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Statistical Convergence of Double Sequences in Intuitionistic Fuzzy Metric Spaces

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Abstract — Statistical convergence has been a prominent research area in mathematics since this concept was independently introduced by Fast and Steinhaus in 1951. Afterward, the statistical convergence of double sequences in metric spaces and fuzzy metric spaces has been widely studied. The main goal of the present study is to introduce the concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy metric spaces. Moreover, this study characterizes the statistical convergence of a double sequence by an ordinary convergent of a subsequence of the double sequence. Besides, the current study theoretically contributes to the mentioned concepts and investigates some of their basic properties. Finally, the paper handles whether the aspects should be further investigated.

Keywords *Statistical convergence, statistical Cauchy sequences, double sequences, intuitionistic fuzzy metric spaces*

Mathematics Subject Classification (2020) 40A05, 40A35

1. Introduction

Statistical convergence, a generalization of ordinary convergence, is based on the natural density of a subset of ℕ, the set of all the natural numbers. Fast [1] and Steinhaus [2] have established this concept separately in 1951. Many mathematicians particularly Salat [3], Freedman and Sember [4], Fridy [5], Connor [6], Kolk [7], Fridy and Orhan [8], and Bulut and Or [9], have contributed to the development of statistical convergence. Pringshem [10] has introduced the convergence and Cauchy sequence of double sequences. After that, Mursaleen and Edely [11] have studied double sequences' statistical convergence.

Fuzzy sets, defined by Zadeh [12], have been used in many fields, such as artificial intelligence, decision-making, image analysis, probability theory, and weather forecasting. In addition, Kramosil and Michalek [13] and Kaleva and Seikkala [14] have first examined fuzzy metric spaces (FMSs). Further, George and Veeramani [15] have redefined the concept of fuzzy metrics to construct the Hausdorff topology. Lately, Mihet [16] has studied the point convergence (p-convergence), a weaker concept than ordinary convergence. Moreover, Gregori et al. [17] have put forward the concept of s-convergence. Morillas and Sapena [18] have defined the standard convergence (std-convergence). Gre-

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gori and Miňana [19] have introduced the concept of strong convergence (st-convergence), a stronger concept than ordinary convergence. Li et al. [20] have proposed the statistical convergence and statistical Cauchy sequence in FMSs and examined some of their basic properties. After that, Park [21] has recently introduced the intuitionistic fuzzy metric spaces (IFMSs) by intuitionistic fuzzy sets, defined by Atanassov [22], and triangular norms and triangular conorms [23]. Moreover, Park [21] studied the convergence sequence concerning in IFMSs. Varol [24] has suggested the statistical convergence in IFMSs and analyzed statistical Cauchy sequences in IFMSs. Besides, Savaş [25] has introduced the statistical convergence and statistical Cauchy sequences for the double sequences in FMSs. Motivated the article [25] and the studies done in the literature on this subject, this paper defines statistical convergence and statistical Cauchy sequences for double sequences in IFMSs.

Section 2 of the handled study provides some basic definitions and properties to be needed in the following sections. Section 3 describes statistical convergence and statistical Cauchy sequences for double sequences in IFMSs. Finally, it discusses the need for further research.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following sections.

Definition 2.1. [4] The natural density of a set $A \subseteq \mathbb{N}$ is defined by

$$\delta(A) = \lim_{n \to \infty} \frac{|\{k \in A : k \leq n\}|}{n}$$

where $|\{k \in A : k \leq n\}|$ denotes the number of elements of A that do not exceed n. It can be observed that if the set A is finite, then $\delta(A) = 0$.

Throughout this paper, $\mathbb{Y}$ denotes $\mathbb{N} \times \mathbb{N}$.

Definition 2.2. [11] The double natural density of a set $A \subseteq \mathbb{Y}$ is defined by

$$\delta_2(A) = \lim_{m,n \to \infty} \frac{|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|}{mn}$$

where $|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|$ denotes the number of elements of A, whose the first and second components do not exceed m and n, respectively. It can be observed that if the set A is finite, then $\delta_2(A) = 0$.

Definition 2.3. [11] Let (x_{jk}) be a double sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Then, (x_{jk}) is called statistically convergent to x_0, if, for all $\varepsilon > 0$, $\delta_2(\{(j,k) \in \mathbb{Y} : |x_{jk} - x_0| \geq \varepsilon\}) = 0$ and is denoted by $st_2 - \lim_{j,k \to \infty} x_{jk} = x_0$.

