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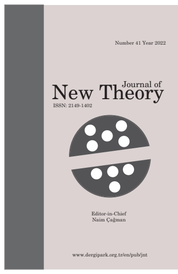

Kibria-Lukman Estimator for General Linear Regression Model with AR(2) Errors: A Comparative Study with Monte Carlo Simulation

Tuğba Söküt Açar[1]

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Abstract — The sensitivity of the least-squares estimation in a regression model is impacted by multicollinearity and autocorrelation problems. To deal with the multicollinearity, Ridge, Liu, and Ridge-type biased estimators have been presented in the statistical literature. The recently proposed Kibria-Lukman estimator is one of the Ridge-type estimators. The literature has compared the Kibria-Lukman estimator with the others using the mean square error criterion for the linear regression model. It was achieved in a study conducted on the Kibria-Lukman estimator's performance under the first-order autoregressive erroneous autocorrelation. When there is an autocorrelation problem with the second-order, evaluating the performance of the Kibria-Lukman estimator according to the mean square error criterion makes this paper original. The scalar mean square error of the Kibria-Lukman estimator under the second-order autoregressive error structure was evaluated using a Monte Carlo simulation and two real examples, and compared with the Generalized Least-squares, Ridge, and Liu estimators. The findings revealed that when the variance of the model was small, the mean square error of the Kibria-Lukman estimator gave very close values with the popular biased estimators. As the model variance grew, Kibria-Lukman did not give fairly similar values with popular biased estimators as in the model with small variance. However, according to the mean square error criterion the Kibria-Lukman estimator outperformed the Generalized Least-Squares estimator in all possible cases.

Keywords — *Autocorrelation, multicollinearity, second-order autoregressive errors, Kibria-Lukman estimator*

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1. Introduction

Regression analysis is widely used to create a functional model based on the relationship between an observed dependent variable (response) and one or more observed independent variables (regressors). A linear regression model is one in which the variables are included in the model as a first-order polynomial. The model is called a simple linear regression model if there is only one independent variable, and a multiple linear regression model if there is more than one independent variable. The following is a formula for a multiple linear regression model with p independent variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_p x_p + \varepsilon. \tag{1}$$

It takes on the matrix form as

$$y = X\beta + \varepsilon$$

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where y is an $n \times 1$ vector of observed response variable, $X = [1, x]$, $x = (x_1, x_2, .., x_p)$ with $x_j = (x_{1j}, x_{2j}, ..., x_{nj})'$ for $j = 1, ..., p$, is an $n \times (p+1)$ vector of known regressor matrix whose first column equals to one, β is an $(p+1) \times 1$ vector of unknown regression parameters and ε is an $n \times 1$ vector of errors with properties $E(\varepsilon) = 0$ and $E(\varepsilon\varepsilon') = \sigma^2 I_n$. The unknown regression parameters must be calculated to determine the functional relationship between the dependent and independent variables. If the following assumptions are met, the Ordinary Least-Squares (OLS) can be used to estimate unknown parameters in a regression model: -Regressor matrix is a non-stochastic matrix - Regressor matrix is a full column rank - Response is a linear function of regressors - The error term is normally distributed with zero mean and constant variance. OLS estimation procedure can be applied which is based on the minimization of the sum of squares error $\min_\beta \left\{ (Y - X\beta)' (Y - X\beta) \right\}$ as

$$\hat{\beta}_{OLS} = \left(X'X \right)^{-1} X'Y$$

In these circumstances, OLS is the best linear unbiased estimator (BLUE) with $E(\hat{\beta}_{OLS}) = \beta$ and $cov(\hat{\beta}_{OLS}) = \sigma^2 \left(X'X \right)^{-1}$.

Multicollinearity is a term used in data analytics to indicate the occurrence of two regressors that are shown to be associated in a linear regression model. If the matrix $X'X$ is not linearly independent, it will not be full column rank. In this case, the matrix $X'X$ becomes ill-conditioned. The condition number, correlation coefficient, and variance inflation factor are utilized to determine the multicollinearity in a dataset. The condition number, κ, is a value calculated from the eigenvalues of the $X'X$ matrix's characteristic roots or eigenvalues. κ, including $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $X'X$, is determined by $\kappa = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}$. According to Belsley et al. [1], there is no substantial problem with multicollinearity if the κ value is less than 10, moderate to strong collinearity if the κ value is between 30 and 100, and severe multicollinearity if the κ value is greater than 100. It is also a sign that the two variables generate multicollinearity when the correlation coefficient between any two independent variables is close to 1 in absolute value or statistically significant. Multicollinearity can be regarded as a result of these variables. In this scenario, some biased estimators defined in Equation 1 to deal with multicollinearity are presented below. Hoerl et al. [2] suggested a ridge estimator under Equation 1 based on the solution of

$$\min_{\beta} \left\{ (Y - X\beta)' (Y - X\beta) + k \left(\beta'\beta - c \right) \right\}$$

where c is a constant and k is a lagrangian multiplier called the biasing parameter as

$$\hat{\beta}_{ridge} = \left(X'X + kI_p \right)^{-1} X'Y, k > 0$$

It is obvious that the ridge regression's effectiveness will change depending on the k-biasing parameter. As a result, the statistical literature includes the proposed biasing parameters according to various criteria (see, [2–4]). Condition number of $X'X + kI_p$ is a decreasing function of k. As a result, as k increases, the condition number decreases dramatically [5]. In practice, however, k is small, and this may not be enough to solve the ill-conditioned.

Liu [6] proposed a liu estimator which is an alternative to ridge based on the solution of $\min_\beta \left\{ (Y - X\beta)' (Y - X\beta) + (\beta - d\hat{\beta})(\beta - d\hat{\beta})' \right\}$ as

$$\hat{\beta}_{liu} = \left(X'X + I_p \right)^{-1} \left(X'X + dI_p \right) \hat{\beta}, 0 < d < 1.$$

Özkale and Kaçıranlar [7] stated that the liu estimator is more advantageous than the ridge because it is a linear function of d. Numerous studies on the choice of the d parameter in the liu estimator can be found in the literature (see, [6–8]).

Kibria and Lukman [9] proposed a new biased estimator called as Kibria-Lukman (KL) estimator to cope with the multicollinearity which is based on the solving of

$$\min_{\beta} \left\{ (Y - X\beta)' (Y - X\beta) + k \left[(\beta + \hat{\beta})'(\beta + \hat{\beta}) - c \right] \right\}$$

as

$$\hat{\beta}_{kl} = \left(X'X + kI_p\right)^{-1}\left(X'X - kI_p\right)\hat{\beta}, k > 0$$

The user of biased estimators must select a biasing parameter (k or d) in order to see improvements in the estimates [10]. There have been numerous studies on the biasing parameter selection processes (see, [3, 7, 9]).

Many KL estimators have been described, each based on a different distribution (inverse Gaussian regression model, Gamma regression model, Poisson regression model, distributed lag model). (see, [11–15]). The goal of the paper is to apply the mean square error (MSE) criterion to extend the KL estimator's performance from non-autoregressive or first-order autoregressive processes which are in the statistical literature to second-order autoregressive process.

The article is structured as follows: The general linear regression model, error structures, and estimators are provided in Section 2. The method for calculating the MSE, which is used to assess model performance for any estimator, is provided in the next section. Section 4 discusses the Monte Carlo simulation's layout and findings. In Section 5, the performance of the KL estimator in the second-order autoregressive model is examined over two real datasets. The paper's findings and recommendations are presented in the final section.

2. General Linear Regression Model, Error Structure and Estimators

When the variance-covariance matrix of the errors is not diagonal form that is $E(\varepsilon\varepsilon') = \sigma_\varepsilon^2 V$, $V \neq I_n$ it is called as general linear regression (GLR) model. There is a violation of the assumption "The error term is normally distributed with zero mean and constant variance", and in this case, the autocorrelation problem arises. Therefore, the errors are correlated. Since V $n \times n$ matrix assumed that known is symmetric and positive definite, then there exits a non-singular $n \times n$ matrix P such that $V^{-1} = P'P$. Premultiplying both sides of Equation 1 by P gives the transformed model as

$$Py = PX\beta + P\varepsilon. \tag{2}$$

In the transformed model new error terms has covariance matrix as $E\left[P\varepsilon\left(P\varepsilon\right)'\right] = \sigma^2 I_n$. Therefore, by applying the least-squares estimation procedure which based on $\min_\beta \left\{(PY - PX\beta)'(PY - X\beta)\right\}$ and by solving the normal equations, generalized least-squares estimator (GLS) obtained as

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}y.$$

GLS is a BLUE estimator with $E[\hat{\beta}_{GLS}] = \beta$ and $cov(\hat{\beta}_{GLS}) = \sigma^2(X'V^{-1}X)^{-1}$ under Equation 2.

In models with an autocorrelation problem, multicollinearity can also occur. Many biased estimators proposed under the linear regression model has also been extended to GLR models.

Ridge Regression (RR) estimator for Equation 2 given by Trenkler [16] as

$$\hat{\beta}_{RR} = (X'V^{-1}X + kI_p)^{-1}X'V^{-1}y, k > 0.$$

RR is a biased estimator with $bias(\hat{\beta}_{RR}, \beta) = -k\left(X'V^{-1}X + kI_n\right)^{-1}\beta$ and variance-covariance matrix is $cov(\hat{\beta}_{RR}, \beta) = \sigma^2\left(X'V^{-1}X + kI_p\right)^{-1}X'V^{-1}X\left(X'V^{-1}X + kI_p\right)^{-1}$.

Noted that, because β and σ^2 population parameters are unknown in real-world applications, GLS estimations which are the best estimators of these parameters under Equation 2, are utilized. Liu estimator for Equation 2 given by Kaçiranlar [17] as

$$\hat{\beta}_{Liu} = (X'V^{-1}X + I_p)^{-1}(X'V^{-1}y + d\hat{\beta}_{GLS}), 0 < d < 1.$$

This is a biased estimator also, and the expected value and the variance-covariance matrix as follows: $bias(\hat{\beta}_{Liu}, \beta) = (d-1)(X'V^{-1}X + I_p)^{-1}\beta$ and $cov(\hat{\beta}_{Liu}, \beta) = \sigma^2\left(X'V^{-1}X + I_p\right)^{-1}(X'V^{-1}X + dI_p)(X'V^{-1}X)^{-1}(X'V^{-1}X + dI_p)\left(X'V^{-1}X + I_p\right)^{-1}$.

In this study, two different structures of autocorrelation, which is another problem apart from multicollinearity, were examined. The first-order autoregressive (AR(1)) model is one of them, while the second-order autoregressive model (AR(2)) is the other. The error terms for the AR(1) process are satisfied

$$\varepsilon_i = \rho\varepsilon_{i-1} + u_i \tag{3}$$

where $E(u_i) = 0$, $E(u_i^2) = \sigma_u^2$ and $E(u_i u_j) = 0$, $i \neq j$. This process will be stationary if $|\rho| < 1$. The matrix P for the AR(1) process is (see [18–20])

$$P = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \dots & 0 & 0 \\ -\rho & 1 & 0 & \dots & 0 & 0 \\ 0 & -\rho & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -\rho & 1 \end{bmatrix}. \tag{4}$$

In the AR(2) process the errors generated by

$$\varepsilon_i = \phi_1\varepsilon_{i-1} + \phi_2\varepsilon_{i-2} + u_i \tag{5}$$

where $E(u_i) = 0$, $E(u_i^2) = \sigma_u^2$, and $E(u_i u_j) = 0$, $i \neq j$. For the AR(2) process to be stationary, the parameters ϕ_1 and ϕ_2 must take values such that $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$ and $-1 < \phi_2 < 1$. The P matrix under the AR(2) structure is given as Judge et al. [21]

$$P = \begin{bmatrix} q_{11} & 0 & 0 & 0 & \dots & 0 & 0 \\ -\rho_1\sqrt{1-\phi_2^2} & \sqrt{1-\phi_2^2} & 0 & 0 & \dots & 0 & 0 \\ -\phi_2 & -\phi_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -\phi_2 & -\phi_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -\phi_1 & 1 \end{bmatrix} \tag{6}$$

where $q_{11} = \left\{\frac{(1+\phi_2)[(1-\phi_2)^2-\phi_1^2]}{1-\phi_2}\right\}^{1/2}$ and $\rho_1 = \frac{\phi_1}{1-\phi_2}$.

KL estimator for the GLR model with AR(1) structure is given by Zubari and Adenomon [22] as

$$\hat{\beta}_{KL_{AR1}} = (X'V^{-1}X + kI_p)^{-1}(X'V^{-1}X - kI_p)X'V^{-1}y, k > 0.$$

This is a biased estimator also, and the expected value and the variance-covariance matrix as follows:

$$bias(\hat{\beta}_{KL_{AR1}}, \beta) = -2k(X'V^{-1}X + kI_p)^{-1}\beta$$

and

$$cov(\hat{\beta}_{KL_{AR1}}, \beta) = \sigma^2\left(X'V^{-1}X + kI_p\right)^{-1}\left(X'V^{-1}X - kI_p\right)\left(X'V^{-1}X\right)^{-1}\left(X'V^{-1}X - kI_p\right)\left(X'V^{-1}X + kI_p\right)^{-1}$$

In the paper, KL is presented as an estimator for the second-order autoregressive model as

$$\hat{\beta}_{KL_{AR2}} = (X'P'PX + kI_p)^{-1}(X'P'PX - kI_p)X'P'Py, \quad k > 0.$$

Unlike $\hat{\beta}_{KL_{AR1}}$ based on the P matrix given in Equaiton 4, $\hat{\beta}_{KL_{AR2}}$ based on the P matrix is in the form of the matrix given Eq. 6. It is to be noted that, the estimator denoted by KL in applications is $\hat{\beta}_{KL_{AR2}}$.

In the paper, the performance of the KL estimator for the P matrix given by 6 is compared with the performance of the alternative estimators. MSE criterion was used to compare the performance. As a result, the following section explains how to calculate the MSE matrix of any estimators.

3. MSE Criterion to Determine the Best Model

MSE of an estimator measures the average of the squares of the errors that is, the average squared difference between the estimated and actual values. MSE is a risk function that represents expected value of squared error loss. MSE is defined for $\tilde{\beta}$ being any estimator as

$$MSE(\tilde{\beta}, \beta) = cov(\tilde{\beta}) + bias(\tilde{\beta})bias(\tilde{\beta})'.$$

It should be noted that if $\tilde{\beta}$ is unbiased, the MSE will be equal to the variance-covariance matrix of $\tilde{\beta}$. The scalar mean square error (sMSE) value is equal to the sum of the diagonal elements of the MSE matrix, namely its trace. Let the two estimator be $\tilde{\beta}_1$ and $\tilde{\beta}_2$. For $\tilde{\beta}_2$ to be superior than $\tilde{\beta}_1$ according to the sMSE criterion, the necessary and sufficient condition is

$$\triangle\left(\tilde{\beta}_1, \tilde{\beta}_2\right) = trace(MSE(\tilde{\beta}_1, \beta)) - trace(MSE(\tilde{\beta}_2, \beta)) > 0.$$

In other words, it defines that the estimator with the smaller sMSE value performs better according to this criterion. The statistical literature has extensively researched the comparison of biased estimators to unbiased estimators using the sMSE criterion for both linear regression and GLR models with AR(1) errors (see, [16,22,23]). In the following section, the biasing parameters (k/d) of the biased estimators were coded to minimize the sMSE in each cycle while evaluating the estimators' performance using the sMSE criterion.

4. Monte Carlo Simulation Study

The Monte Carlo simulation is created based on a number of criteria, which are listed in Table 1.

Table 1. Assumed values of various components for Monte Carlo simulation

Factor	Notation	Values
Sample Size	n	30, 100
Number of the independent variable	p	3
Degree of multicollinearity	γ^2	0.70, 0.80, 0.90
Dispersion parameters	σ	0.1, 0.5, 1
Number of replicates	MCN	1000

Settings: The goal of Monte Carlo simulation research evaluated by Matlab is to examine the GLR model with AR(2) errors fitted by the GLS, RR, Liu, and KL estimators. The sMSE criterion was used to assess the estimators' performance. The correlated regressor variables were generated from McDonald and Galarneau [24] as

$$x_{ij} = \left(1-\gamma^2\right)^{1/2} z_{ij} + \gamma z_{ip}, \quad j = 1, 2, ..., p, \ i = 1, 2, ..., n \tag{7}$$

where z_{ij} are independent standard normal pseudo-random numbers. γ is specified so that the correlation between any two explanatory variables is given by γ^2 [25]. The regressor matrix are centralized and standardized after x_{ij} was produced, so that the $X'X$ becomes the correlation form. The final regressor matrix is written as $Z = (ones(n,1) \;\; X)$ to correspond to the constant parameter. β, $p+1$ vector was written with eigenvectors corresponding to the largest value of $(X'X)$ except the constant parameter which is taken as 0.5.

The simulation loop started with the derivation of the u_i error terms from standard normal distribution. Then, the error terms, ε_i, are generated from Equation 5. Here, to satisfy the stationarity condition different ϕ_1 and ϕ_2 values are as in Table 2.
Finally, observation of the response variable is generated from

$$y_i = \beta_0 + \sum_{j=1}^{p} \beta_j x_{ij} + \varepsilon_i.$$

Table 2. Researcher's parameters for the AR(2) model

	ϕ_1			
$\phi_2 = -0.9$	-1.5	-0.5	0.5	1.5
$\phi_2 = -0.7$	-1.5	-0.5	0.5	1.5
$\phi_2 = 0.7$	-0.2	-0.1	0.1	0.2
$\phi_2 = 0.9$	-0.05	-0.025	0.025	0.05

P matrix in AR(2) structure (Equation 6) was created for different ϕ_1 and ϕ_2 values, and transformed response vector and regressor matrix were obtained. While obtaining the biased estimators, k/d values that minimize the related sMSE were obtained with a Matlab code and assigned as optimum k/d. The experiments were replicated 1000 times and the sMSE of the estimators was calculated for each replicate using the following Equation:

$$sMSE(\tilde{\beta}, \beta) = \frac{1}{MCN} \sum_{r=1}^{MCN} \left(\tilde{\beta}_r - \beta\right)' \left(\tilde{\beta}_r - \beta\right)$$

where $\tilde{\beta}$ is any of the earlier estimators. The estimators with the lowest sMSE has been regarded the best. The simulation results are presented in Tables 3–18.

Table 3. $\phi_1 = -1.5$ *and* $\phi_2 = -0.9$

γ^2	σ_u	**n=30**				**n=100**			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0140	0.0126	0.0131	0.0126	0.0120	0.0109	0.0112	0.0109
	0.5	0.3491	0.2159	0.2109	0.2397	0.2996	0.1876	0.1868	0.2046
	1	1.3964	0.7919	0.7759	0.9395	1.1984	0.7004	0.6987	0.7984
0.8	0.1	0.0188	0.0151	0.0162	0.0149	0.0161	0.0130	0.0137	0.0130
	0.5	0.4702	0.2633	0.2524	0.3107	0.4020	0.2273	0.2261	0.2651
	1	1,8809	0.9991	0.9592	1.2607	1.6082	0.8724	0.8687	1.0785
0.9	0.1	0.0340	0.0219	0.0233	0.0228	0.0292	0.0187	0.0207	0.0195
	0.5	0.8502	0.4557	0.4236	0.5603	0.7312	0.3918	0.3892	0.4895
	1	3,4009	1.7908	1.6631	2.2566	2.9249	1.5447	1.5326	1.9698

Table 4. $\phi_1 = -0.5$ *and* $\phi_2 = -0.9$

γ^2	σ_u	**n=30**				**n=100**			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0259	0.0214	0.0233	0.0210	0.0220	0.0193	0.0203	0.0191
	0.5	0.6471	0.3911	0.3662	0.4566	0.5498	0.3357	0.3355	0.3739
	1	2.5884	1.4329	1.3368	1.8662	2.1992	1.2093	1.2053	1.4852
0.8	0.1	0.0353	0.0252	0.0287	0.0250	0.0294	0.0225	0.0248	0.0222
	0.5	0.8820	0.5178	0.4709	0.6310	0.7360	0.4239	0.4227	0.4927
	1	3.5280	1.9790	1.7800	2.5801	2.9439	1.5975	1.5916	1.9872
0.9	0.1	0.0647	0.0393	0.0426	0.0439	0.0535	0.0329	0.0380	0.0348
	0.5	1.6176	0.9747	0.8604	1.1614	1.3385	0.7777	0.7792	0.8979
	1	6.4703	3.8719	3.3753	4.6652	5.3542	3.0742	3.0668	3.5956

Table 5. $\phi_1 = 0.5$ *and* $\phi_2 = -0.9$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0195	0.0171	0.0181	0.0170	0.0198	0.0184	0.0189	0.0184
	0.5	0.4880	0.2988	0.2892	0.3301	0.4945	0.3236	0.3250	0.3497
	1	1.9520	1.0851	1.0491	1.3234	1.9780	1.1201	1.1184	1.3304
0.8	0.1	0.0264	0.0199	0.0222	0.0195	0.0260	0.0216	0.0233	0.0213
	0.5	0.6588	0.3777	0.3572	0.4426	0.6502	0.3845	0.3853	0.4273
	1	2.6354	1.4311	1.3508	1.7991	2.6010	1.4201	1.4155	1.7331
0.9	0.1	0.0481	0.0296	0.0326	0.0313	0.0467	0.0302	0.0349	0.0303
	0.5	1.2024	0.6965	0.6383	0.8264	1.1685	0.6752	0.6772	0.7756
	1	4.8098	2.7518	2.5044	3.3293	4.6739	2.6510	2.6356	3.1279

Table 6. $\phi_1 = 1.5$ *and* $\phi_2 = -0.9$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0115	0.0111	0.0112	0.0111	0.0106	0.0102	0.0103	0.0102
	0.5	0.2871	0.2029	0.2038	0.2051	0.2643	0.1925	0.1923	0.1991
	1	1.1483	0.7347	0.7359	0.7460	1.0573	0.6928	0.6925	0.7308
0.8	0.1	0.0146	0.0133	0.0137	0.0133	0.0137	0.0123	0.0127	0.0122
	0.5	0.3653	0.2302	0.2278	0.2420	0.3416	0.2163	0.2161	0.2304
	1	1.4613	0.8565	0.8486	0.9618	1.3664	0.8095	0.8090	0.9091
0.9	0.1	0.0247	0.0184	0.0201	0.0180	0.0240	0.0171	0.0186	0.0168
	0.5	0.6182	0.3449	0.3277	0.4127	0.6000	0.3297	0.3294	0.3957
	1	2.4729	1.3291	1.2673	1.6831	2.3998	1.2815	1.2787	1.6109

Table 7. $\phi_1 = -1.5$ *and* $\phi_2 = -0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0152	0.0138	0.0142	0.0137	0.0130	0.0117	0.0120	0.0117
	0.5	0.3801	0.2320	0.2267	0.2559	0.3246	0.2007	0.2000	0.2206
	1	1.5202	0.8480	0.8284	0.9912	1.2983	0.7471	0.7461	0.8672
0.8	0.1	0.0205	0.0163	0.0175	0.0160	0.0174	0.0140	0.0148	0.0139
	0.5	0.5116	0.2828	0.2707	0.3291	0.4359	0.2443	0.2438	0.286
	1	2.0462	1.0732	1.0248	1.3316	1.7434	0.9361	0.9336	1.1744
0.9	0.1	0.0370	0.0233	0.0251	0.0241	0.0317	0.0201	0.0222	0.0210
	0.5	0.9246	0.4940	0.4535	0.5954	0.7933	0.4263	0.4231	0.5261
	1	3.6986	1.9345	1.7780	2.4064	3.1731	1.6759	1.6630	2.1160

Table 8. $\phi_1 = -0.5$ *and* $\phi_2 = -0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0309	0.0251	0.0276	0.0247	0.0260	0.0224	0.0238	0.0222
	0.5	0.7713	0.4623	0.4308	0.5341	0.6510	0.3927	0.3940	0.4406
	1	3.0850	1.6880	1.5651	2.1917	2.6041	1.4154	1.4111	1.7402
0.8	0.1	0.0420	0.0295	0.0339	0.0294	0.0349	0.0260	0.0291	0.0257
	0.5	1.0503	0.6114	0.5601	0.7357	0.8723	0.5002	0.5044	0.5738
	1	4.2012	2.3400	2.1313	2.9897	3.4892	1.9013	1.9038	2.3344
0.9	0.1	0.0770	0.0462	0.0507	0.0516	0.0635	0.0383	0.0449	0.0406
	0.5	1.9244	1.1569	1.0462	1.3541	1.5878	0.9298	0.9390	1.0531
	1	7.6976	4.5945	4.1141	5.4314	6.3511	3.6694	3.6843	4.2093

Table 9. $\phi_1 = 0.5$ *and* $\phi_2 = -0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0231	0.0212	0.0217	0.0197	0.0233	0.0215	0.0221	0.0214
	0.5	0.5769	0.3482	0.3353	0.3909	0.5835	0.3778	0.3796	0.4082
	1	2.3077	1.2645	1.2091	1.5874	2.3339	1.3017	1.2964	1.5537
0.8	0.1	0.0311	0.0231	0.0261	0.0227	0.0307	0.0250	0.0273	0.0247
	0.5	0.7779	0.4447	0.4179	0.5260	0.7670	0.4513	0.4546	0.4988
	1	3.1117	1.6864	1.5731	2.1493	3.0681	1.6690	1.6616	2.0148
0.9	0.1	0.0567	0.0344	0.0385	0.0368	0.0551	0.0349	0.0412	0.0352
	0.5	1.4180	0.8198	0.7542	0.9683	1.3780	0.7984	0.8004	0.8964
	1	5.6719	3.2416	2.9440	3.9004	5.5122	3.1417	3.1128	3.6184

Table 10. $\phi_1 = 1.5$ *and* $\phi_2 = -0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0182	0.0177	0.0177	0.0177	0.0133	0.0129	0.0130	0.0129
	0.5	0.4551	0.2951	0.2959	0.2992	0.3318	0.2428	0.2424	0.2465
	1	1.8204	0.9413	0.9421	0.8856	1.3272	0.8491	0.8488	0.8637
0.8	0.1	0.0217	0.0207	0.0208	0.0207	0.0167	0.0154	0.0158	0.0154
	0.5	0.5415	0.3250	0.3230	0.3350	0.4163	0.2688	0.2683	0.2794
	1	2.1658	1.0766	1.0703	1.0996	1.6654	0.9768	0.9760	1.0813
0.9	0.1	0.0328	0.0287	0.0297	0.0286	0.0279	0.0213	0.0233	0.0208
	0.5	0.8205	0.4528	0.4356	0.4988	0.6985	0.3945	0.3937	0.4582
	1	3.2821	1.6075	1.5529	1.9361	2.7938	1.5073	1.4994	1.8897

Table 11. $\phi_1 = -0.2$ *and* $\phi_2 = 0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0310	0.0267	0.0286	0.0264	0.0271	0.0230	0.0247	0.0228
	0.5	0.7756	0.4712	0.4560	0.5107	0.6767	0.3977	0.3980	0.4365
	1	3.1024	1.6768	1.6115	2.0099	2.7069	1.4271	1.4161	1.7668
0.8	0.1	0.0411	0.0304	0.0351	0.0297	0.0363	0.0266	0.0303	0.0259
	0.5	1.0268	0.5891	0.5642	0.6590	0.9085	0.5041	0.5026	0.5785
	1	4.1071	2.2158	2.0976	2.6590	3.6341	1.8971	1.8795	2.3674
0.9	0.1	0.0732	0.0438	0.0510	0.0457	0.0662	0.0388	0.0462	0.0404
	0.5	1.8299	1.0488	0.9871	1.1816	1.6554	0.9320	0.9295	1.0625
	1	7.3198	4.1652	3.8566	4.7536	6.6215	3.6887	3.6340	4.2780

Table 12. $\phi_1 = -0.1$ *and* $\phi_2 = 0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0319	0.0275	0.0294	0.0272	0.0277	0.0236	0.0253	0.0233
	0.5	0.7963	0.4822	0.4662	0.5218	0.6920	0.4086	0.4086	0.4462
	1	3.1853	1.7140	1.6449	2.0708	2.7681	1.4620	1.4513	1.8034
0.8	0.1	0.0420	0.0312	0.0360	0.0305	0.0371	0.0272	0.0310	0.0265
	0.5	1.0493	0.6012	0.5751	0.6694	0.9270	0.5159	0.5149	0.5892
	1	4.1972	2.2545	2.1260	2.7192	3.7082	1.9426	1.9210	2.4076
0.9	0.1	0.0743	0.0445	0.0522	0.0461	0.0674	0.0395	0.0472	0.0411
	0.5	1.8587	1.0573	0.9970	1.1904	1.6847	0.9498	0.9503	1.0806
	1	7.4347	4.1782	3.8952	4.7788	6.7387	3.7547	3.7086	4.3385

Table 13. $\phi_1 = 0.1$ *and* $\phi_2 = 0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0349	0.0311	0.0327	0.0308	0.0291	0.0253	0.0270	0.0251
	0.5	0.8716	0.5280	0.5127	0.5560	0.7266	0.4400	0.4413	0.4729
	1	3.4863	1.8165	1.7594	2.1701	2.9064	1.5566	1.5468	1.8884
0.8	0.1	0.0447	0.0352	0.0397	0.0344	0.0384	0.0292	0.0330	0.0285
	0.5	1.1172	0.6421	0.6175	0.7051	0.9591	0.5453	0.5448	0.6131
	1	4.4690	2.3426	2.2322	2.8535	3.8364	2.0303	2.0074	2.5107
0.9	0.1	0.0762	0.0482	0.0579	0.0483	0.0684	0.0412	0.0496	0.0421
	0.5	1.9043	1.0851	1.0274	1.2153	1.7096	0.9745	0.9697	1.0970
	1	7.6173	4.2361	3.9368	4.9036	6.8382	3.8348	3.7678	4.4176

Table 14. $\phi_1 = 0.2$ *and* $\phi_2 = 0.7$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0442	0.0410	0.0415	0.0410	0.0338	0.0311	0.0320	0.0310
	0.5	1.1049	0.5997	0.5909	0.6022	0.8459	0.5101	0.5129	0.5266
	1	4.4196	1.8850	1.8579	2.0457	3.3836	1.6864	1.6855	1.9705
0.8	0.1	0.0537	0.0471	0.0488	0.0470	0.0429	0.0361	0.0389	0.0357
	0.5	1.3422	0.7100	0.6935	0.7411	1.0728	0.6112	0.6117	0.6593
	1	5.3687	2.4044	2.3366	2.7871	4.2911	2.1509	2.1412	2.6487
0.9	0.1	0.0841	0.0609	0.0694	0.0598	0.0722	0.0487	0.0576	0.0478
	0.5	2.1027	1.1489	1.0904	1.2812	1.8057	1.0297	1.0205	1.1541
	1	8.4106	4.3172	4.0660	5.1806	7.2230	3.9461	3.8599	4.6895

Table 15. $\phi_1 = -0.05$ *and* $\phi_2 = 0.9$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0330	0.0304	0.0313	0.0303	0.0260	0.0235	0.0245	0.0234
	0.5	0.8261	0.5022	0.4890	0.5213	0.6497	0.4019	0.4017	0.4241
	1	3.3045	1.6754	1.6277	1.9559	2.5988	1.3984	1.3896	1.6646
0.8	0.1	0.0415	0.0350	0.0377	0.0345	0.0338	0.0273	0.0300	0.0268
	0.5	1.0366	0.6018	0.5754	0.6566	0.8442	0.4842	0.4822	0.5358
	1	4.1463	2.1392	2.0268	2.6397	3.3770	1.7717	1.7522	2.2067
0.9	0.1	0.0684	0.0467	0.0550	0.0464	0.0589	0.0376	0.0446	0.0374
	0.5	1.7101	0.9963	0.9222	1.1392	1.4713	0.8328	0.8247	0.9517
	1	6.8402	3.8390	3.5146	4.6251	5.8853	3.2379	3.1698	3.8635

Table 16. $\phi_1 = -0.025$ *and* $\phi_2 = 0.9$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0355	0.0329	0.0336	0.0329	0.0274	0.0251	0.0259	0.0250
	0.5	0.8879	0.5235	0.5106	0.5358	0.6840	0.4228	0.4227	0.4408
	1	3.5516	1.6999	1.6638	1.9319	2.7360	1.4414	1.4327	1.6956
0.8	0.1	0.0439	0.0381	0.0401	0.0378	0.0351	0.0292	0.0317	0.0288
	0.5	1.0981	0.6226	0.5970	0.6688	0.8786	0.5055	0.5031	0.5550
	1	4.3923	2.1631	2.0683	2.6248	3.5143	1.8160	1.7963	2.2567
0.9	0.1	0.0708	0.0500	0.0582	0.0494	0.0602	0.0398	0.0470	0.0393
	0.5	1.7707	1.0146	0.9442	1.1570	1.5059	0.8554	0.8473	0.9726
	1	7.0828	3.8656	3.5559	4.6987	6.0235	3.2971	3.2222	3.9631

Table 17. $\phi_1 = 0.025$ *and* $\phi_2 = 0.9$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0465	0.0434	0.0434	0.0434	0.0342	0.0323	0.0325	0.0323
	0.5	1.1615	0.5842	0.5798	0.5880	0.8552	0.4928	0.4936	0.5006
	1	4.6460	1.7447	1.7333	1.8234	3.4207	1.5365	1.5339	1.6602
0.8	0.1	0.0548	0.0496	0.0500	0.0496	0.0420	0.0378	0.0387	0.0377
	0.5	1.3707	0.6813	0.6675	0.6998	1.0494	0.5756	0.5738	0.6050
	1	5.4826	2.2050	2.1585	2.4485	4.1978	1.9141	1.9034	2.2605
0.9	0.1	0.0816	0.0642	0.0689	0.0635	0.0670	0.0505	0.0562	0.0496
	0.5	2.0400	1.0730	1.0199	1.1810	1.6759	0.9264	0.9178	1.0350
	1	8.1600	3.9407	3.7190	4.6695	6.7035	3.4248	3.3496	4.1995

Table 18. $\phi_1 = 0.05$ *and* $\phi_2 = 0.9$

γ^2	σ_u	n=30				n=100			
		GLS	RR	Liu	KL	GLS	RR	Liu	KL
0.7	0.1	0.0614	0.0563	0.0559	0.0562	0.0450	0.0425	0.0422	0.0424
	0.5	1.5357	0.6245	0.6305	0.6443	1.1259	0.5490	0.5506	0.5627
	1	6.1430	1.7610	1.7634	1.8079	4.5035	1.5716	1.5726	1.5983
0.8	0.1	0.0698	0.0629	0.0621	0.0630	0.0528	0.0487	0.0483	0.0487
	0.5	1.7442	0.7223	0.7243	0.7356	1.3198	0.6320	0.6333	0.6484
	1	6.9766	2.2167	2.2118	2.2865	5.2792	1.9508	1.9495	2.1034
0.9	0.1	0.0964	0.0800	0.0807	0.0801	0.0778	0.0641	0.0663	0.0639
	0.5	2.4110	1.1103	1.0854	1.1626	1.9451	0.9835	0.9814	1.0535
	1	9.6441	3.9398	3.8119	4.3284	7.7806	3.4693	3.4315	4.0780

The simulation tables clearly showed that sMSE value of the RR, Liu, and KL estimators increased as the strength of multicollinearity (γ^2). However, if we compare the biased estimators with the GLS, it is clearly seen that the estimators biased to the strength of the multicollinearity are more robust. In comparison to RR and Liu, the sMSE value of the KL estimator grew as the γ^2 increased. When $\sigma_u = 0.1$, the KL estimator gave much smaller sMSE values than the GLS, while giving very close sMSE values with the RR and Liu biased estimators. However, for $\sigma_u = 1$, the sMSE values of the KL estimator were always smaller than the GLS but gave higher sMSE values compared to RR and Liu. As n increased, sMSE value of the KL estimator decreased. It should be noted, however, that all of these results were interpreted using the stationary AR(2) model. When ϕ_2 constant, the absolute of the ϕ_1 declined in the AR(2) model, sMSE values of the KL estimator and others increased.

5. Application to Real Data

Two real datasets are analyzed to illustrate the sMSE performance of the KL estimator under the GLR model with AR(2) error structure. While performing the analyzes for both datasets, the multicollinearity problem was determined and in the linear regression model, parameter estimation values of OLS, ridge, liu, and kl estimators, sMSE values, k/d values that minimize sMSE for biased estimators, and CPU time is given in seconds. Then, the autocorrelation problem in the datasets was investigated and after the error structures were determined, the similar outputs of GLS, RR, Liu, and KL estimators in the GLR models were given. The examined datasets are available on request from the corresponding author.

5.1. French Economy Data

The French economy data, used by Malinvard [26], consists of one response variable and three regressor variables. The response variable y is imports, x_1 is domestic production, x_2 is stock formation and

x_3 is domestic consumption. All variables cover 1946 to 1845 and are measurements per milliards in French frags. The multiple linear regression model to be estimated as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon. \tag{8}$$

The dataset is used by previous authors to evaluate the performance of the KL estimator under the linear model (see, [9, 27]). Figure 1 depicts the linear correlation (r) between the three regressions,

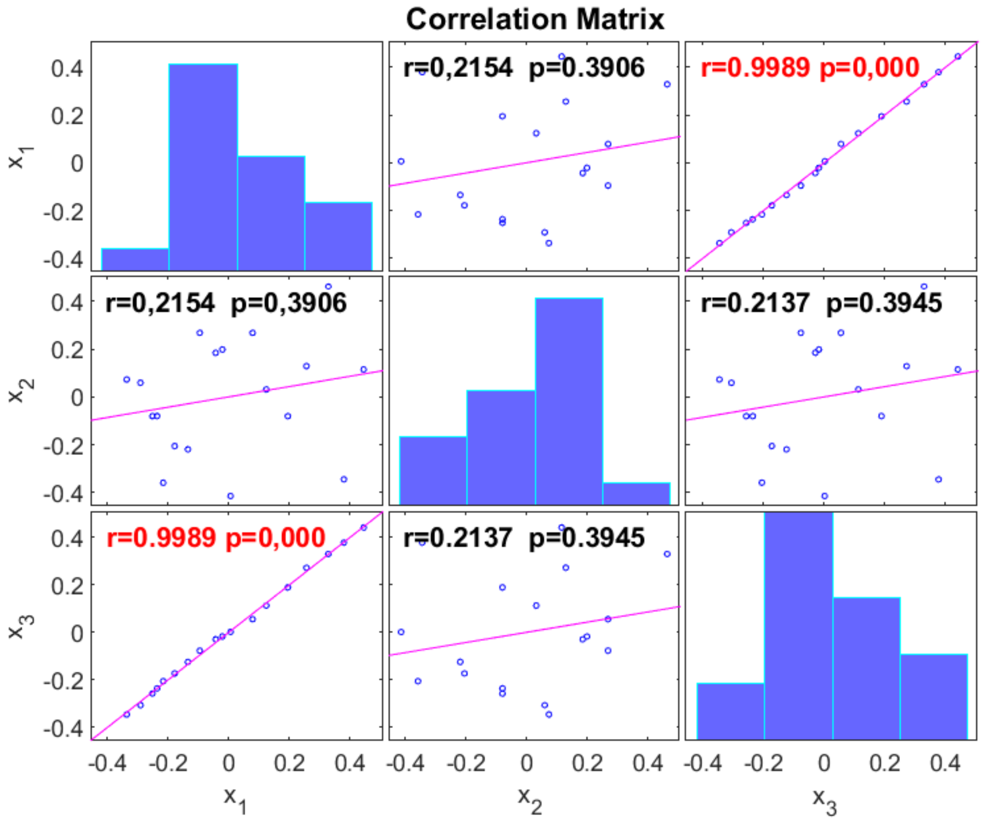

Fig. 1. Correlations between the regressors for French economy data.

as well as their significance at the 0.05 significance level (p). As can be observed, the correlation between the first and third variables is nearly one. In addition, when the condition number is examined, the eigenvalues of the $X'X$ matrix are calculated as 0.2982, 49.6458, 157.9069, 1617795.3772 and $\kappa = 2329.4065$. A strong multicollinearity problem has been detected using these two collinearity diagnostics approaches. As a result, biased estimators were used to estimate the parameters in Equation 1. In the linear regression model with multicollinearity problem, parameter estimates and sMSE values of OLS and alternative biased estimators, and biasing parameters minimizing sMSE for biased estimators are given in Table 19. Also, the times during the minimization of the sMSE values with respect to k/d are given as CPU time.

Table 19 shows that even though the liu estimator fitted to the linear model has the smallest sMSE value, the ridge, liu, and kl estimators are relatively similar in terms of sMSE performance.

Since Kibria and Lukman [9] conducted the paper in a linear regression model, the autocorrelation problem was not included French economy data. However, it has an autocorrelation problem. Autocorrelation function (ACF) and partial autocorrelation function (PACF) are used to determine this status and error structure decision based on ACF and PACF graphs as follows:

Except for the first lag, the other lags fluctuate within the confidence bounds, as shown in the ACF graph of Fig. 2. However, after two lags, the reduction in the PACF graph was abruptly stopped off (Fig. 2). The parameters of the AR(2) model were found to be significant as shown in Table 20.

It can be specify the lag structure, presence of a constant, and innovation distribution of an AR(p) model for this dataset by following Table 20. As shown in Table 20 the constant coefficient

Table 19. Estimation of model coefficients and sMSE values when autocorrelation is neglected for French economy data

Coeff.	**OLS**	**ridge**	**liu**	**kl**
$\hat{\beta}_0$	-19.7251	-18.8982	-18.8957	-18.8994
$\hat{\beta}_1$	0.03220	0.0628	0.0627	0.0628
$\hat{\beta}_2$	0.4142	0.4008	0.4005	0.4008
$\hat{\beta}_3$	0.2427	0.1947	0.1949	0.1948
sMSE	17.2379	16.515619	16.5154	16.515628
k/d	-	0.0131	0.9452	0.0064
CPU time	-	0.2500	0.3750	0.2656

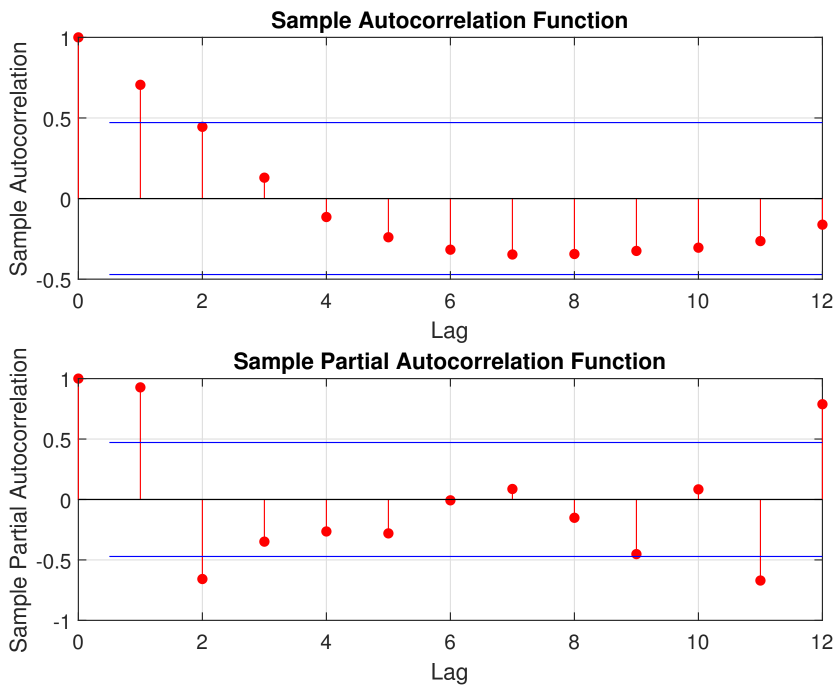

Fig. 2. Correlagram for French economy data

Table 20. Choosing appropriate lag in the AR Model for French economy data

p	**T statistic**	**P Value**
Constant	0.6539	0.5132
AR(1)	6.6345	3.2557e-11
AR(2)	-3.5830	0.0003

can be accepted as statistically zero. Thus, the model to be fit in the AR structure is compatible with Equation 5. The model parameters were estimated in the following order; $\hat{\phi}_1 = 0.7037$ and $\hat{\phi}_2 = 0.0028$.

After the P matrix based on the estimations of the $\hat{\phi}_1$ and $\hat{\phi}_2$ parameters, the response variable and regressor matrix were transformed. The multicollinearity problem still existed in the transformed model ($\kappa = 1271.7991$). Therefore, biased estimators were applied to estimate the regression parameters under the GLR model with AR(2) errors. The GLS, RR, Liu, and KL estimators were used to estimate the parameter estimates in the transformed model, and the sMSE values of the estimators are listed in Table 21.

Table 21 clearly demonstrates that the KL estimator provides an sMSE value that is similar to the well-known RR and Liu estimators when the error structure is AR(2) process. It was also observed that the three-biased estimator had better performance than the unbiased GLS according to the sMSE

Table 21. The parameter estimations and sMSE values in GLR models with AR(2) error structure for French economy data

General Linear (AR(2))	GLS	RR	Liu	KL
Constant	-21.8389	-21.2531	-21.2582	-21.2452
β_1	-0.0137	-0.0024	-0.0027	-0.0022
β_2	0.4840	0.4805	0.4805	0.4805
β_3	0.3234	0.3040	0.3045	0.3038
$sMSE$	13.0806	12.7340	**12.7339**	12.7342
$opt.k/d$	-	0.0032	0.9703	0.0016
CPU time	-	0.2031	0.5781	0.2812

criterion. Tables 19 and 21 are comparable; if the autocorrelation problem is ignored and a linear regression model is fitted, the sMSE value will differ due to differences in the model parameters.

5.2. Weather Data

Weather data received at the station in the Columbia-Pacific Northwest Region for each 15-minute time period on January 1, 2022, was used (https://www.usbr.gov/pn/agrimet/webagdayread.html). Multiple linear regression model to be estimated as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon \tag{8}$$

where y is wind speed, x_1 is humidity, x_2 is vapor pressure x_3 is dew point temperature.

Examining the correlation matrix graph shown in Figure 3, it can be clearly seen that there is a strong correlation between the vapor pressure and the dew point temperature ($r = 0.9359$ and $p = 1.0393e - 44$).

