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CONTENTS

Journal of New Theory

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42 (2023) 1-7
Journal of New Theory
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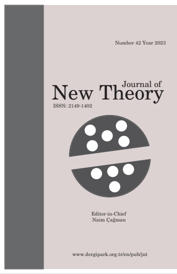

A Short Note on a Mus-Cheeger-Gromoll Type Metric

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Abstract — In this paper, we first show that the complete lift U^c to TM of a vector field U on M is an infinitesimal fiber-preserving conformal transformation if and only if U is an infinitesimal homothetic transformation of (M, g). Here, (M, g) is a Riemannian manifold and TM is its tangent bundle with a Mus-Cheeger-Gromoll type metric $\tilde{g}$. Secondly, we search for some conditions under which $\left(\overset{h}{\nabla}, \tilde{g}\right)$ is a Codazzi pair on TM when (∇, g) is a Codazzi pair on M where $\overset{h}{\nabla}$ is the horizontal lift of a linear connection ∇ on M. We finally discuss the need for further research.

Keywords *Codazzi pair, infinitesimal fiber-preserving conformal transformation, infinitesimal homothetic transformation, Mus-Cheeger-Gromoll type metric, tangent bundle*

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1. Introduction

The Sasaki metric [1] and the Cheeger-Gromoll metric [2] are the well-known metrics on the tangent bundles of Riemannian manifolds. Moreover, many metrics on tangent bundles have been introduced by deforming these two metrics. The rescaled Sasaki metric [3], the twisted Sasaki metric [4], the Mus-Sasaki metric [5], the rescaled Cheeger-Gromoll metric [6], the generalized Cheeger-Gromoll metric [7], and the Cheeger-Gromoll type metric [8] are examples of these deformations. Moreover, Latti and Djaa [9] introduced a new deformation of the Cheeger-Gromoll metric $\tilde{g}$, called the Mus-Cheeger-Gromoll metric. They computed the Levi-Civita connection and studied the curvature properties of a tangent bundle with respect to this metric. This paper will deal with a special case of this metric.

A classical problem on a Riemannian manifold M is to find infinitesimal conformal transformations (conformal vector fields) on M. The vector field U on M is an infinitesimal conformal transformation if and only if there is a function ρ on M satisfying $L_U g = 2\rho g$ where L_U is the Lie derivative with respect to U. If ρ is a nonzero constant (resp. zero), then U is referred to as an infinitesimal homothetic transformation (resp. Killing vector field). Infinitesimal conformal transformations are studied on tangent bundles by many authors [10–14].

Statistical manifolds were studied first by Amari [15] in view of information geometry, and Lauritzen gave applications in [16]. These manifolds have a crucial role in statistics as the statistical model often fashions a geometrical manifold. The geometry of statistical structures on tangent bundles is

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an actual topic. These structures were examined with respect to various Riemannian metrics such as the Sasaki metric [17], the Cheeger-Gromoll metric, and a $g-$natural metric which consists of three classic lifts of the metric g [18], the twisted Sasaki metric, and the gradient Sasaki metric [19].

In this paper, we prove that the complete lift U^c to TM of a vector field U on M is an infinitesimal fiber-preserving conformal transformation (IFPCT) on TM if and only if U is an infinitesimal homothetic transformation (IHT) of (M,g). We also investigate conditions under which $\left(\overset{h}{\nabla},\tilde{g}\right)$ is a Codazzi pair on TM when (∇,g) is a Codazzi pair on M, where $\overset{h}{\nabla}$ is the horizontal lift of a linear connection ∇ on M.

2. Preliminary

Let M be an $n-$dimensional $(n>1)$ Riemannian manifold and ∇ be a linear connection on M. The tangent bundle TM of the manifold M is a $2n-$dimensional differentiable manifold, and it is defined by disjoint tangent spaces at distinct points on M. If $\{N,x^i\}$ is a local coordinate system in M, then $\{\pi^{-1}(N),x^i,x^{\bar{\imath}}=u^i,\bar{\imath}=n+1,...,2n\}$ is a local coordinate system in TM where π is the projection defined by $\pi:TM\to M$. We have a decomposition

$$TTM=VTM\oplus HTM$$

for the tangent bundle of TM where the vertical subspace VTM is spanned by $\left\{\frac{\partial}{\partial u^i}:=\left(\frac{\partial}{\partial x^i}\right)^v\right\}$ and the horizontal subspace HTM is spanned by $\left\{\frac{\delta}{\delta x^i}:=\left(\frac{\partial}{\partial x^i}\right)^h=\frac{\partial}{\partial x^i}-u^m\Gamma^j_{mi}\frac{\partial}{\partial u^j}\right\}$. Here, Γ^j_{mi} are the Christoffel symbols of ∇. The vertical, horizontal, and the complete lift of a vector field $U=U^i\frac{\partial}{\partial x^i}$ are defined by, respectively,

$$U^v=U^i\frac{\partial}{\partial u^i},\quad U^h=U^i\frac{\partial}{\partial x^i}-u^s\Gamma^m_{si}U^i\frac{\partial}{\partial u^m},\quad \text{and}\quad U^c=U^i\frac{\partial}{\partial x^i}+u^s\frac{\partial U^i}{\partial x^s}\frac{\partial}{\partial u^i} \tag{1}$$

where we used Einstein's summation. In the sequel, for brevity, we denote $\frac{\partial}{\partial x^i}$, $\frac{\delta}{\delta x^i}$, and $\frac{\partial}{\partial u^i}$ by ∂_i,δ_i, and $\partial_{\bar{\imath}}$, respectively.

If ∇ is a torsionless linear connection, then the Lie brackets of the vertical lift and the horizontal lift of vector fields fulfill the following relations:

$$\left[U^h,V^h\right]=[U,V]^h-(R(U,V)u)^v,\quad \left[U^h,V^v\right]=(\nabla_UV)^v,\quad \text{and}\quad [U^v,V^v]=0 \tag{2}$$

where R is the curvature of ∇ [20].

The frame $\{E_\lambda\}=\{E_i,E_{\bar{\imath}}\}$ adapted to the torsionless linear connection ∇ is given by

$$E_i=\delta^m_i\partial_m-u^s\Gamma^m_{si}\partial_{\bar{m}}\quad\text{and}\quad E_{\bar{\imath}}=\delta^m_i\partial_{\bar{m}}$$

Moreover, $\left\{dx^h,\delta u^h=du^h+u^c\Gamma^h_{cd}dx^d\right\}$ is the dual frame of $\{E_\lambda\}$. We can rewrite Lie Brackets 2 according to the adapted frame as follows:

$$[E_i,E_j]=u^sR^k_{ijs}E_{\bar{k}},\quad \left[E_i,E_{\bar{j}}\right]=\Gamma^k_{ij}E_{\bar{k}},\quad\text{and}\quad \left[E_{\bar{\imath}},E_{\bar{j}}\right]=0$$

where R^k_{jis} are the components of R. Vector Fields 1 are expressed as, according to the adapted frame,

$$U^v=U^iE_{\bar{\imath}},U^h=U^iE_i\quad\text{and}\quad U^c=U^iE_i+u^s\nabla_sU^iE_{\bar{\imath}} \tag{3}$$

We have the following Lie derivatives with respect to $\tilde{U} = v^k E_k + v^{\bar{k}} E_{\bar{k}}$ [13]

$$\begin{aligned} L_{\tilde{U}} E_k &= -\partial_k v^c E_c + \left\{ u^a v^b R^c_{kba} - v^{\bar{a}} \Gamma^c_{ak} - E_k(v^{\bar{c}}) \right\} E_{\bar{c}} \\ L_{\tilde{U}} E_{\bar{k}} &= \left\{ v^a \Gamma^c_{ak} - E_{\bar{k}}(v^{\bar{c}}) \right\} E_{\bar{c}} \\ L_{\tilde{U}} dx^k &= \partial_n v^k dx^n \\ L_{\tilde{U}} \delta u^k &= -\left\{ u^a v^b R^k_{nba} - v^{\bar{a}} \Gamma^k_{an} - E_n(v^{\bar{k}}) \right\} dx^n - \left\{ v^a \Gamma^k_{an} - E_{\bar{n}}(v^{\bar{k}}) \right\} \delta u^n \end{aligned}$$

The horizontal lift connection $\overset{h}{\nabla}$ of a linear connection ∇ is given by

$$\overset{h}{\nabla}_{U^h} V^h = (\nabla_U V)^h, \quad \overset{h}{\nabla}_{U^h} V^v = (\nabla_U V)^v, \quad \text{and} \quad \overset{h}{\nabla}_{U^v} V^h = \overset{h}{\nabla}_{U^v} V^v = 0$$

Remark that ∇ is a flat and torsionless linear connection if and only if $\overset{h}{\nabla}$ is a torsionless linear connection [20].

The Mus-Cheeger-Gromoll metric G_{mc} on TM is defined by

$$\begin{aligned} G_{mc}\left(U^h, V^h\right) &= g(U,V) \\ G_{mc}\left(U^h, V^v\right) &= 0 \\ G_{mc}\left(U^v, V^v\right) &= f(x)\omega(r^2)(g(U,V) + \alpha(r^2) g(U,u) g(V,u)) \end{aligned}$$

for every vector fields U and V on M where $f : M \to \mathbb{R}_+$ and $\omega, \alpha : \mathbb{R} \to \mathbb{R}_+$ are three functions and $r^2 = g(u,u)$ [9].

Particular cases of the metric G_{mc} are listed below:

i. If $f = 1$, $\omega = \frac{1}{1+r^2}$, and $\alpha = 1$, then G_{mc} is the Cheeger-Gromoll metric [2].

ii. If $f = 1$, $\omega = \left(\frac{1}{1+r^2}\right)^p$, $\alpha = cons.$, then G_{mc} is the generalized Cheeger-Gromoll metric [7].

iii. If $\omega = \left(\frac{1}{1+r^2}\right)^p$ and $\alpha = cons.$, then G_{mc} is the rescaled vertically generalized Cheeger-Gromoll metric [21].

In this paper, we consider a Mus-Cheeger-Gromoll type metric $\tilde{g}$ by assuming $\omega(r^2) = \frac{1}{1+r^2}$.

Definition 2.1. Let (M,g) be a Riemannian manifold and ∇ be a linear connection on M. The couple (g,∇) is called a Codazzi pair if the following Codazzi equations are valid:

$$(\nabla_U g)(V,W) = (\nabla_V g)(W,U) = (\nabla_W g)(U,V)$$

for all vector fields U, V, and W on M. In this case, (M,g,∇) is referred to as a Codazzi manifold and ∇ is called a Codazzi connection. Moreover, if ∇ is torsionless, then (M,g,∇) is a statistical connection.

3. Main Results

Let $g = g_{ij}\ dx^i dx^j$ is the Riemannian metric g on M. Then, the local expression of the Mus-Cheeger-Gromoll type metric $\tilde{g}$ is

$$\tilde{g} = g_{ij} dx^i dx^j + h_{ij} \delta x^i \delta x^j$$

where $h_{ij} = \frac{f}{1+r^2}(g_{ij} + \alpha g_{im} g_{jn} u^m u^n)$. If $G_1 = g_{ij} dx^i dx^j$ and $G_2 = h_{ij} \delta x^i \delta x^j$, then

$$\tilde{g} = G_1 + G_2$$

Besides, recall that a vector field $\tilde{U}$ with components $\left(v^h, v^{\bar{h}}\right)$ on TM is a fibre preserving (FP) if and only if v^h has components $\left(x^h\right)$.

The following lemma states the Lie derivatives of G_1 and G_2.

Lemma 3.1. The Lie derivatives of G_1 and G_2 with respect to a FP vector field $\tilde{U}$ are

$$\begin{aligned} L_{\tilde{U}}G_1 &= (L_U g_{ij})dx^i dx^j \\ L_{\tilde{U}}G_2 &= -2h_{mj}\{u^b v^c R^m_{icb} - v^{\bar{b}}\Gamma^m_{bi} - E_i(v^{\bar{m}})\}dx^i\delta u^j + \{L_U h_{ij} - 2h_{mj}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\overline{m}}) \\ &\quad + \tfrac{1}{1+r^2}v^{\overline{m}}u^s(-2g_{ms}h_{ij} + 2f\alpha' g_{is}g_{jt}g_{mn}u^t u^n + f\alpha(g_{js}g_{im} + g_{jm}g_{is})\}\delta u^i \delta u^j \end{aligned}$$

where $L_U g_{ij}$ is the components of $L_U g$ and $\nabla_i v^m$ is the components of ∇_U.

Proof.
The proof is similar to the proof of Proposition 2.3 in [13]. □

The first main result of the paper is as follows:

Theorem 3.2. If TM is the tangent bundle of (M, g) equipped with the Mus-Cheeger-Gromoll type metric $\tilde{g}$, then the complete lift U^c of a vector field U is an IFPCT of $(TM, \tilde{g})$ if and only if U is an IHT of (M, g).

Proof.
If $\tilde{U}$ is an IFPCT of $(TM, \tilde{g})$, then there exists a smooth function Ω satisfying

$$L_{\tilde{U}}\tilde{g} = 2\Omega\tilde{g}$$

From Lemma 3.1,

$$\begin{aligned} 2\Omega g_{ij}dx^i dx^j + 2\Omega h_{ij}\delta x^i \delta x^j = (L_U g_{ij})dx^i dx^j &- 2h_{mj}\{u^b v^c R^m_{icb} - v^{\bar{b}}\Gamma^m_{bi} - E_i(v^{\bar{m}})\}dx^i \delta u^j \\ &+ \{L_U h_{ij} - 2h_{jm}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\overline{m}}) \\ &+ \tfrac{1}{1+r^2}v^{\overline{m}}u^s(-2g_{ms}h_{ij} + 2f\alpha' g_{is}g_{jt}g_{mn}u^t u^n + f\alpha(g_{js}g_{im} + g_{jm}g_{is})\}\delta u^i \delta u^j \end{aligned}$$

It follows that

$$L_U g_{ij} = 2\Omega g_{ij} \tag{4}$$

$$u^b v^c R^m_{icb} - v^{\bar{b}}\Gamma^m_{bi} - E_i(v^{\bar{m}}) = 0$$

$$L_U h_{ij} - 2h_{jm}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\overline{m}}) + \frac{1}{1+r^2}v^{\overline{m}}u^s(-2g_{ms}h_{ij} + 2f\alpha' g_{is}g_{jt}g_{mn}u^t u^n + f\alpha(g_{js}g_{im} + g_{jm}g_{is})) = 2\Omega h_{ij}$$

From Equation 3, we can write the complete lift a vector field $U = v^k\partial_k$ as $U^c = v^k E_k + u^s\nabla_s v^k E_{\overline{k}}$. Thus,

$$L_U g_{ij} = 2\Omega g_{ij}$$

and

$$u^b(v^c R^m_{icb} - \nabla_i\nabla_b v^m) = 0 \tag{5}$$

Therefore, Equation 5 gives

$$\nabla_i\nabla_b v_j = v^c R_{icbj}$$

Using algebraic properties of the Riemannian curvature tensor,

$$\nabla_i\nabla_b v_j + \nabla_i\nabla_j v_b = 0 \tag{6}$$

Since $L_U g_{ij} = \nabla_i v_j + \nabla_j v_i$, from Equation 4,

$$\nabla_i v_j + \nabla_j v_i = 2\Omega g_{ij}$$

Taking the covariant derivative on both hand sides of the above equation,

$$\nabla_k(\nabla_i v_j) + \nabla_k(\nabla_j v_i) = 2(\nabla_k\Omega)g_{ij} \tag{7}$$

Equations 6 and 7 show that $\nabla_k \Omega = 0$. Hence, Ω is constant. The proof of converse is clear. □

In this part of this section, we deal with Codazzi pairs on M and TM. Let (M, g) be an $n-$dimensional $(n > 1)$ Riemannian manifold and $\left(\overset{h}{\nabla}, \tilde{g}\right)$ be a Codazzi pair on TM. Taking into account Definition 2.1, by direct calculation,

$$\left(\overset{h}{\nabla}_{\delta_i}\widetilde{g}\right)(\delta_j, \partial_{\bar{k}}) = \left(\overset{h}{\nabla}_{\delta_j}\widetilde{g}\right)(\partial_{\bar{k}}, \delta_i) = \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\delta_i, \delta_j) = 0$$

and

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \delta_k) = \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\delta_k, \partial_{\bar{i}}) = \left(\overset{h}{\nabla}_{\delta_k}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0$$

Moreover,

$$\left(\overset{h}{\nabla}_{\delta_i}\widetilde{g}\right)(\delta_j, \delta_k) = \nabla_i g_{jk}$$

and

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = \partial_{\bar{i}}\widetilde{g}(\partial_{\bar{j}}, \partial_{\bar{k}}), \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{i}}) = \partial_{\bar{j}}\widetilde{g}(\partial_{\bar{k}}, \partial_{\bar{i}}), \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = \partial_{\bar{k}}\widetilde{g}(\partial_{\bar{i}}, \partial_{\bar{j}})$$

Furthermore,

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = f\left\{\tfrac{-2u^m}{(1+r^2)^2}(g_{im}g_{jk} + \alpha u^s u^t g_{im}g_{js}g_{kt}) + \tfrac{2\alpha'}{1+r^2}g_{in}g_{js}g_{kt}u^n u^s u^t + \tfrac{\alpha u^s}{1+r^2}(g_{ji}g_{ks} + g_{js}g_{ki})\right\} \quad (8)$$

$$\left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{i}}) = f\left\{\tfrac{-2u^m}{(1+r^2)^2}(g_{jm}g_{ki} + \alpha u^s u^t g_{jm}g_{ks}g_{it}) + \tfrac{2\alpha'}{1+r^2}g_{jn}g_{ks}g_{it}u^n u^s u^t + \tfrac{\alpha u^s}{1+r^2}(g_{kj}g_{is} + g_{ks}g_{ij})\right\} \quad (9)$$

$$\left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = f\left\{\tfrac{-2u^m}{(1+r^2)^2}(g_{km}g_{ij} + \alpha u^s u^t g_{km}g_{is}g_{jt}) + \tfrac{2\alpha'}{1+r^2}g_{kn}g_{is}g_{jt}u^n u^s u^t + \tfrac{\alpha u^s}{1+r^2}(g_{ik}g_{js} + g_{is}g_{jk})\right\} \quad (10)$$

Equations 8-10 yield two cases.

Case 1) If $\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{i}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0$, then, from Equation 8,

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = f\left\{\frac{-2u^m}{1+r^2}(g_{im}g_{jk} + \alpha u^s u^t g_{im}g_{js}g_{kt}) + 2\alpha' g_{in}g_{js}g_{kt}u^n u^s u^t + \alpha u^s(g_{ji}g_{ks} + g_{js}g_{ki})\right\} = 0$$

Taking the derivative in the above equation with respect to $\partial_{\bar{h}}$,

$$0 = f\left\{\tfrac{-2\delta_h^m(1+r^2)+4u^m u^n g_{nh}}{(1+r^2)^2}(g_{im}g_{jk} + \alpha u^s u^t g_{im}g_{js}g_{kt}) - \tfrac{2u^m g_{mi}u^t(2\alpha' g_{hn}g_{js}g_{kt}u^n u^s + \alpha g_{tk}g_{jh} + \alpha g_{jt}g_{hk})}{1+r^2}\right.$$
$$\left. - \tfrac{4u^m \alpha' g_{hn}g_{js}g_{kt}g_{im}u^s u^t u^n}{1+r^2} + (2\alpha' g_{hn}u^n u^s + \alpha\delta_h^s)(g_{ji}g_{ks} + g_{js}g_{ki})\right\}$$

because

$$\begin{aligned} 0 \quad = \quad & f\left\{\frac{-2}{1+r^2}(g_{jk}g_{ih} + \alpha u^s u^t g_{ih}g_{js}g_{kt}) + \frac{4u^m u^n g_{nh}}{(1+r^2)^2}(g_{jk}g_{im} + u^s u^t g_{im}g_{js}g_{kt})\right. \\ & -\frac{2u^m g_{mi}u^t(2\alpha' g_{hn}g_{js}g_{kt}u^n u^s + \alpha g_{tk}g_{jh} + \alpha g_{jt}g_{hk})}{1+r^2} - \frac{4u^m \alpha' g_{hn}u^n g_{js}g_{kt}g_{im}u^s u^t}{1+r^2} \\ & \left. +2\alpha' g_{hn}u^n u^s(g_{ji}g_{ks} + g_{js}g_{ki})\alpha(g_{ji}g_{kh} + g_{jh}g_{ki})\right\} \end{aligned} \quad (11)$$

Equation 11 is satisfied, for all $(x, u) \in TM$. For zero section, i.e., $u = 0$, Equation 11 becomes

$$-2g_{jk}g_{ih} = 0$$

This is a contradiction, when i, j, and k run from 1 to n.

Case 2) If $\left(\overset{h}{\nabla}_{\partial_i}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{\imath}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{\imath}}, \partial_{\bar{j}}) \neq 0$, then, from Equations 8-10,

$$-2g_{jk}g_{ih} = -2g_{ki}g_{jh}$$

Hence, $g_{jk}g_{ih} = g_{ik}g_{jh}$. Multiplying both side of this equation by g^{jh}, $g_{ki} = ng_{ki}$. Thus, $n = 1$. This is a contradiction. Consequently, we can express the following result.

Theorem 3.3. Let TM be the tangent bundle of an $n-$dimensional $(n > 1)$ Riemannian manifold (M, g) equipped with the metric $\tilde{g}$ and ∇ be a linear connection on M. If (∇, g) is a Codazzi pair on M, then $\left(\overset{h}{\nabla}, \widetilde{g}\right)$ is not a Codazzi pair on TM.

4. Conclusion

The Mus-Cheeger-Gromoll metric is a new metric on the tangent bundle of a Riemannian manifold. In this paper, we studied the infinitesimal fiber-preserving property of the complete lift of a vector field and investigated the Codazzi pairs using the horizontal lift of a linear connection. Our findings suggest that these techniques could be applied to more general metrics in tangent bundles, opening up new avenues of research in this area. In addition, we believe that further investigation of the Mus-Cheeger-Gromoll metric could yield even more insights into the nature of Riemannian manifolds and their properties.

Author Contributions

The author read and approved the last version of the article.

Conflicts of Interest

The author declares no conflict of interest.

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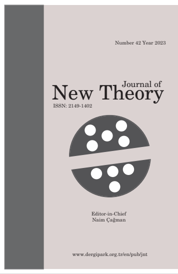

The Effect of the Additive Row Operation on the Permanent

Ahmet Zahid Küçük[1], **Talha Sözer**[2]

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Abstract — The permanent function is not as stable as the determinant function under the elementary row operations. For example, adding a non-zero scalar multiple of a row to another row does not change the determinant of a matrix, but this operation changes its permanent. In this article, the variation in the permanent by applying the operation, which adds a scalar multiple of a row to another row, is examined. The relationship between the permanent of the matrix to which this operation is applied and the permanent of the initial matrix is given by a theorem. Finally, the paper inquires the need for further research.

Keywords *Permanent, Gaussian elimination, elementary row operations, Laplace expansion*

Mathematics Subject Classification (2020) 15A15, 03D15

1. Introduction

The permanent function was introduced first by Binet and Cauchy. According to Binet's definition, provided $m \leq 4$, the permanent of a matrix with order m by n is the sum of all possible products of m elements any two of which are not at the same column or row. Minc [1] emphasized that this definition given by Binet could be generalized for all finite values of m and n and gave the following definition.

Let $A = [a_{i,j}]$ be a matrix of order m by n. Then, per(A), the permanent of A, is defined by

$$\text{per}(A) = \sum_{\sigma} a_{1,\sigma_1} a_{2,\sigma_2} ... a_{m,\sigma_m}$$

where the summation runs over on the set σ, which includes all one-to-one functions defined from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ such that $m \leq n$ [1].

The permanent function can be interpreted as a kind of assessment using all matrix elements. This scalar-valued function of the matrix is best known for its relations with solutions to enumeration problems in combinatorics. For example, the Menage problem is a classical combinatorial enumeration problem, and it has been connected to the permanents of $(0, 1)-$matrices [2]. Another critical problem is computing the permanent of some kind of matrices, for example, the sparse and the circulant. This problem appears in various applications in mathematics, physics, computers, information systems, cryptography, and other fields. It has been studied to obtain various linear recurrence relations for permanents of certain sparse circulant matrices in [3], one of the recent studies in quantum computing.

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By solving the linear recursive system consisting of the obtained recurrences in [3], computing of permanent will be realized in linear time.

The definition of permanent is the same as the definition of determinant except for the factor ± 1 before terms in the summation. Thus, some properties of the permanent have direct analogs for the determinant. The following will give a brief summary of several fundamental properties related to both the permanent and the determinant.

The procedure of reducing anything to a simpler form is frequently used in both the determinant and the permanent. Laplace expansion is an important example of reduction, and it can be applied similarly to these two functions. Let $A(i, j)$ denote the matrix of size $m-1$ by $n-1$ obtained from the matrix $A = [a_{i,j}]$ by deleting row i and column j such that $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. It follows as a direct consequence of the definition of the permanent that

$$\operatorname{per}(A) = \sum_{j=1}^{n} a_{ij} \operatorname{per}(A(i, j))$$

called Laplace expansion of the permanent according to the i^{th} row of matrix A. Another procedure that can be used for reduction is Gaussian elimination. The determinant can be evaluated efficiently using Gaussian elimination (or row reduction) [4]. However, computation of the permanent in this way is much more complicated. The elementary row operations may differ in these functions because the permanent is not as stable as the determinant under the elementary row operations [2]. For example, adding a non-zero scalar multiple of a row to another row does not change the determinant of a matrix, but this operation changes its permanent. The determinant of a matrix with two equal rows is zero, but its permanent does not have to be zero. Multiplying a row by a scalar requires multiplying the determinant by the same scalar. This is also valid for the permanent. Interchanging two rows varies the sign of determinant, but permanent is invariant under this operation [5].

One of the fundamental rules of the determinant is $\det(AB) = \det(A)\det(B)$. This rule is clearly false for permanent. However, in [6], it has been proved that the equality $\operatorname{per}(AB) = \operatorname{per}(A)\operatorname{per}(B)$ holds for the generalized complementary basic (GCB) matrices which have many remarkable properties such as permanental, graph-theoretic, spectral, and inheritance properties [7, 8].

As mentioned above, for a square matrix A, adding a non-zero scalar multiple of a row to another row varies its permanent. To the best of our knowledge, there is no discussion on the effect of this operation on the permanent, in related literature. In this paper, the variation in the permanent has been studied when the operation "adding a scalar multiple of one row to another row" is applied to a square matrix. An equality that gives a relationship between the permanent of the original matrix and the permanent of its changed form is presented. In addition, an algorithm that calculates the variation is also given.

2. Main Results

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & m & n \end{bmatrix}$$

be a square matrix of order 3×3 and $|A|$ denote the determinant of the matrix A. The determinant of a matrix remains unchanged when adding a non-zero scalar k multiple of a row to another row. As an example of this property, the following equations can be written for the matrix A:

$$\begin{vmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g & m & n \end{vmatrix} - |A| = 0$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g+ak & m+bk & n+ck \end{vmatrix} - |A| = 0$$

and

$$\begin{vmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g+ak & m+bk & n+ck \end{vmatrix} - |A| = 0$$

These situations for the permanent function, in contrast to the determinant, are illustrated by the equations below:

$$\text{per}\left(\begin{bmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g & m & n \end{bmatrix}\right) - \text{per}\,(A) = x \tag{1}$$

$$\text{per}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g+ak & m+bk & n+ck \end{bmatrix}\right) - \text{per}\,(A) = y \tag{2}$$

and

$$\text{per}\left(\begin{bmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g+ak & m+bk & n+ck \end{bmatrix}\right) - \text{per}\,(A) = z \tag{3}$$

where $x \neq y \neq z \neq 0$. The following theorem proposes obtaining x and y using $(n-2)$-ordered submatrices of a matrix A with order n by n. Namely, with the following theorem, we express the variation in the permanents for which additive row operation is applied only once, as in x and y. We note that the variation notion used in this study corresponds to x and y in Equalities 1 and 2. We also note that the calculation of variation we suggest is necessary twice to calculate the z value in Equality 3.

Let the notations used in this study clarify before giving on to the theorem. Let $A = [a_{i,j}]$ be a matrix of order n by n. The notation

$$\widetilde{A}_{r|t}$$

denotes the submatrix obtained by deleting the r^{th} row and the t^{th} column of the matrix A. The notation

$$\widetilde{A}_{i,r|j,t}$$

denotes the submatrix obtained by deleting i^{th} row, r^{th} row, j^{th} column, and t^{th} column of the matrix A. The submatrix $\widetilde{A}_{r|t}$ is of order $(n-1)\times(n-1)$ and the submatrix $\widetilde{A}_{i,r|j,t}$ is of order $(n-2)\times(n-2)$. As an example, if we consider the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

then

$$\widetilde{B}_{1|4} = \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \quad \text{and} \quad \widetilde{B}_{1,2|3,4} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

are some submatrices obtained from the matrix B.

Theorem 2.1. Let $A = [a_{i,j}]$ be a matrix of order $n \times n$ and B be the matrix obtained by adding k times of the i^{th} row to the r^{th} row of the matrix A. Then,

$$\text{per}(B) - \text{per}(A) = 2k \sum_{(j,t)\in\Omega} a_{i,j} a_{i,t} \,\text{per}(\widetilde{A}_{i,r|j,t})$$

where the summation extends over the set $\Omega = \{(j,t) \in S \times S \mid j < t\}$ such that $S = \{1, 2, ..., n\}$.

PROOF.
By using the Laplace expansion with respect to r^{th} row of the matrix B, we obtain

$$\text{per}\,(B) = (a_{r,1} + ka_{i,1})\,\text{per}(\widetilde{B}_{r|1}) + \cdots + (a_{r,n} + ka_{i,n})\,\text{per}(\widetilde{B}_{r|n}) \tag{4}$$

where $\widetilde{B}_{r|j}$ denotes the submatrices obtained by deleting r^{th} row and j^{th} column of the matrix B. Equality 4 can be arranged as the form

$$\text{per}\,(B) = \left(a_{r,1}\,\text{per}(\widetilde{B}_{r|1}) + \cdots + a_{r,n} per(\widetilde{B}_{r|n})\right) + k\left(a_{i,1}\,\text{per}(\widetilde{B}_{r|1}) + \cdots + a_{i,n}\,\text{per}(\widetilde{B}_{r|n})\right) \tag{5}$$

By applying the Laplace expansion along by the r^{th} row of the matrix A, it is easily seen that

$$a_{r,1}\,\text{per}(\widetilde{B}_{r|1}) + \cdots + a_{r,n} per(\widetilde{B}_{r|n}) = \text{per}(A)$$

Therefore, from Equality 5, $\text{per}(B) = \text{per}(A) + kV_A$ where

$$V_A = a_{i,1}\,\text{per}(\widetilde{B}_{r|1}) + a_{i,2}\,\text{per}(\widetilde{B}_{r|2}) + \cdots + a_{i,n}\,\text{per}(\widetilde{B}_{r|n}) \tag{6}$$

At this point, the process will be continued by applying the Laplace expansion to every permanent in V_A seen by Equality 6, respectively. Firstly, by expanding the permanent, which in the first term of V_A seen by Equality 6, with respect to i^{th} row, the following equality is obtained:

$$\begin{aligned}\text{per}(\widetilde{B}_{r|1}) =& a_{i,2}\,\text{per}\left(\begin{bmatrix} a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,3} & a_{i-1,4} & \cdots & a_{i-1,n} \\ a_{i+1,3} & a_{i+1,4} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,3} & a_{r-1,4} & \cdots & a_{r-1,n} \\ a_{r+1,3} & a_{r+1,4} & \cdots & a_{r+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,4} & \cdots & a_{n,n} \end{bmatrix}\right) + a_{i,3}\,\text{per}\left(\begin{bmatrix} a_{1,2} & a_{1,4} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,4} & \cdots & a_{i-1,n} \\ a_{i+1,2} & a_{i+1,4} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,2} & a_{r-1,4} & \cdots & a_{r-1,n} \\ a_{r+1,2} & a_{r+1,4} & \cdots & a_{r+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,4} & \cdots & a_{n,n} \end{bmatrix}\right) \\ &+ \cdots + a_{i,n}\,\text{per}\left(\begin{bmatrix} a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n-1} \\ a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,2} & a_{r-1,3} & \cdots & a_{r-1,n-1} \\ a_{r+1,2} & a_{r+1,3} & \cdots & a_{r+1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \end{bmatrix}\right)\end{aligned} \tag{7}$$

Equality 7 can be written briefly as

$$\text{per}(\widetilde{B}_{r|1}) = a_{i,2}\,\text{per}(\widetilde{A}_{i,r|1,2}) + a_{i,3}\,\text{per}(\widetilde{A}_{i,r|1,3}) + \cdots + a_{i,n}\,\text{per}(\widetilde{A}_{i,r|1,n})$$

Similarly, if the permanents in other terms of Equality 6 are expanded along by their i^{th} row, then

$$\text{per}(\widetilde{B}_{r|2}) = a_{i,1}\,\text{per}(\widetilde{A}_{i,r|2,1}) + a_{i,3}\,\text{per}(\widetilde{A}_{i,r|2,3}) + \cdots + a_{i,n}\,\text{per}(\widetilde{A}_{i,r|2,n})$$

$$\text{per}(\widetilde{B}_{r|3}) = a_{i,1}\,\text{per}(\widetilde{A}_{i,r|3,1}) + a_{i,2}\,\text{per}(\widetilde{A}_{i,r|3,2}) + a_{i,4}\,\text{per}(\widetilde{A}_{i,r|3,4}) + \cdots + a_{i,n}\,\text{per}(\widetilde{A}_{i,r|3,n})$$

$$\vdots$$

$$\text{per}(\widetilde{B}_{r|n}) = a_{i,1}\,\text{per}(\widetilde{A}_{i,r|n,1}) + a_{i,2}\,\text{per}(\widetilde{A}_{i,r|n,2}) + \cdots + a_{i,n-1}\,\text{per}(\widetilde{A}_{i,r|n,n-1})$$

If we plug the equalities obtained, for all $\text{per}(\widetilde{B}_{r|\alpha})$, where $\alpha \in S$, into V_A seen by Equality 6, then we get

$$V_A = \sum_{(j,t)\in\Delta} a_{i,j}a_{i,t}\,\text{per}(\widetilde{A}_{i,r|j,t}) \tag{8}$$

where $\Delta = \{(j,t) \in S \times S \mid j \neq t\}$. There are two of each term in the summation in Equality 8 because the terms of the form

$$a_{i,j}a_{i,t}\,\text{per}(\widetilde{A}_{i,r|j,t})$$

equal to the terms of the form

$$a_{i,t}a_{i,j}\,\text{per}(\widetilde{A}_{i,r|t,j})$$

Thus, we can write Equality 8 as

$$V_A = 2\sum_{(j,t)\in\Omega} a_{i,j}a_{i,t}\,\text{per}(\widetilde{A}_{i,r|j,t})$$

where $\Omega = \{(j,t) \in S \times S \mid j < t\}$. □

According to Theorem 2.1, $\text{per}(B) - \text{per}(A)$ which we called as the variation, can be calculated by the following algorithm.

Algorithm 1 Calculation of the variation kV_A

INPUT: Matrix $A = [a_{i,j}]_{n\times n}$ and k, i, and r values.
OUTPUT: Result of the variation kV_A.
Step 1: row_set = The set of row numbers of the matrix A except the rows i and r
Step 2: for $j = 1, 2, \cdots, n-1$ do
    for $t = j+1, \cdots, n$ do
        column_set = The set of column numbers of the matrix A except the columns j and t
        A_tilda = Form a submatrix of the matrix A using row_set and column_set
        Per(A_tilda) = Calculate the permanent of A_tilda
        summation = summation + $a_{i,j} * a_{i,t} *$Per(A_tilda)
    end for
  end for
Step 3: result = $2 * k *$ summation

3. Conclusion

It is important to note that this study does not propose any permanent calculation method. Instead, it provides a theoretical analysis of the variation that results from an additive row operation on the permanent of a square matrix. Moreover, it formulates the variation that occurs in the permanent of a square matrix in which the additive row operation is applied. This formula, called the variation in the permanent, proposes utilizing matrices of order $(n-2) \times (n-2)$ instead of matrices of order $n \times n$. Besides, this paper presents an algorithm to calculate this variation formula. The proposed algorithm needs

$$\frac{(n-2)n!}{2}$$

arithmetic operations if the permanent is calculated by the Naive algorithm, and

$$n(n-1)(n-2)2^{n-3}$$

arithmetic operations if the permanent is calculated by the Ryser-NW algorithm. For the numbers of arithmetic operations of the Naive and the Ryser-NW algorithms, see [9].

This article's findings can be extended to non-square matrices for further investigation. Furthermore, the variation formula suggested herein can be used to study the calculation of any square matrix permanents via the Gaussian elimination process.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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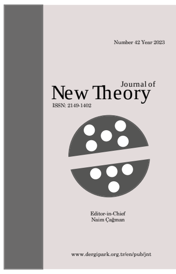

A Characterization of Semiprime Rings with Homoderivations

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Research Article

Abstract — This paper is focused on the commutativity of the laws of semiprime rings, which satisfy some algebraic identities involving homoderivations on ideals. It provides new and notable results that will interest researchers in this field, such as "$\mathfrak{R}$ contains a nonzero central ideal if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that $\delta(\mathfrak{I}) \subseteq Z$ where $\mathfrak{R}$ is a semiprime ring with center Z and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$". Moreover, the research also generalizes some results previously published in the literature, including derivation on prime rings using homoderivation on semiprime rings. It also demonstrates the necessity of hypotheses operationalized in theorems by an example. Finally, the paper discusses how the results herein can be further developed in future research.

Keywords *Semiprime rings, ideals, derivations, homoderivations*

Mathematics Subject Classification (2020) 11T99, 16W25

1. Introduction (Compulsory)

There is a growing literature on strong commutativity preserving (SCP) maps and derivations. Bell and Daif [1] first investigated the derivation of SCP maps on the ideal of a semiprime ring. Bresar [2] generalized this work to the Lie ideal of the ring. In [3], Ma and Xu handled this study for generalized derivations. Moreover, Koç and Gölbaşı [4] have been studied for the multiplicative generalized derivations by generalizing these conditions on the semiprime ring. In [5], Ali et al. showed that if $\mathfrak{R}$ is a semiprime ring and f is an endomorphism which is an SCP map on a nonzero ideal U of $\mathfrak{R}$, then f is commuting on U. Samman [6] proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Researchers have extensively studied derivations and SCP mappings in the context of operator algebras, prime rings, and semiprime rings. In [7], Melaibari et al. examined this condition for homoderivation. This paper investigated SCP maps for homoderivation in the ideal of a semiprime ring. In [8], Herstein showed that if $\mathfrak{R}$ is a prime ring of characteristics different from two and d is a nonzero derivation such that $d(\mathfrak{R}) \subseteq Z$, then $\mathfrak{R}$ must be commutative. This condition on the Lie ideal of the prime ring was discussed by Bergen et al. [9]. Gölbaşı and Koç [10] examined this condition for the (σ,τ)-Lie ideal of the prime ring. This condition is then examined for different subsets of the ring and different derivations. Ashraf et al. [11] proved that a prime ring $\mathfrak{R}$ must be commutative if $\mathfrak{R}$ satisfies the following condition: $f(x)f(y) = xy$ or $f(x)f(y) = yx$ where f is a generalized derivation of $\mathfrak{R}$, and $\mathfrak{I}$ is a nonzero two-sided ideal of $\mathfrak{R}$. In [12], the following conditions are examined by Alharfie and Muthana for homoderivation in the prime ring:

$$i.\, x\delta(y) \pm xy \in Z,\ ii.\, x\delta(y) \pm yx \in Z,\ iii.\, [\delta(x), y] \pm xy \in Z, \text{ and } iv.\, [\delta(x), y] \pm yx \in Z$$

This article aims to generalize the above conditions for homoderivation on an ideal semiprime ring.

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2. Preliminaries

Let $\mathfrak{R}$ be an associative ring with center Z. For any $x, y \in \mathfrak{R}$, the symbol $[x, y]$ stands for the commutator $xy - yx$, and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. Recall that a ring $\mathfrak{R}$ is a semiprime if $x\mathfrak{R}x = 0$ implies $x = 0$. An additive mapping $d: \mathfrak{R} \to \mathfrak{R}$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathfrak{R}$. An additive mapping $\delta: \mathfrak{R} \to \mathfrak{R}$ is called a homoderivation if $\delta(xy) = \delta(x)\delta(y) + \delta(x)y + x\delta(y)$, for all $x, y \in \mathfrak{R}$ in [13]. For example, $\delta(x) = f(x) - x$, for all $x \in \mathfrak{R}$, where f is an endomorphism on $\mathfrak{R}$ if $\delta(x)\delta(y) = 0$, for all $x, y \in \mathfrak{R}$, then the homoderivation δ is a derivation. If $S \subseteq \mathfrak{R}$, then a mapping $f: \mathfrak{R} \to \mathfrak{R}$ preserves S if $f(S) \subseteq S$. A mapping $f: \mathfrak{R} \to \mathfrak{R}$ is zero-power valued on S if f preserves S and if, for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)} = 0$. Let S be a nonempty subset of $\mathfrak{R}$. A mapping F from $\mathfrak{R}$ to $\mathfrak{R}$ is called commutativity preserving on a subset S of $\mathfrak{R}$ if $[x, y] = 0$ implies $[F(x), F(y)] = 0$, for all $x, y \in S$. The mapping F is called an SCP on S if $[x, y] = [F(x), F(y)]$, for all $x, y \in S$.