Definition 2.4. [23] Let $\oplus : [0,1]^2 \to [0,1]$ be a binary operation. Then, $\oplus$ is called a triangular norm (t-norm), if it satisfies the following conditions:

i. $\oplus$ is associative and commutative.

ii. $a \oplus 1 = a$, for all $a \in [0,1]$.

iii. If $a_1 \leq a_3$ and $a_2 \leq a_4$, for each $a_1, a_2, a_3, a_4 \in [0,1]$, then $a_1 \oplus a_3 \leq a_2 \oplus a_4$.

Definition 2.5. [23] Let $\otimes : [0,1]^2 \to [0,1]$ be a binary operation. Then, $\otimes$ is referred to as a triangular conorm (t-conorm), if it satisfies the following conditions:

i. $\otimes$ is associative and commutative

ii. $a \otimes 0 = a$, for all $a \in [0,1]$.

iii. If $a_1 \leq a_3$ and $a_2 \leq a_4$, for each $a_1, a_2, a_3, a_4 \in [0,1]$, then $a_1 \otimes a_3 \leq a_2 \otimes a_4$.

Example 2.6. [21] The following operators are basic examples of t-norms and t-conorms, respectively:

i. $a_1 \oplus a_2 = a_1 a_2$

ii. $a_1 \oplus a_2 = \min\{a_1, a_2\}$

iii. $a_1 \otimes a_2 = \max\{a_1, a_2\}$

iv. $a_1 \otimes a_2 = \min\{a_1 + a_2, 1\}$

Definition 2.7. [21] Let $\mathbb{B}$ be an arbitrary set, $\oplus$ be a continuous t-norm, $\otimes$ be a continuous t-conorm, and φ, ϑ be fuzzy sets on $\mathbb{B}^2 \times (0, \infty)$. For all $x_1, x_2, x_3 \in \mathbb{B}$ and $u, s > 0$, if φ and ϑ satisfy the following conditions:

i. $\varphi(x_1, x_2, u) + \vartheta(x_1, x_2, u) \leq 1$

ii. $\varphi(x_1, x_2, u) > 0$

iii. $\varphi(x_1, x_2, u) = 1 \Leftrightarrow x_1 = x_2$

iv. $\varphi(x_1, x_2, u) = \varphi(x_2, x_1, u)$

v. $\varphi(x_1, x_3, u + s) \geq \varphi(x_1, x_2, u) \oplus \varphi(x_2, x_3, s)$

vi. The function $\varphi_{x_1 x_2} : (0, \infty) \to (0, 1]$, defined by $\varphi_{x_1 x_2} = \varphi(x_1, x_2, u)$, is continuous

vii. $\vartheta(x_1, x_2, u) > 0$

viii. $\vartheta(x_1, x_2, u) = 0 \Leftrightarrow x_1 = x_2$

ix. $\vartheta(x_1, x_2, u) = \vartheta(x_2, x_1, u)$

x. $\vartheta(x_1, x_3, u + s) \leq \vartheta(x_1, x_2, u) \otimes \vartheta(x_2, x_3, s)$

xi. The function $\vartheta_{x_1 x_2} : (0, \infty) \to (0, 1]$, defined by $\vartheta_{x_1 x_2} = \vartheta(x_1, x_2, u)$, is continuous

then a 5-tuple $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ is said to be an intuitionistic fuzzy metric space (IFMS).

The values $\varphi(x_1, x_2, u)$ and $\vartheta(x_1, x_2, u)$ represent the degree of nearness and non-nearness of x_1 and x_2 concerning u, respectively.

Example 2.8. [21] Let $(\mathbb{B}, d)$ be a metric space. Define $a_1 \oplus a_2 = a_1 a_2$ and $a_1 \otimes a_2 = \min\{a_1 + a_2, 1\}$, for all $a_1, a_2 \in [0, 1]$, and suppose that φ and ϑ are fuzzy sets on $\mathbb{B}^2 \times (0, \infty)$ defined by

$$\varphi(x_1, x_2, u) = \frac{u}{u + d(x_1, x_2)} \text{ and } \vartheta(x_1, x_2, u) = \frac{d(x_1, x_2)}{u + d(x_1, x_2)}$$

for all $x_1, x_2 \in \mathbb{B}$ and $u > 0$. Then, $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ is an IFMS.

Remark 2.9. [24] If $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ is an IFMS, then $(\mathbb{B}, \varphi, \oplus)$ is an FMS. Moreover, if $(\mathbb{B}, \varphi, \oplus)$ is an FMS, then $(\mathbb{B}, \varphi, 1 - \varphi, \oplus, \otimes)$ is an IFMS such that $a_1 \otimes a_2 = 1 - [(1 - a_1) \oplus (1 - a_2)]$, for all $a_1, a_2 \in [0, 1]$.