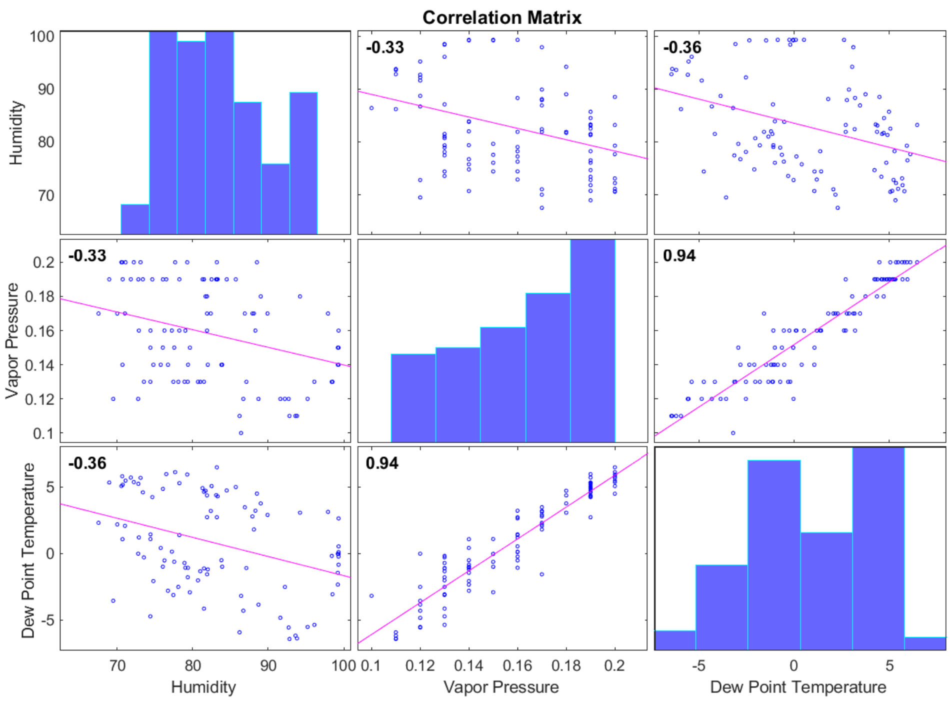

Fig. 3. Correlations between the regressors for weather data

The fact that this correlation coefficient is very close to 1 indicates a multicollinearity problem in the dataset. In addition, the condition number was calculated. The eigenvalues of the $X'X$ matrix are

0.0092, 0.9884, 1258.8777 and 66381.7020, respectively, and the condition number is $\kappa = 8522.8934$. Both the correlation coefficient and the condition number indicated strong multicollinearity. Therefore, while estimating the regression parameters in the linear regression model given by Equation 1, biased estimators were used as alternatives to OLS, and the optimal k/d values that minimize sMSE values were presented in Table 22. Table 22 shows when fitting the weather data with multicollinearity problem with the linear regression model, biased estimators significantly reduce the sMSE value compared to OLS. In addition, when the CPU times spent in the minimization process of optimal biasing parameters according to sMSE are examined, it is seen that the algorithms are completed in close seconds. The existence of the autocorrelation problem was investigated after the multicollinearity problem in the weather data was discovered. The ACF and PACF graphs in Figure 4 and the hypothesis tests in Table 23 were used to represent these investigations.

Table 22. Estimation of model coefficients and sMSE values when autocorrelation is neglected for weather data

Coeff.	**OLS**	**ridge**	**liu**	**kl**
$\hat{\beta}_0$	0.2118	0.3051	0.4471	0.8787
$\hat{\beta}_1$	0.01374	0.0214	0.0192	0.0140
$\hat{\beta}_2$	4.9497	0.0683	0.3735	0.4386
$\hat{\beta}_3$	-0.0429	0.0002	-0.0041	-0.0097
sMSE	349.2349	24.1939	23.2219	25.2441
k/d	-	2.0243	0.0536	0.0081
CPU time	-	0.2368	0.2656	0.2543

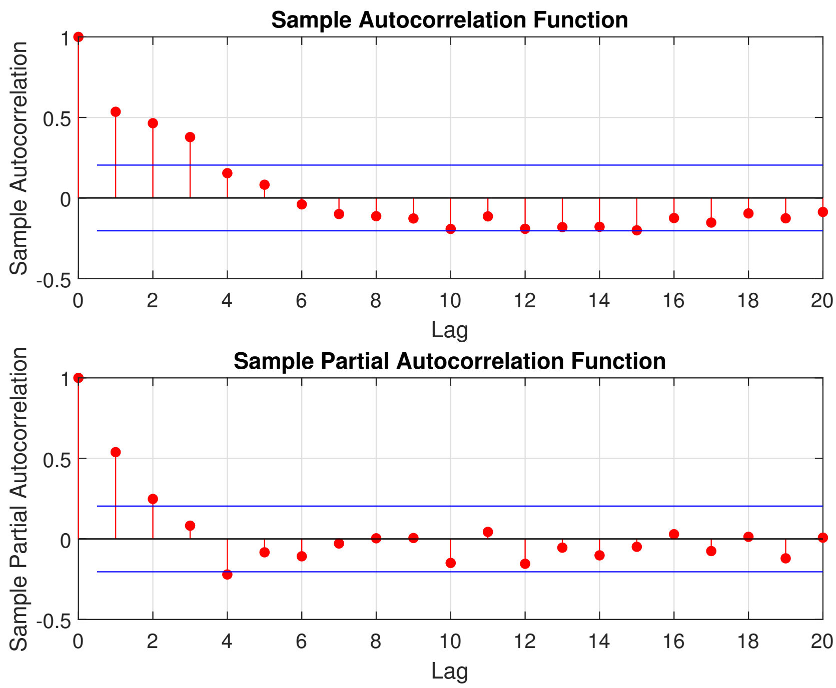

Fig. 4. Correlagram for weather data

Figure 4 clearly illustrated the properties of ACF/PACF of an AR(2) process: Its ACF decreased sharply and PACF was be nearly zero after lag 2.

Table 23 showed that the two delayed functional associations of errors were statistically significant. In this case, $\hat{\phi}_1$ and $\hat{\phi}_2$ parameters in Equation 5 were estimated as 0.3999 and 0.2561, respectively, in the weather dataset which has autocorrelation problem. After it was determined that the functional associations between the errors were modeled with the AR(2) process, the GLR model was fitted by

Table 23. Choosing appropriate lag in the AR Model for weather data

p	**T statistic**	**P Value**
Constant	0.6669	0.5700
AR(1)	4.2189	2.4541e-05
AR(2)	2.1966	0.0280

the unbiased GLS and the biased RR, Liu, and KL estimators. Estimation of regression parameters, sMSE values, optimal k/d biasing parameters minimizing sMSE, and minimization time as CPU were shown in Table 24.

Table 24. The parameter estimations and sMSE values in GLR models with AR(2) error structure for weather data

General Linear (AR(2))	**GLS**	**RR**	**Liu**	**KL**
Constant	-2.1272	-0.5364	-0.4215	-1.5792
β_1	0.04597	0.0326	0.0310	0.0450
β_2	2.9477	-0.0117	0.1115	-0.0690
β_3	-0.0417	-0.0334	-0.0353	-0.0239
$sMSE$	197.278	12.1687	11.9377	16.8049
$opt.k/d$	-	0.5581	0.0420	0.0100
CPU time	-	0.2012	0.2869	0.2831

It can be obviously seen in Table 24 that applying biased estimators in the weather data with multicollinearity and AR(2) autocorrelation problem significantly reduced the sMSE value compared to the unbiased GLS. The fact that the model variance is higher in the weather data compared to the French economy data supports a situation that is visible in the simulation results: In the French economy data with a small model variance, the sMSE value of the KL estimator was very close to the sMSE values of the RR and Liu estimators, but in the weather data with high model variance, the sMSE value of the KL estimator was slightly higher than the sMSE values of the RR and Liu estimators.

6. Conclusion and Recommendation

The simulation results demonstrated that as the variance of the model increases, the sMSE values of the KL estimator and the others increases. Moreover, the sMSE values of the KL estimator and others appear to increase as the severity of multicollinearity increases. When the model variance is small, sMSE values of the KL estimator under the AR(2) error structure are closely similar to the popular biased estimators' values. The sMSE values of the KL estimator and others decreased when the sample size was increased. Examples of two data sets with autocorrelation problems from both multicollinearity and AR(2) processes are also included in the paper. The results of the two different data sets were generally frugal and the findings support the simulation results. Furthermore, CPU times were discovered to be near to each other while determining the optimal biasing parameter over real datasets. In other words, the Kl estimator was close to the popular estimators in terms of CPU time. It was discovered that KL performed substantially better than GLS with optimum biasing parameters, and its results were extremely near to those of Ridge and Liu estimators. In the statistical literature, new unbiased and biased estimators continue to be proposed. As new estimators are proposed, it is critical to examine the assumptions and to make parameter estimations on the correct model for statistical inference.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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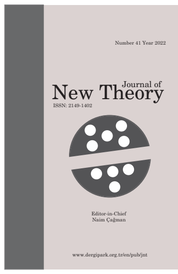

A New Perspective on k-Ideals of a Semiring via Soft Intersection Ideals

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Abstract — In recent years, soft sets have become popular in various fields. For this reason, many studies have been carried out in the field of algebra. In this study, soft intersection k-ideals are defined with the help of a semiring, and some algebraic structures are examined. Moreover, the quotient rings are defined by k-semiring. Isomorphism theorems are examined by quotient rings. Finally, some algebraic properties are investigated by defining soft intersection maximal k-ideals.

Keywords — *Soft sets, soft k-ideals, soft k-semirings, soft maximal k-ideals, quotient k-semirings*

Mathematics Subject Classification (2020) — 03E99, 03E75

1. Introduction

The world is growing and developing rapidly. During the growth and development, we encounter many problems involving uncertainty. Scientists are working quickly to solve such problems. Firstly, Zadeh introduced a different approach to uncertainty by Fuzzy Set Theory [1]. Zadeh defined a function by taking the codomain as $[0, 1]$. With the help of this function, he made an approach to uncertainty by creating a fuzzy set. Rosenfeld dealing with uncertainty developed fuzzy group theory in order to form the algebraic structure of fuzzy set [2]. Then, Pawlak suggested the rough set theory in 1982 [3]. Pawlak made a different approach to uncertainty by defining a rough set with the help of lower and upper approximation. Biswas and Nanda studied algebraic properties of rough sets [4]. After, soft set theory was proposed by Molodtsov as a different approach to uncertainty [5]. Maji et al. applied the theory of soft sets to solve a decision making problem and defined some basic operations on soft sets [6, 7]. Later, Çağman and Enginoğlu redefined operations of soft sets due to some difficulties [8]. Thus, soft sets began to be used by many researchers in various fields like economics, engineering, medical sciences, etc. In addition, many studies have been made by combining soft sets with fuzzy set theory and rough set theory such as fuzzy soft sets and rough soft sets [9–13].

Aktaş and Çağman carried soft sets on a new algebraic structure for the first time. They investigated some algebraic properties by defining soft group [14]. This work paved the way for many studies in algebra. Sun et al. defined soft modules and investigated some algebraic properties [15]. Feng et al. studied soft semirings and soft ideals on soft semirings [16]. Jun et al. applied the soft sets to the theory of BCK/BCI-algebras. Soft BCK/BCI-algebras, soft subalgebras and soft p-ideals of soft BCI-algebras are introduced and their basic properties are derived [17, 18]. Many studies have been added to the literature such as soft relation, soft function, soft mapping, soft BCH-algebras, soft Bl-algebras by soft sets [19–23]. Shabir and Naz introduced soft topological spaces which are defined

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over an initial universe with a fixed set of parameters [25]. Ali et al. defined new operations on soft sets and investigated some important properties associated with these new operations. [24]. In 2012, Çağman et al. studied soft intersection group using the intersection operation on sets [26]. Çıtak and Çağman researched its algebraic properties by defining soft intersection rings [27]. Moreover, Mahmood and Tariq studied generalized k-ideals in semirings using soft intersectional sets [28]. Mahmood et al. carried out concept of soft intersectional ideal on ternary semirings and also discussed some basic results [29]. Studies on soft sets have been increasing in recent years [30–40].

In this paper, we give brief information about semiring and k-semiring and remind some basic concepts in the soft set. Moreover, we present to some algebraic structures on soft sets such as soft intersection group and soft intersection ring. Next, we define soft intersection k-ideals on a semiring. Then, we investigate some algebraic properties of soft intersection k-ideals. In which cases the image and the inverse image of a soft set are soft intersection ideals are investigated. Coset of soft intersection ideal is defined with the help of the extended soft set and its properties were examined. Furthermore, isomorphism theorems are introduced by describing quotient rings with the help of k-semirings. Finally, we define soft intersection maximal k-ideal and research its algebraic properties.

2. Preliminary

Throughout this paper, U is a universal set, E is a set of parameters, $A, B, C \subseteq E$ and $P(U)$ is the power set of U.

Definition 2.1. [41] A nonempty set S together with a binary operation "." is a semigroup if "." is associative in S, that is, $\forall a, b, c \in S, a(bc) = (ab)c$. A semigroup is commutative if "." is commutative in S, that is, $\forall a, b \in S, ab = ba$.

Definition 2.2. [41] Let S be a nonempty set together with two binary operations addition and multiplication denoted by "+", "." respectively. S is called a semiring if

i. $(S, +)$ is a commutative semigroup,

ii. $(S, \cdot)$ is a semigroup,

iii. Distributive law holds, that is, $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$,

iv. There exists $0 \in S$ such that $a + 0 = 0 + a = a$ and $a0 = 0a = 0$ for all $a \in S$.

Definition 2.3. [42] Let S be a semiring. S is called a k-semiring if there exists only $c \in S$ such that $b = a + c$ or $a = b + c$ for all $a, b \in S$.

Definition 2.4. [42] Let S be a semiring and I be a nonempyt subset of S. I is called an ideal of S if

i. $a + b \in I$ for all $a, b \in I$

ii. $ba \in I$ and $ab \in I$ for all $a \in I$ and $b \in S$.

Definition 2.5. [43] Let S be a semiring and I be an ideal of S. I is called a k-ideal of S if $a + r \in I$ for all $a \in I$ and $r \in R$ implies $r \in I$.

Definition 2.6. [44] Let S be a k-semiring and S' be a set of the same cardinality with $S - \{0\}$ such that $S \cap S' = \emptyset$ and $S \cup S' = \overline{S}$. The image of $a \in S - \{0\}$ under a given bijection denoted by a'. Addition and multiplication on $\overline{S}$ are denoted by $\oplus$ and $\odot$, respectively, and are defined as follows:

$$a \oplus b = \begin{cases} a + b, & a, b \in S \\ (x + y)', & a = x', b = y' \in S' \\ c, & a \in S, b = y' \in S', a = y + c \\ c', & a \in S, b = y' \in S', a + c = y \end{cases}$$

and

$$a \odot b = \begin{cases} ab, & a, b \in S \\ xy, & a = x^{'}, b = y^{'} \in S^{'} \\ (ay)^{'}, & a \in S, b = y^{'} \in S^{'} \\ (xb)^{'}, & a = x^{'} \in S^{'}, b = \in S \end{cases}$$

Theorem 2.7. [44] Let S be a semiring. If S is a k-semiring, then $(\overline{S}, \oplus, \odot)$ is a ring and it is called the extended ring of S.

Theorem 2.8. [44] Let $\varphi : S \to M$ be a k-semiring homomorphism and $\overline{S}$ and $\overline{M}$ be extended rings of S and M, respectively. Let $\overline{\varphi} : \overline{S} \to \overline{M}$ be a function such that

$$\overline{\varphi}(a) = \begin{cases} \varphi(a), & a \in S \\ \varphi(a)^{'}, & a \in S - \{0\} \end{cases}$$

$\overline{\varphi}$ is a ring homomorphism. Then $\overline{\varphi}$ is called an extended ring homomorphism.

Theorem 2.9. [44] Let $\varphi : S \to M$ be a k-semiring homomorphism and $\overline{\varphi} : \overline{S} \to \overline{M}$ be an extended ring homomorphism. Then, $Ker(\overline{\varphi}) = \overline{Ker(\varphi)}$.

Definition 2.10. [5] A soft set (F, A) over U is a function defined by $F : E \to P(U)$ such that $F(e) = \emptyset$ if $e \notin A$. A soft set (F, A) over U can be represented by the set of ordered pairs

$$(F, A) = \{(e, F(e)) : e \in E\}$$

Definition 2.11. [7] Let (F, A) be a soft set over U. If $F(e) = \emptyset$ for all $e \in E$, then (F, A) is called an empty soft set, denoted by $\widetilde{\Phi}$.

If $F(e) = U$ for all $e \in E$, then (F, A) is called universal soft set, denoted by $\widetilde{E}$.

Definition 2.12. [8] Let (F, A) and (G, B) be soft sets over U. Then, (F, A) is a soft subset of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, if $F(e) \subseteq G(e)$ for all $e \in E$.

(F, A) is called a soft proper subset of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B)$, if $F(e) \subseteq G(e)$ for all $e \in E$ and $F(e) \neq G(e)$ for at least one $e \in E$.

(F, A) and (G, B) are equal, denoted by $(F, A) = (G, B)$ if $F(e) = G(e)$ for all $e \in E$.

Definition 2.13. [8] Let (F, A) and (G, B) be two soft sets over U. Then, union $(F, A)\widetilde{\cup}(G, B)$ and intersection $(F, A)\widetilde{\cap}(G, B)$ of (F, A) and (G, B) are defined by,

$$(F \cup G)(e) = F(e) \cup G(e), \quad (F \cap G)(e) = F(e) \cap G(e)$$

for all $e \in E$, respectively.

Definition 2.14. [8] Let (F, A) be a soft set over U. Then, complement $(F^{\tilde{c}}, E)$ of (F, A) is defined by,

$$F^{c}(e) = U \setminus F(e)$$

for all $e \in E$.

It is easy to see that $((F, A)^{\tilde{c}})^{\tilde{c}} = (F, A)$ and $\widetilde{\Phi}^{\tilde{c}} = \widetilde{E}$.

Proposition 2.15. [8] Let (F, A) be a soft set over U. Then,

i. $(F, A)\widetilde{\cup}(F, A) = (F, A)$, $(F, A)\widetilde{\cap}(F, A) = (F, A)$

ii. $(F, A)\widetilde{\cup}\widetilde{\Phi} = (F, A)$, $(F, A)\widetilde{\cap}\widetilde{\Phi} = \widetilde{\Phi}$

iii. $(F, A)\widetilde{\cup}\widetilde{E} = \widetilde{E}$, $(F, A)\widetilde{\cap}\widetilde{E} = (F, A)$

iv. $(F, A)\widetilde{\cup}(F, A)^{\tilde{c}} = \widetilde{E}$, $(F, A)\widetilde{\cap}(F, A)^{\tilde{c}} = \widetilde{\Phi}$

Proposition 2.16. [8] Let $(F,A),(G,B)$ and (H,C) be soft sets over U. Then,

i. $(F,A)\widetilde{\cup}(G,B)=(G,B)\widetilde{\cup}(F,A)$, $(F,A)\widetilde{\cap}(G,B)=(G,B)\widetilde{\cap}(F,A)$

ii. $((F,A)\widetilde{\cup}(G,B))^{\tilde{c}}=(G,B)^{\tilde{c}}\widetilde{\cap}(F,A)^{\tilde{c}}$,
$((F,A)\widetilde{\cap}(G,B))^{\tilde{c}}=(G,B)^{\tilde{c}}\widetilde{\cup}(F,A)^{\tilde{c}}$

iii. $((F,A)\widetilde{\cup}(G,B))\widetilde{\cup}(H,C)=(F,A)\widetilde{\cup}((G,B)\widetilde{\cup}(H,C))$,
$((F,A)\widetilde{\cap}(G,B))\widetilde{\cap}(H,C)=(F,A)\widetilde{\cap}((G,B)\widetilde{\cap}(H,C))$

iv. $(F,A)\widetilde{\cup}((G,B)\widetilde{\cap}(H,C))=((F,A)\widetilde{\cup}(G,B))\widetilde{\cap}((F,A)\widetilde{\cup}(H,C))$,
$(F,A)\widetilde{\cap}((G,B)\widetilde{\cup}(H,C))=((F,A)\widetilde{\cap}(G,B))\widetilde{\cup}((F,A)\widetilde{\cap}(H,C))$

Definition 2.17. [26] Let G be a group and (F,G) be a soft set over U. Then, (F,G) is called a soft intersection group over U iff

i. $F(a-b)\supseteq F(a)\cap F(b)$

ii. $F(ab)\supseteq F(a)\cap F(b)$

for all $a,b\in G$.

Proposition 2.18. [27] If (F,G) is a soft intersection group over U, then $F(e_G)\supseteq F(a)$ for all $a\in G$.

Definition 2.19. [27] Let R be a ring and (F,R) be a soft set over U. Then, (F,R) is called a soft intersection ring over U iff

i. $F(a-b)\supseteq F(a)\cap F(b)$

ii. $F(ab)\supseteq F(a)\cap F(b)$

for all $a,b\in R$.

Definition 2.20. [27] Let R be a ring and (F,R) be a soft set over U. Then, (F,R) is called a soft intersection ideal over U iff

i. $F(a-b)\supseteq F(a)\cap F(b)$

ii. $F(ab)\supseteq F(a)\cup F(b)$

for all $a,b\in R$.

Proposition 2.21. [27] Let R be a ring with identity. If (F,R) is a soft intersection ring/ideal over U, then $F(a)\supseteq F(1_R)$ for all $a\in R$.

3. Soft Intersection k-Ideals on a Semiring

In this section, we define soft intersection k-ideals on a semiring and give some basic theory of soft intersection k-ideals on a semiring. Let U be a universal set and E be a set of parameters where $(S,+,\cdot)$ is a semiring.

Definition 3.1. [45] Let (F,S) be a soft set over U. (F,S) is called a soft intersection semiring over U if

i. $F(a+b)\supseteq F(a)\cap F(b)$

ii. $F(ab)\supseteq F(a)\cap F(b)$

for all $a,b\in S$.

Definition 3.2. Let (F,S) be a soft intersection semiring over U. For all $a,b\in S$, *i.* (F,S) is called soft intersection left ideal over U if $F(ab) \supseteq F(b)$.

ii. (F,S) is called soft intersection right ideal over U if $F(ab) \supseteq F(a)$.

If (F,S) is soft intersection left ideal and soft intersection ideal over U, it is called soft intersection ideal over U.

Definition 3.3. Let (F,S) be a soft intersection ideal over U. (F,S) is called soft intersection k-ideal if $F(a) = F(0_S)$ while $F(a+b) = F(0_S)$ and $F(b) = F(0_S)$ for all $a, b \in S$.

Example 3.4. Let Z be a universal set and $(Z_6, +, \cdot)$ be a set of parameters. Let the soft set (F, Z_6) soft set be defined as in the following way:
$F(0) = \{0,1\}$, $F(1) = \{0\}$,$F(2) = \{0,1\}$, $F(3) = \{0\}$, $F(4) = \{0,1\}$, $F(5) = \{0\}$
Then, (F, Z_6) is a soft intersection k-ideal over Z.

Definition 3.5. Let (F,S) be a soft set over U. $F_K = \{a \in S : F(a) \supseteq K, K \in P(U)\}$ is called K-level set of the soft set (F,S).

Theorem 3.6. Let (F,S) be a soft intersection ideal over U. Then, K-level set F_K is an ideal of S where $F(0_S) \supseteq K$.

Proof. Since $F(0_S) \supseteq K$, then $0_S \in F_K$. Thus $F_K \neq \emptyset$ and $F_K \subseteq S$. Now, we provide $a+b \in F_K$ for all $a, b \in F_K$. Since $a, b \in F_K$ then $F(a) \supseteq K$ and $F(b) \supseteq K$. $F(a+b) \supseteq F(a) \cap F(b) \supseteq K$ so $a+b \in F_K$. Now, will provide $as \in F_K$ and $sa \in F_K$ for all $a \in F_K$ and $s \in S$. Since $a \in F_K$, then $F(a) \supseteq K$. Moreover $F(as) \supseteq F(a) \supseteq K$ and $F(sa) \supseteq F(a) \supseteq K$. Then , $as \in F_K$ and $sa \in F_K$. □

Theorem 3.7. Let (F,S) be a soft intersection ideal over U and $F(0_S) = K$. If K-level set F_K is a k-ideal of S, then (F,S) is a soft intersection k-ideal over U.

Proof. Suppose that F_K is a k-ideal of S. Let $F(a+b) = F(0_S)$ and $F(b) = F(0_S)$ for all $a, b \in S$. Therefore, $a+b \in F_K$ and $b \in F_K$. Since F_K is a k-ideal of S, then $a \in F_K$. Thus, $F(a) \supseteq K$ Therefore, (F,S) is a soft intersection k-ideal over U. □

Definition 3.8. Let I be an ideal of semiring S. λ_I is called a soft characteristic function if and only if λ_I is a mapping of S into $P(U)$ where

$$\lambda_I(a) = \begin{cases} U, & a \in I \\ \emptyset, & a \notin I \end{cases}$$

for all $a \in S$.

Theorem 3.9. Let I be a k-ideal of semiring S. Soft caharacteristic function λ_I is a soft intersection k-ideal over U.

Proof. Firstly, we provide that soft characteristic function λ_I is a soft intersection ideal over U. Since I is an ideal, then $a+b \in I$, for all $a, b \in I$. Morever, $\lambda_I(a) = U$ and $\lambda_I(b) = U$ and $\lambda_I(a+b) = U$. Therefore, $\lambda_I(a+b) \supseteq \lambda_I(a) \cap \lambda_I(b)$. In addition, $\lambda_I(a) = \emptyset$ and $\lambda_I(b) = \emptyset$ for all $a, b \notin I$. Thus, $\lambda_I(a) \cap \lambda_I(b) = \emptyset$ and then, $\lambda_I(a+b) \supseteq \lambda_I(a) \cap \lambda_I(b)$. On the other hands, $\lambda_I(a) = U$ and $\lambda_I(b) = \emptyset$ for all $a \in I, b \notin I$. Thus, $\lambda_I(a) \cap \lambda_I(b) = \emptyset$. It follows that $\lambda_I(a+b) \supseteq \lambda_I(a) \cap \lambda_I(b)$. Now, we will provide that $\lambda_I(ab) \supseteq \lambda_I(a)$ and $\lambda_I(ab) \supseteq \lambda_I(b)$ for all $a, b \in S$. Since I is an ideal of S, then $ab \in I$ for all $a, b \in I$. Thus, $\lambda_I(a) = U$ and $\lambda_I(b) = U$ and $\lambda_I(ab) = U$. Therefore, $\lambda_I(ab) \supseteq \lambda_I(a)$ and $\lambda_I(ab) \supseteq \lambda_I(b)$. Since I is an ideal of S, then $ab \in I$ and $ba \in I$ for all $a \in I, b \notin I$. Thus, $\lambda_I(ab) = U$ and $\lambda_I(a) = U$ and$\lambda_I(b) = \emptyset$. Therefore, $\lambda_I(ab) \supseteq \lambda_I(a)$ and $\lambda_I(ab) \supseteq \lambda_I(b)$. Lastly, we will provide that characteristic function λ_I is a soft intersection k-ideal of S. Suppose that $\lambda_I(a+b) = \lambda_I(0_s)$ and $\lambda_I(b) = \lambda_I(0_s)$ for all $a, b \in S$. It follows that $a+b \in I$ and $b \in I$ by definition of soft caharacteristic function. Since I is a k-ideal of S, then $a+b \in I$ and $b \in I$, $a \in I$. Therefore, $\lambda_I(a) = U$. Thus, $\lambda_I(a) = \lambda_I(0_s)$. Consequently, λ_I is a soft intersection k-ideal of S over U. □

Definition 3.10. Let $\varphi : S \to M$ be a semiring homomorphism and (G, M) be a soft set over U. A soft set $(\varphi^{-1}(F), S)$ is defined by $\varphi^{-1}(F(a)) = F(\varphi(a))$ for all $a \in S$. The soft set is called a soft inverse image of (G, M).

Theorem 3.11. Let $\varphi : S \to M$ be a semiring homomorphism and (G, M) be a soft set over U. (G, M) is a soft intersection k-ideal over U if and only if $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U.

Proof. Suppose that (G, M) is a soft intersection k-ideal over U. Firstly, we provide that $(\varphi^{-1}(G), S)$ is a soft intersection ideal,

$$\begin{aligned}\varphi^{-1}(G)(a+b) &= G(\varphi(a+b)) \\ &= G(\varphi(a)+\varphi(b)) \\ &\supseteq G(\varphi(a)) \cap G(\varphi(b)) \\ &= \varphi^{-1}(G)(a) \cap \varphi^{-1}(G)(b)\end{aligned}$$

for all $a, b \in S$. Thus, $\varphi^{-1}(G)(a+b) \supseteq \varphi^{-1}(G)(a) \cap \varphi^{-1}(G)(b)$. For all $a, b \in S$,

$$\begin{aligned}\varphi^{-1}(G)(ab) &= G(\varphi(ab)) \\ &= G(\varphi(a)\varphi(b)) \\ &\supseteq G(\varphi(a)) \\ &= \varphi^{-1}(G)(a)\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(G)(ab) &= G(\varphi(ab)) \\ &= G(\varphi(a)\varphi(b)) \\ &\supseteq G(\varphi(b)) \\ &= \varphi^{-1}(G)(b)\end{aligned}$$

Then, $\varphi^{-1}(G)(ab) \supseteq \varphi^{-1}(G)(a)$ and $\varphi^{-1}(G)(ab) \supseteq \varphi^{-1}(G)(b)$. Therefore, $(\varphi^{-1}(G), S)$ is a soft intersection ideal over U.

Now, we provide that $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U. Suppose that $\varphi^{-1}(G)(a + b) = \varphi^{-1}(G)(0_S)$ and $\varphi^{-1}(G)(b) = \varphi^{-1}(G)(0_S)$ for all $a, b \in S$.

Thus, it follows that $G(\varphi(a + b)) = G(0_M)$ and $G(\varphi(b)) = G(0_M)$. Since φ is a homomorphism then $G(\varphi(a) + \varphi(b)) = G(0_M)$, $G(\varphi(b)) = G(0_M)$ and since (G, M) is a soft intersection k-ideal over U, then $G(\varphi(a)) = G(0_{S'}) = \varphi(0_S)$.

Therefore, $\varphi^{-1}(G)(a) = \varphi^{-1}(G)(0_S)$. Consequently, $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U.

Conversely, suppose that $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U. Firstly, we provide that (G, M) is a soft intersection k-ideal over U. Since φ is an onto homomorphism then there exist $a, b \in S$ such that $x = \varphi(a)$ and $y = \varphi(b)$ for all $x, y \in M$. Then,

$$\begin{aligned}G(x+y) &= G(\varphi(a)+\varphi(b)) \\ &= G(\varphi(a+b)) \\ &= \varphi^{-1}(G)(a+b) \\ &\supseteq \varphi^{-1}(G)(a) \cap \varphi^{-1}(G)(b) \\ &= G(\varphi(a)) \cap G(\varphi(b)) \\ &= G(x) \cap G(y)\end{aligned}$$

Thus, it follows that $G(x+y) \supseteq G(x) \cap G(y)$. Moreover,

$$\begin{aligned} G(xy) &= G(\varphi(a)\varphi(b)) \\ &= G(\varphi(ab)) \\ &= \varphi^{-1}(G)(ab) \\ &\supseteq \varphi^{-1}(G)(a) \\ &= G(\varphi(a)) \\ &= G(x) \end{aligned}$$

and

$$\begin{aligned} G(xy) &= G(\varphi(a)\varphi(b)) \\ &= G(\varphi(ab)) \\ &= \varphi^{-1}(G)(ab) \\ &\supseteq \varphi^{-1}(G)(b) \\ &= G(\varphi(b)) \\ &= G(y) \end{aligned}$$

Therefore, $G(xy) \supseteq G(x)$ and $G(xy) \supseteq G(y)$. Consequently, (G, M) is a soft intersection ideal over U. Now, we provide that (G, M) is a soft intersection k-ideal over U. Suppose that $G(x+y) = G(0_M)$ and $G(y) = G(0_M)$ for all $x, y \in M$. Since φ is an onto homomorphism then there exist $a, b \in S$ such that $x = \varphi(a)$ and $y = \varphi(b)$ for all $x, y \in M$. Thus,

$$\begin{aligned} G(x+y) &= G(\varphi(a)+\varphi(b)) \\ &= G(\varphi(a+b)) \\ &= \varphi^{-1}(G)(a+b) \end{aligned}$$

and

$$\begin{aligned} G(0_M) &= G(\varphi(0_S)) \\ &= \varphi^{-1}(G)(0_S) \end{aligned}$$

Therefore, it follows $\varphi^{-1}(G)(a+b) = \varphi^{-1}(G)(0_S)$. Moreover,

$$G(y) = G(\varphi(b)) = \varphi^{-1}(G)(b)$$

and thus,

$$\varphi^{-1}(G)(b) = \varphi^{-1}(G)(0_S)$$

Since $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U, then $\varphi^{-1}(G)(a) = \varphi^{-1}(G)(0_S)$ while $\varphi^{-1}(G)(a+b) = \varphi^{-1}(G)(0_S)$ and $\varphi^{-1}(G)(b) = \varphi^{-1}(G)(0_S)$

Therefore,

$$\begin{gathered} \varphi^{-1}(G)(a) = \varphi^{-1}(G)(0_S) \\ G(\varphi(a)) = G(\varphi(0_S)) \\ G(x) = G(0_{S'}) \end{gathered}$$

Hence, (G, M) is a soft intersection k-ideal over U. □

Definition 3.12. Let $\varphi : S \to M$ be a semiring homomorphism and (F, S) be a soft set over U. A soft set $(\varphi(F), M)$ is called soft image set of (F, S) defined by

$$\varphi(F)(x) = \begin{cases} \cup\{F(a) : a \in S, \varphi(a) = x\}, & \varphi^{-1}(x) \neq \emptyset \\ \emptyset, & otherwise \end{cases}$$

Definition 3.13. Let A, B be two sets and $\varphi : A \to B$ be a function. Let (F, A) be a soft set over U. (F, A) is called φ-invariant if $F(a) = F(b)$ while $\varphi(a) = \varphi(b)$ for all $a, b \in A$.

Theorem 3.14. Let $\varphi : S \to M$ be a semiring epimorphism and (F, S) be a φ-invariant soft intersection ideal over U. Then, $(\varphi(F), M)$ is a soft intersection ideal over U.

PROOF. Since φ is an onto homomorphism, then there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$ for all $x, y \in M$. Thus, $x + y = \varphi(a) + \varphi(b) = \varphi(a + b)$ and $xy = \varphi(a)\varphi(b) = \varphi(ab)$. Since (F, S) is a φ-invariant then

$$\begin{aligned} x + y = \varphi(a+b) &\Rightarrow \varphi^{-1}(x+y) = a+b \\ &\Rightarrow \varphi(\varphi^{-1}(x+y)) = \varphi(a+b) \\ &\Rightarrow F(\varphi^{-1}(x+y)) = F(a+b) \\ &\Rightarrow \varphi(F)(x+y) = F(a+b) \end{aligned}$$

$\varphi(F)(x) = F(a)$ and $\varphi(F)(y) = F(b)$ and

$$\begin{aligned} xy = \varphi(ab) &\Rightarrow \varphi^{-1}(xy) = ab \\ &\Rightarrow \varphi(\varphi^{-1}(xy)) = \varphi(ab) \\ &\Rightarrow F(\varphi^{-1}(xy)) = F(ab) \\ &\Rightarrow \varphi(F)(xy) = F(ab) \end{aligned}$$

$\varphi(F)(x) = F(a)$ and $\varphi(F)(y) = F(b)$.

Therefore, $\varphi(F)(x + y) = F(a + b) \supseteq F(a) \cap F(b) = \varphi(F)(x) \cap \varphi(F)(y)$ and
$\varphi(F)(xy) = F(ab) \supseteq F(a) = \varphi(F)(x)$, $\varphi(F)(xy) = F(ab) \supseteq F(b) = \varphi(F)(y)$.

Consequently, $(\varphi(F), M)$ is a soft intersection ideal over U. □

Theorem 3.15. Let $\varphi : S \to M$ be a semiring epimorphism and (F, S) be a φ-invariant soft intersection ideal over U. (F, S) is a soft intersection k-ideal over U if and only if $(\varphi(F), M)$ is soft intersection k-ideal over U.

PROOF. Suppose that (F, S) be a soft intersection k-ideal over U. $(\varphi(F), M)$ is a soft intersection ideal over U by Theorem 3.14. Now, we provide that $(\varphi(F), M)$ is a soft intersection k-ideal over U. Suppose that $\varphi(F)(x + y) = \varphi(F)(0_M)$ and $\varphi(F)(y) = \varphi(F)(0_M)$ for all $x, y \in M$. Since φ is an onto homomorphism, then there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$ for all $x, y \in M$. Thus,

$$\begin{aligned} \varphi(F)(x+y) = \varphi(F)(\varphi(a) + \varphi(b)) &= \varphi(F)(\varphi(a+b)) \\ &= F(a+b) \end{aligned}$$

$$\begin{aligned} \varphi(F)(y) &= \varphi(F)(\varphi(b)) \\ &= F(b) \end{aligned}$$

Moreover, since (F, S) is a φ-invariant then $\varphi(F)(0_M) = F(0_S)$. Since $\varphi(F)(x + y) = \varphi(F)(0_M)$ then $F(a + b) = F(0_S)$ and $\varphi(F)(y) = \varphi(F)(0_M)$. Therefore, $F(b) = F(0_S)$. Since (F, S) is a soft intersection k-ideal over U, then $F(a) = F(0_S)$ while $F(a + b) = F(0_S)$ and $F(b) = F(0_S)$. Thus,

$$\begin{aligned} \varphi(F)(x) &= \varphi(F)(\varphi(a)) \\ &= F(a) \\ &= F(0_S) \\ &= \varphi(F)(0_M) \end{aligned}$$

Therefore, $(\varphi(F), M)$ is a soft intersection k-ideal over U. Otherwise, suppose that $(\varphi(F), M)$ be a soft intersection k-ideal over U. Let $F(a + b) = F(0_S)$ and $F(b) = F(0_S)$ for all $a, b \in S$. Since φ is an onto homomorphism, then there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$ for all $x, y \in M$. Then,

$$\begin{aligned} \varphi(F)(x+y) &= \varphi(F)(\varphi(a+b)) \\ &= F(a+b) \\ &= F(0_S) \\ &= \varphi(F)(0_M) \end{aligned}$$

and

$$\begin{aligned} \varphi(F)(y) &= \varphi(F)(\varphi(b)) \\ &= F(b) \\ &= F(0_S) \\ &= \varphi(F)(0_M) \end{aligned}$$

Since $(\varphi(F), M)$ is a soft intersection k-ideal, then $\varphi(F)(x+y) = \varphi(F)(0_M)$ and $\varphi(F)(y) = \varphi(F)(0_M)$ while $\varphi(F)(x) = \varphi(F)(0_M)$.

$$\varphi(F)(x) = \varphi(F)(0_M)$$

$$\varphi(F)(\varphi(a)) = \varphi(F)(0_M)$$

$$F(a) = F(0_S)$$

Hence, (F, S) is a soft intersection k-ideal over U. □

4. Quotient Structure of k-Semiring Over Soft Intersection Ideals

In this section, we define quotient structure of k-semiring and give some basic theory of this. Let S be a semiring and $\overline{S}$ be an extended ring of S. We take a soft intersection ideal (F, S) over U which all level subset is a k-ideal of S. Then, $S = \bigcup_{K \in ImF} F_K$, $\overline{S} = \bigcup_{K \in ImF} \overline{F_K}$ iff $T \supset K$ iff $F_T \subset F_K$ iff $\overline{F_T} \subset \overline{F_K}$.

Definition 4.1. Let S be a semiring and $\overline{S}$ be an extended ring of S. (F, S) be a soft intersection ideal over U where all level sets of (F, S) are k-ideal of S. $(\overline{F}, \overline{S})$ soft set is defined by $\overline{F}(a) = \bigcup\{K : a \in \overline{F_K}, K \in ImF\}$ for all $a \in \overline{S}$. The soft set $(\overline{F}, \overline{S})$ is called extended soft set over U.

Theorem 4.2. Let $(\overline{F}, \overline{S})$ be an extended soft set over U. Then, $(\overline{F}, \overline{S})$ is a soft intersection ideal over U.

Proof. The proof is clear. □

Theorem 4.3. Let $(\overline{F}, \overline{S})$ be an extended soft set over U. Then, $(\overline{F}, \overline{S})$ is an extended of (F, S).

Proof. Suppose that $a \in S$ and $F(a) = K$. Thus, $a \in F_T$ for all $T \subseteq K$. $F(a) \supseteq T \supset K$ for some $T \subseteq K$. This is contradiction with $F(a) = K$ and so $a \notin F_T$ for all $T \supset K$. Since $a \in F_K \subseteq \overline{F_K}$ and $a \notin F_T$ for all $T \supset K$, then $\overline{F}(a) = K = F(a)$. Suppose that $a = b^{'}$ for $a \in S^{'}$ and some $b \in S$. $\overline{F}(a) = \bigcup\{K : a \in \overline{F_K}, K \in ImF\} = V$. Thus, $a = b^{'} \in \overline{F_K}$ for all $K \subseteq V$ and $a = b^{'} \notin \overline{F_K}$ for all $K \supset V$. Therefore, $b \in \overline{F_K}$ for all $K \subseteq V$ and $b \notin \overline{F_K}$ for $K \supset V$. Hence, $\overline{F}(b) = V$ and so $\overline{F}(a) = \overline{F}(b^{'}) = \overline{F}(b) = F(b)$. Thus, $(\overline{F}, \overline{S})$ is an extended of (F, S). □

Theorem 4.4. Let $(\overline{F}, \overline{S})$ be an extended soft set over U. (F, S) is a soft intersection k-ideal over U iff $(\overline{F}, \overline{S})$ is a soft intersection k-ideal over U.

PROOF. Suppose that (F,S) is a soft intersection k-ideal over U. $(\overline{F},\overline{S})$ is a soft intersection ideal over U by Theorem 4.2. Otherwise, suppose that $(\overline{F},\overline{S})$ is a soft intersection ideal over U. Then, let $F(a+b)=F(0_S)$ and $F(b)=F(0_S)$ for all $a,b\in S$. Since (F,S) is a soft intersection ideal, then $(\overline{F},\overline{S})$ is an extended of (F,S). Therefore,

$$\begin{aligned} F(a)=\overline{F}(a) &= \overline{F}(a\oplus 0_S)\\ &= \overline{F}(a\oplus b\oplus b^{'})\\ &\supseteq \overline{F}(a\oplus b)\cap\overline{F}(b^{'})\\ &= F(a+b)\cap F(b)\\ &= F(0_S)\cap F(0_S)\\ &= F(0_S)\end{aligned}$$

Thus, it follows that $F(a)\supseteq F(0_S)$. Since $F(0_S)\supseteq F(a)$ for all $a\in S$, then $F(0_S)=F(a)$. So, (F,S) is a soft intersection k-ideal over U. □

Definition 4.5. Let $(\overline{F},\overline{S})$ be an extended soft set over U. Define $x+(F,S):S\to P(U)$ by $(x+F)(a)=\overline{F}(a\oplus x^{'})$ for all $a,x,x^{'}\in S$. A soft set $x+(F,S)$ is called a coset of soft intersection ideal (F,S).

Theorem 4.6. Let $(\overline{F},\overline{S})$ be an extended of (F,S). $x+(F,S)=y+(F,S)$ for all $x,y\in S$ iff $\overline{F}(x\oplus y^{'})=F(0_S)$.

PROOF. Suppose that $x+(F,S)=y+(F,S)$ for all for $x,y\in S$,

$$\begin{aligned} \overline{F}(x\oplus y^{'}) &= (y+F)(x)\\ &= (x+F)(x)\\ &= \overline{F}(x\oplus x^{'})\\ &= \overline{F}(0_S)\end{aligned}$$

Thus, it follows that $\overline{F}(x\oplus y^{'})=F(0_S)$. Conversely, suppose that $\overline{F}(x\oplus y^{'})=F(0_S)$ for $x,y\in R$.

$$\begin{aligned} (x+F)(a) &= \overline{F}(a\oplus x^{'})\\ &\supseteq \overline{F}(a\oplus x^{'}\oplus y\oplus y^{'}\\ &= \overline{F}(a\oplus y^{'})\cap\overline{F}(y\oplus y^{'})\\ &= \overline{F}(a\oplus y^{'})\cap\overline{F}(0_S)\\ &= \overline{F}(a\oplus Y^{'})\\ &= (y+F)(a)\end{aligned}$$

for all $a\in S$. Therefore, $x+(F,S)\supseteq y+(F,S)$. Similarly, it follows that $y+(F,S)\supseteq x+(F,S)$. Thus, $x+(F,S)=y+(F,S)$. □

Theorem 4.7. Let $(\overline{F},\overline{S})$ be an extended soft set over U. If $x+(F,S)=a+(F,S)$ and $y+(F,S)=b+(F,S)$ for $x,y,a,b\in S$, then

i. $x+y+(F,S)=a+b+(F,S)$

ii. $xy+(F,S)=ab+(F,S)$

PROOF. Suppose that $x+(F,S)=a+(F,S)$ and $y+(F,S)=b+(F,S)$ for $x,y,a,b\in S$.

i. Since $x+(F,S)=a+(F,S)$ and $y+(F,S)=b+(F,S)$ then $\overline{F}(x\oplus a^{'})=F(0_S)$ and $\overline{F}(y\oplus b^{'})=F(0_S)$ for $x,y,a,b\in S$. Then,

$$\begin{aligned} \overline{F}(x\oplus y\oplus a^{'}\oplus b^{'})=F(0_S) &\supseteq \overline{F}(x\oplus a^{'})\cap\overline{F}(y\oplus b^{'})\\ &= F(0_S)\cap F(0_S)\\ &= F(0_S)\\ &= \overline{F(0_S)}\end{aligned}$$

Therefore, $\overline{F}(x \oplus a^{'}) \cap \overline{F}(y \oplus b^{'}) = \overline{F(0_S)}$ and so $x + y + (F,S) = a + b + (F,S)$

ii.

$$\begin{aligned} \overline{F}(ab \oplus (xy)^{'}) &= \overline{F}(ab \oplus (ay)^{'} \oplus ay \oplus (xy)^{'}) \\ &= \overline{F}(a \odot (b \oplus y^{'}) \oplus (a \oplus x^{'}) \odot y) \\ &\supseteq \overline{F}(a \odot (b \oplus y^{'}) \cap F((a \oplus x^{'}) \odot y)) \\ &\supseteq \overline{F}(b \oplus y^{'}) \cap F((a \oplus x^{'})) \\ &= F(0_S) \cap F(0_S) \\ &= F(0_S) \end{aligned}$$

Thus, $\overline{F}(y \oplus b^{'}) = F(0_S)$. And so, $ab + (F,S) = xy + (F,S)$.

□

Define " + " and " · " binary operations on $S/(F,S)$ set of coset of soft intersection ideal (F,S), respectively by

$$[x + (F,S)] + [y + (F,S)] = x + y + (F,S)$$

and

$$[x + (F,S)][y + (F,S)] = xy + (F,S)$$

$S/(F,S)$ is a k-semiring under this operation and identity element of $S/(F,S)$ is $1_S + (F,S)$.

