

$$\kappa^{(\alpha)}(t)=\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}\kappa(t) \quad (24)$$

$\kappa(t)$ in this last equation is the classical curvature and

$$\kappa(t)=\frac{-\dot{x}\ddot{y}+\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \quad (25)$$

Thus, using the arclength definition given by (6), we can give the following theorem.

Theorem 3.3. The fractional curvature of a spacelike plane curve given as $\gamma(t)=\big(x(t),y(t)\big)$ is

$$\kappa^{(\alpha)}(t)=\left\{\frac{\beta(2-\alpha)}{\alpha}\right\}^{\frac{1}{\alpha}}\left[\alpha\int_0^t\sqrt{|-\dot{x}^2+\dot{y}^2|}dt\right]^{1-\frac{1}{\alpha}}\kappa(t) \quad (26)$$

where t is an arbitrary parameter.

The part of $\frac{1}{s^{1-\alpha}}$ in (24) characterizes the effects of the fractional derivative given by the fractional tangent vector (8). The effect of the fractional derivative is strong at the start but becomes less effective over a longer period. This property of the effect influences the change of fractional curvature.

4. Geometry of Timelike Curves with Fractional Derivative

In this section, the geometry of timelike curves is discussed based on the Caputo fractional derivative in the Lorentz plane.

Let us consider a smooth timelike curve β in the 2-dimensional L^2 space is given by

$$\beta: I\subset\mathbb{R}\to L^2,\qquad \beta(t)=\big(\beta_1(t),\beta_2(t)\big)$$

where t is an arbitrary parameter. From the definition of the length σ of a timelike curve β, we can write

$$\sigma=\int_0^t\sqrt{\left|-(\dot{\beta}_1)^2+(\dot{\beta}_2)^2\right|}\,dt,t\in I \quad (27)$$

where $\dot{\beta}_1$ and $\dot{\beta}_2$ denote the ordinary derivatives of β_1 and β_2 concerning *t*, respectively. The above formula is arclength of the timelike curve for the tangent vector: $v_1(\sigma)=\left(\frac{dx}{d\sigma},\frac{dy}{d\sigma}\right)$.

Let us now investigate the effect of the fractional derivative on the curvature of a timelike curve. Since curvature is generally related to the change of the tangent vector of a timelike curve, let's define a fractional tangent vector:

$$v_1{}^{(\alpha)}(\sigma) = \left(\frac{d^\alpha \beta_1(\sigma)}{d\sigma^\alpha}, \frac{d^\alpha \beta_2(\sigma)}{d\sigma^\alpha}\right) \tag{28}$$

Considering the infinite series given in (1), from the fractional derivative of the composite function, we can write $\left\|v_1{}^{(\alpha)}(\sigma)\right\| \neq 1$. This means that the classical arclength given by (27) cannot be used in the geometry of curves with fractional derivatives. To define a fractional unit tangent vector, it is necessary to consider the fractional derivative of the composite function in a simpler form. Therefore, instead of the formula (1), only the first term of the summation is considered in the fractional derivative of the composite function. Thus, both the effect of fractional derivative and first-order derivative are obtained. In this case, we can write

$$\frac{d^\alpha \beta(t(s))}{ds^\alpha} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\beta}{dt} \frac{dt}{ds} \tag{29}$$

Throughout this study, the theory of timelike curves in the Lorentz plane are examined by considering this simple version of the derivative of the composite function. Using equation (29), we can give the following transformation:

$$s = \left[\frac{\alpha^2}{\Gamma(2-\alpha)} \sigma\right]^{\frac{1}{\alpha}} \tag{30}$$

where α denotes the order of the fractional derivative and $0 < \alpha \leq 1$. For the parameter s given by (30), we write

$$\frac{ds}{dt} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \sqrt{\left|-(\dot{\beta}_1)^2 + (\dot{\beta}_2)^2\right|} \tag{31}$$

Since $s > 0$ and $0 < \alpha \leq 1$ in (31), $\frac{ds}{dt}$ is positive. So that parameter t becomes a function dependent on s: $t = t(s)$. In this case, the timelike curve β can be written depending on the parameter s and is denoted by $\beta(s) = \big(\beta_1(s), \beta_2(s)\big)$.