Definition 2.10. [21] Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{B}$ is said to be convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0, 1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n \geq n_\varepsilon$ implies that

$$\varphi(x_n, x_0, u) > 1 - \varepsilon \text{ and } \vartheta(x_n, x_0, u) < \varepsilon$$

or equivalently

$$\lim_{n \to \infty} \varphi(x_n, x_0, u) = 1 \text{ and } \lim_{n \to \infty} \vartheta(x_n, x_0, u) = 0$$

and is denoted by $\frac{\varphi}{\vartheta} - \lim_{n \to \infty} x_n = x_0$ or $x_n \xrightarrow{\frac{\varphi}{\vartheta}} x_0$ as $n \to \infty$.

Definition 2.11. [21] Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a sequence (x_n) is referred to as a Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $u > 0$ and $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n, N \geq n_\varepsilon$ implies that

$$\varphi(x_n, x_N, u) > 1 - \varepsilon \text{ and } \vartheta(x_n, x_N, u) < \varepsilon$$

or equivalently

$$\lim_{n,N\to\infty} \varphi(x_n, x_N, u) = 1 \text{ and } \lim_{n,N\to\infty} \vartheta(x_n, x_N, u) = 0$$

Definition 2.12. [24] Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{B}$ is called statistically convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0,1)$ and $u > 0$,

$$\delta(\{n \in \mathbb{N} : \varphi(x_n, x_0, u) \leq 1 - \varepsilon \text{ or } \vartheta(x_n, x_0, u) \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{|\{k \leq n : \varphi(x_k, x_0, u) \leq 1 - \varepsilon \text{ or } \vartheta(x_k, x_0, u) \geq \varepsilon\}|}{n} = 0$$

Definition 2.13. [24] Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a sequence (x_n) is said to be a statistically Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $N \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : \varphi(x_n, x_N, u) \leq 1 - \varepsilon \text{ or } \vartheta(x_n, x_N, u) \geq \varepsilon\}) = 0$$

Definition 2.14. [25] Let $(\mathbb{B}, \varphi, \oplus)$ be an FMS. Then, a double sequence (x_{jk}) in $\mathbb{B}$ is called statistically convergent to $x_0 \in \mathbb{B}$ concerning fuzzy metric φ, if, for all $u > 0$ and $\varepsilon \in (0,1)$,

$$\delta_2(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) \leq 1 - \varepsilon\}) = 0$$

and is denoted by $st_2 s - \lim_{j,k\to\infty} x_{jk} = x_0$.

Definition 2.15. [25] Let $(\mathbb{B}, \varphi, \oplus)$ be an FMS. Then, a double sequence (x_{jk}) is referred as a statistically Cauchy sequence in $\mathbb{B}$ concerning fuzzy metric φ, if, for all $u > 0$ and $\varepsilon \in (0,1)$, there exists $m, n \in \mathbb{N}$ such that

$$\delta_2(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_{mn}, u) \leq 1 - \varepsilon\}) = 0$$

3. Main Results

This section defines the concepts of statistical convergence and statistical Cauchy sequences for double sequences in IFMSs. In addition, it provides some of their basic properties.

Definition 3.1. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a double sequence (x_{jk}) in $\mathbb{B}$ is said to be convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $j, k \geq n_\varepsilon$ implies that

$$\varphi(x_{jk}, x_0, u) > 1 - \varepsilon \text{ and } \vartheta(x_{jk}, x_0, u) < \varepsilon$$

or equivalently

$$\lim_{j,k\to\infty} \varphi(x_{jk}, x_0, u) = 1 \text{ and } \lim_{j,k\to\infty} \vartheta(x_{jk}, x_0, u) = 0$$

and is denoted by $\frac{\varphi}{\vartheta} - \lim_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{\frac{\varphi}{\vartheta}} x_0$ as $j, k \to \infty$.

Definition 3.2. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a double sequence (x_{jk}) in $\mathbb{B}$ is called statistically convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0,1)$ and

$u > 0$,

$$\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, x_0, u) \geq \varepsilon\}\right) = 0$$

or equivalently

$$\lim_{m,n\to\infty} \frac{|\{(j,k) \in \mathbb{Y} : (j \leq m \text{ and } k \leq n) \text{ and } (\varphi(x_{jk}, x_0, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, x_0, u) \geq \varepsilon)\}|}{mn} = 0$$

and is denoted by ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{{}^{\varphi}_{\vartheta}st_2} x_0$ as $j,k \to \infty$.