Definition 4.8. $S/(F,S)$ set of coset of soft intersection ideal (F,S) is called a quotient ring.

5. Isomorphism Theorem Over Soft Intersection Ideals

In this section, we investigate isomorphism theorem over soft intersection ideals.

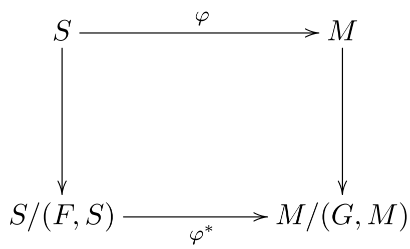
Theorem 5.1. Let $\varphi : S \to M$ be a k-semiring epimorphism. Let$(\overline{F}, \overline{S})$ be extended soft set over U and $F_S \subseteq Ker\varphi$. There is $f : S/(F,S) \to M$ an only epimorphism such that $\varphi = f \circ g$ where $g(x) = x + (F,S)$ for all $x \in S$.

Proof. Define $f : S/(F,S) \to M$ function by $f(x + (F,S)) = \varphi(x)$ for all $x \in S$. Now, we provide that f is well defined. Suppose that $x + (F,S) = y + (F,S)$ for all $x + (F,S), y + (F,S) \in S/(F,S)$. It follows that $\overline{F}(x \oplus y^{'}) = F(0_S) = \overline{F}(0_S)$ by Theorem 4.6. Thus, $x + y^{'} \in \overline{F_{\overline{S}}}$. Since $\overline{F_{\overline{S}}} = \overline{F_S} \subseteq \overline{Ker\varphi} = Ker\overline{\varphi}$ then $\overline{\varphi}(x + y^{'}) = 0_{\overline{S}}$. Thus, $\overline{\varphi}(x) = \overline{\varphi}(y)$. Therefore, $\varphi(x) = \varphi(y)$ and so $\varphi(x + (F,S)) = \varphi(y + (F,S))$. Since φ is onto function then f is onto function, Moreover, it can easily be shown that f is a homomorphism. In addition, $\varphi(x) = f(x + (F,S)) = f(g(x)) = (f \circ g)(x)$ for all $x \in S$. Finally, we will provide that f is a unique. Suppose that $\varphi = h \circ g$ such that $h : S/(F,S) \to M$. Then, $f(x + (F,S)) = \varphi(x) = (h \circ g)(x) = h(x + (F,S))$ for all $x \in S$. It follows that $f = h$. □

Proposition 5.2. Let $\varphi : S \to M$ be a k-semiring epimorphism. Suppose that (F,S) and (G,M) be define two soft intersection ideals over U as level sets of (F,S) and (G,M) be two k-ideals of S and M, respectively. If $(\varphi(F), M)\widetilde{\subset}(G,M)$, then $(\overline{\varphi}(\overline{F})\overline{M})\widetilde{\subseteq}(\overline{G}, \overline{M})$.

Proof. This proof is clear. □

Theorem 5.3. Let $\varphi : S \to M$ be a k-semiring homomorphism. Suppose that (F,S) and (G,M) be define two soft intersection ideals over U as level sets of (F,S) and (G,M) be two k-ideals of S and M, respectively. If $F(0_S) = G(0_M)$ then there exist $\varphi^* : S/(F,S) \to M/(G,M)$ homomorphism such that



Proof. Suppose that $\varphi^* : S/(F,S) \to M/(G,M)$ function is defined by $\varphi^*(x+(F,S)) = \varphi(x) + (G,M)$. If $x+(F,S) = y+(F,S)$, then by Theorem 4.6 $\overline{F}(x \oplus y') = F(0_S)$. Thus,

$$\begin{aligned} \overline{G}(\overline{\varphi}(x) \oplus \bar{(}\varphi(y))') &= \overline{G}(\overline{\varphi}(x \oplus y')) \\ &= \overline{\varphi}^{-1}(\overline{G})(x \oplus y') \\ &= \overline{F}(x \oplus y') \\ &= F(0_S) \\ &= G(0_M) \end{aligned}$$

Therefore,

$$\overline{G}(\overline{\varphi}(x) \oplus \bar{(}\varphi(y))') = \overline{G}(\varphi(x) \oplus \varphi(y)') = G(0_M)$$

Hence, $\varphi(x) + (G,M) = \varphi(y) + (G,M)$. So, φ^* is well defined. Since

$$\begin{aligned} \varphi^*((x+(F,S)+y+(F,S)) &= \varphi^*((x+y+(F,S)) \\ &= \varphi(x+y)+(G,M) \\ &= \varphi(x)+\varphi(x)+(G,M) \\ &= \varphi(x)+(G,M)+\varphi(y)+(G,M) \\ &= \varphi^*(x+(F,S))+\varphi^*(y+(F,S)) \end{aligned}$$

and

$$\begin{aligned} \varphi^*([(x+(F,S)][y+(F,S)]) &= \varphi^*((xy+(F,S)) \\ &= \varphi(xy)+(G,M) \\ &= \varphi(x)\varphi(x)+(G,M) \\ &= \varphi(x)+(G,M)+\varphi(y)+(G,M) \\ &= \varphi^*(x+(F,S))\varphi^*(y+(F,S)) \end{aligned}$$

for all $x+(F,S), y+(F,S) \in S/(F,S)$, then φ^* is a homomorphism. □

Corollary 5.4. Let $\varphi : S \to M$ be a k-semiring epimorphism. φ^* is an isomorphism iff (F,S) is a $\mu \circ \varphi$-invariant where $\mu : M \to M/(G,M)$, $\mu(z) = z + (G,M)$.

Proposition 5.5. Let $\varphi : S \to M$ be a k-semiring epimorphism and (G,M) be a soft intersection ideal over U. Suppose that $(\varphi^{-1}(G),S) = (F,S)$. All level set of (G,M) is a k-ideal iff all level set of $(\varphi^{-1}(G),S)$ is a k-ideal.

Proof. Suppose that all level set of (G,M) soft intersection ideal is a k-ideal. Then, $a+b \in F_K$ and $b \in F_K$ for all $a,b \in S$ such that $K \in P(U)$. Thus, $F(a+b) \supseteq K$ and $F(b) \supseteq K$. Since $(\varphi^{-1}(G),S) = (F,S)$ then $G(\varphi(a+b)) \supseteq K$ and $G(\varphi(b)) \supseteq K$. Since all level set of (G,M) is a k-ideal, then $G(\varphi(a)) \supseteq K$. Hence, $F(a) \supseteq K$ and $a \in F_K$. Therefore, F_K is a k-ideal of S. Conversely, suppose that all level set of soft intersection ideal $(\varphi^{-1}(G),S)$ is a k-ideal. $G(a+b) \supseteq K$ and $G(b) \supseteq K$ for all $K \in P(U)$. Since φ is onto homomorphism, then there are $a,b \in S$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Thus,

$$\begin{aligned} G(a+b) &= G(\varphi(x)+\varphi(y)) \\ &= G(\varphi(x+y)) \\ &= \varphi^{-1}(G)(x+y) \\ &\supseteq K \end{aligned}$$

and

$$\begin{aligned} G(b) &= G(\varphi(y)) \\ &= \varphi^{-1}(G)(y) \\ &\supseteq K \end{aligned}$$

Therefore, it follows that $\varphi^{-1}(G)(x+y) \supseteq K$ and $\varphi^{-1}(G)(y) \supseteq K$. Hence $\varphi^{-1}(G)(x) \supseteq K$. Moreover,

$$\begin{aligned} G(a) &= G(\varphi(x) \\ &= \varphi^{-1}(G)(x) \\ &\supseteq K \end{aligned}$$

Consequently G_K is a k-ideal of M. □

Proposition 5.6. Let $\varphi : S \to M$ be a k-semiring epimorphism and (G, M) be a soft intersection ideal over U. Suppose that $(\varphi^{-1}(G), S) = (F, S)$. Hence, $(\overline{F}, \overline{S}) = (\overline{\varphi}^{-1}(\overline{G}), \overline{S})$.

Proof. It is clear. □

Theorem 5.7. Let $\varphi : S \to M$ be a k-semiring epimorphism and (G, M) be a soft intersection ideal over U. Suppose that $(\varphi^{-1}(G), S) = (F, S)$. Hence, $S/(F, S) \cong M/(G, M)$.

Proof. We know that $(\varphi(F), M) = (\varphi(\varphi^{-1}(G)), M) = (G, M)$ and $F(0_S) = G(0_M)$. Moreover, let $(\mu \circ \varphi)(a) = (\mu \circ \varphi)(b)$ for all $a, b \in S$ such that $\mu(z) = z + (G, M)$ for all $z \in M$. By Proposition 5.6,

$$\begin{aligned} \varphi(a) + (G, M) = \varphi(b) + (G, M) &\Rightarrow \overline{G}(\varphi(a) \oplus \varphi(b)') = G(0_M) = F(0_S) \\ &\Rightarrow \overline{G}(\overline{\varphi}(a) \oplus \overline{\varphi}(b)') = G(0_M = F(0_S) \\ &\Rightarrow \overline{G}(\overline{\varphi}(a + b')) = G(0_M) = F(0_S) \\ &\Rightarrow (\overline{\varphi})^{-1}(\overline{G})(a + b') = G(0_M) = F(0_S) \\ &\Rightarrow \overline{F}(a + b') = F(0_S) \\ &\Rightarrow a + (F, S) = b + (F, S) \end{aligned}$$

Therefore, (F, S) is a $\mu \circ \varphi$-invariant. By Corollary 5.4, it follows that $S/(F, S) \cong M/(G, M)$ □

6. Soft Intersection Maximal k-Ideals

In this section, firstly we define soft intersection maximal k-ideal over a k-semiring. And, we investigate some properties.

Definition 6.1. Let S be a k-semiring and (F, S) be a soft intersection k-ideal over U where all subsets of S are k-ideal. A soft intersection ideal (F, S) is called soft intersection maximal k-ideal if

i. $F(0_S) = U$

ii. $F(1_S) \subset F(0_S)$

iii. $\overline{F}(1_S \oplus (sa)') = F(0_S)$ for some $s \in S$ while $F(a) \subset F(0_S)$ for some $a \in S$.

Theorem 6.2. Let (F, S) be a soft intersection k-ideal over U where all subssets of semiring S be k-ideals. (F, S) is a soft intersection maximal k-ideal over U iff $(\overline{F}, \overline{S})$ is a soft intersection maximal k-ideal over U.

Proof. Suppose that (F, S) is a soft intersection k-ideal over U. Hence

i. $\overline{F}(0_S) = F(0_S) = U$

ii. $\overline{F}(1_S) = F(1_S) \subset F(0_S) = \overline{F}(0_S)$

Let $\overline{F}(a) \subset \overline{F}(0_S)$. Since $a \in \overline{S}$, then $a \in S$ or $a \in S'$. Since $\overline{F}(a) = F(a)$, and $\overline{F}(a) \subset \overline{F}(0_S) = F(0_S)$ and then $F(a) \subset F(0_S)$. Hence, we obtain that $\overline{F}(1_S \oplus (sa)') = F(0_S) = \overline{F}(0_S)$ for some $s \in S$.If $a \in S'$ then $a = x'$ such that there exist $x \in S$ and so $\overline{F}(a) = \overline{F}(x') = \overline{F}(x) = F(x) \subset F(0_S)$. Therefore $\overline{F}(1_S \oplus (sa)') = F(0_S) = \overline{F}(0_S)$ for some $s \in S$. Hence, $(\overline{F}, \overline{S})$ is a soft intersection maximal k-ideal over U.
Conversely, suppose that $(\overline{F}, \overline{S})$ is a soft intersection maximal k-ideal over U. Hence,

i. $\overline{F}(0_S) = F(0_S) = U$

ii. $\overline{F}(1_S) = F(1_S) \subset F(0_S) = \overline{F}(0_S)$

Let $F(a) \subset F(0_S)$. Hence, $F(a) = \overline{F}(a) \subset F(0_S) = \overline{F}(0_S)$. Then, we obtain that $\overline{F}(a) \subset \overline{F}(0_S)$. Therefore, $\overline{F}(1_S \oplus (sa)^{'}) = F(0_S)$. Consequently, (F, S) is a soft intersection k-ideal over U. □

Theorem 6.3. Let $\varphi : S \to M$ be a k-semiring endomorphism and (G, M) be a soft intersection ideal over U. (G, M) is a soft intersection k-ideal over U iff $(\varphi^{-1}(G), S)$ is a soft intersection maximal k-ideal over U.

Proof. Suppose that (G, M) is a soft intersection k-ideal over U. By Theorem 3.11, we know that $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U. Now, we indicate that $(\varphi^{-1}(G), S)$ is a soft intersection maximal k-ideal. Then,

i. $\varphi^{-1}(G)(0_S) = G(\varphi(0_S)) = G(0_M) = U$

ii. $\varphi^{-1}(G)(1_S) = G(\varphi(1_S)) = G(1_M) \subset G(0_M) = G(\varphi(0_S)) = \varphi^{-1}(G)(0_S)$
Hence, $\varphi^{-1}(G)(1_S) \subset \varphi^{-1}(G)(0_S)$

iii. Let $\varphi^{-1}(G)(a) \subset \varphi^{-1}(G)(0_S)$ for some $a \in S$. Hence, $G(\varphi(a)) \subset G(\varphi(0_S)) = G(0_M)$. Therefore, $G(\varphi(a)) \subset G(0_M)$.

Since (G, M) is a soft intersection k-ideal, then

$$\begin{gathered}\overline{G}(1_M \oplus (\varphi(s)\varphi(a))^{'}) = G(0_M)\\ \overline{G}(\varphi(1_S) \oplus \overline{\varphi}(sa)^{'}) = G(0_M)\\ \overline{G}(\overline{\varphi}(1_S) \oplus \overline{\varphi}(sa)^{'}) = G(\varphi(0_S))\\ (\overline{\varphi})^{-1}(\overline{G})(1_S \oplus (sa)^{'}) = \varphi^{-1}(G)(0_S)\end{gathered}$$

Therefore, $(\varphi^{-1}(G), S)$ is a soft intersection maximal k-ideal over U. Conversely, $(\varphi^{-1}(G), S)$ be a soft intersection maximal k-ideal over U. By Theorem 3.11, (G, M) is a soft intersection k-ideal over U.

i. $G(0_M) = G(\varphi(0_S)) = \varphi^{-1}(G)(0_S) = U$

ii. $G(1_M) = G(\varphi(1_S)) = \varphi^{-1}(G)(1_S) \subset \varphi^{-1}(G)(0_S) = G(\varphi(0_S)) = G(0_M)$ Hence, it follows that $G(1_{S'}) \subset G(0_M)$.

iii. Let $G(x) \subset G(0_M)$ for $x \in M$. There exists $a \in S$ such that $\varphi(a) = x$ Therefore, since $G(x) = G(\varphi(a)) = \varphi^{-1}(G)(a) \subset G(0_S)$ then $\varphi^{-1}(G)(1_S \oplus (as)^{'}) = \varphi^{-1}(G)(0_s) = G(\varphi(0_S)) = G(0_M)$

Hence, by Theorem 4.6, it follows that

$$\begin{aligned} G(0_M) &= (\overline{\varphi})^{-1}(\overline{G}(1_S \oplus (as)^{'})\\ &= \overline{G}(\overline{\varphi}(1_S \oplus (as)^{'})\\ &= \overline{G}(\overline{\varphi}(1_S) \oplus \overline{\varphi}(as)^{'})\\ &= \overline{G}(\varphi(1_S) \oplus (\varphi(a)\varphi(s))^{'})\\ &= \overline{G}(1_M \oplus (xk)^{'}))\end{aligned}$$

where $\varphi(s) = k$. Consequently, (G, M) is a soft intersection maximal k-ideal over U. □

7. Conclusion

In this study, we defined soft intersection k-ideals on a semiring and then investigated some algebraic properties of soft intersection k-ideals. Moreover, isomorphism theorems are presented by describing quotient rings with the help of k-semiring, defined soft intersection maximal k-ideal, soft intersection maximal k-ideals are defined and their algebraic properties are investigated. Some other algebraic structures, such as prime rings [46, 47] and semi prime rings [48, 49], are worth studying in future studies.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript. This paper is derived from the first author's master's thesis, supervised by the second author.

Conflicts of Interest

The authors declare no conflict of interest.

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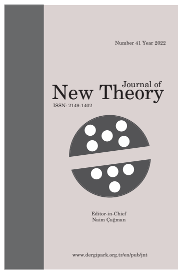

Vertices of Suborbital Graph $F_{u,N}$ under Lorentz Matrix Multiplication

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Abstract — In this study, suborbital graphs, $G_{u,N}$ and $F_{u,N}$ are examined. Modular group Γ and its act on $\widehat{\mathbb{Q}}$ are studied. Lorentz matrix that gives the vertices obtained under the classical matrix multiplication in the suborbital graph $F_{u,N}$ is analysed with the Lorentz matrix multiplication. Lorentz matrix written as Möbius transform is normalized and the type of the transform is researched. Moreover, a different element of Modular group Γ is scrutinized. The vertices on the path starting with ∞ are obtained under this element and the Lorentz matrix multiplication. For this path, it is shown that the vertices obtained in $F_{u,N}$ under the Lorentz matrix multiplication with the Lorentz matrix satisfied the farthest vertex condition for the previous vertex.

Keywords — *Lorentz matrix multiplication, modular group, Mobius transform*

Mathematics Subject Classification (2020) — 22E43, 11F06

1. Introduction

Graph theory and its elements are at the core of our work. With the discovery of non-Euclidean geometry in the 18^{th} and 19^{th} centuries, Graph theory began to be studied in this field as well. Elliptic and hyperbolic geometries are both of non-Euclidean geometries.Moreover, with the discovery of invariant theory and non-Euclidean geometries, linear fractional transformation groups achieve special importance. Since linear fractional transformation groups are suitable for the topological group structure, they have been extensively studied in recent years with different methods. In [1], some ideas were put forward about graph action firstly. These ideas found an important place in the work of [2] and [3] in applications for finite groups. Modular group Γ and its subgroups,which play an important role in the last theorem by proved Fermat, have been researched extensively in recent years. In [4], suborbital graphs, $G_{u,N}$ and $F_{u,N}$ obtained by element of Modular group Γ are examined and they presented some conclusions. In [5], it is provided that suborbital graph is a forest if and only if it does not have triangles. Elliptic elements and elliptic circuits are investigated in [6]. In [7], it is shown that each vertex in the suborbital graph $F_{u,N}$ has a continued fraction structure for $(u, N) = 1$ and $u \leq N$ and investigated the vertices on path with minimal lengths. Suborbital graphs are studied for invariant groups in [8]. The vertices obtained with help of continued fractions and recurrence relations are generalized and associated with Fibonacci numbers in [9]. In [10], Gündoğan and Keçilioğlu defined Lorentz matrix multiplication.

This study investigates corollaries of suborbital graph act by different matrix multiplication.Then,we use Lorentz matrix multiplication and investigate Lorentz matrix (Equation 6) that gives the vertices

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obtained under classical matrix multiplication with Lorentz matrix multiplication. Here, it is demonstrated that Lorentz matrix (Equation 6) is not an element of Modular group Γ.

However, we examined an element (Equation 7) of Modular group Γ. We obtained a path starting with ∞ using Equation 7 in this article.Then, we demonstrated that the vertices on this path satisfy the minimal length condition. In addition,we assumed k as 1 in Equation 7. Therefore, we associated the vertices of path with Fibonacci numbers and n^{th} vertex with golden section.

2. Preliminary

2.1. Suborbital Graphs

Assume that (G, Ω) is a transitive permutation group, $g \in G$ and $\alpha, \beta \in \Omega$. Then G provides

$$g : (\alpha, \beta) \to (g(\alpha), g(\beta))$$

on $\Omega \times \Omega$.The orbitals of this transformation are called suborbitals of G. $O(\alpha, \beta)$ represents the suborbital covering (α, β).

$$O(\alpha, \beta) = \{g(\alpha, \beta) | g \in G\}$$

$$(x,y) \in O(\alpha, \beta) \Leftrightarrow g \in G : (x, y) = g(\alpha, \beta) = (g(\alpha), g(\beta))$$

The suborbital graph $G(\alpha, \beta)$ can be obtained from the suborbital $O(\alpha, \beta)$. Assume that γ and δ vertices in $\widehat{\mathbb{Q}}$, if $(\gamma, \delta) \in O(\alpha, \beta)$ exists, the orbit represents a directional edge from γ to δ and is denoted by $\gamma \longrightarrow \delta$. This edge can be drawn at $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ as a hyperbolic geodesic.

$O(\beta, \alpha)$ is also an orbit and can be equal to or different from the $O(\alpha, \beta)$. If the orbits are different from each other, the suborbital graph $G(\beta, \alpha)$ is the opposite direction of the edges of the suborbital graph $G(\alpha, \beta)$ and in this case the suborbital graphs are called paired suborbital graphs.If the orbits are equal, $G(\beta, \alpha) = G(\alpha, \beta)$ and the graph includes the opposite pair of edges; in this case, by replacing each pair with a simple directed edge, an undirected edge paired with it is obtained.In other words, if $O(\beta, \alpha) = O(\alpha, \beta)$ and $(\gamma, \delta) \in O(\alpha, \beta)$, the edge between γ and δ vertices is denoted by $\gamma - \delta$ instead of $\gamma \leftrightarrow \delta$.

Assume that equivalence relation is "$\approx$ " and for all $\alpha, \beta \in \Omega$, for all $g \in G$, if is provided $g(\alpha) \approx g(\beta)$ when $\alpha \approx \beta$, "$\approx$ " is called "G-invariant equivalence relation "on Ω, and equivalent classes formed in this way are called "blocks". Examples of these relations are identity and universal relation:

i. identity relation, "$\alpha = \beta \Leftrightarrow \alpha \approx \beta$" for all $\alpha, \beta \in \Omega$

ii. universal relation, "$\alpha \approx \beta$" for all $\alpha, \beta \in \Omega$.

Unlike these relations, if there is a G -invariant equivalence relation on Ω, (G, Ω) is called imprimitive, otherwise primitive.The transitive act of the primitive group (G, Ω) is necessary, otherwise the orbits do not constitute a system block and its reverse is not true.

Theorem 2.1. [4] Assume that (G, Ω) is an transitive permutation group. In this case (G, Ω) is primitive $\Leftrightarrow G_\alpha$ the stabilizer of a point $\alpha \in \Omega$ is a maximal subgroup of G for all $\alpha \in \Omega$.

Theorem 2.2. [4] **G** is a suborbital graph for transitive permutation group (G, Ω). In this case,

i. G acts as a group of automorphism of **G**.

ii. G acts as transitive on vertices of **G**.

iii. If **G** is self-paired, then G acts transitively on ordered pairs of consecutive vertices of **G**.

iv. If **G** is not self-paired, then G acts transitively on the edges of **G**.

2.2. Modular Group and its Act on $\widehat{\mathbb{Q}}$ Under Classical Matrix Multiplication

The Modular group is the quotient group of the $SL(2,\mathbb{Z})$ with $\{\mp I\}$. Specially if the Modular group is denoted by Γ, it is written as

$$\Gamma = PSL(2,\mathbb{Z}) \cong SL(2,\mathbb{Z})/\{\mp I\}$$

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

Γ is consist of $\mp \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrices pairs. Here, the $+$ and $-$ symbols are ignored and matrices considered equivalent.

Some equivalent subgroups of Γ are given below:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \, (\text{mod } N) \right\}$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid b \equiv 0 \, (\text{mod } N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \, (\text{mod } N), c \equiv 0 \, (\text{mod } N) \right\}$$

The hyperbolic plane is defined by $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. Möbius transformations are known transformations with the elements of the Modular group in the upper half plane $\mathbb{H}$. Transformation is defined by for all $z \in \mathbb{C}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az + b}{cz + d}$$

f and g are elements of Modular group Γ for $f(z)$ and $g(z)$ linear Möbius transformations.

$$f(z) = \frac{-1}{z} = \frac{0z - 1}{1z + 0} \Rightarrow f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$$

$$g(z) = z + \lambda = \frac{1z + \lambda}{0z + 1} \Rightarrow g = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma$$

In addition, Möbius transformations are used to describe the elements of $\widehat{\mathbb{Q}}$. Especially for $\frac{a}{c} \in \widehat{\mathbb{Q}}$, if $c = 0$, it is accepted as $\frac{a}{c} = \infty$. For $x, y \in \mathbb{Z}$ and $(x, y) = 1$, each element of $\widehat{\mathbb{Q}}$ can be expressed as reduced fraction $\frac{x}{y}$.Since $\frac{x}{y} = \frac{-x}{-y}$, the notation is not uniform. ∞ would be considered as $\frac{1}{0} = \frac{-1}{0}$.For $z = \frac{x}{y}$, Möbius transformation is written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax + by}{cx + dy} \tag{1}$$

The reduced result in Equation 1 is shown as follows:

$$c(ax + by) - a(cx + dy) = cax + cby - acx - ady = (cb - ad)y = -y$$

$$d(ax + by) - b(cx + dy) = dax + dby - bcx - bdy = (ad - bc)x = x$$

$$(ax + by, cx + dy) = 1$$

Lemma 2.3. [4]

i. Act of Γ is transitive on $\widehat{\mathbb{Q}}$.

ii. Element of Γ that fixed a vertex on $\widehat{\mathbb{Q}}$ is infinitely period.□

Assume that examine what has been given so far about suborbital graphs if G is the Modular group Γ and Ω is $\widehat{\mathbb{Q}}$. Γ_∞ which fixed ∞ is a subgroup of Γ produced by $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then we can generate Γ-invariant equivalence relations on $\widehat{\mathbb{Q}}$ by obtaining the subgroups H of Γ containing Γ_∞ or equivalently Z. Since $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the congruence groups $\Gamma_0(N)$ can be selected as H, with $N \in \mathbb{N}$. Clearly, $\Gamma_\infty < \Gamma_0(N) \leqslant \Gamma$ for all $N \in \mathbb{N}$ and $\Gamma_\infty < \Gamma_0(N) < \Gamma$ for $N > 1$. Hence, act of Γ on $\widehat{\mathbb{Q}}$ is imprimitive.

Assume that denote the reduced Γ- invariant equivalence relation on $\widehat{\mathbb{Q}}$ of $\Gamma_0(N)$ with "$\approx_N$". Transformations $v = g(\infty)$ and $w = g'(\infty)$ are provided for $v = \frac{r}{s}$,$w = \frac{x}{y} \in \widehat{\mathbb{Q}}$ and $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$, $g' = \begin{pmatrix} x & * \\ y & * \end{pmatrix} \in \Gamma$.

Since

$$v \approx_N w \Leftrightarrow g(v) \approx_N g'(w) \Leftrightarrow g^{-1}g' \in H = \Gamma_0(N)$$

and

$$g^{-1} = \begin{pmatrix} * & * \\ -s & r \end{pmatrix}$$

$$g^{-1}g' = \begin{pmatrix} * & * \\ -s & r \end{pmatrix} \begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} * & * \\ ry - sx & * \end{pmatrix} \in H = \Gamma_0(N)$$

$$v \approx w \Leftrightarrow ry - sx \equiv 0 \,(\text{mod } N)$$

results are obtained. In other words, $v = \frac{r}{s}$ and $w = \frac{x}{y}$ are equivalent $\Leftrightarrow \exists u \in H : x \equiv ur \,(\text{mod } N)$, $y \equiv us \,(\text{mod } N)$

Similarly, $\Gamma^0(N)$ which fixed 0 is a subgroup of Γ produced by $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus, we can generate Γ-invariant equivalence relations on $\widehat{\mathbb{Q}}$ by finding the subgroups K of Γ containing $\Gamma^0(N)$ or equivalently B.

The number of equivalence classes under "$\approx_N$" is given by

$$\Psi(N) =\mid \Gamma : \Gamma_0(N) \mid$$

equation.

2.3. Investigation of $G_{u,N}$ and $F_{u,N}$

Since the act of Γ on $\widehat{\mathbb{Q}}$ is transitive, each suborbit contains the pair (∞, v) for $v \in \widehat{\mathbb{Q}}$. If $v = \frac{u}{N}$ for $N \geqslant 0$ and $(u, N) = 1$, suborbit is denoted by $O_{u,N}$, suborbital graph $G(\infty, v)$ corresponding to $O_{u,N}$ is denoted by $G_{u,N}$. If $v = \infty$, $G_{1,0} = G_{-1,0}$ is trivial suborbital graph, so we can assume $v \in \widehat{\mathbb{Q}}.v, v' \in \widehat{\mathbb{Q}}$ and $O(\infty, v) = O(\infty, v') \Leftrightarrow v$ and v' are in orbit of Γ_∞. Since Γ_∞ produced by $Z : v \longrightarrow v + 1$, equivalent to $v' = \frac{u'}{N}$ for $u \equiv u'$ (mod N).

Theorem 2.4. [4] $\frac{r}{s} \rightarrow \frac{x}{y} \in G_{u,N}$ if and only if

i. $x \equiv ur \,(\text{mod } N)$, $y \equiv us \,(\text{mod } N)$ and $ry - sx = N$

ii. $x \equiv -ur \,(\text{mod } N)$, $y \equiv -us \,(\text{mod } N)$ and $ry - sx = -N$

Corollary 2.5. [4] Suborbital graph which is paired with $G_{u,N}$ is $G_{-\overline{u},N}$ for $\overline{u}$ providing $u\overline{u} \equiv 1 \,(\text{mod } N)$.

Corollary 2.6. [4] $G_{u,N}$ is self-paired $\Leftrightarrow u^2 \equiv -1 \,(\text{mod } N)$.

$G_{u,N}$ is the discrete union of $\Psi(N)$ subgraphs and the vertices of each subgraph form a single block according to the $\approx_N$ Γ-invariant equivalence relation defined by $ry - sx \equiv 0 \pmod N$. Since Γ acts transitively on $\widehat{\mathbb{Q}}$, Γ permutates these blocks as transitive and all subgraphs are isomorphic. $F_{u,N}$ be the subgraph of $G_{u,N}$ consisting ∞ on vertices and

$$[\infty] = \left\{ \frac{x}{y} | y \equiv 0 \pmod N, x, y \in \mathbb{Q} \right\}$$

Thus, $G_{u,N}$ consists of $\Psi(N)$ discrete copies of $F_{u,N}$.

Theorem 2.7. [4] $\frac{r}{s} \rightarrow \frac{x}{y} \in F_{u,N}$ if and only if

i. $x \equiv ur \pmod N$ and $ry - sx = N$

ii. $x \equiv -ur \pmod N$ and $ry - sx = -N$

Theorem 2.8. [4] $\Gamma_0(N)$ permutates vertices and edges of $F_{u,N}$ transitively.

2.4. Continued Fractions

Continued fractions are basically divided into two groups as finite and infinite.

2.4.1. Finite Continued Fractions

A finite continued fraction is defined as follow

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{m-2} + \cfrac{1}{a_{m-1} + \cfrac{1}{a_m}}}}}}$$

for $a_1 \geqslant 0$,$i \geqslant 2$ and a_i positive integer. It can be written as notation $x = [a_1; a_2, a_3, \ldots, a_k]$.

2.4.2. Infinite Continued Fractions

An infinite continued fraction is defined as follow

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{m-2} + \cfrac{1}{\ddots}}}}}$$

for $a_1 \geqslant 0$,$i \geqslant 2$ and $a_i \geqslant 1$. It can be written as $x = [a_1; a_2, a_3, \ldots]$ [11].

More generally, a continued fraction is defined

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots}}} \tag{2}$$

where $\mathbb{N}$ is set of natural numbers, $\mathbb{Z}$ is set of integer numbers and $a_i \in \mathbb{Z} - \{0\}$, $b_i \in \mathbb{Z}$ for all $i \in \mathbb{N} \cup \{0\}$.

Continued fraction in Equation 2 can be written as

$$b_0 + \mathbf{K}_{i=1}^{\infty} \left(\frac{a_i}{b_i} \right) \tag{3}$$

However, n. approach for continued fraction in Equation 3 is denoted by f_n and it is written as

$$f_n = b_0 + \mathbf{K}_{i=1}^{n} \left(\frac{a_i}{b_i} \right)$$

In additon, $\{f_n\}$ sequence is obtained by $(\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}\cup\{0\}})$ for $i \geqslant 1$, $a_i \neq 0$ and linear fractional transformation sequences $\{t_n(s)\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{T_n(s)\}_{n\in\mathbb{N}\cup\{0\}}$ where

$$t_0(s) = s, t_n(s) = \frac{a_n}{b_n + s}, n = 1, 2, 3, \ldots$$

$$T_0(s) = t_0(s), T_n(s) = T_{n-1}(t_n(s)), n = 1, 2, 3, \ldots$$

$$f_n = T_n(0) \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}, n = 1, 2, 3, \ldots$$

From here,

$$((\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}\cup\{0\}}), \{f_n\})$$

can be written. This obtained sequence corresponds to the continued fraction given in Equation 2. a_i is called the partial numerator and b_i is called the partial denominator.

In accordance with the above, the linear fractional transformation $T_n(s)$ can be expressed by

$$T_n(s) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + s}}}}$$

From the representation of continued fractions,

$$T_n(s) = (t_0 \circ t_1 \circ t_2 \circ \ldots \circ t_n)$$

can be written where $\circ$ compound function.We get

$$(t_0 \circ t_1)(s) = t_0(t_1(s))$$

and

$$t^n(s) = (t \circ t \circ t \circ \ldots \circ t)(s)$$

The number of n^{th} modified approaches is denoted by

$$T_n(S_n) \in \mathbb{R}$$

for $\{S_n\}_{n\in\mathbb{N}\cup\{0\}}$ sequence.

2.5. Paths of Minimal Length on Suborbital Graphs

In this section, some definitions and theorems are given about paths of minimal length on suborbital graph.

Definition 2.9. [7] $v_0, v_1, v_2, \ldots, v_m$ is a sequence of different vertices of suborbital graph $F_{u,N}$. If $m \geqslant 2$, $v_0 \to v_1 \to v_2 \to \ldots \to v_m \to v_0$ is called directed circuit (or closed path).If at least one (but not all) edge in this path are is the opposite direction, this path is called an undirected circuit (or reverse directed circuit). If $m = 2$, the circuit is called a triangle, directed or not.If $m = 1$, the path $v_0 \to v_1 \to v_0$ is called a self-matched edge.

Definition 2.10. [7] Since the elements of the Modular group represent Hyperbolic lines to Hyperbolic lines, the elements of the graph $F_{u,N}$ for proper visualization are shown half lines perpendicular to the real axis in the upper half plane of $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ and half lines with the center on $\mathbb{R}$ as hyperbolic geodesics.

Definition 2.11. [7] The path $v_0 \to v_1 \to v_2 \to \ldots \to v_m$ and $v_0 \to v_1 \to v_2 \to \ldots$ is called a path and an infinite path in the graph $F_{u,N}$ respectively.

Definition 2.12. [7] $\frac{r}{s} \overrightarrow{<} \frac{x}{y} \in F_{u,N}$ ($\frac{r}{s} \overleftarrow{>} \frac{x}{y} \in F_{u,N}$), if there is no vertex greater (or smaller) than the $\frac{x}{y}$ vertex connected to the $\frac{r}{s}$ vertex in the graph $F_{u,N}$, the $\frac{x}{y}$ vertex is called the farthest (nearest) vertex.

Definition 2.13. [7] For the path $v_0 \to v_1 \to v_2 \to \ldots \to v_m$ in the graph $F_{u,N}$ to have minimal length,$v_i \not\leftrightarrow v_j$ where $i < j-1$, $i \in \{0, 1, \ldots, m-2\}$,$j \in \{2, 3, \ldots, m\}$ and vertex v_{i+1} should be the farthest vertex that connects to vertex v_i.

Definition 2.14. [7] If $F_{u,N}$ contains no circuits it is called a forest. A connected non-empty graph with no circuit is a tree.

Lemma 2.15. [12] If $(u, N) = 1$, there is an integer k that satisfies the equation $u^2 + ku + 1 \equiv 0 \pmod N$.

For $k \geqslant 2$ and $k \in \mathbb{Z}$, $\begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix} \in \Gamma_0(N)$ that the element of an equivalent subgroup of the Modular group connects the vertices in order on an infinite minimal length path in suborbital graph $F_{u,N}$ and each vertex forms a continued fractional structure.

$$\infty = \frac{1}{0} \to \frac{u}{N} \to \frac{u+\frac{1}{k}}{N} \to \frac{u+\frac{1}{k-\frac{1}{k}}}{N} \to \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k}}}}{N} \to \ldots$$

This path is right directed. Each vertex that can be connected to the previous vertex is the farthest vertex.

It can be defined as

$$v_q = \begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix}^q (v_0) \tag{4}$$

for all $q \in \mathbb{Z}^+$,where $v_0 = \frac{u}{N}$.

Theorem 2.16. [12] Assume that $u^2 + ku + 1 \equiv 0 \pmod N$ and $1 < k < N$ in Farey graph.

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u+\frac{1}{k}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u+\frac{1}{k}}{N}$ can be connected becomes $\frac{u+\frac{1}{k-\frac{1}{k}}}{N}$ and there is no similar nearest vertex.

Theorem 2.17. [7] Assume that $u^2+ku+1 \equiv 0 \pmod N$ and $u^2-lu+1 \equiv 0 \pmod N$ for $1 \leqslant k, l \leqslant N$. If the suborbital graph $F_{u,N}$ is paired with itself, it is $k = l = N$ and otherwise $l = N - k$.

Theorem 2.18. [7] Assume that $u^2 - lu + 1 \equiv 0 \pmod N$ and $1 < l \leqslant N$ in Farey graph.

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u-\frac{1}{l}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u-\frac{1}{l}}{N}$ can be connected becomes $\frac{u-\frac{1}{l-\frac{1}{l}}}{N}$ and there is no similar nearest vertex.

Corollary 2.19. [7] If $u^2 - u + 1 \equiv 0 \pmod N$ then $F_{u,N}$ has a triangle as $\frac{1}{0} \leftarrow \frac{u-1}{N} \leftarrow \frac{u}{N} \leftarrow \frac{1}{0}$.

2.6. Lorentz Matrix Multiplication

In this section, we will investigate Lorentz matrix multiplication and related concepts, which have an important place in our study.

2.6.1. Lorentz Transform

Definition 2.20. [13] Linear transform $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transform $\Leftrightarrow \theta(x) \circ \theta(y) = x \circ y$ for all $x, y \in \mathbb{R}^n$.

Base $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$ is orthonormal if and only if $x_1 \circ x_1 = 1$, for other cases $x_i \circ x_j = \delta_{ij}$.

Theorem 2.21. [13] Linear transform $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transform if and only if θ is linear and $\{\theta(e_1), \theta(e_2), \ldots, \theta(e_n)\}$ is an Lorentz orthonormal base of $\mathbb{R}^n$.

Assume that θ is linear and $\{\theta(e_1), \theta(e_2), \ldots, \theta(e_n)\}$ is an Lorentz orthonormal base of $\mathbb{R}^n$. Since θ Lorentz transform,

$$\begin{aligned} \theta(x) \circ \theta(y) &= \theta(\textstyle\sum_{i=1}^n x_i e_i) \circ \theta(\sum_{j=1}^n x_j e_j) \\ &= (\textstyle\sum_{i=1}^n x_i \theta(e_i)) \circ (\sum_{j=1}^n y_j \theta(e_j)) \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^n x_i y_j \theta(e_i) \circ \theta(e_j) \\ &= -x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = x \circ y \end{aligned}$$

Definition 2.22. [13] $x, y \in \mathbb{R}^n$ is Lorentz orthogonal $\Leftrightarrow x \circ y = 0$.

2.6.2. Some Properties of Lorentz Matrix Multiplication

Assume that R_n^m be the set of matrices of type $m \times n$ and R_p^n be the set of matrices of type $n \times p$. Between the rows of the matrix $A = (a_{ij}) \in R_n^m$ and the columns of the matrix $B = (b_{jk}) \in R_p^n$, $A._L B = (-a_{i1} b_{1k} + \sum_{j=2}^n a_{ij} b_{jk})$ is defined with "$._L$" and this product is called the "Lorentz matrix product". $A._L B$ is a matrix of type $m \times p$. Besides, if we assume A_i as i^{th} row of A and B^j as j^{th} column of B, $\langle A_i, B^j \rangle_L$ is dot product $(i,j)^{th}$ of $A._L B$. L_n^m is denoted R_n^m that Lorentz matrix multiplication applied. $A._L B$ can be given more generally as follows:

$$A._L B = \begin{pmatrix} \langle A_1, B^1 \rangle & \ldots & \langle A_1, B^j \rangle \\ \vdots & \ddots & \vdots \\ \langle A_j, B^1 \rangle & \ldots & \langle A_j, B^j \rangle \end{pmatrix}$$

Theorem 2.23. [10] The following equations are obtained under Lorentz matrix multiplication.

i. For all $A \in L_n^m$, $B \in L_p^n$, $C \in L_r^p$, $A._L(B._L C) = (A._L B)._L C$

ii. For all $A \in L_n^m$, $B, C \in L_p^n$, $A._L(B + C) = A._L B + A._L C$

iii. For all $A, B \in L_n^m$, $C \in L_p^n$, $(A + B)._L C = A._L C + B._L C$

iv. For all $k \in \mathbb{R}$,$A \in L_n^m$, $B \in L_p^n$, $k(A._L B) = (kA)._L B = A._L(kB)$

Theorem 2.24. [10] The Lorentz unit matrix can be represented as

$$I._L = \begin{pmatrix} -1 & \ldots & 0 \\ \vdots & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \ldots & 1 \end{pmatrix}_{n \times n}$$

Definition 2.25. [10] A is a matrix of type $n \times n$, if there is a B matrix of type $n \times n$ such that $A._L B = B._L A = I_n$, A is called reversible and denoted by A^{-1}.

Definition 2.26. [10] Transpose of $A = [a_{ij}] \in L_n^m$ demonstrations with A^T and define with $A^T = [a_{ji}] \in L_m^n$.

Definition 2.27. [10] If $A^{-1} = A^T$ for $A \in L_n^n$ matrix, A is called $L-$ orthogonal.

2.7. Pseudo Matrix Multiplication

Throughout this section $R^{m,n}$ is denoted the set of matrices of type $m \times n$. $R^{m,n}$ is a real vector space by addition and scalar multiplication. Each element of the matrix $A \cdot_v B$ is the inner product defined by Equation 5 where "$\cdot_v$" is the Pseudo matrix product between two matrices. The set of matrices defined pseudo matrix multiplication is denoted by $R_v^{m,n}$. $(i,j)^{th}$ Element of matrice's $A \cdot_v B$ is defined by

$$\langle x, y\rangle_v = -\sum_{j=1}^{v} a_{ij} b_{jk} + \sum_{j=v+1}^{n} a_{ij} b_{jk} \tag{5}$$

i. If $v = 0$ then it is equivalent to classic matrix multiplication.

$$\begin{aligned} \langle x, y\rangle_0 &= -\textstyle\sum_{j=1}^{0} a_{ij} b_{jk} + \sum_{j=0+1}^{n} a_{ij} b_{jk} \\ &= a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk} \end{aligned}$$

ii. If $v = 1$ then it is equivalent to Lorentz matrix multiplication.

$$\begin{aligned} \langle x, y\rangle_1 &= -\textstyle\sum_{j=1}^{1} a_{ij} b_{jk} + \sum_{j=1+1}^{n} a_{ij} b_{jk} \\ &= -a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk} \end{aligned}$$

in [14].

Theorem 2.28. [14] $det(A \cdot_v B) = (-1)^v det A \cdot det B$, for all $A, B \in R_v^{n,n}$.

Since it is equivalent to Lorentz matrix multiplication for $v = 1$,$det(A \cdot_1 B) = -det A \cdot det B$ is obtained.

2.8. Coordinate Transformations in Two Dimensional Lorentz Space

In this section, obtaining the Lorentz matrix using the displacement between two points in $\mathbb{R}^2$ is examined. Assume that $m(CAx) = \alpha$,$m(BAC) = \beta$, $m(BAx) = \theta$, $B(\sinh\theta, \cosh\theta)$ and $C(\sinh\alpha, \cosh\alpha)$. If point $C(x, y)$ is rotated counter clockwise around the origin by an angle of β, it becomes point $B(x', y')$.Since the coordinates of point C are taken as $x = r\sinh\alpha$ and $y = r\cosh\alpha$, the coordinates of point B are written as $x' = r\sinh(\alpha+\beta)$ and $y' = r\cosh(\alpha+\beta)$.

$$\begin{aligned} x' &= r\sinh(\alpha+\beta) \\ &= r(\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha) \\ &= r\sinh\alpha\cosh\beta + r\sinh\beta\cosh\alpha \\ &= x\cosh\beta + y\sinh\beta \end{aligned}$$

$$\begin{aligned} y' &= r\cosh(\alpha+\beta) \\ &= r(\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta) \\ &= r\cosh\alpha\cosh\beta + r\sinh\alpha\sinh\beta \\ &= y\cosh\beta + x\sinh\beta \\ &= x\sinh\beta + y\cosh\beta \end{aligned}$$

are obtained.

The trigonometric expansions of x' and y' can be rewritten in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cosh\beta + y\sinh\beta \\ x\sinh\beta + y\cosh\beta \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If this matrix product is written according to the Lorentz matrix multiplication

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cosh\beta + y\sinh\beta \\ x\sinh\beta + y\cosh\beta \end{pmatrix} = \begin{pmatrix} -\cosh\beta & \sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \cdot_L \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -\cosh\beta & \sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \in L_2^2$$

matrix is obtained [10].