Now let us define the tangent vector of a given timelike curve using the parameter s and the Caputo fractional derivative:

$$v_1{}^{(\alpha)}(s) \equiv \frac{d^\alpha \beta(s)}{ds^\alpha} = \left(\frac{d^\alpha \beta_1(s)}{d\sigma^\alpha}, \frac{d^\alpha \beta_2(s)}{d\sigma^\alpha}\right) \tag{32}$$

Considering Equation (29), the norm of the tangent vector of the timelike curve is

$$\left\|v_1{}^{(\alpha)}(s)\right\| = \sqrt{\left|-\left(\frac{d^\alpha \beta_1}{ds^\alpha}\right)^2 + \left(\frac{d^\alpha \beta_2}{ds^\alpha}\right)^2\right|} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{dt}{ds} \sqrt{\left|-(\dot{\beta}_1)^2 + (\dot{\beta}_2)^2\right|} = 1 \tag{33}$$

Then using (29), a unit vector of the timelike curve β orthogonal to $\boldsymbol{v_1}$ can be defined as follows:

$$v_2{}^{(\alpha)}(s) \equiv \left(\frac{d^\alpha \beta_2}{ds^\alpha}, \frac{d^\alpha \beta_1}{ds^\alpha}\right) \tag{34}$$

So, we can give the following theorem.

Theorem 4.1. Let β be a timelike curve in the Lorentz plane that satisfies the condition (29) and has the parameter s given by (30). Then $v_1{}^{(\alpha)}(s)$ and $v_2{}^{(\alpha)}(s)$ given by (32) and (34) are the unit tangent vector and unit normal vector of the timelike curve β, respectively, and s is the arclength.

In the following, we is constructed the geometry of a timelike curve with the fractional $\left(v_1{}^{(\alpha)}(s), v_2{}^{(\alpha)}(s)\right)$ Frenet-Serret frame using the Caputo derivative.

Let us take a smooth timelike curve $\beta(s) = \big(\beta_1(s), \beta_2(s)\big)$ given by the arclength parameter (30) in the Lorentz plane. Based on the fractional frenet frame, let us define the frenet-serret formulas and the curvature of the timelike curve β. From Theorem 4.1, the tangent vector ${v_1}^{(\alpha)}(s)$ of the timelike curve provides $g({v_1}^{(\alpha)}(s), {v_1}^{(\alpha)}(s)) = -1$. If we take the derivative of both sides of this equation concerning s, we get

$$ {v_1}^{(\alpha)}(s)\frac{d{v_1}^{(\alpha)}(s)}{ds} = 0 \tag{35} $$

which means that $\frac{d{v_1}^{(\alpha)}(s)}{ds}$ can be expressed with the normal vector ${v_2}^{(\alpha)}(s)$:

$$ \frac{d{v_1}^{(\alpha)}(s)}{ds} = k^{(\alpha)}(s){v_2}^{(\alpha)}(s) \tag{36} $$

where $k^{(\alpha)}(s)$ is the fractional curvature of the timelike curve β. Then the norm of the normal vector ${v_2}^{(\alpha)}(s)$ is also equal to one, $\|{v_2}^{(\alpha)}(s)\| = 1$. Then we write $g({v_2}^{(\alpha)}(s), {v_2}^{(\alpha)}(s)) = 1$. If we take the derivative of both sides of this last equation concerning s, we have

$$ g\left({v_2}^{(\alpha)}(s), \frac{d{v_2}^{(\alpha)}(s)}{ds}\right) = 0 \tag{37} $$

From (37), $\frac{d{v_2}^{(\alpha)}(s)}{ds}$ can be given using a certain arclength function $\mu^{(\alpha)}$:

$$ \frac{d{v_2}^{(\alpha)}(s)}{ds} = \mu^{(\alpha)}(s){v_1}^{(\alpha)}(s) \tag{38} $$

Considering the orthogonality relation $g({v_1}^{(\alpha)}(s), {v_2}^{(\alpha)}(s)) = 0$, taking the derivative of both sides of this relation concerning s, we get the following expression:

$$ {v_1}^{(\alpha)}(s)\frac{d{v_2}^{(\alpha)}}{ds} + \frac{d{v_1}^{(\alpha)}}{ds}{v_2}^{(\alpha)}(s) = 0 \tag{39} $$

If the equations (36) and (38) are substituted in the expression (39), we get $\mu^{(\alpha)} = k^{(\alpha)}$. Thus, the following theorem is obtained.

Theorem 4.2. Let β be a timelike curve in the Lorentz plane that satisfies the condition (5) and has the parameter s given by (30). Let us consider $\left({v_1}^{(\alpha)}(s), {v_2}^{(\alpha)}(s)\right)$ as the fractional frame of this timelike curve β. Then the Frenet-Serret formulas for β can be given as

$$ \frac{d{v_1}^{(\alpha)}(s)}{ds} = k^{(\alpha)}(s){v_2}^{(\alpha)}(s) \tag{40} $$

$$ \frac{d{v_2}^{(\alpha)}(s)}{ds} = k^{(\alpha)}(s){v_1}^{(\alpha)}(s) \tag{41} $$