Proposition 2.1. Let $\mathfrak{R}$ be a semiprime ring. Then,

i. The center of $\mathfrak{R}$ contains no nonzero nilpotent elements.

ii. If P is a nonzero prime ideal of $\mathfrak{R}$ and $a, b \in \mathfrak{R}$ such that $a\,\mathfrak{R}\,b \subseteq P$, then either $a \in P$ or $b \in P$.

iii. The center of a nonzero one-sided ideal is in the center of $\mathfrak{R}$. In particular, any one-sided commutative ideal is included in the center of $\mathfrak{R}$.

Lemma 2.2. [14] If $\mathfrak{R}$ is a semiprime ring, then the center of a nonzero ideal of $\mathfrak{R}$ is contained in the center of $\mathfrak{R}$.

3. Main Results

This section investigates the aforesaid commutativity conditions for homoderivations in the semiprime ring.

Theorem 3.1. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that $\delta(\mathfrak{I}) \subseteq Z$.

PROOF.

By the hypothesis, we have

$$\delta(v_1) \in Z, \text{ for all } v_1 \in \mathfrak{I}$$

Commuting this term with $r \in \mathfrak{R}$, we obtain that

$$[\delta(v_1), r] = 0, \text{ for all } v_1 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing v_1 by v_1v_2, $v_2 \in \mathfrak{I}$, in this equation and using the hypothesis, we get

$$\delta(v_1)[v_2, r] + [v_1, r]\delta(v_2) = 0$$

Taking r by v_1, we obtain that

$$\delta(v_1)[v_2, v_1] = 0 \tag{1}$$

Replacing v_2 by v_2r, $r \in \mathfrak{R}$, in the last equation and using Equation 1, we have

$$\delta(v_1)v_2[r, v_1] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}, r \in \mathfrak{R} \tag{2}$$

Taking v_2 by $[r, v_1]t\delta(v_1)$, $t \in \mathfrak{R}$, we observe that

$$\delta(v_1)[r, v_1]t\delta(v_1)[r, v_1] = 0$$

and

$$\delta(v_1)[r,v_1]\mathfrak{R}\,\delta(v_1)[r,v_1] = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get

$$\delta(v_1)[r,v_1] = 0$$

That is,

$$\delta(v_1)\mathfrak{R}[\mathfrak{R},v_1] = (0),\text{ for all } v_1 \in \mathfrak{I}$$

Since $\mathfrak{R}$ is a semiprime ring, it must contain a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$. If P is a typical member of $\wp$ and $v_1 \in \mathfrak{I}$, we have $[\mathfrak{R},v_1] \subseteq P$ or $\delta(v_1) \subseteq P$ by Proposition 2.1 (*ii*). Define two additive subgroups $\mathcal{L} = \{v_1 \in \mathfrak{I} \mid [\mathfrak{R},v_1] \subseteq P\}$ and $\mathcal{F} = \{v_1 \in \mathfrak{I} \mid \delta(v_1) \subseteq P\}$.

It is clear that $\mathfrak{I} = \mathcal{L} \cup \mathcal{F}$. Since a group cannot be a union of two of its subgroups, either $\mathcal{L} = \mathfrak{I}$ or $\mathcal{F} = \mathfrak{I}$. Therefore, we have

$$[\mathfrak{R},\mathfrak{I}] \subseteq P \text{ or } \delta(\mathfrak{I}) \subseteq P$$

Thus, both cases together yield

$$[\mathfrak{R},\mathfrak{I}]\delta(\mathfrak{I}) \subseteq P,\text{ for any } P \in \wp$$

Therefore,

$$[\mathfrak{R},\mathfrak{I}]\delta(\mathfrak{I}) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$$

and

$$[\mathfrak{R},\mathfrak{I}]\delta(\mathfrak{I}) = (0)$$

Hence,

$$[\mathfrak{R},\mathfrak{R}\mathfrak{I}\delta(\mathfrak{I})\mathfrak{R}]\mathfrak{R}\mathfrak{R}\mathfrak{I}\delta(\mathfrak{I})\mathfrak{R} = (0)$$

This implies that $[\mathfrak{R},\Pi]\mathfrak{R}\Pi = (0)$ where $\Pi = \mathfrak{R}\mathfrak{I}\delta(\mathfrak{I})\mathfrak{R}$ is a nonzero ideal of $\mathfrak{R}$ since $\delta(\mathfrak{I}) \neq (0)$. Then,

$$[\mathfrak{R},\Pi]\mathfrak{R}[\mathfrak{R},\Pi] = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get $[\mathfrak{R},\Pi] = (0)$. Hence, $\Pi \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

The theorem below is proved for prime rings in Theorem 5 [15]. Here, it is generalized using semiprime rings.

Theorem 3.2. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that

i. $\delta([\mathfrak{I},\mathfrak{I}]) = (0)$ or

ii. $[\delta(\mathfrak{I}),\mathfrak{I}] \subseteq Z$ or

iii. $[\delta(\mathfrak{I}),\delta(\mathfrak{I})] = (0)$, $\mathfrak{I}\delta^2(\mathfrak{I}) \neq (0)$, and $\delta(\mathfrak{I}) \subseteq \mathfrak{I}$

PROOF.

i. By the hypothesis, we have

$$\delta([v_1,v_2]) = 0,\text{ for all } v_1,v_2 \in \mathfrak{I}$$

Replacing v_2 by v_2v_1 in this equation, we get

$$\delta([v_1,v_2]v_1) = 0$$

Since δ is a homoderivation, we have

$$\delta([v_1, v_2])\delta(v_1) + \delta([v_1, v_2])v_1 + [v_1, v_2]\delta(v_1) = 0$$

By the hypothesis, we get

$$[v_1, v_2]\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Using the same arguments in the proof of Theorem 3.1, we find that $\mathfrak{R}$ contains a nonzero central ideal.

ii. By the hypothesis, we get

$$[\delta(v_1), v_2] \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is,

$$\big[[\delta(v_1), v_2], r\big] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing v_2 by $\delta(v_1)v_2$ in the last equation and using this equation, we have

$$0 = [\delta(v_1)[\delta(v_1), v_2], r] = [\delta(v_1), r][\delta(v_1), v_2]$$

Taking r by $v_2 r$ in this equation, we see that

$$[\delta(v_1), v_2]r[\delta(v_1), v_2] = 0$$

By the semiprimeness of $\mathfrak{R}$, we get

$$[\delta(v_1), v_2] = 0$$

That is, $\delta(v_1) \in Z(\mathfrak{I})$, for all $v_1 \in \mathfrak{I}$. By Lemma 2.2, we get $\delta(v_1) \in Z$, for all $v_1 \in \mathfrak{I}$. By Theorem 3.1, we conclude that $\mathfrak{R}$ contains a nonzero central ideal.

iii. By the hypothesis, we get

$$[\delta(v_1), \delta(v_2)] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $v_2\delta(v_1)$ in this equation and using this equation, we have

$$[\delta(v_1), v_2]\delta^2(v_1) = 0$$

Replacing v_2 by $rv_2, r \in \mathfrak{R}$ in the last equation and using this equation, we have

$$[\delta(v_1), r]v_2\delta^2(v_1) = 0$$

That is,

$$[\delta(v_1), \mathfrak{R}]\mathfrak{R}\mathfrak{I}\delta^2(v_1) = (0), \text{ for all } v_1 \in \mathfrak{I}$$

Since $\mathfrak{R}$ is a semiprime ring, we must contain a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$. If P is a typical member of $\wp$ and $v_1 \in \mathfrak{I}$, by Proposition 2.1 (*ii*), we have

$$[\delta(v_1), \mathfrak{R}] \subseteq P \text{ or } \mathfrak{I}\delta^2(v_1) \subseteq P$$

Define two additive subgroups

$$\mathcal{L} = \{v_1 \in \mathfrak{I} \mid [\delta(v_1), \mathfrak{R}] \subseteq P\} \text{ and } \mathcal{F} = \{v_1 \in \mathfrak{I} \mid \mathfrak{I}\delta^2(v_1) \subseteq P\}$$

It is clear that $\mathfrak{I} = \mathcal{L} \cup \mathcal{F}$. Since a group cannot be a union of its two subgroups, either $\mathcal{L} = \mathfrak{I}$ or $\mathcal{F} = \mathfrak{I}$. Then,

$$[\delta(\mathfrak{I}), \mathfrak{R}] \subseteq P \text{ or } \mathfrak{I}\delta^2(\mathfrak{I}) \subseteq P$$

Thus, both cases together yield

$$[\delta(\mathfrak{I}), \mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) \subseteq P, \text{ for any } P \in \wp$$

Therefore,

$$[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$$

and

$$[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

That is,

$$\begin{aligned}(0) &= [\delta(\mathfrak{I}\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &= [\delta(\mathfrak{I})\delta(\mathfrak{I}) + \delta(\mathfrak{I})\mathfrak{I} + \mathfrak{I}\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &= [\delta(\mathfrak{I})\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + [\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + [\mathfrak{I}\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &= [\delta(\mathfrak{I}),\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I}) + \delta(\mathfrak{I})[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + [\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + \mathfrak{I}[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &\quad +[\mathfrak{I},\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I})\end{aligned}$$

Using $\delta(\mathfrak{I}) \subseteq \mathfrak{I}$, we have

$$[\delta(\mathfrak{I}),\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I}) + \delta(\mathfrak{I})[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + [\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + \mathfrak{I}[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) + [\delta(\mathfrak{I}),\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

Since $[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$, we get

$$[\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

This implies that

$$\left[\delta\big(\mathfrak{I}\delta(\mathfrak{I})\big)\mathfrak{I},\mathfrak{R}\right]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

and

$$[\mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

Hence,

$$[\mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{R}\mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I} = (0)$$

This implies that $[\Pi,\mathfrak{R}]\mathfrak{R}\Pi = (0)$ where $\Pi = \mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I}$ is a nonzero ideal of $\mathfrak{R}$ since $\mathfrak{I}\delta^2(\mathfrak{I}) \neq (0)$. Then,

$$[\Pi,\mathfrak{R}]\mathfrak{R}[\Pi,\mathfrak{R}] = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get $[\Pi,\mathfrak{R}] = (0)$. Hence, $\Pi \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

The proof of the following result differs from that of Theorem 2.2 in [7].

Corollary 3.3. Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$, and δ a nonzero and zero-power valued homoderivation on $\mathfrak{I}$. If δ is an SCP on $\mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal.

PROOF.

By the hypothesis, we get

$$[\delta(v_1),\delta(v_2)] = [v_1,v_2], \text{ for all } v_1,v_2 \in \mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathfrak{I}$, in the last equation, we have

$$[\delta(v_1),\delta(v_2)]\delta(v_3) + \delta(v_2)[\delta(v_1),\delta(v_3)] + [\delta(v_1),\delta(v_2)]v_3 + \delta(v_2)[\delta(v_1),v_3] + [\delta(v_1),v_2]\delta(v_3) + v_2[\delta(v_1),\delta(v_3)] = [v_1,v_2]v_3 + v_2[v_1,v_3]$$

Using the hypothesis, we obtain that

$$[v_1,v_2]\delta(v_3) + \delta(v_2)[v_1,v_3] + \delta(v_2)[\delta(v_1),v_3] + [\delta(v_1),v_2]\delta(v_3) = 0$$

That is,

$$[v_1 + \delta(v_1), v_2]\delta(v_3) + \delta(v_2)[\delta(v_1) + v_1, v_3] = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(v_1) = 0$, for all $v_1 \in \mathfrak{I}$. Replacing v_1 by $v_1 - \delta(v_1) + \delta^2(v_1) + \cdots + (-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we get

$$[v_1, v_2]\delta(v_3) + \delta(v_2)[v_1, v_3] = 0$$

Replacing v_3 by v_1 in the last equation, we get

$$[v_1, v_2]\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

The rest of the proof is the same as Equation 1. This completes the proof.

Theorem 3.4. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that

i. $\delta(\mathfrak{I} \circ \mathfrak{I}) = (0)$ or

ii. $\delta(\mathfrak{I}) \circ \mathfrak{I} \subseteq Z$

Proof.

i. We have

$$\delta(v_1 \circ v_2) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Relacing v_2 by $v_2 v_1$ in the above equation, we get

$$\delta(v_1 \circ v_2)\delta(v_1) + \delta(v_1 \circ v_2)v_1 + (v_1 \circ v_2)\delta(v_1) = 0$$

Using the hypothesis, we get

$$(v_1 \circ v_2)\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by rv_2, $r \in \mathfrak{R}$, in the last equation, we get

$$[v_1, r]v_2\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}, r \in \mathfrak{R}$$

Using the same arguments in the proof of Theorem 3.1, we find that $\mathfrak{R}$ contains a nonzero central ideal.

ii. We get

$$\delta(v_1) \circ v_2 \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in this equation, we have

$$(\delta(v_1) \circ v_2)v_3 + v_2[v_3, \delta(v_1)] \in Z$$

That is,

$$[(\delta(v_1) \circ v_2)v_3 + v_2[v_3, \delta(v_1)], r] = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

and

$$[(\delta(v_1) \circ v_2), r]v_3 + (\delta(v_1) \circ v_2)[v_3, r] + v_2\big[[v_3, \delta(v_1)], r\big] + [v_2, r][v_3, \delta(v_1)] = 0$$

Using the hypothesis, we observe that

$$(\delta(v_1) \circ v_2)[v_3, r] + v_2\big[[v_3, \delta(v_1)], r\big] + [v_2, r][v_3, \delta(v_1)] = 0$$

Taking r by v_3 in the above equation, we have

$$v_2\big[[v_3, \delta(v_1)], v_3\big] + [v_2, v_3][v_3, \delta(v_1)] = 0$$

Replacing v_2 by $\delta(v_1)[\delta(v_1), v_3]$ in the last equation, we get

$$\delta(v_1)[\delta(v_1),v_3]\big[[v_3,\delta(v_1)],v_3\big] + [\delta(v_1)[\delta(v_1),v_3],v_3][v_3,\delta(v_1)] = 0$$

That is,

$$\delta(v_1)[\delta(v_1),v_3]\big[[v_3,\delta(v_1)],v_3\big] + \delta(v_1)\big[[\delta(v_1),v_3],v_3\big][v_3,\delta(v_1)] + [\delta(v_1),v_3][\delta(v_1),v_3][v_3,\delta(v_1)] = 0$$

Using the above equation, we get

$$[\delta(v_1),v_3][\delta(v_1),v_3][v_3,\delta(v_1)] = 0$$

and

$$([\delta(v_1),v_3])^3 = 0$$

The semiprime ring contains no nonzero nilpotent elements. Thus, $[\delta(v_1),v_3] = 0$, for all $v_1, v_3 \in \mathfrak{I}$. By Theorem 3.2 (*ii*), we get $\mathfrak{R}$ contains a nonzero central ideal.

Theorem 3.5. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero and zero-power valued homoderivation δ on $\mathfrak{I}$ such that, for all $v_1, v_2 \in \mathfrak{I}$,

i. $\delta(v_1)\delta(v_2) = v_1v_2$ or

ii. $\delta(v_1)\delta(v_2) = v_2v_1$ or

iii. $\delta(v_1)\delta(v_2) = [v_1,v_2]$ or

iv. $\delta(v_1)\delta(v_2) = v_1 \circ v_2$ or

v. $\delta([v_1,v_2]) = [\delta(v_1),v_2]$ or

vi. $\delta(v_1 \circ v_2) = \delta(v_1) \circ v_2$

PROOF.

i. By the hypothesis, we get

$$\delta(v_1)\delta(v_2) = v_1v_2, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathfrak{I}$, in this equation, we have

$$\delta(v_1)\delta(v_2)v_3 + \delta(v_1)\delta(v_2)\delta(v_3) + \delta(v_1)v_2\delta(v_3) = v_1v_2v_3$$

Using the hypothesis, we see that

$$v_1v_2\delta(v_3) + \delta(v_1)v_2\delta(v_3) = 0 \tag{3}$$

That is,

$$\big(v_1 + \delta(v_1)\big)v_2\delta(v_3) = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n(x) > 1$ such that $\big(\delta^{n(x)}\big)(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_1 by $v_1 - \delta(v_1) + \delta^2(v_1) + \cdots + (-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we get

$$v_1v_2\delta(v_3) = 0$$

Replacing v_1 by $v_1v_2\delta(v_3)rv_1$ in this equation, we get

$$v_1v_2\delta(v_3)rv_1v_2\delta(v_3) = 0$$

That is,

$$v_1v_2\delta(v_3)\mathfrak{R}v_1v_2\delta(v_3) = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$v_1 v_2 \delta(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_1 by $[r, v_3]$ in the last equation, we have

$$[r, v_3] v_2 \delta(v_3) = 0, \text{ for all } v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

and

$$[\mathfrak{R}, v_3] \mathfrak{I} \delta(v_3) = 0, \text{ for all } v_3 \in \mathfrak{I}$$

The rest of the proof is the same as Equation 2. This completes the proof.

ii. By the hypothesis, we have

$$\delta(v_1)\delta(v_2) = v_2 v_1, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in this equation, we get

$$\delta(v_1)\delta(v_2)\delta(v_3) + \delta(v_1)\delta(v_2)v_3 + \delta(v_1)v_2\delta(v_3) = v_2 v_3 v_1$$

and

$$\delta(v_1)\delta(v_2)\delta(v_3) + \delta(v_1)\delta(v_2)v_3 + \delta(v_1)v_2\delta(v_3) = v_2 v_1 v_3 - v_2 v_1 v_3 + v_2 v_3 v_1$$

Using the hypothesis, we obtain that

$$\delta(v_1) v_2 \delta(v_3) = -v_2 v_1 v_3 + v_2 v_3 v_1 - v_2 v_1 \delta(v_3) \tag{4}$$

Replacing v_2 by $r v_2$, $r \in \mathfrak{R}$, in this equation, we obtain that

$$\delta(v_1) r v_2 \delta(v_3) = -r v_2 v_1 v_3 + r v_2 v_3 v_1 - r v_2 v_1 \delta(v_3)$$

Using Equation 4 in this equation, we get

$$\delta(v_1) r v_2 \delta(v_3) = r \delta(v_1) v_2 \delta(v_3), \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing r by $\delta(v_3)$ in the above equation, we find that

$$\delta(v_1)\delta(v_3) v_2 \delta(v_3) = \delta(v_3)\delta(v_1) v_2 \delta(v_3)$$

Using the hypothesis, we observe that

$$v_3 v_1 v_2 \delta(v_3) = v_1 v_3 v_2 \delta(v_3), \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

That is,

$$[v_1, v_3] v_2 \delta(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_1 by $r v_1$ in the last equation and using this equation, we get

$$[r, v_3] v_1 v_2 \delta(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Taking v_1 by $v_2 \delta(v_3) t [r, v_3]$, $t \in \mathfrak{R}$, in the last equation, we observe that

$$[r, v_3] v_2 \delta(v_3) t [r, v_3] v_2 \delta(v_3) = 0$$

and

$$[r, v_3] v_2 \delta(v_3) \mathfrak{R} [r, v_3] v_2 \delta(v_3) = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get

$$[r, v_3] v_2 \delta(v_3) = 0, \text{ for all } v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

The rest of the proof is the same as Equation 2. This completes the proof.

iii. By the hypothesis, we get

$$\delta(v_1)\delta(v_2) = [v_1, v_2], \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_1 by v_1v_3 in this equation, we have

$$\delta(v_1)\delta(v_3)\delta(v_2) + \delta(v_1)v_3\delta(v_2) + v_1\delta(v_3)\delta(v_2) = [v_1, v_2]v_3 + v_1[v_3, v_2]$$

Using the hypothesis, we see that

$$[v_1, v_3]\delta(v_2) + \delta(v_1)v_3\delta(v_2) = [v_1, v_2]v_3$$

Taking v_2 by v_1 in the above equation, we find that

$$[v_1, v_3]\delta(v_1) + \delta(v_1)v_3\delta(v_1) = 0$$

By the hypothesis, we have

$$\delta(v_1)\delta(v_3)\delta(v_1) + \delta(v_1)v_3\delta(v_1) = 0 \tag{5}$$

That is,

$$\delta(v_1)(\delta(v_3) + v_3)\delta(v_1) = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_3 by $v_3 - \delta(v_3) + \delta^2(v_3) + \cdots + (-1)^{n-1}\delta^{n-1}(v_3)$ in this equation, we get

$$\delta(v_1)v_3\delta(v_1) = 0$$

That is,

$$\delta(v_1)v_3\mathfrak{R}\delta(v_1)v_3 = (0), \text{ for all } v_1, v_3 \in \mathfrak{I}$$

By the semiprimeness of $\mathfrak{R}$, we have

$$\delta(v_1)v_3 = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}$$

Taking v_3 by $r[\delta(v_1), v_3]$, $r \in \mathfrak{R}$, in this equation, we obtain that

$$\delta(v_1)r[\delta(v_1), v_3] = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R} \tag{6}$$

Replacing r by v_3r in Equation 6, we find that

$$\delta(v_1)v_3r[\delta(v_1), v_3] = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R} \tag{7}$$

Multiplying Equation 6 on the left by v_3, we have

$$v_3\delta(v_1)r[\delta(v_1), v_3] = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R} \tag{8}$$

Subtracting Equation 7 from Equation 8, we arrive at

$$[\delta(v_1), v_3]r[\delta(v_1), v_3] = 0$$

Since $\mathfrak{R}$ is a semiprime ring, we get $[\delta(v_1), v_3] = 0$, for all $v_1, v_3 \in \mathfrak{I}$. By Theorem 3.2 (*ii*), we have $\mathfrak{R}$ contains a nonzero central ideal.

iv. By the hypothesis, we have

$$\delta(v_1)\delta(v_2) = v_1 \circ v_2, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_1 by v_1v_3, $v_3 \in \mathfrak{I}$, in this equation, we get

$$\delta(v_1)\delta(v_3)\delta(v_2) + \delta(v_1)v_3\delta(v_2) + v_1\delta(v_3)\delta(v_2) = v_1(v_3 \circ v_2) - [v_1, v_2]v_3$$

Using the hypothesis, we obtain that

$$\delta(v_1)\delta(v_3)\delta(v_2) + \delta(v_1)v_3\delta(v_2) = -[v_1, v_2]v_3$$

Replacing v_2 by v_1 in this equation, we obtain that

$$\delta(v_1)\delta(v_3)\delta(v_1) + \delta(v_1)v_3\delta(v_1) = 0$$

The rest of the proof is the same as Equation 5. This completes the proof.

v. We obtain that

$$\delta([v_1, v_2]) = [\delta(v_1), v_2], \text{ for all } v_1, v_2 \in \mathfrak{I}$$

This implies that

$$[\delta(v_1), \delta(v_2)] + [\delta(v_1), v_2] + [v_1, \delta(v_2)] = [\delta(v_1), v_2], \text{ for all } v_1, v_2 \in \mathfrak{I}$$

and

$$[\delta(v_1), \delta(v_2)] + [v_1, \delta(v_2)] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is,

$$[\delta(v_1) + v_1, \delta(v_2)] = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_1 by $v_1 - \delta(v_1) + \delta^2(v_1) + \cdots + (-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we get

$$[v_1, \delta(v_2)] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Theorem 3.2 (*ii*) concludes that $\mathfrak{R}$ contains a nonzero central ideal.

vi. We get

$$\delta(v_1 \circ v_2) = \delta(v_1) \circ v_2, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

If this expression is edited, we have

$$\delta(v_1) \circ \delta(v_2) + \delta(v_1) \circ v_2 + v_1 \circ \delta(v_2) = \delta(v_1) \circ v_2$$

and

$$\delta(v_1) \circ \delta(v_2) + v_1 \circ \delta(v_2) = 0$$

That is,

$$(\delta(v_1) + v_1) \circ \delta(v_2) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Since δ is zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_1 by $v_1 - \delta(v_1) + \delta^2(v_1) + \cdots + (-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we obtain that

$$v_1 \circ \delta(v_2) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

By Theorem 3.4 (*ii*), we get $\mathfrak{R}$ contains a nonzero central ideal.

Theorem 3.6. Let $\mathfrak{R}$ be a 2-torsion free semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero and zero-power valued homoderivation δ on $\mathfrak{I}$ such that, for all $v_1, v_2 \in \mathfrak{I}$,

i. $v_1\delta(v_2) + v_1v_2 \in Z$ or

ii. $v_1\delta(v_2) + v_2v_1 = 0$ or

iii. $v_1\delta(v_2) \pm v_1 \circ v_1 = 0$ or

iv. $[\delta(v_1), v_2] \pm v_1v_2 = 0$ or *v.* $[\delta(v_1), v_2] \pm v_2 v_1 = 0$

PROOF.

i. By the hypothesis, we get

$$v_1\delta(v_2) + v_1 v_2 \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is,

$$v_1(\delta(v_2) + v_2) \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_2 by $v_2 - \delta(v_2) + \delta^2(v_2) + \cdots + (-1)^{n-1}\delta^{n-1}(v_2)$ in this equation, we obtain that

$$v_1 v_2 \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Commuting this term with $r \in \mathfrak{R}$, we obtain that

$$0 = [v_1 v_2, r] = [v_1, r]v_2 + v_1[v_2, r]$$

Replacing v_1 by $v_3 v_1$, $v_3 \in \mathfrak{I}$, in this equation and using this equation, we get

$$[v_3, r]v_1 v_2 = 0$$

Taking v_2 by $t[v_3, r]v_1$ in this equation, we have

$$[v_3, r]v_1 t[v_3, r]v_1 = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r, t \in \mathfrak{R}$$

That is,

$$[v_3, r]v_1\mathfrak{R}[v_3, r]v_1 = (0), \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Since $\mathfrak{R}$ is a semiprime, we have

$$[v_3, r]v_1 = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing v_1 by $t[v_3, r]$ in the last equation, we get

$$[v_3, r]t[v_3, r] = 0, \text{ for all } v_3 \in \mathfrak{I}, r, t \in \mathfrak{R}$$

By the semiprimeness of $\mathfrak{R}$, we have $[v_3, r] = 0$, for all $v_3 \in \mathfrak{I}, r \in \mathfrak{R}$. Thus, $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

ii. We get

$$v_1\delta(v_2) + v_2 v_1 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_1 v_2$, $v_2 \in \mathfrak{I}$, in this equation and using this equation, we have

$$v_1\delta(v_1)(\delta(v_2) + v_2) = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_2 by $v_2 - \delta(v_2) + \delta^2(v_2) + \cdots + (-1)^{n-1}\delta^{n-1}(v_2)$ in this equation, we obtain that

$$v_1\delta(v_1)v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $rv_1\delta(v_1)$, $r \in \mathfrak{R}$, in the last equation, we have

$$v_1\delta(v_1)rv_1\delta(v_1) = 0, \text{ for all } v_1 \in \mathfrak{I}, r \in \mathfrak{R}$$

By the semiprimeness of $\mathfrak{R}$, we get

$$v_1\delta(v_1) = 0, \text{ for all } v_1 \in \mathfrak{I} \quad (9)$$

By the hypothesis, we get

$$v_1\delta(v_1) + v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Using Equation 9, we obtain that

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I} \quad (10)$$

Replacing v_1 by $v_1 + v_2$ in this equation, we observe that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal. This completes the proof.

iii. By the hypothesis, we get

$$v_1\delta(v_2) \pm v_1 \circ v_2 = 0$$

Replacing v_2 by v_2v_1 in this equation, we get

$$v_1\delta(v_2)\delta(v_1) + v_1\delta(v_2)v_1 + v_1v_2\delta(v_1) \pm (v_1 \circ v_2)v_1 = 0$$

Using the hypothesis, we get

$$v_1\delta(v_2)\delta(v_1) + v_1v_2\delta(v_1) = 0$$

That is,

$$v_1(\delta(v_2) + v_2)\delta(v_1) = 0$$

Since δ is zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_2 by $v_2 - \delta(v_2) + \delta^2(v_2) + \cdots + (-1)^{n-1}\delta^{n-1}(v_2)$ in this equation, we obtain that

$$v_1v_2\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $\delta(v_1)rv_1$, $r \in \mathfrak{R}$, in the above equation, we see that

$$v_1\delta(v_1)rv_1\delta(v_1) = 0$$

By the semiprimeness of $\mathfrak{R}$, we have

$$v_1\delta(v_1) = 0, \text{ for all } v_1 \in \mathfrak{I}$$

By the hypothesis and using this equation, we have

$$v_1 \circ v_1 = 0, \text{ for all } v_1 \in \mathfrak{I} \quad (11)$$

and

$$2v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Since $\mathfrak{R}$ is a 2-torsion free, we have

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we see that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

iv. We get

$$[\delta(v_1), v_2] \pm v_1 v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $v_2 v_1$ in this equation, we get

$$[\delta(v_1), v_2]v_1 + v_2[\delta(v_1), v_1] \pm v_1 v_2 v_1 = 0$$

By the hypothesis, we get

$$v_2[\delta(v_1), v_1] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I} \tag{12}$$

Replacing v_2 by $[\delta(v_1), v_1]r$, $r \in \mathfrak{R}$, in this equation, we get

$$[\delta(v_1), v_1]r[\delta(v_1), v_1] = 0$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$[\delta(v_1), v_1] = 0, \text{ for all } v_1 \in \mathfrak{I}$$

By the hypothesis, we get

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we see that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

v. By the hypothesis, we get

$$[\delta(v_1), v_2] \pm v_2 v_1 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_1 v_2$ in this equation, we have

$$v_1[\delta(v_1), v_2] + [\delta(v_1), v_1]v_2 \pm v_1 v_2 v_1 = 0$$

That is,

$$[\delta(v_1), v_1]v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $r[\delta(v_1), v_1]$, $r \in \mathfrak{R}$, in this equation, we get

$$[\delta(v_1), v_1]r[\delta(v_1), v_1] = 0$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$[\delta(v_1), v_1] = 0, \text{ for all } v_1 \in \mathfrak{I}$$

By the hypothesis, we get

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we see that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal. We complete the proof.

Example 3.7. Let $\mathfrak{R} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$, $\mathfrak{I} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix} | a, b \in \mathbb{R} \right\}$, and $\delta: \mathfrak{R} \to \mathfrak{R}$ be a map defined by

$$\delta \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

Then, it is easy to verify that δ is an homoderivation of $\mathfrak{R}$, $\mathfrak{I}$ is an ideal of $\mathfrak{R}$, and $\mathfrak{R}$ is not a semiprime ring. The commutativity conditions given in Theorem 3.5 are satisfied. However, we have $\mathfrak{I} \nsubseteq Z$. We conclude that $\mathfrak{R}$ does not contain a nonzero central ideal.

4. Conclusion

The present study has shown some essential properties of a nonzero ideal of a semiprime ring with homoderivation. Moreover, it has exemplified the necessity of the hypotheses used in theorems. In future research, some well-known results in derivation can be applied to homoderivation and generalized homoderivation. Besides, the findings herein could help to uncover properties of homoderivations in Lie ideals or square-closed Lie ideals.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

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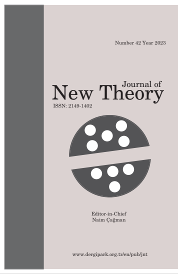

Multiplicity of Scator Roots and the Square Roots in $\mathbb{S}^{1+2}$

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Abstract — This paper presents the roots of elliptic scator numbers in $\mathbb{S}^{1+n}$, which includes both the fundamental 2π symmetry and the π-pair symmetry for $n \geq 2$. Here, the scator set $\mathbb{S}^{1+n}$ is a subset of $\mathbb{R}^{1+n}$ with the scator product and the multiplicative representation. These roots are expressed in terms of both additive (rectangular) and multiplicative (polar) variables. Additionally, the paper provides a comprehensive description of square roots in $\mathbb{S}^{1+2}$, which includes a geometrical representation in three-dimensional space that provides a clear visualization of the concept and makes it easier to understand and interpret. Finally, the paper handles whether the aspects should be further investigated.

Keywords *Roots, non-distributive algebras, hypercomplex numbers*

Mathematics Subject Classification (2020) 30G35, 20M14

1. Introduction

Due to the extra number of dimensions, hypercomplex number systems generally have a larger set of roots than those obtained in the complex plane. However, the number of roots varies considerably depending on the algebraic system [1]. The existence of roots and the obtention of their actual value are two different problems, just as in polynomial real algebra. Formulae to find the roots in diverse systems is subject to active research [2].

Scator algebra is an extension of complex algebra to higher dimensions where the real axis is unique, but there can be an arbitrary number n of hyperimaginary units. In the scator context, the scalar component corresponds to the real part, and each of the n director components corresponds to the imaginary part of a different complex set. In this sense, there are n copies of the complex set embedded in a $1+n$ dimensional scator algebra, just as in Clifford algebras. A geometric representation of scator elements is possible in Argand type diagrams with the appropriate increase of extra imaginary axes. In accordance with Froebenius Theorem and accord with other algebraic systems, not all group properties can be satisfied for scators for $n \geq 2$. Scator algebra is endowed with addition and product operations and a main second-order involution. However, a peculiarity of the scator system is that the product is generally not distributive over addition. The scator product definition gives rise to two branches, elliptic and hyperbolic [3], that are, to some extent, related to Clifford algebras and higher dimensional versions of complex and perplex algebras [4]. This communication is devoted to the description of roots of elliptic scators, also referred to as imaginary or cuspheric scators.

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A general description of the roots of elliptic scators relies on two main theorems that give rise to the Victoria equations in the multiplicative and additive representations. The former establishes that an exponent $\frac{1}{q}$ distributes over the scator component factors. The latter translates this result to the additive representation retaining the multiplicative (angle) variables. The Victoria equation in the additive representation may be viewed as a higher dimensional version of the de Moivre theorem. In [5], these theorems were presented, and several cases were expounded with particular emphasis in the roots of unity. An asset of scator roots is that their number is always finite, contrasting with some infinite solutions obtained in Clifford algebras [6].

In the present communication, the multiplicity of roots is treated in general in the multiplicative (polar) and additive (rectangular) representations in Sections 2 and 3. Particular attention is given to the π-pair symmetry overseen in the seminal publication [5]. The reader may choose to skip the two initial sections in the first approach, where arbitrary $\mathbb{S}^{1+n}$ dimensions and q roots are undertaken. In the remaining manuscript, square roots in $\mathbb{S}^{1+2}$ are treated in detail. In Section 4, square roots in the additive representation are expounded using multiplicative angle variables and additive rectangular variables. In Section 5, the geometric representation of scators and their construction via the addition or product of their components is described. In Subsection 5.1, the geometric visualization of the square root in $\mathbb{S}^{1+2}$ is presented. Conclusions are drawn in the last section.

2. Scators Roots

Scator elements in the multiplicative representation are written as a product of exponentials

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \in \mathbb{S}^{1+n}$$

where the multiplicative scalar φ_0 and the multiplicative director coefficients φ_j, for j from 1 to $n \in \mathbb{N}$, are real quantities and $\check{\mathbf{e}}_j \notin \mathbb{R}$. The scator set $\mathbb{S}^{1+n}$ is a subset of $\mathbb{R}^{1+n}$ where the scator product and the multiplicative representation exist. The product of two scators is evaluated by performing the multiplicative scalars product and the addition of the multiplicative director coefficients with the same director unit,

$$\overset{o}{\alpha}\overset{o}{\beta} = \left(\alpha_0 \prod_{j=1}^{n} \exp\left(\alpha_j \check{\mathbf{e}}_j\right)\right)\left(\beta_0 \prod_{j=1}^{n} \exp\left(\beta_j \check{\mathbf{e}}_j\right)\right) = \alpha_0\beta_0 \prod_{j=1}^{n} \exp\left[\left(\alpha_j + \beta_j\right)\check{\mathbf{e}}_j\right]$$

The components having the same director $\check{\mathbf{e}}_j$ satisfy the addition theorem for exponents. In contrast, components with different director units $\check{\mathbf{e}}_l$ and $\check{\mathbf{e}}_m$ $(l \neq m)$ do not, i.e., $\exp\left(\alpha_l \check{\mathbf{e}}_l\right)\exp\left(\beta_m \check{\mathbf{e}}_m\right) \neq \exp\left(\alpha_l \check{\mathbf{e}}_l + \beta_m \check{\mathbf{e}}_m\right)$. An expression for the exponential of a scator with 1+2 components has been derived in [7]. The conjugate of the scator $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j}$ is obtained by taking the negative of the director components $\overset{o}{\varphi}{}^{*} \equiv \varphi_0 \prod_{j=1}^{n} e^{-\varphi_j \check{\mathbf{e}}_j}$. The magnitude of a scator $\overset{o}{\varphi}$ is $\|\overset{o}{\varphi}\| = \sqrt{\overset{o}{\varphi}\overset{o}{\varphi}{}^{*}} = \varphi_0$, the multiplicative scalar thus represents the scator magnitude. The multiplicative inverse $\overset{o}{\varphi}{}^{-1} = \overset{o}{\varphi}{}^{*}\|\overset{o}{\varphi}\|^{-2}$ exists, if the scator magnitude is not zero. The additive representation of scator elements is

$$\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j$$

where the additive scalar component f_0 and the additive director components f_j, for j from 1 to $n \in \mathbb{N}$, are real quantities and $\check{\mathbf{e}}_j \notin \mathbb{R}$. The scator set $\mathbb{S}^{1+n}$ requires that the additive scalar component must be different from zero, if two or more additive director components are not zero,

$$\mathbb{S}^{1+n} = \left\{\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j : f_0 \neq 0 \text{ if } \exists\ f_j f_l \neq 0, \text{ for any } j \neq l\right\}$$

The multiplicative and additive representations are related by

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} = \varphi_0 \prod_{k=1}^{n} \cos(\varphi_k) + \varphi_0 \sum_{j=1}^{n} \prod_{k \neq j}^{n} \cos(\varphi_k) \sin(\varphi_j) \check{\mathbf{e}}_j = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j$$

If $f_0 \neq 0$, then the magnitude in terms of additive variables is given by

$$\|\overset{o}{\varphi}\| = |f_0| \prod_{j=1}^{n} \sqrt{1 + \frac{f_j^2}{f_0^2}} \tag{1}$$

and if $f_0 = 0$, then $\|\overset{o}{\varphi}\| = |f_j|$. A constant magnitude generates the cusphere isometric surface. Other relevant properties of elliptic scator algebra are summarized in [7].

In $\mathbb{S}^{1+1}$, the multiplicity of roots is due to the trigonometric functions 2π periodicity. Scators with a single director component are isomorphic to the set of complex numbers, i.e., $\mathbb{S}^{1+1} \cong \mathbb{C}$. Thus, the q roots familiar from complex algebra are reproduced, for each $\check{\mathbf{e}}_j$, if all the other director components vanish. In $\mathbb{S}^{1+2}$ or higher dimensions ($\mathbb{S}^{1+n}, n \geq 2$), the 2π trigonometric functions periodicity can be applied to each of the n φ_j's. Then, there are q roots per each of the n hypercomplex director directions. According to this reasoning, Corollary 1 in [5] stated incorrectly: "There are at most q^n different roots for a scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ raised to the power $\frac{1}{q}$". In scator algebra, when two or more hyper-imaginary units are present, the arguments of two multiplicative components can be simultaneously modified by π. Their product leaving the element invariant. This symmetry increases the multiplicity of the roots. These assertions are formulated in the following propositions.

Definition 2.1. The π-pair transformation symmetry requires the simultaneous displacement by π of the argument of two multiplicative director components of a scator element. Given $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \in \mathbb{S}^{1+n}$, a π-pair transformation $\overset{o}{\varphi} \to \overset{o}{\varphi}'$ is

$$\overset{o}{\varphi}' = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{(\varphi_l \pm \pi)\check{\mathbf{e}}_l} e^{(\varphi_m \pm \pi)\check{\mathbf{e}}_m}$$

for any l, m pair from 1 to n.

Proposition 2.2. Elliptic scators are invariant under π-pair transformations.

PROOF.
For the components $\check{\mathbf{e}}_l$ and $\check{\mathbf{e}}_m$ of a scator $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j}$,

$$\overset{o}{\varphi} = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{\varphi_l \check{\mathbf{e}}_l} e^{\varphi_m \check{\mathbf{e}}_m} \in \mathbb{S}^{1+n}$$

Perform a π-pair displacement of the components $\check{\mathbf{e}}_l$ and $\check{\mathbf{e}}_m$

$$\overset{o}{\varphi}' = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{(\varphi_l \pm \pi)\check{\mathbf{e}}_l} e^{(\varphi_m \pm \pi)\check{\mathbf{e}}_m} = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{\varphi_l \check{\mathbf{e}}_l} e^{\varphi_m \check{\mathbf{e}}_m} e^{\pm\pi \check{\mathbf{e}}_l} e^{\pm\pi \check{\mathbf{e}}_m}$$

then

$$\overset{o}{\varphi}' = \overset{o}{\varphi} e^{\pm\pi \check{\mathbf{e}}_l} e^{\pm\pi \check{\mathbf{e}}_m} = \overset{o}{\varphi} (-1)(-1) = \overset{o}{\varphi}$$

This π-pair displacement can be carried over an arbitrary pair of components. Therefore,

$$\overset{o}{\varphi}' = \varphi_0 \prod_{j=1}^{n} e^{(\varphi_j + \sigma_j \pi)\check{\mathbf{e}}_j} = \overset{o}{\varphi}$$

for $\sigma_j = 1$ applied in pairs. If $\sigma_j = 0$, then the j^{th} component is unaltered. □

Corollary 2.3. The π-pair displacement of components k and l and subsequent π-pair displacement of k and m is equal to the π-pair displacement of l and m.