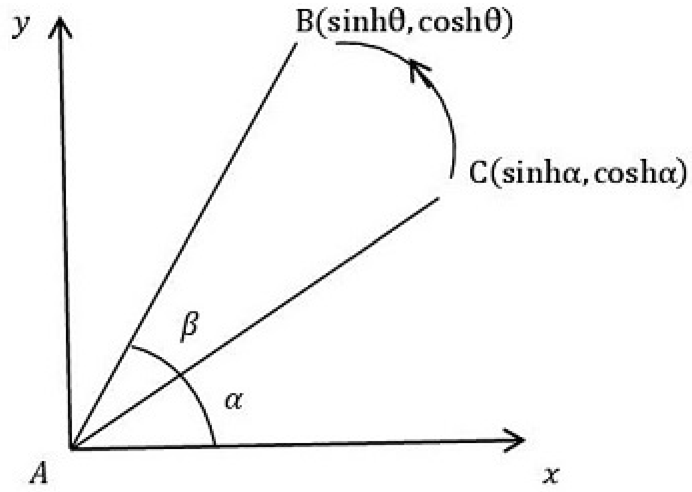

Fig. 1. Rotation diagram between two points

3. Main Results

3.1. Obtaining Vertices of a Suborbital Graph $F_{u,N}$ Under Lorentz Matrix Multiplication

In this section, we examine that the Lorentz matrix in Equation 6 that gives the vertices obtained under the classical matrix multiplication in the suborbital graph $F_{u,N}$ under the Lorentz matrix multiplication and see that the Lorentz matrix in Equation 6 is not a member of the Modular group. In Subsection 2.8, obtaining the Lorentz matrix using the displacement between two points in $\mathbb{R}^2$ was examined. We know that from Equation 4

$$v_q = \begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix}^q (v_0)$$

From Subsection 2.8, the Lorentz matrix giving the same vertices on the path of minimal length can be given by

$$\begin{pmatrix} u & \frac{u^2+ku+1}{N} \\ N & u+k \end{pmatrix} \in L_2^2 \tag{6}$$

From here, the vertices of the path with minimal length are provided as follows:

$$v_q = \begin{pmatrix} u & \frac{u^2+ku+1}{N} \\ N & u+k \end{pmatrix}^q \cdot_L (v_0)$$

for all $q \in \mathbb{Z}^+$,where $v_0 = \frac{u}{N}$.

Corollary 3.1. Lorentz matrix given in Equation 6 is not member of Modular group.

PROOF.

$$\begin{aligned} \begin{vmatrix} u & \frac{u^2+ku+1}{N} \\ N & u+k \end{vmatrix} &= u(u+k) - N(\tfrac{u^2+ku+1}{N}) \\ &= u^2 + uk - u^2 - uk - 1 \\ &= -1 \end{aligned}$$

□

Since the determinant is -1 in Corollary 3.1., we can normalize the relevant matrix. The relevant matrix can be written as the Möbius transform as follows:

$$m(z) = \frac{uz + \frac{u^2+ku+1}{N}}{Nz + u + k}$$

For $\alpha \in \widehat{\mathbb{C}}$,

$$m(z) = \frac{\alpha uz + \alpha\frac{u^2+ku+1}{N}}{\alpha Nz + \alpha(u + k)}$$

$$\begin{aligned} det(m(z)) &= \alpha u\alpha(u+k) - \alpha N(\alpha\frac{u^2+ku+1}{N}) \\ &= \alpha^2 u(u+k) - \alpha^2(u^2+ku+1) \\ &= \alpha^2u^2 + \alpha^2uk - \alpha^2u^2 - \alpha^2uk - \alpha^2 \\ &= 1 \end{aligned}$$

$\alpha^2 = -1, \alpha = \mp i$ If $\alpha = i$, then

$$m(z) = \frac{iuz + i\frac{u^2+ku+1}{N}}{iNz + i(u + k)}$$

Möbius transform can be writtten for $i \in \widehat{\mathbb{C}}$,

$$\begin{pmatrix} ui & \frac{u^2+ku+1}{N}i \\ Ni & (u+k)i \end{pmatrix}$$

as an element of Modular group.Similar operations can be done for $\alpha = -i$.

From here, the type of Möbius transformation can be determined. Trace of Möbius transformation $m(z) = \frac{az+b}{cz+d}$ can be written as $\tau(m) = (a + d)^2$. From the above matrix,

$$\begin{aligned} \tau(m) &= (a+d)^2 \\ &= (ui + i(u+k))^2 \\ &= (ui)^2 + 2uii(u+k) + (i(u+k))^2 \\ &= -u^2 - 2u^2 - 2uk + (-u^2 - 2uk - k^2) \\ &= -4u^2 - 4uk - k^2 = -(2u+k)^2 \end{aligned}$$

trace is obtained. m is elliptic since $\tau(m) = 0$ real for $u = -\frac{k}{2}$ when $k \geqslant 2$,$k \in \mathbb{Z}$ and u are arbitrary and m is loxodromic for $u \neq -\frac{k}{2}$ and $\tau(m)$ is real.

Corollary 3.2. Assume that $u^2 + ku + 1 \equiv 0 \pmod N$ under Lorentz multiplication in Farey graph provided for $(u, N) = 1$ and $k \geqslant 2$,$k \in \mathbb{Z}$. Under Lorentz matrix multiplication, i and ii are provided for the vertices obtained by the matrix given in [7].

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u+\frac{1}{k}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u+\frac{1}{k}}{N}$ can be connected becomes $\frac{u+\frac{1}{k-\frac{1}{k}}}{N}$ and there is no similar nearest vertex.

Since the vertices obtained under Lorentz matrix multiplication in the suborbital graph $F_{u,N}$ with vertices obtained in Theorem 2.16. are the same, then the proof of this Corollary is the same with proof of Theorem 2.16.

Corollary 3.3. For $(u, N) = 1$ and $k \in \mathbb{Z}$, where $u^2 + ku - 1 \equiv 0 \pmod N$,

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \in \Gamma_0(N) \tag{7}$$

provides under Lorentz matrix multiplication,

$$\infty = \frac{1}{0} \to \frac{u}{N} \to \frac{u - \frac{1}{k}}{N} \to \frac{u - \frac{1}{k+\frac{1}{k}}}{N} \to \frac{u - \frac{1}{k+\frac{1}{k+\frac{1}{k}}}}{N} \to \dots$$

path in suborbital graph $F_{u,N}$.

Proof. Here the first four vertices of the path are found, the other vertices are obtained in a similar way.

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ N \end{pmatrix}$$

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} u \\ N \end{pmatrix} = \begin{pmatrix} ku-1 \\ Nk \end{pmatrix} = \begin{pmatrix} u - \frac{1}{k} \\ N \end{pmatrix}$$

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} ku-1 \\ Nk \end{pmatrix} = \begin{pmatrix} k^2u+u-k \\ Nk^2+N \end{pmatrix} = \begin{pmatrix} u - \frac{1}{k+\frac{1}{k}} \\ N \end{pmatrix}$$

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} k^2u+u-k \\ Nk^2+N \end{pmatrix} = \begin{pmatrix} k^3u-k^2+2uk-1 \\ Nk^3+2Nk \end{pmatrix} = \begin{pmatrix} u - \frac{1}{k+\frac{1}{k+\frac{1}{k}}} \\ N \end{pmatrix}$$

□

Example 3.4. If $u = 1$, $N = 5$ and $k = 3$, from Corollary 3.3.

$$\infty = \frac{1}{0} \to \frac{1}{5} \to \frac{2}{15} \to \frac{7}{50} \to \frac{1 - \frac{1}{3+\frac{1}{3+\frac{1}{3}}}}{5} \to \dots$$

is obtained.

Corollary 3.5. Assume that $u^2 + ku - 1 \equiv 0 \pmod N$ under Lorentz multiplication in Farey graph provided for $(u, N) = 1$ and $k \in \mathbb{Z}$. Under Lorentz matrix multiplication, i and ii are provided for the vertices obtained by the matrix given in Equation 7.

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u-\frac{1}{k}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u-\frac{1}{k}}{N}$ can be connected becomes $\frac{u-\frac{1}{k+\frac{1}{k}}}{N}$ and there is no similar nearest vertex.

Proof. *i.* Assume that $\frac{u-\frac{r}{m}}{N}$ be the farthest vertex that can be connected with $\frac{u}{N}$, where $\frac{u}{N}$ is a vertex in $F_{u,N}$ under Lorentz multiplication. Hence

$$\frac{u}{N} \to \frac{u - \frac{r}{m}}{N} = \frac{um-r}{mN}$$

is obtained. $um - r \equiv u^2 \pmod N$,$umN - N(um - r) = N$ must be provided for edge condition. Thus,

$$umN - N(um - r) = N, umN - Num + Nr = N \Rightarrow r = 1$$

If the value of $r = 1$ is substituted in the congruence equation,

$$um - 1 \equiv u^2 \,(\text{mod } N)$$

is obtained. Then $-u^2 + um - 1 \equiv 0 \,(\text{mod } N)$. If equations $-u^2 + um - 1 \equiv 0 \,(\text{mod } N)$ and $u^2 + ku - 1 \equiv 0 \,(\text{mod } N)$ are added, $(um-1) + (uk-1) \equiv 0 \,(\text{mod } N)$. From here, $m \equiv k \,(\text{mod } N)$ is reached. As a result,

$$m = k + Nx, x \in \mathbb{N} \cup \{0\}$$

Thus, $\frac{r}{m} = \frac{1}{k+Nx}$. Here, a function can be defined as follows:

$$f(x) = \frac{u - \frac{r}{m}}{N} = \frac{u - \frac{1}{k+Nx}}{N}, f : \mathbb{N} \cup \{0\} \to \mathbb{R}$$

If the derivative of the function is taken, it is seen that it is strictly increasing.

$$f^{'}(x) = \frac{1}{(k+Nx)^2} > 0$$

Since it is a strictly increasing function, it takes the minimum value for $x = 0$.If the value of $x = 0$ is written in the relevant function, it becomes $f(0) = \frac{u-\frac{1}{k}}{N}$. It is obvious that $(uk-1, kN) = 1$. Consequently, $\frac{u-\frac{1}{k}}{N}$ is a vertex at $F_{u,N}$ and is the farthest vertex to which $\frac{u}{N}$ can connect.

ii. Assume that $\frac{u-\frac{r}{m}}{N}$ be the farthest vertex that can be connected with $\frac{u-\frac{1}{k}}{N}$, where $\frac{u-\frac{1}{k}}{N}$ is a vertex in $F_{u,N}$ under Lorentz multiplication. Hence

$$\frac{u - \frac{1}{k}}{N} = \frac{uk-1}{kN} \to \frac{u - \frac{r}{m}}{N} = \frac{um-r}{mN}$$

is obtained. $um - r \equiv -u(uk-1) \,(\text{mod } N)$,$(uk-1)mN - kN(um-r) = -N$ must be provided for edge condition. So, $(uk-1)mN - kN(um-r) = -N, ukmN - mN - knuM + kNr = -N, -mN + kNr = -N, -m + kr = -1 \Rightarrow m = kr + 1$. If the value of $m = kr+1$ is substituted in the congruence equation, $u(kr+1) - r \equiv -u(uk-1) \,(\text{mod } N)$ is obtained. Then $u(kr+1) - r \equiv -u(uk-1) \,(\text{mod } N)$, $ukr + u - r + u^2k - u \equiv 0 \,(\text{mod } N)$ is provided for $ukr + u - r \equiv -u^2k + u \,(\text{mod } N)$. From here, $(uk-1)r + u^2k \equiv 0 \,(\text{mod } N)$ and $uk - 1 \equiv -u^2 \,(\text{mod } N)$ are reached.

$$(uk-1)r + u^2k \equiv 0 \,(\text{mod } N), -u^2r + u^2k \equiv 0 \,(\text{mod } N)$$

$$-r + k \equiv 0 \,(\text{mod } N), r \equiv k \,(\text{mod } N), r = k + Nx, x \in \mathbb{N} \cup \{0\}$$

Hence, $\frac{r}{m} = \frac{k+Nx}{k(k+Nx)+1}$. Here, a function can be defined as follows:

$$\begin{aligned} f(x) &= \frac{u - \frac{k+Nx}{k(k+Nx)+1}}{N} \\ &= \frac{u(k(k+Nx)+1) - (k+Nx)}{N(k(k+Nx)+1)} \\ &= \frac{uk^2 + ukNx + u - k - Nx}{Nk^2 + kN^2x + N}, f : \mathbb{N} \cup \{0\} \to \mathbb{R} \end{aligned}$$

If the derivative of the function is taken, it is seen that it is strictly decreasing.

$$f^{'}(x) = \frac{-1}{(k^2 + kNx + 1)^2} < 0$$

Since it is a strictly decreasing function, it takes the maximum value for $x = 0$. If the value of $x = 0$ is written in the relevant function, it becomes $f(0) = \frac{u - \frac{1}{k+\frac{1}{k}}}{N}$. We have to show that $(uk^2 + u - k, k^2 + 1) = 1$. Assume that $(u(k^2+1) - k, k^2 + 1) = g$. From here, $g \setminus k^2 + 1$ and

$g \setminus (u(k^2+1))$. Then $g \setminus (u(k^2+1)) - k$, $g \setminus -k$. Hence $g \setminus k^2+1$, $g = 1$. We have to demonstrate that $(uk^2+u-k, N) = 1$. For $(u(k^2+1)-k, N) = z$, $u(k^2+1)-k = k(uk-1)+u \equiv 0(\text{mod } z)$ and $N \equiv 0(\text{mod } z)$ are obtained. So, $uk-1 \equiv -u^2\,(\text{mod } N)$. Then, $k(uk-1)+u = k(-u^2)+u \equiv 0(\text{mod } z)$, $u(-ku+1) \equiv 0(\text{mod } z)$, $u \equiv 0(\text{mod } z)$ or $-ku+1 \equiv 0(\text{mod } z)$. This is a contradiction. Hence $z = 1$.

Consequently, $\frac{u-\frac{1}{k+\frac{1}{k}}}{N}$ is a vertex at $F_{u,N}$ and is the farthest vertex to which $\frac{u-\frac{1}{k}}{N}$ can connect. Since $f(0) = \frac{u-\frac{1}{k+\frac{1}{k}}}{N}$ is, there is no nearest vertex to which $\frac{u-\frac{1}{k}}{N}$ can connect.

□

Corollary 3.6. For $(u, N) = 1$,where $u^2+u-1 \equiv 0\,(\text{mod } N)$ for $k = 1$, $\begin{pmatrix} u & \frac{u^2+u-1}{N} \\ N & u+1 \end{pmatrix} \in \Gamma_0(N)$ provides under Lorentz matrix multiplication,

$$\infty = \frac{1}{0} \to \frac{u}{N} \to \frac{u-1}{N} \to \frac{u-\frac{1}{2}}{N} \to \frac{u-\frac{2}{3}}{N} \to \dots \to \frac{u-\frac{F_n}{F_{n+1}}}{N} \to \dots$$

path in suborbital graph $F_{u,N}$.

PROOF. $(n+1)^{th}$ vertex is obtained as follows where n^{th} vertex is $\frac{u-\frac{F_n}{F_{n+1}}}{N}$:

$$\begin{aligned}
\begin{pmatrix} u & \frac{u^2+u-1}{N} \\ N & u+1 \end{pmatrix} \cdot_L \begin{pmatrix} u-\frac{F_n}{F_{n+1}} \\ N \end{pmatrix} &= \begin{pmatrix} -u(u-\frac{F_n}{F_{n+1}}) + N(\frac{u^2+u-1}{N}) \\ -N(u-\frac{F_n}{F_{n+1}}) + N(u+1) \end{pmatrix} \\
&= \begin{pmatrix} -u^2 + u\frac{F_n}{F_{n+1}} + u^2 + u - 1 \\ -Nu + N\frac{F_n}{F_{n+1}} + Nu + N \end{pmatrix} \\
&= \begin{pmatrix} u\frac{F_n}{F_{n+1}} + u - 1 \\ N\frac{F_n}{F_{n+1}} + N \end{pmatrix} \\
&= \begin{pmatrix} uF_{n+2} - F_{n+1} \\ NF_{n+2} \end{pmatrix} \\
&= \begin{pmatrix} u - \frac{F_{n+1}}{F_{n+2}} \\ N \end{pmatrix}
\end{aligned}$$

□

Example 3.7. If $u = 2$, $N = 5$ and $k = 1$, from Corollary 3.6.,

$$\infty = \frac{1}{0} \to \frac{2}{5} \to \frac{1}{5} \to \frac{3}{10} \to \dots \to \frac{2-\frac{F_{n+1}}{F_{n+2}}}{5} \to \dots$$

is obtained.

Corollary 3.8. For $(u, N) = 1$, where $u^2+u-1 \equiv 0\,(\text{mod } N)$ for $k = 1$ and α is golden ratio, $\begin{pmatrix} u & \frac{u^2+u-1}{N} \\ N & u+1 \end{pmatrix} \in \Gamma_0(N)$ provides value of vertex as $\frac{u-\frac{1}{\alpha}}{N}$ for $n \to \infty$ under Lorentz matrix multiplication in suborbital graph $F_{u,N}$.

PROOF. From Corollary 3.6.,

$$\begin{aligned}\lim_{n\to\infty} \frac{u-\frac{F_n}{F_{n+1}}}{N} &= \frac{u-\lim_{n\to\infty}\frac{F_n}{F_{n+1}}}{N}\\ &= \frac{u-\frac{1}{\lim_{n\to\infty}\frac{F_{n+1}}{F_n}}}{N}\\ &= \frac{u-\frac{1}{\alpha}}{N}\end{aligned}$$

□

Example 3.9. If $u = 2$, $N = 5$, $k = 1$, and $\alpha = 1.618$ from Corollary 3.8.,

$$\frac{2-\frac{1}{1.618}}{5} = 0,276$$

is obtained.

4. Conclusion

In this study, we especially examined suborbital graphs obtained by Lorentz matrix multiplication. It is seen that Lorentz matrix which gave vertices that are obtained by classical matrix multiplication is not an element of Modular group Γ. Moreover, we defined a matrix that is an element of Modular group Γ. Furthermore, we investigated vertices, edges and path obtained under Lorentz matrix multiplication by this matrix. It is indicated that vertices on the path provide the minimal length condition. The vertices are associated with Fibonacci numbers for $k = 1$ and value of vertex is found for $n \to \infty$.

Authors Contributions

All the authors declare that they contributed equally and adequately to this paper. They all read and approved the last version of the paper. This study was derived from the first author's PhD dissertation supervised by the second author.

Conflict of Interest

All the authors declare that they have no conflict of interest.

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New Inequalities for Hyperbolic Lucas Functions

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Abstract — This article introduces the classic Wilker's, Wu-Srivastava, Hugyen's, Cusa-Hugyen's, and Wilker's-Anglesio type inequalities for hyperbolic Lucas functions with some new refinements.

Keywords — *Hyperbolic Lucas functions, Wilker's inequality, Cusa-Huygen's inequality, Wu-Srivastava inequality*

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1. Introduction

As it is known, the theory of inequality is one of the most important branches of mathematics. Especially in functional analysis, differential equations and mathematical analysis, inequalities have a great impact. Fundamental research in this area belongs to the great mathematicians such as Hardy, Cauchy, Hölder, Littlewood, Minkowski and others. One of the curious topics of the theory of inequality are inequalities related to trigonometric and hyperbolic functions. The most famous studies on this subject belong to mathematicians such as Wilker, Huygen's, Mitrinovic, Wu, Srivastava, Adamovic, and Cusa. In this study, we will give analogues and some new improvements of these inequalities for hyperbolic Lucas functions. Hyperbolic Lucas functions are defined by inspiring the Binet formula for Lucas numbers, which are interesting in number theory and on which many studies have been made. The reason that makes these functions special is that they are related to the golden ratio. Because the golden ratio has many incredible applications in nature. Therefore, it would be interesting to give analogues of theorems related to classical hyperbolic and trigonometric functions for hyperbolic Lucas functions.

Now we will give some famous inequalities:

i. The Wilker's inequality is given as (see [1–15]).

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2 \tag{1}$$

ii. The Huygens inequality is given as (see [3, 4, 11, 12]).

$$\frac{2\sin x}{x}+\frac{\tan x}{x}>3 \tag{2}$$

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iii. The Cusa-Huygens inequality is given as (see [12, 16]).

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3} \tag{3}$$

iv. The Wu-Srivastava inequality is given as (see [9]).

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \tag{4}$$

v. The Wilker's-Anglesio inequality is given as (see [5] [17]).

$$\left(\frac{\sinh(x)}{x}\right)^2 + \frac{\tanh(x)}{x} > 2 + \frac{8}{45}x^3 \tanh(x) \tag{5}$$

Inequalities (1), (2), (3), (4) and (5) are satisfied for $x \in \left(0, \frac{\pi}{2}\right)$.

2. Preliminaries

This section provides some of the basic notions needed for the following sections. The classical hyperbolic functions are as follows.

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \sinh(x) = \frac{e^x - e^{-x}}{2}, \text{ and } \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{6}$$

Similarly, Stakhov and Rozin described hyperbolic Lucas functions in 2005. see([8, 18, 19]).

Definition 2.1. The symmetrical hyperbolic Lucas sine, cosine and tangent functions are defined as follows, respectively.

$$sLh(x) = \alpha^x - \alpha^{-x}, cLh(x) = \alpha^x + \alpha^{-x}, \text{ and } tLh(x) = \frac{\alpha^x - \alpha^{-x}}{\alpha^x + \alpha^{-x}} \text{ for all } x \in R \tag{7}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Definition 2.2. [20] [21] The generalized hyperbolic sine, cosine and tangent functions are defined as follows, respectively.

i. $\sinh_\varphi(x) = \dfrac{\varphi^x - \varphi^{-x}}{2}$

ii. $\tanh_\varphi(x) = \dfrac{\varphi^x - \varphi^{-x}}{\varphi^x + \varphi^{-x}}$

iii. $\cosh_\varphi(x) = \dfrac{\varphi^x + \varphi^{-x}}{2}$

Some basic properties of hyperbolic Lucas functions are as following:

i. $cLh(x) = cLh(-x)$

ii. $sLh(x) = -sLh(-x)$

iii. $tLh(x) = -tLh(-x)$

iv. $sLh'(x) = cLh(x)\ln(\alpha)$

v. $cLh'(x) = sLh(x)\ln(\alpha)$

vi. $tLh^{'}(x) = \dfrac{4\ln(\alpha)}{cLh^2(x)}$

Lemma 2.3. If $x \in [0, \infty)$, then the following inequalities hold:

i. $sLh(x) \geq 2x\ln(\alpha)$

ii. $x\ln(\alpha) \geq tLh(x)$

PROOF. part *i)* Let $f : R^+ \to R$ be a function defined by

$$f(x) = sLh(x) - 2x\ln(\alpha)$$

The derivative of $f(x)$ is

$$f^{'}(x) = \ln(\alpha)(cLh(x) - 2) \geq 0$$

Because $cLh(x) \geq 2$ .Then we obtain $f(x)$ is an increasing function on the interval $[0, \infty)$, this means that $f(x) \geq f(0) = 0$ Therefore

$$sLh(x) \geq 2x\ln(\alpha)$$

Similarly, we can proof part *ii*. □

Lemma 2.4. [8] If $x \neq 0$, then the following inequality holds:

$$cLh(x) < \frac{1}{4(\ln(\alpha))^3}\left(\frac{sLh(x)}{x}\right)^3 \tag{8}$$

PROOF. From the properties of hyperbolic Lucas functions, it is clear that it is sufficient to prove the theorem for $x > 0$.
Let $f : R^+ \to R$ be a function defined by $f(x) = \dfrac{sLh^3(x)}{x^3 cLh(x)}$
The derivative of $f(x)$ is

$$f^{'}(x) = \frac{sLh^2(x)}{x^4 cLh^2(x)}\left[2xcLh^2(x)\ln(\alpha) + 4x\ln(\alpha) - 3cLh(x)sLh(x)\right]$$

Now let $g : R^+ \to R$ be a function defined by

$$g(x) = 2xcLh^2(x)\ln(\alpha) + 4x\ln(\alpha) - 3cLh(x)sLh(x)$$

The derivative of $g(x)$ is

$$g^{'}(x) = 2\ln(\alpha)[2xsLh(2x) - 3cLh(2x)) + cLh^2(x) + 2]$$

Now let $h : R^+ \to R$ be a function defined by

$$h(x) = 2xsLh(2x) - 3cLh(2x)) + cLh^2(x) + 2$$

The derivative of $h(x)$ is

$$h^{'}(x) = 2sLh(2x)[1 - 2\ln(\alpha)] + 4xcLh(2x)$$

this show $h^{'}(x) > 0$ Then we obtain $h(x), g(x)$ are increasing and positive functions on $(0, \infty)$. Hence, we get $f(x)$ is an increasing on $(0, \infty)$, by using $\lim_{x\to 0^+} f(x) = 4(\ln(\alpha))^3$. We conclude that

$$f(x) > 4(\ln(\alpha))^3$$

□

Lemma 2.5. [22,23] If $x, y > 0$, and $\mu \in [0, 1]$, then

$$\mu x + (1 - \mu)y \geq x^{\mu} y^{1-\mu}$$

Lemma 2.6. [22,23] (Cauchy-Schwarz inequality) If $x_i, y_i > 0$, then

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$

Lemma 2.7. If $x_i, y_i > 0$, $i = 1, 2, ..., n$, then

$$\left(\sum_{i=1}^{n}(x_i + y_i)\right)^2 \ge 4\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right) \tag{9}$$

PROOF. We know that: $4xy \le (x+y)^2, \forall x, y > 0$

$$4\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right) \le \left[\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right) + \left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right)\right]^2$$

$$= \left[\sum_{i=1}^{n}\left(\sqrt{x_i y_i} + \sqrt{\frac{x_i^2 + y_i^2}{2}}\right)\right]^2$$

And by Lemma 2.6, we get

$$\left[\sum_{i=1}^{n}\left(\sqrt{x_i y_i} + \sqrt{\frac{x_i^2 + y_i^2}{2}}\right)\right]^2 \le \left[\sum_{i=1}^{n}\sqrt{(1+1)\left(x_i y_i + \frac{x_i^2 + y_i^2}{2}\right)}\right]^2 = \left(\sum_{i=1}^{n}(x_i + y_i)\right)^2$$

□

Lemma 2.8. If $x, y > 0$, $x \ge y$ and $\mu \in [\frac{1}{2}, 1]$, then the following inequality holds:

$$\mu x + (1-\mu)y \ge x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \ge x^{\mu}y^{1-\mu} \tag{10}$$

PROOF. We obtain the first part of the inequality directly from Lemma 2.5

$$\begin{aligned} \mu x + (1-\mu)y &= (2\mu - 1)(x - y) + (1-\mu)x + \mu y \\ &\ge x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \end{aligned}$$

Now, we have to illustrate that the below inequality holds :

$$x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \ge x^{\mu}y^{1-\mu}$$

For this let's define a function $f : [1, \infty) \to R$

$$f(t) = t^{1-\mu} + (2\mu - 1)(t - 1) - t^{\mu}$$
$$f'(t) = (1-\mu)t^{-\mu} + (2\mu - 1) - \mu t^{\mu - 1}$$
$$f''(t) = (1-\mu)(-\mu)t^{-\mu-1} - \mu(\mu - 1)t^{\mu-2} = \mu(\mu - 1)\left[\frac{1}{t^{\mu+1}} - \frac{1}{t^{2-\mu}}\right] \ge 0, \forall t \ge 1$$

then we obtain :

$$\forall t \ge 1, f'(t) \ge f'(1) = 0$$

thus $f(t)$ is an increasing and positive function for all $t \ge 1$. If we take $t = \dfrac{x}{y}$ and multiply both sides of the inequality by y, then we obtain :

$$\left(\frac{x}{y}\right)^{1-\mu} y + (2\mu - 1)(x - y) \ge \left(\frac{x}{y}\right)^{\mu} y$$

or

$$x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \ge x^{\mu}y^{1-\mu}$$

□

Lemma 2.9. If $x, y > 0$, $x \geq y$ and $\mu \in [\frac{1}{2}, \frac{3}{4}]$, then the following inequality is satisfied

$$\mu x + (1-\mu)y \geq x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{x-y}{2} \geq x^{\mu} y^{1-\mu} \tag{11}$$

Proof. by Lemma 2.5, we obtain

$$\mu x + (1-\mu)y = (\mu - \frac{1}{2})x + (\frac{3}{2} - \mu)y + \frac{x-y}{2} \geq x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{x-y}{2}$$

Now we have to demonstrate that the following inequality is satisfied:

$$x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{x-y}{2} \geq x^{\mu} y^{1-\mu} \tag{12}$$

by Lemma 2.5, we obtain

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] \geq \left(\frac{x}{y}\right)^{\frac{1}{4}}$$

Also we know for all $\mu \in [\frac{1}{2}, \frac{3}{4}]$

$$\left(\frac{x}{y}\right)^{\frac{3}{4}-\mu} \geq 1$$

or

$$\left(\frac{x}{y}\right)^{\frac{1}{4}} \geq \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}$$

is true. Then we get:

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] \geq \left(\frac{x}{y}\right)^{\frac{1}{4}} \geq \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}} \tag{13}$$

It is clear that the inequality (12) is equivalent to the following inequality:

$$\frac{x-y}{2} \geq x^{\mu-\frac{1}{2}} y^{1-\mu} \left[\sqrt{x} - \sqrt{y}\right]$$

If $x = y$, the inequality is trivial. So let's assume $x > y$ and divide both side of the inequality by $\sqrt{y}(\sqrt{x} - \sqrt{y})$ then we get the following inequality:

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] > \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}$$

This inequality is true according to the (13). □

Ibrahimov [24] proved the below inequalities for generalized hyperbolic functions

Theorem 2.10. If $x \geq 0$ and $s > f > 1$ then the following inequalities are satisfied:

i. $\sinh_s(x) \ln f \geq \sinh_f(x) \ln s$

ii. $\tanh_s(x) \ln f \leq \tanh_f(x) \ln s$

iii. $\cosh_s(x) \ln f \geq \cosh_f(x) \ln s$

3. Main Results

Theorem 3.1. (Wu-Srivastava type inequality) If x nonzero real number then the following inequality holds:

$$\left(\frac{x}{sLh(x)}\right)^2 + \frac{x}{tLh(x)} > \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right) \tag{14}$$

PROOF. From the properties of hyperbolic Lucas functions, obviously that it is sufficient to prove the theorem for $x > 0$.
Let $f : R^+ \to R$ be a function defined by

$$f(x) = x^2 + xsLh(2x) - \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right) sLh^2(x)$$

The derivatives of $f(x)$ are

$$f^{'}(x) = 2x - \left(1+\frac{1}{2\ln\alpha}\right) sLh(2x) + 2xcLh(2x)\ln\alpha$$

$$f^{''}(x) = (2 - cLh(2x) + 4xsLh(2x))\,(\ln(\alpha))^2$$

$$f^{'''}(x) = -2sLh(2x)\ln(\alpha) + 4sLh(2x)(\ln(\alpha))^2 + 8xcLh(2x)(\ln(\alpha))^3$$

$$\begin{aligned} f^{(4)}(x) &= -4cLh(2x)(\ln(\alpha))^2 + 16cLh(2x)(\ln(\alpha))^3 + 16xsLh(2x)(\ln(\alpha))^4 \\ &= 4cLh(2x)(\ln(\alpha))^2(4\ln(\alpha)-1) + 16xsLh(2x)(\ln(\alpha))^4 \end{aligned}$$

This means that $f^{(4)}(x) \geq f^{(4)}(0) = 8(\ln(\alpha))^2(4\ln(\alpha)-1) > 0$, $f^{'''}(0) = 0$, $f^{''}(0) = 0$, $f^{'}(0) = 0$. Thus $f^{'''}(x)$, $f^{''}(x)$, $f^{'}(x)$ and $f(x)$ are increasing and positive functions on the interval $[0,\infty)$, this means that $f(x) \geq f(0) = 0$, for all $x \geq 0$. Therefore

$$x^2 + xsLh(2x) \geq \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right) sLh^2(x)$$

By dividing both sides of the inequality by $sLh^2(x)$ for $x > 0$, we obtain

$$\left(\frac{x}{sLh(x)}\right)^2 + \frac{x}{tLh(x)} > \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right)$$

□

In addition we give Cusa-Huygens type inequality for hyperbolic Lucas functions.

Theorem 3.2. If $x \neq 0$, then the following inequality is satisfied:

$$\frac{sLh(x)}{x} < \left(\frac{cLh(x)}{2}+1\right)\ln(\alpha) \tag{15}$$

PROOF. From the properties of hyperbolic Lucas functions, obviously that it is sufficient to prove the theorem for $x > 0$.

Let $f : R^+ \to R$ be a function defined by

$$f(x) = \left(\frac{cLh(x)}{2}+1\right)\ln(\alpha) - \frac{sLh(x)}{x}$$

The derivative of $f(x)$ is

$$\begin{aligned} f^{'}(x) &= \frac{sLh(x)}{2x}(\ln(\alpha))^2 - \frac{xcLh(x)\ln(\alpha) - sLh(x)}{x^2} \\ &= \frac{x^2 sLh(x)(\ln(\alpha))^2 - 2xcLh(x)\ln(\alpha) + 2sLh(x)}{2x^2} \end{aligned}$$

In addition, let $g : R^+ \to R$ be a function defined by

$$g(x) = x^2 sLh(x)(\ln(\alpha))^2 - 2xcLh(x)\ln(\alpha) + 2sLh(x)$$

The derivative of $g(x)$ is

$$g'(x) = x^2 cLh(x)(\ln(\alpha))^3 > 0$$

Then we obtain $g(x)$ is an increasing function on $(0, \infty)$. This means that $g(x) > g(0) = 0$. Hence, we get $f(x)$ is an increasing on $(0, \infty)$, by using

$$\lim_{x\to 0^+} \left[\left(\frac{cLh(x)}{2} + 1 \right) \ln(\alpha) - \frac{sLh(x)}{x} \right] = 0$$

We conclude that $f(x) > 0$. □

Furthermore, we give Huygens type inequality for hyperbolic Lucas functions.

Theorem 3.3. If $x \neq 0$, then the following inequality is satisfied:

$$2\frac{sLh(x)}{x} + \frac{tLh(x)}{x} > 3\,(4)^{\frac{1}{3}} \ln(\alpha) \tag{16}$$

Proof. By Lemmas 2.4, 2.5, we get

$$\frac{2}{3}\frac{sLh(x)}{x} + \frac{1}{3}\frac{tLh(x)}{x} > \left(\frac{sLh(x)}{x}\right)^{\frac{2}{3}} \left(\frac{tLh(x)}{x}\right)^{\frac{1}{3}} = \frac{sLh(x)}{x}\frac{1}{\sqrt[3]{cLh(x)}} > (4)^{\frac{1}{3}} \ln(\alpha)$$

□

Besides, we give two Refinements of Huygens inequality for hyperbolic Lucas functions.

Theorem 3.4. If $x \neq 0$, then the following inequality is satisfied:

$$2\frac{sLh(x)}{x} + \frac{tLh(x)}{x} > 3\left(\frac{sLh(x)}{x}\right)^{\frac{1}{3}} \left(\frac{tLh(x)}{x}\right)^{\frac{2}{3}} + \frac{sLh(x) - tLh(x)}{x} > 3(4)^{\frac{1}{3}} \ln(\alpha)$$

Proof. From the properties of hyperbolic Lucas functions, obviously that it is sufficient to prove the theorem for $x > 0$. By using Lemmas 2.4, 2.8 we get

$$\begin{aligned}\frac{2}{3}\frac{sLh(x)}{x} + \frac{1}{3}\frac{tLh(x)}{x} &> \left(\frac{sLh(x)}{x}\right)^{\frac{1}{3}} \left(\frac{tLh(x)}{x}\right)^{\frac{2}{3}} + \frac{1}{3}\left(\frac{sLh(x) - tLh(x)}{x}\right) > \\ &> \left(\frac{sLh(x)}{x}\right)^{\frac{2}{3}} \left(\frac{tLh(x)}{x}\right)^{\frac{1}{3}} = \frac{sLh(x)}{x}\frac{1}{\sqrt[3]{cLh(x)}} > (4)^{\frac{1}{3}} \ln(\alpha)\end{aligned}$$

□

Theorem 3.5. If $x \neq 0$, then the following inequality is satisfied:

$$2\frac{sLh(x)}{x} + \frac{tLh(x)}{x} > 3\left[\left(\frac{sLh(x)}{x}\right)^{\frac{1}{6}} \left(\frac{tLh(x)}{x}\right)^{\frac{5}{6}} + \frac{sLh(x) - tLh(x)}{2x}\right] > 3(4)^{\frac{1}{3}} \ln(\alpha)$$

Proof. From the properties of hyperbolic Lucas functions, clearly that it is sufficient to prove the theorem for $x > 0$. By Lemmas 2.4, 2.9 we get

$$\begin{aligned}\frac{2}{3}\frac{sLh(x)}{x} + \frac{1}{3}\frac{tLh(x)}{x} &> \left(\frac{sLh(x)}{x}\right)^{\frac{1}{6}} \left(\frac{tLh(x)}{x}\right)^{\frac{5}{6}} + \frac{sLh(x) - tLh(x)}{2x} > \\ &> \left(\frac{sLh(x)}{x}\right)^{\frac{2}{3}} \left(\frac{tLh(x)}{x}\right)^{\frac{1}{3}} = \frac{sLh(x)}{x}\frac{1}{\sqrt[3]{cLh(x)}} > (4)^{\frac{1}{3}} \ln(\alpha)\end{aligned}$$

□

Next, we give Wilker's inequality for hyperbolic Lucas functions.

Theorem 3.6. If $x \neq 0$ then the following inequality is satisfied:

$$\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x} > 4(\ln \alpha)^{\frac{3}{2}} \tag{17}$$

PROOF. From the properties of hyperbolic Lucas functions, clearly that it is sufficient to prove the theorem for $x > 0$. By Lemmas 2.4, 2.5 we get

$$\frac{1}{2}(\frac{sLh(x)}{x})^2 + \frac{1}{2}\frac{tLh(x)}{x} > \frac{sLh(x)}{x}\sqrt{\frac{tLh(x)}{x}} = \sqrt{\left(\frac{sLh(x)}{x}\right)^3 . \frac{1}{cLh(x)}} > 2(\ln \alpha)^{\frac{3}{2}}$$

□

Finally, we give Wilker's-Anglesio inequality for hyperbolic Lucas functions.

Theorem 3.7. If $x \neq 0$ then the following inequality is satisfied:

$$\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x} > 2\ln \alpha + \frac{8}{45}(\ln \alpha)^4 x^3 tLh(x) \tag{18}$$

PROOF. From the properties of hyperbolic Lucas functions, clearly that it is sufficient to prove the theorem for $x > 0$. Let $B : R^+ \to R$ be a function defined by

$$B(x) = \frac{\frac{1}{4(\ln \alpha)^2}\left(\frac{sLh(x)}{x}\right)^2 + \frac{1}{\ln \alpha}\frac{tLh(x)}{x} - 2}{x^3 tLh(x)}$$

Bahşi [8] proved that this function is increasing on $(0, \infty)$ and

$$\lim_{x \to 0+} B(x) = \frac{8(\ln \alpha)^3}{45} \tag{19}$$

This means that

$$\frac{1}{4(\ln \alpha)^2}\left(\frac{sLh(x)}{x}\right)^2 + \frac{1}{\ln \alpha}\frac{tLh(x)}{x} > 2 + \frac{8(\ln \alpha)^3}{45}x^3 tLh(x)$$

it is obvious

$$\frac{1}{\ln \alpha}\left[\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x}\right] > \frac{1}{4(\ln \alpha)^2}\left(\frac{sLh(x)}{x}\right)^2 + \frac{1}{\ln \alpha}\frac{tLh(x)}{x} > 2 + \frac{8(\ln \alpha)^3}{45}x^3 tLh(x)$$

Hence

$$\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x} > 2\ln \alpha + \frac{8}{45}(\ln \alpha)^4 x^3 tLh(x) \tag{20}$$

□

Corollary 3.8. If $x \neq 0$, then the following inequalities are satisfied:

$$\frac{2}{x}sLh(x) + \frac{1}{x}tLh(x) > \frac{sLh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cLh^2(x)}{2cLh^3(x)}}\right) \tag{21}$$

$$\frac{sLh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cLh^2(x)}{2cLh^3(x)}}\right) > \frac{sLh(x)}{x}\left(1 + \frac{2}{\sqrt{cLh(x)}}\right) \tag{22}$$

$$\frac{sLh(x)}{x}\left(1 + \frac{2}{\sqrt{cLh(x)}}\right) > 3\sqrt[3]{4}\ln(\alpha) \tag{23}$$

PROOF. By Lemma 2.7 we obtain:

$$2sLh(x) + tLh(x) > sLh(x) + 2\sqrt[4]{\frac{sLh(x)tLh(x)}{2}\left(sLh^2(x) + tLh^2(x)\right)}$$

$$= sLh(x)\left(1 + 2\sqrt[4]{\frac{1 + cLh^2(x)}{2cLh^3(x)}}\right)$$

Hence, (21) is proved.
By Lemma 2.5 we obtain:

$$1 + cLh^2(x) \geq 2cLh(x) \tag{24}$$

$$1 + \frac{2}{\sqrt{cLh(x)}} \geq \frac{3}{\sqrt[3]{cLh(x)}} \tag{25}$$

and by inequality (24), we get the inequality (22). Also by using Lemma 2.4 and inequality (25) we obtain the inequality (23). □

Now we calculate the limit using Theorem 3.2 and Lemma 2.4 without using L'Hôpital's rule.

Corollary 3.9.

$$\lim_{x \to 0} \frac{sLh(x)}{x} = 2\ln(\alpha) \tag{26}$$

PROOF. By Lemma 2.4 and Theorem 3.2, we obtain

$$\sqrt[3]{4cLh(x)}ln(\alpha) < \frac{sLh(x)}{x} < \left(\frac{cLh(x)}{2} + 1\right)\ln(\alpha)$$

Take $f(x) = \sqrt[3]{4cLh(x)}\ln(\alpha)$; $g(x) = \frac{sLh(x)}{x}$; $h(x) = \left(\frac{cLh(x)}{2} + 1\right)\ln(\alpha)$
Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt[3]{4cLh(x)}\ln(\alpha) = 2\ln(\alpha)$$

and

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \left(\frac{cLh(x)}{2} + 1\right)\ln(\alpha) = 2\ln(\alpha)$$

By Sandwich theorem,

$$\lim_{x \to 0} \frac{sLh(x)}{x} = 2\ln(\alpha)$$

□

Corollary 3.10. If $x \geq 0$, then the following inequality is satisfied:

$$2\sinh(x)(\ln\alpha)^2 + tLh(x)\ln\alpha \geq sLh(x) + \tanh(x) \tag{27}$$

PROOF. Let f be a function defined by

$$f(x) = 2\sinh(x)(\ln\alpha)^2 + tLh(x)\ln\alpha - sLh(x) - \tanh(x)$$

The derivative of $f(x)$ is

$$f^{'}(x) = \ln\alpha\left(2\cosh(x)\ln\alpha - clh(x)\right) + \left(\frac{4(\ln\alpha)^2}{(clh(x))^2} - \frac{1}{(\cosh(x))^2}\right) \geq 0$$

According to Theorem 2.10 $f(x)$ is an increasing function on $[0, \infty)$, this means that

$$f(x) \geq f(0) = 0$$

□

4. Conclusion

In this study, analogues of some important inequalities related to hyperbolic and trigonometric functions are obtained for hyperbolic Lucas functions. In addition, some modifications of Young's inequality have been proved and new results have been obtained for Lucas functions as a result of these modifications. In the future studies can investigate improvements and generalizations of these inequalities.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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On Quasi Quadratic Modules of Lie Algebras

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Abstract — This study introduces the category of quasi-quadratic modules of Lie algebras and discusses the functorial relations between quasi-quadratic modules and quadratic modules of Lie algebras.

Keywords — *Crossed modules, quasi quadratic modules, quadratic modules*

Mathematics Subject Classification (2020) — 18A40, 18G45

1. Introduction

The concept of crossed modules first appears in Whitehead's work on groups, [1, 2], related to the homotopy type of 3-dimensional complexes. Its Lie algebra version has been introduced by Kassel and Loday in [3]. Since then, many algebraic contents have been created based on this concept. Some of them are considered two-dimensional analogous to crossed modules. One of them is the notion of 2-crossed modules for groups introduced by Conduché, [4], with the objective of generalizing the well known result by which the category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length 2. The Lie algebra version of this notion belongs to Ellis, [5]. Another notion is quadratic modules of groups defined by Baues, [6], which is a special case of it. Ulualan and Uslu have adapted this algebraic 3-type model to the Lie algebras, [7] as well as Arvasi and Ulualan have worked on that of commutative algebra case, [8], [9]. Many studies have been conducted in these contexts for various algebraic cases, including the homotopy theory and some categorical results, [10–14]. Carrasco and Porter have initially introduced 2-quasi-crossed modules of groups in [15] as an auxiliary tool that they are in between 2-pre-crossed modules and 2-crossed modules. It is well known that we can get a crossed module associated to a pre-crossed module. It is seen in [15] that it is possible to construct the 2-crossed module associated to a 2-quasi-crossed module of groups by similar ideas. Thus, we can obtain some functorial relations and some categorical results. In [16], the second author and Kaplan have also given the concept of quasi 2-crossed modules of Lie algebras and some functorial results. In this work we focus on the concept of the quadratic module which is another two-dimensional analogous of crossed modules. We will introduce the definition of a quasi-quadratic module as an auxiliary concept in the construction of some categorical content and also examine how this relates to quadratic modules as functorially.

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2. Preliminaries

A pre-crossed module of Lie algebras (Y, Z, ∂) is given by a Lie homomorphism $\partial : Y \to Z$, together with a left Lie algebra action "$*_1$" of Z on Y such that the condition

XMod$_L$1: $\partial(z *_1 y) = [z, \partial(y)]$,

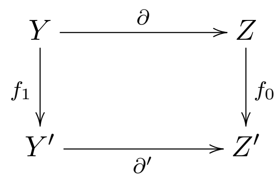
is satisfied for each $z \in Z$ and $y \in Y$.

A crossed module of Lie algebras (Y, Z, ∂) is a pre-crossed module satisfying, in addition condition:

XMod$_L$2: $\partial(y) *_1 y' = [y, y']$,

for all $y_1, y_2 \in Y$.

A crossed module morphism $f : (Y, Z, \partial) \to (Y', Z', \partial')$ consists of Lie algebra morphisms f_1 and f_0 such that the following diagram is commutative and preserves the action of Z on Y:

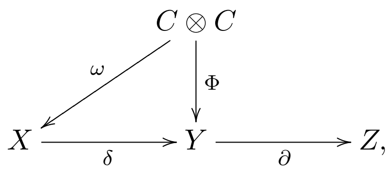


Therefore we can define the category of crossed modules of Lie algebras denoting it as $\mathbf{XMod_L}$. If we fix the bottom part, the Z Lie algebra, then $\mathbf{XMod_L}/\mathbf{Z}$ will be the category of crossed Z-modules.

2.1. Quadratic modules of Lie algebras

We recall the definition of quadratic modules of Lie algebras given [7].

A quadratic module $(\omega, \delta, \partial)$ of Lie algebras consists of Lie algebra homomorphisms as illustrated in the below diagram, satisfying following conditions:



$\mathbf{QM_L1}$: The homomorphism $\partial : Y \to Z$ is a $nil(2)$-module and $Y \twoheadrightarrow C = Y^{cr}/[Y^{cr}, Y^{cr}]$ is defined by $y \mapsto [y]$ and Φ is defined by

$$\Phi([y] \otimes [y']) = \partial(y) *_1 y' - [y, y'],$$

for $y, y' \in Y$,

$\mathbf{QM_L2}$: The boundary Lie homomorphisms composition of ∂ and δ is zero map and $\delta\omega = \Phi$.