Example 3.3. Let $\mathbb{B} = \mathbb{R}$, $a_1 \oplus a_2 = a_1 a_2$ and $a_1 \otimes a_2 = \min\{a_1 + a_2, 1\}$ for all $a_1, a_2 \in [0,1]$. Define φ and ϑ by

$$\varphi(x_1, x_2, u) = \frac{u}{u + |x_1 - x_2|} \text{ and } \vartheta(x_1, x_2, u) = \frac{|x_1 - x_2|}{u + |x_1 - x_2|}$$

for all $x_1, x_2 \in \mathbb{R}$ and $u > 0$. Then, $(\mathbb{R}, \varphi, \vartheta, \oplus, \otimes)$ is an IFMS [21]. Moreover, define a sequence (x_{jk}) by

$$x_{jk} := \begin{cases} 1, & j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise} \end{cases}$$

Then, for all $\varepsilon \in (0,1)$ and for any $u > 0$, let

$$K = \{(j,k) \in \mathbb{Y} : (j \leq m \text{ and } k \leq n) \text{ and } (\varphi(x_{jk}, 0, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, 0, u) \geq \varepsilon)\}$$

Hence,

$$\begin{aligned} K &= \left\{(j,k) \in \mathbb{Y} : (j \leq m \text{ and } k \leq n) \text{ and } \left(\tfrac{u}{u+|x_{jk}|} \leq 1-\varepsilon \text{ or } \tfrac{|x_{jk}|}{u+|x_{jk}|} \geq \varepsilon\right)\right\} \\ &= \{(j,k) \in \mathbb{Y} : j \leq m,\ k \leq n, \text{ and } x_{jk} = 1\} \\ &= \{(j,k) \in \mathbb{Y} : j \leq m,\ k \leq n, \text{ and } j \text{ and } k \text{ are squares }\} \end{aligned}$$

and thus

$$\frac{|K|}{mn} = \frac{|\{(j,k) \in \mathbb{Y} : j \leq m,\ k \leq n, \text{ and } j \text{ and } k \text{ are squares }\}|}{mn} \leq \frac{\sqrt{m}\sqrt{n}}{mn} \to 0 \text{ as } m,n \to \infty$$

Consequently, (x_{jk}) is statistically convergent to 0 concerning intuitionistic fuzzy metric (φ, ϑ).

Lemma 3.4. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS, (x_{jk}) be a double sequence in $\mathbb{B}$, and $x_0 \in \mathbb{B}$. Then, for all $\varepsilon \in (0,1)$ and $u > 0$, the following statements are equivalent:

i. ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_0$

ii. $\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) > 1-\varepsilon\}\right) = \delta_2\left(\{(j,k) \in \mathbb{Y} : \vartheta(x_{jk}, x_0, u\} < \varepsilon\}\right) = 1$

iii. $\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) \leq 1-\varepsilon\}\right) = \delta_2\left(\{(j,k) \in \mathbb{Y} : \vartheta(x_{jk}, x_0, u) \geq \varepsilon\}\right) = 0$

Proof.
The proof is straightforward using Definition 3.2 and the density function's properties. □

Theorem 3.5. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. If a double sequence (x_{jk}) in $\mathbb{B}$ is statistically convergent concerning intuitionistic fuzzy metric (φ, ϑ), then the statistically limit is unique.

Proof.
Suppose that ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_1$, ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_2$, and $x_1 \neq x_2$. For a given $\varepsilon \in (0,1)$, choose $\eta \in (0,1)$ such that $(1-\eta) \oplus (1-\eta) > 1-\varepsilon$ and $\eta \otimes \eta < \varepsilon$. Then, define the following sets, for any $u > 0$,

$$K_1(\eta, u) := \{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_1, u) \leq 1-\eta\}$$

$$K_2(\eta, u) := \{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_2, u) \leq 1-\eta\}$$

$$T_1(\eta, u) := \{(j,k) \in \mathbb{Y} : \vartheta\left(x_{jk}, x_1, u\right) \geq \eta\}$$

and

$$T_2(\eta, u) := \{(j,k) \in \mathbb{Y} : \vartheta\left(x_{jk}, x_2, u\right) \geq \eta\}$$