PROOF.
The k and l π-pair displacement is

$$\varphi_0 \prod_{j\neq k,l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \exp\left(\left(\varphi_k+\pi\right)\check{\mathbf{e}}_k\right)\exp\left(\left(\varphi_l+\pi\right)\check{\mathbf{e}}_l\right)\exp\left(\varphi_m\check{\mathbf{e}}_m\right)$$

and the subsequent k and m π-pair displacement is

$$\varphi_0 \prod_{j\neq k,l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \exp\left(\left(\varphi_k+\pi+\pi\right)\check{\mathbf{e}}_k\right)\exp\left(\left(\varphi_l+\pi\right)\check{\mathbf{e}}_l\right)\exp\left(\left(\varphi_m+\pi\right)\check{\mathbf{e}}_m\right)$$

Due to the 2π symmetry this scator is equal to

$$\varphi_0 \prod_{j\neq k,l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \exp\left(\varphi_k\check{\mathbf{e}}_k\right)\exp\left(\left(\varphi_l+\pi\right)\check{\mathbf{e}}_l\right)\exp\left(\left(\varphi_m+\pi\right)\check{\mathbf{e}}_m\right)$$

that is the l and m π-pair displacement. □

In the multiplicative representation, Theorem 1 in [5] established the roots of a scator due to the 2π trigonometric periodicity. This theorem can now be extended to include the roots arising from the π-pair symmetry.

Theorem 2.4. In the multiplicative representation, for a scator $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \in \mathbb{S}^{1+n}$ raised to the power $\frac{1}{q}$ such that $q \in \mathbb{Z}$, the exponent $\frac{1}{q}$ distributes over the scator component factors

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j}\right)^{\frac{1}{q}} = \varphi_0^{\frac{1}{q}} \prod_{j=1}^{n} e^{\frac{1}{q}\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\check{\mathbf{e}}_j} \tag{2}$$

where $r_j \in \mathbb{Z}$, from 0 to $q-1$ and σ_j is 0 or 1, the sum of all σ_j is even, for j from 1 to n.

PROOF.
Let $\overset{o}{\varphi} = \overset{o}{\zeta}^{q}$. From the distributivity of an integer exponent over the scator factors, stated in Theorem 3 [7],

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} \exp\left(\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\check{\mathbf{e}}_j\right) = \left(\zeta_0 \prod_{j=1}^{n} \exp\left(\zeta_j\check{\mathbf{e}}_j\right)\right)^{q} = \zeta_0^q \prod_{j=1}^{n} \exp\left(q\zeta_j\check{\mathbf{e}}_j\right)$$

equating components, $\varphi_0 = \zeta_0^q$ and $\varphi_j + 2\pi r_j + \sigma_j\pi = q\zeta_j$, where $r_j \in \mathbb{Z}$ takes values from 0 to $q-1$ and $\sigma_j = 0$ or 1 in pairs, for each subindex j. The $2\pi r_j$ addend in the argument makes explicit the fundamental symmetry of the exponential function with unit directors that satisfy $\check{\mathbf{e}}_j\check{\mathbf{e}}_j = -1$. Whereas the even sum of σ_j exhibits the π-pair symmetry of components couples. Evaluate the above equation to the power $\frac{1}{q}$,

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 \prod_{j=1}^{n} \exp\left(\varphi_j\check{\mathbf{e}}_j\right)\right)^{\frac{1}{q}} = \zeta_0 \prod_{j=1}^{n} \exp\left(\zeta_j\check{\mathbf{e}}_j\right)$$

Substitute $\zeta_j = \frac{\varphi_j+2\pi r_j+\sigma_j\pi}{q}$ and $\zeta_0 = \varphi_0^{\frac{1}{q}}$, Equation 2,

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 \prod_{j=1}^{n} \exp\left(\varphi_j\check{\mathbf{e}}_j\right)\right)^{\frac{1}{q}} = \varphi_0^{\frac{1}{q}} \prod_{j=1}^{n} \exp\left(\frac{1}{q}\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\check{\mathbf{e}}_j\right)$$

is obtained. □

Corollary 2.5. A scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ to the power $\frac{1}{q}$ has at most $2pq^n$ different roots, where p is the number of different π-pair possibilities.

Proof.
For $r_j \in \mathbb{Z}$, from 0 to $q-1$, there are q possible arguments for each of the n director components. Therefore, there are q^n possible permutations. For each of them, there is a number p of π-pair possibilities, where every π-pair has two possible values. Thus, there are $2p\,q^n$ possible configurations. □

In many cases, the number of different roots is less than $2pq^n$, either because some root values are repeated or involve only a single director component. Restricted to $\mathbb{S}^{1+2}$, $\overset{o}{\varphi} = \varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}$, where in low dimensions x, y, z lower case roman letters are used instead of numbering the subindices. The multiplicative Victoria Equation 2 is then

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}\right)^{\frac{1}{q}} = \varphi_0^{\frac{1}{q}} e^{\frac{1}{q}(\varphi_x + 2\pi r_x + \sigma\pi)\check{\mathbf{e}}_x} e^{\frac{1}{q}(\varphi_y + 2\pi r_y + \sigma\pi)\check{\mathbf{e}}_y} \tag{3}$$

where $r_x, r_y \in \mathbb{Z}$, from 0 to $q-1$ and $\sigma = 0, 1$. For $\mathbb{S}^{1+2}$, there is only one π-pair possibility, both components with either 0 or π phase shift. Thus, there are at most $2q^2$ roots in $\mathbb{S}^{1+2}$.

3. Roots in the Additive Representation

Theorem 2 in [5] establishes the equation for the roots of scator numbers with the multiplicity due to the fundamental 2π symmetry of the trigonometric functions. This theorem is extended here in order to encompass the roots arising from the π-pair symmetry.

Theorem 3.1. A scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ raised to the power $\frac{1}{q}$, $q \in \mathbb{Z}$, in the additive representation satisfies the Victoria equation

$$\begin{aligned}
\overset{o}{\varphi}^{\frac{1}{q}} &= \left(\varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j\right)^{\frac{1}{q}} \\
&= \overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n} \\
&= \varphi_0^{\frac{1}{q}} \prod_{k=1}^{n} \cos\left(\tfrac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k \pi\right)\right) \\
&\quad + \sum_{j=1}^{n} \varphi_0^{\frac{1}{q}} \prod_{k\neq j}^{n} \cos\left(\tfrac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k \pi\right)\right) \sin\left(\tfrac{1}{q}\left(\varphi_j + 2\pi r_j + \sigma_j \pi\right)\right) \check{\mathbf{e}}_j
\end{aligned} \tag{4}$$

for $r_j \in \mathbb{Z}$, from 0 to $q-1$, and $\sigma_k, \sigma_j = 0$ or 1, where $\sum_{k=1}^{n} \sigma_k$ is even for j, k from 1 to n. Provided that the q products of $\overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ and its n components are associative for a given set of r_j's and σ-pairs , $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ is a root of $\overset{o}{\varphi} = \overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$.

Proof.
The sum of σ_k is even if the value of 1 is always assigned in pairs. For each pair $\sigma_l = \sigma_m = 1$, $l \neq m$ from 1 to n, the φ_l and φ_m arguments of the trigonometric functions are displaced by π. The scator $\overset{o}{\varphi} = \varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j$ is left unchanged by this transformation since $\cos(\varphi_l + \pi)\cos(\varphi_m + \pi) = \cos\varphi_l \cos\varphi_m$, $\cos(\varphi_m + \pi)\sin(\varphi_l + \pi) = \cos\varphi_m \sin\varphi_l$ and $\cos(\varphi_l + \pi)\sin(\varphi_m + \pi) = \cos\varphi_l \sin\varphi_m$. For odd n director dimension, the remaining unpaired φ_j should not be displaced to leave $\overset{o}{\varphi}$ invariant. This π-pair symmetry is carried through to the RHS of Equation 4. Write the scator $\overset{o}{\varphi} = \overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ in the additive representation with multiplicative variables

$$\varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j = \left(\zeta_0 \prod_{k=1}^{n} \cos\zeta_k + \sum_{j=1}^{n} \zeta_0 \prod_{k\neq j}^{n} \cos\zeta_k \sin\zeta_j \check{\mathbf{e}}_j\right)^{q} \tag{5}$$

From Theorem 4 in [7], that generalizes De Moivre formula to $\mathbb{S}^{1+n}$ scator space, provided that the product of the factors and its components are associative,

$$\varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j = \zeta_0^q \prod_{k=1}^{n} \cos(q\zeta_k) + \sum_{j=1}^{n} \zeta_0^q \prod_{k\neq j}^{n} \cos(q\zeta_k)\sin(q\zeta_j)\check{\mathbf{e}}_j \tag{6}$$

Equating the additive scalar components

$$\varphi_0 \prod_{k=1}^{n} \cos(\varphi_k + 2\pi r_k + \sigma_k\pi) = \zeta_0^q \prod_{k=1}^{n} \cos(q\zeta_k) \tag{7}$$

whereas for each j director component

$$\varphi_0 \prod_{k\neq j}^{n} \cos(\varphi_k + 2\pi r_k + \sigma_k\pi)\sin(\varphi_j + 2\pi r_j + \sigma_j\pi)\check{\mathbf{e}}_j = \zeta_0^q \prod_{k\neq j}^{n} \cos(q\zeta_k)\sin(q\zeta_j)\check{\mathbf{e}}_j$$

where the fundamental 2π symmetry of the trigonometric functions as well as the π-pair symmetry are written explicitly, each $r_j \in \mathbb{Z}$ goes from 0 to $q-1$ and $\sigma_l = \sigma_m = 0, 1$ are set in pairs with equal values, any unpaired σ is set equal to zero. If all $\check{\mathbf{e}}_j$ coefficients are zero except one, say the $\check{\mathbf{e}}_l$ coefficient, $\varphi_0 \sin(\varphi_l + 2\pi r_l + \sigma_l\pi) = \zeta_0^q \sin(q\zeta_l)$ and the relationship between angles is straightforward. In this case, σ_l is unpaired and equal to zero. If two or more $\check{\mathbf{e}}_j$ coefficients are different from zero, $\cos(\varphi_j + 2\pi r_j + \sigma_j\pi) \neq 0$ and $\cos(q\zeta_j) \neq 0$ for all j, since $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$. The products can then be completed for all k and each of the $\check{\mathbf{e}}_j$ equations become

$$\varphi_0 \prod_{k=1}^{n} \cos(\varphi_k + 2\pi r_k + \sigma_k\pi)\tan(\varphi_j + 2\pi r_j + \sigma_j\pi)\check{\mathbf{e}}_j = \zeta_0^q \prod_{k=1}^{n} \cos(q\zeta_k)\tan(q\zeta_j)\check{\mathbf{e}}_j$$

With the use of Equation 7,

$$\tan(q\zeta_j) = \tan(\varphi_j + 2\pi r_j + \sigma_j\pi) \Rightarrow \zeta_j = \frac{1}{q}(\varphi_j + 2\pi r_j + \sigma_j\pi) \tag{8}$$

for all j from 1 to n. Replace the angles $\zeta_k = \frac{1}{q}(\varphi_k + 2\pi r_k + \sigma_j\pi)$ in Equation 7, to find $\zeta_0 = \varphi_0^{\frac{1}{q}}$. Evaluate Equation 5 to the power $\frac{1}{q}$,

$$\begin{aligned} \overset{o}{\varphi}^{\frac{1}{q}} &= \left(\varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j\right)^{\frac{1}{q}} \\ &= \overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n} \\ &= \zeta_0 \prod_{k=1}^{n} \cos\zeta_k + \sum_{j=1}^{n} \zeta_0 \prod_{k\neq j}^{n} \cos\zeta_k \sin\zeta_j \check{\mathbf{e}}_j \end{aligned}$$

Rewrite the ζ_j variables in terms of φ_j from Equation 8 to obtain,

$$\begin{aligned} \overset{o}{\varphi}^{\frac{1}{q}} = \overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n} = \Bigg(&\varphi_0^{\frac{1}{q}} \prod_{k=1}^{n} \cos\left(\frac{1}{q}(\varphi_k + 2\pi r_k + \sigma_k\pi)\right) \\ &+ \sum_{j=1}^{n} \varphi_0^{\frac{1}{q}} \prod_{k\neq j}^{n} \cos\left(\frac{1}{q}(\varphi_k + 2\pi r_k + \sigma_k\pi)\right)\sin\left(\frac{1}{q}(\varphi_j + 2\pi r_j + \sigma_j\pi)\right)\check{\mathbf{e}}_j\Bigg) \end{aligned}$$

The derivation of Equation 6 from Equation 5 required associativity of the $\overset{o}{\zeta}{}^q_{r_1 r_2 \cdots r_n}$ products. However, due to the multi-valued inversion that followed, it is possible that for certain $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n}$ solutions, some of the q products of $\overset{o}{\zeta}{}^q_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n}$ do not satisfy associativity. From [8] Theorem 2.1, associativity is insured if all possible product pairs have a non vanishing additive scalar component. For each $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n}$ to be a root,

$$\begin{aligned}\overset{o}{\zeta}{}^{q}_{r_1r_2\cdots r_n,\sigma_1\sigma_2\cdots\sigma_n} &= \Big[\varphi_0^{\frac{1}{q}}\prod_{k=1}^{n}\cos\left(\tfrac{1}{q}\left(\varphi_k+2\pi r_k+\sigma_k\pi\right)\right) \\ &\quad +\sum_{j=1}^{n}\varphi_0^{\frac{1}{q}}\prod_{k\neq j}^{n}\cos\left(\tfrac{1}{q}\left(\varphi_k+2\pi r_k+\sigma_k\pi\right)\right)\sin\left(\tfrac{1}{q}\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\right)\check{\mathbf{e}}_j\Big]^q \\ &= \prod_{j=1}^{n}\left(\cos\left(\tfrac{1}{q}\left(\varphi_k+2\pi r_k+\sigma_k\pi\right)\right)+\sin\left(\tfrac{1}{q}\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\right)\check{\mathbf{e}}_j\right)^q\end{aligned} \tag{9}$$

must be associative. Thus, none of the $q\times n$ products should give a scator with zero additive scalar component if two or more director coefficients are different from zero, then $\overset{o}{\zeta}_{r_1r_2\cdots r_n,\sigma_1\sigma_2\cdots\sigma_n}$ satisfies $\overset{o}{\zeta}{}^{q}_{r_1r_2\cdots r_n,\sigma_1\sigma_2\cdots\sigma_n}=\overset{o}{\varphi}$. □

The scator roots are identical in the multiplicative representation (Theorem 2.4) or the additive representation (Theorem 3.1), unless obstructed by the lack of associativity. Recall that associativity is not an issue in the multiplicative representation. However, in the additive representation, non-associative products can lead to spurious roots. This problem is discussed at length in [5].

4. Square Roots in 1+2 Dimensions

Lemma 4.1. The square roots of $\overset{o}{\varphi}=\varphi_0\cos\varphi_x\cos\varphi_y+\varphi_0\cos\varphi_y\sin\varphi_x\,\check{\mathbf{e}}_x+\varphi_0\cos\varphi_x\sin\varphi_y\,\check{\mathbf{e}}_y$, are

$$\overset{o}{\varphi}{}^{\frac{1}{2}}=\overset{o}{\zeta}_{\pm,0}=\pm\varphi_0^{\frac{1}{2}}\left(\cos\frac{\varphi_x}{2}\cos\frac{\varphi_y}{2}+\cos\frac{\varphi_y}{2}\sin\frac{\varphi_x}{2}\check{\mathbf{e}}_x+\cos\frac{\varphi_x}{2}\sin\frac{\varphi_y}{2}\check{\mathbf{e}}_y\right) \tag{10}$$

and from the π-pair symmetry

$$\overset{o}{\varphi}{}^{\frac{1}{2}}=\overset{o}{\zeta}_{\pm,1}=\pm\varphi_0^{\frac{1}{2}}\left(\sin\frac{\varphi_x}{2}\sin\frac{\varphi_y}{2}-\sin\frac{\varphi_y}{2}\cos\frac{\varphi_x}{2}\check{\mathbf{e}}_x-\sin\frac{\varphi_x}{2}\cos\frac{\varphi_y}{2}\check{\mathbf{e}}_y\right) \tag{11}$$

PROOF.
For $q=2$ in $\mathbb{S}^{1+2}$, from the Victoria Equation 4 in Theorem 3.1, the roots of $\overset{o}{\varphi}$ are

$$\begin{aligned}\overset{o}{\varphi}{}^{\frac{1}{2}} &= \varphi_0^{\frac{1}{2}}\cos\left(\tfrac{\varphi_x}{2}+\tfrac{\sigma\pi}{2}+\pi r_x\right)\cos\left(\tfrac{\varphi_y}{2}+\tfrac{\sigma\pi}{2}+\pi r_y\right)+\varphi_0^{\frac{1}{2}}\cos\left(\tfrac{\varphi_y}{2}+\tfrac{\sigma\pi}{2}+\pi r_y\right)\sin\left(\tfrac{\varphi_x}{2}+\tfrac{\sigma\pi}{2}+\pi r_x\right)\check{\mathbf{e}}_x \\ &\quad +\varphi_0^{\frac{1}{2}}\cos\left(\tfrac{\varphi_x}{2}+\tfrac{\sigma\pi}{2}+\pi r_x\right)\sin\left(\tfrac{\varphi_y}{2}+\tfrac{\sigma\pi}{2}+\pi r_y\right)\check{\mathbf{e}}_y \\ &= \overset{o}{\zeta}_{r_xr_y,\sigma}\end{aligned}$$

for $r_x,r_y,\sigma=0,1$. In this particular case, the r_x,r_y different values change the sign of all components, $-$ for $r_x\neq r_y$ and $+$ for $r_x=r_y$. This degeneracy halves the number of roots arising from the 2π symmetry from $q^n=4$ to 2,

$$\begin{aligned}\overset{o}{\varphi}{}^{\frac{1}{2}} &= \pm\varphi_0^{\frac{1}{2}}\left[\cos\left(\tfrac{\varphi_x}{2}+\tfrac{\sigma\pi}{2}\right)\cos\left(\tfrac{\varphi_y}{2}+\tfrac{\sigma\pi}{2}\right)+\cos\left(\tfrac{\varphi_y}{2}+\tfrac{\sigma\pi}{2}\right)\sin\left(\tfrac{\varphi_x}{2}+\tfrac{\sigma\pi}{2}\right)\check{\mathbf{e}}_x\right. \\ &\quad \left.+\cos\left(\tfrac{\varphi_x}{2}+\tfrac{\sigma\pi}{2}\right)\sin\left(\tfrac{\varphi_y}{2}+\tfrac{\sigma\pi}{2}\right)\check{\mathbf{e}}_y\right] \\ &= \overset{o}{\zeta}_{\pm,\sigma}\end{aligned} \tag{12}$$

The multiplicity coming from the π-pair symmetry is $2p=2$, since there is only one possible pairing. Hence, the total number of possibly different square roots is 4. If $\sigma=0$ in Equation 12, two of the square roots are given by Equation 10. If $\sigma=1$, the other two square roots, since $\cos\left(\frac{\varphi_j}{2}+\frac{\pi}{2}\right)=-\sin\frac{\varphi_j}{2}$ and $\sin\left(\frac{\varphi_j}{2}+\frac{\pi}{2}\right)=\cos\frac{\varphi_j}{2}$, are given by Equation 11. □

In the subsets $\mathbb{S}^{1+1}$, where either the $\check{\mathbf{e}}_x$ or $\check{\mathbf{e}}_y$ coefficient is different from zero, there are two roots since these subspaces are isomorphic to the complex plane. In [5], it was wrongly stated that "For elements $\overset{o}{\varphi}\in\mathbb{S}^{1+n}\setminus\mathbb{S}^{1+0}$, there are only two different square roots in the additive representation".

The π-pair symmetry now included, has corrected this mistake. There can be, as shown above, up to four different square roots in $\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$. The number of $2pq^n$ roots in $\mathbb{S}^{1+n}$ for $n > 2$ increase considerably due to the π-pair p different pairing possibilities as well as the n dimensions.

4.1. Square roots with additive variables

Lemma 4.2. The square roots of $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ are

$$\begin{aligned}\sqrt{\overset{o}{\varphi}} &= \overset{o}{\zeta}_{\pm,0} \\ &= \pm\frac{1}{2}\sqrt{\frac{1}{|s|}}\left[\sqrt{\left(\sqrt{s^2+x^2}+s\right)\left(\sqrt{s^2+y^2}+s\right)}\right. \\ &\quad +\mathrm{sgn}x\sqrt{\left(\sqrt{s^2+x^2}-s\right)\left(\sqrt{s^2+y^2}+s\right)}\,\check{\mathbf{e}}_x \\ &\quad \left.+\mathrm{sgn}y\sqrt{\left(\sqrt{s^2+x^2}+s\right)\left(\sqrt{s^2+y^2}-s\right)}\,\check{\mathbf{e}}_y\right]\end{aligned} \tag{13}$$

and for the π-pair symmetry

$$\begin{aligned}\sqrt{\overset{o}{\varphi}} &= \overset{o}{\zeta}_{\pm,1} \\ &= \pm\frac{1}{2}\sqrt{\frac{1}{|s|}}\left[\mathrm{sgn}x\,\mathrm{sgn}y\sqrt{\left(\sqrt{s^2+x^2}-s\right)\left(\sqrt{s^2+y^2}-s\right)}\right. \\ &\quad -\mathrm{sgn}y\sqrt{\left(\sqrt{s^2+x^2}+s\right)\left(\sqrt{s^2+y^2}-s\right)}\check{\mathbf{e}}_x \\ &\quad \left.-\mathrm{sgn}x\sqrt{\left(\sqrt{s^2+x^2}-s\right)\left(\sqrt{s^2+y^2}+s\right)}\check{\mathbf{e}}_y\right]\end{aligned} \tag{14}$$

PROOF.
The scator $\overset{o}{\varphi} = \varphi_0 \cos\varphi_x \cos\varphi_y + \varphi_0 \cos\varphi_y \sin\varphi_x\,\check{\mathbf{e}}_x + \varphi_0 \cos\varphi_x \sin\varphi_y\,\check{\mathbf{e}}_y$ in terms of additive variables is $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$. The relationship between multiplicative and additive variables is

$$s = \varphi_0 \cos\varphi_x \cos\varphi_y, \quad x = \varphi_0 \cos\varphi_y \sin\varphi_x, \quad y = \varphi_0 \cos\varphi_x \sin\varphi_y.$$

From the quotient of the directors over the scalar coefficient

$$\frac{x}{s} = \tan\varphi_x, \Rightarrow \cos\varphi_x = \frac{s}{\sqrt{s^2+x^2}} \text{ and } \frac{y}{s} = \tan\varphi_y, \Rightarrow \cos\varphi_y = \frac{s}{\sqrt{s^2+y^2}} \tag{15}$$

In order to write the square roots in terms of the additive variables, rewrite the half angles in terms of angles $\cos\frac{\varphi}{2} = \frac{1}{\sqrt{2}}\sqrt{1+\cos\varphi}$ and $\sin\frac{\varphi}{2} = \frac{1}{\sqrt{2}}\sqrt{1-\cos\varphi}$. These substitutions put together give

$$\cos\left(\frac{\varphi_x}{2}\right)\cos\left(\frac{\varphi_y}{2}\right) = \frac{1}{2}\sqrt{\left(1+\frac{s}{\sqrt{s^2+x^2}}\right)\left(1+\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{16}$$

$$\sin\left(\frac{\varphi_x}{2}\right)\cos\left(\frac{\varphi_y}{2}\right) = \frac{\mathrm{sgn}x}{2}\sqrt{\left(1-\frac{s}{\sqrt{s^2+x^2}}\right)\left(1+\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{17}$$

$$\sin\left(\frac{\varphi_y}{2}\right)\cos\left(\frac{\varphi_x}{2}\right) = \frac{\mathrm{sgn}y}{2}\sqrt{\left(1+\frac{s}{\sqrt{s^2+x^2}}\right)\left(1-\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{18}$$

$$\sin\left(\frac{\varphi_x}{2}\right)\sin\left(\frac{\varphi_y}{2}\right) = \frac{\mathrm{sgn}x\,\mathrm{sgn}y}{2}\sqrt{\left(1-\frac{s}{\sqrt{s^2+x^2}}\right)\left(1-\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{19}$$

where sgn is the sign function. The scator magnitude from Equation 1 is

$$\|\overset{o}{\varphi}\| = \varphi_0 = |s|\sqrt{1+\frac{x^2}{s^2}}\sqrt{1+\frac{y^2}{s^2}} = \frac{1}{|s|}\sqrt{s^2+x^2}\sqrt{s^2+y^2}$$

Evaluate the product of the magnitude's square root

$$\sqrt{\varphi_0} = \sqrt{\frac{1}{|s|}\sqrt{s^2+x^2}\sqrt{s^2+y^2}}$$

times Equations 16-19. Substitution in Equation 10, Lemma 4.1, gives Equation 13. The π-pair multiplicity is obtained from substitution in Equation 11.

If x or y are zero, the usual square root of a complex number is recovered. For example, from Equation 13, if $y=0$, $\sqrt{\overset{o}{\varphi}} = \pm\left(\frac{1}{\sqrt{2}}\sqrt{\sqrt{s^2+x^2}+s}+\frac{\text{sgn}x}{\sqrt{2}}\sqrt{\sqrt{s^2+x^2}-s}\,\check{\mathbf{e}}_x\right)$. The x or y zero limit does not make sense for $\overset{o}{\zeta}_{\pm,1}$, because these novel roots arise from the π-pair symmetry that requires at least two nonvanishing director components. □

Corollary 4.3. The square roots of $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ lie on the plane

$$(\cos\varphi_x - \cos\varphi_y)\,S + \sin\varphi_x\,X - \sin\varphi_y\,Y = 0 \tag{20}$$

that in additive variables is

$$s\left(\sqrt{s^2+x^2}-\sqrt{s^2+y^2}\right)S - x\sqrt{s^2+y^2}\,X + y\sqrt{s^2+x^2}\,Y = 0 \tag{21}$$

Proof.
Since the roots come in $\pm$ pairs, zero must be on the plane where the roots lie. Let this plane be $a_0S + a_xX + a_yY = 0$. Substitute the positive value of the roots Equations 10 and 11, upon division by $\cos\frac{\varphi_x}{2}\cos\frac{\varphi_y}{2}$, the equations are

$$a_0 + \tan\frac{\varphi_x}{2}a_x + \tan\frac{\varphi_y}{2}a_y = 0 \text{ and } a_0 - \cot\frac{\varphi_x}{2}a_x - \cot\frac{\varphi_y}{2}a_y = 0$$

Write the semiangles in terms of angles $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$ and $\cot\frac{\theta}{2} = \frac{\sin\theta}{1-\cos\theta}$. Isolate $\sin\varphi_y\,a_y$ and add the two equations

$$2a_0 + \left[\frac{(1+\cos\varphi_y)}{1+\cos\varphi_x} - \frac{(1-\cos\varphi_y)}{1-\cos\varphi_x}\right]\sin\varphi_x a_x = 0$$

Upon rearrangement

$$a_x = \frac{-\sin\varphi_x}{(\cos\varphi_y - \cos\varphi_x)}a_0, \quad a_y = \frac{-\sin\varphi_y}{(\cos\varphi_x - \cos\varphi_y)}a_0$$

where a_y follows an analogous procedure. From

$$a_0S - \frac{\sin\varphi_x}{(\cos\varphi_y - \cos\varphi_x)}X - \frac{\sin\varphi_y}{(\cos\varphi_x - \cos\varphi_y)}a_0Y = 0$$

Equation 20 is obtained. A similar procedure starting with Equations 13 and 14 gives Equation 21.
□

This result is particular to square roots. Higher order roots no longer lie on a plane as evinced by cube and higher order roots in [5].

In the multiplicative representation, the square roots in $\mathbb{S}^{1+2}$ from Equation 3 with $q=2$, are

$$\overset{o}{\varphi}^{\frac{1}{2}} = \left(\varphi_0 e^{\varphi_x\check{\mathbf{e}}_x}e^{\varphi_y\check{\mathbf{e}}_y}\right)^{\frac{1}{2}} = \varphi_0^{\frac{1}{2}}e^{\left(\frac{\varphi_x}{2}+\sigma\frac{\pi}{2}+\pi r_x\right)\check{\mathbf{e}}_x}e^{\left(\frac{\varphi_y}{2}+\sigma\frac{\pi}{2}+\pi r_y\right)\check{\mathbf{e}}_y} \tag{22}$$

for $\sigma = 0,1$ and $r_x = 0,1$ and $r_y = 0,1$. The $e^{\pi r_x\check{\mathbf{e}}_x}$ and $e^{\pi r_y\check{\mathbf{e}}_y}$ factors introduce a minus sign if

$r_x \neq r_y$. The four possibly distinct square roots of a scator $\overset{o}{\varphi} = \varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}$ are

$$\overset{o}{\varphi}^{\frac{1}{2}} = \pm\varphi_0^{\frac{1}{2}} e^{\frac{\varphi_x}{2}\check{\mathbf{e}}_x} e^{\frac{\varphi_y}{2}\check{\mathbf{e}}_y}, \qquad \pm\varphi_0^{\frac{1}{2}} e^{\left(\frac{\varphi_x}{2}+\frac{\pi}{2}\right)\check{\mathbf{e}}_x} e^{\left(\frac{\varphi_y}{2}+\frac{\pi}{2}\right)\check{\mathbf{e}}_y} \tag{23}$$

5. Geometric visualization

The scalar and the two director components of a scator $\overset{o}{\varphi} \in \mathbb{S}^{1+2}$ can be depicted in orthogonal directions in a three dimensional space as shown in Figure 1.

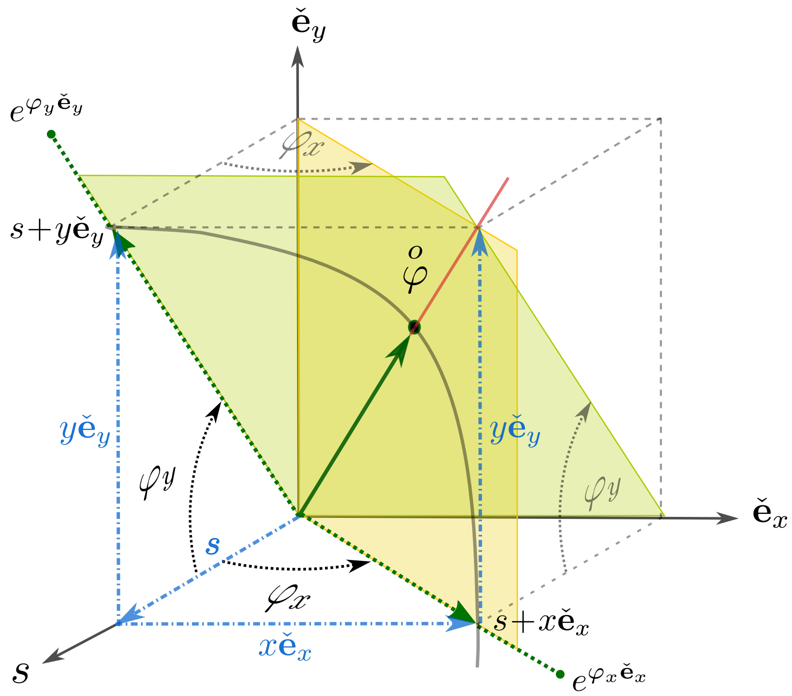

Figure 1. Geometrical representation of the unit magnitude ($\varphi_0 = 1$) scator $\overset{o}{\varphi} = \cos\varphi_x \cos\varphi_y + \cos\varphi_y \sin\varphi_x \, \check{\mathbf{e}}_x + \cos\varphi_x \sin\varphi_y \, \check{\mathbf{e}}_y$. In additive variables $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$.

1. The additive components of a scator $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ can be represented as directed line segments in the s, $\check{\mathbf{e}}_x$, $\check{\mathbf{e}}_y$ axes respectively.

2. In terms of the multiplicative variables:

(a) φ_0 is the distance given by the scator magnitude of the point $\overset{o}{\varphi}$ to the origin,

(b) φ_x is the angle that the projection of the point $\overset{o}{\varphi}$ onto the s, x plane makes with the positive scalar axis and,

(c) φ_y is the angle of the projection onto the s, y plane with the positive scalar axis.

3. A scator can be constructed from the sum of its components, $\overset{o}{\varphi} = (s) + (x\,\check{\mathbf{e}}_x) + (y\,\check{\mathbf{e}}_y)$. This procedure is visualized with dash-dot blue lines in Figure 1. The tip of the $\overset{o}{\varphi}$ scator does not match the sum of the three components because the scator magnitude is not an Euclidean magnitude.

4. A scator can also be constructed by the sum of two scators with a scalar component, $\overset{o}{\varphi}_{sx} = s + x\,\check{\mathbf{e}}_x = \cos\varphi_y e^{\varphi_x \check{\mathbf{e}}_x}$ and $\overset{o}{\varphi}_{sy} = s + y\,\check{\mathbf{e}}_y = \cos\varphi_x e^{\varphi_y \check{\mathbf{e}}_y}$. $\overset{o}{\varphi}_{sx}$ and $\overset{o}{\varphi}_{sy}$ are depicted by green dashed lines with arrows in Figure 1. However, the additive inverse of the scalar component has to be added to achieve the appropriate result $\overset{o}{\varphi}_{sx} + \overset{o}{\varphi}_{sy} - s = 2s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y - s = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$. Having a scalar and a director component, permits the representation in polar coordinates. Notice that the $\overset{o}{\varphi}_{sx}$ and $\overset{o}{\varphi}_{sy}$ scators have a somewhat smaller magnitude than their counterparts in the multiplicative representation. The scators $e^{\varphi_x \check{\mathbf{e}}_x} = \cos\varphi_x + \sin\varphi_x \check{\mathbf{e}}_x$ and $e^{\varphi_y \check{\mathbf{e}}_y} = \cos\varphi_y + \sin\varphi_y \check{\mathbf{e}}_y$ have been depicted in the figure ending with a dot. The scators $\overset{o}{\varphi}_{sx} = s + x\,\check{\mathbf{e}}_x$ and $e^{\varphi_x \check{\mathbf{e}}_x}$ are collinear, thus are $\overset{o}{\varphi}_{sy} = s + y\,\check{\mathbf{e}}_y$ and $e^{\varphi_y \check{\mathbf{e}}_y}$.

5. Scators can also be constructed from the product of their components,

$$\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y = \frac{1}{s}\overset{o}{\varphi}_{sx}\overset{o}{\varphi}_{sy} = \frac{1}{s}\left(s + x\,\check{\mathbf{e}}_x\right)\left(s + y\,\check{\mathbf{e}}_y\right)$$

provided that product with the multiplicative inverse of the additive scalar component is included. In the multiplicative representation, $\overset{o}{\varphi} = \varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}$ is equal to the product of the multiplicative director components $\overset{o}{\varphi} = \overset{o}{\varphi}_x \overset{o}{\varphi}_y = \left(\sqrt{\varphi_0} e^{\varphi_x \check{\mathbf{e}}_x}\right)\left(\sqrt{\varphi_0} e^{\varphi_y \check{\mathbf{e}}_y}\right)$, where the magnitude has been symmetrically split between components. This product can be seen as a rotation of the $\sqrt{\varphi_0} e^{\varphi_x \check{\mathbf{e}}_x}$ scator by φ_y in the $\check{\mathbf{e}}_y$ direction (plane depicted in semitransparent yellow). It can of course be seen the other way around, a rotation in the $\check{\mathbf{e}}_x$ direction (plane depicted in semitransparent green) of the scator $\sqrt{\varphi_0} e^{\varphi_y \check{\mathbf{e}}_y}$.

This last property of scators is in sharp contrast with vector elements, where the product of components cannot be used to construct a several component vector.

Scator products geometrically represent rotations, although the term rotation is a bit of an abuse. The product of an arbitrary scator with a unit scator geometrically represents a rotation that preserves the scator magnitude. However, they are not rotations in the Euclidean sense because the Euclidean metric is not preserved. For this reason, the end point of the scator $\overset{o}{\varphi}$ is not equal to the end point of the three components sum. In contrast, the tip of each of the $\overset{o}{\varphi}_{sx}$ or $\overset{o}{\varphi}_{sy}$ scators coincides with the sum $s + x\check{\mathbf{e}}_x$ or $s + y\check{\mathbf{e}}_x$ respectively, since the scator magnitude in $1+1$ dimensions is equal to the Euclidean magnitude.

5.1. Geometric representation of the square root

The scator square roots involve halving the φ_x and φ_y angles and taking the square root of the magnitude φ_0. Some of the square roots also involve adding a π and/or a $\frac{\pi}{2}$ term to the argument. The scator angles and half angles without any other terms are geometrically depicted in Figure 2.

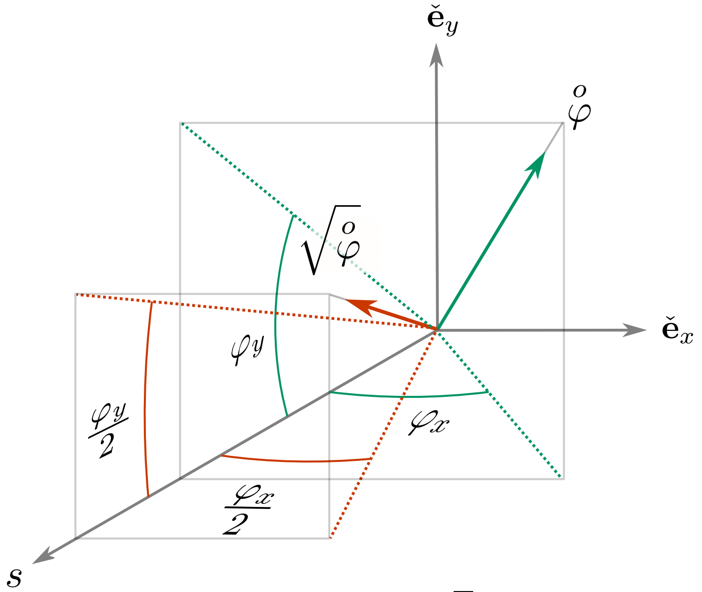

Figure 2. Geometrical representation of a square root $\sqrt{\overset{o}{\varphi}}$ of a scator $\overset{o}{\varphi}$. The φ_x and φ_y angles are halved and the square root of the magnitude is evaluated.

A scator $\overset{o}{\varphi}$ (in green) is projected in the $s, \check{\mathbf{e}}_x$ and $s, \check{\mathbf{e}}_y$ planes (green dotted lines). The angles φ_x and φ_y are the angles that these projections make with respect to the scalar s axis. These angles are halved and represent the projections of the resultant scator $\sqrt{\overset{o}{\varphi}}$ (in red). A unit magnitude scator is assumed, so that the tip of both scators must lie on the unit cusphere surface.

The projection of $\overset{o}{\varphi}$ in the $s, \check{\mathbf{e}}_x$ plane (green dotted line) is the scator $\overset{o}{\varphi}_x = e^{\varphi_x \check{\mathbf{e}}_x}$. It is a unit magnitude hypotenuse with projections $\cos\varphi_x$ and $\sin\varphi_x$ in the s and $\check{\mathbf{e}}_x$ axes, that correspond to the additive representation of this scator $\overset{o}{\varphi}_x = e^{\varphi_x \check{\mathbf{e}}_x} = \cos\varphi_x + \sin\varphi_x\, \check{\mathbf{e}}_x$. The scator magnitude of $\overset{o}{\varphi}_x$, from Equation 1 is $\|\overset{o}{\varphi}\| = |f_0|\sqrt{1+\frac{f_x^2}{f_0^2}} = \sqrt{f_0^2+f_x^2}$, equal to the Pythagorean identity. The scator magnitude $\sqrt{s^2+x^2}$ in $\mathbb{S}^{1+1}$ is identical to the Euclidean magnitude. Thus, a right angle triangle where the tip of the hypotenuse matches the sum of the directed catheti is obtained.

An analogous result is obtained for the projection of $\overset{o}{\varphi}$ in the $s, \check{\mathbf{e}}_y$ plane (green dotted line), $\overset{o}{\varphi}_y = e^{\varphi_y \check{\mathbf{e}}_y} = \cos\varphi_y + \sin\varphi_y\, \check{\mathbf{e}}_y$. Again, a unit magnitude hypotenuse is made up from a right angle triangle, but this time in the $s, \check{\mathbf{e}}_y$ plane.

The product of these two projections $\overset{o}{\varphi} = \overset{o}{\varphi}_x \overset{o}{\varphi}_y = \left(e^{\varphi_x \check{\mathbf{e}}_x}\right)\left(e^{\varphi_y \check{\mathbf{e}}_y}\right)$, construct the $\overset{o}{\varphi}$ scator. Its additive representation is $\overset{o}{\varphi} = \cos\varphi_x \cos\varphi_y + \cos\varphi_y \sin\varphi_x\, \check{\mathbf{e}}_x + \cos\varphi_x \sin\varphi_y\, \check{\mathbf{e}}_y$. Its magnitude, from Equation 1 is

$$\|\overset{o}{\varphi}\| = |s|\sqrt{1+\frac{x^2}{s^2}}\sqrt{1+\frac{y^2}{s^2}} = \sqrt{s^2+x^2+y^2+\frac{x^2y^2}{s^2}}$$

It is no longer the sum of three squares but has the $\frac{x^2y^2}{s^2}$ term. The magnitude of this scator is one,

$$\|\overset{o}{\varphi}\| = \sqrt{\cos^2\varphi_x \cos^2\varphi_y + \cos^2\varphi_y \sin^2\varphi_x + \cos^2\varphi_x \sin^2\varphi_y + \sin^2\varphi_x \sin^2\varphi_y} = 1$$

where the last term $\frac{x^2y^2}{s^2} = \frac{\cos^2\varphi_y \sin^2\varphi_x \cos^2\varphi_x \sin^2\varphi_y}{\cos^2\varphi_x \cos^2\varphi_y}$ is crucial to attain this result. The tip of the scator $\overset{o}{\varphi} = s + x\, \check{\mathbf{e}}_x + y\, \check{\mathbf{e}}_y$, cannot match the tip of the directed sum of the three components (That would imply a magnitude $\sqrt{s^2+x^2+y^2}$).