$\mathbf{QM_L3}$: X is a Lie Z-algebra, all of the homomorphisms in the diagram are Z-equivariant, and the left Lie algebra action "$*_3$" of Z on X also holds the following equality

$$\partial(y) *_3 x = \omega([\delta(x)] \otimes [y] + [y] \otimes [\delta(x)]),$$

for $x \in X$ and $y \in Y$,

$\mathbf{QM_L4}$:

$$\omega([\delta(x)] \otimes [\delta(x')]) = [x', x],$$

for $x, x' \in X$.

Remark 2.1. It should be noted that $X \xrightarrow{\delta} Y$ is a crossed module, with a left Lie algebra action "$*_2$"

$$y *_2 x = \omega([\delta(x)] \otimes [y]),$$

for each $y \in Y$ and $x \in X$. On the other hand, generally, $Y \xrightarrow{\partial} Z$ is only a $nil(2)$ module.

Remark 2.2. By $\mathbf{QM}_L\mathbf{3}$, we have:

$$\partial(y) *_3 x - y *_2 x = \omega([y] \otimes [\delta(x)]).$$

Lemma 2.3. Let $\mathcal{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ be a quadratic module and consider the "$*_2$" and "$*_3$" Lie algebra actions. Then for all $z \in Z$ and $y_1, y_2, y_3 \in Y$, we have:

$$z *_3 \omega([y_1] \otimes [y_2]) = \omega([z *_1 y_1] \otimes [y_2]) + \omega([y_1] \otimes [z *_1 y_2]), \tag{1}$$

$$\begin{aligned} \omega([[y_1, y_2]] \otimes [y_3]) &= \partial(y_1) *_3 \omega([y_2] \otimes [y_3]) + \omega([y_1] \otimes [[y_2, y_3]]) \\ &\quad -\partial(y_2) *_3 \omega([y_1] \otimes [y_3]) - \omega([y_2] \otimes [[y_1, y_3]]), \end{aligned} \tag{2}$$

$$\omega([y_1] \otimes [[y_2, y_3]]) = y_2 *_2 \omega([y_1] \otimes [y_3]) - y_3 *_2 \omega([y_1] \otimes [y_2]). \tag{3}$$

Example 2.4. Let $Y \xrightarrow{\partial} Z$ be $nil(2)$-module. We can define the following (Id, Φ, ∂) quadratic module

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{Id} & \downarrow{\scriptstyle \Phi} & & \\ C \otimes C & \xrightarrow[\Phi]{} & Y & \xrightarrow[\partial]{} & Z. \end{array}$$

Example 2.5. Let $(\omega, \delta, \partial)$ be a quadratic module. Then we can define the quadratic module $(0, 0, \partial)$ with trivial quadratic map such as

$$1 \xrightarrow{0} Y \xrightarrow{\partial} Z.$$

Example 2.6. If $\mathcal{L} = (X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-]))$ is a quadratic module, then $Im\delta$ is a Lie ideal of Y and we have that

$$Y/Im\delta \xrightarrow{\partial} Z,$$

is an induced crossed module.

Definition 2.7. Let $\mathcal{L} = (\omega, \delta, \partial)$ and $\mathcal{L}' = (\omega', \delta', \partial')$ be two quadratic modules of Lie algebras. A morphism of quadratic modules from $\mathcal{L}$ to $\mathcal{L}'$ is illustrated by the following commutative diagram:

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{\omega} & X & \xrightarrow{\delta} & Y & \xrightarrow{\partial} & Z \\ {\scriptstyle \varphi \otimes \varphi}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C' \otimes C' & \xrightarrow[\omega']{} & X' & \xrightarrow[\delta']{} & Y' & \xrightarrow[\partial']{} & Z', \end{array}$$

where (f_1, f_0) is a morphism of $nil(2)$ modules which induces $\varphi : C \to C'$ and also following equations are satisfied:

$$\begin{gathered} f_1(z *_1 y) = f_0(z) *'_1 f_1(y), \\ f_2(z *_3 x) = f_0(z) *'_3 f_2(x), \\ f_2(\omega([y_1] \otimes [y_2])) = \omega'([f_1(y_1)] \otimes [f_1(y_2)]), \end{gathered}$$

for all $z \in Z$, $y, y_1, y_2 \in Y$ and $x \in X$.

We will denote by $\mathbf{QM_L}$ the category of quadratic modules of Lie algebras and by $\mathbf{QM_L}/(Y \overset{\partial}{\to} Z)$ the subcategory of quadratic modules over fixed $nil(2)$ module (Y, Z, ∂).

Remark 2.8. We can define another canonically associated crossed module as $\widehat{\delta} : X \to Y \rtimes Z, \widehat{\delta}(x) = (\delta(x), 0_Z)$ with $(y, z) \bullet x = y *_2 x + z *_3 x$. $(X, Y \rtimes Z, \widehat{\delta})$ is a crossed modules because of

XMod$_L$1

$$\begin{aligned} \widehat{\delta}((y, z) \bullet x) &= \widehat{\delta}(y *_2 x + z *_3 x) \\ &= (\delta(y *_2 x + z *_3 x), 0_Z) \\ &= (\delta(y *_2 x) + \delta(z *_3 x), 0_Z) \\ &= ([y, \delta(x)] + z *_1 \delta(x) - 0_Z *_1 y, [z, 0_Z]) \\ &= [(y, z), (\delta(x), 0_Z)] \\ &= [(y, z), \widehat{\delta}(x)], \end{aligned}$$

XMod$_L$2

$$\begin{aligned} \widehat{\delta}(x') \bullet x'' &= (\delta(x'), 0_Z) \bullet x'' \\ &= \delta(x') *_2 x'' + 0_z *_3 x'' \\ &= [x', x''], \end{aligned}$$

for each $z \in Z$, $y \in Y$ and $x, x', x'' \in X$.

3. Quasi-Quadratic Modules of Lie Algebras

Definition 3.1. A quasi-quadratic module of Lie algebras is a semi-exact sequence

$$C \xrightarrow{\delta} D \xrightarrow{\partial} E,$$

of E-Lie algebras together with a μ quadratic map

$$\mu([-] \otimes [-]) : B \otimes B \longrightarrow C$$

where $B = D^{cr}/[D^{cr}, D^{cr}]$, such that $\mathcal{QQM}_L1$, $\mathcal{QQM}_L2$, $\mathcal{QQM}_L3$ and $\mathcal{QQM}_L4$ hold:

$\mathcal{QQM}_L1$:

$$\delta\mu([d_1] \otimes [d_2]) = \Phi([d_1] \otimes [d_2]) = \partial(d_1) *_1 d_2 - [d_1, d_2],$$

$\mathcal{QQM}_L2$:

$$\begin{aligned} \mu([[d_1, d_2]] \otimes [d_3]) = {} & \partial(d_1) *_3 \mu([d_2] \otimes [d_3]) + \mu([d_1] \otimes [[d_2, d_3]])| \\ & -\partial(d_2) *_3 \mu([d_1] \otimes [d_3]) - \mu([d_2] \otimes [[d_1, d_3]]), \end{aligned}$$

$\mathcal{QQM}_L3$:

$$\mu([d_1] \otimes [[d_2, d_3]]) = d_2 *_2 \mu([d_1] \otimes [d_3]) - d_3 *_2 \mu([d_1] \otimes [d_2]),$$

$\mathcal{QQM}_L4$:

$$[\mu([d_1] \otimes [d_2]), \partial(d_1) *_3 (d_2 *_2 c)] = \mu([\partial(d_1) *_1 [d_2, \delta c]] \otimes [\delta\mu([d_1] \otimes [d_2])]),$$

for all $d_1, d_2, d_3 \in D$ and $c \in C$.

Quasi-quadratic module morphisms are defined in the same way as quadratic module morphisms. We will denote the category of quasi-quadratic module of Lie algebras by $\mathcal{QQM}_L$.

Lemma 3.2. Any quadratic module is a quasi-quadratic module.

Proof. Let $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\big)$ be a quadratic module of Lie algebras. We need to show that it satisfies the quasi-quadratic module axioms. As can be seen, the quasi-quadratic module $\mathcal{QQM}_L1$ axiom and the quadratic module $\mathbf{QM_L}1$ axiom are the same. In addition, $\mathcal{QQM}_L2$ and $\mathcal{QQM}_L3$ axioms are provided due to Lemma 2.3. Therefore, we only need to provide the axiom $\mathcal{QQM}_L4$.

$\mathcal{QQM}_L4$:

$$\begin{aligned}
&[\mu([d_1] \otimes [d_2]), \partial(d_1) *_3 (d_2 *_2 c)] \\
&\qquad = \mu([\delta(\partial(d_1) *_3 (d_2 *_2 c))] \otimes [\delta\mu([d_1] \otimes [d_2])]) \\
&\qquad = \mu([\partial(d_1) *_1 \delta(d_2 *_2 c)] \otimes [\delta\mu([d_1] \otimes [d_2])]) \\
&\qquad = \mu([\partial(d_1) *_1 [d_2, \delta(x)]] \otimes [\partial(d_1) *_1 d_2 - [d_1, d_2]]),
\end{aligned}$$

for all $d_1, d_2 \in D$ and $c \in C$. □

Now, we will construct a quadratic module of Lie algebras associated with a quasi-quadratic module of Lie algebras:

Let $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\big)$ be quasi-quadratic module and I be defined as the Lie subalgebra of C generated by the elements of the form:

- $d\#_1 c = \partial(d) *_3 c - \mu([d] \otimes [\delta(c)]) - \mu([\delta(d)] \otimes [c])$,
- $c_1 \#_2 c_2 = [c_2, c_1] - \mu([\delta(c_1)] \otimes [\delta(c_2)])$,

$d \in D$ and $c, c_1, c_2 \in C$.

Lemma 3.3. For any quasi-quadratic module $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\big)$, the Lie subalgebra I of C is an E-invariant Lie ideal.

Proof. I is an E-invariant Lie ideal of C:

$$\begin{aligned}
e *_3 (d\#_1 c) &= e *_3 (\partial(d) *_3 c - \mu([d] \otimes [\delta(c)]) - \mu([\delta(c)] \otimes [d])) \\
&= e *_3 (\partial(d) *_3 c) - e *_3 \mu([d] \otimes [\delta(c)]) - e *_3 \mu([\delta(c)] \otimes [d]) \\
&= [e, \partial(d)] *_3 c + \partial(d) *_3 (e *_3 c) - e *_3 \mu([d] \otimes [\delta(c)]) \\
&\quad - e *_3 \mu([\delta(c)] \otimes [d]) \\
&= \partial(e *_1 d) *_3 c - \mu([e *_1 d] \otimes [\delta(c)]) - \mu([\delta(c)] \otimes [e *_1 d]) \\
&\quad + \partial(d) *_3 (e *_3 c) - \mu([d] \otimes [\delta(e *_3 c)]) - \mu([\delta(e *_3 c)] \otimes [d]) \\
&= (e *_1 d\#_1 c) + (d\#_1 e *_3 c),
\end{aligned}$$

Also:

$$\begin{aligned}
e *_3 (c_1 \#_2 c_2) &= e *_3 ([c_2, c_1] - \mu([\delta(c_1)] \otimes [\delta(c_2)])) \\
&= e *_3 [c_2, c_1] - e *_3 \mu([\delta(c_1)] \otimes [\delta(c_2)]) \\
&= [e *_3 c_2, c_1] + [c_2, e *_3 c_1] - \mu([e *_1 \delta(c_1)] \otimes [\delta(c_2)]) \\
&\quad - \mu([\delta(c_1)] \otimes [e *_1 \delta(c_2)]) \\
&= [c_2, e *_3 c_1] - \mu([\delta(e *_3 c_1)] \otimes [\delta(c_2)]) + [e *_3 c_2, c_1] \\
&\quad - \mu([\delta(c_1)] \otimes [\delta(e *_3 c_2)]) \\
&= (e *_3 c_1 \#_2 c_2) + (c_1 \#_2 e *_3 c_2).
\end{aligned}$$

□

For any quasi-quadratic module $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\big)$, we can define the quotient Lie algebra $C^{cr} = C/I$ and quotient homomorphism:

$$\delta^{cr} : C^{cr} = C/I \longrightarrow D$$

$$(c + I) \longmapsto \delta^{cr}((c + I)) = \delta(c).$$

Lemma 3.4. Let $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\big)$ be quasi-quadratic module, then

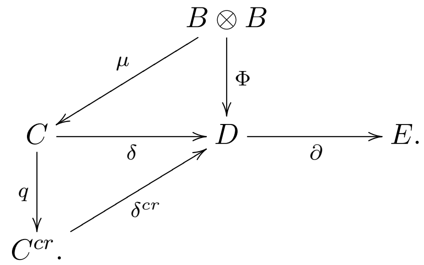
1-) the induced map gives a quadratic module

$$\mathcal{L}^{cr} = \left(C^{cr} \xrightarrow{\delta^{cr}} D \xrightarrow{\partial} E \ , \mu^{cr}([-]\otimes[-]) \right)$$

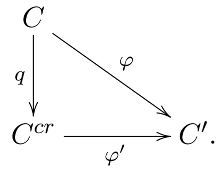
where the quadratic map $\mu^{cr}([-]\otimes[-])$ is the composition $q\mu = \mu^{cr}$,

$$B\otimes B \xrightarrow{\mu} C \xrightarrow{q} C^{cr},$$

for all $d_1, d_2 \in D$; $\mu^{cr}([d_1]\otimes[d_2]) = (\mu([d_1]\otimes[d_2]) + I)$. All these data are summarized diagrammatically as follows:



2-) Let $\mathcal{L}' = \left(C' \xrightarrow{\delta'} D \xrightarrow{\partial} E, \mu([-]\otimes[-])\right)$ be an object in $\mathbf{QM}_L/(D \xrightarrow{\partial} E)$. If $\varphi : C \longrightarrow C'$ is a morphism of quasi-quadratic module over $\left(D \xrightarrow{\partial} E\right)$, then φ determines a unique Lie morphism $\varphi' : C^{cr} \longrightarrow C'$ such that $\varphi' q = \varphi$:



Proof. 1-) For $d\#_1 c$ and $c_1\#_2 c_2$ elements in C, $\delta(d\#_1 c) = \delta(c_1\#_2 c_2) = 0$;

$$\begin{aligned} \delta(d\#_1 c) &= \delta(\partial(d) *_3 c - \mu([d]\otimes[\delta(c)]) - \mu([\delta(c)]\otimes[d])) \\ &= \delta(\partial(d) *_3 c) - \delta(\mu([d]\otimes[\delta(c)])) - \delta(\mu([\delta(c)]\otimes[d])) \\ &= \partial(d) *_1 \delta(c) - \partial(d) *_1 \delta(c) + [d, \delta(c)] - \delta(d *_2 c) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \delta(c_1\#_2 c_2) &= \delta([c_2, c_1] - \mu([\delta(c_1)]\otimes[\delta(c_2)])) \\ &= \delta([c_2, x_1]) - \delta(\mu([\delta(c_1)]\otimes[\delta(c_2)])) \\ &= \delta([c_2, x_1]) - \delta(\delta(c_2) *_2 c_1) \\ &= \delta([c_2, x_1]) - \delta([c_2, x_1]) \\ &= 0. \end{aligned}$$

We have $\delta(I) = 0$, for all $d \in D$ and $c, c_1, c_2 \in C$.
Also, this construct needs to satisfy the $\mathbf{QM_L}$ axioms:

$\mathbf{QM_L 1}$ The axiom $\mathbf{QM_L 1}$ is provided directly.

$\mathbf{QM_L 2}$:

$$\begin{aligned} \delta^{cr}\mu^{cr}([d_1]\otimes[d_2]) &= \delta^{cr}(\mu([d_1]\otimes[d_2]) + I) \\ &= \delta\mu([d_1]\otimes[d_2]) \\ &= \partial(d_1) *_1 d_2 - [d_1, d_2], \end{aligned}$$

$\mathbf{QM_L 3}$:

$$\begin{aligned} &\mu^{cr}([\delta^{cr}(c+I)]\otimes[d] + [d]\otimes[\delta^{cr}(c+I)]) \\ &= \mu^{cr}([\delta(c)]\otimes[d] + [d]\otimes[\delta(c)]) \\ &= (\mu([\delta(c)]\otimes[d] + [d]\otimes[\delta(c)]) + I) \\ &= \mu([\delta(c)]\otimes[d]) + \mu([d]\otimes[\delta(c)]) + I \\ &= \partial(y) *_3 c + I, \end{aligned}$$

$\mathbf{QM_L4}$:

$$\begin{aligned} \mu^{cr}([\delta^{cr}(c_1+I)]\otimes[\delta^{cr}(c_2+I)]) &= \mu^{cr}([\delta(c_1)]\otimes[\delta(c_2)]) \\ &= (\mu([\delta(c_1)]\otimes[\delta(c_2)])+I) \\ &= [c_2,c_1]+I, \end{aligned}$$

for each $d, d_1, d_2 \in D$, $c, c_1, c_2 \in C$.

2-) It is clear since all elements in the form $d\#_1c$ and $c_1\#_2c_2$ vanish.

□

Thus, we can define a functor from the association of the quadratic module $\mathcal{L}^{cr} = \big(C^{cr} \xrightarrow{\delta^{cr}} D \xrightarrow{\partial} E, \mu^{cr}([-]\otimes[-])\big)$ to a quasi-quadratic module $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-]\otimes[-])\big)$ as follows:

$$(-)^{cr} : \mathcal{QQM}_L \longrightarrow \mathbf{QM_L}.$$

Therefore, a morphism

$$\begin{array}{ccccc} C & \xrightarrow{\delta} & D & \xrightarrow{\partial} & E \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C'' & \xrightarrow[\delta'']{} & D'' & \xrightarrow[\partial'']{} & E'', \end{array}$$

gives a (f^{cr}, f_1, f_0) of the associated quadratic modules. In more detail, $f^{cr} : C^{cr} \longrightarrow C''^{cr}$ is well defined by $f^{cr}(c+I) = f_2(c) + I''$, since $f_2(d\#_1c) = f_1(d)\#_1f_2(c)$ and $f_2(c_1\#_2c_2) = f_2(c_1)\#_2f_2(c_2))$. Thus, it satisfies the functorial rules. As a result, we get the following adjunction

$$\mathcal{QQM}_L \underset{\hookleftarrow}{\overset{(-)^{cr}}{\rightleftarrows}} \mathbf{QM_L}.$$

4. Conclusion

In this paper, the category of quasi-quadratic modules of Lie algebras has been introduced. It is concluded that the existence of an adjunction between this category and that of quadratic modules of Lie algebras is also valid as in the category of 2-crossed modules of groups and Lie algebras. Thus, it can be expected that quasi-quadratic modules are a useful tool in the construction of some categorical content, such as coproduct objects in the category of quadratic modules for Lie algebras. Furthermore, in future research, the answer to the question of which object corresponds to “quasi” in the category of crossed squares, which is another two-dimensional analogous of crossed modules, can be searched.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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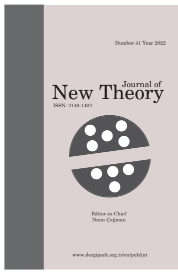

Soft A-Metric Spaces

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Abstract — This paper draws on the theory of soft A-metric space using soft points of soft sets and the concept of A-metric spaces. This new space has great importance as a new type of generalisation of metric spaces since it includes various known metric spaces. In this paper, we introduce the concept of soft A-metric space and examine the relations with known spaces. Then, we examine various basic properties of these spaces: soft Hausdorffness, a soft Cauchy sequence, and soft convergence.

Keywords — *Soft metric, A-metric, soft A-metric*

Mathematics Subject Classification (2020) — 54E45, 47H10

1. Introduction

Metric spaces have major importance in both mathematics and other sciences. The first study of metric spaces was initiated by Fréchet [1] at the beginning of the 20th century. Since that day, a great many generalisations of metric space have been obtained by different authors. Firstly, in 1963, 2-metric spaces were studied by Gahler [2]. In 1984, Dhage [3] introduced the notion of D-metric using basic modifications in the definition of 2-metric. After that, Mustafa and Sims [4] initiated the theory of G-metric since they found various mistakes in the definition of open sets in D-metric spaces. Later, because of the same reasons, Sedghi et al. [5] gave the theory of D^*-metric space. In 2012, Sedghi et al. [6] introduced the structure of S-metric spaces by modifying some conditions in the definition of D^*-metric spaces. Finally, Ahmed et al. [7] examined A-metric spaces as a general version of S-metric spaces.

Soft set theory was presented as a significant tool by Molodtsov [8] for dealing with uncertainties. Maji et al. [9] examined the primary properties of this space. Babitha and Sunil [10] investigated soft set relations and functions in this concept. Gündüz and Poşul [11] introduced the probabilistic soft sets. Many researchers applied this new concept to their studies [12–24].

The concept of soft metric space was studied by Das and Samanta [25] as a generalisation of metric spaces in 2013. This new metric caught the attention of authors, and many studies have been done on this topic [26–32].

In this study, we work on the notion of soft A-metric space. We design this theory using the soft points of soft sets and the concept of A-metric spaces. This study gives a new general form of metric spaces, and the resulting structure is a larger family from soft metric spaces. This paper is organised into 4 sections. In section 2, we recall some important definitions in soft set theory. In section 3, we

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introduce the concept of soft A-metric space as a new generalisation of metric spaces and examine the relations of soft metric spaces, soft S-metric spaces and soft A-metric spaces. After that, we present various important properties of this space: soft Hausdorffness, being a soft Cauchy sequence, soft convergence, and soft completeness. In section 4, we describe our results and point to the studies that can be done about this new theory.

2. Preliminaries

This section provides various basic definitions and properties before moving on to the main topic.

Definition 2.1. [8] Consider that X is an initial universe, E is the set of all the parameters, and $P(X)$ is the power set of X. Define a mapping $F{:}E \to P(X)$. Then, an ordered pair (F, E) is called a soft set over X. In that case, it can be thought that if (F, E) is a soft set over X, then it is a parameterized family of subsets of the set X.

From here, assume that X is an initial universe, E is the set of all the parameters, $P(X)$ is the power set of X, and (F, E) and (G, E) are soft sets over X.

Definition 2.2. [12] (F, E) is a soft subset of (G, E), if $F(a) \subseteq G(a)$, for every $a \in E$. This is written by $(F, E)\widetilde{\subseteq}(G, E)$. In addition, (G, E) is a soft superset of (F, E).

Definition 2.3. [12] (F, E) and (G, E) are soft equal, if $(F, E)\widetilde{\subseteq}(G, E)$ and $(G, E)\widetilde{\subseteq}(F, E)$.

Definition 2.4. [24] A soft set (H, E) is called the soft intersection of (F, E) and (G, E) over X, if $H(a) = F(a) \cap G(a)$, for every $a \in E$. This is written by $(H, E) = (F, E)\widetilde{\cap}(G, E)$.

Definition 2.5. [24] A soft set (U, E) is called the soft union of (F, E) and (G, E) over X, if $U(a) = F(a) \cup G(a)$, for every $a \in E$. This is written by $(U, E) = (F, E)\widetilde{\cup}(G, E)$.

Definition 2.6. [9] A soft set (F, E) is null soft set over X, if $F(a) = \varnothing$, for every $a \in E$. This is written by Φ.

Definition 2.7. [9] A soft set (F, E) is absolute soft set over X, if $F(a) = X$, for every $a \in E$. This is written by $\widetilde{X}$.

Definition 2.8. [24] A soft set (K, E) is called the soft difference of (F, E) and (G, E) over X, if $K(a) = F(a)\backslash G(a)$, for every $a \in E$. This is written by $(K, E) = (F, E)\widetilde{\backslash}(G, E)$.

Definition 2.9. [24] Consider that a mapping $F^c : E \to P(X)$ defined by $F^c(a) = X\backslash F(a)$, for every $a \in E$. Then, $(F, E)^c = (F^c, E)$ is called the soft complement of (F, E).

Definition 2.10. [15] Let $\widetilde{\tau}$ be the collection of soft sets over X. $\widetilde{\tau}$ is called a soft topology on X, if the followings hold:

i. Φ and $\widetilde{X}$ belong to $\widetilde{\tau}$.

ii. The intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

iii. The union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The ordered triplet $(X, \widetilde{\tau}, E)$ is called a soft topological space over X.

Definition 2.11. [15] Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. Then, elements of $\widetilde{\tau}$ are called soft open sets in X. Moreover, (F, E) is a soft closed set in X, if $(F, E)^c$ belongs to $\widetilde{\tau}$.

Definition 2.12. [25] A soft set (F, E) is called a soft point, if $F(a) = \{x\}$ and $F(a') = \varnothing$, for the element $a \in E$ and for every $a' \in E\backslash\{a\}$. The soft point is written by (x_a, E) or x_a. Note that every soft set can be defined as a union of soft points.

From now on, the collection of all soft points of the absolute soft set will be denoted by $SP(\widetilde{X})$.

Definition 2.13. [25] Let x_a and $y_{a'}$ be soft points over X. It is said to be x_a and $y_{a'}$ are equal soft points, if $x = y$ and $a = a'$.

Definition 2.14. [25] Let x_a be a soft point over X. If $x_a(a)$ is an element of $F(a)$, i.e., $\{x\} \subseteq F(a)$, then x_a belongs to (F, E). This is written by $x_a \widetilde{\in} (F, E)$.

Proposition 2.15. [25] The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it.

Proposition 2.16. [25] Let x_a be a soft point over X. Then,

i. $x_a \widetilde{\in} (F, E) \Leftrightarrow x_a \widetilde{\notin} (F, E)^c$.

ii. $x_a \widetilde{\in} (F, E) \widetilde{\cup} (G, E) \Leftrightarrow x_a \widetilde{\in} (F, E)$ or $x_a \widetilde{\in} (G, E)$.

iii. $x_a \widetilde{\in} (F, E) \widetilde{\cap} (G, E) \Leftrightarrow x_a \widetilde{\in} (F, E)$ and $x_a \widetilde{\in} (G, E)$.

Remark 2.17. [25] The collection of all soft points of (F, E) will be expressed by $SP(F, E)$.

Definition 2.18. [25] Consider that $\mathbb{R}$ is the set of real numbers. In addition, the collection of all the non-empty bounded subset of $\mathbb{R}$ stands for $B(\mathbb{R})$. A soft real set is also denoted by (F, E), where F is a mapping from E to $B(\mathbb{R})$. If (F, E) has a only one element, then it is a soft real number and this is written by $\widetilde{r}, \widetilde{s}, \widetilde{p}$ etc. In this study, the soft real number $\widetilde{r}$ satisfies $\widetilde{r}(a) = r$, for all $a \in E$.

Definition 2.19. [25] Consider soft real numbers $\widetilde{r}$ and $\widetilde{s}$. Then, for all $a \in E$, the followings hold:

i. $\widetilde{r} \widetilde{\leq} \widetilde{s}$, if $\widetilde{r}(a) \leq \widetilde{s}(a)$.

ii. $\widetilde{r} \widetilde{\geq} \widetilde{s}$, if $\widetilde{r}(a) \geq \widetilde{s}(a)$.

iii. $\widetilde{r} \widetilde{<} \widetilde{s}$, if $\widetilde{r}(a) < \widetilde{s}(a)$.

iv. $\widetilde{r} \widetilde{>} \widetilde{s}$, if $\widetilde{r}(a) > \widetilde{s}(a)$.

Definition 2.20. [25] Let $\mathbb{R}(E)^*$ be the set of all the positive soft real numbers. A soft metric on $\widetilde{X}$ is a mapping $d : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$ that satisfies the following conditions: for every soft points $x_a, y_b, z_c \in SP(\widetilde{X})$,

i. $d(x_a, y_b) \geq \widetilde{0}$.

ii. $d(x_a, y_b) = \widetilde{0}$ if and only if $x_a = y_b$.

iii. $d(x_a, y_b) = d(y_b, x_a)$.

iv. $d(x_a, z_c) \leq d(x_a, y_b) + d(y_b, z_c)$.

Then, the ordered triplet $\left(\widetilde{X}, d, E\right)$ is called a soft metric space.

Definition 2.21. [25] Let $\left(\widetilde{X}, d, E\right)$ be a soft metric space, $\left\{x^k_{a_k}\right\}$ be a soft sequence of soft points in $\left(\widetilde{X}, d, E\right)$ and y_b is a soft point over $\widetilde{X}$. Then,

i. $\left\{x^k_{a_k}\right\}$ is called a soft convergent sequence, if for $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $d\left(x^k_{a_k}, y_b\right) < \widetilde{\varepsilon}$, for each natural number $k \geq k_0$. Moreover, it is said that $\left\{x^k_{a_k}\right\}$ converges to y_b.

ii. $\left\{x^k_{a_k}\right\}$ is called a soft Cauchy sequence, if for $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $d\left(x^k_{a_k}, x^m_{a_m}\right) < \widetilde{\varepsilon}$, for each natural numbers $k, m \geq k_0$.

iii. If every soft Cauchy sequence is soft convergent in a soft metric space, then this space is called soft complete metric space.

Definition 2.22. [25] Let $\left(\widetilde{X}, d, E\right)$ be a soft metric space. For a soft real number $\widetilde{r} > \widetilde{0}$ and a soft point $x_a \in SP(\widetilde{X})$, the soft open ball $B\left(x_a, \widetilde{r}\right)$ and soft closed ball $\mathbf{B}\left(x_a, \widetilde{r}\right)$ with center x_a and a radius $\widetilde{r}$ are defined as follows:

$$\begin{aligned} B\left(x_a, \widetilde{r}\right) &= \left\{y_b \in SP(\widetilde{X}) : d(y_b, x_a) < \widetilde{r}\right\} \\ \mathbf{B}\left(x_a, \widetilde{r}\right) &= \left\{y_b \in SP(\widetilde{X}) : d\left(y_b, x_a\right) \leq \widetilde{r}\right\} \end{aligned}$$

Definition 2.23. [25] A soft metric space $\left(\widetilde{X}, d, E\right)$ is soft Hausdorff space, if for every different soft points x_a, y_b in $SP(\widetilde{X})$, there exist two soft open balls $B\left(x_a, \widetilde{r}\right)$ and $B\left(y_b, \widetilde{r}\right)$ such that their soft intersection is null soft set.

Definition 2.24. [32] A soft S-metric on $SP(\widetilde{X})$ is a mapping $S : \left(SP(\widetilde{X})\right)^3 \to [0, \infty)$ that satisfies the following conditions: for every soft points x_a, y_b, z_c, t_d in $SP(\widetilde{X})$,

i. $S\left(x_a, y_b, z_c\right) = \widetilde{0} \Leftrightarrow x_a = y_b = z_c$.

ii. $S\left(x_a, y_b, z_c\right) \leq S(x_a, x_a, t_d) + S\left(y_b, y_b, t_d\right) + S\left(z_c, z_c, t_d\right)$.

The ordered pair (X, S) is called a soft S-metric space.

Definition 2.25. [7] Let $X \neq \varnothing$ be a set and $n \geqslant 2$ be a natural number. A A-metric on X is a mapping $A : X^n \to [0, \infty)$ that satisfies the following conditions: for every $x_i \in X, i = 1, 2, ..., n$,

i. $A\left(x_1, x_2, \ldots, x_{n-1}, x_n\right) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_n$.

ii. $A\left(x_1, x_2, \ldots, x_{n-1}, x_n\right) \leq A\left(x_1, x_1, \ldots, x_1, a\right) + A\left(x_2, x_2, \ldots, x_2, a\right) + ... + A\left(x_n, x_n, \ldots, x_n, a\right)$.

The ordered pair (X, A) is called a A-metric space.

3. Soft A-Metric Spaces

This section presents the theory of soft A-metric space, which uses soft points of soft sets and A-metric spaces. In this study, $\mathbb{R}(E)^*$ stands for the set of all the positive soft real numbers.

Definition 3.1. If a mapping which is defined from $\left(SP(\widetilde{X})\right)^n$ to $\mathbb{R}(E)^*$ satisfies the followings, then it is said to be a soft A-metric on $SP(\widetilde{X})$, where $n \geqslant 2$ is a natural number: for each soft points $x_{ia_i}, y_b \in SP(\widetilde{X}), i = 1, 2, ..., n$,

S1. $A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{n-1a_{n-1}}, x_{na_n}\right) = \widetilde{0} \Leftrightarrow x_{1a_1} = x_{2a_2} = \ldots = x_{na_n}$.

S2. $$\begin{aligned} A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{n-1a_{n-1}}, x_{na_n}\right) \leq\ & A\left(x_{1a_1}, x_{1a_1}, \ldots, x_{1a_1}, y_b\right) + A\left(x_{2a_2}, x_{2a_2}, \ldots, x_{2a_2}, y_b\right) \\ & + ... + A\left(x_{na_n}, x_{na_n}, \ldots, x_{na_n}, y_b\right). \end{aligned}$$

Then, the ordered triplet $(\widetilde{X}, A, E)$ is said to be a soft A-metric space.

Remark 3.2. Note that if $n = 3$ is taken in the definition of soft A-metric spaces, then the definition of the soft S-metric spaces is obtained. Similarly, if $n = 2$ is taken in the definition of soft A-metric spaces, then the definition of the soft metric spaces is obtained. Therefore, soft A-metric space is a general version of soft S-metric spaces and soft metric spaces. In other words,

i. For $n = 3$, every soft A-metric space is a soft S-metric space.

ii. For $n = 2$, every soft A-metric space is a soft metric space.

Example 3.3. Let $E \neq \varnothing$ be a set of parameters, $E \subset \mathbb{R}$ and d be an ordinary metric on a non-empty set $X \subset \mathbb{R}$. Then, $d_A\left(x_{ia_i}, y_{ib_i}\right) = |a_i - b_i| + d\left(x_i, y_i\right), i = 1, 2, ..., n,$ is a soft metric [32]. Now, we define a mapping $A : \left(SP(\widetilde{X})\right)^n \to \mathbb{R}(E)^*$ as follow:

$$\begin{aligned} A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{n-1a_{n-1}}, x_{na_n}\right) &= d_A\left(x_{1a_1}, x_{na_n}\right) + d_A\left(x_{2a_2}, x_{na_n}\right) \\ &\quad + \ldots + d_A\left(x_{n-1a_{n-1}}, x_{na_n}\right) \end{aligned}$$

for all $x_{ia_i} \in SP(\widetilde{X})$ and $i = 1, 2, ..., n$. Then, A is a soft A-metric on $SP(\widetilde{X})$. For this, let's show that the condition S2 is satisfied:

$$\begin{aligned} A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{na_n}\right) &= d_A\left(x_{1a_1}, x_{na_n}\right) + d_A\left(x_{2a_2}, x_{na_n}\right) + \ldots + d_A(x_{n-1a_{n-1}}, x_{na_n}) \\ &= |a_1 - a_n| + |a_2 - a_n| + ... + |a_{n-1} - a_n| + d\left(x_1, x_n\right) + d\left(x_2, x_n\right) \\ &\quad +... + d\left(x_{n-1}, x_n\right) \\ &\leq |a_1 - b| + |b - a_n| + |a_2 - b| + |b - a_n| + ... + |a_{n-1} - b| + |b - a_n| \\ &\quad + d\left(x_1, y\right) + d\left(y, x_n\right) + d\left(x_2, y\right) + d\left(y, x_n\right) + ... + d\left(x_{n-1}, y\right) + d\left(y, x_n\right) \\ &\leq |a_1 - b| + |a_1 - b| + ... + |a_1 - b| + d\left(x_1, y\right) + d\left(x_1, y\right) + ... + d\left(x_1, y\right) \\ &\quad + |a_2 - b| + |a_2 - b| + ... + |a_2 - b| + d\left(x_2, y\right) + d\left(x_2, y\right) + ... + d\left(x_2, y\right) \\ &\quad +... + |a_n - b| + |a_n - b| + ... + |a_n - b| + d\left(x_n, y\right) + d\left(x_n, y\right) \\ &\quad +... + d\left(x_n, y\right) \\ &= A\left(x_{1a_1}, x_{1a_1}, \ldots, x_{1a_1}, y_b\right) + A\left(x_{2a_2}, x_{2a_2}, \ldots, x_{2a_2}, y_b\right) \\ &\quad +... + A\left(x_{na_n}, x_{na_n}, \ldots, x_{na_n}, y_b\right). \end{aligned}$$

Remark 3.4. It is obvious that every one of soft A-metrics is a family of parametrized A-metric. Namely, if we consider a soft A-metric space $\left(\widetilde{X}, A, E\right)$, then (X, A_a) is an A-metric space, for every a in E. But it is not true converse of this statement. Here, A_a stands for the $A-$metric for only parameter a and (X, A_a) is a crisp $A-$metric space.

Example 3.5. Let $E = \mathbb{R}$ and $\left(X, \widetilde{A}\right)$ be an A-metric space. Define a mapping

$$A : \left(SP(\widetilde{X})\right)^n \to \mathbb{R}(E)^*$$

$$A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{na_n}\right) = \widetilde{A}\left(x_1, x_2, \ldots, x_{n-1}, x_n\right)^{1+|a_1-a_2|+|a_1-a_3|+\ldots+|a_1-a_n|}$$

for all $x_{ia_i} \in SP(\widetilde{X})$ and $i = 1, 2, ..., n$. Then, for every $a \in \mathbb{R}$, A_a is an A-metric on X, but A is not a soft A-metric on $SP(\widetilde{X})$.

Lemma 3.6. Let A be a soft A-metric on $SP(\widetilde{X})$. Then,

$$A\left(x_a, x_a, \ldots, x_a, y_b\right) = A\left(y_b, y_b, \ldots, y_b, x_a\right)$$

Proof. Because of conditions S1 and S2 in the definition of soft A-metrics,

$$\begin{aligned} A\left(x_a, x_a, \ldots, , x_a, y_b\right) &\leq (n-1)\, A\left(x_a, x_a, \ldots, x_a, x_a\right) + A\left(y_b, y_b, \ldots, y_b, x_a\right) \\ &= A\left(y_b, y_b, \ldots, y_b, x_a\right) \end{aligned}$$

Thus,

$$A\left(x_a, x_a, \ldots, , x_a, y_b\right) \leq A\left(y_b, y_b, \ldots, y_b, x_a\right) \tag{1}$$

Similarly,

$$\begin{aligned} A\left(y_b, y_b, \ldots, y_b, x_a\right) &\leq (n-1)\, A\left(y_b, y_b, \ldots y_b, y_b\right) + A\left(x_a, x_a, \ldots, x_a, y_b\right) \\ &= A\left(x_a, x_a, \ldots, x_a, y_b\right) \end{aligned}$$

Therefore,

$$A\left(y_b, y_b, \ldots, y_b, x_a\right) \leq A\left(x_a, x_a, \ldots, x_a, y_b\right) \tag{2}$$

Hence, from inequality (1) and (2),

$$A\left(x_a, x_a, \ldots, x_a, y_b\right) = A\left(y_b, y_b, \ldots, y_b, x_a\right)$$

□

Definition 3.7. Let A be a soft A-metric on $SP(\widetilde{X})$. The soft open ball $B_A\left(x_a, \widetilde{r}\right)$ is defined as follows:

$$B_A\left(x_a, \widetilde{r}\right) = \left\{y_b \in SP(\widetilde{X}) : A\left(y_b, y_b, \ldots, y_b, x_a\right) < \widetilde{r}\right\}$$

where $x_a \in SP(\widetilde{X})$ is the center of the soft open ball and the non-negative soft real number $\widetilde{r}$ is the radius of the soft open ball. Moreover,

$$\mathbf{B}_A\left(x_a, \widetilde{r}\right) = \left\{y_b \in SP(\widetilde{X}) : A\left(y_b, y_b, \ldots, y_b, x_a\right) \leq \widetilde{r}\right\}$$

is the soft closed ball with the center x_a and the radius $\widetilde{r}$.

Example 3.8. Let $n = 5$ in the definition of soft A-metric spaces, $E = \mathbb{Z}$, and $X = \mathbb{R}^n$. Denote

$$\begin{aligned} A\left(x_{1a_1}, x_{2a_2}, x_{3a_3}, x_{4a_4}, x_{5a_5}\right) &= \left|a_1 - a_5\right| + \left|a_2 - a_5\right| + \left|a_3 - a_5\right| + \left|a_4 - a_5\right| + d\left(x_1, x_5\right) \\ &\quad + d\left(x_2, x_5\right) + d\left(x_3, x_5\right) + d\left(x_4, x_5\right) \end{aligned}$$

for all $x_{ia_i} \in SP(\widetilde{X}), i = 1, 2, ..., 5$. Then, for $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^5$,

$$\begin{aligned} B_A\left(\theta_0, \widetilde{9}\right) &= \left\{y_b \in SP(\widetilde{X}) : A\left(y_b, y_b, y_b, y_b, \theta_0\right) < \widetilde{9}\right\} \\ &= \left\{y_b \in SP(\widetilde{X}) : 4\left|b\right| + 4d\left(y, \theta\right) < \widetilde{9}\right\} \\ &= \left\{y_b \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{9}}{4} - \left|b\right|\right\} \\ &= \left\{y_b \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{9} - 4\left|b\right|}{4}\right\} \\ &= \left\{y_0 \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{9}}{4}\right\} \cup \left\{y_1 \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{5}}{4}\right\} \\ &\quad \cup \left\{y_2 \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{1}}{4}\right\} \cup \left\{y_{-1} \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{5}}{4}\right\} \\ &\quad \cup \left\{y_{-2} \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{1}}{4}\right\} \end{aligned}$$

Definition 3.9. Let $(\widetilde{X}, A, E)$ be a soft A-metric space and (F, E) be a soft set on X. If, for all $x_a \in (F, E)$, there exists a $\widetilde{r} > \widetilde{0}$ such that $B_A\left(x_a, \widetilde{r}\right) \subset SP\left(F, E\right)$, then (F, E) is said to be a soft open set in $\left(\widetilde{X}, A, E\right)$.

Proposition 3.10. The soft open ball $B_A\left(x_a, \widetilde{r}\right)$ is a soft open set in a soft A-metric space $\left(\widetilde{X}, A, E\right)$.

Proof. Let $y_b \in B_A(x_a, \widetilde{r})$. Then, $A(y_b, y_b, \ldots, y_b, x_a) < \widetilde{r}$. Let $\widetilde{d} = A(x_a, x_a, \ldots, x_a, y_b)$ and $\widetilde{r}'(e) = \frac{\widetilde{r}(e)-\widetilde{d}}{n-1}$, for all $e \in E$. We claim that $B_A(y_b, \widetilde{r}') \subset B_A(x_a, \widetilde{r})$. For this, let $z_c \in B_A(y_b, \widetilde{r}')$. Then, $A(z_c, z_c, \ldots, z_c, y_b) < \widetilde{r}'$. Owing to the condition S2 in the definition of soft A-metrics,

$$\begin{aligned} A(z_c, z_c, \ldots, z_c, x_a) &\leq A(z_c, z_c, \ldots, z_c, y_b) + A(z_c, z_c, \ldots, z_c, y_b) \\ &\quad + \ldots + A(z_c, z_c, \ldots, z_c, y_b) + A(x_a, x_a, \ldots, x_a, y_b) \\ &= (n-1)\, A(z_c, z_c, \ldots, z_c, y_b) + A(x_a, x_a, \ldots, x_a, y_b) \\ &< (n-1)\, \widetilde{r}' + \widetilde{d} \\ &= \widetilde{r} \end{aligned}$$

Then, $z_c \in B_A(x_a, \widetilde{r})$ and so, $B_A(y_b, \widetilde{r}') \subset B_A(x_a, \widetilde{r})$. □

Theorem 3.11. Every soft A-metric space produces a soft topology as follows:

$$\tau = \left\{(F,E) : \text{ For every } x_a \in SP(\widetilde{X}), \text{ there exists a } \widetilde{r} > \widetilde{0} \text{ such that } B_A(x_a, \widetilde{r}) \subset SP(F,E)\right\}$$

This topology is said to be soft topology produced by soft A-metric.

Proof. Firstly, we will show that the intersection of two open soft sets is also a soft open set. Let us consider the soft open sets (F,E) and (G,E). Let $x_a \in (F,E)\widetilde{\cap}(G,E)$. Then, since $x_a \in (F,E)$ and $x_a \in (G,E)$, there exists a $\widetilde{r_1} > \widetilde{0}$ such that $B_A(x_a, \widetilde{r_1}) \subset SP(F,E)$ and there exists a $\widetilde{r_2} > \widetilde{0}$ such that $B_A(x_a, \widetilde{r_2}) \subset SP(G,E)$. Take $\widetilde{r}(e) = \min\{\widetilde{r_1}(e), \widetilde{r_2}(e)\}$, for all $e \in E$. Hence, $B_A(x_a, \widetilde{r}) \subset B_A(x_a, \widetilde{r_1})$ and $B_A(x_a, \widetilde{r}) \subset B_A(x_a, \widetilde{r_2})$. Then, we have

$$x_a \in B_A(x_a, \widetilde{r}) \subset B_A(x_a, \widetilde{r_1}) \cap B_A(x_a, \widetilde{r_2}) \subset SP(F,E) \cap SP(G,E)$$

Thus, $(F,E)\widetilde{\cap}(G,E)$ is a soft open set. Secondly, we will show that the arbitrary union of soft open sets is also a soft open set. Let (F_λ, E) be a soft open set, for all λ in I, an index set. Let $x_a \in \widetilde{\bigcup_\lambda}(F_\lambda, E)$. Then, $x_a \in (F_{\lambda_0}, E)$, for a λ_0 in I. Since (F_{λ_0}, E) is a soft open set, there exists a $\widetilde{r} > \widetilde{0}$ such that $B_A(x_a, \widetilde{r}) \subset SP(F_{\lambda_0}, E)$. Then, we have

$$x_a \in B_A(x_a, \widetilde{r}) \subset SP(F_{\lambda_0}, E) \subset \bigcup_\lambda SP(F_\lambda, E)$$

Hence, $\widetilde{\bigcup_\lambda}(F_\lambda, E)$ is a soft open set. In addition, obviously, Φ and $\widetilde{X}$ are soft open sets. Therefore, τ is a soft topology. □

Theorem 3.12. Every soft A-metric space is a soft Hausdorff space. Namely, for every different soft points $x_a,\ y_b \in SP(\widetilde{X})$, there exist two soft open balls such that their soft intersection is null soft set.