By Lemma 3.4, since (x_{jk}) is statistically convergent to x_1 and x_2 concerning intuitionistic fuzzy metric (φ, ϑ), for $u > 0$,

$$\delta_2\left(K_1(\eta, u)\right) = \delta_2\left(T_1(\eta, u)\right) = 0 = \delta_2\left(K_2(\eta, u)\right) = \delta_2\left(T_2(\eta, u)\right)$$

Let

$$A(\eta, u) := \left(K_1(\eta, u) \cup K_2(\eta, u)\right) \cap \left(T_1(\eta, u) \cup T_2(\eta, u)\right)$$

for $u > 0$. Hence, $\delta_2\left(A(\eta, u)\right) = 0$ which implies that $\delta_2\left(\mathbb{Y} \backslash A(\eta, u)\right) = 1$. If $(j,k) \in \mathbb{Y} \backslash A(\eta, u)$, then

$$(j,k) \in \mathbb{Y} \backslash \left(K_1(\eta, u) \cup K_2(\eta, u)\right) \text{ or } (j,k) \in \mathbb{Y} \backslash \left(T_1(\eta, u) \cup T_2(\eta, u)\right)$$

Let $(j,k) \in \mathbb{Y} \backslash \left(K_1(\eta, u) \cup K_2(\eta, u)\right)$. Then,

$$\varphi\left(x_1, x_2, u\right) \geq \varphi\left(x_1, x_{jk}, \frac{u}{2}\right) \oplus \varphi\left(x_{jk}, x_2, \frac{u}{2}\right) > (1-\eta) \oplus (1-\eta) > 1 - \varepsilon$$

Therefore, $\varphi\left(x_1, x_2, u\right) > 1 - \varepsilon$. Since $\varepsilon \in (0,1)$ is arbitrary, for all $u > 0$, $\varphi\left(x_1, x_2, u\right) = 1$ and thus $x_1 = x_2$.

Let $(j,k) \in \mathbb{Y} \setminus \left(T_1(\eta, u) \cup T_2(\eta, u)\right)$. Then,

$$\vartheta\left(x_1, x_2, u\right) \leq \vartheta\left(x_1, x_{jk}, \frac{u}{2}\right) \otimes \vartheta\left(x_{jk}, x_2, \frac{u}{2}\right) < \eta \otimes \eta < \varepsilon$$

Therefore, $\vartheta\left(x_1, x_2, u\right) < \varepsilon$. Since $\varepsilon \in (0,1)$ is arbitrary, for all $u > 0$, $\vartheta\left(x_1, x_2, u\right) = 0$ and thus $x_1 = x_2$. □

Theorem 3.6. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. If (x_{jk}) is convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), then it is statistically convergent to x_0 concerning intuitionistic fuzzy metric (φ, ϑ).

Proof.
Let (x_{jk}) be convergent to x_0 concerning intuitionistic fuzzy metric (φ, ϑ). Then, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that $j, k \geq n_0$ implies that $\varphi\left(x_{jk}, x_0, u\right) > 1 - \varepsilon$ and $\vartheta\left(x_{jk}, x_0, u\right) < \varepsilon$. Hence, the set

$$\{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, u\right) \leq 1 - \varepsilon \text{ or } \vartheta\left(x_{jk}, x_0, u\right) \geq \varepsilon\}$$

has a finite number of terms. Therefore,

$$\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, u\right) \leq 1 - \varepsilon \text{ or } \vartheta\left(x_{jk}, x_0, u\right) \geq \varepsilon\}\right) = 0$$

Consequently, (x_{jk}) is statistically convergent to x_0 concerning intuitionistic fuzzy metric (φ, ϑ). □

The converse of Theorem 3.6 is not always correct.

Example 3.7. For the IFMS provided in Example 3.3, define a double sequence (x_{jk}) by

$$x_{jk} = \begin{cases} jk, & j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise} \end{cases}$$

(x_{jk}) is statistically convergent to 0 concerning intuitionistic fuzzy metric (φ, ϑ). However, (x_{jk}) is not convergent to 0 concerning intuitionistic fuzzy metric (φ, ϑ).