The $\overset{o}{\varphi}$ scator root (in red) is now the product of the two projection scators $e^{\frac{\varphi_x}{2}\check{\mathbf{e}}_x}$ and $e^{\frac{\varphi_y}{2}\check{\mathbf{e}}_y}$. It also has unit magnitude and is leaned closer to the s axis in both $\check{\mathbf{e}}_x$ and $\check{\mathbf{e}}_y$ as expected for smaller angles.

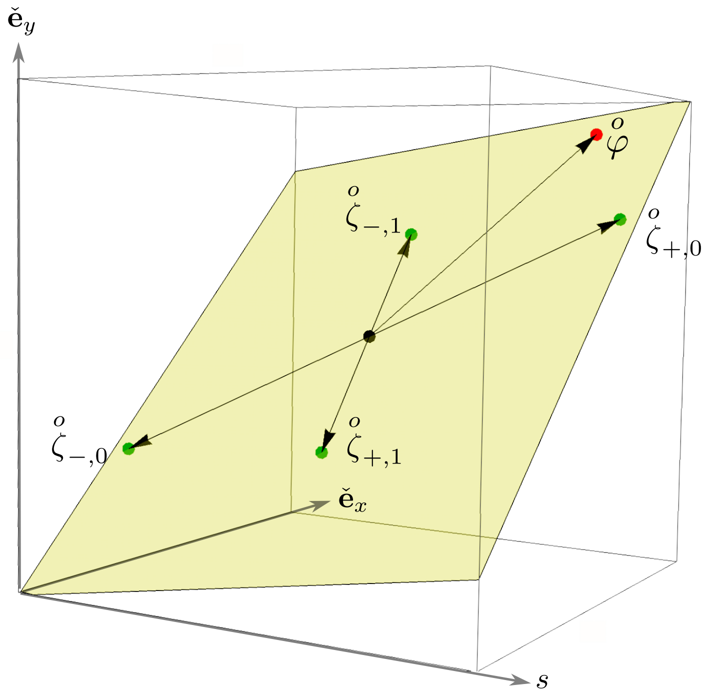

Figure 3. Roots (green points) of $\overset{o}{\varphi} = \cos\frac{\pi}{6}\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\sin\frac{\pi}{6}\, \check{\mathbf{e}}_x + \cos\frac{\pi}{6}\sin\frac{\pi}{5}\, \check{\mathbf{e}}_y$ (red point). The origin is located at the black point. The $\left(2\sqrt{3}-1-\sqrt{5}\right)s + 2\,x - 2\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}y = 0$ plane is shown in semitransparent yellow. The four roots lie on this plane but not the $\overset{o}{\varphi}$ scator (red).

Consider as a numeric example, the scator

$$\overset{o}{\varphi} = \frac{\sqrt{3}}{8}\left(\sqrt{5}+1\right) + \frac{1}{8}\left(\sqrt{5}+1\right)\check{\mathbf{e}}_x + \frac{1}{4}\sqrt{\frac{3}{2}\left(5-\sqrt{5}\right)}\,\check{\mathbf{e}}_y$$

where the values have been chosen so that the two different angles are rational (relative primes) functions of π. From Equation 13, two roots are given by

$$\overset{o}{\zeta}_{\pm,0} = \pm\left(\frac{\left(\sqrt{3}+1\right)\sqrt{5+\sqrt{5}}}{8} + \frac{\left(\sqrt{3}-1\right)\sqrt{5+\sqrt{5}}}{8}\check{\mathbf{e}}_x + \frac{\left(\sqrt{3}+1\right)\left(\sqrt{5}-1\right)}{8\sqrt{2}}\check{\mathbf{e}}_y\right)$$

and the other two π-pair symmetry roots from Equation 14 are

$$\overset{o}{\zeta}_{\pm,\pi} = \pm\left(\frac{\left(\sqrt{3}-1\right)\left(\sqrt{5}-1\right)}{8\sqrt{2}} - \frac{\left(\sqrt{3}+1\right)\left(\sqrt{5}-1\right)}{8\sqrt{2}}\check{\mathbf{e}}_x - \frac{\left(\sqrt{3}-1\right)}{8}\sqrt{5+\sqrt{5}}\,\check{\mathbf{e}}_y\right)$$

This scator in multiplicative variables is

$$\overset{o}{\varphi} = \cos\frac{\pi}{6}\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\sin\frac{\pi}{6}\check{\mathbf{e}}_x + \cos\frac{\pi}{6}\sin\frac{\pi}{5}\check{\mathbf{e}}_y$$

Its roots from Equations 10 and 11 are

$$\overset{o}{\varphi}{}^{\frac{1}{2}} = \overset{o}{\zeta}_{\pm,0} = \pm\left(\cos\frac{\pi}{12}\cos\frac{\pi}{10} + \cos\frac{\pi}{10}\sin\frac{\pi}{12}\check{\mathbf{e}}_x + \cos\frac{\pi}{12}\sin\frac{\pi}{10}\check{\mathbf{e}}_y\right)$$

and

$$\overset{o}{\varphi}{}^{\frac{1}{2}} = \overset{o}{\zeta}_{\pm,1} = \pm\left(\sin\frac{\pi}{12}\sin\frac{\pi}{10} - \sin\frac{\pi}{10}\cos\frac{\pi}{12}\check{\mathbf{e}}_x - \sin\frac{\pi}{12}\cos\frac{\pi}{10}\check{\mathbf{e}}_y\right)$$

These roots are depicted in Figure 3. The equation of the plane where the four roots lie, from Equation 21 is

$$\left(2\sqrt{3}-1-\sqrt{5}\right)s + 2\,x - 2\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}\,y = 0$$

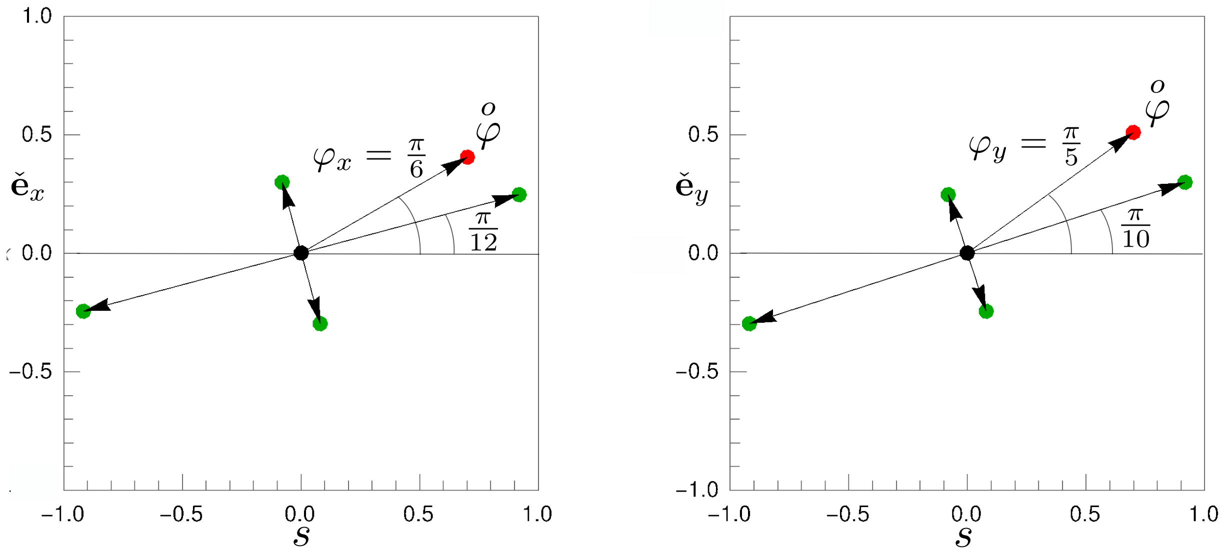

Figure 4. Projections of the roots (green points) of $\overset{o}{\varphi}$ (red point) in the $s, \check{\mathbf{e}}_x$ (left) and $s, \check{\mathbf{e}}_y$ (right) planes.

The halving of the angles is not at all evident in the three dimensional plot. However, in Figure 4, where the projections in the $s, \check{\mathbf{e}}_x$ and $s, \check{\mathbf{e}}_y$ planes are shown, the angle division is clearly depicted. Furthermore, the other three roots, $\overset{o}{\zeta}_{-,0}$, $\overset{o}{\zeta}_{+,1}$ and $\overset{o}{\zeta}_{-,1}$ are seen as π, $\frac{\pi}{2}$ and $\pi+\frac{\pi}{2}$ rotations respectively of the first root.

6. Conclusion

Scator roots exist in $\mathbb{S}^{1+n}$, and their values in the multiplicative and additive representations are given in closed forms by the Victoria equations in Theorems 2.4 and 3.1. These extended versions of previous theorems exhaust all possible values for the roots of a scator. The multiplicity of the q^{th} root of a scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ is at most $2pq^n$, where p is the number of different π-pair possibilities (Corollary 2.5). The q^{th} root of a scator involves the division of the scator angles by q. The φ_j angles are the multiplicative director components of the scator. They can be represented geometrically as the angle of the projections in the $s, \check{\mathbf{e}}_j$ planes. The square root of a scator in $\mathbb{S}^{1+2}$ has at most four different values that are contained in a plane (Corollary 4.3). Their values are given by Lemmas 4.1 and 4.2. These roots can be nicely depicted in three-dimensional space with the s, $x\,\check{\mathbf{e}}_x$ and $y\,\check{\mathbf{e}}_y$ components drawn in orthogonal axes. The geometric construction of a scator by adding its components is not surprising since vectors and other algebraic structures exhibit this feature. However, the construction of a $1+n$ dimensional scator via the product of its $1+1$ components is quite novel and uncommon in most algebraic structures.

In future studies, the square roots obtained here may be successfully used to find the inverse orbits in the quadratic iteration dynamic scator space. Thus, an algorithm for visualizing the scator fractal Julia sets in 1+2D may be provided. Moreover, this square roots inverse visualization procedure may be implemented. We believe the present results also pave the way for studies on considering higher-order roots and evaluating square roots in higher dimensional scator spaces.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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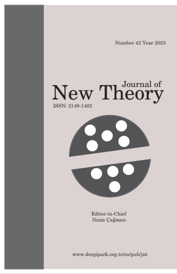

Characterizations of Unit Darboux Ruled Surface with Quaternions

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Abstract — This paper presents a quaternionic approach to generating and characterizing the ruled surface drawn by the unit Darboux vector. The study derives the Darboux frame of the surface and relates it to the Frenet frame of the base curve. Moreover, it obtains the quaternionic shape operator and its matrix representation using the normal and geodesic curvatures to provide a more detailed analysis. To illustrate the concepts discussed, the paper offers a clear example that will help readers better understand the concepts and showcases the quaternionic shape operator, Gauss curvature, mean curvature, and rotation matrix. Finally, it emphasizes the need for further research on this topic.

Keywords *Gauss curvature, mean curvature, ruled surfaces, quaternions, quaternionic shape operator*

Mathematics Subject Classification (2020) 11R52, 53A05

1. Introduction (Compulsory)

Quaternions include diverse fields such as game programming, robotics, animation, and navigation systems [1–3]. In addition to these areas, quaternions are important for the theory of curves and surfaces. Bharathi and Nagaraj have introduced the Serret-Frenet formulae for quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ [4]. By utilizing this study, numerous studies have examined quaternionic curves. One of them, the authors have proved that if the bitorsion of a quaternionic curve does not vanish, then there is no quaternionic curve in E^4. Therefore, they have expressed (1, 3) type Bertrand curves for quaternionic curves [5]. Babaarslan and Yaylı have examined constant slope surfaces with quaternions [6]. In [7], the authors have expressed the ruled surface as quaternionic and computed some properties of the ruled surface. Moreover, they have investigated the dual ruled surface using dual quaternion [8]. In light of these studies, Çalışkan have examined the quaternionic and dual quaternionic Darboux ruled surfaces [9]. In [10], the authors have examined the advantage of the dual number of Clifford algebra to make the singular ruled surfaces transform into dual singular curves. Aslan and Yaylı have defined the quaternionic shape operator by the quaternion. Their article has aimed to find a way to the invariants of the surface using Darboux frames and quaternions [11]. In [12,13], the connection between split quaternions and surfaces with the constant slope in Minkowski 3-space has been explored. It is demonstrated that these surfaces can be transformed using rotation matrices associated with quaternions and homothetic motions. A surface is said to be ruled if it is generated by moving a straight line continuously in $\mathbb{R}^3$. Thus, a ruled surface has a parameterization

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in the form $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{x}(s)$ where we call α the base curve and $\vec{x}$ the generator vector of the ruled surface [14]. Ruled surfaces are important for robot kinematics. Ryuh has suggested that ruled surfaces play an important role in robot end-effectors [15]. In another study, the ruled surfaces' differential properties drawn by the developed trihedron's generator vector have been examined [16]. In [17], the authors have introduced a new type of ruled surface defined using an orthonormal Sannia frame on a base curve. They have studied the properties of these surfaces using the first and second fundamental forms, as well as the mean and Gaussian curvatures. They have provided conditions for when these surfaces are developable and minimal and present some examples of these ruled surfaces. Eren et al. have introduced new types of ruled surfaces in Euclidean 3-space. These surfaces have been obtained using the evolution of an involute-evolute curve pair and studied with the modified orthogonal frame. They have provided some results on these surfaces' Gaussian and mean curvatures [18, 19]. Bilici has examined ruled surfaces produced by a Frenet trihedron of closed dual involute for a specific dual curve. He has specifically focused on the relations between the pitch, the angle of the pitch, and the drall of these surfaces [20]. Some ruled surfaces produced using the Frenet trihedron, Blaschke frame, and the surface family are studied in [21–24].

In Section 2, we provide some necessary background information about the problem of the paper that was mentioned in the introduction. In Section 3, we give characterizations of ruled surfaces drawn by the unit Darboux vector using quaternions. We obtain the quaternionic shape operators and their matrix representations using normal and geodesic curvatures. In the last section, we exemplify the findings.

2. Preliminaries

Let $\alpha : I \to \mathbb{R}^3$ be a unit-speed curve. Then, the three vector fields $\vec{t}(s), \vec{n}(s)$, and $\vec{b}(s)$ on the curve α are unit vector fields that are mutually orthogonal at each point. We call $\vec{t}(s), \vec{n}(s)$, and $\vec{b}(s)$ the Frenet vectors on the curve α. The Frenet formulas can be given

$$\vec{t}'(s) = \kappa(s)\vec{n}(s), \quad \vec{n}'(s) = -\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s), \quad \text{and} \quad \vec{b}'(s) = -\tau(s)\vec{n}(s)$$

where $\kappa(s)$ and $\tau(s)$ are the first and second curvature of the unit-speed curve, respectively [25]. For any unit-speed curve $\alpha : I \to \mathbb{R}^3$, the vector $\vec{W}(s)$ is called Darboux vector defined by

$$\vec{W}(s) = \tau(s)\vec{t}(s) + \kappa(s)\vec{b}(s)$$

If consider the normalization of the Darboux $\vec{C}(s) = \dfrac{1}{\|\vec{W}(s)\|}\vec{W}(s)$, we have

$$\vec{C}(s) = \sin\xi(s)\vec{t}(s) + \cos\xi(s)\vec{b}(s)$$

where $\cos\xi(s) = \frac{\kappa(s)}{\|\vec{W}(s)\|}$, $\sin\xi(s) = \frac{\tau(s)}{\|\vec{W}(s)\|}$, and $\angle(\vec{W}(s), \vec{b}(s)) = \xi(s)$. A quaternion is a unit length of four-vectors $q = d + a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$ characterized by the following properties:

$$\begin{cases} \vec{e_1}^2 = \vec{e_2}^2 = \vec{e_3}^2 = \vec{e}_1 \times \vec{e}_2 \times \vec{e}_3 = -1, \\ \vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \vec{e}_3 \times \vec{e}_1 = \vec{e}_2 \end{cases} \quad \vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^3$$

The quaternion product of two quaternions q_1 and q_2, which we write as $q_1 \times q_2$, takes the form

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)\vec{e}_1 \\ &\quad +(d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)\vec{e}_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)\vec{e}_3 \end{aligned}$$

The complex conjugate of a quaternion q is denoted as $\bar{q} = d - a\vec{e}_1 - b\vec{e}_2 - c\vec{e}_3$. The norm of q is

$$\mathbf{N}(q) = \sqrt{d^2 + a^2 + b^2 + c^2}$$

Pure quaternion is denoted as $\bar{q}+q=0=a\vec{e}_1+b\vec{e}_2+c\vec{e}_3$. The quaternion multiplication of two pure quaternions is $q_1\times q_2=-\langle q_1,q_2\rangle+q_1\wedge q_2$. The unit quaternion can be written in the form as $q=\cos\varphi+\sin\varphi\vec{v}$ where $\vec{v}\in\mathbb{R}^3$ and $\|\vec{v}\|=1$. Let q be a unit quaternion and $\vec{w}$ be a pure quaternion. Then,

$$\vec{w}'=q\times\vec{w}\times q^{-1}$$

is rotated 2φ about the axis $\vec{v}$. We say finally that the desired rotation matrix fixing the direction v is

$$R=\begin{bmatrix} 1+\sin^2\varphi(u_1^2-u_2^2-u_3^2-1) & -\sin 2\varphi u_3+2\sin^2\varphi u_1u_2 & \sin 2\varphi u_2+2\sin^2\varphi u_1u_3 \\ \sin 2\varphi u_3+2\sin^2\varphi u_1u_2 & 1+\sin^2\varphi(u_2^2-u_1^2-u_3^2-1) & 2\sin^2\varphi u_2u_3-\sin 2\varphi u_1 \\ 2\sin^2\varphi u_1u_3-\sin 2\varphi u_2 & \sin 2\varphi u_1+2\sin^2\varphi u_2u_3 & 1+\sin^2\varphi(u_3^2-u_2^2-u_1^2-1) \end{bmatrix}$$

where R is an orthogonal matrix. For detailed information on the theory of quaternion, see the references [1,3,26].

If p is a point of M, for each tangent vector $\vec{X}$ to M at p, $S_p(\vec{X})=-\nabla_{\vec{X}}\vec{Z}$. S_p is defined as the shape operator of M at p. The shape operator is the symmetric linear map. Here, Z is the unit normal vector field. A surface M in $\mathbb{R}^3$ is flat provided its Gauss curvature is zero, and minimal provided its mean curvature is zero. Moreover, the Gauss and minimal curvatures are independent of the choice of basis. These curvatures are found in the equations

$$K=\frac{\|S(\vec{T}(u))\wedge S(\vec{T}(t))\|}{\|\vec{T}(u)\wedge\vec{T}(t)\|},\ H=\frac{\|S(\vec{T}(u))\wedge\vec{T}(t)+\vec{T}(u)\wedge S(\vec{T}(t))\|}{2\|\vec{T}(u)\wedge\vec{T}(t)\|}$$

where $\vec{T}(u)$ and $\vec{T}(t)$ are the tangent vectors of $\beta(u)$ and $\zeta(t)$, respectively [25]. Let β be a curve that is traced on a surface and Darboux frame $\{\vec{T}(u),\vec{Y}(u),\vec{Z}(u)\}$ is an orthogonal frame. The equations of motion of the Darboux frame can be written as

$$\begin{bmatrix}\vec{T}'(u)\\ \vec{Y}'(u)\\ \vec{Z}'(u)\end{bmatrix}=\|\Lambda_u\|\begin{bmatrix}0 & k_g(u) & k_n(u)\\ -k_g(u) & 0 & t_g(u)\\ -k_n(u) & -t_g(u) & 0\end{bmatrix}\begin{bmatrix}\vec{T}(u)\\ \vec{Y}(u)\\ \vec{Z}(u)\end{bmatrix} \tag{1}$$

Here, k_n, k_g, and t_g are the normal curvature, the geodesic curvature, and the geodesic torsion, respectively [14,25].

Theorem 2.1. [11] Let M be a surface with parameter u and $\beta(u)$ be a unit speed curve in M. Using the quaternion operator $Q(u)=k_n(u)+t_g(u)\vec{Z}(u)$, the shape operator can be given as

$$S(\vec{T}(u))=Q(u)\times\vec{T}(u) \tag{2}$$

The quaternion Q will be called a quaternionic shape operator.

Quaternionic shape operator can be given by the unit quaternion $p(u)=\cos 2\varphi(u)+\sin 2\varphi(u)\vec{Z}(u)$ as

$$Q(u)=\sqrt{{k_n}^2(u)+{t_g}^2(u)}\big(\cos 2\varphi(u)+\sin 2\varphi(u)\vec{Z}(u)\big)$$

Then, we can say that the vector $Q(u)\times\vec{T}(u)$ is obtained by revolving $\vec{T}(u)$ around the normal vector $\vec{Z}(u)$ of the surface through twice the angle of φ [11].

Theorem 2.2. [11] Let M be a surface, X be a local parameterization of M, N be the unit normal vector field of M, and $Q(t)$ and $Q(u)$ be the quaternionic shape operators. Then, the Gauss curvature K and mean curvature H of M are as follows:

$$K = \frac{\|(Q(u)\times\vec{T}(u))\wedge(Q(t)\times\vec{T}(t))\|}{\|\vec{T}(u)\wedge\vec{T}(t)\|} \tag{3}$$

and

$$H = \frac{\|(Q(u)\times\vec{T}(u))\wedge\vec{T}(t)+\vec{T}(u)\wedge(Q(t)\times\vec{T}(t))\|}{2\|\vec{T}(u)\wedge\vec{T}(t)\|} \tag{4}$$

3. Main Results

In this section, the quaternionic expression of the ruled surfaces drawn by the unit Darboux vector and the striction curve on the surface are given. We obtain some interesting results, such as rotation matrices, Gauss, and mean curvatures of the surface.

Theorem 3.1. Let $\bar{\alpha}$ be a striction curve belonging to a ruled surface $\vec{\Lambda}(s,v)=\vec{\alpha}(s)+v\vec{C}(s)$. The quaternionic equations of the ruled surface and striction curve are given by

$$\vec{\Lambda}(s,v)=\vec{\alpha}(s)+vp(s)\times\vec{t}(s)$$

and

$$\bar{\alpha}=\vec{\alpha}(s)-v\frac{\langle C'(s),\vec{t}(s)\rangle}{\|C'(s)\|^2}p(s)\times\vec{t}(s)$$

Proof.
By taking into account the unit quaternion $p(s)=\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}}-\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}}\vec{n}(s)$ and the pure quaternion $\vec{t}(s)$, we obtain the ruled surface as follows:

$$\begin{aligned}\vec{\Lambda}(s,v) &= \vec{\alpha}(s)+vp(s)\times\vec{t}(s)\\ &= \vec{\alpha}(s)+v\left(\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}}-\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}}\vec{n}(s)\right)\times\vec{t}(s)\\ &= \vec{\alpha}(s)+v\vec{C}(s).\end{aligned}$$

Similarly, we can obtain a striction curve using quaternion. □

Theorem 3.2. Let $\vec{\Lambda}(s,v)=\vec{\alpha}(s)+v\vec{C}(s)$ be a ruled surface. There exists a frame of the curve $\alpha(s)$ which is called Frenet frame and denoted by $\{\vec{t}(s),\vec{n}(s),\vec{b}(s)\}$. The relations among frames can be given by

$$\begin{bmatrix}\vec{T}(s)\\ \vec{Y}(s)\\ \vec{Z}(s)\end{bmatrix}=\begin{bmatrix}\frac{m}{\sqrt{m^2+l^2}} & 0 & \frac{l}{\sqrt{m^2+l^2}}\\ \frac{-l}{\sqrt{m^2+l^2}} & 0 & \frac{m}{\sqrt{m^2+l^2}}\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}\vec{t}(s)\\ \vec{n}(s)\\ \vec{b}(s)\end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix}\vec{T}(v)\\ \vec{Y}(v)\\ \vec{Z}(v)\end{bmatrix}=\begin{bmatrix}\frac{\tau(s)}{\|\vec{W}(s)\|} & 0 & \frac{\kappa(s)}{\|\vec{W}(s)\|}\\ -\frac{\kappa(s)}{\|\vec{W}(s)\|} & 0 & \frac{\tau(s)}{\|\vec{W}(s)\|}\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}\vec{t}(s)\\ \vec{n}(s)\\ \vec{b}(s)\end{bmatrix} \tag{6}$$

where $m=1+v\Big(\frac{\tau(s)}{\|\vec{W}(s)\|}\Big)'$ and $l=v\Big(\frac{\kappa(s)}{\|\vec{W}(s)\|}\Big)'$.

Proof.
The partial derivative is taken according to s and v for the ruled surface Λ, we obtain

$$\vec{\Lambda}_s=\vec{t}(s)+v\left(\Big(\frac{\tau(s)}{\|\vec{W}(s)\|}\Big)'\vec{t}(s)+\Big(\frac{\kappa(s)}{\|\vec{W}(s)\|}\Big)'\vec{b}(s)\right)$$

and

$$\vec{\Lambda}_v=\vec{C}(s)$$

Arriving at this equation, we reach the tangent vectors of the parameters curve as follows:

$$\vec{T}(s) = \frac{\vec{\Lambda}_s}{\|\vec{\Lambda}_s\|} = \frac{\left(1 + v(\frac{\tau(s)}{\|\vec{W}(s)\|})'\right)\vec{t}(s) + v(\frac{\kappa(s)}{\|\vec{W}(s)\|})'\vec{b}(s)}{\sqrt{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{T}(v) = \frac{\vec{\Lambda}_v}{\|\vec{\Lambda}_v\|} = \vec{C}(s)$$

For $v < \frac{\kappa(s)\sqrt{\kappa^2(s)+\tau(s)^2(s)}}{\tau(s)\kappa'(s)-\kappa(s)\tau'(s)}$, the unit normal vector of the ruled surface Λ is given as

$$\vec{Z} = \frac{\vec{\Lambda}_s \wedge \vec{\Lambda}_v}{\|\vec{\Lambda}_s \wedge \vec{\Lambda}_v\|} = -\vec{n}(s)$$

$\vec{Y}(s)$ and $\vec{Y}(v)$ depending on Frenet frame at point $\alpha(s)$ can be obtained as

$$\vec{Y}(s) = \vec{Z}(s) \wedge \vec{T}(s) = \frac{-v(\frac{\kappa(s)}{\|\vec{W}(s)\|})'\vec{t}(s) + \left(1 + v(\frac{\tau(s)}{\|\vec{W}(s)\|})'\right)\vec{b}(s)}{\sqrt{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{Y}(v) = \vec{Z}(v) \wedge \vec{T}(v) = -\frac{\kappa(s)}{\|\vec{W}(s)\|}\vec{t}(s) + \frac{\tau(s)}{\|\vec{W}(s)\|}\vec{b}(s)$$

If the Darboux frame denoted by $\{\vec{T}(s), \vec{Y}(s), \vec{Z}(s)\}$ is written in matrix form, this completes the proof of the theorem. $\square$

Theorem 3.3. Let $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{C}(s)$ be a ruled surface. The singular point of the ruled surface is given by $P(s_0, v_0) = \left(s_0, -\frac{\|W(s_0)\|\kappa(s_0)}{\kappa(s_0)\tau'(s_0)+\tau(s_0)\kappa'(s_0)}\right)$.

Proof.

The unit normal vector field of the ruled surface Λ is defined by $Z = \frac{\Lambda_s \times \Lambda_v}{\|\Lambda_s \times \Lambda_v\|}$ at those points $(s_0, v_0) \in Z$ at which $\Lambda_s \times \Lambda_v$ does not vanish. Then, Λ is a regular surface if and only if the unit normal vector field Z is everywhere well defined. The points for which $\Lambda_s \times \Lambda_v$ vanishes can be called singular points. The equation

$$\|\Lambda_s \times \Lambda_v\|(s_0, v_0) = \frac{1}{\|W(s_0)\|^2}\sqrt{(\|W(s_0)\|\kappa(s_0) + v_0(\kappa(s_0)\tau'(s_0) - \tau(s_0)\kappa'(s_0))^2} = 0$$

can be written to have singular points. Hence, we can write

$$v_0 = -\frac{\|W(s_0)\|\kappa(s_0)}{\kappa(s_0)\tau'(s_0) + \tau(s_0)\kappa'(s_0)}$$

Then, the singular point of the ruled surface is $P(s_0, v_0) = \left(s_0, -\frac{\|W(s_0)\|\kappa(s_0)}{\kappa(s_0)\tau'(s_0)+\tau(s_0)\kappa'(s_0)}\right)$. $\square$

Taking into account Equation 1, we can write proposition as follows:

Proposition 3.4. The normal and the geodesic curvatures of the ruled surface Λ can be given by

$$k_n(s) = \frac{-\kappa(s)\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right) + \tau(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'}{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}, \quad k_n(v) = 0$$

and

$$t_g(s)=\frac{\tau(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\kappa(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'}{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2},\quad t_g(v)=0$$

Theorem 3.5. Let Λ be a ruled surface and $Q(s)$ and $Q(v)$ be quaternionic shape operators. The shape operators $S(\vec{T}(s))$ and $S(\vec{T}(v))$ are obtained by

$$S(\vec{T}(s))=\frac{-\kappa(s)\vec{t}(s)+\tau(s)\vec{b}(s)}{\sqrt{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$S(\vec{T}(v))=\vec{0}$$

Proof.
By using Proposition 3.4, quaternionic shape operators are given by

$$\begin{aligned}Q(s)&=k_n(s)+t_g(s)\vec{Z}(s)\\&=\frac{-\kappa(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\tau(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'-\left[\tau(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\kappa(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right]\vec{n}}{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}\end{aligned}$$

and

$$Q(v)=k_n(v)+t_g(v)\vec{Z}(v)=\vec{0}$$

By considering Equation 2, the shape operators

$$\begin{aligned}S(\vec{T}(s))&=Q(s)\times\vec{T}(s)=k_n(s)\vec{T}(s)+t_g(s)\vec{Y}(s)\\&=\frac{-\kappa(s)\vec{t}(s)+\tau(s)\vec{b}(s)}{\sqrt{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}\end{aligned}$$

and

$$S(\vec{T}(v))=Q(v)\times\vec{T}(v)=k_n(v)\vec{T}(v)+t_g(v)\vec{Y}(v)=\vec{0}$$

are expressed. □

Corollary 3.6. The operator $Q(s)$ rotates the tangent vector $\vec{T}(s)$ in the tangent plane of the ruled surface and around the normal vector $\vec{Z}(s)$ of the surface. The rotation matrix which provides that rotation is

$$R=\begin{bmatrix}1+\sin^2\varphi(n_1^2-n_2^2-n_3^2-1) & \sin 2\varphi n_3+2\sin^2\varphi n_1n_2 & -\sin 2\varphi n_2+2\sin^2\varphi n_1n_3\\ -\sin 2\varphi n_3+2\sin^2\varphi n_1n_2 & 1+\sin^2\varphi(n_2^2-n_1^2-n_3^2-1) & 2\sin^2\varphi n_2n_3+\sin 2\varphi n_1\\ 2\sin^2\varphi n_1n_3+\sin 2\varphi n_2 & -\sin 2\varphi n_1+2\sin^2\varphi n_2n_3 & 1+\sin^2\varphi(n_3^2-n_2^2-n_1^2-1)\end{bmatrix}$$

where $Z(s)=-n(s)=(-n_1,-n_2,-n_3)$ and the cosine and sine of the angle of between $\vec{T}(s)$ and $Q(s)\times\vec{T}(s)$ are

$$\cos 2\varphi(s)=\frac{-m\kappa(s)+l\tau(s)}{\sqrt{(m^2+l^2)(\kappa^2(s)+\tau(s)^2(s))}}\quad\text{and}\quad\sin 2\varphi(s)=\frac{l\kappa(s)+m\tau(s)}{\sqrt{(m^2+l^2)(\kappa^2(s)+\tau(s)^2(s))}}$$

Theorem 3.7. The ruled surface Λ is flat. The mean curvature H of this surface is obtained by

$$H = \frac{\kappa^2(s) + \tau(s)^2(s)}{2|-m\kappa(s) + l\tau(s)|}$$

Proof.
The Gauss curvature is a measure of the intrinsic curvature of a surface, and it is defined as quaternionic as follows:

$$K = \frac{\|(Q(s) \times \vec{t}(s)) \wedge (Q(v) \times \vec{T}(v))\|}{\|\vec{t}(s) \wedge \vec{T}(v)\|}$$

If we substitute the quaternionic shape operators and tangent vector of parameter curves, we obtain

$$\begin{aligned} K &= \frac{1}{\sqrt{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}} \frac{\|(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)) \wedge \vec{0})\|}{\|\vec{t}(s) \wedge \vec{T}(v)\|} \\ &= 0 \end{aligned}$$

This means the surface is flat. The mean curvature is a measure of the extrinsic curvature of the ruled surface, and the curvature is calculated as quaternionic as follows:

$$H = \frac{\|(Q(s) \times \vec{t}(s)) \wedge \vec{T}(v) + \vec{t}(s) \wedge (Q(v) \times \vec{T}(v))\|}{2\|\vec{t}(s) \wedge \vec{T}(v)\|}$$

If we substitute the quaternionic shape operators $Q(v)$, we get

$$H = \frac{\|(Q(s) \times \vec{t}(s)) \wedge \vec{T}(v)\|}{2\|\vec{t}(s) \wedge \vec{T}(v)\|}$$

By taking into consideration Equations 5 and 6 and Theorem 3.5,

$$H = \frac{\sqrt{\kappa^2(s) + \tau(s)^2(s)}}{2|-m\kappa(s) + l\tau(s)|}$$

is arrived. □

Considering the above theorem, we reach the following corollary.

Corollary 3.8. If the base curve of the ruled surface Λ drawn by the unit Darboux vector is a line and planar, then the surface is minimal.

In differential geometry, the Darboux vector is a vector-valued function that measures the rate of change of the tangent vector of a curve as it moves along the curve. The ruled surface generated by the unit Darboux vector can be expressed as a function of the $\xi(s)$, which is the angle between Darboux and binormal vectors. This means some surface characterizations can be studied and analyzed as a function of $\xi(s)$.

Theorem 3.9. Let $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{C}(s)$ be a ruled surface. There exists a frame of the curve $\alpha(s)$ called Frenet frame and denoted by $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$. The relations among frames in terms of $\xi(s)$ are given by

$$\begin{bmatrix} \vec{T}(s) \\ \vec{Y}(s) \\ \vec{Z}(s) \end{bmatrix} = \begin{bmatrix} \frac{1+v\xi'(s)\cos\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} & 0 & \frac{-v\xi'(s)\sin\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} \\ \frac{v\xi'(s)\sin\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} & 0 & \frac{1+v\xi'(s)\cos\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}$$

and

$$\begin{bmatrix} \vec{T}(v) \\ \vec{Y}(v) \\ \vec{Z}(v) \end{bmatrix} = \begin{bmatrix} \sin\xi(s) & 0 & \cos\xi(s) \\ -\cos\xi(s) & 0 & \sin\xi(s) \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}$$

Proof.

If the partial derivative is taken according to s and v using the angle $\xi(s)$, we have

$$\vec{\Lambda}_s = \vec{t}(s) + v\left(\xi'(s)\cos\xi(s)\vec{t}(s) - \xi'(s)\sin\xi(s)\vec{b}(s)\right)$$

and

$$\vec{\Lambda}_v = \vec{C}(s)$$

Arriving at this equation, we reach to

$$\vec{T}(s) = \frac{\vec{\Lambda}_s}{\|\vec{\Lambda}_s\|} = \frac{\left(1 + v\xi'(s)\cos\xi(s)\right)\vec{t}(s) - v(\xi'(s)\sin\xi(s)\vec{b}(s))}{\sqrt{(1 + v\xi'(s)\cos\xi(s))^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{T}(v) = \frac{\vec{\Lambda}_v}{\|\vec{\Lambda}_v\|} = \sin\xi(s)\vec{t}(s) + \cos\xi(s)\vec{b}(s)$$

For $v < \frac{\cos\xi(s)}{\xi'(s)}$, the unit normal vector of the ruled surface Λ is given as

$$\vec{Z} = \frac{\vec{\Lambda}_s \wedge \vec{\Lambda}_v}{\|\vec{\Lambda}_s \wedge \vec{\Lambda}_v\|} = -\vec{n}(s)$$

$\vec{Y}(s)$ and $\vec{Y}(v)$ depending on Frenet frame at point $\alpha(s)$ can be obtained

$$\vec{Y}(s) = \vec{Z}(s) \wedge \vec{T}(s) = \frac{v\xi'(s)\sin\xi(s)\vec{t}(s) + \left(1 + v\xi'(s)\cos\xi(s)\right)\vec{b}(s)}{\sqrt{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{Y}(v) = \vec{Z}(v) \wedge \vec{T}(v) = -\sin\xi(s)\vec{t}(s) + \cos\xi(s)\vec{b}(s)$$

This completes the proof of the theorem. □

Theorem 3.10. Let Λ be a ruled surface and $\xi(s)$ be the angle between the Darboux vector and the binormal vector. The singular point belonging to the ruled surface is given by $P(s_0, v_0) = \left(s_0, \frac{\cos\xi(s_0)}{\xi'(s_0)}\right)$.

Proof.

To determine the normal vector for a singular point, the denominator of the normal vector must be zero. As a result, when performing the necessary operations, the singular point becomes $P(s_0, v_0) = \left(s_0, \frac{\cos\xi(s_0)}{\xi'(s_0)}\right)$. □

Taking into account Equation 1, we have the following result.

Corollary 3.11. The normal and the geodesic curvatures of the ruled surface in terms of angle $\xi(s)$ can be expressed as follows:

$$k_n(s) = \frac{-(\cos\xi(s) + v\xi'(s))\|\vec{W}(s)\|}{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}, \; k_n(v) = 0$$

and

$$t_g(s) = \frac{\sin\xi(s)\|\vec{W}(s)\|}{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}, \; t_g(v) = 0$$

Using the angle $\xi(s)$, quaternionic shape operators are given by

$$Q(s)=\frac{-(\cos\xi(s)+v\xi'(s))\|\vec{W}(s)\|}{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}+\frac{\sin\xi(s)\|\vec{W}(s)\|}{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}\vec{n}(s)$$

and

$$Q(v)=k_n(v)+t_g(v)\vec{Z}(v)=0$$

Theorem 3.12. Let Λ be a ruled surface and $\xi(s)$ be the angle between the Darboux vector and the binormal vector. Using the quaternionic operators, the shape operators $S(\vec{T}(s))$ and $S(\vec{T}(v))$ are obtained by

$$S(\vec{T}(s))=Q(s)\times\vec{T}(s)=\frac{-(\cos\xi(s)\|\vec{W}(s)\|+v\xi'(s)\|\vec{W}(s)\|(\cos 2\xi(s)+v\xi'(s)\cos\xi(s)+1))\vec{t}(s)}{(1+v\xi'(s)(2\cos\xi(s)+v\xi'(s)))^{3/2}}$$

$$+\frac{(\sin\xi(s)\|\vec{W}(s)\|+v\xi'(s)\|\vec{W}(s)\|(\sin 2\xi(s)+v\xi'(s)\sin\xi(s)))\vec{b}(s)}{(1+v\xi'(s)(2\cos\xi(s)+v\xi'(s)))^{3/2}}$$

and

$$S(\vec{T}(v))=Q(v)\times\vec{T}(v)=\vec{0}$$

Proof.
The proof of the theorem is similar to the proof of Theorem 3.3. □

Corollary 3.13. According to $\xi(s)$, the angle between Darboux and binormal vectors, the cosine and sine of the angle of between $\vec{T}(s)$ and $Q(s)\times\vec{T}(s)$ are as follows:

$$\cos 2\varphi(s)=\frac{-\cos\xi(s)-v\xi'(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}}$$

and

$$\sin 2\varphi(s)=\frac{\sin\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}}$$

Theorem 3.14. The ruled surface Λ is flat. The mean curvature H of this surface is obtained by

$$H=\frac{\|\vec{W}(s)\|}{2|1+v\xi'(s)|}$$

Proof.
The proof of the theorem is similar to the proof of Theorem 3.7. □

Example 3.15. The various position of the generating unit Darboux vector is obtained from the ruled surface. Such a surface has a parameterization,

$$\vec{\Lambda}(s,v)=\left(\sqrt{1+s^2},\ln(s+\sqrt{1+s^2}),s+v\right)$$

If we choose the quaternion as $p(s)=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2\sqrt{1+s^2}}(1,s,0)$, then we can write the surface with quaternions as follows:

$$\vec{\Lambda}(s,v)=\left(\sqrt{1+s^2},\ln(s+\sqrt{1+s^2}),s\right)+\frac{1}{\sqrt{2+2s^2}}vp(s)\times(s,1,\sqrt{1+s^2})$$

This ruled surface is provided in Figure 1.