Proof. Let $x_a,\ y_b \in SP(\widetilde{X})$ and $x_a \neq y_b$. Then, $A(x_a, x_a, \ldots, x_a, y_b) > \widetilde{0}$. For a soft real number $\widetilde{r},\ \widetilde{0} < \widetilde{r} < \widetilde{1},\ A(x_a, x_a, \ldots, x_a, y_b) = \widetilde{r}$. Now, consider the soft open balls $B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right)$ and $B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$. We claim that $B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right) \cap B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$ is null soft set. For this, we suppose that $B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right) \cap B_A\left(y_b, \frac{\widetilde{r}}{2}\right) \neq \emptyset$. Then, there exists a $z_c \in SP(\widetilde{X})$ such that $z_c \in B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right) \cap B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$. Since $z_c \in B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right)$ and $z_c \in B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$, then $A(z_c, z_c, \ldots, z_c, x_a) < \frac{\widetilde{r}}{2(n-1)}$ and $A(z_c, z_c, \ldots, z_c, y_b) < \frac{\widetilde{r}}{2}$, respectively. Because of the condition S2 of the definition of soft A-metrics,

$$\begin{aligned} A(x_a, x_a, \ldots, x_a, y_b) &\leq A(x_a, x_a, \ldots, x_a, z_c) + A(x_a, x_a, \ldots, x_a, z_c) + \\ &\quad + \ldots + A(x_a, x_a, \ldots, x_a, z_c) + A(y_b, y_b, \ldots, y_b, z_c) \\ &= (n-1)A(x_a, x_a, \ldots, x_a, z_c) + A(y_b, y_b, \ldots, y_b, z_c) \\ &< (n-1)\, \frac{\widetilde{r}}{2(n-1)} + \frac{\widetilde{r}}{2} \\ &= \widetilde{r} \end{aligned}$$

Since this is a contradiction, the claim is true. Then, soft A-metric spaces are soft Hausdorff spaces. □

Definition 3.13. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space, $\left\{x_{a_k}^k\right\}$ be a soft sequence of soft points in $\left(\widetilde{X}, A, E\right)$, and y_b is a soft point of over $\widetilde{X}$. Then,

i. $\left\{x_{a_k}^k\right\}$ is called a soft convergent sequence, if for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_b\right) < \widetilde{\varepsilon}$, for each natural number $k \geq k_0$. This is denoted by $\lim\limits_{k\to\infty} x_{a_k}^k = y_b$. Moreover, it is said that $\left\{x_{a_k}^k\right\}$ converges to y_b.

ii. $\left\{x_{a_k}^k\right\}$ is called a soft Cauchy sequence, if for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_{a_m}^m\right) < \widetilde{\varepsilon}$, for each natural numbers $k, m \geq k_0$.

iii. If every soft Cauchy sequence is soft convergent in a soft A-metric space, then this space is said to be soft complete A-metric space.

Lemma 3.14. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space. Every soft convergent sequence in this space converges a unique soft point.

PROOF. Let $\left\{x_{a_k}^k\right\}$ be a soft sequence of soft points in $\left(\widetilde{X}, A, E\right)$ and it soft converges to both y_b and z_c. Then, for each $\widetilde{\varepsilon} > \widetilde{0}$, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_b\right) < \frac{\widetilde{\varepsilon}}{2\,(n-1)}$$

for each natural number $k \geq k_1$, and

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, z_c\right) < \frac{\widetilde{\varepsilon}}{2}$$

for each natural number $k \geq k_2$. We take $k_0 = \max\{k_1, k_2\}$. Then, for each natural number $k \geq k_0$, from Lemma 3.6 and the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A(y_b, y_b, \ldots, y_b, z_c) &\leq (n-1)\,A\left(y_b, y_b, \ldots, y_b, x_{a_k}^k\right) + A\left(z_c, z_c, \ldots, z_c, x_{a_k}^k\right) \\ &= (n-1)\,A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_b\right) + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, z_c\right) \\ &< (n-1)\,\frac{\widetilde{\varepsilon}}{2\,(n-1)} + \frac{\widetilde{\varepsilon}}{2} \\ &= \widetilde{\varepsilon} \end{aligned}$$

Thus, we get $A(y_b, y_b, \ldots, y_b, z_c) = \widetilde{0}$ and this means that $y_b = z_c$. □

Lemma 3.15. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space. In this space, every soft convergent sequence is a soft Cauchy sequence.

PROOF. A soft sequence $\left\{x_{a_k}^k\right\}$ of soft points in $\left(\widetilde{X}, A, E\right)$ soft converges to y_b. Then, for each $\widetilde{\varepsilon} > \widetilde{0}$, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_b\right) < \frac{\widetilde{\varepsilon}}{2\,(n-1)}$$

for each natural number $k \geq k_1$, and

$$A\left(x_{a_m}^m, x_{a_m}^m, \ldots, x_{a_m}^m, y_b\right) < \frac{\widetilde{\varepsilon}}{2}$$

for each natural number $m \geq k_2$. We take $k_0 = \max\{k_1, k_2\}$. Then, for each natural numbers $k, m \geq k_0$, from the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_{a_m}^m\right) &\leq (n-1)\, A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_b\right) + A\left(x_{a_m}^m, x_{a_m}^m, \ldots, x_{a_m}^m, y_b\right) \\ &< (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + \frac{\widetilde{\varepsilon}}{2} = \widetilde{\varepsilon} \end{aligned}$$

Therefore, $\{x_{a_k}^k\}$ is a soft Cauchy sequence. □

Lemma 3.16. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space and $\{x_{a_k}^k\}$ and $\left\{y_{b_k}^k\right\}$ be soft sequences of soft points in this space. If $\{x_{a_k}^k\}$ converges to x_a, and $\left\{y_{b_k}^k\right\}$ converges to y_b, then

$$\lim_{k\to\infty} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) = A\left(x_a, x_a, \ldots, x_a, y_b\right)$$

Proof. Since $\lim_{k\to\infty} x_{a_k}^k = x_a$, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $k_1 \in \mathbb{N}$ such that

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_a\right) < \frac{\widetilde{\varepsilon}}{2(n-1)}$$

for each natural number $k \geq k_1$. Similarly, since $\lim_{k\to\infty} y_{b_k}^k = y_b$, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $k_2 \in \mathbb{N}$ such that

$$A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, y_b\right) < \frac{\widetilde{\varepsilon}}{2(n-1)}$$

for each natural number $k \geq k_2$. If we take $k_0 = \max\{k_1, k_2\}$, then for every natural number $k \geq k_0$, from the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) &\leq (n-1)\, A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_a\right) + A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, x_a\right) \\ &\leq (n-1)\, A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_a\right) + (n-1)\, A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, y_b\right) \\ &\quad + A\left(x_a, x_a, \ldots, x_a, y_b\right) \\ &< (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + A\left(x_a, x_a, \ldots, x_a, y_b\right) \end{aligned}$$

Thus,

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) - A\left(x_a, x_a, \ldots, x_a, y_b\right) < \widetilde{\varepsilon} \tag{3}$$

Similarly, from Lemma 3.6 and the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A\left(x_a, x_a, \ldots, x_a, y_b\right) &\leq (n-1)\, A\left(x_a, x_a, \ldots, x_a, x_{a_k}^k\right) + A\left(y_b, y_b, \ldots, y_b, x_{a_k}^k\right) \\ &\leq (n-1)\, A\left(x_a, x_a, \ldots, x_a, x_{a_k}^k\right) + (n-1)\, A\left(y_b, y_b, \ldots, y_b, y_{b_k}^k\right) \\ &\quad + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) \\ &= (n-1)\, A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k x_a\right) + (n-1)\, A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, y_b\right) \\ &\quad + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) \\ &< (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) \end{aligned}$$

Hence,

$$A\left(x_a, x_a, \ldots, x_a, y_b\right) - A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) < \widetilde{\varepsilon} \tag{4}$$

Hence, from inequalities (3) and (4),

$$\left| A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) - A\left(x_a, x_a, \ldots, x_a, y_b\right) \right| < \widetilde{\varepsilon}$$

Therefore, $\lim_{k\to\infty} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) = A\left(x_a, x_a, \ldots, x_a, y_b\right).$ □

4. Conclusion

This study looked into soft A-metric space which is built by soft points of soft sets and A-metric spaces. Soft A-metric space is the general form of soft S-metric spaces, and it is valuable in this respect. Moreover, it is a generalisation of soft metric spaces. Therefore, soft A-metric spaces are a larger family of soft metric spaces. Many studies can be done on soft A-metric spaces, and important results can be obtained. Especially various well-known fixed point studies and fixed circle studies in this concept will contribute to science. In all these respects, this study presents a new line of vision to generalised metric spaces.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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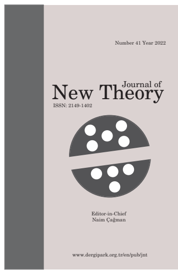

Existence, Uniqueness, and Stability of Solutions to Variable Fractional Order Boundary Value Problems

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Abstract — This paper investigates the sufficient conditions for the existence and uniqueness of a class of Riemann-Liouville fractional differential equations of variable order with fractional boundary conditions. The problem is converted into differential equations of constant orders by combining the concepts of generalized intervals and piecewise constant functions. We derive the required conditions for ensuring the uniqueness of the problem in order to utilize the Banach fixed point theorem. The stability of the obtained solution in the Ulam-Hyers-Rassias (UHR) sense is also investigated, and we finally provide an illustrative example.

Keywords — *Existence, uniqueness, stability, boundary value problems, fractional calculus*

Mathematics Subject Classification (2020) — 26A33, 34D20

1. Introduction

Fractional calculus, which includes differentiation and integration has a history dating back over three centuries [1]. It has extended integration and differentiation operations to any fractional orders that might take any real or complex value. Thus, it is possible to think of the order of the fractional integrals and derivatives as a function of time or another variable. In this context, Samko and Ross examined the first study regarding the idea of variable order (VO) differentiation in [2, 3]. Based on using the R-L derivative and the Fourier transform, they have defined and interpreted the integration and differentiation of functions to a variable order $(\frac{d}{dx})^{\alpha(t)}f(x)$. The notion of variable and distributed order fractional operators is then developed by Lorenzo and Hartley. They reviewed the VO fractional operator research results and then studied the concepts of variable order fractional operators in various forms [4, 5].

The memory and heredity aspects of numerous physical processes and events can be used to characterize by the variable order fractional operators thanks to their non-stationary power-law kernel. As a result, fractional calculus with variable order was used as a prospective option to provide an appropriate mathematical framework for precisely modelling complicated physical systems and processes. Having followed that, VO-FDEs have attracted increasing attention, owing to their compatibility with describing a wide range of phenomena, including anomalous diffusion, medicine, viscoelasticity, control system, and many other branches of physics and engineering, to name a few [6–13]. Many publications have been devoted to finding numerical solutions for fractional differential equations of VO due to the difficulty in obtaining explicit solutions. See also [14–18] and the references therein. Nonetheless,

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some recent publications discuss the existence, uniqueness, and stability features of variable fractional order differential equations [19–32].

We aim to study following fractional boundary value problem (BVP) of variable order in Riemann-Liouville sense (VORLFDE) with fractional variable order boundary conditions as well.

$$\begin{cases} \mathcal{D}^{w(t)}y(t) = g(t, y(t)), & t \in \mathcal{J} \\ I^{2-w(t)}y(0) = \alpha_0 I^{2-w(t)}y(a), & \mathcal{D}^{w(t)-1}y(0) = \alpha_1 \mathcal{D}^{w(t)-1}y(a) \end{cases} \tag{1}$$

where $\mathcal{J} = [0, a]$, $0 < a < \infty$, $w(t) : \mathcal{J} \to (1, 2]$ is the variable order of the fractional derivatives, $g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\mathcal{D}^{w(t)}$, $I^{w(t)}$ denotes the Riemann-Liouville fractional derivative and integral of order $w(t)$ respectively and α_0, α_1 real numbers such that $\alpha_0 \neq 1$ and $\alpha_1 \neq 1$.

We will be concerned with the existence and uniqueness of solution of problem 1 and further study the stability of the obtained solution of problem 1 in the Ulam-Hyers-Rassias (UHR) sense.

2. Mathematical Preliminaries

This section introduces several important notions and lemmas that are required to grasp the main theorems covered in the next parts. We also present additional features for variable order operators.

Let $C(\mathcal{J}, R)$ be the set of all continuous real-valued functions from $\mathcal{J}$ into $\mathbb{R}$. Setting the standard norm $\|\varkappa\| = sup\{|\varkappa(t)| : t \in \mathcal{J}\}$ for an element in $C(\mathcal{J}, \mathbb{R})$, then $C(\mathcal{J}, \mathbb{R})$ has become a Banach space with such a norm.

For $-\infty < t_1 < t_2 < +\infty$, we consider the mappings $w(t) : [t_1, t_2] \to (0, +\infty)$ and $\theta(t) : [t_1, t_2] \to (n-1, n)$. Then, the left Riemann-Liouville fractional integral (RLFI) of variable-order $w(t)$ for function $y(t)$ is given [3] by

$$I_{t_1^+}^{w(t)}y(t) = \int_{t_1}^{t} \frac{(t-s)^{w(t)-1}}{\Gamma(w(t))} y(s)ds, \ \ t > a_1 \tag{2}$$

and the left Riemann-Liouville fractional derivative (RLFD) of variable-order $\theta(t)$ for function $y(t)$ is defined by

$$\mathcal{D}_{t_1^+}^{\theta(t)}y(t) = \left(\frac{d}{dt}\right)^n I_{t_1^+}^{n-\theta(t)}y(t) = \left(\frac{d}{dt}\right)^n \int_{t_1}^{t} \frac{(t-s)^{n-\theta(t)-1}}{\Gamma(n-\theta(t))} y(s)ds, \ \ t > t_1 \tag{3}$$

As expected, RLFI and RLFD coincide with the conventional Riemann-Liouville fractional derivative and integral, respectively [1,3], when replacing constant values by $w(t)$ and $\theta(t)$.

Remark 2.1. [28] It should be emphasized that for R-L fractional integrals with constant orders, the semi-group property is satisfied, but not for those with variable orders, i.e.,

$$I_{t_1^+}^{w(t)} I_{t_1^+}^{\theta(t)} y(t) \neq I_{t_1^+}^{w(t)+\theta(t)} y(t)$$

Definition 2.2. [18] Let I be a subset of $\mathbb{R}$. Then we define the followings:

- If the set I is an interval, a point or an empty set, it is referred to as a generalized interval.
- If each x in I lies in precisely one of the generalized intervals E in $\mathcal{P}$, then the finite set $\mathcal{P}$ of generalized intervals is known as a partition of I
- If for any $E \in \mathcal{P}$, g is constant on E, the function $g : I \to \mathbb{R}$ is said to be piecewise constant with regard to partition $\mathcal{P}$ of I

Theorem 2.3. [34] Suppose E is a Banach space. If $T : E \to E$ is a completely continuous operator and $\Omega = \{x \in E : x = \eta Tx, 0 < \eta < 1\}$ is bounded, then T has a fixed point in E.

Definition 2.4. [33] BVP 1 is said to be Hyers-Ulam-Rassias stable (UHR) with regard to the function $\kappa \in C(\mathcal{J}, \mathbb{R}_+)$ if a constant $c_g > 0$ exists such that for any $\epsilon > 0$ and for each function $z \in C(\mathcal{J}, \mathbb{R})$ satisfying

$$|\mathcal{D}_{0^+}^{w(t)} z(t) - g(t, z(t))| \le \epsilon\kappa(t),\ t \in \mathcal{J} \tag{4}$$

there exists a solution $y \in C(\mathcal{J}, \mathbb{R})$ of BVP 1 with

$$|z(t) - y(t)| \le c_g \epsilon\kappa(t),\ t \in \mathcal{J}$$

3. Existence of Solutions

Let us proceed by stating the following hypothesis:

(H1) Assume that $\{a_k\}_{k=0}^n$ is the finite sequence of points such that $0 = a_0 < a_k < a_n = a,\ k = 1, \dots, n-1$ where $n \in \mathbb{N}$. Let $\mathcal{J}_k := (a_{k-1}, a_k],\ k = 1, 2, \dots, n$. Then, $\mathcal{P} = \cup_{k=1}^n \mathcal{J}_k$ is a partition of the interval $\mathcal{J}$.

For each $m = 1, \dots, n$, the symbol $E_m = C(\mathcal{J}_m, \mathbb{R})$, indicates the Banach space of continuous functions $y : \mathcal{J}_m \to \mathbb{R}$ equipped with sup-norm $\|y\|_{E_m} = \sup_{t \in \mathcal{J}_m} |y(t)|$.

Let $w(t) : \mathcal{J} \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$w(t) = \sum_{m=1}^{n} w_m I_m(t)$$

where $1 < w_m \le 2$ are constants, and I_m stands for the indicator of the interval $\mathcal{J}_m$, $m = 1, 2, \dots, n$, that is,

$$I_m(t) = \begin{cases} 1, & t \in \mathcal{J}_m \\ 0, & elsewhere \end{cases}$$

For any $t \in \mathcal{J}_m,\ m = 1, \dots, n$, one can represent R-L fractional variable-order derivative $w(t)$ of the function $y(t) \in C(\mathcal{J}, \mathbb{R})$, defined by (3), as the sum of left R-L fractional derivatives of integer orders $w_k,\ k = 1, \dots, m$

$$\begin{aligned} \mathcal{D}_{0^+}^{w(t)} y(t) &= \frac{1}{\Gamma(2 - w(t))} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-w(t)} y(s) ds \\ &= \frac{1}{\Gamma(2 - w(t))} \left(\sum_{k=1}^{m-1} \frac{d^2}{dt^2} \int_{a_{k-1}}^{a_k} (t-s)^{1-w_k} y(s) ds + \frac{d^2}{dt^2} \int_{a_{m-1}}^{t} (t-s)^{1-w_m} y(s) ds \right) \end{aligned} \tag{5}$$

As a consequence, BVP 1 can be expressed on $\mathcal{J}_m$ for each $m = 1, \dots, n$ in the manner shown below

$$\frac{1}{\Gamma(2 - w(t))} \Big(\sum_{k=1}^{m-1} \frac{d^2}{dt^2} \int_{a_{k-1}}^{a_k} (t-s)^{1-w_k} y(s) ds + \frac{d^2}{dt^2} \int_{a_{m-1}}^{t} (t-s)^{1-w_m} y(s) ds \Big) = g(t, y(t)) \tag{6}$$

Let the function $\tilde{y} \in C(\mathcal{J}_m, \mathbb{R})$ be such that $\tilde{y}(t) \equiv 0$ on $t \in [0, a_{m-1}]$ and it solves integral Equation 6. Then, it is reduced to

$$\mathcal{D}_{a_{m-1}^+}^{w_m} \tilde{y}(t) = g(t, \tilde{y}(t)),\quad t \in \mathcal{J}_m$$

We consider the auxiliary BVP given below for integer order Riemann-Liouville fractional differential equations while regarding the aforementioned statement above for any $m = 1, 2, \dots, n$.

$$\begin{cases} \mathcal{D}_{a_{m-1}^+}^{w_m} y(t) = g(t, y(t)), & t \in \mathcal{J}_m \\ \mathcal{D}_{a_{m-1}^+}^{w_m - 2} y(a_{m-1}) = \alpha_0 \mathcal{D}_{a_{m-1}^+}^{w_m - 2} y(a_m), & \mathcal{D}_{a_{m-1}^+}^{w_m - 1} y(a_{m-1}) = \alpha_1 \mathcal{D}_{a_{m-1}^+}^{w_m - 1} y(a_m) \end{cases} \tag{7}$$

Lemma 3.1. Let $\alpha_0, \alpha_1 \neq 1$, $g \in C(\mathcal{J}_m \times \mathbb{R}, \mathbb{R})$ for $m = 1, \dots, n$, and $\gamma \in (0,1)$ be a number such that $t^\gamma g \in C(\mathcal{J}_m \times \mathbb{R}, \mathbb{R})$.

Then, the function $x \in E_m$ satisfies problem 7 iff y solves the integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(w_m)} \int_{a_{m-1}}^{t} (t-s)^{w_m-1} g(s,x(s))ds + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)} \int_{a_{m-1}}^{a_m} g(s,x(s))ds \\ & \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)} \int_{a_{m-1}}^{a_m} [(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s] g(s,x(s))ds \end{aligned} \tag{8}$$

Proof. Let $x \in E_m$ be a solution of BVP 7. Using the operator $I^{w_m}_{a^+_{m-1}}$ on each sides of BVP 7, we find

$$x(t) = \lambda_1 t^{w_m-1} + \lambda_0 t^{w_m-2} + I^{w_m}_{a^+_{m-1}} g(t,x(t)) \tag{9}$$

where λ_0, λ_1 are constants.

Using funciton 9, we have

$$\mathcal{D}^{w_m-1} x(t) = \lambda_1 \Gamma(w_m) + I^1 g(t,x(t))$$

In view of assumptions on the function g and by the boundary condition

$$\mathcal{D}^{w_m-1}_{a^+_{m-1}} x(a_{m-1}) = \alpha_1 \mathcal{D}^{w_m-1}_{a^+_{m-1}} x(a_m)$$

we conclude that

$$\lambda_1 = \frac{\alpha_1}{(1-\alpha_1)\Gamma(w_m)} \int_{a_{m-1}}^{a_m} g(s,x(s))ds$$

since $I^{2-w_m}(t^{w_m-1}) = \Gamma(w_m)t$ and $I^{2-w_m}(t^{w_m-2}) = \Gamma(w_m-1)$, from the boundary condition

$$\mathcal{D}^{w_m-2}_{a^+_{m-1}} x(a_{m-1}) = \alpha_0 \mathcal{D}^{w_m-2}_{a^+_{m-1}} x(a_m)$$

we get

$$\lambda_0 = \frac{\alpha_1(\alpha_0 a_m - a_{m-1})}{(1-\alpha_1)(1-\alpha_0)\Gamma(w_m-1)} \int_{a_{m-1}}^{a_m} g(s,x(s))ds + \frac{\alpha_0}{(1-\alpha_0)\Gamma(w_m-1)} \int_{a_{m-1}}^{a_m} (a_m - s) g(s,x(s))ds$$

Thus,

$$x(t) = \int_{a_{m-1}}^{a_m} G_m(t,s) g(s,x(s))ds$$

where $G_m(t,s)$ is Green's function defined by:

$$G_m(t,s) = \begin{cases} \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)} + \frac{t^{w_m-2}[\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)} + \frac{1}{\Gamma(w_m)}(t-s)^{w_m-1} \\ \qquad a_{m-1} \leq s \leq t \leq a_m \\ \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)} + \frac{t^{w_m-2}[\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)} \\ \qquad a_{m-1} \leq t \leq s \leq a_m \end{cases}$$

where $m = 1, 2, ..., n$.

Then, we get x that solves Integral Equation (8).

In contrast, consider $x \in E_m$ to be a solution of Integral Equation 8. We conclude that x is the solution to the BVP 7 by virtue of the continuity of function $t^\gamma g$. □

We shall demonstrate the existence result for the BVP of R-L fractional differential equations of constant order (7). The proof will be carried out by the aid of Theorem 2.3.

Theorem 3.2. Suppose that the conditions of Lemma 3.1 hold and there exists a constant $N > 0$ such that

$$t^{\gamma}|g(t,y)| \leq N, \quad \forall t \in \mathcal{J}_m, y \in \mathbb{R}$$

with $\gamma = 2 - w$. Then, BVP 7 for Riemann-Liouville fractional differential equations of integer order has at least one solution in $C_{\gamma}[a_{m-1}, a_m]$.

Proof. For any function $y \in C_{\gamma}[a_{m-1}, a_m]$, we construct the operator

$$\begin{aligned} Sy(t) &= \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^{t}(t-s)^{w_m-1}g(s,y(s))ds + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,y(s))ds \\ &+ \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,y(s))ds \end{aligned} \tag{10}$$

It results in immediately by the properties of fractional integrals and the continuity of function $t^{\gamma}g$ that the operator $S : C_{\gamma}[a_{m-1}, a_m] \to C_{\gamma}[a_{m-1}, a_m]$ given by equality 10 is well defined. Let

$$\begin{aligned} R_m &\geq \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \Big|\frac{\alpha_1 a_m}{1-\alpha_1}\Big| + \Big|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big| \\ &+ \Big|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\Big|\Big] \end{aligned}$$

Consider the set

$$B_{R_m} = \{y \in C_{\gamma}[a_{m-1}, a_m], \ \|y\|_{\gamma} \leq R_m\}$$

For all $m \in \{1, 2, ..., n\}$, the ball B_{R_m} is a nonempty closed convex subset of $C_{\gamma}[a_{m-1}, a_m]$.

We are in position to examine the assumption of the Theorem 3.2 for the operator S. We shall demonstrate it in three stages.

Step 1: Let B_{R_m} be a bounded set in $C_{\gamma}[a_{m-1}, a_m]$. Hence, B_{R_m} is bounded on $C[a_{m-1}, a_m]$ and there exists a constant N such that $t^{\gamma}|g(t, y(t))| \leq N, \forall y \in B_{R_m}, t \in [a_{m-1}, a_m]$. Thus,

$$\begin{aligned} t^{\gamma}|(Sy)(t)| &\leq \frac{Nt^{\gamma}}{\Gamma(w_m)}\int_{a_m-1}^{t} s^{-\gamma}(t-s)^{w_m-1}ds + \Big|\frac{\alpha_1 N t}{(1-\alpha_1)\Gamma(w_m)}\Big|\int_{a_{m-1}}^{a_m} s^{-\gamma}ds \\ &+ \Big|\frac{N}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\Big|\int_{a_{m-1}}^{a_m}\Big|\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s\Big|s^{-\gamma}ds \\ &\leq \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \Big|\frac{\alpha_1 a_m}{1-\alpha_1}\Big| + \Big|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big| \\ &+ \Big|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\Big|\Big] \end{aligned}$$

which implies that

$$\begin{aligned} \|(Sy)\|_{\gamma} \leq & \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \Big|\frac{\alpha_1 a_m}{1-\alpha_1}\Big| + \Big|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big| \\ &+ \Big|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\Big|\Big] \end{aligned}$$

Hence, $S(B_{R_m})$ is uniformly bounded.

Step 2: Let $t_1, t_2 \in \mathcal{J}_m, \ t_1 < t_2$and $y \in B_{R_m}$. Then, we have

$$\begin{aligned} \Big|t_1^{\gamma}(Sy)(t_1) - t_2^{\gamma}(Sy)(t_2)\Big| = & \Big|\tfrac{1}{\Gamma(w_m)}\int_{a_{m-1}}^{t_1}\Big[t_1^{\gamma}(t_1-s)^{w_m-1} - t_2^{\gamma}(t_2-s)^{w_m-1}\Big]g(s,y(s))ds \\ & -\tfrac{1}{\Gamma(w_m)}\int_{t_1}^{t_2} t_2^{\gamma}(t_2-s)^{w_m-1}g(s,u(s))ds + \tfrac{\alpha_1(t_1-t_2)}{(1-\alpha_1)\Gamma(w_m-1)}\int_{a_m-1}^{a_m} g(s,y(s))ds\Big| \\ \leq & N\Big(\Big|\tfrac{1}{\Gamma(w_m)}\int_{a_{m-1}}^{t_1}\Big[t_1^{\gamma}(t_1-s)^{w_m-1} - t_2^{\gamma}(t_2-s)^{w_m-1}\Big]ds \\ & -\tfrac{1}{\Gamma(w_m)}\int_{t_1}^{t_2} t_2^{\gamma}(t_2-s)^{w_m-1}ds\Big| + \Big|\tfrac{\alpha_1(t_1-t_2)}{(1-\alpha_1)\Gamma(w_m-1)}\int_{a_m-1}^{a_m} ds\Big|\Big) \end{aligned}$$

Hence, $\left|t_1^{\gamma}(Sy)(t_1)-t_2^{\gamma}(Sy)(t_2)\right| \to 0$ as $|t_1-t_2| \to 0$. Thus, $t^{\gamma}S(B_{R_m})$ is equicontinuous. Consequently, the operator S is compact.

Step3: Consider the set

$$\Omega = \{y \in \mathbb{R} \setminus y = \eta Sy, 0 < \eta < 1\}$$

and show that the set Ω is bounded. Let $y \in \Omega$, then $y = \eta Sy, 0 < \eta < 1$. For any $t \in [a_m - 1, a_m]$, we have

$$|y(t)| \leq \eta\Big[\tfrac{1}{\Gamma(w_m)}\int_{a_m-1}^{t}(t-s)^{w_m-1}|g(s,y(s))|ds + \left|\tfrac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\right|\int_{a_{m-1}}^{a_m}|g(s,y(s))|ds$$
$$+\left|\tfrac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1}\right|\int_{a_{m-1}}^{a_m}\left|\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s\right|ds\Big]$$

We have

$$\|(Sy)\|_{\gamma} \leq \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \left|\frac{\alpha_1 a_m}{1-\alpha_1}\right| + \left|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\right|$$
$$+\left|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\right|\Big]$$

This implies that the set Ω is bounded independently of $\eta \in (0,1)$. On account of Theorem 2.3, we find that the operator S has at least one fixed point, which follows that problem 7 possesses at least one solution. □

Consider the next assumption:

(H2) Let $g \in C(\mathcal{J} \times \mathbb{R}, \mathbb{R})$ and there exist a constant $K > 0$ such that

$$t^{\gamma}|g(t,u) - g(t,v)| \leq K|u - v|$$

for any $u,\ v \in \mathbb{R}$, $t \in \mathcal{J}$ and $\gamma = 2 - w$.

Theorem 3.3. Assume that conditions (H1) and (H2) hold. Then, problem 7 has a unique solution in $C_{\gamma}[a_{m-1}, a_m]$ if

$$K < \frac{1}{\rho} \tag{11}$$

where

$$\rho \;=\; \tfrac{(a_m^{1-2\gamma} - a_{m-1}^{1-2\gamma})}{(1-2\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \left|\tfrac{\alpha_1}{1-\alpha_1}\right|a_m + \left|\tfrac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\right|$$
$$+\left|\tfrac{\alpha_0(1-\alpha_1)(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\right|\Big(\tfrac{(1-2\gamma)(a_m^{2-2\gamma} - a_{m-1}^{2-2\gamma})}{(2-2\gamma)(a_m^{1-2\gamma} - a_{m-1}^{1-2\gamma})}\Big)\Big]$$

Proof. In view of (H2), for every $t \in [a_{m-1}, a_m]$, we have

$$t^{\gamma}|(Su)(t) - (Sv)(t)| \leq \tfrac{t^{\gamma}}{\Gamma(w_m)}\int_{a_m-1}^{t}(t-s)^{w_m-1}|g(s,u(s)) - g(s,v(s))|ds$$
$$+\left|\tfrac{\alpha_1 t}{(1-\alpha_1)\Gamma(w_m)}\right|\int_{a_{m-1}}^{a_m}|g(s,u(s)) - g(s,v(s))|ds$$
$$+\left|\tfrac{1}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\right|\int_{a_{m-1}}^{a_m}\left|\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s\right||g(s,u(s)) - g(s,v(s))|ds$$
$$\leq K\Big[\tfrac{t^{\gamma}}{\Gamma(w_m)}\int_{a_m-1}^{t}s^{-\gamma}(t-s)^{w_m-1}|u(s) - v(s)|ds + \left|\tfrac{\alpha_1 t}{(1-\alpha_1)\Gamma(w_m)}\right|\int_{a_{m-1}}^{a_m}s^{-\gamma}|u(s) - v(s)|ds$$
$$+\left|\tfrac{1}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\right|\int_{a_{m-1}}^{a_m}s^{-\gamma}\left|\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s\right||u(s) - v(s)|ds\Big]$$

By the definition of $\|.\|_\gamma$, we obtain

$$\|(Su)(t)-(Sv)(t)\|_\gamma \le K\Big[\tfrac{t^\gamma}{\Gamma(w_m)}\int_{a_m-1}^{t} s^{-2\gamma}(t-s)^{w_m-1}ds+\Big|\tfrac{\alpha_1 t}{(1-\alpha_1)\Gamma(w_m)}\Big|\int_{a_{m-1}}^{a_m} s^{-2\gamma}ds$$
$$+\Big|\tfrac{1}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\Big|\int_{a_{m-1}}^{a_m} s^{-2\gamma}\Big|\alpha_0 a_m-\alpha_1 a_{m-1}-\alpha_0(1-\alpha_1)s\Big|ds\Big]\|u-v\|_\gamma$$
$$\le \tfrac{K(a_m^{1-2\gamma}-a_{m-1}^{1-2\gamma})}{(1-2\gamma)\Gamma(w_m)}\Big[a_m^\gamma(a_m-a_{m-1})^{w_m-1}+\Big|\tfrac{\alpha_1}{1-\alpha_1}\Big|a_m+\Big|\tfrac{(\alpha_0 a_m-\alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big|$$
$$+\Big|\tfrac{\alpha_0(1-\alpha_1)(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big|\Big(\tfrac{(1-2\gamma)(a_m^{2-2\gamma}-a_{m-1}^{2-2\gamma})}{(2-2\gamma)(a_m^{1-2\gamma}-a_{m-1}^{1-2\gamma})}\Big)\Big]\|u-v\|_\gamma$$

□

From the above estimate, it follows by condition 11 that the operator S is a contraction. As a result of the Banach fixed point theorem, we may derive that S has a unique fixed point, which corresponds to a unique solution to problem 7.

Now, we will prove the existence result for problem 1.

Theorem 3.4. Let the assumptions (H1), (H2) and inequality 11 hold for all $m \in \{1,2,\dots,n\}$. Then, problem 1 has unique solution in $C_\gamma[0,a]$.

Proof. For each $m \in \{1,2,\dots,n\}$, owing to Theorem 3.3 the BVP 7 for R-L fractional differential equations of integer order possesses unique solution $\widetilde{y}_m \in C_\gamma[a_{m-1},a_m]$. For any $m \in \{1,2,\dots,n\}$, we define the function

$$y_m=\begin{cases}0, & t\in[0,a_{m-1}]\\ \widetilde{y}_m, & t\in\mathcal{J}_m\end{cases} \tag{12}$$

As a result, the function $y_m \in C_\gamma[a_{m-1},a_m]$ satisfies the integral problem 6 on $\mathcal{J}_m$, implying that $y_m(0)=0,\ y_m(a_m)=\widetilde{y}_m(a_m)=0$ and solves problem 6 for $t\in\mathcal{J}_m,\ \ m\in\{1,2,\dots,n\}$. Then, the function

$$y(t)=\begin{cases} y_1(t), t\in\mathcal{J}_1\\ y_2(t)=\begin{cases}0, & t\in\mathcal{J}_1\\ \widetilde{y}_2, & t\in\mathcal{J}_2\end{cases}\\ \vdots\\ y_n(t)=\begin{cases}0, & t\in[0,a_{n-1}]\\ \widetilde{y}_n, & t\in\mathcal{J}_n\end{cases}\end{cases}$$

is a solution of BVP 1 in $C_\gamma[0,a]$. □

4. Ulam-Hyers-Rassias Stability of VORLFDE

Theorem 4.1. Let the conditions (H1), (H2) and inequality 11 be satisfied. Further assume that

(H3) $\kappa\in C(\mathcal{J},\mathbb{R}_+)$ is increasing and there exists $\lambda_\kappa>0$ such that

$$I_{a_{m-1}^+}^{w_m}\kappa(t)\le\lambda_\kappa\ \kappa(t)$$

hold for $t\in\mathcal{J}_m$, $m=1,2,\dots$. Then, BVP 1 is UHR stable with respect to κ.

Proof. Let $\epsilon>0$ be an arbitrary number and the function $z(t)$ from $C(\mathcal{J},\mathbb{R})$ satisfy inequality 4. For any $m\in\{1,2,\dots,n\}$ we define the functions $z_1(t)\equiv z(t),\ t\in[0,a_1]$ and for $m=2,3,\dots,n$:

$$z_m(t)=\begin{cases}0, & t\in[0,a_{m-1}]\\ z(t), & t\in\mathcal{J}_m\end{cases}$$

For any $m\in\{1,2,\dots,n\}$ according to equality 3 for $t\in\mathcal{J}_m$ we get

$$\mathcal{D}_{0^+}^{w(t)}z_m(t)=\frac{1}{\Gamma(2-w(t))}\frac{d^2}{dt^2}\int_{a_{m-1}}^{t}(t-s)^{1-w_m}\frac{z(s)}{s}ds \tag{13}$$

we take the RLFI $I^{w_m}_{a_{m-1}^+}$ of both sides of the inequality 4, apply (H3) and obtain

$$\begin{aligned}\Big|z_m(t) - \tfrac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1} g(s,z_m(s))ds - \tfrac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,z_m(s))ds \\ -\tfrac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,z_m(s))ds\Big| &\le \epsilon\, I^{w_m}_{a_{m-1}^+}\kappa(t) \\ &\le \epsilon\ \lambda_\kappa\ \kappa(t)\end{aligned}$$

According to Theorem 3.4, BVP 1 has a solution $y \in C(\mathcal{J},\mathbb{R})$ defined by $y(t) = y_m(t)$ for $t \in \mathcal{J}_m,\ m = 1, 2, \ldots, n$, where

$$y_m = \begin{cases} 0, & t \in [0, a_{m-1}] \\ \widetilde{y}_m, & t \in \mathcal{J}_m \end{cases} \tag{14}$$

and $\widetilde{y}_m \in E_m$ is a solution of problem 7. According to Lemma 3.1, the integral equation

$$\begin{aligned}\widetilde{y}_m(t) = &\frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1} g(s,\widetilde{y}_m(s))ds + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,\widetilde{y}_m(s))ds \\ &+\frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,\widetilde{y}_m(s))ds\end{aligned} \tag{15}$$

holds. Let $t \in \mathcal{J}_m$ where $m \in \{1, 2, \ldots, n\}$. Then, we obtain

$$\begin{aligned}|z(t)-y(t)| &= |z(t) - y_m(t)| \\ &= |z_m(t) - \widetilde{y}_m(t)| \\ &\le \Big|z_m(t) - \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1} g(s,z_m(s))ds - \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,z_m(s))ds \\ &\quad -\frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,z_m(s))ds\Big| \\ &\quad +\frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1}\Big|g(s,z_m(s)) - g(s,\widetilde{y}_m(s))\Big|ds \\ &\quad +\frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m}\Big|g(s,z_m(s)) - g(s,\widetilde{y}_m(s))\Big|ds \\ &\quad +\frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]\Big|g(s,z_m(s)) - g(s,\widetilde{y}_m(s))\Big|ds \\ &\le \lambda_\kappa\ \epsilon\ \kappa(t) + \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t s^{-\gamma}(t-s)^{w_m-1}(K|z_m(s) - \widetilde{y}_m(s)|)ds \\ &\quad +\frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} s^{-\gamma}(K|z_m(s) - \widetilde{y}_m(s)|)ds \\ &\quad +\frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]s^{-\gamma}(K|z_m(s) - \widetilde{y}_m(s)|)ds \\ &\le \lambda_\kappa\ \epsilon\ \kappa(t) + \frac{(t-a_{m-1})^{w_m-1}(t^{1-\gamma} - {a_{m-1}}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}(K\|z_m - \widetilde{y}_m\|_{E_m}) \\ &\quad +\frac{\alpha_1 t^{w_m-1}(a_m^{1-\gamma} - {a_{m-1}}^{1-\gamma})}{(1-\gamma)(1-\alpha_1)\Gamma(w_m)}(K\|z_m - \widetilde{y}_m\|_{E_m}) \\ &\quad +\frac{t^{w_m-2}a_{m-1}^{-\gamma}(a_m - a_{m-1})}{2(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}[2(\alpha_0 a_m - \alpha_1 a_{m-1}) - \alpha_0(1-\alpha_1)(a_m - a_{m-1})](K\|z_m - \widetilde{y}_m\|_{E_m}) \\ &\le \lambda_\kappa\ \epsilon\ \kappa(t) + \eta\|z-y\|_{\mathcal{J}}\end{aligned}$$

Then,

$$\|z-y\|_{\mathcal{J}}\Big(1-\eta\Big) \;\leq\; \lambda_\kappa\ \epsilon\ \kappa(t)$$

or for any $t \in \mathcal{J}$

$$|z(t)-y(t) \leq \|z-y\|_{\mathcal{J}} \leq \frac{\lambda_\kappa}{1-\eta}\epsilon\kappa(t)$$

Therefore, BVP 1 is UHR stable with respect to κ. □

5. Example

Let $\mathcal{J} := [0,4]$, $a_0 = 0$, $a_1 = 2$, $a_2 = 4$. Consider the following fractional problem with fractional boundary conditions of variable order

$$\begin{cases} \mathcal{D}_{0^+}^{w(t)} y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t\in\mathcal{J}, \\ \mathcal{D}_{0^+}^{w(t)-2} y(0) = \alpha_0 \mathcal{D}_{0^+}^{w(t)-2} y(4), \;\; \mathcal{D}_{0^+}^{w(t)-1} y(0) = \alpha_1 \mathcal{D}_{0^+}^{w(t)-2} y(4) \end{cases} \tag{16}$$

where

$$w(t) = \begin{cases} \frac{3}{2}, & t\in \mathcal{J}_1 := [0,2] \\ \frac{10}{9}, & t\in\mathcal{J}_2 :=]2,4] \end{cases} \tag{17}$$

Denote

$$g(t,y) = \frac{\sin y + 2\cos(t)}{t^2}, \;(t,y)\in[0,4]\times\mathbb{R}$$

Taking into account function 17, we can construct two auxiliary BVPs by means of problem 7 for Riemann-Liouville fractional differential equations of integer order

$$\begin{cases} \mathcal{D}_{0+}^{\frac{3}{2}} y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t\in\mathcal{J}_1, \\ \mathcal{D}_{0+}^{\frac{-1}{2}} y(0) = \alpha_0 \mathcal{D}_{0+}^{\frac{-1}{2}} y(2), \qquad \mathcal{D}_{0+}^{\frac{1}{2}} y(0) = \alpha_1 \mathcal{D}_{0+}^{\frac{1}{2}} y(2) \end{cases} \tag{18}$$

and

$$\begin{cases} \mathcal{D}_{2+}^{\frac{10}{9}} y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t\in\mathcal{J}_2, \\ \mathcal{D}_{2+}^{\frac{-8}{9}} y(2) = \alpha_0 \mathcal{D}_{2+}^{\frac{-8}{9}} y(4), \qquad \mathcal{D}_{2+}^{\frac{1}{9}} y(2) = \alpha_1 \mathcal{D}_{2+}^{\frac{1}{9}} y(4) \end{cases} \tag{19}$$

For $m = 1$. Clearly,

$$t^\gamma\Big|g(t,y)\Big| \;\leq\; t^{\frac{1}{2}}\Big|\frac{\sin y(t)+2\cos(t)}{t^2}\Big| \leq \frac{3}{2\sqrt{2}} = N$$

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we obtain

$$\begin{aligned} I_{0^+}^{w_1}\kappa(t) &= \tfrac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}} s^{\frac{1}{2}} ds \\ &\leq \tfrac{\sqrt{2}}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}} ds \\ &\leq \tfrac{2\sqrt{2}}{3\Gamma(\frac{3}{2})}\kappa(t) := \lambda_{\kappa(t)}\kappa(t) \end{aligned}$$

where $\lambda_\kappa = \frac{2\sqrt{2}}{3\Gamma(\frac{3}{2})}$. Thus, it implies that condition (H3) holds.

By Theorem 3.2, BVP 18 has a solution $\widetilde{y}_1 \in E_1$. For $m = 2$. Clearly,

$$t^\gamma\Big|g(t,y)\Big| \;\leq\; t^{\frac{8}{9}}\Big|\frac{\sin y(t)+2\cos(t)}{t^2}\Big| \leq 3.4^{\frac{-10}{9}} = N$$

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we obtain

$$\begin{array}{rcl} I_{2^+}^{w_2}\kappa(t) & = & \frac{1}{\Gamma(\frac{10}{9})}\int_2^t (t-s)^{\frac{1}{9}} s^{\frac{1}{2}} ds \\ & \leq & \frac{2}{\Gamma(\frac{10}{9})}\int_2^t (t-s)^{\frac{1}{9}} ds \\ & \leq & \frac{9}{5\Gamma(\frac{10}{9})}\kappa(t) := \lambda_{\kappa(t)}\kappa(t) \end{array}$$

where $\lambda_\kappa = \frac{9}{5\Gamma(\frac{10}{9})}$. Thus, condition (H3) is satisfied.

According to Theorem 3.2, BVP 19 possesses a solution $\widetilde{y}_2 \in E_2$. Thus, according to Theorem 3.4 the BVP 16 has a solution

$$y(t) = \begin{cases} \widetilde{y}_1(t), & t \in \mathcal{J}_1 \\ y_2(t), & t \in \mathcal{J}_2 \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1 \\ \widetilde{y}_2(t), & t \in \mathcal{J}_2 \end{cases}$$

In view of Theorem 4.1, BVP 16 is also UHR stable with respect to κ.

6. Conclusion

This study examines the necessary and sufficient conditions for the existence and uniqueness of a class of variable order differential equations with fractional boundary conditions. Combining the ideas of generalized intervals and piecewise constant functions, the problem is transformed into differential equations of constant orders. Using the Banach fixed point theorem, we develop the conditions for guaranteeing the problem's uniqueness. We also discuss the stability of the obtained solution in the Ulam-Hyers-Rassias (UHR) sense.

For the future investigations, one can study to get similar results by considering systems of differential equations as well as employing variable order fractional operators in Caputo's sense. Moreover, the obtained results may be extended to variable order fractional boundary or initial value problems in which the non-linear term may have discontinuities at some interior points.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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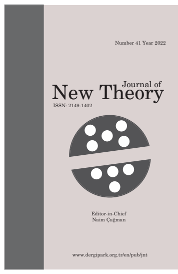

Cesàro Summability Involving δ-Quasi-Monotone and Almost Increasing Sequences

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Abstract — This paper generalises a well-known theorem on $\mid C, \rho \mid_\kappa$ summability to the $\varphi - \mid C, \rho; \beta \mid_\kappa$ summability of an infinite series using an almost increasing and a δ-quasi monotone sequence.

Keywords — *Cesàro summability, δ-quasi-monotone sequence, summability factors, almost increasing sequence, infinite series*

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1. Introduction

The absolute summability of some infinite series (ISs) is an interesting topic. Especially, absolute Cesàro and absolute Riesz summability methods have different applications dealing with some well-known classes of sequence such as δ-quasi monotone, (ϕ, δ) monotone, almost increasing and quasi power increasing sequences. In [1], Bor and Özarslan proved two theorems on $\mid C, \rho; \beta \mid_\kappa$ and $\mid \bar{N}, p_n; \beta \mid_\kappa$ summability methods. In [2–4], the authors obtained theorems on absolute Cesàro and absolute Riesz summability via almost increasing and δ-quasi monotone sequences. Özarslan [5–9], Bor [10], Kartal [11] proved theorems on absolute Cesàro summability factors. Kartal [12, 13] used almost increasing sequences to absolute Riesz summability, Bor and Agarwal [14], Kartal [15] applied almost increasing sequences to absolute Cesàro summability, also Bor et al. [16], Özarslan [17] operated quasi power increasing sequences. In [18], Özarslan and Şakar used (ϕ, δ) monotone sequences to get sufficient conditions for absolute Riesz summability of an ISs.