Theorem 3.8. Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. Then, (x_{jk}) is statistically convergent to $x_0\in\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ) if and only if there exists a subset $K\subset\mathbb{Y}$ such that

$$\delta_2(K)=1 \quad \text{and} \quad {}^{\varphi}_{\vartheta}-\lim_{\substack{m,n\to\infty\\(m,n)\in K}} x_{mn}=x_0$$

PROOF.
$(\Rightarrow:)$ Let ${}^{\varphi}_{\vartheta}st_2-\lim\limits_{j,k\to\infty} x_{jk}=x_0$ and

$$K_r(u)=\left\{(j,k)\in\mathbb{Y}:\varphi\left(x_{jk},x_0,u\right)>1-\frac{1}{r} \text{ and } \vartheta\left(x_{jk},x_0,u\right)<\frac{1}{r}\right\},\quad r\in\mathbb{N}$$

Then, according to Definition 3.2,

$$\delta_2\left(K_r(u)\right)=1 \tag{1}$$

such that $r\in\mathbb{N}$. From the definition of $K_r(u)$, it is clear that

$$K_1(u)\supset K_2(u)\supset\cdots\supset K_r(u)\supset K_{r+1}(u)\supset\cdots \tag{2}$$

such that $r\in\mathbb{N}$. Choose an arbitrary element $(t_1,s_1)\in K_1(u)$. According to Equation 1, there exists such that a $(t_2,s_2)\in K_2(u)$ satisfying the conditions $t_2>t_1$ and $s_2>s_1$, for each (m,n) such that $m>t_2$ and $n>s_2$,

$$\frac{K_2(u)(m,n)}{mn}>\frac{1}{2}$$

where

$$K_2(u)(m,n)=\sum_{\substack{k\le m\\ l\le n\\ (k,l)\in K_2(u)}}1$$

Further, according to Equation 1, there exists such that a $(t_3,s_3)\in K_3(u)$ satisfying the conditions $t_3>t_2$ and $s_3>s_2$, for each (m,n) such that $m>t_3$ and $n>s_3$,

$$\frac{K_3(u)(m,n)}{mn}>\frac{2}{3}$$

etc. Therefore, by induction, construct a sequence (t_r,s_r) of the set $\mathbb{Y}$ such that

$$t_1<t_2<\ldots<t_r<\ldots \text{ and } s_1<s_2<\ldots<s_r<\ldots$$

$(t_r,s_r)\in K_r(u)$, for all $r\in\mathbb{N}$, and

$$\frac{K_r(u)(m,n)}{mn}>\frac{r-1}{r},\quad r\in\mathbb{N} \tag{3}$$

for each (m,n) where $m\ge t_r$ and $n\ge s_r$. Form the set K as follows: Each element between $(1,1)$ and (t_1,s_1) belongs to the set K, further, any element between (t_r,s_r) and (t_{r+1},s_{r+1}) belongs to K if and only if it belongs to $K_r(u)$ such that $r\in\mathbb{N}$. According to Equations 1 and 3, for each (m,n) such that $t_r\le m<t_{r+1}$ and $s_r\le n<s_{r+1}$,

$$\frac{K(m,n)}{mn}>\frac{K_r(u)(m,n)}{mn}>\frac{r-1}{r}$$

Thus, it is obvious that $\delta_2(K)=1$. Let $u>0$ and $\varepsilon\in(0,1)$. Choose an r such that $\frac{1}{r}<\varepsilon$. Let $(m,n)\in K$ such that $m\ge t_r$ and $n\ge s_r$. Then, there exists a number $l\ge r$ such that $t_l\le m<t_{l+1}$, $s_l\le n<s_{l+1}$, and $(m,n)\in K_l$. Hence,

$$\varphi\left(x_{mn},x_0,u\right)>1-\frac{1}{l}\ge 1-\frac{1}{r}>1-\varepsilon \text{ and } \vartheta\left(x_{mn},x_0,u\right)<\frac{1}{l}\le\frac{1}{r}<\varepsilon$$

Thereby, $\varphi\left(x_{mn}, x_0, u\right) > 1-\varepsilon$ and $\vartheta\left(x_{mn}, x_0, u\right) < \varepsilon$, for each $(m,n) \in K$ where $m \geq t_r$ and $n \geq s_r$, i.e.,

$$\tfrac{\varphi}{\vartheta} - \lim_{\substack{m,n\to\infty \\ (m,n)\in K}} x_{mn} = x_0$$

($\Leftarrow$:) Suppose that there exists a set $K = \{(m,n) \in \mathbb{Y} : m,n = 1,2,\ldots\}$ such that $\delta_2(K) = 1$ and $\tfrac{\varphi}{\vartheta} - \lim_{\substack{m,n\to\infty \\ (m,n)\in K}} x_{mn} = x_0$, i.e., for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $m,n \geq n_\varepsilon$ implies that $\varphi\left(x_{mn}, x_0, u\right) > 1-\varepsilon$ and $\vartheta\left(x_{mn}, x_0, u\right) < \varepsilon$. Hence,

$$A(\varepsilon,u) := \{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, u\right) \leq 1-\varepsilon \text{ or } \vartheta\left(x_{jk}, x_0, u\right) \geq \varepsilon\} \subseteq \mathbb{Y} \setminus \{\left(j_{n_\varepsilon+1}, k_{n_\varepsilon+1}\right), \left(j_{n_\varepsilon+2}, k_{n_\varepsilon+2}\right), \cdots\}$$