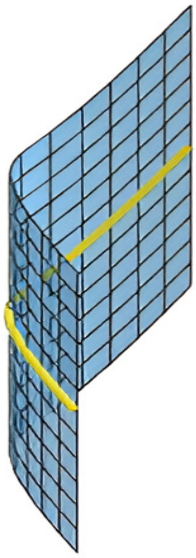

Figure 1. The ruled surface drawn by unit Darboux vector and the base curve of the surface

The quaternionic shape operator, denoted as $Q(s)$, is a mathematical construct that can be used to analyze the shape of the surface. It is calculated as

$$Q(s) = k_n(s) + t_g(s)\vec{Z} = \frac{1}{2\sqrt{2}(1+s^2)} - \frac{1}{2\sqrt{2}(1+s^2)^{3/2}}(1, -s, 0)$$

The shape operator, denoted as $S(\vec{T}(s))$, is obtained by taking the quaternionic product of the quaternionic shape operator and the tangent vector $\vec{T}(s)$. The tangent vector is a vector that is tangent to the surface at a particular point and points in the direction of the surface at that point. The shape operator is then calculated as

$$S(\vec{T}(s)) = Q(s) \times \vec{T}(s) = \frac{1}{2\sqrt{2}(1+s^2)^{3/2}}(s, 1, 0)$$

The shape operator for the parameter v, denoted as $S(\vec{T}(v))$, is equal to $\vec{0}$. Hence, by Equations 3 and 4, it is easy to express Gauss and mean curvatures as

$$K = 0 \quad \text{and} \quad H = \frac{\sqrt{2}}{4(1+s^2)^{5/2}}$$

This means that the surface is developable and is not a minimal surface. The operator $Q(s)$ rotates the tangent vector $\vec{T}(s)$ in the tangent plane of the ruled surface and around the normal vector $\vec{Z}(s)$ of the ruled surface. In this case, the rotation matrix for the unit quaternion $q = \cos\varphi + \sin\varphi\vec{Z}(s)$ is given as

$$R_1 = \frac{1}{\sqrt{2}(1+s^2)} \begin{bmatrix} \sqrt{2} - s^2(\sqrt{2}-1) & -s(\sqrt{2}-1) & s\sqrt{1+s^2} \\ -s(\sqrt{2}-1) & 1 & 1 \\ -s\sqrt{1+s^2} & -1 & \frac{1}{1+s^2} \end{bmatrix}$$

where $\vec{Z}(s) = \left(\frac{1}{\sqrt{1+s^2}}, -\frac{s}{\sqrt{1+s^2}}, 0\right)$.

4. Conclusion

Quaternions are an essential topic in animation, robot kinematics, and rotational motion in 3-dimensional space. Ruled surfaces have a vital role in technology (especially robot end-effectors). Moreover, it is known that Gauss and mean curvatures and the shape operator are the invariants in the surface of theory. These invariants are quaternionically calculated for the unit Darboux ruled surface.

In this study, we combine some points on two critical subjects. Besides, we provide some theorems related to the invariants and then show how to find a rotation matrix. Based on the quaternionic shape operator and the rotation matrix, we derive different situations of the invariants and rotations: one from the curvatures of the base curve and the other one by the angle $\xi(s)$ between $\vec{W}(s)$ and $\vec{b}(s)$. Thus, we observe what happens when we express the relation form of the frame equations using ξ instead of the curvatures. Furthermore, we obtain the shape operators by revolving tangent vectors of parameter curves around the surface's normal vector through twice the angle of φ and then get rotation matrices.

In further research, it would be valuable to replicate similar approaches in different spaces, such as Galileo or Lorentz spaces. These alternative spaces could potentially yield different results and provide a deeper understanding of the results herein being studied.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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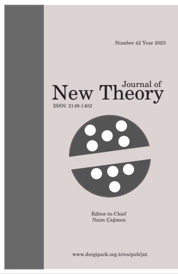

A Hybridization of Modified Rough Bipolar Soft Sets and TOPSIS for MCGDM

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Abstract — Uncertain data is a challenge to decision-making (DM) problems. Multi-criteria group decision-making (MCGDM) problems are among these problems that have received much attention. MCGDM is difficult because the existing alternatives frequently conflict with each other. In this article, we suggest a novel hybrid model for an MCGDM approach based on modified rough bipolar soft sets (MRBSs) using a well-known method of technique for order of preference by similarity to ideal solution (TOPSIS), which combines MRBSs theory and TOPSIS for the prioritization of alternatives in an uncertain environment. In this technique, we first introduce an aggregated parameter matrix with the help of modified bipolar soft lower and upper matrices to identify the positive and negative ideal solutions. After that, we define the separation measurements of these two solutions and compute relative closeness to choose the best alternative. Next, an application of the proposed technique in the MCGDM problem is introduced. Afterward, an algorithm for this application is developed, which is illustrated by a case study. The application demonstrates the usefulness and efficiency of the proposal. Compared to some existing studies, we additionally present several merits of our proposed technique. Eventually, the paper handles whether additional studies on these topics are needed.

Keywords *Bipolar soft sets, bipolar soft rough sets, MRBSs, TOPSIS, MCGDM*

Mathematics Subject Classification (2020) 03B52, 68T27

1. Introduction

Various issues in social sciences, engineering, medical sciences, economics, and other domains include uncertainty. It is impossible to address these issues using traditional mathematical approaches. The traditional mathematical model is a rational model of decision-making (DM) that depends on the hypothesis that decision-makers have access to complete knowledge and can make the best decision by weighing every alternative. Due to this, the mathematical model is highly complicated, and an accurate solution cannot be obtained. To overcome this trouble, scholars are endeavoring to discover suitable methodologies and mathematical theories to address data uncertainty. These theories include fuzzy sets (FSs), rough set (RS) theory, vague set theory, automata theory, etc., but they have only partially been successful in solving the problems. These theories diminished the space between traditional mathematical concepts and ambiguous real-world data.

Zadeh [1] developed the FS theory to characterize fuzzy data mathematically. But, in FSs, finding

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the membership function might sometimes be challenging. Consequently, Molodtsov [2] developed the soft set (SS) concept as a new strategy for modeling uncertainty, liberated from this trouble. SS theory needs an approximate description of an object as its initial viewpoint. The selection of suitable parameters like numbers, functions, words, etc., makes SS theory very advantageous and straightforward to use in reality. Maji et al. [3] established several operations on SS. Ali et al. [4] offered a variety of novel operations on SS. Çağman et al. [5] projected the idea of fuzzy SS theory. Al-Shami and Mhemdi [6] offered to belong and non-belong relations on double-framed SS.

The RS theory [7, 8] is an effective mathematical strategy for handling uncertainties. In RSs, uncertainty is characterized by a set's boundary region. Pawlak examined how close a bunch of objects are to the information associated with them using their lower and upper approximations.

The connections between SS theory, RSs, and FSs were provided by Feng et al. [9,10], leading to three kinds of hybrid models: rough SS (RSS), soft RSs (SRSs), and soft-rough FSs (SRFSs). Shabir et al. [11] redefined a version of an SRS called a modified SRS (MSRS). Shaheen et al. [12] established the concept of dominance-based SRSs.

Bipolarity is critical in various kinds of data when establishing mathematical modeling for specific problems. Bipolarity takes both the positive and negative characteristics of the data into account. The positive data delivers what is conceivable, whereas the negative data emphasizes the impossibility. The idea behind the existence of bipolar information is that a large variety of human DM relies upon bipolar judgmental cognition.

Shabir and Naz [13] put the groundwork for bipolar SSs (BSSs) due to the significance of bipolarity. Following this research, the BSS theory gained much fame among scholars. Karaaslan and Karataş [14] reformulate BSS with a novel approximation, offering a prospect to explore the topological structures of BSS. Mahmood [15] redesigned a form of BSS, known as T-BSS, and employed this concept for DM problems. Moreover, Naz and Shabir [16] established the idea of fuzzy BSS and investigated their algebraic structures. Al-Shami [17] came up with the idea of bipolar soft sets and the relations between them and ordinary points, along with applications.

Karaaslan and Çağman [18] originally suggested bipolar SRSs (BSRSs) tackle the roughness of BSS, which was then changed and improved by establishing the conception of MRBSs by Shabir and Gul [19]. Gul et al. [20] established a new strategy of the roughness for BSS with applications in MCGDM. Gul and Shabir [21] pioneered the concept (α, β)-bipolar fuzzified RS using bipolar fuzzy tolerance relation.

In decision analysis, several multi-criteria DM (MCDM) frameworks have been carried out in the literature. TOPSIS is one of the classical MCDM methods offered by Hwang and Yoon [22] in 1981. The fundamental notion of TOPSIS is to measure the distance between every alternative and ideal solution. The optimal alternative should be the one that has to have the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). PIS addresses the scenario for the best possible decision, whereas NIS shows the scenario for the worst. Chen [23] generalized the TOPSIS approach for taking the MCDM problem in a fuzzy context. Afterward, Chen and Tsao [24] proposed the interval-valued fuzzy TOPSIS. Boran et al. [25] fostered the TOPSIS for MCDM problems based on intuitionistic FS. Ali et al. [26] offered the TOPSIS model for probabilistic interval-valued hesitant fuzzy sets with application to healthcare facilities in public hospitals. Eraslan [27] gave a DM method using TOPSIS on SS theory. Eraslan and Karaaslan [28] gave a group DM method based on TOPSIS under a fuzzy SS environment.

Shabir et al. [29] proposed an algebraic approach to N-SS with application in DM via TOPSIS. Akram et al. [30] generalize the TOPSIS and ELECTRE-I methods in a bipolar fuzzy framework. Akram and Adeel [31] extended the TOPSIS for MCGDM via an interval-valued hesitant fuzzy N-SS context. Xu and Zhang [32] constructed a strategy based on maximizing deviation and the TOPSIS to explain multi-attribute DM problems. In 2014, Zhang and Xu [33] extended the TOPSIS in MCDM using Pythagorean FSs. Mousavi-Nasab and Sotoudeh-Anvari [34] gave an MCDM-based method using TOPSIS, COPRAS, and DEA for material selection problems. Mahmood et al. [35] pioneered a novel TOPSIS method based on lattice-ordered T-BSS with applications in DM.

Inspired by the previously mentioned earlier studies and the basic principle of MRBSs, we have observed that the BSS can manage the bipolarity of the data concerning specific alternatives with the assistance of two mappings. The positive side of the data is addressed by one mapping, whereas the other mapping measures the negative side. Keeping in mind the relationship between RSs and BSS, Karaaslan and Çağman [18] attempted to explore the roughness of BSS, which has certain shortcomings. To overcome these shortcomings, Shabir and Gul [19] pioneered the idea of MRBSs.

Moreover, to per best of our knowledge, there does not exist any investigation on the appropriate fusion of TOPSIS with MRBSs. This gap motivates the current research to propose a novel TOPSIS approach using MRBSs and discuss their application in DM.

In a nutshell, to expand the theory of MRBSs, the primary goal of this study is to establish a novel TOPSIS approach for MCGDM problems via the MRBSs environment. We introduce a DM algorithm that determines the best and worst decision among some alternatives, with implementation on selecting the optimal candidate for a particular post.

This article is structured as follows: Section 2 introduces basic notations related to RS, SS, BSS, BSRS, and MRBSs. These notions will assist us in discussing our work and suffice the paper for the reader. After this, we give the general procedure of the TOPSIS technique. Section 3 puts forward the new TOPSIS-based strategy for addressing MCGDM problems using MRBSs. Section 4 states our suggested algorithm for choosing the optimal alternative, which we validate through a fully developed case study in Section 5. Section 6 represents a comparative analysis between the proposed technique and the other existing methods in solving MCGDM problems. Finally, Section 7 ends with an outline of the current work and a few perspectives for the future.

2. Preliminaries

In this section, we recapitulate a few essential notions associated with the background of this study. Throughout this article, unless stated otherwise, we will use $\mho$ for an initial universe, $\mathfrak{A}$ for the set of all the parameters related to the objects in $\mho$, and $2^{\mho}$ for the power set of $\mho$.

Definition 2.1. [7] Let $\varnothing \neq \mho$ be a finite universe, and σ be an equivalence relation of $\mho \times \mho$. Then, $(\mho, \sigma)$ is stated to be an approximation space.

If $\varnothing \neq \mathcal{Q} \subseteq \mho$, then $\mathcal{Q}$ may or may not be expressed as a union of some equivalence classes of $\mho$. If $\mathcal{Q}$ is expressed as a union of some equivalence classes, then $\mathcal{Q}$ is said to be σ-definable; in any other case, it is referred to as σ-undefinable. If $\mathcal{Q}$ is σ-undefinable, then the lower and upper approximations of $\mathcal{Q}$ concerning σ are given as follows:

$$\underline{apr}_{\sigma}(\mathcal{Q}) = \{q \in \mho : [q]_{\sigma} \subseteq \mathcal{Q}\} \tag{1}$$

and

$$\overline{apr}_{\sigma}(\mathcal{Q}) = \{q \in \mho : [q]_{\sigma} \cap \mathcal{Q} \neq \emptyset\} \tag{2}$$

where

$$[q]_{\sigma} = \{r \in \mho : (q, r) \in \sigma\}$$

The boundary region of the RS is characterized as:

$$Bnd_{\sigma}(\mathcal{Q}) = \overline{apr}_{\sigma}(\mathcal{Q}) - \underline{apr}_{\sigma}(\mathcal{Q})$$

From Equations (1) and (2) we can see that

i. An element q belongs to the lower approximation $\underline{apr}_{\sigma}(\mathcal{Q})$ if all elements equivalent to q belong to $\mathcal{Q}$.

ii. An element q belongs to the upper approximation $\overline{apr}_{\sigma}(\mathcal{Q})$ if at least one element equivalent to q belongs to $\mathcal{Q}$.

Let $\mho$ be a non-void universe and $\mathfrak{A}$ be a set of parameters. Then, an SS is defined through a set-valued map, as described below.

Definition 2.2. [2] A pair $(\hat{f}, \mathfrak{A})$ is called an SS over $\mho$, where $\hat{f} : \mathfrak{A} \longrightarrow 2^{\mho}$ is a set-valued map.

In other words, an SS over $\mho$ gives a parameterized collection of subsets of $\mho$. An SS over $\mho$ may also be represented as:

$$(\hat{f}, \mathfrak{A}) = \{(\wp, \hat{f}(\wp)) : \wp \in \mathfrak{A}, \hat{f}(\wp) \in 2^{\mho}\}$$

A BSS is obtained through two set-valued maps by considering not only a set of parameters but also an associated set of parameters with an opposite meaning known as "not set of parameters".

Definition 2.3. [3] By a "NOT set of parameters" of $\mathfrak{A}$, we mean a set having the form $\widetilde{\mathfrak{A}} = \{\neg\wp : \wp \in \mathfrak{A}\}$ where $\neg\wp =$ not $\wp$, for all $\wp \in \mathfrak{A}$.

Definition 2.4. [13] A triplet $(\hat{f}, \hat{g} : \mathfrak{A})$ is termed as a BSS over $\mho$ where $\hat{f} : \mathfrak{A} \longrightarrow 2^{\mho}$ and $\hat{g} : \widetilde{\mathfrak{A}} \longrightarrow 2^{\mho}$ such that, for all $\wp \in \mathfrak{A}$, $\hat{f}(\wp) \cap \hat{g}(\neg\wp) = \varnothing$.

In other words, a BSS over $\mho$ offers a couple of parameterized families of subsets of $\mho$ and $\hat{f}(\wp) \cap \hat{g}(\neg\wp) = \varnothing$, for all $\wp \in \mathfrak{A}$, is used as a consistency constraint. A BSS might be characterized as:

$$(\hat{f}, \hat{g} : \mathfrak{A}) = \left\{(\wp, \hat{f}(\wp), \hat{g}(\neg\wp)) : \wp \in \mathfrak{A}, \neg\wp \in \widetilde{\mathfrak{A}} \text{ and } \hat{f}(\wp) \cap \hat{g}(\neg\wp) = \varnothing\right\}$$

After this, the collection of all the BSSs over $\mho$ will be denoted by $\mathcal{BSS}^{\mho}$.

Definition 2.5. [18] Let $(\hat{f}, \hat{g} : \mathfrak{A}) \in \mathcal{BSS}^{\mho}$. Then, $\beta = \langle \mho, (\hat{f}, \hat{g} : \mathfrak{A}) \rangle$ is termed as a $\mathcal{BSA}_s$ (bipolar soft approximation space). For any $\mathcal{Q} \subseteq \mho$, the bipolar soft rough approximations based on β are defined:

$$\underline{BS}_{\beta}(\mathcal{Q}) = \left(\underline{S}_{\beta^P}(\mathcal{Q}), \underline{S}_{\beta^N}(\mathcal{Q})\right)$$

and

$$\overline{BS}_{\beta}(\mathcal{Q}) = \left(\overline{S}_{\beta^P}(\mathcal{Q}), \overline{S}_{\beta^N}(\mathcal{Q})\right)$$

where

$$\underline{S}_{\beta^P}(\mathcal{Q}) = \left\{q \in \mho : \exists\ \wp \in \mathfrak{A}, [q \in \hat{f}(\wp) \subseteq \mathcal{Q}]\right\}$$

$$\underline{S}_{\beta^N}(\mathcal{Q}) = \left\{q \in \mho : \exists\ \neg\wp \in \widetilde{\mathfrak{A}}, [q \in \hat{g}(\neg\wp), \hat{g}(\neg\wp) \cap \mathcal{Q}^c \neq \varnothing]\right\}$$

$$\overline{S}_{\beta^P}(\mathcal{Q}) = \left\{q \in \mho : \exists\ \wp \in \mathfrak{A}, [q \in \hat{f}(\wp), \hat{f}(\wp) \cap \mathcal{Q} \neq \varnothing]\right\}$$

and

$$\overline{S}_{\beta^N}(\mathcal{Q}) = \left\{q \in \mho : \exists\ \neg\wp \in \widetilde{\mathfrak{A}}, [q \in \hat{g}(\neg\wp) \subseteq \mathcal{Q}^c]\right\}$$

Moreover, if $\underline{BS}_{\beta}(\mathcal{Q}) \neq \overline{BS}_{\beta}(\mathcal{Q})$, then $\mathcal{Q}$ is called a BSRS; else $\mathcal{Q}$ is called bipolar soft β-definable.

The boundary region of a BSRS is described as:

$$BND_\beta(\mathcal{Q}) = (\overline{S}_{\beta^P}(\mathcal{Q}) \setminus \underline{S}_{\beta^P}(\mathcal{Q}), \underline{S}_{\beta^N}(\mathcal{Q}) \setminus \overline{S}_{\beta^N}(\mathcal{Q}))$$

BSRSs were originally initiated by Karaaslan and Çağman [18] to manage the roughness of BSSs, which was subsequently altered and improved by Shabir and Gul [19] by launching the idea of MRBSs. MRBSs are characterized as follows:

Definition 2.6. [19] Let $(\hat{f}, \hat{g} : \mathfrak{A}) \in \mathcal{BSS}^{\mho}$ such that $\hat{f} : \mathfrak{A} \longrightarrow 2^{\mho}$ and $\hat{g} : \widetilde{\mathfrak{A}} \longrightarrow 2^{\mho}$. Construct two different maps as follows:

$$\begin{aligned} \Phi : \mho &\longrightarrow 2^{\mathfrak{A}} \\ q &\longmapsto \Phi(q) = \{\wp : q \in \hat{f}(\wp)\} \end{aligned}$$

and

$$\begin{aligned} \Psi : \mho &\longrightarrow 2^{\widetilde{\mathfrak{A}}} \\ q &\longmapsto \Psi(q) = \{\neg\wp : q \in \hat{g}(\neg\wp)\} \end{aligned}$$

Then, $\Omega = \langle \mho, (\Phi, \Psi) \rangle$ is called a modified rough bipolar soft approximation space (MRBS-AS).

For any $\varnothing \neq \mathcal{Q} \subseteq \mho$, the lower modified bipolar pair (LMBP) and the upper modified bipolar pair (UMBP) concerning Ω are defined in the following manner, respectively:

$$\underline{MBS}_\Omega(\mathcal{Q}) = (\underline{\mathcal{Q}}_{\Phi^+}, \underline{\mathcal{Q}}_{\Psi^-})$$

and

$$\overline{MBS}_\Omega(\mathcal{Q}) = (\overline{\mathcal{Q}}^{\Phi^+}, \overline{\mathcal{Q}}^{\Psi^-})$$

where

$$\begin{aligned} \underline{\mathcal{Q}}_{\Phi^+} &= \{p \in \mathcal{Q} : \Phi(p) \neq \Phi(r), \text{ for all } r \in \mathcal{Q}^c\} \\ \overline{\mathcal{Q}}^{\Phi^+} &= \{p \in \mho : \Phi(p) = \Phi(r), \text{ for some } r \in \mathcal{Q}\} \\ \underline{\mathcal{Q}}_{\Psi^-} &= \{p \in \mho : \Psi(p) = \Psi(r), \text{ for some } r \in \mathcal{Q}\} \end{aligned}$$

and

$$\overline{\mathcal{Q}}^{\Psi^-} = \{p \in \mathcal{Q} : \Psi(p) \neq \Psi(r), \text{ for all } r \in \mathcal{Q}^c\}$$

Here, $\mathcal{Q}^c = \mho - \mathcal{Q}$. Generally, $\underline{\mathcal{Q}}_{\Phi^+}$, $\overline{\mathcal{Q}}^{\Phi^+}$, $\underline{\mathcal{Q}}_{\Psi^-}$, and $\overline{\mathcal{Q}}^{\Psi^-}$ will be called Φ-lower positive, Φ-upper positive, Ψ-lower negative, and Ψ-upper negative MRBS-approximations of $\mathcal{Q} \subseteq \mho$, respectively. If $\underset{\sim}{MBS}_\Omega(\mathcal{Q}) \neq \widetilde{MBS}_\Omega(\mathcal{Q})$, then $\mathcal{Q}$ is said to be an MRBSs; otherwise, $\mathcal{Q}$ is said to be MRBS-definable.

The corresponding positive, boundary, and negative regions under MRBSs are listed as follows:

$$Pos_\Omega(\mathcal{Q}) = (\underline{\mathcal{Q}}_{\Phi^+}, \overline{\mathcal{Q}}^{\Psi^-})$$

$$Bnd_\Omega(\mathcal{Q}) = \left(\overline{\mathcal{Q}}^{\Phi^+} \setminus \underline{\mathcal{Q}}_{\Phi^+}, \underline{\mathcal{Q}}^{\Psi^-} \setminus \overline{\mathcal{Q}}_{\Psi^-}\right)$$

and

$$Neg_\Omega(\mathcal{Q}) = \left(\left(\overline{\mathcal{Q}}^{\Phi^+}\right)^c, \left(\underline{\mathcal{Q}}_{\Psi^-}\right)^c\right)$$

TOPSIS is one of the most frequently utilized techniques for MCDM because it ranks alternatives and chooses optimal alternatives in the concept evaluation procedure using Euclidean distances. Suppose that for any DM problem, there are n criteria and m alternatives. Then, a decision matrix is described as $\mathfrak{D} = [\delta_{ij}]_{m\times n}$ where $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, n\}$, and δ_{ij} demonstrates the preference value of an alternative for design criteria.

The procedure of TOPSIS described in [36] is as follows:

i. Construct the normalized decision matrix $\mathfrak{D}_{nor} = [r_{ij}]_{m\times n}$ where $r_{ij} = \dfrac{\delta_{ij}}{\sqrt{\sum_{i=1}^{m} \delta_{ij}^2}}$ and weighted normalized decision matrix. Here, $v_{ij} = \omega_j r_{ij}$ is a weighted normalized value where ω_j is the weight of a criteria.

ii. Evaluate the positive ideal solution (PIS) and negative ideal solution (NIS) as:

$$v_i^+ = \Big\{\Big(\bigvee_i (v_{ij}) \mid j \in \mathcal{I}\Big), \Big(\bigwedge_i (v_{ij}) \mid j \in \mathcal{J}\Big)\Big\}$$

and

$$v_i^- = \Big\{\big(\bigwedge_i (v_{ij}) \mid j \in \mathcal{I}), (\bigvee_i (v_{ij}) \mid j \in \mathcal{J})\Big\}$$

where $\mathcal{I}$ and $\mathcal{J}$ are related to the benefit and cost criterion, respectively.

iii. Determine the separation measure of each alternative from PIS and NIS by using n-dimensional Euclidean distance:

$$\delta_i^+ = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_i^+\right)^2}, \quad i \in \{1, 2, \cdots, m\}$$

and

$$\delta_i^- = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_i^-\right)^2}, \quad i \in \{1, 2, \cdots, m\}$$

iv. Evaluate the relative closeness coefficient of each alternative to the ideal solution, given as:

$$\mathcal{C}_i^* = \frac{\delta_i^-}{\delta_i^- + \delta_i^+}, \quad i \in \{1, 2, \cdots, m\}$$

v. Sort the alternative concerning the value of $\mathcal{C}_i^*$. The optimal alternative is the object with the highest value of $\mathcal{C}_i^*$. That alternative would have the least distance from the PIS and the largest distance from the NIS.

3. An Integrated Model of MCGDM using TOPSIS Technique and MRBSs

The MCGDM is one of the substantial components of modern decision theory. MCGDM aims to select the optimal from finite alternatives by incorporating the evaluation information of various alternatives acquired from a group of experts(decision-makers). It is instrumental in economic evaluation, clustering analysis, site selection, medical diagnosis, etc. In MCGDM, the primary step is to consider a finite number of alternatives in terms of multiple conflicting criteria based on the experts' opinions. Characterizing the evaluation information for several attributes is a significant problem in the MCGDM. In real-life MCGDM problems, uncertainty is inevitable because of imprecise judgment by decision-makers. TOPSIS is a practical and extensively used multi-criteria DM (MCDM) technique for sorting alternatives and determining the optimal alternative in the concept evaluation procedure. The aggregating function computed in TOPSIS indicates "closeness to ideal solution". To make criteria with the same units, TOPSIS employs vector normalization. The critical concept of TOPSIS is that the alternative that has been selected as the optimal should have the smallest distance from the PIS and the greatest from the NIS.

In this section, we utilize the TOPSIS technique for MCGDM based on the MRBSs. The systematic procedure of the TOPSIS under the MRBSs is explained as follows:

3.1. Description of Problem

In this subsection, we first give the essential explanation of the MCGDM problem under consideration. Suppose that $\mho = \{\mu_1, \mu_2, \dots, \mu_n\}$ be the set consisting of n alternatives in which the best object is to be selected and $\mathfrak{A} = \{\wp_1, \wp_2, \dots, \wp_m\}$ be the set of parameters (criterion) of objects. Assume that we have a group of independent experts $\mathcal{G} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k\}$ consisting of k decision-makers to evaluate the objects in $\mho$. Each expert needs to review all the objects of $\mho$ and will be requested to only choose "the optimal alternatives" as their evaluation result. Hence each expert's primary evaluation result is a subset of $\mho$. For the sake of simplicity, we assume that the evaluations of these experts in $\mathcal{G}$ are of the same importance. Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_k$ are non-void subsets of $\mho$, indicate primary evaluations of experts $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$, about n alternatives concerning m parameters, respectively, and $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r \in \mathcal{BSS}^{\mho}$ are the real results previously captured for the same problems in various locations or various periods. Specifically, we can take the MRBS-approximations of the expert $\mathcal{P}_i$'s primary evaluation result $\mathcal{Q}_i$ concerning the MRBS-AS $\Omega = \langle \mho, (\Phi, \Psi) \rangle$. The Φ-lower positive approximation $\underline{\mathcal{Q}_i}_{\Phi+}$ can be interpreted as the set consisting of the objects which are undoubtedly the optimum candidates according to the expert $\mathcal{P}_i$'s primary evaluation. Similarly, the Φ-upper positive approximation $\overline{\mathcal{Q}_i}^{\Phi+}$ can be interpreted as the set consisting of the objects which are possibly the optimum candidates according to the expert $\mathcal{P}_i$'s primary evaluation. The Ψ-lower positive approximation $\underline{\mathcal{Q}_i}_{\Psi-}$ can be interpreted as the set consisting of the objects which are possibly the worst candidates according to the expert $\mathcal{P}_i$'s primary evaluation. Likewise, the Ψ-upper negative approximation $\overline{\mathcal{Q}_i}^{\Psi-}$ can be interpreted as the set consisting of the objects which are surely the worst candidates according to the expert $\mathcal{P}_i$'s primary evaluation. Then, the DM for this MCGDM problem is: "how to resolve differences of the evaluation conveyed by the individual experts to determine the object which is highly favorable by the entire group of experts".

3.2. Methodology

Here, we present the step-by-step mathematical formulation and process of the TOPSIS technique under the framework of MRBSs for the MCGDM problem.

Definition 3.1. Let $\underline{MBS}_{\mathcal{B}_q}(\mathcal{Q}_j) = (\underline{\mathcal{Q}_j}_{\Phi_q^+}, \underline{\mathcal{Q}_j}_{\Psi_q^-})$ be the LMBP and $\overline{MBS}_{\mathcal{B}_q}(\mathcal{Q}_j) = (\overline{\mathcal{Q}_j}^{\Phi_q^+}, \overline{\mathcal{Q}_j}^{\psi_q^-})$ be UMBP of $\mathcal{Q}_j$ such that $j \in \{1, 2, \cdots, k\}$ concerning $\mathcal{B}_q = (\hat{f}_q, \hat{g}_q : \mathfrak{A}) \in \mathcal{BSS}^{\mho}$, for $q \in \{1, 2, \dots, r\}$. Then,

$$\underline{M} = \begin{pmatrix} \langle \underline{\mathcal{Q}_1}_{\Phi_1^+}, \underline{\mathcal{Q}_1}_{\Psi_1^-} \rangle & \langle \underline{\mathcal{Q}_2}_{\Phi_1^+}, \underline{\mathcal{Q}_2}_{\Psi_1^-} \rangle & \cdots & \langle \underline{\mathcal{Q}_k}_{\Phi_1^+}, \underline{\mathcal{Q}_k}_{\Psi_1^-} \rangle \\ \langle \underline{\mathcal{Q}_1}_{\Phi_2^+}, \underline{\mathcal{Q}_1}_{\Psi_2^-} \rangle & \langle \underline{\mathcal{Q}_2}_{\Phi_2^+}, \underline{\mathcal{Q}_2}_{\Psi_2^-} \rangle & \cdots & \langle \underline{\mathcal{Q}_k}_{\Phi_2^+}, \underline{\mathcal{Q}_k}_{\Psi_2^-} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \underline{\mathcal{Q}_1}_{\Phi_r^+}, \underline{\mathcal{Q}_1}_{\Psi_r^-} \rangle & \langle \underline{\mathcal{Q}_2}_{\Phi_r^+}, \underline{\mathcal{Q}_2}_{\Psi_r^-} \rangle & \cdots & \langle \underline{\mathcal{Q}_k}_{\Phi_r^+}, \underline{\mathcal{Q}_k}_{\Psi_r^-} \rangle \end{pmatrix}_{r \times k}$$

and

$$\overline{M} = \begin{pmatrix} \langle \overline{\mathcal{Q}_1}^{\Phi_1^+}, \overline{\mathcal{Q}_1}^{\Psi_1^-} \rangle & \langle \overline{\mathcal{Q}_2}^{\Phi_1^+}, \overline{\mathcal{Q}_2}^{\Psi_1^-} \rangle & \cdots & \langle \overline{\mathcal{Q}_k}^{\Phi_1^+}, \overline{\mathcal{Q}_k}^{\Psi_1^-} \rangle \\ \langle \overline{\mathcal{Q}_1}^{\Phi_2^+}, \overline{\mathcal{Q}_1}^{\Psi_2^-} \rangle & \langle \overline{\mathcal{Q}_2}^{\Phi_2^+}, \overline{\mathcal{Q}_2}^{\Psi_2^-} \rangle & \cdots & \langle \overline{\mathcal{Q}_k}^{\Phi_2^+}, \overline{\mathcal{Q}_k}^{\Psi_2^-} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \overline{\mathcal{Q}_1}^{\Phi_r^+}, \overline{\mathcal{Q}_1}^{\Psi_r^-} \rangle & \langle \overline{\mathcal{Q}_2}^{\Phi_r^+}, \overline{\mathcal{Q}_2}^{\Psi_r^-} \rangle & \cdots & \langle \overline{\mathcal{Q}_k}^{\Phi_r^+}, \overline{\mathcal{Q}_k}^{\Psi_r^-} \rangle \end{pmatrix}_{r \times k}$$

are stated to be modified bipolar soft lower and upper approximation matrices, respectively, where

$$\underline{\mathcal{Q}_j}_{\Phi_q^+} = (\underline{\mu_{1j}}_{\Phi_q^+}, \underline{\mu_{2j}}_{\Phi_q^+}, \dots, \underline{\mu_{nj}}_{\Phi_q^+})$$

$$\underline{\mathcal{Q}_j}_{\Psi_q^-} = (\underline{\mu_{1j}}_{\Psi_q^-}, \underline{\mu_{2j}}_{\Psi_q^-}, \dots, \underline{\mu_{nj}}_{\Psi_q^-})$$

$$\overline{\mathcal{Q}_j}^{\Phi_q^+} = (\overline{\mu_{1j}}^{\Phi_q^+}, \overline{\mu_{2j}}^{\Phi_q^+}, \dots, \overline{\mu_{nj}}^{\Phi_q^+})$$

and

$$\overline{\mathcal{Q}_j}^{\Psi_q^-} = (\overline{\mu_{1j}}^{\Psi_q^-}, \overline{\mu_{2j}}^{\Psi_q^-}, \dots, \overline{\mu_{nj}}^{\Psi_q^-})$$

Here,

$$\underline{\mu_{ij}}_{\Phi_q^+} = \begin{cases} 1, & \mu_i \in \underline{\mathcal{X}_j}_{\Phi_q^+} \\ 0, & \mu_i \notin \underline{\mathcal{X}_j}_{\Phi_q^+} \end{cases}$$

$$\underline{\mu_{ij}}_{\Psi_q^-} = \begin{cases} -\frac{1}{2}, & \mu_i \in \underline{\mathcal{X}_j}_{\Psi_q^-} \\ 0, & \mu_i \notin \underline{\mathcal{X}_j}_{\Psi_q^-} \end{cases}$$

$$\overline{\mu_{ij}}^{\Phi_q^+} = \begin{cases} \frac{1}{2}, & \mu_i \in \overline{\mathcal{X}_j}^{\Phi_q^+} \\ 0, & \mu_i \notin \overline{\mathcal{X}_j}^{\Phi_q^+} \end{cases}$$

and

$$\overline{\mu_{ij}}^{\Psi_q^-} = \begin{cases} -1, & \mu_i \in \overline{\mathcal{X}_j}^{\Psi_q^-} \\ 0, & \mu_i \notin \overline{\mathcal{X}_j}^{\Psi_q^-} \end{cases}$$

Remark 3.2. From Definition 3.1, we have

i. $\underline{\mathcal{Q}_j}_{\Phi_q^+}$ and $\overline{\mathcal{Q}_j}^{\Phi_q^+}$ show the Φ-lower and Φ-upper positive MRBS-approximation of the evaluation $\mathcal{Q}_j \subseteq \mho$ by the j^{th} expert related to q^{th} actual result represented by the BSS $\mathcal{B}_q = (\hat{f}_q, \hat{g}_q : \mathfrak{A})$.

ii. $\underline{\mathcal{Q}_j}_{\Psi_q^-}$ and $\overline{\mathcal{Q}_j}^{\Psi_q^-}$ show the Ψ-lower and Ψ-upper negative MRBS-approximation of the evaluation $\mathcal{Q}_j \subseteq \mho$ by the j^{th} expert related to q^{th} actual result represented by the BSS $\mathcal{B}_q = (\hat{f}_q, \hat{g}_q : \mathfrak{A})$.

Definition 3.3. Let $\underline{M}$ and $\overline{M}$ be modified bipolar soft lower and upper approximation matrices concerning $\underline{MBS}_{\mathcal{B}_q}(\mathcal{Q}_j)$ and $\overline{MBS}_{\mathcal{B}_q}(\mathcal{Q}_j)$. Then,

$$A = \underline{M} + \overline{M} = (\alpha_{ij})_{r \times k} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rk} \end{pmatrix}$$

is regarded as aggregated parameter matrix, where every element has the form:

$$\alpha_{ij} = \langle \alpha_{ij}^{\Phi^+}, \alpha_{ij}^{\Psi^-} \rangle = \left\langle \underline{\mathcal{X}_j}_{\Phi_q^+} \oplus \overline{\mathcal{X}_j}_{\Phi_q^+}, \underline{\mathcal{X}_j}_{\Psi_q^-} \oplus \overline{\mathcal{X}_j}_{\Psi_q^-} \right\rangle$$

such that $\alpha_{ij}^{\Phi^+} = \underline{\mathcal{X}_j}_{\Phi_q^+} \oplus \overline{\mathcal{X}_j}_{\Phi_q^+} = (\dots, \underline{\mu_{mj}}_{\Phi_i^+} + \overline{\mu_{mj}}^{\Phi_i^+}, \dots)$ and $\alpha_{ij}^{\Psi^-} = \underline{\mathcal{X}_j}_{\Psi_q^-} \oplus \overline{\mathcal{X}_j}_{\Psi_q^-} = (\dots, \underline{\mu_{mj}}_{\Psi_i^-} + \overline{\mu_{mj}}^{\Psi_i^-}, \dots)$. Here, the operation $\oplus$ stands for the vector addition.

Definition 3.4. Assume that A is an aggregated parameter matrix. Then,

$$S = \left(\langle s_{ij}^{\Phi^+}, s_{ij}^{\Psi^-} \rangle\right)_{r \times k}$$

is said to be a standardized decision matrix where $s_{ij}^{\Phi^+} = \left(\sum_{m=1}^{k} \alpha_{im}^{\Phi^+}\right)_j$ and $s_{ij}^{\Psi^-} = \left(\sum_{m=1}^{k} \alpha_{im}^{\Psi^-}\right)_j$ such that $i \in \{1, 2, \dots, r\}$ and $j \in \{1, 2, \dots, n\}$.

Remark 3.5. From Definition 3.4, we have noticed that $s_{ij}^{\Phi^+}$ is the positive information for the jth coordinate of the vector sum of the i^{th} row of the matrix A and $s_{ij}^{\Psi^-}$ is the negative information for the j^{th} coordinate of the vector sum of the i^{th} row of the matrix A. ın other words, each row in the matrix S is a vector obtained by taking the column sum of A. Thus, $\langle s_{ij}^{\Phi^+}, s_{ij}^{\Psi^-}\rangle$ represents the standardized MRBS-approximation of alternative u_i under the scenario of j^{th} real result previously acquired for the same problems in various locations or various periods.

Definition 3.6. Let S be a standardized decision matrix. Then,

$$\aleph = (n_{ij})_{r\times k} = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1k} \\ n_{21} & n_{22} & \cdots & n_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ n_{r1} & n_{r2} & \cdots & n_{rk} \end{pmatrix}$$

is called a normalized decision matrix where each entry is of the form $n_{ij} = \langle \eta_{ij}^{\Phi^+}, \eta_{ij}^{\Psi^-}\rangle$ with the following conditions:

$$\eta_{ij}^{\Phi^+} = \frac{s_{ij}^{\Phi^+}}{\sqrt{\sum_{\ell=1}^{r}\left(s_{\ell j}^{\Phi^+}\right)^2}}$$

and

$$\eta_{ij}^{\Psi^-} = \frac{s_{ij}^{\Psi^-}}{\sqrt{\sum_{\ell=1}^{r}\left(s_{\ell j}^{\Psi^-}\right)^2}}$$

Definition 3.7. Let $\aleph$ be a normalized decision matrix. Then,

$$\mathfrak{D} = (\delta_{ij})_{r\times k} = \begin{pmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1k} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{r1} & \delta_{r2} & \cdots & \delta_{rk} \end{pmatrix}$$

is called an average weighted normalized decision matrix where each entry is of the form:

$$\delta_{ij} = \frac{\left|\eta_{ij}^{\Phi^+}\right| + \left|\eta_{ij}^{\Psi^-}\right|}{2}$$

Definition 3.8. Let $\mathfrak{D}$ be an average weighted normalized decision matrix. Then, the expressions:

$$MPIS = \{\delta_1^{\Phi^+}, \delta_2^{\Phi^+}, \cdots, \delta_k^{\Phi^+}\} = \{\max(\delta_{ij}) : i \in I_r\} \quad \text{such that} \quad I_r = \{1, 2, \cdots, r\}$$

and

$$MNIS = \{\delta_1^{\Psi^-}, \delta_2^{\Psi^-}, \cdots, \delta_k^{\Psi^-}\} = \{\min(\delta_{ij}) : i \in I_r\} \quad \text{such that} \quad I_r = \{1, 2, \cdots, r\}$$

are called modified PIS and modified NIS, respectively.