This article is organized as following: preliminaries on some sequences and the absolute Cesàro summability methods are given in Section 2, a known result about absolute Cesàro summability of a factored ISs is stated in Section 3, a generalisation of the theorem stated in Section 3 is proved in Section 4.

2. Preliminaries

In this section, several fundamental notions which will be used throughout this paper are defined. Throughout this paper, let $\sum a_n$ be an ISs with the partial sums (s_n) and (t_n^ρ) be the nth Cesàro

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mean of order ρ, with $\rho > -1$, of the sequence (na_n), that is [19]

$$t_n^{\rho} = \frac{1}{A_n^{\rho}} \sum_{r=1}^{n} A_{n-r}^{\rho-1} r a_r$$

where

$$A_n^{\rho} \simeq \frac{n^{\rho}}{\Gamma(\rho+1)}, \quad A_0^{\rho} = 1, \text{ and } A_{-n}^{\rho} = 0 \text{ for } n > 0$$

Here, Γ is gamma function defined by $\Gamma(\rho) = \int_0^{\infty} x^{\rho-1} e^{-x} dx$.
Let (ω_n^{ρ}) be a sequence defined as below [20]

$$\omega_n^{\rho} = \begin{cases} |t_n^{\rho}|, & \rho = 1 \\ \max_{1 \le r \le n} |t_r^{\rho}|, & 0 < \rho < 1 \end{cases} \tag{1}$$

Definition 2.1. [21] Let (φ_n) be any positive sequence. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \rho; \beta \mid_{\kappa}$, $\kappa \ge 1$, $\rho > -1$, $\beta \ge 0$, if

$$\sum_{n=1}^{\infty} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid t_n^{\rho} \mid^{\kappa} < \infty$$

Remark 2.2. $\varphi - \mid C, \rho; \beta \mid_{\kappa}$ summability reduces to $\mid C, \rho \mid_{\kappa}$ summability [22] in case of $\varphi_n = n$ and $\beta = 0$.

Definition 2.3. [23] A sequence (G_n) is said to be δ-quasi-monotone if $G_n \to 0$, $G_n > 0$ ultimately and $\Delta G_n = G_n - G_{n+1} \ge -\delta_n$ where $\delta = (\delta_n)$ is a sequence of positive numbers.

Definition 2.4. [24] A positive sequence (c_n) is said to be almost increasing if there exist a positive increasing sequence (d_n) and two positive constants M and N such that $M d_n \le c_n \le N d_n$.

Lemma 2.5. [25] If $0 < \rho \le 1$ and $1 \le v \le n$, then

$$\left| \sum_{r=0}^{v} A_{n-r}^{\rho-1} a_r \right| \le \max_{1 \le m \le v} \left| \sum_{r=0}^{m} A_{m-r}^{\rho-1} a_r \right| \tag{2}$$

Lemma 2.6. [26] Let (H_n) be an almost increasing sequence such that $n|\Delta H_n| = O(H_n)$. If (G_n) is a δ-quasi-monotone, and $\sum n\delta_n H_n < \infty$, $\sum G_n H_n$ is convergent, then

$$n H_n G_n = O(1) \quad as \quad n \to \infty \tag{3}$$

$$\sum_{n=1}^{\infty} n H_n |\Delta G_n| < \infty \tag{4}$$

3. Known Result

In [27], $\mid C, \rho \mid_{\kappa}$ method has been used to obtain following theorem.

Theorem 3.1. Let (H_n) be an almost increasing sequence and (γ_n) be any sequence with $|\Delta H_n| = O(H_n/n)$ such that

$$|\gamma_n| H_n = O(1) \quad as \qquad n \to \infty \tag{5}$$

Assuming also that there exists a sequence of numbers (G_n) such that it is δ-quasi-monotone such that $\sum n\delta_n H_n < \infty$, $\sum G_n H_n$ is convergent, and $|\Delta\gamma_n| \le |G_n|$ for all n. If the sequence (ω_n^{ρ}) satisfies the condition

$$\sum_{n=1}^{u} \frac{(\omega_n^{\rho})^{\kappa}}{n H_n^{\kappa-1}} = O(H_u) \quad \text{as} \quad u \to \infty \tag{6}$$

then the series $\sum a_n \gamma_n$ is summable $\mid C, \rho \mid_{\kappa}$, $0 < \rho \le 1$, $\kappa \ge 1$.

4. Main Result

The main concern of the article is to generalise Theorem 3.1 for the more general $\varphi - \mid C, \rho; \beta \mid_\kappa$ method.

Theorem 4.1. Let (H_n), (G_n) and (γ_n) be the sequences satisfying the same conditions as given in Theorem 3.1. Assuming that there is an $\epsilon > 0$ such that the sequence $(n^{\epsilon-\kappa}\varphi_n^{\beta\kappa+\kappa-1})$ is non-increasing. If

$$\sum_{n=1}^{u} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \frac{(\omega_n^\rho)^\kappa}{H_n^{\kappa-1}} = O(H_u) \quad \text{as} \quad u \to \infty \tag{7}$$

then the series $\sum a_n\gamma_n$ is summable $\varphi - \mid C, \rho; \beta \mid_\kappa$, $\beta \geq 0$, $0 < \rho \leq 1$, $\epsilon + (\rho - 1)\kappa > 0$, $\kappa \geq 1$.

Proof. Let $0 < \rho \leq 1$. Let (I_n^ρ) be the nth (C, ρ) mean of the sequence $(na_n\gamma_n)$. Using Abel's formula, we write

$$\begin{aligned} I_n^\rho &= \frac{1}{A_n^\rho} \sum_{i=1}^{n} A_{n-i}^{\rho-1} i a_i \gamma_i \\ &= \frac{1}{A_n^\rho} \sum_{i=1}^{n-1} \Delta\gamma_i \sum_{r=1}^{i} A_{n-r}^{\rho-1} r a_r + \frac{\gamma_n}{A_n^\rho} \sum_{i=1}^{n} A_{n-i}^{\rho-1} i a_i \end{aligned}$$

By Lemma 2.5, we achieve

$$\begin{aligned} |I_n^\rho| &\leq \frac{1}{A_n^\rho} \sum_{i=1}^{n-1} \mid \Delta\gamma_i \mid \left| \sum_{r=1}^{i} A_{n-r}^{\rho-1} r a_r \right| + \frac{\mid \gamma_n \mid}{A_n^\rho} \left| \sum_{i=1}^{n} A_{n-i}^{\rho-1} i a_i \right| \\ &\leq \frac{1}{A_n^\rho} \sum_{i=1}^{n-1} A_i^\rho \omega_i^\rho \mid \Delta\gamma_i \mid + \mid \gamma_n \mid \omega_n^\rho \\ &= I_{n,1}^\rho + I_{n,2}^\rho \end{aligned}$$

To prove Theorem 4.1, we need to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,j}^\rho \mid^\kappa < \infty \quad \text{for} \quad j = 1, 2$$

First, for $j = 1$, we have

$$\sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^\rho \mid^\kappa \leq \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} (A_n^\rho)^{-\kappa} \left\{ \sum_{i=1}^{n-1} A_i^\rho \omega_i^\rho \, |\Delta\gamma_i| \right\}^\kappa$$

Using the fact that $|\Delta\gamma_n| \leq |G_n|$ and Hölder's inequality, we achieve

$$\begin{aligned} \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^\rho \mid^\kappa &\leq \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} (A_n^\rho)^{-\kappa} \sum_{i=1}^{n-1} (A_i^\rho)^\kappa (\omega_i^\rho)^\kappa \, |G_i|^\kappa \left\{ \sum_{i=1}^{n-1} 1 \right\}^{\kappa-1} \\ &= O(1) \sum_{i=1}^{u} i^{\rho\kappa} (\omega_i^\rho)^\kappa \, |G_i| \, |G_i|^{\kappa-1} \sum_{n=i+1}^{u+1} \frac{\varphi_n^{\beta\kappa+\kappa-1} n^{\epsilon-\kappa}}{n^{1+\epsilon+(\rho-1)\kappa}} \end{aligned}$$

Here, using (3), we get $|G_i| = O(1/iH_i)$, therefore

$$\begin{aligned} \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^\rho \mid^\kappa &= O(1) \sum_{i=1}^{u} i^{\rho\kappa} (\omega_i^\rho)^\kappa \, |G_i| \, \frac{1}{(iH_i)^{\kappa-1}} \varphi_i^{\beta\kappa+\kappa-1} i^{\epsilon-\kappa} \int_i^\infty \frac{dx}{x^{1+\epsilon+(\rho-1)\kappa}} \\ &= O(1) \sum_{i=1}^{u} i \, |G_i| \, \varphi_i^{\beta\kappa+\kappa-1} i^{-\kappa} \frac{(\omega_i^\rho)^\kappa}{H_i^{\kappa-1}} \end{aligned}$$

Now, by applying Abel's formula, and by the hypotheses of Theorem 4.1 and Lemma 2.6, we achieve

$$\begin{aligned}\sum_{n=2}^{u+1}\varphi_n^{\beta\kappa+\kappa-1}n^{-\kappa}\mid I_{n,1}^{\rho}\mid^{\kappa} &= O(1)\sum_{i=1}^{u-1}\Delta(i\,|G_i|)\sum_{r=1}^{i}\varphi_r^{\beta\kappa+\kappa-1}r^{-\kappa}\frac{(\omega_r^{\rho})^{\kappa}}{H_r^{\kappa-1}}\\ &+ O(1)u\,|G_u|\sum_{i=1}^{u}\varphi_i^{\beta\kappa+\kappa-1}i^{-\kappa}\frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}}\\ &= O(1)\left(\sum_{i=1}^{u-1}i|\Delta G_i|H_i+\sum_{i=1}^{u-1}|G_i|\,H_i+u\,|G_u|\,H_u\right)\\ &= O(1)\quad as\quad u\to\infty\end{aligned}$$

Now, we write that $|\gamma_n| = O(1/H_n)$ from (5). Therefore, for $j = 2$, we get

$$\begin{aligned}\sum_{n=1}^{u}\varphi_n^{\beta\kappa+k\kappa-1}n^{-\kappa}\mid I_{n,2}^{\rho}\mid^{\kappa} &= \sum_{n=1}^{u}\varphi_n^{\beta\kappa+\kappa-1}n^{-\kappa}\,|\gamma_n|\,|\gamma_n|^{\kappa-1}\,(\omega_n^{\rho})^{\kappa}\\ &= O(1)\sum_{n=1}^{u}\varphi_n^{\beta\kappa+\kappa-1}n^{-\kappa}\,|\gamma_n|\,\frac{1}{H_n^{\kappa-1}}(\omega_n^{\rho})^{\kappa}\end{aligned}$$

From Abel's formula, we get

$$\begin{aligned}\sum_{n=1}^{u}\varphi_n^{\beta\kappa+\kappa-1}n^{-\kappa}\mid I_{n,2}^{\rho}\mid^{\kappa} &= O(1)\sum_{n=1}^{u-1}\Delta\,|\gamma_n|\sum_{i=1}^{n}\varphi_i^{\beta\kappa+\kappa-1}i^{-\kappa}\frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}}\\ &+ O(1)\,|\gamma_u|\sum_{i=1}^{u}\varphi_i^{\beta\kappa+\kappa-1}i^{-\kappa}\frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}}\end{aligned}$$

Then, we achieve

$$\begin{aligned}\sum_{n=1}^{u}\varphi_n^{\beta\kappa+\kappa-1}n^{-\kappa}\mid I_{n,2}^{\rho}\mid^{\kappa} &= O(1)\left(\sum_{n=1}^{u-1}|\Delta\gamma_n|\,H_n+|\gamma_u|\,H_u\right)\\ &= O(1)\left(\sum_{n=1}^{u-1}|G_n|\,H_n+|\gamma_u|\,H_u\right)\\ &= O(1)\quad as\quad u\to\infty\end{aligned}$$

□

5. Conclusions

In this paper, a theorem dealing with generalised absolute Cesàro summability has been introduced which reduces to Theorem 3.1 for $\varphi_n = n$, $\beta = 0$ and $\epsilon = 1$. Hence, the equality (7) reduces to the equality (6). Furthermore, a known result on $|C,1|_{\kappa}$ summability can be deducted whenever $\varphi_n = n$, $\beta = 0$, $\rho = 1$ and $\epsilon = 1$ [27]. In the light of this study, one can generalise these results for using either different summability methods or different sequence classes.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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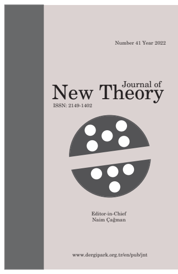

The Source of Primeness of Rings

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Abstract — In this study, we define a new concept, i.e., source of primeness of a ring R, as $P_R := \bigcap_{a \in R} S_R^a$ such that $S_R^a := \{b \in R \mid aRb = (0)\}$. We then examine some basic properties of P_R related to the ring's idempotent elements, nilpotent elements, zero divisor elements, and identity elements. Finally, we discuss the need for further research.

Keywords — *Prime ring, semiprime ring, source of primeness, source of semiprimeness*

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1. Introduction

Our primary aim in this study is to describe the ring types that are not included in the literature. These definitions were derived in light of the original existing definitions in ring theory and can be seen as generalizations of division rings, reduced rings and domain, respectively. To define these new concepts in rings, we will first examine the structure of the subset, which we call the source of primality in rings, in consideration of articles [1,2]. Before we get to the point, let us summarize the terminology we will use throughout the study.

We will now give the basic definitions in [3–5]. An element with a right (left) multiplicative inverse in a unit ring is called a right (left) unit, and accordingly, by the unit is meant a two-sided unit. An element a of a ring R is called a right (left) zero divisor if there is a nonzero element $b \in R$ such that $ba = 0$ ($ab = 0$, respectively). A nonzero-divisor element is neither a left nor a right zero-divisor. A domain is a ring with no nonzero right or left zero-divisors. A ring with nonzero elements, which are all units, is called a division ring. An element a of a ring R is called a nilpotent element of index n if n is the least positive integer such that $a^n = 0$. A reduced ring has no nonzero nilpotent elements. An idempotent element e of R is $e = e^2$. An element of R is called central if it commutes with every element of R.

R is a prime ring if $aRb = (0)$ with $a, b \in R$ requires $a = 0$ or $b = 0$, and R is a semiprime ring if $aRa = (0)$ with $a \in R$ requires $a = 0$. Here, the ideal generated by the zero element is shown by (0). In [1] authors, defined the source of semiprimeness of the subset A in R where A is a nonempty subset of ring R as follows: $\mathcal{S}_R(A) = \{a \in R \mid aAa = (0)\}$ and $\mathcal{S}_R$ is written in place of $\mathcal{S}_R(R)$ for a ring R. As is known, every ring is isomorphic to the subdirect sum of the prime rings. Now let us introduce our primary instrument, which we have focused on throughout the article. We define the subset S_R^a of R as

$$S_R^a = \{b \in R \mid aRb = (0)\}$$

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Moreover, we call the following set as the source of primeness of R

$$P_R = \bigcap_{a \in R} S_R^a$$

This set is always different from empty because it contains $\{0\}$. If P_R consists only of $\{0\}$, then P_R is trivial, and at another end, P_R may be equal R. P_R is trivial if only if R is a prime ring. If we ignore both situations, our general interest will be situations between these two ends. A diligent reader must have intuited that P_R has always been located at the prime radical of R because $P_R \subseteq S_R$.

Thus, by examining the elements of the $R - P_R$ set more closely, we have a chance to examine the P_R subset to obtain the structural properties of R. We choose the term "the source of primeness of R" because every element in $R - P_R$ acts as a nonzero element in any prime ring. For every $b \in R - P_R$, $RRb \neq (0)$ comes from the "primeness" part.

At that, we examined the properties of P_R for the arbitrary ring R . The first important property is that while I is a right ideal, P_R is an ideal. Another important consequence is that the source of primeness of R is preserved under the ring isomorphisms.

2. Results

Definition 2.1. Let R be a ring, $\emptyset \neq A \subseteq R$ and $a \in R$. We define $S_R^a(A)$ as follows:

$$S_R^a(A) = \{b \in R \mid aAb = (0)\}$$

$P_R(A) = \bigcap_{a \in R} S_R^a(A)$ is called the source of primeness of the subset A in R. We write S_R^a instead of $S_R^a(R)$. In particular, we can similarly define the source of primeness of the semigroup R as follows:

$$P_R = \bigcap_{a \in R} S_R^a$$

Some of the basic inferences that will help understand the concept are as follows:

i. Since R be a ring, we obtain $aA0 = (0)$ for all $a \in R$. Hence, $P_R = \bigcap_{a \in R} S_R^a \neq \emptyset$.

ii. $S_R^0(A) = R$

iii. $S_A^a \subseteq S_R^a(A)$. If $b \in S_A^a$, then $b \in A$ such that $aAb = (0)$. Since $A \subseteq R$, we have $b \in R$ and $aAb = (0)$. This means that $b \in S_R^a(A)$.

If $x \in P_R(A)$, then $aAx = (0)$, for all $a \in R$. Hence, $RAx = (0)$. Therefore, $P_R(A) = \{x \in R : RAx = (0)\}$.

Proposition 2.2. Let R be a ring and $\emptyset \neq A, B \subseteq R$. Then, $P_{R\times R}(A \times B) = P_R(A) \times P_R(B)$.

Proof. $P_{R\times R}(A \times B) = \{(x, y) \in R \times R \mid (R \times R)(A \times B)(x, y) = (0, 0)\}$. Assume that $(x, y) \in P_{R\times R}(A \times B)$. Then, $(R \times R)(A \times B)(x, y) = (0, 0)$. Namely, $RAx = (0)$, $RBy = (0)$. Hence, $x \in P_R(A)$, $y \in P_R(B)$. Thus, $(x, y) \in P_R(A) \times P_R(B)$. Similarly, the reverse side is also seen. □

Proposition 2.3. Let R be a ring. If $1_R \in R$, then $P_R \subseteq \{x \in R \mid x^2 = 0\}$.

Proof. Let $K = \{x \in R \mid x^2 = 0\}$. If $x \in P_R$, then $RRx = (0)$. Since $1_R \in R$, then $0 = x1_Rx = x^2$. Hence, $x \in K$ is satisfied. Thus, $P_R \subseteq K$. □

Proposition 2.4. Let A and B be two nonempty subsets of a ring R. Then, the following conditions hold:

i. If $A \subseteq B$, then $P_R(B) \subseteq P_R(A)$. In particular, $P_R \subseteq P_R(A)$.

ii. If A is a subring of R, then $A \cap P_R(A) \subseteq P_A$.

Proof.

i. Let $x \in P_R(B)$. We have $x \in \bigcap_{a\in R} S_R^a(B)$ and $aBx = (0)$, for all $a \in R$. Since $A \subseteq B$, we get $aAx = (0)$ for all $a \in R$. This means that $x \in S_R^a(A)$ for all $a \in R$. Hence, we get $x \in \bigcap_{a\in R} S_R^a(A)$ and $x \in P_R(A)$. This gives up $P_R(B) \subseteq P_R(A)$. Specially, $P_R \subseteq P_R(A)$ is satisfied for $A \subseteq R$.

ii. Let $x \in A \cap P_R(A)$. Then, $x \in A$ and $x \in P_R(A)$. Hence, we get $x \in A$ and $x \in \bigcap_{a\in R} S_R^a(A)$. Using $x \in A$, $x \in S_A^a$ for all $a \in A$. This expression gives us $x \in \bigcap_{a\in R} S_A^a = P_A$. Thus, $A \cap P_R(A) \subseteq P_A$.

□

It is well known that every prime ring is a semiprime ring. Therefore, we present a relationship between the source of primeness and the source of semiprimeness as follows:

Proposition 2.5. Let R be a ring, $\emptyset \neq A \subseteq R$. Then, $P_R(A) \subseteq S_R(A)$.

Proof. If $b \in P_R(A)$, then $b \in \bigcap_{a\in R} S_R^a(A)$. In particular, $b \in S_R^b(A)$. Therefore, $bAb = (0)$. Hence, $b \in S_R(A)$. □

Proposition 2.6. Let R be a ring, $a \in R$ and I be a nonempty subset of R. In this case, the following properties are provided.

i. $S_R^a(I)$ is a right ideal of R.

ii. If I is a right ideal of R, then $S_R^a(I)$ is a left ideal of R.

iii. If I is a right ideal of R, then $S_R^a(I)$ is an ideal of R.

Proof.

i. Let $x, y \in S_R^a(I)$. Then, $aIx = aIy = (0)$ for $a \in R$. From here $aI(x - y) = aIx - aIy = (0)$, we obtain $x - y \in S_R^a(I)$. Besides that, we have $aI(xr) = (aIx)r = (0)$ for any $r \in R$. Thus, we get $xr \in S_R^a(I)$. Hereby, $S_R^a(I)$ is a right ideal of R.

ii. Let I is a right ideal of R and $x \in S_R^a(I)$, $r \in R$. Then, we get $aI(rx) = a(Ir)x \subseteq aIx = (0)$. Hence, we have $rx \in S_R^a(I)$ and $S_R^a(I)$ is a left ideal of R.

iii. We can easily see that if I is a right ideal of R, then $S_R^a(I)$ is an ideal of R from *i* and *ii*.

□

Theorem 2.7. Let R be a ring and I be a nonempty subset of R. If I is a right ideal of R, then $P_R(I)$ is an ideal of R.

Proof. Let I be a right ideal of R and $x, y \in P_R(I)$. Therefore, $x, y \in \bigcap_{a\in R} S_R^a(I)$ and $x, y \in S_R^a(I)$ for all $a \in R$. Then, $aIx = aIy = (0)$ for all $a \in R$. Since $S_R^a(I)$ is an ideal of R for all $a \in R$ from Proposition 2.6, we write $x - y, xr, rx \in S_R^a(I)$ for all $a, r \in R$. Consequently, we get $x - y, xr, rx \in \bigcap_{a\in R} S_R^a(I) = P_R(I)$. For this reason, $P_R(I)$ is an ideal of R. □

In the following theorem, if the ring R is prime, its relation to the set the source of primeness examined.

Theorem 2.8. Let R be a ring. Thus the followings are provided.

i. If R is a prime ring, then $P_R = \{0\}$.

ii. The source of primeness P_R is contained by every prime ideal of the R.

PROOF.

i. Let R be a prime ring and $x \in P_R$. From definition of the set P_R, we have $RRx = (0)$. Since R is prime ring, we obtained $x = 0$. Namely, $P_R = \{0\}$.

ii. Let P be a prime ideal in R. If $x \in P_R$, then $RRx = (0) \subseteq P$. Since P is prime, we get $x \in P$. Hence, we get $P_R \subseteq P$. This gives us that every prime ideal of R includes P_R.

□

We know that the properties of idempotent, nilpotent, and zero-divisor elements in a ring are also related to the primality of that ring. Some of the relationships between the source of primeness and these special elements are as follows:

Proposition 2.9. Let R be a ring. Then, the following holds.

i. If R is a Boolean ring, then $P_R = \{0\}$.

ii. If $0 \neq a \in P_R$, then a is a zero divisor element of R.

iii. If R has identity element, then $P_R = \{0\}$.

PROOF.

i. Let R be a Boolean ring and $a \in P_R$. Since $RRa = (0)$, then $aaa = 0$. Moreover, a is an idempotent element, thus $a = 0$. Then, $P_R = \{0\}$.

ii. If $0 \neq a \in P_R$, then $RRa = (0)$. First case is if $RR = (0)$, then $aa = 0$. Thus, a is a zero divisor element. The other case is $RR \neq (0)$. In this case, a is a right zero divisor element since $RRa = (0)$. Besides, it is either $Ra = 0$ or $Ra \neq (0)$. If $Ra = 0$, then a is a zero divisor element. Let $Ra \neq (0)$. Since $RRa = (0)$, $aRa = (0)$. This means that a is a left zero divisor element. Thus, a is a zero divisor element.

iii. Let R has identity element 1_R and $a \in P_R$. Then, we have $RRa = (0)$. In particular, we write $1_R 1_R a = 0$. Then, we obtain $P_R = \{0\}$.

□

From Proposition 2.9, it is easy to see that the following corollary.

Corollary 2.10. For any ring R the following is always true.

i. There is no idempotent element other than zero in P_R.

ii. Every element in P_R is nilpotent.

PROOF.

i. Let $a \in P_R$ be an idempotent element. Since $RRa = (0)$, then $aaa = 0$. Moreover, a is an idempotent element, thus $a = 0$.

ii. If $0 \neq a \in P_R$. Since $RRa = (0)$, then we get $aaa = 0$. Thus, a is a nilpotent element.

□

Theorem 2.11. Let R and T be two rings and $f : R \to T$ a ring homomorphism. Therefore, $f(P_R) \subseteq P_{f(R)}$. If f is injective, then $f(P_R) = P_{f(R)}$.

PROOF. Let $x \in f(P_R)$. In this case, there is $a \in P_R$ such that $x = f(a)$. Namely, $RRa = (0)$. Since

$$(0) = f(RRa) = f(R)f(R)f(a)$$

we get $f(a) \in P_{f(R)}$ and so $x \in P_{f(R)}$. Hence, $f(P_R) \subseteq P_{f(R)}$.

Let $y \in P_{f(R)}$. From the set definition, $f(R)f(R)y = (0)$ for $y \in f(R)$. Since $y \in f(R)$, we have $y = f(r)$ for $r \in R$. Hence, $f(R)f(R)f(r) = (0)$ and since f is a homomorphism $f(RRr) = (0)$ is satisfied. Using f is injective, we obtain $RRr = (0)$ for $r \in R$. So, $r \in P_R$. From here, $y = f(r) \in f(P_R)$ is obtained. Thus, $P_{f(R)} \subseteq f(P_R)$. □

3. Conclusions

We first defined the concept of the source of primeness of an associative ring. We have shown that when R is a prime ring, the source of primeness set consists only of zero element, which is very useful for examining the work done on the prime ring for the source of primeness of R. Furthermore, we adapt some well-known results in prime ring to the source of primeness of R. The properties of the set source of primeness of a ring R can be investigated in the sense of the articles in [6–8].

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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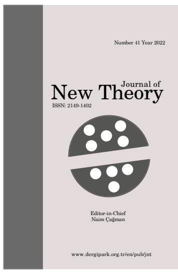

Bivariate Variations of Fibonacci and Narayana Sequences and Universal Codes

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Abstract − In this study, we worked on the third-order bivariate variant of the Fibonacci universal code and the second-order bivariate variant of the Narayana universal code, depending on two negative integer variables u and v. We then showed in tables these codes for $1 \leq k \leq 100$, $u = -1, -2, \dots, -20$, and $v = -2, -3, \dots, -21$ (u and v are consecutive, $v < u$). Moreover, we obtained some significant results from these tables. Furthermore, we compared the use of these codes in cryptography. Finally, we obtained the third-order bivariate variant of Fibonacci codes is more valuable than the second-order bivariate variant of Narayana codes.

Keywords − *Fibonacci sequence, Narayana sequence, cryptography, variant Fibonacci code, variant Narayana code*

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1. Introduction

Number sequences have been popular with scientists for centuries. The most popular of these sequences is the Fibonacci sequence. The Fibonacci sequence, $\{F_k\}_0^\infty$, is a series of numbers, starting with the integers 0 and 1, in which the value of any element is computed by taking the summation of the two consecutive numbers. Here, for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$ [1]. In 1202, this sequence was introduced by Fibonacci. The Fibonacci numbers in the range of $-8 \leq k \leq 8$ are as follows:

$$-21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21.$$

A sequence that based on the Fibonacci sequence and is a generalization of the Fibonacci sequence is the Tribonacci sequence. The Tribonacci sequence, $\{T_k\}_0^\infty$, is a series of numbers, starting with the integers 0, 1, and 1, in which the value of any element is computed by taking the summation of the preceding three terms [2]. If so, for $k \geq 3$, $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ [1]. In 1914, this number series was presented by Agronomof. The Tribonacci numbers in the range of $-8 \leq k \leq 8$ are as follows:

$$4, 1, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, 24, 44.$$

Another sequence that based on a similar problem with Fibonacci numbers is the Narayana sequence. The Narayana sequence $\{N_k\}_0^\infty$, is a series of numbers, starting with the integers 1, 1, and 1, in which the value of any element is computed by taking the summation of the previous term and term two places before. If so, for

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$k \geq 3$, $N_{k+1} = N_k + N_{k-2}$ [1]. In 1356, this number series was presented by Narayana in the treatise named Gaṇita Kaumudi. The Narayana numbers in the range of $-8 \leq k \leq 8$ are as follows:

$$-2, 1, 1, -1, 0, 1, 0, 0, 1, 1, 1, 2, 3, 4, 6, 9, 13.$$

There are also different variants of the Fibonacci sequence. The author [3] defined a variation of the Fibonacci sequence that is the more general of Gopala Hemachandra (GH) sequence [4]:

$$\{u, v, u+v, u+2v, 2u+3v, 3u+5v\}$$

where u, v are integers and a second order variant Fibonacci sequence, as the Gopala Hemachandra sequence above such that $v = 1 - u$.

In addition, different universal codes have been defined in the literature. Codes that are obtained by using number sequences are involving to them. The Fibonacci universal code, GH code (or variant Fibonacci code) and the Narayana code are examples of these can be given [3-9]. The most used of these codes is the Fibonacci code [8]. Fibonacci code is a universal code that encodes positive integers with binary representations according to Zeckendorf's Theorem [9]. To find the Fibonacci code for any positive integer, the following algorithm must hold.

One Fibonacci representation or code can be obtained for each positive integer C with a binary string of length t, $g_1 g_2 \dots g_{t-1} g_t$, such that $C = \sum_{i=1}^{t} g_i F_i^{(m)}$. The representation is unique if and only if it is used the following algorithm: When it is given the integer C, it is selected the largest Fibonacci number $F_k^{(2)}$ equal to C or smaller to C; after that, it is continued repeating with $C - F_k^{(2)}$ [8]. Finally, to find the Fibonacci code for any positive integer, it is added $1-$bit to the Fibonacci representation of the positive integer. For example, $31 = 2 + 8 + 21 = F_2^{(2)} + F_5^{(2)} + F_7^{(2)}$.

Hence, its Fibonacci representation is 0100101 and its Fibonacci code is 01001011.

According to the above algorithm, there is no contiguous 1-bit in the binary representation. If we apply this algorithm to higher orders, it is seen that the same operations are carried out. But in the higher orders, it is added 1, $m-1$ bits to the m^{th} order Fibonacci representation of k to build the m^{th} order Fibonacci code. Therefore, there are no successive of m bits [8].

Narayana and GH universal code are two generalizations of Fibonacci universal code. Thus, these codes also encode positive integers with binary representations according to above algorithm. That is, these universal codes and the variants of these codes can be obtained by using the same rule used to generate the standard Fibonacci code [3].

Moreover, cryptographic applications can be made using these codes or representations. There are many works on these codes in literature (see for example [5-10]).

This paper research the second-order bivariate variant of Narayana codes and the third-order bivariate GH codes and the properties of these codes. Furthermore, it also gives some results regarding the use of these codes in cryptography.

2. Preliminaries

In this section, the basic definitions and theorems that used in this study will be given.

Definition 2.1. [8] The m^{th} order Fibonacci numbers, represented by $F_k^{(m)}$, are described with iteration relation as follows:

$$F_k^{(m)} = F_{k-1}^{(m)} + F_{k-2}^{(m)} + \cdots + F_{k-m}^{(m)}$$

for $k > 0$ and the boundary conditions $F_0^{(m)} = 1$ and $F_l^{(m)} = 0$ $(-m < l < 0)$.

Thus, in 2007, a second-order variation of the Fibonacci code was presented for $m = 2$ as follows:

Definition 2.2. [3] The second-order variant Fibonacci sequences, $GH_u^{(2)}(k)$ is described with the sequences $\{u, v, u+v, u+2v, 2u+3v, 3u+5v\}$ where $v = 1-u$, that is , $GH_u^{(2)}(1) = u$; $GH_u^{(2)}(2) = 1-u$; and for $k \geq 3$, $GH_u^{(2)}(k) = GH_u^{(2)}(k-1) + GH_u^{(2)}(k-2)$.

Afterwards, in 2010, the authors [9] studied on the second-order variation Fibonacci code and obtained some results about the second-order variation Fibonacci codes of some positive integers.

Then, in 2015, a third-order variation of the Fibonacci code was presented for $m = 3$ as follows:

Definition 2.3. [10] The third-order variant Fibonacci sequences, $GH_u^{(3)}(k)$ is described with the sequences $\{u, v, u+v, 2\,u+2v, 3u+4v, 6u+7v\}$ where $v = 1-u$, that is , $GH_u^{(3)}(1) = u$, $GH_u^{(3)}(2) = 1-u$, $GH_u^{(3)}(3) = 1$, and for $k \geq 4$, $GH_u^{(3)}(k) = GH_u^{(3)}(k-1) + GH_u^{(3)}(k-2) + GH_u^{(3)}(k-3)$.

In this situation, the second and the third order variant Fibonacci (or GH) codes can be defined above definitions. And we know that these universal codes can be obtained by using the same rule used to generate the standard Fibonacci code [3]. Thus, it is obvious that as the value of u changes, a different sequence is obtained. Daykin [11] proved that only the Fibonacci sequence forms a unique Fibonacci code for all positive integers. Moreover, some integers have many GH codes, while others have no GH code. For instance, if $VF_{-5}^{(2)}(k) = \{-5,6,1,7,8,15,23,38,\dots\}$, then there is no GH code for $k = 20$. Similarly, if $VF_{-2}^{(2)}(k) = \{-2,3,1,4,5,9,14,23,\dots\}$, then there are two GH codes for $k = 13$. Because

$13 = 4 + 9 = VF_{-2}^{(2)}(4) + VF_{-2}^{(2)}(6) = -2 + 1 + 14 = VF_{-2}^{(2)}(1) + VF_{-2}^{(2)}(3) + VF_{-2}^{(2)}(7)$,

these codes are 0001011 and 10100011. Furthermore, if $VF_{-11}^{(3)}(k) = \{-11,12,1,2,15,18,35,68,\dots\}$, then there is no GH code for $k = 11$ [10], and if $VF_{-5}^{(3)}(k) = \{-5,6,1,2,9,12,23,44,\dots\}$, then there are two GH codes for integer $k = 9$. Because

$9 = VF_{-5}^{(3)}(5) = -5 + 2 + 12 = VF_{-5}^{(3)}(1) + VF_{-5}^{(3)}(4) + VF_{-5}^{(3)}(6)$,

these codes are 0000111 and 10010111.

Moreover, in 2016, it was obtained and proved the following theorem by Basu et al [12]. According to following theorem, it is obtained that the relationship between the second-order variant Fibonacci sequence and the Fibonacci sequence.

Theorem 2.4. [12] Let $VF_u(k)$ is the second-order variant Fibonacci sequence and $F(k)$ is the Fibonacci sequence. In this case, for $k \geq 1$,

$VF_u(k) = F(k-2) - uF(k-4)$

where k is an integer.

Definition 2.5. [5] A second-order variant of the Narayana sequence, $VN_u(k)$, is described the sequence $\{u, 3-u, 1-u, 1, 4-u, 5-2u, 6-2u, 10-3u, \dots\}$ with initial conditions $VN_u(0) = u$; $VN_u(1) = 3-u$; $VN_u(2) = 1-u$; and for $k \geq 3$, $VN_u(k) = VN_u(k-1) + VN_u(k-3)$.

Similarly, the second order variant Narayana codes can be defined above definition. And, we know that these universal codes can be obtained by using the same algorithm used to generate the standard Fibonacci code [3]. For instance, there is no variant Narayana code for 2 for $VN_{-3}(k) = \{-3,6\,4,1,7,11,12,19\}$. And, we obtain that there are two variant Narayana codes of 12 for $VN_{-1}(k) = \{-1,4,2,1,5,7,8,13,20,\dots\}$. Because

$12 = 4 + 8 = VN_{-1}(2) + VN_{-1}(7) = -1 + 13 = VN_{-5}^{(3)}(1) + VN_{-5}^{(3)}(8)$,

these variant Narayana codes of 12 are 01000011 and 100000011.

3. The Research Findings and Discussion

In this study, firstly, we obtained and demonstrated the following theorem, which is the relation between the third-order variant Fibonacci sequence and the Tribonacci sequence.

Theorem 3.1. Let $VF_u^{(3)}(k)$ is the third-order variant Fibonacci sequence and $T(k)$ is the Tribonacci sequence. In this case, for $k \geq 1$,

$VF_u^{(3)}(k) = T(k-1) - uT(k-4)$

where k is an integer.

Proof. We obtain the following set from the third-order variant of the Fibonacci sequence.

$$VF_u^{(3)}(k) = \{u, 1-u, 1, 2, 4-u, 7-u, 13-2u, 24-4u, \dots\}.$$

$$VF_u^{(3)}(1) = u = 0 - u(-1) = T(-1) - uT(-3),$$

$$VF_u^{(3)}(2) = 1 - u = 1 - u(1) = T(1) - uT(-2),$$

$$VF_u^{(3)}(3) = 1 = 1 - u(0) = T(2) - uT(-1).$$

The results are correct for $k = 1, k = 2$, and $k = 3$, as seen above. Suppose that the results are correct for $k = 1,2 \dots, n$. In that case, $VF_u^{(3)}(n-2) = T(n-3) - uT(n-6)$, $VF_u^{(3)}(n-1) = T(n-2) - uT(n-5)$, and $VF_u^{(3)}(n) = T(n-1) - uT(n-4)$. Let's show that this equation is true for n +1.

$$VF_u^{(3)}(n+1) = T(n-1) - uT(n-4) + T(n-2) - uT(n-5) + T(n-3) - uT(n-6)$$

$$= T(n-1) + T(n-2) + T(n-3) - uT(n-4) + T(n-5) + T(n-6)$$

$$= T(n) - uT(n-3) = T(n+1-1) - uT(n+1-4).$$

Therefore, by the induction, we can write

$$VF_u^{(3)}(k) = T(k-1) - uT(k-4), k \geq 1.$$

3.1. The Third Order Bivariate Gopala Hemachandra Sequences and Codes

We will describe a new variation depending on two negative variables of the GH sequence for $m = 3$ as follows:

Definition 3.1.1. The third-order bivariate GH sequence $GH_{(u,v)}^{(3)}(k)$ is described the sequence $\{u, v, s, u+v+s, u+2v+2s, 2u+3v+4s, 4u+6v+7s, \dots \}$ where $s = 1-u-v$, with initial conditions $GH_{(u,v)}^{(3)}(1) = u$, $GH_{(u,v)}^{(3)}(2) = v$, $GH_{(u,v)}^{(3)}(3) = 1-u-v$ and for $k \geq 4$, $GH_{(u,v)}^{(3)}(k) = GH_{(u,v)}^{(3)}(k-1) + GH_{(u,v)}^{(3)}(k-2) + GH_{(u,v)}^{(3)}(k-3)$.

For instance, from the above definition for $u = -2, v = -3$, we have $\{-2, -3, 6, 1, 4, 11, 16, 31, \dots\}$. It is obviously seen that different sequences will be obtained for different values of u and v.

With these bivariate variants of GH sequences, we may compose a new universal source code, which we named the bivariate GH code or bivariate variant of Fibonacci universal code. To make, it can be used the same rule used to generate the standard Fibonacci code as in the second and the third order variant Fibonacci code. In addition, we know that GH sequences authorize having more than one Zeckendorf representation of any integer. Similarly, here, we have obtained the third-order bivariate variant of the GH sequences authorizing having more than one Zeckendorf representation of any integer. For example, for $GH_{(-2,-3)}^{(3)}(k) = \{-2, -3, 6, 1, 4, 11, 16, 31, \dots\}$,

$$17 = 6 + 11 = GH_{(-2,-3)}^{(3)}(3) + GH_{(-2,-3)}^{(3)}(6) = 1 + 16 = GH_{(-2,-3)}^{(3)}(4) + GH_{(-2,-3)}^{(3)}(7).$$

Therefore, the third order bivariate GH code of 17 would be both 00100111 and 000100111.

In this study, we investigated for which values u and v, the third-order bivariate GH codes precisely exist or for which k positive integers they do not exist. For instance, we obtained that each positive integer has at least one bivariate GH code for $GH^{(3)}_{(-2,-6)}(k) = \{-2, -6, 9, 1, 4, 14, 19, 37, \dots\}$. But there is no bivariate GH code of 5 for $GH^{(3)}_{(-6,-2)}(k) = \{-6, -2, 9, 1, 8, 18, 27, 53, \dots\}$.

Basu and Prasad [9] obtained the second-order GH codes $GH^{(2)}_{u}(k)$ or undetectable values of the positive integer k for $k = 1, 2, \dots, 100$ and for $u = -2, \dots, -20$. Besides, Nallı and Özyılmaz [10] obtained the third-order GH codes $GH^{(3)}_{u}(k)$ or undetectable values of the positive integer k, for $k = 1, 2, \dots, 100$ and for $u = -2, \dots, -20$.

In this section, we obtained the third-order bivariate GH codes $VF^{(3)}_{(u,v)}(k)$ or undetectable values (–) of the positive integer k for $k = 1, 2, \dots, 100$ and for $u = -1, \dots, -20$ and $v = -2, \dots, -21$ (u and v are consecutive, $v < u$) Tables 1 and 2. From Tables 1 and 2, we got the following results for the third-order bivariate GH codes.

i. For the positive integers $k = 1,2,3$, the third-order bivariate GH code $VF^{(3)}_{(u,v)}(k)$ exactly exists, for $u = -1, -2, \dots, -20$ and $v = -2, -3, \dots, -21$ (u and v are consecutive, $v < u$).

ii. For $1 \le k \le 100$, there are at most j consecutive undetectable (--) values in the third-order bivariate GH code in $VF^{(3)}_{(-3+j,-(4+j))}(k)$ column in which $1 \le j \le 17$.

iii. For $1 \le k \le 100$, as long as j raises, the detectable of GH code is reduced in $VF^{(3)}_{(-3+j,-(4+j))}(k)$ column in which $1 \le j \le 17$.