Therefore, $\delta_2\left(A(\varepsilon,u)\right) = 0$. Consequently, $\tfrac{\varphi}{\vartheta} st_2 - \lim_{j,k\to\infty} x_{jk} = x_0$. □

Definition 3.9. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a double sequence (x_{jk}) is referred to as a Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $u > 0$ and $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that $j \geq p \geq n_\varepsilon$ and $k \geq q \geq n_\varepsilon$ implies that

$$\varphi(x_{jk}, x_{pq}, u) > 1-\varepsilon \text{ and } \vartheta(x_{jk}, x_{pq}, u) < \varepsilon$$

or equivalently

$$\lim_{p,q\to\infty} \varphi(x_{jk}, x_{pq}, u) = 1 \text{ and } \lim_{p,q\to\infty} \vartheta(x_{jk}, x_{pq}, u) = 0$$

Definition 3.10. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a double sequence (x_{jk}) is called a statistically Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $(p,q) \in \mathbb{Y}$ such that

$$\delta_2(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_{pq}, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, x_{pq}, u) \geq \varepsilon\}) = 0$$

Theorem 3.11. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. If (x_{jk}) is statistically convergent concerning intuitionistic fuzzy metric (φ, ϑ), then it is a statistically Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ).

PROOF.
Let $\tfrac{\varphi}{\vartheta} st_2 - \lim_{j,k\to\infty} x_{jk} = x_0$. Then, for all $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $(1-\varepsilon_2) \oplus (1-\varepsilon_2) > 1-\varepsilon_1$ and $\varepsilon_2 \otimes \varepsilon_2 < \varepsilon_1$, and for all $u > 0$,

$$\delta_2\left(\left\{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, \frac{u}{2}\right) \leq 1-\varepsilon_1 \text{ or } \vartheta\left(x_{jk}, x_0, \frac{u}{2}\right) \geq \varepsilon_1\right\}\right) = 0$$

In particular, for $j = p$ and $k = q$,

$$\delta_2\left(\left\{(p,q) \in \mathbb{Y} : \varphi\left(x_{pq}, x_0, \frac{u}{2}\right) \leq 1-\varepsilon_1 \text{ or } \vartheta\left(x_{pq}, x_0, \frac{u}{2}\right) \geq \varepsilon_1\right\}\right) = 0$$

Since

$$\varphi\left(x_{jk}, x_{pq}, u\right) \geq \varphi\left(x_{jk}, x_0, \frac{u}{2}\right) \oplus \varphi\left(x_{pq}, x_0, \frac{u}{2}\right) \geq (1-\varepsilon_2) \oplus (1-\varepsilon_2) > 1-\varepsilon_1$$

and

$$\vartheta\left(x_{jk}, x_{pq}, u\right) \leq \vartheta\left(x_{jk}, x_0, \frac{u}{2}\right) \otimes \vartheta\left(x_{pq}, x_0, \frac{u}{2}\right) < \varepsilon_2 \otimes \varepsilon_2 < \varepsilon_1$$

then

$$\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_{pq}, u\right) \leq 1-\varepsilon_1 \text{ or } \vartheta\left(x_{jk}, x_{pq}, u\right) \geq \varepsilon_1\}\right) = 0$$

That is, (x_{jk}) is a statistically Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ). □

Theorem 3.12. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. Then, the following statements are equivalent.

i. (x_{jk}) is a statistically Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ).

ii. There exists a subset $K \subset \mathbb{Y}$ such that $\delta_2(K) = 1$ and the subsequence (x_{mn}), indexed by elements in K, of the sequence (x_{jk}) is a Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ).