Definition 3.9. Let MPIS and MNIS be positive and negative ideal solutions. Then, the separation measurement of each alternative to MPIS is determined as follows:

$$\mathcal{S}_i^{\Phi^+} = \sqrt{\sum_{j=1}^{k}\left(\delta_{ij} - \delta_j^{\Phi^+}\right)^2}, \quad i \in \{1, 2, \cdots, r\}$$

Similarly, the separation measurement of each alternative to MNIS is evaluated as follows:

$$\mathcal{S}_i^{\Psi^-} = \sqrt{\sum_{j=1}^{k}\left(\delta_{ij} - \delta_j^{\Psi^-}\right)^2}, \quad i \in \{1, 2, \cdots, r\}$$

Definition 3.10. Let $\mathcal{S}_i^{\Phi^+}$ and $\mathcal{S}_i^{\Psi^-}$ be separation measurements of MPIS and MNIS, respectively. The relative closeness of alternatives to the ideal solution is defined as:

$$\mathfrak{C}_i^{(\Phi^+,\Psi^-)} = \frac{S_i^{\Psi^-}}{S_i^{\Psi^-} + S_i^{\Phi^+}}, \quad i \in \{1, 2, \cdots, r\}$$

Here, $0 \leq \mathfrak{C}_i^{(\Phi^+,\Psi^-)} \leq 1$, for all $i \in \{1, 2, \cdots, r\}$. The larger value of $\mathfrak{C}_i^{(\Phi^+,\Psi^-)}$ corresponds to the most desirable alternative. It has the least distance from the MPIS and the highest distance from the MNIS.

4. An Algorithm for the Proposed MCGDM Problem

In this section, we present an algorithm for the developed TOPSIS-based MCGDM problem considered in Section 3. The related steps are outlined as follows:

Step 1. Take primary evaluations $\mathcal{Q}_i$ of experts $\mathcal{P}_i$ such that $i \in \{1, 2, \cdots, k\}$.

Step 2. Construct $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ using the real results.

Step 3. Determine $\underline{MBS}_{\mathcal{B}_q}(\mathcal{Q}_j)$ and $\overline{MBS}_{\mathcal{B}_q}(\mathcal{Q}_j)$, for $j \in \{1, 2, \cdots, k\}$ and $q \in \{1, 2, \cdots, r\}$, from Definition 2.6.

Step 4. Construct $\underline{M}$ and $\overline{M}$ according to Definition 3.1.

Step 5. Construct the aggregated parameter matrix from Definition 3.3.

Step 6. Compute the standardized decision matrix using Definition 3.4.

Step 7. Compute the normalized decision matrix according to Definition 3.6.

Step 8. Construct the average weighted normalized decision matrix using Definition 3.7.

Step 9. Determine the MPIS and the MNIS using Definition 3.8.

Step 10. According to Definition 3.9, calculate separation measurements of MPIS and MNIS for every alternative.

Step 11. Determine relative closeness of alternatives to ideal solutions using Definition 3.10.

Step 12. Ranking the preference order.

The flowchart of the above algorithm is displayed in Figure 1.

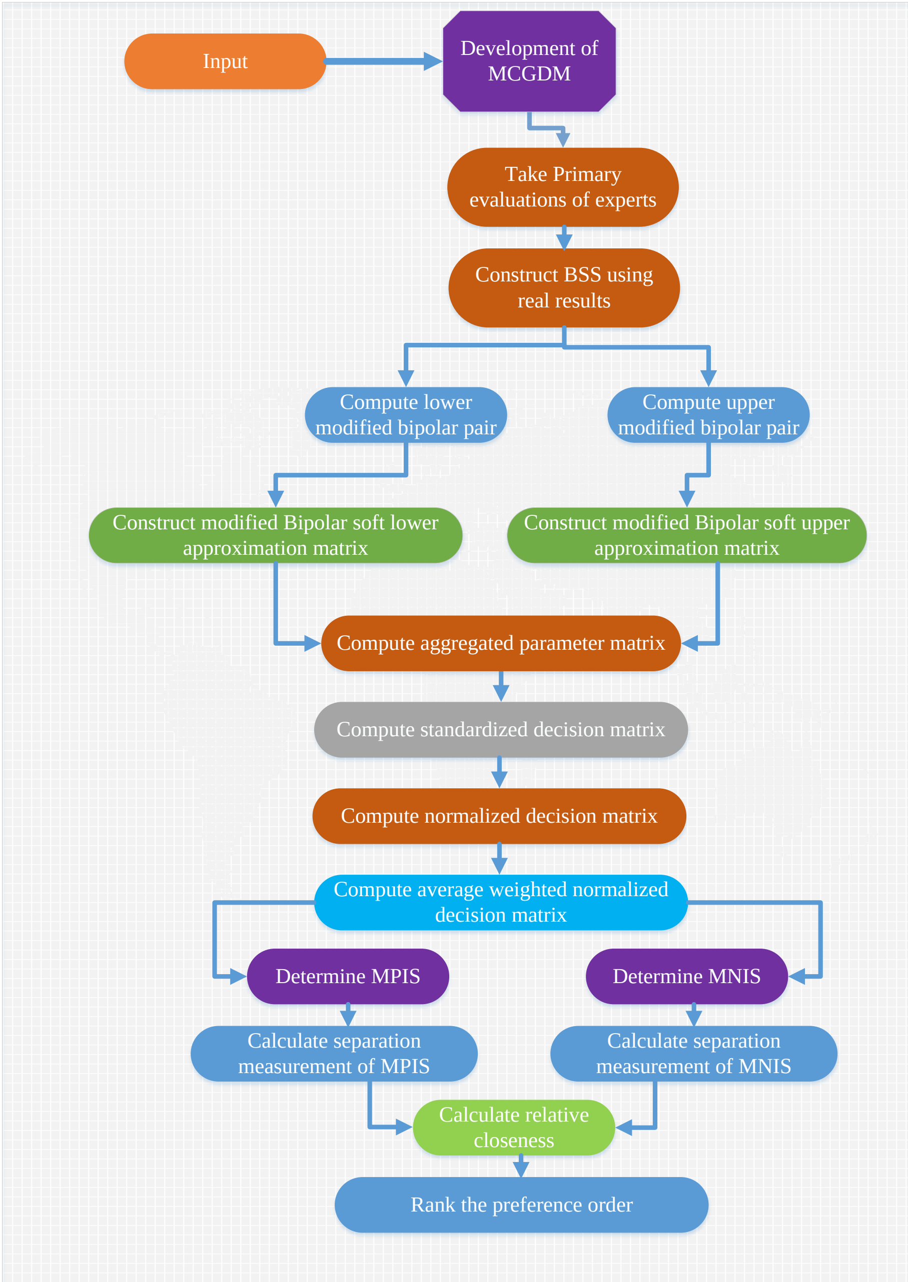

Figure 1. Flowchart of TOPSIS using MRBSs

5. Case Study

In this section, we discuss a design example of the MCGDM problem in MRBSs to illustrate the potential of the above-formulated TOPSIS method.

Example 5.1. Due to globalization's growing competition and mechanical upgrades, global markets are forcing companies to deliver top-quality things and services. This must be achieved through the participation of suitable employees. Employee selection is a procedure selection of people with the essential capabilities to perform a specific job at best. It chooses the information nature of employees and performs a crucial role in personnel management. Growing rivalry in worldwide markets encourages organizations to put greater emphasize on the recruitment process. Several companies determine the best job-hunter using rigorous and expensive identification methodologies. A candidate may be judged by various parameters such as managerial skills, ability to work under pressure, fluency in English, etc. It is wise to consult experts to accurately judge the candidates based on these parameters.

Assume that a production corporation wants to hire a marketing manager for a vacant post. Let $\mho = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ be the set of five candidates who might fit the marketing manager position at the production company. A panel of experts $\mathcal{G} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is set up to hire the most suitable candidate for this job. The panel will evaluate the candidates according to the set of parameters $\mathfrak{A} = \{\wp_1, \wp_2, \wp_3\}$ such that $\wp_1$ = managerial skills, $\wp_2$ = ability to work under pressure, and $\wp_3$ = fluency in English. The following calculations are performed to solve the MCGDM problem using the proposed methodology.

Step 1. The panel of experts $\mathcal{G} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ gives their primary evaluations for the candidates as:

$$\mathcal{Q}_1 = \{\mu_1, \mu_2, \mu_5\}, \quad \mathcal{Q}_2 = \{\mu_1, \mu_3, \mu_5\}, \quad \text{and} \quad \mathcal{Q}_3 = \{\mu_2, \mu_4, \mu_5\}$$

Step 2. Real results in three various times and places are displayed as BSSs $\mathcal{B}_1 = (\hat{f}_1, \hat{g}_1 : \mathfrak{A})$, $\mathcal{B}_2 = (\hat{f}_2, \hat{g}_2 : \mathfrak{A})$, and $\mathcal{B}_3 = (\hat{f}_3, \hat{g}_3 : \mathfrak{A})$ as follows:

$$\begin{aligned} \hat{f}_1 : \mathfrak{A} &\longrightarrow 2^{\mho} \\ \wp &\longmapsto \hat{f}_1(\wp) = \begin{cases} \{\mu_1\}, & \wp = \wp_1 \\ \{\mu_1, \mu_5\}, & \wp = \wp_2 \\ \{\mu_4, \mu_5\}, & \wp = \wp_3 \end{cases} \end{aligned} \qquad \begin{aligned} \hat{g}_1 : \widetilde{\mathfrak{A}} &\longrightarrow 2^{\mho} \\ \neg\wp &\longmapsto \hat{g}_1(\neg\wp) = \begin{cases} \{\mu_3, \mu_5\}, & \neg\wp = \neg\wp_1 \\ \{\mu_3\}, & \neg\wp = \neg\wp_2 \\ \{\mu_1, \mu_3\}, & \neg\wp = \neg\wp_3 \end{cases} \end{aligned}$$

$$\begin{aligned} \hat{f}_2 : \mathfrak{A} &\longrightarrow 2^{\mho} \\ \wp &\longmapsto \hat{f}_2(\wp) = \begin{cases} \{\mu_2\}, & \wp = \wp_1 \\ \{\mu_2, \mu_4\}, & \wp = \wp_2 \\ \{\mu_3, \mu_4\}, & \wp = \wp_3 \end{cases} \end{aligned} \qquad \begin{aligned} \hat{g}_2 : \widetilde{\mathfrak{A}} &\longrightarrow 2^{\mho} \\ \neg\wp &\longmapsto \hat{g}_2(\neg\wp) = \begin{cases} \{\mu_1, \mu_4\}, & \neg\wp = \neg\wp_1 \\ \{\mu_5\}, & \neg\wp = \neg\wp_2 \\ \{\mu_1, \mu_5\}, & \neg\wp = \neg\wp_3 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \hat{f}_3 : \mathfrak{A} &\longrightarrow 2^{\mho} \\ \wp &\longmapsto \hat{f}_3(\wp) = \begin{cases} \{\mu_3, \mu_5\}, & \wp = \wp_1 \\ \{\mu_2\}, & \wp = \wp_2 \\ \{\mu_2, \mu_5\}, & \wp = \wp_3 \end{cases} \end{aligned} \qquad \begin{aligned} \hat{g}_3 : \widetilde{\mathfrak{A}} &\longrightarrow 2^{\mho} \\ \neg\wp &\longmapsto \hat{g}_3(\wp) = \begin{cases} \{\mu_1, \mu_2\}, & \neg\wp = \neg\wp_1 \\ \{\mu_4\}, & \neg\wp = \neg\wp_2 \\ \{\mu_1, \mu_3\}, & \neg\wp = \neg\wp_3 \end{cases} \end{aligned}$$

Step 3. Using Definition 2.6, the LMBP and the UMBP for $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$ concerning $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are as follows:

$$\begin{aligned} \underline{MBS}_{\mathcal{B}_1}(\mathcal{Q}_1) &= (\{\mu_1, \mu_5\}, \{\mu_1, \mu_2, \mu_4, \mu_5\}) \\ \underline{MBS}_{\mathcal{B}_1}(\mathcal{Q}_2) &= (\{\mu_1, \mu_5\}, \{\mu_1, \mu_3, \mu_5\}) \\ \underline{MBS}_{\mathcal{B}_1}(\mathcal{Q}_3) &= (\{\mu_4, \mu_5\}, \{\mu_2, \mu_4, \mu_5\}) \end{aligned} \qquad \begin{aligned} \overline{MBS}_{\mathcal{B}_1}(\mathcal{Q}_1) &= (\{\mu_1, \mu_2, \mu_3, \mu_5\}, \{\mu_1, \mu_5\}) \\ \overline{MBS}_{\mathcal{B}_1}(\mathcal{Q}_2) &= (\{\mu_1, \mu_2, \mu_3, \mu_5\}, \{\mu_1, \mu_3, \mu_5\}) \\ \overline{MBS}_{\mathcal{B}_1}(\mathcal{Q}_3) &= (\{\mu_2, \mu_3, \mu_4, \mu_5\}, \{\mu_2, \mu_4, \mu_5\}) \end{aligned}$$

$$\begin{aligned} \underline{MBS}_{\mathcal{B}_2}(\mathcal{Q}_1) &= (\{\mu_1, \mu_2, \mu_5\}, \{\mu_1, \mu_5\}) \\ \underline{MBS}_{\mathcal{B}_2}(\mathcal{Q}_2) &= (\{\mu_1, \mu_3, \mu_5\}, \{\mu_1, \mu_2, \mu_3, \mu_5\}) \\ \underline{MBS}_{\mathcal{B}_2}(\mathcal{Q}_3) &= (\{\mu_2, \mu_4\}, \{\mu_2, \mu_3, \mu_4, \mu_5\}) \end{aligned} \qquad \begin{aligned} \overline{MBS}_{\mathcal{B}_2}(\mathcal{Q}_1) &= (\{\mu_1, \mu_2, \mu_5\}, \{\mu_1, \mu_2, \mu_3, \mu_5\}) \\ \overline{MBS}_{\mathcal{B}_2}(\mathcal{Q}_2) &= (\{\mu_1, \mu_3, \mu_5\}, \{\mu_1, \mu_5\}) \\ \overline{MBS}_{\mathcal{B}_2}(\mathcal{Q}_3) &= (\{\mu_1, \mu_2, \mu_4, \mu_5\}, \{\mu_4, \mu_5\}) \end{aligned}$$

and

$$\begin{aligned} \underline{MBS}_{\mathcal{B}_3}(\mathcal{Q}_1) &= (\{\mu_2, \mu_5\}, \{\mu_2, \mu_4, \mu_5\}) \\ \underline{MBS}_{\mathcal{B}_3}(\mathcal{Q}_2) &= (\{\mu_3, \mu_5\}, \{\mu_1, \mu_3, \mu_5\}) \\ \underline{MBS}_{\mathcal{B}_3}(\mathcal{Q}_3) &= (\{\mu_2, \mu_5\}, \{\mu_2, \mu_4, \mu_5\}) \end{aligned} \qquad \begin{aligned} \overline{MBS}_{\mathcal{B}_3}(\mathcal{Q}_1) &= (\{\mu_1, \mu_2, \mu_4, \mu_5\}, \{\mu_2, \mu_4, \mu_5\}) \\ \overline{MBS}_{\mathcal{B}_3}(\mathcal{Q}_2) &= (\{\mu_1, \mu_3, \mu_4, \mu_5\}, \{\mu_1, \mu_3, \mu_5\}) \\ \overline{MBS}_{\mathcal{B}_3}(\mathcal{Q}_3) &= (\{\mu_1, \mu_2, \mu_4, \mu_5\}, \{\mu_2, \mu_4, \mu_5\}) \end{aligned}$$

Step 4. Using Definition 3.1, the modified bipolar soft lower upper approximation matrices are obtained as:

$$\underline{M}=\begin{pmatrix}\left\langle(1,0,0,0,1),(-\frac{1}{2},-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2})\right\rangle & \left\langle(1,0,0,0,1),(-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2})\right\rangle & \left\langle(0,0,0,1,1),(0,-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2})\right\rangle\\ \left\langle(1,1,0,0,1),(-\frac{1}{2},0,0,0,-\frac{1}{2})\right\rangle & \left\langle(1,0,1,0,1),(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,-\frac{1}{2})\right\rangle & \left\langle(0,1,0,1,0),(0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})\right\rangle\\ \left\langle(0,1,0,0,1),(0,-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2})\right\rangle & \left\langle(0,0,1,0,1),(-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2})\right\rangle & \left\langle(0,1,0,0,1),(0,-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2})\right\rangle\end{pmatrix}$$

and

$$\overline{M}=\begin{pmatrix}\left\langle(\frac{1}{2},\frac{1}{2},\frac{1}{2},0,\frac{1}{2}),(-1,0,0,0,-1)\right\rangle & \left\langle(\frac{1}{2},\frac{1}{2},\frac{1}{2},0,\frac{1}{2}),(-1,0,-1,0,-1)\right\rangle & \left\langle(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),(0,-1,0,-1,-1)\right\rangle\\ \left\langle(\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}),(-1,-1,-1,0,-1)\right\rangle & \left\langle(\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}),(-1,0,0,0,-1)\right\rangle & \left\langle(\frac{1}{2},\frac{1}{2},0,\frac{1}{2},\frac{1}{2}),(0,0,0,-1,-1)\right\rangle\\ \left\langle(\frac{1}{2},\frac{1}{2},0,\frac{1}{2},\frac{1}{2}),(0,-1,0,-1,-1)\right\rangle & \left\langle(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2}),(-1,0,-1,0,-1)\right\rangle & \left\langle(\frac{1}{2},\frac{1}{2},0,\frac{1}{2},\frac{1}{2}),(0,-1,0,-1,-1)\right\rangle\end{pmatrix}$$

Step 5. According to Definition 3.3, the aggregated parameter matrix is constructed as:

$$A=\begin{pmatrix}\left\langle(1.5,0.5,0.5,0,1.5),(-1.5,-0.5,0,-0.5,-1.5)\right\rangle & \left\langle(1.5,0.5,0.5,0,1.5),(-1.5,0,-1.5,0,-1.5)\right\rangle & \left\langle(0,0.5,0.5,1.5,1.5),(0,-1.5,0,-1.5,-1.5)\right\rangle\\ \left\langle(1.5,1.5,0,0,1.5),(-1.5,-1,-1,0,-1.5)\right\rangle & \left\langle(1.5,0,1.5,0,1.5),(-1.5,-0.5,-0.5,0,-1.5)\right\rangle & \left\langle(0.5,1.5,0,1.5,0.5),(0,-0.5,-0.5,-1.5,-1.5)\right\rangle\\ \left\langle(0.5,1.5,0,0.5,1.5),(0,-1.5,0,-1.5,-1.5)\right\rangle & \left\langle(0.5,0,1.5,0.5,1.5),(-1.5,0,-1.5,0,-1.5)\right\rangle & \left\langle(0.5,1.5,0,0.5,1.5),(0,-1.5,0,-1.5,-1.5)\right\rangle\end{pmatrix}$$

Step 6. Compute standardized decision matrix using Definition 3.4, we have

$$S=\begin{pmatrix}\langle 3,-3\rangle & \langle 1.5,-2\rangle & \langle 1.5,-1.5\rangle & \langle 1.5,-2\rangle & \langle 4.5,-4.5\rangle\\ \langle 3.5,-3\rangle & \langle 3,-2\rangle & \langle 1.5,-2\rangle & \langle 1.5,-1.5\rangle & \langle 3.5,-4.5\rangle\\ \langle 1.5,-1.5\rangle & \langle 3,-3\rangle & \langle 1.5,-1.5\rangle & \langle 1.5,-3\rangle & \langle 4.5,-4.5\rangle\end{pmatrix}$$

Step 7. According to Definition 3.6, the normalized decision matrix can be determined as:

$$\aleph=\begin{pmatrix}\langle 0.619,-0.666\rangle & \langle 0.333,-0.485\rangle & \langle 0.577,-0.514\rangle & \langle 0.577,-0.512\rangle & \langle 0.620,-0.577\rangle\\ \langle 0.722,-0.666\rangle & \langle 0.666,-0.485\rangle & \langle 0.577,-0.686\rangle & \langle 0.577,-0.384\rangle & \langle 0.482,-0.577\rangle\\ \langle 0.309,-0.333\rangle & \langle 0.666,-0.728\rangle & \langle 0.577,-0.514\rangle & \langle 0.577,-0.768\rangle & \langle 0.620,-0.577\rangle\end{pmatrix}$$

Step 8. Using Definition 3.7, the weighted normalized decision matrix is obtained as follows:

$$\mathfrak{D}=\begin{pmatrix}0.643 & 0.818 & 0.546 & 0.545 & 0.599\\ 0.694 & 0.576 & 0.632 & 0.481 & 0.530\\ 0.321 & 0.697 & 0.546 & 0.673 & 0.599\end{pmatrix}$$

Step 9. According to Definition 3.8, MPIS and MNIS are obtained as follows:

$$MPIS=\{0.818, 0.694, 0.697\}$$

and

$$MNIS=\{0.545, 0.481, 0.321\}$$

Step 10. By Definition 3.9, the separation measurements of MPIS and MNIS for every parameter are calculated as:

$$\mathcal{S}_1^{\Phi^+} = 0.415 \qquad \mathcal{S}_1^{\Psi^-} = 0.234$$
$$\mathcal{S}_2^{\Phi^+} = 0.118 \qquad \mathcal{S}_2^{\Psi^-} = 0.474$$
$$\mathcal{S}_3^{\Phi^+} = 0.317 \qquad \mathcal{S}_3^{\Psi^-} = 0.271$$
$$\mathcal{S}_4^{\Phi^+} = 0.347 \qquad \mathcal{S}_4^{\Psi^-} = 0.352$$

and

$$\mathcal{S}_5^{\Phi^+} = 0.291 \qquad \mathcal{S}_5^{\Psi^-} = 0.287$$

Step 11. The relative closeness of each alternative to the ideal solution according to Definition 3.10 can be calculated as:

$$\mathfrak{C}_1^{(\Phi^+,\Psi^-)} = 0.361$$
$$\mathfrak{C}_2^{(\Phi^+,\Psi^-)} = 0.801$$
$$\mathfrak{C}_3^{(\Phi^+,\Psi^-)} = 0.461$$
$$\mathfrak{C}_4^{(\Phi^+,\Psi^-)} = 0.465$$

and

$$\mathfrak{C}_5^{(\Phi^+,\Psi^-)} = 0.497$$

Step 12. Ranking the preference order is given as:

$$\mu_2 \succeq \mu_5 \succeq \mu_4 \succeq \mu_3 \succeq \mu_1$$

This indicates that μ_2 is the optimal candidate for the marketing manager position. We also note that although the initial selection of three experts favored candidate μ_5 more, considering the previous three evaluations regarding BSS and the proposed TOPSIS method revealed a different ranking with more intelligence and insight into the given scenario. Note that "$\succeq$" is the symbol of the preference order of alternatives. The graphical display for the ranking of the candidates is also given in Figure 2.

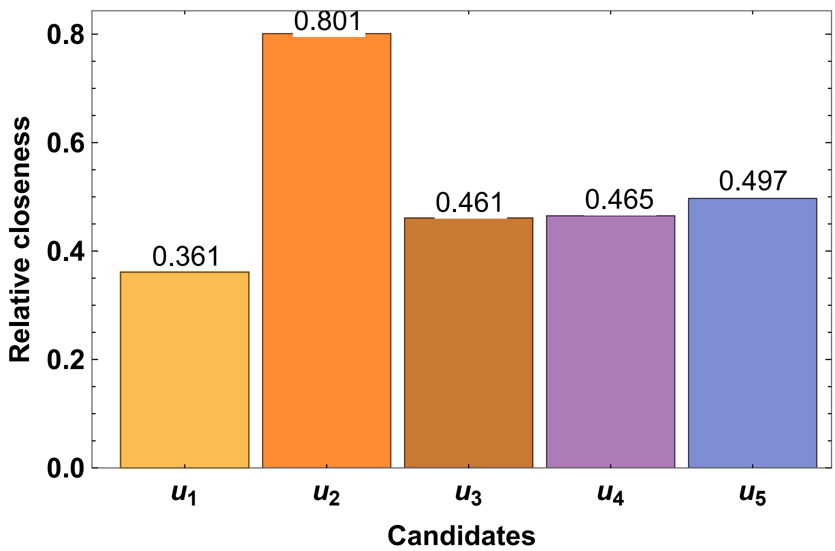

Figure 2. Graph for the ranking of candidates

6. Comparative Analysis and Discussion

In this section, we discuss the merits and drawbacks of the proposed technique and compare the suggested study with a few existing approaches.

6.1. Merits of the Proposed Model

Real-world MCGDM issues typically arise in a complex environment under ambiguous and imprecise data, which is tough to handle. The suggested model is highly appropriate for the considered problem when the information is complicated and uncertain, especially when the current information depends on the bipolar data by experts. Some advantages of the suggested approach are summarized as follows:

i. The suggested technique replicates each alternative's positive and negative characteristics as BSS. To manage aggressive DM, this integrated model is more comprehensive and suitable.

ii. This approach is also preferable because, in this method, the experts are free from any external constraints and requirements.

iii. There is no possibility of losing collective information throughout the process since aggregation is done in the final step.

iv. The established strategy not only takes experts' assessments but also integrates the previous experiences by the MRBS-approximations in real circumstances. Therefore, it is a more generalized approach for a better understanding available data and using artificial intelligence to make decisions.

6.2. Drawbacks of the Proposed Model

The suggested model has a few minor shortcomings, including its complicated structure and the massive information in the form of BSS. Such huge information is challenging to address because of enormous calculations, which are difficult to handle. However, one may establish MATLAB programming to ease these calculations simpler. Moreover, in the proposed model, parameters are independent of the environment. Therefore it cannot produce a ranking result when the parameters are dependent.

6.3. Comparison with Other Models

In this subsection, we compare the suggested strategy with TOPSIS approaches in fuzzy and bipolar fuzzy settings. Among the various MCDM approaches, the TOPSIS technique is the most favored one.

In the fuzzy TOPSIS technique, linguistic evaluations are used instead of numerical values. That is, the rating of the objects and the weights of criteria within the problem are evaluated utilizing fuzzy linguistic variables. Although the TOPSIS technique is the most effective approach in a fuzzy setting, it just gives us a mechanism to estimate the truth membership. On the other hand, the suggested TOPSIS technique offers a modified method for coping with MCGDM problems in which the subjective data is provided via a decision-maker in the form of BSS.

The researchers initiated and investigated bipolar fuzzy TOPSIS [30, 37] and extended the TOPSIS method based on IVHFNSSs [31]. It is generally known that the models can manage some DM problems to convey the idea of experts by using a crisp number. But, due to the uncertainty of the objective world and the complexity of the decision problems, they cannot address some group DM problems. For instance, some experts argue the membership degree of an object to a set and cannot compromise each other. One wants to assign 0.3, but the other prefers to choose 0.5. In this situation, MRBSs can be a perfect solution to this problem.

We explore the following points if we compare our proposed model with the TOPSIS techniques described in [27, 28, 38]. Firstly, these methods cannot address the bipolarity in the DM process, which is a critical feature of human cognition. Secondly, these techniques do not ensure harmony in decision-makers' opinions. Applying the most recent techniques presented in [18, 19] to Example 5.1 yields the following ranking results among the alternatives, displayed in Table 1.

Table 1. The ranking results of various methods to Example 5.1

Current Methods	Ranking Orders
Karaaslan and Çağman [18]	$\mu_5 \succeq \mu_2 \succeq \mu_3 \approx \mu_4 \succeq \mu_1$
Shabir and Gul [19]	$\mu_2 \approx \mu_1 \approx \mu_3 \succeq \mu_5 \succeq \mu_4$
Our proposed approach	$\mu_2 \succeq \mu_5 \succeq \mu_4 \succeq \mu_3 \succeq \mu_1$

A characteristics comparison of various approaches with suggested technique is given in Table 2. The comparison is evaluated with features: membership function (MF), non-membership function (NMF), parametrization, number of decision-makers, and ranking of alternatives.

Table 2. Characteristics comparison of different methods with proposed method

Methods	Characteristics				
	Handle MF	Handle NMF	Manage parametrization	Decision-makers	Ranking
Akram et al. [30]	Yes	Yes	No	One	Yes
Alghamdi et al. [37]	Yes	Yes	No	One	Yes
Eraslan and Karaaslan [28]	Yes	No	Yes	More than one	Yes
Feng [39]	Yes	No	Yes	More than one	Yes
Saeed et al. [40]	Yes	No	Yes	More than one	Yes
Sarwar [41]	Yes	No	No	More than one	Yes
Proposed Method	Yes	Yes	Yes	More than one	Yes

7. Conclusion

MRBSs are treated as practical tools for portraying the uncertainties and vagueness involved with the MCGDM problems. Thus decision-makers become more flexible in representing their judgment using MRBSs. In this work, we have presented a novel application of the MCGDM problem with the data having bipolarity and uncertainty. The framework is based on the TOPSIS method and MRBSs. We have defined a detailed mathematical procedure for the TOPSIS-based MRBSs method. The proposed approach integrates the strength of MRBSs theory in handling uncertainty and the advantage of the TOPSIS evaluation technique in MCGDM. An algorithm of DM is also established, which has two key benefits. Firstly, it evaluates the bipolarity of the data, containing uncertainty. Secondly, it considers the opinions of any (finite) number of experts about any (finite) number of objects. Additionally, we provide an application to demonstrate that the proposed strategy can effectively apply to specific issues, including uncertainty. At last, a comparative study of the suggested approach is conducted.

Numerous topics require further investigation. Bearing in mind the above, future perspectives will focus on the following:

i. The hybridization of the MRBS theory and more comprehensive selection models, such as VIKOR, ELECTRE, AHP, COPRAS, and PROMETHEE.

ii. The proposed method can be generalized to a fuzzy environment, and useful DM methods could be established.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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On the Hyperbolic Leonardo and Hyperbolic Francois Quaternions

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Abstract — In this paper, we present a new definition, referred to as the Francois sequence, related to the Lucas-like form of the Leonardo sequence. We also introduce the hyperbolic Leonardo and hyperbolic Francois quaternions. Afterward, we derive the Binet-like formulas and their generating functions. Moreover, we provide some binomial sums, Honsberger-like, d'Ocagne-like, Catalan-like, and Cassini-like identities of the hyperbolic Leonardo quaternions and hyperbolic Francois quaternions that allow an understanding of the quaternions' properties and their relation to the Francois sequence and Leonardo sequence. Finally, considering the results presented in this study, we discuss the need for further research in this field.

Keywords *Fibonacci numbers, Leonardo numbers, Lucas numbers, Francois numbers, hyperbolic quaternions*

Mathematics Subject Classification (2020) 11B39, 05A15

1. Introduction

The algebra of hyperbolic quaternions in abstract algebra is a non-associative algebra over real numbers with elements of the form

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

where q_0, q_1, q_2, and q_3 are real numbers and e_0, e_1, e_2, and e_3 are the standard basis in $\mathbb{R}^4$. The hyperbolic quaternion multiplication is defined using the rules

$$e_0^2 = e_1^2 = e_2^2 = e_3^2 = 1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2$$

This algebra is also non-commutative. Let $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ and $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ be any two hyperbolic quaternions. Then, the addition and subtraction of the hyperbolic quaternions are

$$q \mp p = (q_0 \mp p_0)e_0 + (q_1 \mp p_1)e_1 + (q_2 \mp p_2)e_2 + (q_3 \mp p_3)e_3$$

and multiplication of the hyperbolic quaternions is

$$\begin{aligned} qp &= (q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3)(p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3) \\ &= (q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3)e_0 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)e_1 \\ &+ (q_0p_2 - q_2p_0 + q_1p_3 + q_3p_1)e_2 + (q_0p_3 + q_3p_0 - q_1p_2 + q_2p_1)e_3 \end{aligned}$$

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Moreover, for $k \in \mathbb{R}$, the multiplication by scalar is

$$kq = kq_0e_0 + kq_1e_1 + kq_2e_2 + kq_3e_3$$

and conjugate and norm of the hyperbolic quaternion q are

$$\overline{q} = q_0e_0 - q_1e_1 - q_2e_2 - q_3e_3$$

and

$$||q|| = \sqrt{|q\overline{q}|} = \sqrt{q_0^2 - q_1^2 - q_2^2 - q_3^2}$$

respectively. One of the non-associative hyperbolic number systems, ideal for studying space-time theories in relativities, is the hyperbolic quaternions. Many studies have been published on hyperbolic quaternions. Macfarlane yields the hyperbolic counterpart of the spherical quaternions in [1]. Kösal introduces hyperbolic quaternions and their algebraic properties in [2]. The four-dimensional real algebra of bihyperbolic numbers is studied by Bilgin and Ersoy in [3]. An alternative representational method is proposed for the formulation of classical and generalized electromagnetism in the case of the existence of magnetic monopoles and massive photons after presenting the hyperbolic quaternion formalism by Demir et al. in [4]. Kuruz introduces hyperbolic matrices with hyperbolic number entries in [5]. Assis presents some properties of mathematical and physical interest in generalized algebras of two, three, and four dimensions in [6]. The Fibonacci and Lucas sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are defined by two order recurrences, respectively,

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \tag{1}$$

and

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \tag{2}$$

Here, F_n and L_n are the nth Fibonacci and Lucas numbers. First few terms of these sequences are, respectively,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

and

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199$$

The Recurrences 1 or 2 involve the characteristic equation

$$x^2 - x - 1 = 0 \tag{3}$$

The roots of Equation 3 are

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} \tag{4}$$

Then, the following relations can be derived

$$\alpha + \beta = 1 \quad \alpha - \beta = \sqrt{5} \quad \alpha\beta = -1$$

Therefore, the Binet formulas for the Fibonacci and Lucas sequences are, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

More information for the Fibonacci and Lucas numbers are given in [7, 8]. The Leonardo sequence $\{\mathcal{L}_n\}_{n\geq 0}$ is defined by recurrence

$$\mathcal{L}_0 = 1, \quad \mathcal{L}_1 = 1, \quad \text{and} \quad \mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n + 1$$

where $\mathcal{L}_n$ is the nth Leonardo number. An expression of the relationship between Leonardo and Fibonacci numbers is

$$\mathcal{L}_n = 2F_{n+1} - 1, \quad n \geq 0$$

The Binet-like formula for the Leonardo sequence is

$$\mathcal{L}_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1 \tag{5}$$

where α and β are given Equation 4. Other studies about Leonardo numbers can be listed in [9–15].

2. The Francois Numbers

This section presents a new definition, called the Francois sequence, related to the Lucas-like form of the Leonardo sequence as follows:

Definition 2.1. The Francois sequence $\{\mathcal{F}_n\}$ is defined by

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + 1, \quad n \geq 2 \tag{6}$$

with initial conditions $\mathcal{F}_0 = 2$ and $\mathcal{F}_1 = 1$. Here, $\mathcal{F}_n$ is the nth Francois number.

First few terms of this sequence are 2, 1, 4, 6, 11, 18, 30, 49, 80, 130, 211. The Recurrence 6 can also be written as follows

$$\mathcal{F}_{n+3} = 2\mathcal{F}_{n+2} - \mathcal{F}_n \tag{7}$$

In fact, by the equalities $\mathcal{F}_{n+3} = \mathcal{F}_{n+2} + \mathcal{F}_{n+1} + 1$ and $\mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n + 1$, we reach Equation 7.

Equation 7 satisfies the characteristic equation

$$t^3 - 2t^2 + 1 = 0 \tag{8}$$

The roots of Equation 8 are 1, α, and β. Here, the other roots except 1 are the same as those of Equation 3. Taking $\mathcal{F}_0 = 2$, $\mathcal{F}_1 = 1$, and $\mathcal{F}_2 = 4$, we can easily reach the following result.

Theorem 2.2. The Binet-like formula for the Francois sequence is

$$\mathcal{F}_n = \alpha^n + \beta^n + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 1, \quad n \geq 0 \tag{9}$$

where α and β are given in Equation 4.

Proof.
Assume that $\mathcal{F}_n = a\alpha^n + b\beta^n + c$. Thus, we have

$$\mathcal{F}_0 = a + b + c = 2$$

$$\mathcal{F}_1 = a\alpha + b\beta + c = 1$$

and

$$\mathcal{F}_2 = a\alpha^2 + b\beta^2 + c = 4$$

By performing the solution with the Gaussian elimination method, we can find that

$$a = 1 + \frac{\alpha}{\alpha - \beta}, \quad b = 1 - \frac{\beta}{\alpha - \beta}, \quad \text{and} \quad c = -1$$

This proof is complete. □

Theorem 2.3. For $n \geq 0$, the following identity is valid:

$$\mathcal{F}_n = L_n + F_{n+1} - 1, \quad n \geq 0$$

Proof.
The proof is clear by Theorem 2.2. □

Studies similar to the Leonardo and Francois numbers can be seen in [9, 16–18]. The hyperbolic Fibonacci and hyperbolic Lucas quaternions are defined as follows, respectively,

$$HF_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3$$

and

$$HL_n = L_n e_0 + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3$$

The Binet-like formulas for the hyperbolic Fibonacci and hyperbolic Lucas quaternions are as the form, respectively,

$$HF_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \tag{10}$$

and

$$HL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n \tag{11}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

and

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$$

The hyperbolic Fibonacci and hyperbolic Lucas quaternions and some of their generalizations are given in [19–23].

3. Hyperbolic Leonardo and Hyperbolic Francois Quaternions

In this section, we define the hyperbolic Leonardo and hyperbolic Francois quaternions, and we provide their Binet-like formulas and generating functions. Then, we obtain certain binomial sums, Honsberger-like, d'Ocagne-like, Catalan-like, and Cassini-like identities of the hyperbolic Leonardo quaternions.

Definition 3.1. The hyperbolic Leonardo quaternion sequence $\{\mathcal{HL}_n\}_{n\geq 0}$ is defined by

$$\mathcal{HL}_n = \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3 \tag{12}$$

where Le_n is the nth Leonardo number and e_0, e_1, e_2, and e_3 are units of the hyperbolic quaternions.

Definition 3.2. The hyperbolic Francois quaternion sequence $\{\mathcal{HF}_n\}_{n\geq 0}$ is defined by

$$\mathcal{HF}_n = \mathcal{F}_n e_0 + \mathcal{F}_{n+1} e_1 + \mathcal{F}_{n+2} e_2 + \mathcal{F}_{n+3} e_3$$

where $\mathcal{F}_n$ is the nth Francois number and e_0, e_1, e_2, and e_3 are units of the hyperbolic quaternions.

Theorem 3.3. (Binet-like Formula) The Binet-like formula for the hyperbolic Leonardo quaternions is

$$\mathcal{HL}_n = 2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}, \quad n \geq 0 \tag{13}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$$

and

$$\hat{1} = e_0 + e_1 + e_2 + e_3$$

PROOF.
From Identities 5 and 12,

$$\begin{aligned}
\mathcal{HL}_n &= \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3 \\
&= \left[2\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)-1\right]e_0 + \left[2\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)-1\right]e_1 + \left[2\left(\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\right)-1\right]e_2 \\
&\quad + \left[2\left(\frac{\alpha^{n+4}-\beta^{n+4}}{\alpha-\beta}\right)-1\right]e_3 \\
&= 2\left(\frac{\alpha^{n+1}}{\alpha-\beta}(e_0+\alpha e_1+\alpha^2 e_2+\alpha^3 e_3) - \frac{\beta^{n+1}}{\alpha-\beta}(e_0+\beta e_1+\beta^2 e_2+\beta^3 e_3)\right) - (e_0+e_1+e_2+e_3) \\
&= 2\left(\frac{\hat{\alpha}\alpha^{n+1}-\hat{\beta}\beta^{n+1}}{\alpha-\beta}\right) - \hat{1}
\end{aligned}$$

is obtained. □

Note that the hyperbolic Leonardo quaternion sequence can be expressed in terms of the hyperbolic Fibonacci quaternion as:

$$\mathcal{HL}_n = 2HF_{n+1} - \hat{1}, \qquad n \geq 0$$

where HF_n is nth the hyperbolic Fibonacci quaternion.

Theorem 3.4. (Binet-like Formula) The Binet-like formula for the hyperbolic Francois quaternions is

$$\mathcal{HF}_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \frac{\hat{\alpha}\alpha^{n+1}-\hat{\beta}\beta^{n+1}}{\alpha-\beta} - \hat{1}, \quad n \geq 0 \tag{14}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$$

and

$$\hat{1} = e_0 + e_1 + e_2 + e_3$$

PROOF.
It is proved similarly to the proof of Theorem 3.3. □

Note that the hyperbolic Francois quaternion sequence can be expressed in terms of the hyperbolic Fibonacci and hyperbolic Lucas quaternion as:

$$\mathcal{HF}_n = HL_n + HF_{n+1} - \hat{1}, \quad n \geq 0$$

where HL_n and HF_n is nth the hyperbolic Lucas and hyperbolic Fibonacci quaternions, respectively.