Let us now investigate the relationship between the fractional curvature and classical curvature of a given timelike curve β. Considering (32) and (40), we can write

$$ \frac{d}{ds}\left(\frac{d^{\,\alpha}\beta(s)}{ds^{\,\alpha}}\right) = k^{(\alpha)}(s){v_2}^{(\alpha)}(s) \tag{42} $$

If the normal vector ${v_2}^{(\alpha)}$ is applied to both sides of (42), we can write the fractional curvature as

$$ k^{(\alpha)}(s) = g\left({v_2}^{(\alpha)}(s), \frac{d}{ds}\left(\frac{d^{\,\alpha}\beta(s)}{ds^{\,\alpha}}\right)\right) \tag{43} $$

Considering the normal vector ${v_2}^{(\alpha)}(s)$ in (43), $k^{(\alpha)}(s)$ according to the fractional derivative is written as

$$k^{(\alpha)}(s) = -\frac{d^{\,\alpha}\beta_2}{ds^{\,\alpha}}\frac{d}{ds}\left(\frac{d^{\,\alpha}\beta_1}{ds^{\,\alpha}}\right) + \frac{d^{\,\alpha}\beta_1}{ds^{\,\alpha}}\frac{d}{ds}\left(\frac{d^{\,\alpha}\beta_2}{ds^{\,\alpha}}\right) \tag{44}$$

If a curve is given by an arbitrary parameter t and not by the arc length s, then we must calculate the fractional curvature according to an arbitrary parameter t. Then let us calculate the fractional curvature according to an arbitrary parameter t. From the expression (29) for the composite function $t = t(s)$, we can write

$$\frac{d}{ds}\left(\frac{d^{\,\alpha}\beta_1}{ds^{\,\alpha}}\right) = \frac{\alpha s^{-\alpha}\ddot{\beta_1}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds} + s\frac{d^2t}{ds^2}\right) + \frac{\alpha s^{1-\alpha}\ddot{\beta_1}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \tag{45}$$

$$\frac{d}{ds}\left(\frac{d^{\,\alpha}\beta_2}{ds^{\,\alpha}}\right) = \frac{\alpha s^{-\alpha}\ddot{\beta_2}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds} + s\frac{d^2t}{ds^2}\right) + \frac{\alpha s^{1-\alpha}\ddot{\beta_2}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \tag{46}$$

where $\ddot{\beta_1} = \frac{d^2\beta_1}{dt^2}$ and $\ddot{\beta_2} = \frac{d^2\beta_2}{dt^2}$. If the expressions (45) and (46) are written instead in (44), the fractional-order curvature is

$$k^{(\alpha)}(t) = \left\{\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right\}^2 \left(\dot{\beta_1}\ddot{\beta_2} - \ddot{\beta_1}\dot{\beta_2}\right)\left(\frac{dt}{ds}\right)^3 \tag{47}$$

Moreover, from (31), (47) it can be rewritten by

$$k^{(\alpha)}(t) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}k(t) \tag{48}$$

$k(t)$ in this last equation is the classical curvature and

$$k(t) = \frac{-\dot{\beta_1}\ddot{\beta_2} + \ddot{\beta_1}\dot{\beta_2}}{\left(\left|-(\dot{\beta_1})^2 + (\dot{\beta_2})^2\right|\right)^{\frac{3}{2}}} \tag{49}$$

Thus, using the arclength definition given by (30), we can give the following theorem.

Theorem 4.3. The fractional curvature of a timelike plane curve given as $\beta(t) = \big(x(t), y(t)\big)$ is

$$k^{(\alpha)}(t) = \left\{\frac{\Gamma(2-\alpha)}{\alpha}\right\}^{\frac{1}{\alpha}}\left[\alpha\int_0^t \sqrt{\left|-(\dot{\beta_1})^2 + (\dot{\beta_2})^2\right|}dt\right]^{1-\frac{1}{\alpha}} k(t) \tag{50}$$

where t is an arbitrary parameter.

The part of $\frac{1}{s^{1-\alpha}}$ in (48) characterizes the effects of the fractional derivative given by the fractional tangent vector (32). The effect of the fractional derivative is strong at the start but becomes less effective over a longer period. This property of the effect influences the change of fractional curvature.

5. Conclusion

In this paper, firstly, the tangent vector of a spacelike (timelike) curve in the Lorentz plane are defined in terms of the fractional derivative. Then, by considering a spacelike (timelike) curve in the Lorentz plane, the arc length and fractional ordered frame of this curve are obtained. Later, the Caputo fractional derivative is considered and the relations between the standard curvature and fractional curvature of the spacelike (timelike) curves in the Lorentz plane are obtained. It has been observed that these relations geometrically overlap with the results obtained using the derivative in the classical sense.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interests

The author declares no conflict of interest.

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