Table 1. The third order bivariate GH codes of k for $1 \le k \le 100$, and for $u = -1, -2, \dots, -10$ and $v = -2, -3, \dots, -11$ (u and v are consecutive, $v < u$).

k	$(u=-1)$ $(v=-2)$	$(u=-2)$ $(v=-3)$	$(u=-3)$ $(v=-4)$	$(u=-4)$ $(v=-5)$	$(u=5)$ $(v=-6)$	$(u=-6)$ $(v=-7)$	$(u=-7)$ $(v=-8)$	$(u=-8)$ $(v=-9)$	$(u=-9)$ $(v=-10)$	$(u=-10)$ $(v=-11)$
1	000111	000111	000111	000111	000111	000111	000111	000111	000111	000111
2	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111
3	0000111	1001111	1001111	1001111	1001111	1001111	1001111	1001111	1001111	1001111
4	00111	0000111	01111	–	–	–	–	–	–	–
5	001111	0001111	0000111	01111	–	–	–	–	–	–
6	01000111	00111	0001111	0000111	01111	–	–	–	–	–
7	10000111	0110111	11000111	0001111	0000111	01111	–	–	–	–
8	00000111	01000111	00111	11000111	0001111	0000111	01111	–	–	–
9	00010111	10000111	001111	11010111	11000111	0001111	0000111	01111	–	–
10	010000111	10010111	01000111	00111	11010111	11000111	0001111	0000111	01111	–
11	100000111	00000111	10000111	001111	–	11010111	11000111	0001111	0000111	01111
12	000000111	00010111	10010111	01000111	00111	–	11010111	11000111	0001111	0000111
13	000100111	010000111	0010111	10000111	001111	–	–	11010111	11000111	0001111
14	100010111	100000111	00000111	10010111	01000111	00111	–	–	11010111	11000111
15	000010111	100100111	00010111	110000111	10000111	001111	–	–	–	11010111
16	001000111	000000111	010000111	0010111	10010111	01000111	00111	–	–	–
17	001100111	000100111	100000111	00000111	110000111	10000111	001111	–	–	–
18	101010111	100010111	100100111	00010111	110100111	10010111	01000111	00111	–	–
19	100001111	100110111	00001111	010000111	0010111	110000111	10000111	001111	–	–
20	000001111	000010111	000000111	100000111	00000111	110100111	10010111	01000111	00111	–
21	0100000111	000110111	000100111	100100111	00010111	–	110000111	10000111	001111	–
22	0101000111	001000111	100010111	01100111	010000111	0010111	110100111	10010111	01000111	00111
23	0000000111	001100111	100110111	00001111	100000111	00000111	–	110000111	10000111	001111
24	0001000111	101010111	011000111	000000111	100100111	00010111	–	110100111	10010111	01000111
25	1000100111	100001111	000010111	000100111	110110111	010000111	0010111	–	110000111	10000111

Table 1. Continued

k	$(u=-1)$ $(v=-2)$	$(u=-2)$ $(v=-3)$	$(u=-3)$ $(v=-4)$	$(u=-4)$ $(v=-5)$	$(u=5)$ $(v=-6)$	$(u=-6)$ $(v=-7)$	$(u=-7)$ $(v=-8)$	$(u=-8)$ $(v=-9)$	$(u=-9)$ $(v=-10)$	$(u=-10)$ $(v=-11)$
26	0000100111	100101111	000110111	100010111	01100111	100000111	00000111	–	110100111	10010111
27	0010000111	000001111	00101111	100110111	00001111	100100111	00010111	–	–	110000111
28	0011000111	0100000111	001000111	00110111	000000111	110110111	010000111	0010111	–	110100111
29	0100010111	0101000111	001100111	011000111	000100111	–	100000111	00000111	–	–
30	0101010111	1001000111	101010111	000010111	100010111	01100111	100100111	00010111	–	–
31	0000010111	0101000111	100001111	000110111	100110111	00001111	110110111	010000111	0010111	–
32	0001010111	0001000111	100101111	110001111	00100111	000000111	–	100000111	00000111	–
33	0100001111	1000100111	001010111	00101111	00110111	000100111	–	100100111	00010111	–
34	0101001111	1001100111	000001111	001000111	011000111	100010111	01100111	110110111	010000111	0010111
35	0000001111	0000100111	0100000111	001100111	000010111	100110111	00001111	–	100000111	00000111
36	0001001111	0001100111	0101000111	101010111	000110111	–	000000111	–	100100111	00010111
37	1000101111	0010000111	1001000111	100001111	110001111	00100111	000100111	–	110110111	010000111
38	0000101111	0011000111	011001111	100101111	110101111	00110111	100010111	01100111	–	100000111
39	0010001111	0100010111	0101000111	1101000111	00101111	011000111	100110111	00001111	–	100100111
40	0011001111	1000010111	0001000111	001010111	001000111	000010111	–	000000111	–	110110111
41	01000000111	1001010111	1000100111	000001111	001100111	000110111	–	000100111	–	–
42	01010000111	0000010111	1001100111	0100000111	101010111	110001111	00100111	100010111	01100111	–
43	00000000111	0001010111	0110000111	0101000111	100001111	110101111	00110111	100110111	00001111	–
44	00010000111	0100001111	0000100111	1001000111	100101111	–	011000111	–	000000111	–
45	10001000111	1000001111	0001100111	–	1101000111	00101111	000010111	000010111	000100111	–
46	00001000111	1001001111	1100010111	011001111	–	001000111	000110111	–	100010111	01100111
47	00100000111	0000001111	0010000111	0101000111	001010111	001100111	110001111	00100111	100110111	00001111
48	00110000111	0001001111	0011000111	0001000111	000001111	101010111	110101111	00110111	–	000000111
49	01000100111	1000101111	0100010111	1000100111	0100000111	100001111	–	011000111	–	000100111
50	10000100111	1001101111	1000010111	–	0101000111	100101111	–	000010111	000010111	100010111
51	00000100111	0000101111	1001010111	001001111	1001000111	1101000111	00101111	000110111	–	100110111
52	00010100111	0001101111	1100001111	0110000111	1101100111	–	001000111	110001111	00100111	–
53	10001100111	11000000111	0000010111	0000100111	–	–	001100111	110101111	00110111	–
54	00001100111	0110101111	0001010111	0001100111	011001111	001010111	101010111	–	011000111	–
55	00000010111	01000000111	1000110111	1100010111	0101000111	000001111	100001111	–	000010111	000010111
56	00010010111	10000000111	1000001111	1101010111	0001000111	0100000111	100101111	–	000110111	
57	10001010111	10010000111	1001001111	0010000111	1000100111	0101000111	1101000111	00101111	110001111	00100111
58	00001010111	00000000111	0000110111	0011000111	1001100111	1001000111	–	001000111	110101111	00110111
59	00100010111	00010000111	0000001111	0100010111	–	1101100111	–	001100111	–	011000111
60	00110010111	10001000111	0001001111	1000010111	001001111	–	–	101010111	–	000010111
61	01000110111	01100000111	1000101111	1001010111	0110000111	–	001010111	100001111	–	000110111
62	10000110111	00001000111	1001101111	1100001111	0000100111	011001111	000001111	100101111	–	110001111
63	00000110111	00011000111	0110001111	0010100111	0001100111	0101000111	0100000111	1101000111	00101111	110101111
64	01000001111	00100000111	0000101111	0000010111	1100010111	0001000111	0101000111	–	001000111	–
65	10000001111	00110000111	0001101111	0001010111	1101010111	1000100111	1001000111	–	001100111	–
66	00000001111	10101000111	11000000111	1000110111	–	–	1101100111	–	101010111	–
67	00010001111	10000100111	11010000111	1000001111	0010000111	–	–	–	100001111	–
68	10001001111	10010100111	0110101111	1001001111	0011000111	–	–	001010111	100101111	–
69	00001001111	00000100111	01000000111	0110010111	0100010111	001001111	–	000001111	1101000111	00101111
70	00100001111	00010100111	10000000111	0000110111	1000010111	0110000111	011001111	0100000111	1001100111	001000111
71	00110001111	10001100111	10010000111	0000001111	1001010111	0000100111	0101000111	0101000111	–	001100111
72	01000101111	10000010111	11011000111	0001001111	1100001111	0001100111	0001000111	1001000111	–	101010111

Table 1. Continued

k	$(u=-1)$ $(v=-2)$	$(u=-2)$ $(v=-3)$	$(u=-3)$ $(v=-4)$	$(u=-4)$ $(v=-5)$	$(u=5)$ $(v=-6)$	$(u=-6)$ $(v=-7)$	$(u=-7)$ $(v=-8)$	$(u=-8)$ $(v=-9)$	$(u=-9)$ $(v=-10)$	$(u=-10)$ $(v=-11)$
73	01010101111	00001100111	00000000111	1000101111	1101001111	1100010111	1000100111	1101100111	–	100001111
74	000001001111	00000010111	00010000111	1001101111	0010100111	1101010111	1001100111	–	–	100101111
75	110000000111	00010010111	10001000111	0011010111	0000010111	–	–	–	001010111	1101000111
76	010000000111	10001010111	10011000111	0110001111	0001010111	–	–	–	000001111	–
77	100000000111	01100010111	01100000111	0000101111	1000110111	0010000111	–	–	0100000111	–
78	000000000111	00001010111	00001000111	0001101111	1000001111	0011000111	001001111	011001111	0101000111	–
79	000100000111	00011010111	00011000111	11000000111	1001001111	0100010111	0110000111	0101000111	1001000111	–
80	100010000111	00100010111	11000100111	11010000111	1101101111	1000010111	0000100111	0001000111	1101100111	–
81	000010000111	00110010111	00100000111	0010001111	0110010111	1001010111	0001100111	1000100111	–	–
82	001000000111	01000110111	00110000111	0110101111	0000110111	1100001111	1100010111	1001100111	–	001010111
83	001100000111	10000110111	10101000111	01000000111	0000001111	1101001111	1101010111	–	–	000001111
84	010001000111	11000001111	10000100111	10000000111	0001001111	–	–	–	–	0100000111
85	100001000111	00000110111	10010100111	10010000111	1000101111	0010100111	–	–	–	0101000111
86	000001000111	01000001111	00101000111	11011000111	1001101111	0000010111	–	–	011001111	1001000111
87	000101000111	10000001111	00000100111	0010101111	0010010111	0001010111	0010000111	001001111	0101000111	1101100111
88	010000100111	10010001111	00010100111	00000000111	0011010111	1000110111	0011000111	0110000111	0001000111	–
89	100000100111	00000001111	10001100111	00010000111	0110001111	1000001111	0100010111	0000100111	1000100111	–
90	000000100111	00010001111	10000010111	10001000111	0000101111	1001001111	1000010111	0001100111	1001100111	–
91	000100100111	10001001111	10010010111	10011000111	0001101111	–	1001010111	1100010111	–	–
92	100010100111	01100001111	00001100111	–	11000000111	–	1100001111	1101010111	–	–
93	000010100111	00001001111	00000010111	01100000111	11010000111	0110010111	1101001111	–	–	–
94	001000100111	00011001111	00010010111	00001000111	0010110111	0000110111	–	–	–	011001111
95	001100100111	00100001111	10001010111	00011000111	0010001111	0000001111	–	–	–	0101000111
96	010001100111	00110001111	10011010111	11000100111	0110101111	0001001111	0010100111	–	001001111	0001000111
97	100001100111	01000101111	01100010111	11010100111	01000000111	1000101111	0000010111	0010000111	0110000111	1000100111
98	000001100111	01010101111	00001010111	00100000111	01010000111	1001101111	0001010111	0011000111	0000100111	1001100111
99	000101100111	00101001111	00011010111	00110000111	10010000111	–	1000110111	0100010111	0001100111	–
100	100000010111	110000000111	11000110111	10101000111	11011000111	0010010111	1000001111	1000010111	1100010111	–

Table 2. The third order bivariate GH codes of k for $1 \leq k \leq 100$, and for $u=-11,-12,\dots,-20$ and $v=-12,-13,\dots,-21$ (u and v are consecutive, $v<u$).

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
1	000111	000111	000111	000111	000111	000111	000111	000111	000111	000111
2	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111
3	1001111	1001111	1001111	1001111	1001111	1001111	1001111	1001111	1001111	1001111
4	–	–	–	–	–	–	–	–	–	–
5	–	–	–	–	–	–	–	–	–	–
6	–	–	–	–	–	–	–	–	–	–
7	–	–	–	–	–	–	–	–	–	–
8	–	–	–	–	–	–	–	–	–	–
9	–	–	–	–	–	–	–	–	–	–
10	–	–	–	–	–	–	–	–	–	–
11	–	–	–	–	–	–	–	–	–	–
12	01111	–	–	–	–	–	–	–	–	–
13	0000111	01111	–	–	–	–	–	–	–	–
14	0001111	0000111	01111	–	–	–	–	–	–	–

Table 2. Continued

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
15	11000111	0001111	0000111	01111	–	–	–	–	–	–
16	11010111	11000111	0001111	0000111	01111	–	–	–	–	–
17	–	11010111	11000111	0001111	0000111	01111	–	–	–	–
18	–	–	11010111	11000111	0001111	0000111	01111	–	–	–
19	–	–	–	11010111	11000111	0001111	0000111	01111	–	–
20	–	–	–	–	11010111	11000111	0001111	0000111	01111	–
21	–	–	–	–	–	11010111	11000111	0001111	0000111	01111
22	–	–	–	–	–	–	11010111	11000111	0001111	0000111
23	–	–	–	–	–	–	–	11010111	11000111	0001111
24	00111	–	–	–	–	–	–	–	11010111	11000111
25	001111	–	–	–	–	–	–	–	–	11010111
26	01000111	00111	–	–	–	–	–	–	–	–
27	10000111	001111	–	–	–	–	–	–	–	–
28	10010111	01000111	00111	–	–	–	–	–	–	–
29	110000111	10000111	001111	–	–	–	–	–	–	–
30	110100111	10010111	01000111	00111	–	–	–	–	–	–
31	–	110000111	10000111	001111	–	–	–	–	–	–
32	–	110100111	10010111	01000111	00111	–	–	–	–	–
33	–	–	110000111	10000111	001111	–	–	–	–	–
34	–	–	110100111	10010111	01000111	00111	–	–	–	–
35	–	–	–	110000111	10000111	001111	–	–	–	–
36	–	–	–	110100111	10010111	01000111	00111	–	–	–
37	0010111	–	–	–	110000111	10000111	001111	–	–	–
38	00000111	–	–	–	110100111	10010111	01000111	00111	–	–
39	00010111	–	–	–	–	110000111	10000111	001111	–	–
40	010000111	0010111	–	–	–	110100111	10010111	01000111	00111	–
41	100000111	00000111	–	–	–	–	110000111	10000111	001111	–
42	100100111	00010111	–	–	–	–	110100111	10010111	01000111	00111
43	110110111	010000111	0010111	–	–	–	–	110000111	10000111	001111
44	–	100000111	00000111	–	–	–	–	110100111	10010111	01000111
45	–	100100111	00010111	–	–	–	–	–	110000111	10000111
46	–	110110111	010000111	0010111	–	–	–	–	110100111	10010111
47	–	–	100000111	00000111	–	–	–	–	–	110000111
48	–	–	100100111	00010111	–	–	–	–	–	110100111
49	–	–	110110111	010000111	0010111	–	–	–	–	–
50	01100111	–	–	100000111	00000111	–	–	–	–	–
51	00001111	–	–	100100111	00010111	–	–	–	–	–
52	000000111	–	–	110110111	010000111	0010111	–	–	–	–
53	000100111	–	–	–	100000111	00000111	–	–	–	–
54	100010111	01100111	–	–	100100111	00010111	–	–	–	–
55	100110111	00001111	–	–	110110111	010000111	0010111	–	–	–
56	–	000000111	–	–	–	100000111	00000111	–	–	–
57	–	000100111	–	–	–	100100111	00010111	–	–	–
58	–	100010111	01100111	–	–	110110111	010000111	0010111	–	–
59	–	100110111	00001111	–	–	–	100000111	00000111	–	–
60	–	–	000000111	–	–	–	100100111	00010111	–	–
61	–	–	000100111	–	–	–	110110111	010000111	0010111	–

Table 2. Continued

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
62	00100111	–	100010111	01100111	–	–	–	100000111	00000111	–
63	00110111	–	100110111	00001111	–	–	–	100100111	00010111	–
64	011000111	–	–	000000111	–	–	–	110110111	010000111	0010111
65	000010111	–	–	000100111	–	–	–	–	100000111	00000111
66	000110111	–	–	100010111	01100111	–	–	–	100100111	00010111
67	110001111	00100111	–	100110111	00001111	–	–	–	110110111	010000111
68	110101111	00110111	–	–	000000111	–	–	–	–	100000111
69	–	011000111	–	–	000100111	–	–	–	–	100100111
70	–	000010111	–	–	100010111	01100111	–	–	–	110110111
71	–	000110111	–	–	100110111	00001111	–	–	–	–
72	–	110001111	00100111	–	–	000000111	–	–	–	–
73	–	110101111	00110111	–	–	000100111	–	–	–	–
74	–	–	011000111	–	–	100010111	01100111	–	–	–
75	00101111	–	000010111	–	–	100110111	00001111	–	–	–
76	001000111	–	000110111	–	–	–	000000111	–	–	–
77	011010111	–	110001111	00100111	–	–	000100111	–	–	–
78	101010111	–	110101111	00110111	–	–	100010111	01100111	–	–
79	100001111	–	–	011000111	–	–	100110111	00001111	–	–
80	100101111	–	–	000010111	–	–	–	000000111	–	–
81	1101000111	00101111	–	000110111	–	–	–	000100111	–	–
82	–	001000111	–	110001111	00100111	–	–	100010111	01100111	–
83	–	001100111	–	110101111	00110111	–	–	100110111	00001111	–
84	–	101010111	–	–	011000111	–	–	–	000000111	–
85	–	100001111	–	–	000010111	–	–	–	000100111	–
86	–	100101111	–	–	000110111	–	–	–	100010111	01100111
87	–	1101000111	00101111	–	110001111	00100111	–	–	100110111	00001111
88	–	–	001000111	–	110101111	00110111	–	–	–	000000111
89	001010111	–	001100111	–	–	011000111	–	–	–	000100111
90	000001111	–	101010111	–	–	000010111	–	–	–	100010111
91	000101111	–	100001111	–	–	000110111	–	–	–	100110111
92	0101000111	–	100101111	–	–	110001111	00100111	–	–	–
93	1001000111	–	1101000111	00101111	–	110101111	00110111	–	–	–
94	1101100111	–	–	001000111	–	–	011000111	–	–	–
95	–	–	–	001100111	–	–	000010111	–	–	–
96	–	001010111	–	101010111	–	–	000110111	–	–	–
97	–	000001111	–	100001111	–	–	110001111	00100111	–	–
98	–	0100000111	–	100101111	–	–	110101111	00110111	–	–
99	–	1000000111	–	1101000111	00101111	–	–	011000111	–	–
100	–	1001000111	–	–	001000111	–	–	000010111	–	–

In this section, we have also obtained the following theorem, which is the relation between the third-order bivariate variant of the Fibonacci sequence and the Tribonacci sequence.

Theorem 3.1.2. Let $VF^{(3)}_{(u,v)}(k)$ is the third-order bivariate variant of the Fibonacci sequence and $T(k)$ is the Tribonacci sequence. In this case, for $k \geq 1$,

$$VF^{(3)}_{(u,v)}(k) = T(k-2) - u\big(T(k-4) + T(k-5)\big) - vT(k-5)$$

where k is an integer.

PROOF. We have the following set from the third-order bivariate variant of the Fibonacci sequence.

$$VF^{(3)}_{(u,v)}(k) = \{u, v, 1-u-v, 1, 2-u, 4-2u-v, 7-3u-v, 13-6u-2v, 24-11u-4v, \cdots\}.$$

$$VF^{(3)}_{(u,v)}(1) = u = 0 - u\big(0+(-1)\big) - v.0 = T(-1) - u.\big(T(-3)+T(-4)\big) - vT(-4),$$

$$VF^{(3)}_{(u,v)}(2) = v = 0 - u\big((-1)+1\big) - v.(-1) = T(0) - u.\big(T(-2)+T(-3)\big) - vT(-3),$$

$$VF^{(3)}_{(u,v)}(3) = 1-u-v = 1 - u(0+1) - v.1 = T(1) - u.\big(T(-1)+T(-2)\big) - vT(-2).$$

The results are correct for $k = 1, k = 2$, and $k = 3$, as seen above. Suppose that the results are correct for $k = 1,2 \dots, n$. In that case,

$$VF^{(3)}_{(u,v)}(n-2) = T(n-4) - u\big(T(n-6)+T(n-7)\big) - vT(n-7).$$

$$VF^{(3)}_{(u,v)}(n-1) = T(n-3) - u\big(T(n-5)+T(n-6)\big) - vT(n-6).$$

$$VF^{(3)}_{(u,v)}(n) = T(n-2) - u\big(T(n-4)+T(n-5)\big) - vT(n-5).$$

Let us show that this equation is true for $n+1$.

$$\begin{aligned} VF^{(3)}_{(u,v)}(n+1) &= T(n-4) - u\big(T(n-6)+T(n-7)\big) - vT(n-7) + T(n-3) - u(T(n-5)+T(n-6)) \\ &\quad -vT(n-6) + T(n-2) - u(T(n-4)+T(n-5) - vT(n-5) \\ &= T(n-4) + T(n-3) + T(n-2) - u(T(n-4)+T(n-5) + T(n-5) \\ &\quad +T(n-6) + T(n-6) + T(n-7)) - v(T(n-7)+T(n-6)+T(n-5)) \\ &= T(n-1) - u\big(T(n-3)+T(n-4)\big) - vT(n-4) \\ &= T(n+1-2) - u\big(T(n+1-4)+T(n+1-5)\big) - vT(n+1-5). \end{aligned}$$

Therefore, by the induction, we can write

$$VF^{(3)}_{(u,v)}(k) = T(k-2) - u\big(T(k-4)+T(k-5)\big) - vT(k-5), k \geq 1.$$

3.2. The Second-Order Bivariate Variant of Narayana Sequences and Codes

This section describes a new variation depends on two negative variables of the Narayana sequence as follows:

Definition 3.2.1. The second-order bivariate variant of the Narayana sequence $VN^{(2)}_{(u,v)}(k)$ is described the sequence $\{u, v, s, u+s, v+u+s, v+u+2s, v+2u+3s, \dots\}$ where $s = 1-u-v$, with initial conditions $VN^{(2)}_{(u,v)}(1) = u$; $VN^{(2)}_{(u,v)}(2) = v$; $VN^{(2)}_{(u,v)}(3) = 1-u-v$; and for $k \geq 4$, $VN^{(2)}_{(u,v)}(k) = VN^{(2)}_{(u,v)}(k-1) + VN^{(2)}_{(u,v)}(k-3)$.

Here, if $u = -2$ and $v = -3$, then the second-order bivariate variant of the Narayana sequence is $\{-2, -3, 6, 4, 1, 7, 11, 12, 19, \dots\}$. It is obviously seen that different sequences will be obtained for different values of u and v.

With these bivariate variant of Narayana sequences, we may compose a new universal source code, which we have named the bivariate variant of the Narayana code. To make, it can be used the same rule used to generate the standard Fibonacci code as in the second and the third order bivariate variant of Fibonacci code.

Moreover, as in the variations of Fibonacci sequences, the bivariate variant of Narayana sequences has more than one Zeckendorf representation of any integer, too. For example, the second order bivariate variant of Narayana code of 17 would be 1000000011 or 00100011 for $VN^{(2)}_{(-2,-3)}(k) = \{-2,-3,6,4,1,7,11,12,19,\dots\}$. Because

$17 = -2 + 19 = VN^{(2)}_{(-2,-3)}(1) + VN^{(2)}_{(-2,-3)}(9) = 6 + 11 = VN^{(2)}_{(-2,-3)}(3) + VN^{(2)}_{(-2,-3)}(7)$.

In this study, we have also investigated for which values u and v the second-order bivariate variant of Narayana codes precisely exist or for which k positive integers they don't exist. For instance, we have obtained that there is no bivariate variant of the Narayana code of 2 for $VN^{(2)}_{(-1,-3)}(k) = \{-1,-3,5,4,1,6,10,11,17\}$.

Das and Sinha have obtained the second-order variant of Narayana codes $VN^{(2)}_{u}(k)$ or undetectable values of the positive integer k for $k = 1,2,\dots,50$ and for $u = -1,-2,\dots,-20$ in [13]. In this section, we have obtained the second-order bivariate variant of Narayana codes $VN^{(2)}_{(u,v)}(k)$ or undetectable values (–) of the positive integer k for $k = 1,2,\dots,100$ and for $u = -1,-2,\dots,-20$ and $v = -2,-3,\dots,-21$ (u and v are consecutive, $v < u$) Tables 3 and 4.

From Tables 3 and 4, we got the following results for the second-order bivariate variant of Narayana codes.

i. For the positive integers $k = 1$, the second-order bivariate variant of the Narayana code $VN^{(2)}_{(u,v)}(k)$ exactly exists for $u = -1,-2,\dots,-20$ and $v = -2,-3,\dots,-21$ (u and v are consecutive, $v < u$).

ii. For $1 \le k \le 100$, there are at most j consecutive undetectable (--) values in the second-order bivariate variant of Narayana code in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \le j \le 19$.

iii. For $1 \le k \le 100$, as long as j raises, the detectable of Narayana code is reduced in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \le j \le 19$.

Table 3. The second order bivariate variant of Narayana codes of k for $1 \le k \le 50$, and for $u = -1,-2,\dots,-10$ and $v = -2,-3,\dots,-11$ (u and v are consecutive, $v < u$).

k	$(u=-1)$ $(v=-2)$	$(u=-2)$ $(v=-3)$	$(u=-3)$ $(v=-4)$	$(u=-4)$ $(v=-5)$	$(u=5)$ $(v=-6)$	$(u=-6)$ $(v=-7)$	$(u=-7)$ $(v=-8)$	$(u=-8)$ $(v=-9)$	$(u=-9)$ $(v=-10)$	$(u=-10)$ $(v=-11)$
1	000011	000011	000011	000011	000011	000011	000011	000011	000011	000011
2	10011	10011	10011	10011	10011	10011	10011	10011	10011	10011
3	00011	–	–	–	–	–	–	–	–	–
4	0011	00011	–	–	–	–	–	–	–	–
5	0000011	1000011	00011	–	–	–	–	–	–	–
6	01000011	0011	1000011	00011	–	–	–	–	–	–
7	10000011	0000011	–	1000011	00011	–	–	–	–	–
8	00000011	01000011	0011	–	1000011	00011	–	–	–	–
9	000000011	100100011	0000011	–	–	1000011	00011	–	–	–
10	000010011	100000011	01000011	0011	–	–	1000011	00011	–	–
11	100100011	00000011	100100011	0000011	–	–	–	1000011	00011	–
12	0100000011	000000011	100000011	01000011	0011	–	–	–	1000011	00011
13	1000000011	000010011	100010011	1001011	0000011	–	–	–	–	1000011
14	0000000011	100100011	00000011	100000011	01000011	0011	–	–	–	–
15	0000100011	00010011	000000011	100010011	1001011	0000011	–	–	–	–
16	1001000011	0100000011	000010011	–	100000011	01000011	0011	–	–	–
17	0001000011	1000000011	0010011	00000011	100010011	1001011	0000011	–	–	–
18	0010000011	1000100011	–	000000011	–	100000011	01000011	0011	–	–
19	0000010011	0000000011	00010011	000010011	–	100010011	1001011	0000011	–	–
20	01000000011	0000100011	0100000011	100100011	00000011	–	100000011	01000011	0011	–
21	10000000011	1001000011	1000000011	0010011	000000011	–	100010011	1001011	0000011	–
22	00000000011	–	1000100011	–	000010011	–	–	100000011	01000011	0011
23	00001000011	0001000011	001000011	00010011	100100011	00000011	–	100010011	1001011	0000011

Table 3. Continued.

k	$(u=-1)$ $(v=-2)$	$(u=-2)$ $(v=-3)$	$(u=-3)$ $(v=-4)$	$(u=-4)$ $(v=-5)$	$(u=5)$ $(v=-6)$	$(u=-6)$ $(v=-7)$	$(u=-7)$ $(v=-8)$	$(u=-8)$ $(v=-9)$	$(u=-9)$ $(v=-10)$	$(u=-10)$ $(v=-11)$
24	10010000011	1000010011	0000000011	0100000011	–	000000011	–	–	100000011	01000011
25	00010000011	0010000011	0000100011	1000000011	0010011	000010011	–	–	100010011	1001011
26	00100000011	0000010011	1001000011	1000100011	–	100100011	00000011	–	–	100000011
27	00000100011	01000000011	–	00100011	00010011	–	000000011	–	–	100010011
28	01000010011	10000000011	–	001000011	0100000011	–	000010011	–	–	–
29	010000000011	10001000011	0001000011	0000000011	1000000011	0010011	100100011	00000011	–	–
30	100000000011	00000000011	1000010011	0000100011	1000100011	–	–	000000011	–	–
31	000000000011	00001000011	–	1001000011	–	00010011	–	000010011	–	–
32	000010000011	10010000011	0010000011	–	00100011	0100000011	–	100100011	00000011	–
33	100100000011	–	0000010011	–	00101011	1000000011	0010011	–	000000011	–
34	000100000011	00010000011	01000000011	–	0000000011	1000100011	–	–	000010011	–
35	001000000011	10000100011	10000000011	0001000011	0000100011	–	00010011	–	100100011	00000011
36	000001000011	00100000011	10001000011	1000010011	1001000011	–	0100000011	–	–	000000011
37	010000100011	00000100011	–	–	–	00100011	1000000011	0010011	–	000010011
38	100000100011	01000010011	00000000011	–	–	001000011	1000100011	–	–	100100011
39	000000100011	010000010011	00001000011	0010000011	–	0000000011	–	00010011	–	–
40	000000010011	100000000011	10010000011	0010100011	–	0000100011	–	0100000011	–	–
41	000010010011	100010000011	–	01000000011	0001000011	1001000011	–	1000000011	0010011	–
42	100100010011	000000000011	–	10000000011	1000010011	–	00100011	1000100011	–	–
43	0100000000011	000010000011	00010000011	10001000011	–	–	001000011	–	00010011	–
44	1000000000011	100100000011	10000100011	–	–	–	0000000011	–	0100000011	–
45	0000000000011	00010010011	–	–	–	–	0000100011	–	1000000011	0010011
46	0000100000011	000100000011	00100000011	00000000011	0010000011	–	1001000011	–	1000100011	–
47	1001000000011	100001000011	00000100011	00001000011	0010100011	0001000011	–	00100011	–	00010011
48	0001000000011	001000000011	01000010011	10010000011	01000000011	1000010011	–	00101011	–	0100000011
49	0010000000011	000001000011	01001010011	–	10000000011	–	–	0000000011	–	1000000011
50	0000010000011	010000100011	10001010011	–	10001000011	–	–	0000100011	–	1000100011
51	0100001000011	100000100011	10001001011	–	–	–	–	1001000011	–	–
52	1000001000011	100000010011	00000010011	00010000011	–	–	–	–	00100011	–
53	0000001000011	000000100011	000000000011	10000100011	–	0010000011	0001000011	–	001000011	–
54	0000000100011	000000010011	000010000011	–	00000000011	0010100011	1000010011	–	0000000011	–
55	0000100100011	000010010011	100100000011	–	00001000011	01000000011	–	–	0000100011	–
56	1001000100011	100100010011	–	00100000011	10010000011	10000000011	–	–	1001000011	–
57	0100000010011	000100100011	–	00000100011	–	10001000011	–	–	–	00100011
58	1000000010011	0100000000011	000100000011	01000010011	–	–	–	–	–	001000011
59	0000000010011	1000000000011	100001000011	010000000011	–	–	–	0001000011	–	0000000011
60	0000100010011	1000100000011	–	100000000011	–	–	0010000011	1000010011	–	0000100011
61	1001000010011	0000000000011	001000000011	100010000011	00010000011	–	0010100011	–	–	1001000011
62	0001000010011	0000100000011	000001000011	–	10000100011	00000000011	01000000011	–	–	–
63	0010000010011	1001000000011	010000100011	00000010011	–	00001000011	10000000011	–	–	–
64	0000010010011	–	100000100011	000000000011	–	10010000011	10001000011	–	–	–
65	01000000000011	0001000000011	100000010011	000010000011	–	–	–	–	0001000011	–
66	10000000000011	1000010000011	100010010011	100100000011	00100000011	–	–	–	1000010011	–
67	00000000000011	0010000000011	000000100011	–	00000100011	–	–	0010000011	–	–
68	00001000000011	0000010000011	000000010011	–	01000010011	–	–	0010100011	–	–
69	10010000000011	0100001000011	000010010011	00010010011	010000000011	–	–	01000000011	–	–
70	00010000000011	1000001000011	100100010011	000100000011	100000000011	00010000011	00000000011	10000000011	–	–

Table 3. Continued.

k	$(u=-1)$ $(v=-2)$	$(u=-2)$ $(v=-3)$	$(u=-3)$ $(v=-4)$	$(u=-4)$ $(v=-5)$	$(u=5)$ $(v=-6)$	$(u=-6)$ $(v=-7)$	$(u=-7)$ $(v=-8)$	$(u=-8)$ $(v=-9)$	$(u=-9)$ $(v=-10)$	$(u=-10)$ $(v=-11)$
71	00100000000011	1000000100011	–	100001000011	100010000011	10000100011	00001000011	10001000011	–	0001000011
72	00000100000011	0000001000011	000100100011	–	–	–	10010000011	–	–	1000010011
73	01000010000011	0000000100011	0100000000011	–	–	–	–	–	–	–
74	10000010000011	0000100100011	1000000000011	001000000011	00000010011	–	–	–	0010000011	–
75	00000010000011	1001000100011	1000100000011	000001000011	000000000011	–	–	–	0010100011	–
76	00000001000011	0001001000011	–	010000100011	000010000011	00100000011	–	–	01000000011	–
77	00001001000011	0001000100011	0000000000011	100000100011	100100000011	00000100011	–	–	10000000011	–
78	10010001000011	1000000010011	0000100000011	100000010011	–	01000010011	–	00000000011	10001000011	–
79	01000000100011	1000100010011	1001000000011	100010010011	00100100011	010000000011	00010000011	00001000011	–	–
80	10000000100011	0000000010011	–	–	–	100000000011	10000100011	10010000011	–	–
81	00000000100011	0000100010011	–	000000100011	00010010011	100010000011	–	–	–	0010000011
82	00001000100011	1001000010011	0001000000011	000000010011	000100000011	–	–	–	–	0010100011
83	10010000100011	–	1000010000011	000010010011	–	–	–	–	–	01000000011
84	00010000100011	0001000010011	–	100100010011	–	–	–	–	–	10000000011
85	00100000100011	1000010010011	0010000000011	001001000011	–	00000010011	–	–	–	10001000011
86	00000100100011	0010000010011	0000010000011	–	–	000000000011	00100000011	–	00000000011	–
87	01000000010011	0000010010011	0100001000011	000100100011	001000000011	000010000011	00000100011	–	00001000011	–
88	10000000010011	01000000000011	1000001000011	0100000000011	000001000011	100100000011	01000010011	00010000011	10010000011	–
89	00000000010011	10000000000011	1000000100011	1000000000011	010000100011	–	010000000011	10000100011	–	–
90	00001000010011	10001000000011	1000100100011	1000100000011	100000100011	–	100000000011	–	–	–
91	10010000010011	00000000000011	0000001000011	–	100010100011	–	100010000011	–	–	–
92	00010000010011	00001000000011	0000000100011	–	100010010011	–	–	–	–	–
93	00100000010011	10010000000011	0000100100011	0000000000011	–	–	–	–	–	–
94	00000100010011	–	1001000100011	0000100000011	–	000100000011	–	–	–	00000000011
95	01000010010011	00010000000011	–	1001000000011	000000100011	100001000011	–	–	0010010011	00001000011
96	010000000000011	10000100000011	0001001000011	–	000000010011	–	00000010011	00100000011	–	10010000011
97	100000000000011	00100000000011	0100000010011	–	000010010011	–	000000000011	00000100011	00010000011	–
98	000000000000011	00000100000011	0100100010011	–	–	–	000010000011	01000010011	10000100011	–
99	000010000000011	01000010000011	1000100010011	0001000000011	–	–	100100000011	010000000011	–	–
100	100100000000011	10000010000011	0010101000011	1000010000011	001001000011	001000000011	–	100000000011	–	–

Table 4. The second order bivariate variant of Narayana codes of k for $1 \leq k \leq 50$, and for $u = -11, -12, \dots, -20$ and $v = -12, -13, \dots, -21$ (u and v are consecutive, $v < u$).

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
1	000011	000011	000011	000011	000011	000011	000011	000011	000011	000011
2	10011	10011	10011	10011	10011	10011	10011	10011	10011	10011
3	–	–	–	–	–	–	–	–	–	–
4	–	–	–	–	–	–	–	–	–	–
5	–	–	–	–	–	–	–	–	–	–
6	–	–	–	–	–	–	–	–	–	–
7	–	–	–	–	–	–	–	–	–	–
8	–	–	–	–	–	–	–	–	–	–
9	–	–	–	–	–	–	–	–	–	–
10	–	–	–	–	–	–	–	–	–	–
11	–	–	–	–	–	–	–	–	–	–

Table 4. Continued.

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
12	–	–	–	–	–	–	–	–	–	–
13	00011	–	–	–	–	–	–	–	–	–
14	1000011	00011	–	–	–	–	–	–	–	–
15	–	1000011	00011	–	–	–	–	–	–	–
16	–	–	1000011	00011	–	–	–	–	–	–
17	–	–	–	1000011	00011	–	–	–	–	–
18	–	–	–	–	1000011	00011	–	–	–	–
19	–	–	–	–	–	1000011	00011	–	–	–
20	–	–	–	–	–	–	1000011	00011	–	–
21	–	–	–	–	–	–	–	1000011	00011	–
22	–	–	–	–	–	–	–	–	1000011	00011
23	–	–	–	–	–	–	–	–	–	1000011
24	0011	–	–	–	–	–	–	–	–	–
25	0000011	–	–	–	–	–	–	–	–	–
26	01000011	0011	–	–	–	–	–	–	–	–
27	10000011	0000011	–	–	–	–	–	–	–	–
28	100000011	01000011	0011	–	–	–	–	–	–	–
29	100010011	10000011	0000011	–	–	–	–	–	–	–
30	–	100000011	01000011	0011	–	–	–	–	–	–
31	–	100010011	10000011	0000011	–	–	–	–	–	–
32	–	–	100000011	01000011	0011	–	–	–	–	–
33	–	–	100010011	10000011	0000011	–	–	–	–	–
34	–	–	–	100000011	01000011	0011	–	–	–	–
35	–	–	–	100010011	10000011	0000011	–	–	–	–
36	–	–	–	–	100000011	01000011	0011	–	–	–
37	–	–	–	–	100010011	10000011	0000011	–	–	–
38	00000011	–	–	–	–	100000011	01000011	0011	–	–
39	000000011	–	–	–	–	100010011	10000011	0000011	–	–
40	000010011	–	–	–	–	–	100000011	01000011	0011	–
41	100100011	00000011	–	–	–	–	100010011	10000011	0000011	–
42	–	000000011	–	–	–	–	–	100000011	01000011	0011
43	–	000010011	–	–	–	–	–	100010011	10000011	0000011
44	–	100100011	00000011	–	–	–	–	–	100000011	01000011
45	–	–	000000011	–	–	–	–	–	100010011	10000011
46	–	–	000010011	–	–	–	–	–	–	100000011
47	–	–	100100011	00000011	–	–	–	–	–	100010011
48	–	–	–	000000011	–	–	–	–	–	–
49	0010011	–	–	000010011	–	–	–	–	–	–
50	–	–	–	100100011	00000011	–	–	–	–	–
51	0100000011	–	–	–	000010011	–	–	–	–	–
52	1000000011	0010011	–	–	100100011	00000011	–	–	–	–
53	1000100011	–	–	–	–	000000011	–	–	–	–
54	–	00010011	–	–	–	000010011	–	–	–	–
55	–	0100000011	–	–	–	100100011	00000011	–	–	–
56	–	1000000011	0010011	–	–	–	000000011	–	–	–
57	–	1000100011	–	–	–	–	000010011	–	–	–
58	–	–	00010011	–	–	–	100100011	00000011	–	–

Table 4. Continued.

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
59	–	–	0100000011	–	–	–	–	000000011	–	–
60	–	–	1000000011	0010011	–	–	–	000010011	–	–
61	00100011	–	1000100011	–	–	–	–	100100011	00000011	–
62	001000011	–	–	00010011	–	–	–	–	000000011	–
63	0000000011	–	–	0100000011	–	–	–	–	000010011	–
64	0000100011	–	–	1000000011	0010011	–	–	–	100100011	00000011
65	1001000011	–	–	1000100011	–	–	–	–	–	000000011
66	–	00100011	–	–	00010011	–	–	–	–	000010011
67	–	001000011	–	–	0100000011	–	–	–	–	100100011
68	–	0000000011	–	–	1000000011	0010011	–	–	–	–
69	–	0000100011	–	–	1000100011	–	–	–	–	–
70	–	1001000011	–	–	–	00010011	–	–	–	–
71	–	–	00100011	–	–	0100000011	–	–	–	–
72	–	–	001000011	–	–	1000000011	0010011	–	–	–
73	–	–	0000000011	–	–	1000100011	–	–	–	–
74	–	–	0000100011	–	–	–	00010011	–	–	–
75	–	–	1001000011	–	–	–	0100000011	–	–	–
76	–	–	1001000011	–	–	–	0100000011	–	–	–
77	0001000011	–	–	00100011	–	–	1000000011	0010011	–	–
78	–	–	–	001000011	–	–	1000100011	–	–	–
79	–	–	–	0000000011	–	–	–	00010011	–	–
80	–	–	–	0000100011	–	–	–	0100000011	–	–
81	–	–	–	1001000011	–	–	–	1000000011	0010011	–
82	–	–	–	–	00100011	–	–	1000100011	–	–
83	–	0001000011	–	–	001000011	–	–	–	00010011	–
84	–	–	–	–	0000000011	–	–	–	0100000011	–
85	–	–	–	–	0000100011	–	–	–	1000000011	0010011
86	–	–	–	–	1001000011	–	–	–	1000100011	–
87	–	–	–	–	–	00100011	–	–	–	00010011
88	0010000011	–	–	–	–	001000011	–	–	–	0100000011
89	0000010011	–	0001000011	–	–	0000000011	–	–	–	1000000011
90	01000000011	–	–	–	–	0000100011	–	–	–	1000100011
91	10000000011	–	–	–	–	1001000011	–	–	–	–
92	10001000011	–	–	–	–	–	00100011	–	–	–
93	–	–	–	–	–	–	001000011	–	–	–
94	–	–	–	–	–	–	0000000011	–	–	–
95	–	0010000011	–	0001000011	–	–	0000100011	–	–	–
96	–	0000010011	–	–	–	–	1001000011	–	–	–

Table 4. Continued.

k	$(u=-11)$ $(v=-12)$	$(u=-12)$ $(v=-13)$	$(u=-13)$ $(v=-14)$	$(u=-14)$ $(v=-15)$	$(u=15)$ $(v=-16)$	$(u=-16)$ $(v=-17)$	$(u=-17)$ $(v=-18)$	$(u=-18)$ $(v=-19)$	$(u=-19)$ $(v=-20)$	$(u=-20)$ $(v=21)$
97	–	01000000011	–	–	–	–	–	00100011	–	–
98	–	10000000011	–	–	–	–	–	001000011	–	–
99	–	10001000011	–	–	–	–	–	0000000011	–	–
100	–	–	–	–	–	–	–	0000100011	–	–

Moreover, this section obtains the following theorem, which is the relation between the second-order bivariate variant of the Narayana sequence and the Narayana sequence.

Theorem 3.2.3. Let $VN^{(2)}_{(u,v)}(k)$ is the second-order bivariate variant of the Narayana sequence and $N(k)$ is the Narayana sequence. In this case, for $k \geq 1$,

$$VN^{(2)}_{(u,v)}(k) = N(k-3) - uN(k-6) - v\big(N(k-6) + N(k-7)\big),$$

where k is an integer.

PROOF. We have the following set from the second-order bivariant variant of the Narayana sequence.

$$VN^{(2)}_{(u,v)}(k) = \{u, v, 1-u-v, 1-v, 1, 2-u-v, 3-u-2v, 4-u-2v, 6-2u-3v, \dots\}.$$

$$VN^{(2)}_{(u,v)}(1) = u = 0 - u(-1) - v\big((-1)+1\big) = N(-2) - uN(-5) - v(N(-5) + N(-6)),$$

$$VN^{(2)}_{(u,v)}(2) = v = 0 - u0 - v(0-1) = N(-1) - uN(-4) - v(N(-4) + N(-5)).$$

The results are correct for $k=1, k=2$, and $k=3$, as seen above. Suppose that the results are correct for $k = 1,2 \dots, n$. In that case,

$$VN^{(2)}_{(u,v)}(n-2) = N(n-5) - uN(n-8) - v(N(n-8) + N(n-9)),$$

$$VN^{(2)}_{(u,v)}(n-1) = N(n-4) - uN(n-7) - v(N(n-7) + N(n-8)),$$

$$VN^{(2)}_{(u,v)}(n) = N(n-3) - uN(n-6) - v(N(n-6) + N(n-7)).$$

Let's show that this equation is true for $n+1$.

$$\begin{aligned} VN^{(2)}_{(u,v)}(n+1) = {} & N(n-5) - uN(n-8) - v\big(N(n-8) + N(n-9)\big) + N(n-4) - uN(n-7) \\ & -v\big(N(n-7) + N(n-8)\big) + N(n-3) - uN(n-6) - v\big(N(n-6)\big) + N(n-7)) \\ = {} & N(n-5) + N(n-4) + N(n-3) - u\big(N(n-8) + N(n-7) + N(n-6)\big) \\ & -v\Big(\big(N(n-6) + N(n-7) + N(n-7)\big) + N(n-8) + N(n-8) + N(n-9)\Big) \\ = {} & N(n-2) - uN(n-5) - v\big(N(n-5) + N(n-6)\big) \\ = {} & N(n+1-3) - uN(n+1-6) - v\big(N(n+1-6) + N(n+1-7)\big). \end{aligned}$$

Therefore, by the induction, we can write

$$VN^{(2)}_{(u,v)}(k) = N(k-3) - uN(k-6) - v\big(N(k-6) + N(k-7)\big), k \geq 1.$$

3.3. Cryptographic Comparison of Bivariate Gopala Hemachandra and Variant Narayana Codes

Cryptography is used to make sense of incomprehensible messages [14]. What cryptography focuses on is privacy. Cryptography aims to provide secure communication between two people. The information is named plaintext [15].

Cryptography is divided into two according to a key structure. These are symmetric cryptography and asymmetric cryptography. DES and AES are examples of symmetric cryptography, while RSA and ElGamal are examples of asymmetric cryptography. Different cryptographic systems are also built, too. One of them is the system obtained using source coding. A few examples made with the system can be found in [10], [12]. Hence, new source codes of third and fourth order identified in this study can also be used in cryptographic applications. But columns with undetectable codes can't be used in cryptography. While for the third-order bivariate GH code, there are no undetectable codes in $GH^{(3)}_{(-1,-2)}(k)$, $GH^{(3)}_{(-2,-3)}(k)$, $GH^{(3)}_{(-3,-4)}(k)$ columns, for the second-order bivariate variant of the Narayana code, there are no undetectable codes in only $VN^{(2)}_{(-1,-2)}(k)$ column. In addition, for the positive integers $k = 1, 2, 3$, the third-order bivariate GH code $GH^{(3)}_{(u,v)}(k)$ precisely exists, for a bivariate variant of Narayana code, for the positive integers $k = 1$, the second order bivariate variant of Narayana code $VN^{(2)}_{(u,v)}(k)$ exactly exists. Similarly, for $1 \leq k \leq 100$, while there is at most j consecutive undetectable (--) values, the third-order bivariate GH code in $GH^{(3)}_{(-(3+j),-(4+j))}(k)$ column in which $1 \leq j \leq 17$, there is at most j consecutive undetectable $(--)$ values the second order bivariate variant of Narayana code in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \leq j \leq 19$.

At that rate, we obtained that the third-order bivariate variant of the Fibonacci code is more valuable than the second-order bivariate variant of the Narayana code in terms of cryptography.

4. Conclusion

In this study, firstly, we examined Fibonacci, Tribonacci, and Narayana sequences and investigated the third-order variant of the Fibonacci sequence, and also obtained which is the relation between the third-order variant of the Fibonacci sequence $VF^{(3)}_u(k)$ and the Tribonacci sequence $T(k)$ for any integer $k \geq 1$ with a theorem.

Then, we described a new variant of the Fibonacci sequence and a new variant of the Narayana sequence, depending on two negative integer variables u and v. Moreover, we named these new sequences the third-order bivariate variant of the Fibonacci sequence and the second-order bivariate variant of the Narayana sequence, respectively. Then, we obtained a bivariate variant of the GH code and a bivariate variant of the Narayana universal code based on these bivariate variant sequences we described.

In addition, we obtained the relation between the third-order bivariate variant of the Fibonacci sequence $VF^{(3)}_{(u,v)}(k)$ and the Tribonacci sequence $T(k)$, and the second-order bivariate variant of the Narayana sequence $VN^{(2)}_{(u,v)}(k)$ and the Narayana sequence $N(k)$ for any integer $k \geq 1$.

Afterwards, we showed in tables $VF^{(3)}_{(u,v)}(k)$, $VN^{(2)}_{(u,v)}(k)$ we have defined for $1 \leq k \leq 100$ and $u = -1, -2, \ldots, -20$ and $v = -2, -3, \ldots, -21$ (u and v are consecutive, $v < u$). We got some important results from the tables for these bivariate variants of universal codes. For $k = 1, 2, 3$, the third-order bivariate GH code $GH^{(3)}_{(u,v)}(k)$ exactly exists. There are at most j consecutive undetectable (-) values in the third-order bivariate GH code in $GH^{(3)}_{(-(3+j),-(4+j))}(k)$ column in which $1 \leq j \leq 17$ and for $1 \leq k \leq 100$. As long as j raises, the detectable of GH code is reduced in $GH^{(3)}_{(-(3+j),-(4+j))}(k)$ column in which $1 \leq j \leq 17$ and for $1 \leq k \leq 100$. For $k = 1$, the second-order bivariate variant of the Narayana code $VN^{(2)}_{(u,v)}(k)$ exactly exists. For $1 \leq k \leq 100$, there are at most j consecutive undetectable (--) values in the second-order bivariate variant of Narayana code in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \leq j \leq 19$. And for $1 \leq k \leq 100$, as long as j raises, the detectable of Narayana code is reduced in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \leq j \leq 19$.

Finally, we compared the two universal codes defined here regarding the existence of these codes for positive integers k and the use of these sequences in cryptography. And, according to the above results, in cryptographic applications, columns $GH_{(-1,-2)}^{(3)}(k)$, $GH_{(-2,-3)}^{(3)}(k)$, $GH_{(-3,-4)}^{(3)}(k)$ can be precisely used for a bivariate GH code for $m = 3$, the column $VN_{(-1,-2)}^{(2)}(k)$ can be precisely used for a bivariate variant of the Narayana code for $m = 2$. And hence, we acquired the third-order bivariate variant of the Fibonacci code is more valuable than the second-order bivariate variant of the Narayana code.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

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