Theorem 3.5. (Generating Function) The generating function for the hyperbolic Leonardo quaternions is

$$\mathcal{G}_{\mathcal{HL}}(x)=\frac{A-Bx+Cx^2}{1-2x+x^3}$$

where

$$A=e_0+e_1+3e_2+5e_3$$

$$B=e_0-e_1+e_2+e_3$$

and

$$C=e_0-e_1-e_2-3e_3$$

Proof.
Let

$$\mathcal{G}_{\mathcal{HL}}(x)=\sum_{n=0}^{\infty}\mathcal{HL}_n x^n=\mathcal{HL}_0+\mathcal{HL}_1x+\mathcal{HH}_2x^2+\mathcal{HL}_3x^3+\ldots+\mathcal{HL}_nx^n+\ldots$$

be generating function of the hyperbolic Leonardo quaternions. Assume that multiply every side of the expansions above by $-2x$ and x^3 as follows:

$$-2x\mathcal{G}_{\mathcal{HL}}(x)=-2\mathcal{HL}_0x-2\mathcal{HL}_1x^2-2\mathcal{HL}_2x^3-2\mathcal{HL}_3x^4-\ldots-2\mathcal{HL}_nx^{n+1}-\ldots$$

and

$$x^3\mathcal{G}_{\mathcal{HL}}(x)=\mathcal{HL}_0x^3+\mathcal{HL}_1x^4+\mathcal{HH}_2x^5+\mathcal{HL}_3x^6+\ldots+\mathcal{HL}_nx^{n+3}+\ldots$$

Then, we write

$$\begin{aligned}(1-2x+x^3)\mathcal{G}_{\mathcal{HL}}(x)=&\mathcal{HL}_0+(\mathcal{HL}_1-2\mathcal{HL}_0)x+(\mathcal{HL}_2-2\mathcal{HL}_1)x^2+(\mathcal{HL}_3-2\mathcal{HL}_2+\mathcal{HL}_0)x^3+\ldots\\&+(\mathcal{HL}_n-2\mathcal{HL}_{n-1}+\mathcal{HL}_{n-3})x^n+\ldots\end{aligned}$$

By using the values,

$$\mathcal{HL}_0=e_0+e_1+3e_2+5e_3$$

$$\mathcal{HL}_1=e_0+3e_1+5e_2+9e_3$$

$$\mathcal{HL}_2=3e_0+5e_1+9e_2+15e_3$$

$$\mathcal{HL}_3=5e_0+9e_1+15e_2+25e_3$$

and

$$\mathcal{HL}_n-2\mathcal{HL}_{n-1}+\mathcal{HL}_{n-3}=0$$

are obtained. □

Theorem 3.6. (Generating Function) The generating function for the hyperbolic Francois quaternions is

$$\mathcal{G}_{\mathcal{HF}}(x)=\frac{E-Fx+Gx^2}{1-2x+x^3}$$

where

$$E=2e_0+e_1+4e_2+6e_3$$

$$F=3e_0-2e_1+2e_2+e_3$$

and

$$G=2e_0-2e_1-e_2-4e_3$$

Proof.
The proof is similar to one of Theorem 3.5. □

Theorem 3.7. (Exponential Generating Function) The exponential generating function for the hyperbolic Leonardo quaternions is

$$\mathcal{E}_{\mathcal{HL}}(x) = 2\left(\frac{\hat{\alpha}\alpha e^{\alpha x} - \hat{\beta}\beta e^{\beta x}}{\alpha - \beta}\right) - \hat{1}e^{x}$$

Proof.
Using Equation 13,

$$\begin{aligned}
\mathcal{E}_{\mathcal{HL}}(x) &= \sum_{n=0}^{\infty} \mathcal{HL}_n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}\right)\frac{x^n}{n!} \\
&= \frac{2\hat{\alpha}\alpha}{\alpha - \beta}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!} - \frac{2\hat{\beta}\beta}{\alpha - \beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!} - \hat{1}\sum_{n=0}^{\infty}\frac{x^n}{n!} \\
&= 2\left(\frac{\hat{\alpha}\alpha e^{\alpha x} - \hat{\beta}\beta e^{\beta x}}{\alpha - \beta}\right) - \hat{1}e^{x}
\end{aligned}$$

is obtained. □

Theorem 3.8. (Exponential Generating Function) The exponential generating function for the hyperbolic Francois quaternions is

$$\mathcal{E}_{\mathcal{HF}}(x) = \hat{\alpha}\alpha e^{\alpha x} + \hat{\beta}\beta e^{\beta x} + \frac{\hat{\alpha}\alpha e^{\alpha x} - \hat{\beta}\beta e^{\beta x}}{\alpha - \beta} - \hat{1}e^{x}$$

Proof.
The proof is similar to one of Theorem 3.7. □

Theorem 3.9. (Binomial Sum) Let m be a positive integer. Then,

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{HL}_n = \mathcal{HL}_{2m} + \hat{1}(1 - 2^m)$$

Proof.
Considering Equations 3 and 13 and the binomial formula,

$$\begin{aligned}
\sum_{n=0}^{m}\binom{m}{n}\mathcal{HL}_n &= \sum_{n=0}^{m}\binom{m}{n}\left[2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}\right] \\
&= \frac{2\hat{\alpha}\alpha}{\alpha - \beta}\sum_{n=0}^{m}\binom{m}{n}\alpha^n - \frac{2\hat{\beta}\beta}{\alpha - \beta}\sum_{n=0}^{m}\binom{m}{n}\beta^n - \hat{1}\sum_{n=0}^{m}\binom{m}{n} \\
&= \frac{2\hat{\alpha}\alpha}{\alpha - \beta}(\alpha + 1)^m - \frac{2\hat{\beta}\beta}{\alpha - \beta}(\beta + 1)^m - \hat{1}2^m \\
&= 2\left(\frac{\hat{\alpha}\alpha^{2m+1} - \hat{\beta}\beta^{2m+1}}{\alpha - \beta}\right) - \hat{1} + \hat{1} - \hat{1}2^m \\
&= \mathcal{HL}_{2m} + \hat{1}(1 - 2^m)
\end{aligned}$$

is obtained. □

Corollary 3.10. Let m be a positive integer. Then,

$$\sum_{k=0}^{m}\binom{m}{k}\mathcal{HL}_{n-k}=\mathcal{HL}_{n+m}+\hat{1}(1-2^m), \qquad n\geq 0$$

Proof.
Considering Equations 3 and 13 and the binomial formula,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\mathcal{HL}_{n-k} &= \sum_{k=0}^{m}\binom{m}{k}\left[2\left(\frac{\hat{\alpha}\alpha^{n-k+1}-\hat{\beta}\beta^{n-k+1}}{\alpha-\beta}\right)-\hat{1}\right]\\ &= \sum_{k=0}^{m}\binom{m}{k}\left[2\left(\frac{\hat{\alpha}\alpha^{m-k}\alpha^{n-m+1}-\hat{\beta}\beta^{m-k}\beta^{n-m+1}}{\alpha-\beta}\right)-\hat{1}\right]\\ &= \frac{2\hat{\alpha}\alpha^{n-m+1}}{\alpha-\beta}\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}1^k-\frac{2\hat{\beta}\beta^{n-m+1}}{\alpha-\beta}\sum_{k=0}^{m}\binom{m}{k}\beta^{m-k}1^k-\hat{1}\sum_{k=0}^{m}\binom{m}{k}\\ &= \frac{2\hat{\alpha}\alpha^{n-m+1}}{\alpha-\beta}(\alpha+1)^m-\frac{2\hat{\beta}\beta^{n-m+1}}{\alpha-\beta}(\beta+1)^m-\hat{1}2^m\\ &= 2\left(\frac{\hat{\alpha}\alpha^{n+m+1}-\hat{\beta}\beta^{n+m+1}}{\alpha-\beta}\right)-\hat{1}+\hat{1}-\hat{1}2^m\\ &= \mathcal{HL}_{n+m}+\hat{1}(1-2^m)\end{aligned}$$

is obtained. □

Some identities, such as Honsberger, dOcagne, Catalan, and Cassini identities for Fibonacci and its generating, have been studied by many authors (see [19,24,25]). Here, we obtain similar identities for the hyperbolic Leonardo quaternion.

Theorem 3.11. (Honsberger-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. The following relation is satisfied:

$$\mathcal{HL}_{n+1}\mathcal{HL}_m+\mathcal{HL}_n\mathcal{HL}_{m-1}=4\left(\frac{\hat{\alpha}^2\alpha^{n+m}-\hat{\beta}^2\beta^{n+m}}{\alpha-\beta}\right)-\hat{1}\left(\mathcal{HL}_{n+1}+\mathcal{HL}_m\right), \quad n,m\geq 0$$

Proof.
Using Equation 13,

$$\begin{aligned}\mathcal{HL}_{n+1}\mathcal{HL}_m+\mathcal{HL}_n\mathcal{HL}_{m-1} &= \left[2\left(\tfrac{\hat{\alpha}\alpha^{n+1}-\hat{\beta}\beta^{n+1}}{\alpha-\beta}\right)-\hat{1}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{m}-\hat{\beta}\beta^{m}}{\alpha-\beta}\right)-\hat{1}\right]\\ &\quad+\left[2\left(\tfrac{\hat{\alpha}\alpha^{n}-\hat{\beta}\beta^{n}}{\alpha-\beta}\right)-\hat{1}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{m-1}-\hat{\beta}\beta^{m-1}}{\alpha-\beta}\right)-\hat{1}\right]\\ &= 4\left(\tfrac{(\hat{\alpha})^2\alpha^{n+m+1}-\hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^m-\hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^m+(\hat{\beta})^2\beta^{n+m+1}}{(\alpha-\beta)^2}\right)\\ &\quad-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n+1}-\hat{\beta}\beta^{n+1}}{\alpha-\beta}\right)-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{m}-\hat{\beta}\beta^{m}}{\alpha-\beta}\right)+\hat{1}^2\\ &\quad+4\left(\tfrac{(\hat{\alpha})^2\alpha^{n+m-1}-\hat{\alpha}\hat{\beta}\alpha^{n}\beta^{m-1}-\hat{\beta}\hat{\alpha}\beta^{n}\alpha^{m-1}+(\hat{\beta})^2\beta^{n+m-1}}{(\alpha-\beta)^2}\right)\\ &\quad-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n}-\hat{\beta}\beta^{n}}{\alpha-\beta}\right)-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{m-1}-\hat{\beta}\beta^{m-1}}{\alpha-\beta}\right)+\hat{1}^2\\ &= 4\left(\frac{\hat{\alpha}^2\alpha^{n+m}-\hat{\beta}^2\beta^{n+m}}{\alpha-\beta}\right)-\hat{1}\left(\mathcal{HL}_{n+1}+\mathcal{HL}_m\right)\end{aligned}$$

is obtained. □

Theorem 3.12. (d'Ocagne-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. For $n, m \geq 0$,

$$\mathcal{HL}_m\mathcal{HL}_{n+1} - \mathcal{HL}_{m+1}\mathcal{HL}_n = \hat{1}\left(\mathcal{HL}_{m+1} + \mathcal{HL}_n - \mathcal{HL}_m - \mathcal{HL}_{n+1}\right) + 4\left(\frac{\hat{\alpha}\hat{\beta}\alpha^m\beta^n - \hat{\beta}\hat{\alpha}\beta^m\alpha^n}{\alpha-\beta}\right) + 2\hat{1}^2$$

Proof.
Using Equation 13,

$$\begin{aligned}
\mathcal{HL}_m\mathcal{HL}_{n+1} - \mathcal{HL}_{m+1}\mathcal{HL}_n &= \left[2\left(\frac{\hat{\alpha}\alpha^{m+1} - \hat{\beta}\beta^{m+1}}{\alpha-\beta}\right) - \hat{1}\right]\left[2\left(\frac{\hat{\alpha}\alpha^{n+2} - \hat{\beta}\beta^{n+2}}{\alpha-\beta}\right) - \hat{1}\right] \\
&\quad - \left[2\left(\frac{\hat{\alpha}\alpha^{m+2} - \hat{\beta}\beta^{m+2}}{\alpha-\beta}\right) - \hat{1}\right]\left[2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha-\beta}\right) - \hat{1}\right] \\
&= 4\left(\frac{(\hat{\alpha})^2\alpha^{n+m+3} - \hat{\alpha}\hat{\beta}\alpha^{m+1}\beta^{n+2} - \hat{\beta}\hat{\alpha}\beta^{m+1}\alpha^{n+2} + (\hat{\beta})^2\beta^{n+m+3}}{(\alpha-\beta)^2}\right) \\
&\quad - 2\hat{1}\left(\frac{\hat{\alpha}\alpha^{m+1} - \hat{\beta}\beta^{m+1}}{\alpha-\beta}\right) - 2\hat{1}\left(\frac{\hat{\alpha}\alpha^{n+2} - \hat{\beta}\beta^{n+2}}{\alpha-\beta}\right) + \hat{1}^2 \\
&\quad - 4\left(\frac{(\hat{\alpha})^2\alpha^{n+m+3} - \hat{\alpha}\hat{\beta}\alpha^{m+2}\beta^{n+1} - \hat{\beta}\hat{\alpha}\beta^{m+2}\alpha^{n+1} + (\hat{\beta})^2\beta^{n+m+3}}{(\alpha-\beta)^2}\right) \\
&\quad + 2\hat{1}\left(\frac{\hat{\alpha}\alpha^{m+2} - \hat{\beta}\beta^{m+2}}{\alpha-\beta}\right) + 2\hat{1}\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha-\beta}\right) - \hat{1}^2 \\
&= \hat{1}\left(\mathcal{HL}_{m+1} + \mathcal{HL}_n - \mathcal{HL}_m - \mathcal{HL}_{n+1}\right) + 4\left(\frac{\hat{\alpha}\hat{\beta}\alpha^m\beta^n - \hat{\beta}\hat{\alpha}\beta^m\alpha^n}{\alpha-\beta}\right) + 2\hat{1}^2
\end{aligned}$$

is obtained. □

Theorem 3.13. (Catalan-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. For $n \geq r \geq 0$, the following relation is satisfied:

$$\mathcal{HL}_{n-r}\mathcal{HL}_{n+r} - \mathcal{HL}_n^2 = (-1)^{n-r}\left(\frac{\hat{\alpha}\hat{\beta}\beta^r + \hat{\beta}\hat{\alpha}\alpha^r}{\alpha-\beta}\right)F_r + \hat{1}\left(2\mathcal{HL}_n - \mathcal{HL}_{n-r} - \mathcal{HL}_{n+r}\right) + 2\hat{1}^2$$

Proof.
Using Equality 13,

$$\begin{aligned}
\mathcal{HL}_{n-r}\mathcal{HL}_{n+r} - \mathcal{HL}_n^2 &= \left[2\left(\tfrac{\hat{\alpha}\alpha^{n-r} - \hat{\beta}\beta^{n-r}}{\alpha-\beta}\right) - \hat{1}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha-\beta}\right) - \hat{1}\right] \\
&\quad - \left[2\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) - \hat{1}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) - \hat{1}\right] \\
&= 4\left(\tfrac{(\hat{\alpha})^2\alpha^{2n} - \hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r} - \hat{\beta}\hat{\alpha}\beta^{n-r}\alpha^{n+r} + (\hat{\beta})^2\beta^{2n}}{(\alpha-\beta)^2}\right) \\
&\quad - 2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n-r} - \hat{\beta}\beta^{n-r}}{\alpha-\beta}\right) - 2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha-\beta}\right) + \hat{1}^2 \\
&\quad - 4\left(\tfrac{(\hat{\alpha})^2\alpha^{2n} - \hat{\alpha}\hat{\beta}\alpha^{n}\beta^{n} - \hat{\beta}\hat{\alpha}\beta^{n}\alpha^{n} + (\hat{\beta})^2\beta^{2n}}{(\alpha-\beta)^2}\right) \\
&\quad + 2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) + 2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) - \hat{1}^2 \\
&= (-1)^{n-r}\left(\frac{\hat{\alpha}\hat{\beta}\beta^r + \hat{\beta}\hat{\alpha}\alpha^r}{\alpha-\beta}\right)F_r + \hat{1}\left(2\mathcal{HL}_n - \mathcal{HL}_{n-r} - \mathcal{HL}_{n+r}\right) + 2\hat{1}^2
\end{aligned}$$

is obtained. □

Corollary 3.14. (Cassini-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. The following relation is satisfied:

$$\mathcal{HL}_{n-1}\mathcal{HL}_{n+1} - \mathcal{HL}_n^2 = (-1)^{n-1}\left(\frac{\hat{\alpha}\hat{\beta}\beta + \hat{\beta}\hat{\alpha}\alpha}{\alpha - \beta}\right) + \hat{1}\left(2\mathcal{HL}_n - \mathcal{HL}_{n-1} - \mathcal{HL}_{n+1}\right) + 2\hat{1}^2, \quad n \geq 0$$

Proof.
We take 1 instead of r in Theorem 3.13 to prove this theorem. □

Proofs of the following propositions can be easily proved using Equations 5, 9–11, 13, and 14.

Proposition 3.15. For $n \geq 0$, the following identities are valid:

i. $\mathcal{HL}_{n+1}e_0 - \mathcal{HL}_{n+2}e_1 - \mathcal{HL}_{n+3}e_2 - \mathcal{HL}_{n+4}e_3 = -2\mathcal{L}_{n+5} - 3$

ii. $\mathcal{HL}_{n+1}e_0 + \mathcal{HL}_{n+2}e_1 + \mathcal{HL}_{n+3}e_2 + \mathcal{HL}_{n+4}e_3 = 2\mathcal{HL}_n + 2\mathcal{L}_{n+5} + 3$

Proposition 3.16. The following identities are valid:

i. $\mathcal{HL}_{n+r}F_{n+r} = \frac{2}{5}\left(HL_{2n+2r+1} - (-1)^{n+r}HL_1\right) - \hat{1}F_{n+r}, \quad n, r \geq 0$

ii. $\mathcal{HL}_{n-r}F_{n-r} = \frac{2}{5}\left(HL_{2n-2r+1} - (-1)^{n-r}HL_1\right) - \hat{1}F_{n-r}, \quad n \geq r \geq 0$

iii. $\mathcal{HL}_{n-r}F_{n+r} = \frac{2}{5}\left(HL_{2n+1} - (-1)^{n+r}HL_{1-2r}\right) - \hat{1}F_{n-r}, \quad n \geq r \geq 0$

iv. $\mathcal{HL}_{n+r}F_{n-r} = \frac{2}{5}\left(HL_{2n+1} - (-1)^{n-r}HL_{2r+1}\right) - \hat{1}F_{n+r}, \quad n \geq r \geq 0$

v. $\mathcal{HL}_{n+r}L_{n+r} = 2\left(HF_{2n+2r+1} + (-1)^{n+r}HF_1\right) - \hat{1}L_{n+r}, \quad n, r \geq 0$

vi. $\mathcal{HL}_{n-r}L_{n-r} = 2\left(HF_{2n-2r+1} + (-1)^{n-r}HF_1\right) - \hat{1}L_{n-r}, \quad n \geq r \geq 0$

vii. $\mathcal{HL}_{n-r}L_{n+r} = 2\left(HF_{2n+1} + (-1)^{n-r}HF_{2r+1}\right) - \hat{1}L_{n+r}, \quad n \geq r \geq 0$

viii. $\mathcal{HL}_{n+r}L_{n-r} = 2\left(HF_{2n+1} + (-1)^{n+r}HF_{1-2r}\right) - \hat{1}L_{n-r}, \quad n \geq r \geq 0$

ix. $\mathcal{HF}_{n+r}F_{n+r} = HF_{2n+2r} - (-1)^{n+r}HF_0 + \frac{1}{5}\left(HL_{2n+2r+1} - (-1)^{n+r}HL_1\right) - \hat{1}F_{n+r}, \quad n, r \geq 0$

x. $\mathcal{HF}_{n-r}F_{n-r} = HF_{2n-2r} - (-1)^{n-r}HF_0 + \frac{1}{5}\left(HL_{2n-2r+1} - (-1)^{n-r}HL_1\right) - \hat{1}F_{n+r}, \quad n \geq r \geq 0$

xi. $\mathcal{HF}_{n-r}F_{n+r} = HF_{2n} - (-1)^{n+r}HF_{-2r} + \frac{1}{5}\left(HL_{2n+1} - (-1)^{n+r}HL_{1-2r}\right) - \hat{1}F_{n+r}, \quad n \geq r \geq 0$

xii. $\mathcal{HF}_{n+r}F_{n-r} = HF_{2n} - (-1)^{n-r}HF_{2r} + \frac{1}{5}\left(HL_{2n+1} - (-1)^{n-r}HL_{2r+1}\right) - \hat{1}F_{n-r}, \quad n \geq r \geq 0$

xiii. $\mathcal{HF}_{n+r}L_{n+r} = HL_{2n+2r} + (-1)^{n+r}HL_0 + HF_{2n+2r+1} - (-1)^{n+r}HF_1 - \hat{1}L_{n+r}, \quad n, r \geq 0$

xiv. $\mathcal{HF}_{n-r}L_{n-r} = HL_{2n-2r} + (-1)^{n-r}HL_0 + HF_{2n-2r+1} - (-1)^{n-r}HF_1 - \hat{1}L_{n+r}, \quad n \geq r \geq 0$

xv. $\mathcal{HF}_{n-r}L_{n+r} = HL_{2n} + (-1)^{n+r}HL_{-2r} + HF_{2n+1} - (-1)^{n+r}HF_{1-2r} - \hat{1}L_{n+r}, \quad n \geq r \geq 0$

xvi. $\mathcal{HF}_{n+r}L_{n-r} = HL_{2n} + (-1)^{n-r}HL_{2r} + HF_{2n+1} - (-1)^{n-r}HF_{2r+1} - \hat{1}L_{n-r}, \quad n \geq r \geq 0$

Proposition 3.17. For $n \geq 0$, the following identities are valid:

i. $\mathcal{HL}_n + \mathcal{HF}_n = 3HF_{n+1} + HL_n - 2\hat{1}$

ii. $\mathcal{HF}_n - \mathcal{HL}_n = HF_{n+1} + HL_n$

iii. $\mathcal{HL}_nL_n + \mathcal{HF}_nF_n = 2\left(HF_{2n+1} + (-1)^nHF_1\right) + HF_{2n} - (-1)^nHF_0 + \frac{1}{5}\left(HL_{2n+1} - (-1)^nHL_1\right) - \hat{1}\left(L_n + F_n\right)$

iv. $\mathcal{HL}_nL_n - \mathcal{HF}_nF_n = 2\left(HF_{2n+1} + (-1)^nHF_1\right) - HF_{2n} + (-1)^nHF_0 - \frac{1}{5}\left(HL_{2n+1} - (-1)^nHL_1\right) - \hat{1}\left(L_n - F_n\right)$

4. Conclusion

In the present study, we consider the Leonardo and Francois numbers related to the Fibonacci and Lucas numbers, respectively. We define and investigate the hyperbolic Leonardo and hyperbolic Francois quaternions. We derive the Binet-like formulas, generating and exponential generating functions for these new quaternions. We provide certain binomial sums. Finally, we establish well-known identities for these quaternions, such as the Honsberger-like, d'Ocagne-like, Catalan-like, and Cassini-like identities. In the future, researchers may examine many more identities of the hyperbolic Leonardo and Francois quaternions. In addition, these quaternions can be used in interdisciplinary studies.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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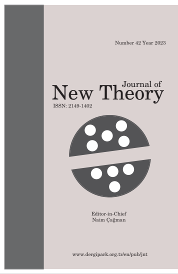

A Note on 4-Dimensional 2-Crossed Modules

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Abstract — The study presents the direct product of two objects in the category of 4-dimensional 2-crossed modules. The structures of the domain, kernel, image, and codomain can be related using isomorphism theorems by defining the kernel and image of a morphism in a category. It then establishes the kernel and image of a morphism in the category of 4-dimensional 2-crossed modules to apply isomorphism theorems. These isomorphism theorems provide a powerful tool to understand the properties of this category. Moreover, isomorphism theorems in 4-dimensional 2-crossed modules allow us to establish connections between different algebraic structures and simplify complicated computations. Lastly, the present research inquires whether additional studies should be conducted.

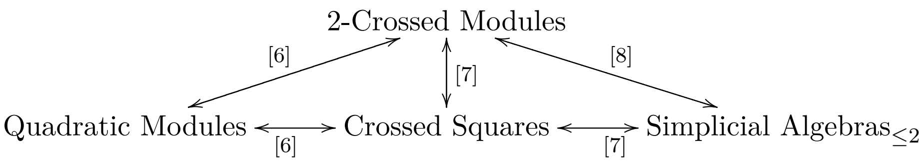
Keywords *Crossed module, kernel, image, category*

Mathematics Subject Classification (2020) 18D99, 55P15

1. Introduction

Crossed modules were first used for groups in Whitehead's work [1] and presented for commutative algebras by Porter in [2]. The idea of crossed modules modeling homotopy 2-types is well-known for becoming useful in a wide range of situations. Conduché [3] presented the idea of 2-crossed modules of groups as an algebraic model of homotopy 3-types. Algebra adaptation of 2-crossed modules is given in [4].

As an algebraic model for homotopy 3-types, Baues [5] established the concept of a quadratic module of groups and provided a relationship from simplicial groups. Actually, a quadratic module structure is a 2-crossed module structure with extra nilpotency conditions. The connection between quadratic and 2-crossed modules was demonstrated in [6]. The relations between the category of 2-crossed modules and related categories such as simplicial groups, quadratic modules, and crossed squares are given in the following diagram:



The 2-truncation of the Moore complex with a simplicial group results in a 2-crossed module. Since the 2-crossed module can be obtained from a simplicial group's Moore complex, it makes sense to consider

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this model while researching algebraic topology from the perspective of simplicial groups. 2-crossed modules have various uses in category theory, including their universal properties, representations, cohomology, and relations with other categorical structures. For more information on 2-crossed modules, see [9–13].

Baues and Bleile developed the idea of 4-dimensional quadratic complexes [14] to examine the presentation of a space X as the mapping cone of a map $\partial(X)$ beneath a space D for the algebraic description of pointed relative CW-complexes with cells in dimension 4. Based on the work of Baues and Bleile, the idea of 4-Dimensional 2-crossed modules was developed in [15] to examine any probable equivalence between homotopy 4-types. Moreover, subobjects and quotient objects in this category are defined in [15].

In this work, we give fundamental properties for a given 4-dimensional 2-crossed module morphism, including the kernel and the image. The isomorphism theorems explain the connection between quotients, homomorphisms, and subobjects. For distinct algebraic structures, there are different iterations of the isomorphism theorem. We also define the direct product to generalize the isomorphism theorems for 4-Dimensional 2-crossed modules.

2. Direct Product of 4-Dimensional 2-Crossed Modules

In this section, we will obtain the direct product of two given 4-dimensional 2-crossed modules. A 4-dimensional 2-crossed module [15] is a complex of algebras

$$\sigma : C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

such that

i. $(C_2, C_1, C_0, \partial_2, \partial_1)$ is a 2-crossed module where $\{-,-\} : C_1 \times C_1 \to C_2$ is the Peiffer lifting,

ii. C_3 is a C_1-module where $\partial_1(C_1)$ acts trivially, and

iii. ∂_3 is a homomorphism of C_1-modules where $\partial_2\partial_3 = 1$.

A morphism between 4-dimensional 2-crossed modules, $f : \sigma_1 \to \sigma_2$, is a commutative diagram

$$\begin{array}{ccccccc}
\sigma_1 : C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\sigma_2 : D_3 & \xrightarrow[\delta_3]{} & D_2 & \xrightarrow[\delta_2]{} & D_1 & \xrightarrow[\delta_1]{} & D_0
\end{array}$$

where (f_2, f_1, f_0) is a 2-crossed module morphism and f_3 is an f_0-equivariant homomorphism of modules. We will denote this category with X_2Mod^{4D}.

Let $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ and $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be two objects in X_2Mod^{4D}. We will define the direct product of 4-dimensional 2-crossed modules σ_1 and σ_2. For this, we first define the product of pre-crossed modules ∂_1 and ∂_2.

Proposition 2.1. The algebra homomorphism

$$\begin{array}{rccl}
\Phi_1 : & C_1 \times D_1 & \to & C_0 \times D_0 \\
& (c_1, d_1) & \mapsto & \phi_1(c_1, d_1) = (\partial_1(c_1), \delta_1(d_1))
\end{array}$$

is a pre-crossed module of algebras.

PROOF.
For direct product algebras $C_1 \times D_1$ and $C_0 \times D_0$, let the action of $C_0 \times D_0$ on $C_1 \times D_1$ be defined as

$$(c_0, d_0) \cdot (c_1, d_1) = (c_0 \cdot c_1, d_0 \cdot d_1)$$

for $(c_i, d_i) \in C_i \times D_i$ such that $i = 0, 1$. Then, for $(c_0, d_0), (c_0', d_0') \in C_0 \times D_0$ and $(c_1, d_1) \in C_1 \times D_1$, we have

$$\begin{aligned}
[(c_0, d_0) + (c_0', d_0')] \cdot (c_1, d_1) &= (c_0 + c_0', d_0 + d_0') \cdot (c_1, d_1) \\
&= ((c_0 + c_0') \cdot c_1, (d_0 + d_0') \cdot d_1) \\
&= (c_0 \cdot c_1 + c_0' \cdot c_1, d_0 \cdot d_1 + d_0' \cdot d_1) \\
&= (c_0 \cdot c_1, d_0 \cdot d_1) + (c_0' \cdot c_1, d_0' \cdot d_1) \\
&= (c_0, d_0) \cdot (c_1, d_1) + (c_0', d_0') \cdot (c_1, d_1)
\end{aligned}$$

and

$$\begin{aligned}
[(c_0, d_0)(c_0', d_0')] \cdot (c_1, d_1) &= (c_0 c_0', d_0 d_0') \cdot (c_1, d_1) \\
&= ((c_0 c_0') \cdot c_1, (d_0 d_0') \cdot d_1) \\
&= (c_0 \cdot (c_0' \cdot c_1), d_0 \cdot (d_0' \cdot d_1)) \\
&= (c_0, d_0) \cdot (c_0' \cdot c_1, d_0' \cdot d_1) \\
&= (c_0, d_0) \cdot [(c_0', d_0') \cdot (c_1, d_1)]
\end{aligned}$$

Therefore, with this action, $C_1 \times D_1$ is an $(C_0 \times D_0)$-algebra. Moreover, for $(c_0, d_0) \in C_0 \times D_0$ and $(c_1, d_1) \in C_1 \times D_1$,

$$\begin{aligned}
\Phi_1((c_0, d_0) \cdot (c_1, d_1)) &= \Phi_1(c_0 \cdot c_1, d_0 \cdot d_1) \\
&= (\partial_1(c_0 \cdot c_1), \delta_1(d_0 \cdot d_1)) \\
&= (c_0 \cdot \partial_1(c_1), d_0 \delta_1(d_1)) \\
&= (c_0, d_0) \cdot (\partial_1(c_1), \delta_1(d_1)) \\
&= (c_0, d_0) \cdot \Phi_1(c_1, d_1)
\end{aligned}$$

is obtained. Therefore, $\Phi_1 : C_1 \times D_1 \to C_0 \times D_0$ is a pre-crossed module. □

$C_1 \times D_1$ acts on $C_2 \times D_2$ and $C_3 \times D_3$ via Φ_1. Define the Peiffer Lifting as

$$\begin{aligned}
\{-,-\}_P : \quad (C_1 \times D_1) \times (C_1 \times D_1) &\to C_1 \times D_1 \\
((c_1, d_1), (c_1', d_1')) &\mapsto \{(c_1, d_1) \otimes (c_1', d_1')\}_P = (\{c_1 \otimes c_1'\}_C, \{d_1 \otimes d_1'\}_D)
\end{aligned}$$

and

$$\begin{aligned}
\Phi_i : \quad (C_i \times D_i) &\to C_{i-1} \times D_{i-1} \\
(c_i, d_i) &\mapsto (\partial_i(c_i), \delta_i(d_i))
\end{aligned}$$

for $i = 2, 3$.

Proposition 2.2. Let $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ and $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be two objects in X_2Mod^{4D}. Then, the direct product of σ_1 and σ_2

$$\sigma_P := (C_3 \times D_3, C_2 \times D_2, C_1 \times D_1, C_0 \times D_0, \Phi_3, \Phi_2, \Phi_1)$$

is an object in X_2Mod^{4D}.

PROOF.
Let $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ and $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be two objects in X_2Mod^{4D}.

Then,

i. PL1. For $(c_1,d_1),(c'_1,d'_1)\in C_1\times D_1$,

$$\begin{aligned}\Phi_2\{(c_1,d_1)\otimes(c'_1,d'_1)\}_P &= \Phi_2\left(\{c_1\otimes c'_1\}_C,\{d_1\otimes d'_1\}_D\right)\\ &= (\partial_2\{c_1\otimes c'_1\}_C,\delta_2\{d_1\otimes d'_1\}_D)\\ &= (c_1c'_1-c_1\cdot\partial_1(c'_1),d_1d'_1-d_1\cdot\delta_1(d'_1))\\ &= (c_1,d_1)(c'_1,d'_1)-(c_1,d_1)\cdot\Phi_1(c'_1,d'_1)\end{aligned}$$

PL2. For $(c_2,d_2),(c'_2,d'_2)\in C_2\times D_2$,

$$\begin{aligned}\{\Phi_2(c_2,d_2)\otimes\Phi_2(c'_2,d'_2)\}_P &= \{(\partial_2(c_2),\delta_2(d_2))\otimes(\partial_2(c'_2),\delta_2(d'_2))\}_P\\ &= (\{\partial_2(c_2)\otimes\partial_2(c'_2)\}_C,\{\delta_2(d_2)\otimes\delta_2(d'_2)\}_D)\\ &= (c_2c'_2,d_2d'_2)\\ &= (c_2,d_2)(c'_2,d'_2)\end{aligned}$$

PL3. For $(c_1,d_1),(c'_1,d'_1),(c''_1,d''_1)\in C_1\times D_1$,

$$\begin{aligned}\{(c_1,d_1)\otimes(c'_1,d'_1)(c''_1,d''_1)\}_P &= (\{c_1\otimes c'_1c''_1\}_C,\{d_1\otimes d'_1d''_1\}_D)\\ &= (\{c_1c'_1\otimes c''_1\}_C+\partial_1(c''_1)\cdot\{c_1\otimes c'_1\}_C,\{d_1d'_1\otimes d''_1\}_C+\partial_1(d''_1)\cdot\{d_1\otimes d'_1\}_D)\\ &= \{(c_1,d_1)(c'_1,d'_1)\otimes(c''_1,d''_1)\}_P+\Phi_1\left((c''_1,d''_1)\right)\{(c_1,d_1),(c'_1,d'_1)\}_P\end{aligned}$$

PL4.a. For $(c_2,d_2)\in C_2\times D_2$ and $(c_1,d_1)\in C_1\times D_1$,

$$\begin{aligned}\{\Phi_2\left(c_2,d_2\right)\otimes(c_1,d_1)\}_P &= \{(\partial_2(c_2),\delta_2(d_2))\otimes(c_1,d_1)\}_P\\ &= (\{(\partial_2(c_2)\otimes c_1)\}_C,(\delta_2(d_2)\otimes d_1)\}_D)\\ &= (c_2\cdot c_1-\partial_2(c_2)\cdot c_1,d_2\cdot d_1-\delta_2(d_2)\cdot d_1)\\ &= (c_2,d_2)\cdot(c_1,d_1)-\Phi_2\left(c_2,d_2\right)\cdot(c_1,d_1)\end{aligned}$$

b. For $(c_2,d_2)\in C_2\times D_2$ and $(c_1,d_1)\in C_1\times D_1$,

$$\begin{aligned}\{(c_1,d_1)\otimes\Phi_2\left(c_2,d_2\right)\}_P &= \{(c_1,d_1)\otimes(\partial_2(c_2),\delta_2(d_2))\}_P\\ &= (\{c_1\otimes\partial_2(c_2)\}_C,\{d_1\otimes\delta_2(d_2)\}_D)\\ &= \{c_1\otimes\partial_2(c_2)\}_C,\{d_1\otimes\delta_2(d_2)\}_D\\ &= (c_1\cdot c_2,d_1\cdot d_2)\\ &= (c_1,d_1)\cdot(c_2,d_2)\end{aligned}$$

PL5. For $(c_1,d_1),(c'_1,d'_1)\in C_1\times D_1$ and $(c_0,d_0)\in C_0\times D_0$,

$$\begin{aligned}\{(c_1,d_1)\otimes(c'_1,d'_1)\}_P\cdot(c_0,d_0) &= (\{c_1\otimes c'_1\}_C,\{d_1\otimes d'_1\}_D)\cdot(c_0,d_0)\\ &= (\{c_1\otimes c'_1\}_C\cdot(c_0,d_0),\{d_1\otimes d'_1\}_D)\\ &= (\{c_1\cdot c_0\otimes c'_1\cdot d_0\}_C,\{d_1\otimes d'_1\}_D)\\ &= \{(c_1\cdot c_0,d_1)\otimes(c'_1\cdot d_0,d'_1)\}_P\\ &= \{(c_1,d_1)\cdot(c_0,d_0)\otimes(c'_1,d'_1)\}_P\end{aligned}$$

and

$$\begin{aligned}
\{(c_1,d_1)\otimes(c_1',d_1')\}_P\cdot(c_0,d_0) &= (\{c_1\otimes c_1'\}_C,\{d_1\otimes d_1'\}_D)\cdot(c_0,d_0)\\
&= (\{c_1\otimes c_1'\}_C,\{d_1\otimes d_1'\}_D\cdot(c_0,d_0))\\
&= (\{c_1\otimes c_1'\}_C,\{d_1\cdot c_0\otimes d_1'\cdot d_0\}_D)\\
&= \{(c_1,d_1\cdot c_0)\otimes(c_1',d_1'\cdot d_0)\}_P\\
&= \{(c_1,d_1)\otimes(c_1',d_1')\cdot(c_0,d_0)\}_P
\end{aligned}$$

ii. $C_2\times D_2$ acts on $C_3\times D_3$ via $C_1\times D_1$ trivially. Therefore, $C_3\times D_3$ is a $C_2\times D_2$-module.

iii. For $(c_3,d_3)\in C_3\times D_3$,

$$\begin{aligned}
\Phi_2\Phi_3(c_3,d_3) &= \Phi_2\left(\partial_3\left(c_3\right),\delta_3\left(d_3\right)\right)\\
&= \left(\partial_2\partial_3\left(c_3\right),\delta_2\delta_3\left(d_3\right)\right)\\
&= (1_{C_1},1_{D_1})\\
&= 1_{C_1\times D_1}
\end{aligned}$$

is obtained. □

3. Kernel and Image of a Morphism in X_2Mod^{4D}

In this section, we give the notions of the kernel and image of a morphism in the category X_2Mod^{4D}. Throughout this section, we will consider the morphism

$$f=(f_3,f_2,f_1,f_0):\sigma_1:=(C_3,C_2,C_1,C_0,\partial_3,\partial_2,\partial_1)\to\sigma_2:=(D_3,D_2,D_1,D_0,\delta_3,\delta_2,\delta_1)$$

in X_2Mod^{4D}.

Proposition 3.1. The object

$$\begin{array}{ccccccc}
 & & \ker f_1\times\ker f_1 & & & & \\
 & & \downarrow{\scriptstyle \{-,-\}_{\ker}} & & & & \\
\ker f_3 & \xrightarrow[\overline{\partial_3}]{} & \ker f_2 & \xrightarrow[\overline{\partial_2}]{} & \ker f_1 & \xrightarrow[\overline{\partial_1}]{} & \ker f_0
\end{array}$$

in X_2Mod^{4D} is an ideal of σ_1 where $\overline{\partial_i}$ are restrictions, for $i=0,1,2$.

Proof.

$\ker f_i$ is an ideal of C_i, for $i=0,1,2,3$. The commutativity of the following diagram

$$\begin{array}{ccccccc}
\ker f_3 & \xrightarrow{\partial_3|_{\ker f_3}} & \ker f_2 & \xrightarrow{\partial_2|_{\ker f_2}} & \ker f_1 & \xrightarrow{\partial_1|_{ker f_1}} & \ker f_0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
C_3 & \xrightarrow[\partial_3]{} & C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0
\end{array}$$

implies $\partial_i(\ker f_{i+1})\subset\ker f_i$, for $i=1,2,3$. The axioms of an ideal in X_2Mod^{4D} can be calculated using the definition of a kernel, i.e.,

$$f_2(x_0\cdot c_1)=f_1(x_0)\cdot f_2(c_1)=0\cdot f_2(c_1)=0 \text{ implies that } x_0\cdot c_1\in\ker f_2$$

for $x_0\in\ker f_0$ and $c_1\in C_1$. □

Definition 3.2. The ideal

$$\begin{array}{ccccccc} & & ker f_1 \times \ker f_1 & & & & \\ & & \big\downarrow {\scriptstyle \{-,-\}_{\ker}} & & & & \\ \ker f_3 & \xrightarrow[\overline{\partial_3}]{} & \ker f_2 & \xrightarrow[\overline{\partial_2}]{} & \ker f_1 & \xrightarrow[\overline{\partial_1}]{} & \ker f_0 \end{array}$$

is called the kernel of the morphism f in X_2Mod^{4D} .

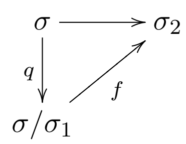
Definition 3.3. The subobject

$$\begin{array}{ccccccc} & & f_1(C_1) \times f_1(C_1) & & & & \\ & & \big\downarrow {\scriptstyle \{-,-\}_{Im}} & & & & \\ f_3(C_3) & \xrightarrow[\delta_3|_{f_3}]{} & f_2(C_2) & \xrightarrow[\partial_2|_{f_2}]{} & f_1(C_1) & \xrightarrow[\partial_1|_{f_1}]{} & f_0(C_0) \end{array}$$

of σ_2 is called the image of the morphism f in X_2Mod^{4D} .

4. Universal Property in X_2Mod^{4D}

In this section, we show the existence and uniqueness of a morphism $f : \sigma/\sigma_1 \to \sigma_2$ in X_2Mod^{4D} that makes the following diagram



commutative where σ_1 is an ideal of σ_1 and σ_2 is another object in X_2Mod^{4D}.

Theorem 4.1. [15] Let

$$\sigma_1 : D_3 \xrightarrow{\delta_3} D_2 \xrightarrow{\delta_2} D_1 \xrightarrow{\delta_1} D_0$$

be an ideal of

$$\sigma : C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

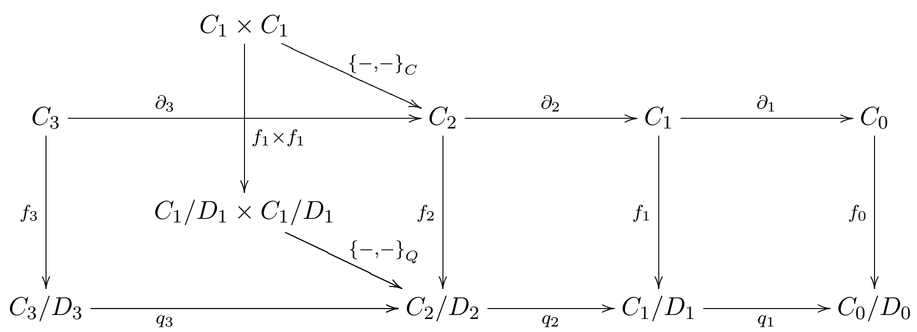
in X_2Mod^{4D}. Then,

$$\sigma/\sigma_1 : C_3/D_3 \xrightarrow{q_3} C_2/D_2 \xrightarrow{q_2} C_1/D_1 \xrightarrow{q_1} C_0/D_0$$

is an object in X_2Mod^{4D}.

Theorem 4.2. Let $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be an ideal of $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ in X_2Mod^{4D}. If $\beta : \sigma_1 \to \sigma_3 := (E_3, E_2, E_1, E_0, \eta_3, \eta_2, \eta_1)$ is a morphism, then there exists a unique morphism $\alpha : \sigma_1/\sigma_2 \to \sigma_3$ in X_2Mod^{4D}.

PROOF.
If σ_2 is an ideal of σ_1, then



(1)

there exists $f_{i+1} : C_i \to C_i/D_i$, for $i = 0, 1, 2, 3$. Since, for $c_0 \in C_0$ and $d_i \in D_i$, we have

$$f_{i+1}(c_0 \cdot d_0) = (c_0 \cdot d_i) + D_i = (c_0 + D_i) \cdot (d_i + D_i) = f_0(c_0) \cdot f_i(d_i), \quad i = 0, 1, 2, 3$$

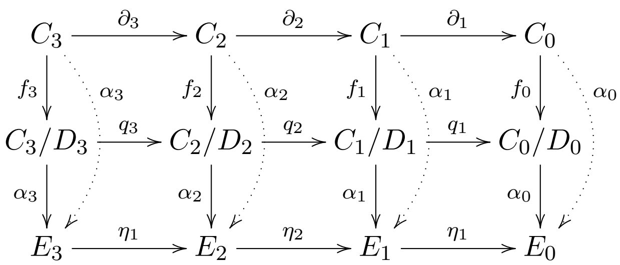
and Diagram 1 is commutative. Moreover, $f = (f_1, f_2, f_3, f_4)$ is a morphism in X_2Mod^{4D}. Since, for all $c + D_i, c' + D_i \in C_i/D_i$,

$$\begin{aligned} c + D_i = c' + D_i &\Rightarrow c - c' \in D_i \\ &\Rightarrow \beta_i(c - c') \in \beta_i(D_i) \\ &\Rightarrow \beta_i(c) - \beta_i(c') \in \beta_i(D_i) \\ &\Rightarrow D_i \subset Ker\beta_i \\ &\Rightarrow \beta_i(D_i) = 1 \end{aligned}$$

$\alpha_i : C_i/D_i \to E_i$ $(i = 1, 2, 3, 4)$ are well defined. Furthermore, for $i = 1, 2, 3$,

$$q_i\alpha_{i+1}(c_i + D_i) = q_i\beta_{i+1}(c_i) = \beta_i(\partial_i(c_i)) = \alpha_i(\partial_i(c_i) + D_i) = \alpha_i q_i(c_i + D_i)$$

As a result, α is the unique morphism in X_2Mod^{4D}, making the following diagram



commutative. □

5. Conclusion

In this work, we defined the direct products in the category of 4-Dimensional 2-crossed modules. We provide the kernel and image of a morphism in this category in order to adapt the isomorphism theorems. Given a morphism $f = (f_3, f_2, f_1, f_0) : \sigma \to \sigma'$ in X_2Mod^{4D}, the mappings $f_i^* : \sigma/\ker f \to f(\sigma)$ defined as $f_i^*(x + \ker f_i) = f_i(x)$, for $i = 0, 1, 2, 3$, are isomorphisms in this category. Using the isomorphism theorem on groups, it can be easily shown that f^* is an isomorphism. As a result, we get

$$\sigma/\ker f \cong f(\sigma)$$

In future studies, isomorphism theorems can be given for this category. Moreover, using semi-direct product groupoid adaptations can be obtained.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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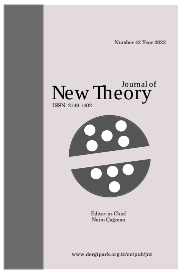

Generalized Cross Product in $(2+s)$-Dimensional Framed Metric Manifolds with Application to Legendre Curves

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Abstract — This study generalizes the cross product defined in 3-dimensional almost contact metric manifolds and describes a new generalized cross product for $n = 1$ in $(2n + s)$-dimensional framed metric manifolds. Moreover, it studies some of the proposed product's basic properties. It also performs characterizations of the curvature of a Legendre curve on an S-manifold and calculates the curvature of a Legendre curve. Furthermore, it shows that Legendre curves are also biharmonic curves. Next, this study observes that a Legendre curve of osculating order 5 on S-manifolds is imbedded in the 3-dimensional K-contact space. Lastly, the current paper discusses the need for further research.

Keywords *Generalized cross product, S-manifolds, Legendre curves*

Mathematics Subject Classification (2020) 53D10, 53A04

1. Introduction

Yano [1] has led up to the groundwork of S-manifolds and has defined the concept of f-structures in M^{2n+s} manifolds. Almost complex $(s = 0)$ and almost contact $(s = 1)$ structures are examples of f-structures. Goldberg and Yano [2] have defined the concept of the framed f-structures. Moreover, they have suggested a complex structure by the concept of f-structures. Furthermore, they have proposed the concept of framed metric manifolds by examining the normality condition of a metric framed structure. Blair [3] has introduced S-manifolds, generalizing almost complex Kaehler and almost contact Sasakian structures. Sarkar et al. [4] have found the curvature and torsion of Legendre curves in 3-dimensional trans-Sasakian manifolds with respect to the semisymmetric metric connection. Özgür and Güvenç [5] have propounded biharmonic Legendre curves in S-space forms. They have analyzed characterizations of the curvature of the biharmonic Legendre curves in 4 cases.

This paper is organized as follows: Section 2 provides the concept of S-manifolds and some of their basic properties. Section 3 generalizes the new cross-product in a 3-dimensional almost contact metric manifolds defined by Camcı [6] and defines a generalized cross-product in $(2 + s)$-dimensional S-manifolds. Besides, it demonstrates that this cross-product in $\mathbb{R}^4$ is coherent with a triple product [7] by an example. In addition, Section 3 provides the basic properties of the generalized cross-product. Section 4 calculates the curvature of Legendre curves using the generalized cross-product and demonstrated that $(2 + 3)$-dimensional S-manifolds are imbedded in 3-dimensional space. Finally, the need for further research is discussed.

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2. Preliminaries

This section provides the concept of S-manifolds and some of their basic properties.

Definition 2.1. [1] Let (M,g) be a Riemannian manifold with $\dim M = 2n+s$. It is said to be a framed f-structure if f is a tensor field of type $(1,1)$ and rank $2n$ satisfying $f^3+f=0$.

Definition 2.2. [2] Let M^{2n+s} be a manifold with an f-structure of rank $2n$. Then, f-structure is said to be has completed frames if there exists vector fields $\xi_1,\xi_2,\cdots,\xi_s$ on M^{2n+s}, and $\eta_1,\eta_2,\cdots,\eta_s$ are 1-forms, then $\eta_i \circ f = 0$, $f\circ \xi_i = 0$, and $f^2 = -I + \sum_{i=1}^{s} \eta_i \otimes \xi_i$.

Definition 2.3. [8] Let M^{2n+s} be an f-structure with completed frames. The framed f-structure is normal if the tensor field S of type (1,2) given by

$$S = [f,f] + 2\sum_{i=1}^{s} d\eta_i \otimes \xi_i$$

vanishes.

Definition 2.4. [2] Let M^{2n+s} be a manifold. Then, M is said to be an S-manifold if the f-structure is normal.

Definition 2.5. [1] Let M^{2n+s} be a Riemannian manifold. The distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$, and its complementary orthogonal distribution is denoted by D. Consequently, $TM = D \oplus \mathcal{M}$. Moreover, if $X\in D$, then $\eta_i(X)=0$, for any $i\in\{1,2,\cdots,s\}$, and if $X\in\mathcal{M}$, then $fX=0$.

Definition 2.6. [2] Let M^{2n+s} has an f-structure with completed frames. If there exists a Riemannian metric g on M^{2n+s} such that

$$g(X,Y) = g(fX,fY) + \sum_{i=1}^{s} \eta_i(X)\eta_i(Y)$$

and $X,Y\in\chi(M^{2n+s})$, then M^{2n+s} is called that has a metric f-structure.

From now on, the notation ϕ is used instead of f.

Definition 2.7. [3] Let M^{2n+s} be an S-manifold. The covariant differentiation ∇ of M^{2n+s} satisfies

$$\nabla_X \xi_i = -\phi X, \quad i\in\{1,2,\cdots,s\}$$

and

$$(\nabla_X\phi)Y = \sum_{i=1}^{s} [g(\phi X,\phi Y)\xi_i + \eta_i(Y)\phi^2 X]$$

for all $X,Y\in\chi(M)$. Besides, for all $i\in\{1,2,\cdots,s\}$, $\eta_1\wedge\eta_2\wedge\cdots\wedge\eta_s\wedge(d\eta_i)^n \neq 0$ and $\Phi = d\eta_i$ on an S-manifold such that Φ is the fundamental 2-form defined by

$$\Phi(X,Y) = g(X,\phi Y), \quad X,Y\in TM$$

Definition 2.8. [9] A submanifold of an S-manifold is called an integral submanifold if

$$\eta_i(X) = 0, \quad i\in\{1,2,\cdots,s\}, \text{ for all } X\in\chi(M)$$

Definition 2.9. [5] A 1-dimensional integral submanifold of an S-manifold $\left(M^{2n+s},\phi,\xi_i,\eta_j,g\right)$, $i,j\in\{1,2,\cdots,s\}$ is called a Legendre curve of M. In other words, a curve $\gamma: I\to M$ is called a Legendre curve if $\eta_i(T)=0$, for all $i\in\{1,2,\cdots,s\}$ such that T is the tangent vector field of γ.

3. Generalized Cross product in $(2+s)$-dimensional Framed Metric Manifolds

This section, firstly, generalizes the cross-product in a 3-dimensional almost contact metric manifolds defined by Camcı [5] and defines a generalized cross-product in $(2+s)$-dimensional S-manifolds.

Definition 3.1. Let A be a matrix of type $s\times(s+1)$. Then,

i. $\tilde{A}_{ijk}$ such that $i,k\in\{1,2,\cdots,s\}$ and $j\in\{i+1,i+2,\cdots,s+1\}$ is the matrix obtained by deleting the i^{th} and j^{th} columns and k^{th} row of the matrix A. Specially, for $s=1$, $\det\tilde{A}_{121}=1$.
ii. $\tilde{\tilde{A}}_{mn}$ such that $m\in\{2,3,\cdots,s\}$ and $n\in\{m+1,m+2,\cdots,s+1\}$ is the matrix obtained by deleting the i^{th} and j^{th} columns of the matrix A and adding the first column of the matrix A to the left of the first column as a new column.

For example, for $s=3$, let

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}.$$

Then,

$$\tilde{A}_{231}=\begin{bmatrix} a_{21} & a_{24}\\ a_{31} & a_{34}\end{bmatrix}$$

and

$$\tilde{\tilde{A}}_{23}=\begin{bmatrix} a_{11} & a_{11} & a_{14}\\ a_{21} & a_{21} & a_{24}\\ a_{31} & a_{31} & a_{34}\end{bmatrix}$$

Definition 3.2. Let $M=\left(M^{2+s},\phi,\xi_i,\eta_j,g\right)$, $i,j\in\{1,2,\cdots,s\}$, be a framed metric manifold in $(2+s)$-dimensional space. We define a generalized cross-product $\times$ by

$$X_1\times X_2\times\cdots\times X_{s+1}=\sum_{k=1}^{s}\left(\sum_{i=1}^{s}\sum_{j=i+1}^{s+1}(-1)^{i+j+k}\Phi\left(X_i,X_j\right)\det\tilde{A}_{ijk}\right)\xi_k+\begin{vmatrix} \phi X_1 & \phi X_2 & \cdots & \phi X_{s+1}\\ \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1})\\ \vdots & \vdots & \ddots & \vdots\\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1})\end{vmatrix} \quad (1)$$

such that $X_1,X_2,\cdots,X_{s+1}\in TM$ and

$$A=\begin{bmatrix} \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1})\\ \eta_2(X_1) & \eta_2(X_2) & \cdots & \eta_2(X_{s+1})\\ \vdots & \vdots & \ddots & \vdots\\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1})\end{bmatrix}$$

Moreover, $\tilde{A}_{ijk}$, $i,k\in\{1,2,\cdots,s\}$ and $j\in\{i+1,i+2,\cdots,s+1\}$, is as in Definition 3.1. For example, for $s=1$, we obtain

$$X_1\times X_2=\Phi(X_1,X_2)\xi_1+\eta_1(X_2)\phi X_1-\eta_1(X_1)\phi X_2 \quad (2)$$

Equation 2 gives the cross product in 3-dimensional almost contact metric manifolds defined in [6]. For $s=2$, the generalized cross-product is

$$X_1 \times X_2 \times X_3 = \big(\Phi(X_1,X_2)\eta_2(X_3) - \Phi(X_1,X_3)\eta_2(X_2) + \Phi(X_2,X_3)\eta_2(X_1)\big)\xi_1 + \big(-\Phi(X_1,X_2)\eta_1(X_3) + \Phi(X_1,X_3)\eta_1(X_2) - \Phi(X_2,X_3)\eta_1(X_1)\big)\xi_2 + \begin{vmatrix} \phi X_1 & \phi X_2 & \phi X_3 \\ \eta_1(X_1) & \eta_1(X_2) & \eta_1(X_3) \\ \eta_2(X_1) & \eta_2(X_2) & \eta_2(X_3) \end{vmatrix}$$

Secondly, it demonstrates that this cross-product in $\mathbb{R}^4$ is coherent with a triple product [7] by an example.

Example 3.3. For the subspace $V = \{(x_1, x_2, 0,0) \mid x_1, x_2 \in \mathbb{R}\}$ of the 4-dimensional Euclidean space $\mathbb{R}^4(x_1,x_2,x_3,x_4)$, the projection function $\pi(x_1,x_2,x_3,x_4) = (x_1,x_2,0,0)$, and the almost complex plane $J(x_1,x_2,x_3,x_4) = (x_2,-x_1,-x_4,x_3)$, $\big(\mathbb{R}^4(x_1,x_2,x_3,x_4), \phi, \xi_i, \eta_j, g\big)$, $i,j \in \{1,2\}$, is a framed metric manifold such that $\phi = J \circ \pi$, $\eta_1 = dx_3$, $\eta_2 = dx_4$, $\xi_1 = (0,0,1,0)$, $\xi_2 = (0,0,0,1)$, and g is the standard Euclidean metric. As a result, $X_1 \times X_2 \times X_3 = X_1 \wedge X_2 \wedge X_3$ such that $X_1 \wedge X_2 \wedge X_3$ is the triple product in $\mathbb{R}^4$ provided in [7].

Finally, this section provides some of the basic properties of the generalized cross-product.

Theorem 3.4. Let $M = \big(M^{2+s}, \phi, \xi_i, \eta_j, g\big)$, $i,j \in \{1,2,\cdots,s\}$ be a framed metric manifold in $(2+s)$-dimensional space. Then, for all $X_1, X_2, \cdots, X_{s+1} \in TM$, the generalized cross-product has the following properties:

i. The generalized cross-product is bilinear and antisymmetric.
ii. $X_1 \times X_2 \times \cdots \times X_{s+1}$ is perpendicular to each of $X_1, X_2, \cdots, X_{s+1}$.
iii. $\phi X = \xi_1 \times \xi_2 \times \cdots \times \xi_s \times X$.

Proof.

i. The proof is straightforward from the fundamental 2-form Φ and the determinant function's bilinearity.

ii. We need to show that $g(X_1 \times X_2 \times \cdots \times X_{s+1}, X_t) = 0$, for $t \in \{1,2,\cdots,s+1\}$. For $t = 1$, from Equation 1, we obtain

$$g(X_1 \times X_2 \times \cdots \times X_{s+1}, X_1) = \sum_{k=1}^{s}\left(\sum_{i=1}^{s}\sum_{j=i+1}^{s+1}(-1)^{i+j+k}\Phi(X_i,X_j)\det\tilde{A}_{ijk}\right)\eta_k(X_1) + \begin{vmatrix} 0 & g(\phi X_2, X_1) & \cdots & g(\phi X_{s+1}, X_1) \\ \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1}) \end{vmatrix} \tag{3}$$

Thus,

$$\begin{vmatrix} 0 & g(\phi X_2, X_1) & \cdots & g(\phi X_{s+1}, X_1) \\ \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1}) \end{vmatrix} = \sum_{k=1}^{s}\left(\sum_{j=2}^{s+1}(-1)^{j+k}\Phi(X_1,X_j)\det\tilde{A}_{1jk}\right)\eta_k(X_1) \tag{4}$$

If we substitute Equation 4 for Equation 3, we get

$$g(X_1 \times X_2 \times X_3 \times \cdots \times X_{s+1}, X_1) = \sum_{i=2}^{s}\sum_{j=i+1}^{s+1}(-1)^{i+j}\Phi(X_i,X_j)\det\tilde{\tilde{A}}_{ij}$$

Here, the $\tilde{\tilde{A}}_{ij}$, $i \in \{2,3,\cdots,s\}$ and $j \in \{i+1, i+2, \cdots, s+1\}$, is as in Definition 3.1. Thus, $\det \tilde{\tilde{A}}_{ij} = 0$. Therefore,

$$g(X_1 \times X_2 \times X_3 \times \cdots \times X_{s+1}, X_1) = 0$$

Similarly, it is proved that $g(X_1 \times X_2 \times X_3 \times \cdots \times X_{s+1}, X_t) = 0$, for $t \in \{2,3,\cdots,s+1\}$.

iii. If $\xi_i = Y_i$, for all $i \in \{1,2,\cdots,s\}$, and $X = Y_{s+1}$, for all $X, \xi_1, \xi_2, \cdots, \xi_s \in TM$, from Equation 1,

$$\xi_1 \times \xi_2 \times \ldots \times \xi_s \times X = Y_1 \times Y_2 \times \ldots \times Y_{s+1} = \sum_{k=1}^{s} \left(\sum_{i=1}^{s} \sum_{j=i+1}^{s+1} (-1)^{i+j+k} \Phi(Y_i, Y_j) \det \tilde{A}_{ijk} \right) \xi_k + \begin{vmatrix} \phi Y_1 & \phi Y_2 & \cdots & \phi Y_{s+1} \\ \eta_1(Y_1) & \eta_1(Y_2) & \cdots & \eta_1(Y_{s+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \eta_s(Y_1) & \eta_s(Y_2) & \cdots & \eta_s(Y_{s+1}) \end{vmatrix} \quad (5)$$

From Equation 5, $\Phi(Y_i, Y_j) = 0$, $i,j \in \{1,2,\cdots,s+1\}$. Then, if we substitute the expressions $\phi Y_i = \phi \xi_i = 0$ and $Y_{s+1} = X$ in Equation 5, we get $\xi_1 \times \xi_2 \times \ldots \times \xi_s \times X = \phi X$. □

4. Legendre Curves in S-manifolds

Let $\gamma: I \to M$ be a unit-speed curve in an n-dimensional Riemannian manifold (M, g) and $k_1, k_2, \cdots, k_r$ be positive functions on I. If there is an orthonormal basis $\{V_1, V_2, \cdots, V_r\}$ along γ that satisfies the following Frenet equations, γ is called a Frenet curve of osculating order r:

$$V_1 = \gamma'$$

$$\nabla_{V_1} V_1 = k_1 V_2$$

$$\nabla_{V_1} V_2 = -k_1 V_1 + k_2 V_3$$

$$\vdots$$

$$\nabla_{V_1} V_r = -k_{r-1} V_{r-1}$$

Theorem 4.1. Let $M = (M^{2+s}, \phi, \xi_i, \eta_j, g)$, $i,j \in \{1,2,\cdots,n\}$ be an S-manifold and $\gamma: I \to M$ be a Legendre curve of osculating order $(2+s)$. Then, for $\varepsilon = \pm 1$, the following equations are obtained:

$$V_2 = \varepsilon \phi V_1$$

$$V_3 = \frac{\varepsilon}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_s$$

$$k_2 = \sqrt{s}$$

and

$$k_3 = 0$$

Proof.

For the function $\sigma_{ij}: I \to \mathbb{R}$ defined by $\sigma_{ij}(s) = g(V_i, \xi_j)$, for $i \in \{1,2,\cdots,s+2\}$ and $j \in \{1,2,\cdots,s\}$,

$$\xi_j = \sum_{i=1}^{s+2} \sigma_{ij} V_i, \quad j \in \{1,2,\cdots,s\} \quad (6)$$

Let γ be a Legendre curve. Then,

$$\sigma_{11} = \sigma_{12} = \cdots = \sigma_{1s} = 0 \tag{7}$$

Moreover, from Theorem 3.4,

$$\phi V_1 = (-1)^{s+1} \begin{vmatrix} V_2 & V_3 & \cdots & V_{s+1} & V_{s+2} \\ \sigma_{21} & \sigma_{31} & \cdots & \sigma_{(s+1)1} & \sigma_{(s+2)1} \\ \sigma_{22} & \sigma_{32} & \cdots & \sigma_{(s+1)2} & \sigma_{(s+2)2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{2s} & \sigma_{3s} & \cdots & \sigma_{(s+1)s} & \sigma_{(s+2)s} \end{vmatrix} \tag{8}$$

If we take the derivative from $\sigma_{11} = 0 = g(V_1, \xi_1)$, then

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = \cdots = \sigma_{2s} = 0 \tag{9}$$

From Equations 8-10,

$$\phi V_1 = (-1)^{s+1} \begin{vmatrix} \sigma_{31} & \cdots & \sigma_{(s+1)1} & \sigma_{(s+2)1} \\ \sigma_{32} & \cdots & \sigma_{(s+1)2} & \sigma_{(s+2)2} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{3s} & \cdots & \sigma_{(s+1)s} & \sigma_{(s+2)s} \end{vmatrix} V_2 \tag{10}$$

If we substitute Equations 7 and 9 in Equation 6, then

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_s \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{31} & \sigma_{41} & \cdots & \sigma_{(s+2)1} \\ \sigma_{32} & \sigma_{42} & \cdots & \sigma_{(s+2)2} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{3s} & \sigma_{42} & \cdots & \sigma_{(s+2)s} \end{bmatrix}}_{B} \begin{bmatrix} V_3 \\ V_4 \\ \vdots \\ V_{s+2} \end{bmatrix} \tag{11}$$

Since $\{\xi_1, \xi_2, \ldots, \xi_s\}$ and $\{V_3, \ldots, V_{s+2}\}$ are orthonormal systems, B is an orthonormal matrix. In this case, $|B| = \pm 1$ where $|B|$ is the determinant of the matrix B. From Equation 10, for $\varepsilon = \pm 1$,

$$\phi V_1 = \varepsilon V_2 \tag{12}$$

From Equation 12,

$$\phi V_2 = \varepsilon V_1 \tag{13}$$

If we take the derivative from $\sigma_{2i} = g(V_2, \xi_i) = 0$, $i \in \{1,2,\cdots,s\}$,

$$k_2 \sigma_{3k} = g(V_2, \phi V_1), \quad k \in \{1,2,\cdots,s\} \tag{14}$$

From Equation 8,

$$g(V_2, \phi V_1) = 1 \tag{15}$$

Then, from Equations 14 and 15,

$$k_2\sigma_{3k} = 1, \quad k \in \{1,2,\cdots,s\}$$

In this case,

$$\sigma_{3k} = \frac{1}{k_2}, \quad k \in \{1,2,\cdots,s\} \tag{16}$$

Since B is an orthonormal matrix,

$$\sum_{k=1}^{s} (\sigma_{3k})^2 = 1 \tag{17}$$

From Equations 16 and 17, for $\varepsilon = \pm 1$,

$$\sigma_{3k} = \frac{\varepsilon}{\sqrt{s}}, \quad k \in \{1,2,\cdots,s\} \tag{18}$$

and

$$k_2 = \sqrt{s}$$

If we take the derivative from Equation 17, then

$$-k_2 g(V_2, \xi_1) + k_3 g(V_4, \xi_1) + g(V_3, \phi V_1) = 0 \tag{19}$$

From Equations 9 and 12, Equation 19 becomes

$$k_3\sigma_{41} = 0 \tag{20}$$

Similarly,

$$k_3\sigma_{42} = k_3\sigma_{43} = \cdots = k_3\sigma_{4s} = 0 \tag{21}$$

As the matrix B is orthonormal,

$$(\sigma_{41})^2 + (\sigma_{42})^2 + \cdots + (\sigma_{4s})^2 = 1 \tag{22}$$

From Equations 20-22,

$$k_3 = 0$$

Since B is an orthonormal matrix, we can write Equation 11 as follows:

$$\begin{bmatrix} V_3 \\ V_4 \\ \vdots \\ V_{s+2} \end{bmatrix} = B^T \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_s \end{bmatrix} \tag{23}$$

From Equations 11 and 18,

$$B^T = \begin{bmatrix} \frac{\varepsilon}{\sqrt{s}} & \frac{\varepsilon}{\sqrt{s}} & \cdots & \frac{\varepsilon}{\sqrt{s}} \\ \sigma_{41} & \sigma_{42} & \cdots & \sigma_{4s} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{(s+2)1} & \sigma_{(s+2)2} & \cdots & \sigma_{(s+2s,)s} \end{bmatrix} \tag{24}$$

and from Equations 23 and 24,

$$V_3 = \frac{\varepsilon}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha$$

is obtained. □

Remark 4.2. In [5], Özgür and Güvenç assumed that $V_2 = \phi V_1$ and obtained the same results as in Theorem 4.1 for biharmonic Legendre curves. Thus, Legendre curves are also biharmonic.

In [10], Hasegawa et al. have introduced $\mathbb{R}^{2+s}(-3s)$ space as follows: Let the coordinate functions' set of $M = \mathbb{R}^{2+s}$ be $\{x, y, z_1, \cdots, z_s\}$. In this space,

$$\xi_i = 2\frac{\partial}{\partial z_i}, \quad i \in \{1,2,\cdots,s\}$$

$$\eta_j = \frac{1}{2}\left(dz_j - y dx\right), \quad j \in \{1,2,\cdots,s\} \tag{25}$$

and

$$g = \frac{1}{4}(dx^2 + dy^2) + \sum_{j=1}^{s} \eta_j \otimes \eta_j$$

with

$$X = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + \sum_{i=1}^{s} \xi_i \frac{\partial}{\partial z_i} \in \chi(M)$$

Thus, $(M = \mathbb{R}^{2+s}, \phi, \xi_i, \eta_j, g)$ is an S-space form with the constant φ-sectional curvature $c = -3s$. Besides, $\{e, \phi e, \xi_1, \xi_2, \cdots, \xi_s\}$ is an orthonormal basis on $\mathbb{R}^{2+s}(-3s)$ such that

$$e_1 = e = 2\frac{\partial}{\partial y}, \qquad e_2 = \phi e = 2\left(\frac{\partial}{\partial x} + y \sum_{i=1}^{s} \xi_i\right), \qquad \text{and} \qquad \xi_i = 2\frac{\partial}{\partial z_i}, \quad i \in \{1,2,\cdots,s\}$$

According to this basis, the Levi-Civita connection is calculated as

$$\nabla_e e = \nabla_{\phi e} \phi e = 0$$

$$\nabla_e \phi e = \sum_{i=1}^{s} \xi_i \tag{26}$$

$$\nabla_{\phi e} e = -\sum_{i=1}^{s} \xi_i$$

$$\nabla_e \xi_i = \nabla_{\xi_i} e = -\phi e \tag{26}$$

and

$$\nabla_{\phi e} \xi_i = \nabla_{\xi_i} \phi e = e$$

We will examine Legendre curves in $\mathbb{R}^{2+s}(-3s)$. Let $\gamma: I \to \mathbb{R}^{2+s}(-3s)$ be a Legendre curve. Let

$$\gamma(t) = \big(x(t), y(t), z_1(t), \dots, z_s(t)\big)$$

such that t is the arc-length parameter. If the tangent vector field of γ is V_1, then $\eta_j(V_1) = 0$, $j \in \{1,2,\cdots,s\}$ since γ is a Legendre curve. From Equation 25,

$$z_1'(t) = z_2'(t) = \cdots = z_s'(t) = y(t)x'(t)$$

If $z_i(t) = f(t)$, then

$$\gamma(t) = (x(t), y(t), f(t) + c_1, f(t) + c_2, \cdots, f(t) + c_s)$$

If the tangent vector field of the curve γ in terms of basis $\{e, \phi e, \xi_1, \xi_2, \dots, \xi_s\}$ is as follows:

$$V_1 = \frac{1}{2}(y'e + x'\phi e)$$

Since γ is a unit speed curve,

$$(x')^2 + (y')^2 = 4$$

Hence, we have the following example:

Example 4.3. $\gamma: I \to \mathbb{R}^{2+s}(-3s), \gamma(t) = (x(t), y(t), f(t) + c_1, f(t) + c_2, \dots, f(t) + c_s)$ is a unit speed Legendre curve such that

$$x(t) = \frac{2}{c}\cos\theta(t) + x_0$$

$$y(t) = \frac{2}{c}\sin\theta(t) + y_0$$

$$f(t) = \frac{1}{c^2}\sin 2\theta + \frac{2y_0}{c}\cos\theta - \frac{2}{c}t$$

and

$$\theta(t) = ct + c_0$$

The tangent vector field of γ in terms of basis $\{e, \phi e, \xi_1, \xi_2, \dots, \xi_s\}$ is $V_1 = \cos\theta\, e - \sin\theta\, \phi e$. From 26,

$$\nabla_{V_1} e = \sin\theta \sum_{i=1}^{s} \xi_i$$

$$\nabla_{V_1} \phi e = \cos\theta \sum_{i=1}^{s} \xi_i$$

$$\nabla_{V_1} \xi_i = -\sin\theta\, e - \cos\theta\, \phi e, \quad i \in \{1,2,\cdots,s\}$$

$$\nabla_{V_1} V_1 = -c\sin\theta\, e - c\cos\theta\, \phi e$$

and

$$k_1 = c$$

Then, $V_2 = -(\sin\theta\, e + \cos\theta\, \phi e)$ and $\phi V_1 = \cos\theta\, \phi e + \sin\theta\, e = -V_2$. If

$$E_1 = \gamma' = V_1$$

$$E_2 = \nabla_{V_1} V_1$$

and

$$E_3 = \nabla_{V_1}(\nabla_{V_1} V_1) - \frac{\langle \nabla_{V_1}(\nabla_{V_1} V_1), \nabla_{V_1} V_1 \rangle}{\langle \nabla_{V_1} V_1, \nabla_{V_1} V_1 \rangle} \nabla_{V_1} V_1 - \frac{\langle \nabla_{V_1}(\nabla_{V_1} V_1), V_1 \rangle}{\langle V_1, V_1 \rangle} V_1$$

then

$$V_3 = \frac{E_3}{\|E_3\|}$$

and

$$\nabla_{V_1}(\nabla_{V_1} V_1) = -c^2 \cos\theta\, e + c^2 \sin\theta\, \phi e - c \sum_{i=1}^{s} \xi_i$$

Then,

$$\langle \nabla_{V_1}(\nabla_{V_1} V_1), \nabla_{V_1} V_1 \rangle = 0$$

$$\langle \nabla_{V_1}(\nabla_{V_1} V_1), V_1 \rangle = -c^2$$

$$E_3 = -c \sum_{i=1}^{s} \xi_i$$

and

$$V_3 = \frac{-\sum_{i=1}^{s} \xi_i}{\sqrt{s}}$$

From $k_2 = \langle \nabla_{V_1} V_2, V_3 \rangle$ and

$$\nabla_{V_1} V_2 = -\left(c\cos\theta\, e - c\sin\theta\, \phi e + \sum_{i=1}^{s} \xi_i \right)$$

we obtain $k_2 = \sqrt{s}$. Similarly, since

$$\nabla_{V_1} V_3 = \sqrt{s}(\sin\theta\, e + \cos\theta\, \phi e)$$

and

$$\nabla_{V_1} V_3 = -k_2 V_2 + k_3 V_4$$

we obtain $k_3 V_4 = 0$. Thus, $k_3 = 0$. □

Let $\alpha: I \to M$ be a unit-speed curve in a 4-dimensional Riemannian manifold (M, g). The Frenet vectors of the curve α are

$$V_1 = \alpha', \quad V_2 = \frac{\alpha''}{\|\alpha''\|}, \quad V_4 = -\frac{\alpha' \times \alpha'' \times \alpha'''}{\|\alpha' \times \alpha'' \times \alpha'''\|}, \quad \text{and} \quad V_3 = V_4 \times V_1 \times V_2 \tag{27}$$

and the system $\{V_1, V_2, V_3, V_4\}$ is an orthonormal system in 4-dimensional space [11]. From Equation 27,

$$V_4 = -V_1 \times V_2 \times V_3 \tag{28}$$

is obtained.

Theorem 4.4. Let $M = \left(M^{2+2}, \phi, \xi_i, \eta_j, g\right)$, $i, j \in \{1,2\}$, be an S-manifold and $\gamma: I \to M^{2+2}$ be a Legendre curve of osculating order 4. Then, $V_4 = \frac{\varepsilon}{\sqrt{2}}(\xi_1 - \xi_2)$.

Proof.

From Equations 1 and 28,

$$\begin{aligned} V_4 = &\left(g(V_1, \phi V_2)\eta_2(V_3) - g(V_1, \phi V_3)\eta_2(V_2) + g(V_2, \phi V_3)\eta_2(V_1)\right)\xi_1 \\ &+\left(-g(V_1, \phi V_2)\eta_1(V_3) + g(V_1, \phi V_3)\eta_1(V_2) - g(V_2, \phi V_3)\eta_1(V_1)\right)\xi_2 + \begin{vmatrix} \phi V_1 & \phi V_2 & \phi V_3 \\ \eta_1(V_1) & \eta_1(V_2) & \eta_1(V_3) \\ \eta_2(V_1) & \eta_2(V_2) & \eta_2(V_3) \end{vmatrix} \end{aligned} \tag{29}$$

From Equations 9, 12, 13, 18, 28, and 29,

$$V_4 = \frac{\varepsilon}{\sqrt{2}}(\xi_1 - \xi_2)$$

is obtained. □

Example 4.5. Let $c_1, c_2 \in \mathbb{R}$. Then, the curve $\gamma: I \to \mathbb{R}^{2+2}(-6)$ defined by

$$\gamma(t) = \left(2\ln\left|\sqrt{1+t^2} + t\right|, 2\sqrt{1+t^2}, 4t + c_1, 4t + c_2\right)$$

is a unit speed Legendre curve. The tangent vector field of γ in terms of basis $\{e, \phi e, \xi_1, \xi_2\}$ is

$$V_1 = \frac{t}{\sqrt{1+t^2}}e + \frac{1}{\sqrt{1+t^2}}\phi e$$

From Equation 26,

$$\nabla_{V_1} e = -\frac{1}{\sqrt{1+t^2}}(\xi_1 + \xi_2)$$

$$\nabla_{V_1} \phi e = \frac{t}{\sqrt{1+t^2}}(\xi_1 + \xi_2)$$

$$\nabla_{V_1} \xi_i = \frac{1}{\sqrt{1+t^2}}(e - t\phi e), \quad i \in \{1,2\}$$

$$\nabla_{V_1} V_1 = \frac{1}{(1+t^2)^{3/2}}e - \frac{t}{(1+t^2)^{3/2}}\phi e$$

and

$$k_1 = \frac{1}{1+t^2}$$

then

$$V_2 = \frac{1}{\sqrt{1+t^2}}e - \frac{t}{\sqrt{1+t^2}}\phi \text{ and } \phi V_1 = -V_2$$

Thus,

$$\nabla_{V_1}V_2 = -\frac{t}{(1+t^2)^{3/2}}e - \frac{1}{(1+t^2)^{3/2}}\phi e - (\xi_1 + \xi_2)$$

and

$$\nabla_{V_1}V_2 = -k_1V_1 + k_2V_3$$

Therefore, we obtain

$$k_2 = \sqrt{2} \text{ and } V_3 = -\frac{\xi_1+\xi_2}{\sqrt{2}}$$

Since $\nabla_{V_1}V_3 = -\frac{\sqrt{2}}{\sqrt{1+t^2}}(e + t\phi e)$ and $\nabla_{V_1}V_3 = -k_2V_2 + k_3V_4$, we obtain $k_3 = 0$. From Equation 28,

$$V_4 = \begin{vmatrix} e & \phi e & \xi_1 & \xi_2 \\ \frac{1}{\sqrt{1+t^2}} & -\frac{t}{\sqrt{1+t^2}} & 0 & 0 \\ \frac{t}{\sqrt{1+t^2}} & \frac{1}{\sqrt{1+t^2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}(\xi_2 - \xi_1)$$

is obtained. □

Theorem 4.6. Let $M = \left(M^{2+3}, \phi, \xi_i, \eta_j, g\right)$, $i, j \in \{1,2,3\}$, be an S-manifold and $\gamma: I \to M^{2+3}$ be a Legendre curve of osculating order 5. γ is imbedded in the 3-dimensional K-contact space.

PROOF.

Let

$$U_1 = \cos\theta\, V_4 - \sin\theta\, V_5$$
$$U_2 = \sin\theta\, V_4 + \cos\theta\, V_5 \tag{30}$$

such that

$$\theta(s) = \int k_4 ds \tag{31}$$

From Equations 30 and 31,

$$\nabla_{V_1}U_1 = 0$$

and

$$\nabla_{V_1}U_2 = 0$$

Therefore, U_1 and U_2 are constant. From Equation 30, $\{V_1, V_2, V_3, U_1, U_2\}$ is an orthonormal basis. For the functions

$$f_i : I \to \mathbb{R}$$

$$s \to f_i(s) = \langle \gamma(s) - \gamma(0), U_i \rangle$$

such that $i \in \{1,2\}$ from Equations 30 and 31, we get, for all $s \in I$,

$$f_i'(s) = \langle V_1, U_i \rangle = 0$$
$$f_i(s) = c \in \mathbb{R}$$

and

$$f_i(0) = \langle \gamma(0) - \gamma(0), U_i \rangle = c = 0$$

Hence, for all $s \in I$,

$$f_i(s) = 0, \quad i \in \{1,2\}$$

Then, $\gamma(s) - \gamma(0) \in \mathrm{Sp}\{V_1, V_2, V_3\}$. Let $\omega = \{X \in \chi(M) : \ g(X, U_1) = 0 \wedge g(X, U_2) = 0\}$. Since

$$\omega = \mathrm{Sp}\{V_1, V_2, V_3\} \text{ and } \gamma(s) - \gamma(0) \in \omega$$

For the function

$$\pi : \chi(M) \to \omega$$

$$X \quad \to \pi(X) = \bar{X} + \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\xi$$

such that $\bar{X} \in D = \{X \in \chi(M) : \ \eta_i(X) = 0, \forall i \in \{1,2,3\}\}$, $X = \bar{X} + \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$, and $\xi = \frac{\xi_1+\xi_2+\xi_3}{3}$, if we get $\eta = \eta_1 + \eta_2 + \eta_3$, $\tilde{\phi} = \phi$, and $\tilde{g} = 3g$, then $(w^3, \tilde{\phi}, \xi, \eta, \tilde{g})$ is a K-contact space. Since $\tilde{g} = 3g$, then $\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$ and $\tilde{\nabla} = \nabla$. Because $d\eta = d\eta_1 + d\eta_2 + d\eta_3 = 3d\eta_1 = 3\Phi = 3g = \tilde{g}$, then $\tilde{\Phi} = d\eta$. Moreover,

$$\tilde{\nabla}_{\pi X}\xi = \nabla_{\pi X}\left(\frac{\xi_1 + \xi_2 + \xi_3}{3}\right) = \frac{1}{3}(\nabla_{\pi X}\xi_1 + \nabla_{\pi X}\xi_2 + \nabla_{\pi X}\xi_3) = \frac{-3}{3}\phi(\pi X) = -\phi(\pi X) = -\tilde{\phi}(\pi X)$$

As

$$\eta(V_1) = (\eta_1 + \eta_2 + \eta_3)(V_1) = \eta_1(V_1) + \eta_2(V_1) + \eta_3(V_1) = 0$$

The curve γ is also a Legendre curve at ω. □

5. Conclusion

This study generalized the cross product defined in 3-dimensional almost contact metric manifolds and defined a new generalized cross product in $(2n+s)$-dimensional framed metric manifolds such that $n = 1$. Moreover, it characterized the curvatures of Legendre curves in S-manifolds. Moreover, this study proved that Legendre curves are biharmonic. Besides, it demonstrated that $(2+3)$-dimensional S-manifolds are imbedded in 3-dimensional space. In future studies, researchers can investigate Slant curves in S-manifolds using the generalized cross product herein.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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