PROOF.
The proof is similar to the proof of Theorem 3.8. □

4. Conclusion

This paper investigated the concept of statistical convergence, a generalization of ordinary convergence, for the double sequences in IFMSs. Additionally, it researched statistical Cauchy sequences and revealed characterizations of these concepts for double sequences. In further works, researchers can study the concept of Lacunary convergence in an IFMS using the concepts and results herein and analyze some of its basic properties.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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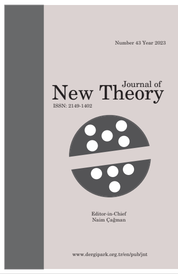

On Soft Normed Quasilinear Spaces

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Abstract — In this study, we investigate some properties of soft quasi-sequences and present new results. We then study the completeness of soft normed quasilinear space and present an analog of convergence and boundness results of soft quasi sequences in soft normed quasilinear spaces. Moreover, we define regular and singular subspaces of soft quasilinear spaces and draw several conclusions related to these notions. Afterward, we provide examples of these results in soft normed quasilinear spaces generalizing well-known results in soft linear normed spaces. Additionally, we offer new results concerning soft quasi subspaces of soft normed quasilinear spaces. Finally, we discuss the need for further research.

Keywords *Soft set, soft quasilinear space, soft normed quasilinear space, soft quasi vector, convergence*

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1. Introduction

Quasilinear spaces and normed quasilinear spaces were introduced by Aseev [1]. Then, several results on normed quasilinear spaces were obtained by defining proper quasilinear spaces in [2–4]. Later on, the quasilinear functions with bounded interval values were studied, and the Hahn-Banach extension theorem was analyzed in [5, 6], respectively. Then, quasilinear inner product spaces, generalizations of inner product spaces, were defined to develop quasilinear functional analysis in [7–10]. In addition, Yılmaz et al. [11] demonstrated that Hilbert quasilinear spaces are a special class of fuzzy number sequences. In [12, 13], Levent and Yılmaz included some quasilinear applications, such as signal processing.

Molodtsov [14] introduced soft sets in 1999. His next presentation covered a variety of applications of this theory in economics, engineering, and medicine. Following that, Maji et al. [15] presented several operations on soft sets. After, Das and Samanta introduced soft elements [16] and soft real numbers [17]. Additionally, they worked on soft linear operators, soft linear spaces, soft inner product spaces, and some of their features in [18–21]. Afterward, they introduced soft normed spaces in a novel perspective, along with soft inner product spaces and soft Hilbert spaces in [22, 23], respectively.

Bozkurt [24] introduced soft quasilinear spaces and soft normed quasilinear spaces, being more generalized than the previous notions of soft linear spaces and quasilinear spaces. Afterward, Bozkurt and Gönci defined soft inner product quasilinear spaces and soft Hilbert quasilinear spaces in [25]. Moreover, they worked on some properties of soft inner product quasilinear spaces.

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In this study, we give some properties of soft quasi-sequences and several new theorems related to their convergence in soft normed quasilinear spaces. Moreover, we study the completeness of soft normed quasilinear spaces. Besides, we define the regular and singular subspaces of a soft quasilinear space and find several conclusions that are related to these notions. In addition, we provide a few examples of soft normed quasilinear spaces. Finally, we discuss the need for further research.

2. Preliminaries

The objective of this section is to introduce some concepts in soft set theory and some basic notions, such as soft quasilinear spaces and soft normed quasilinear spaces, concerning soft set theory. Let Q be a universe, P be a set of parameters, $P(Q)$ be the power set of Q, and B be a non-empty subset of P.

Definition 2.1. [14] A pair (G, P) is called a soft set over Q, where G is a mapping defined by $G : P \to P(Q)$.

Definition 2.2. [19] A soft set (G, P) over Q is said to be an absolute soft set represented by $\widetilde{Q}$, if $G(\gamma) = Q$, for every $\gamma \in P$. A soft set (G, P) over Q is said to be a null soft set represented by Φ, if $G(\gamma) = \emptyset$, for every $\gamma \in P$.

Definition 2.3. [17] Let Q be a non-empty set and P be a non-empty parameter set. Then, a function $q : P \to Q$ is said to be a soft element of Q. A soft element q of Q is said to belong to a soft set G of Q, which is denoted by $q \in Q$, if $q(\gamma) \in G(\gamma)$, $\gamma \in P$. Thus, for a soft set G of Q with respect to the index set P, we get $G(\gamma) = \{q(\gamma), \gamma \in P\}$. A soft set (G, P) for which $G(\gamma)$ is a singleton set, for all $\gamma \in P$, can be determined with a soft element by simply determining the singleton set with the element that it contains, for all $\gamma \in P$.

The set of all the soft sets (G, P) over Q will be described by $S\left(\widetilde{Q}\right)$ for which $G(\gamma) \neq \emptyset$, for all $\gamma \in P$ and the collection of all the soft elements of (G, P) over Q will be denoted by $SE\left(\widetilde{Q}\right)$.

Definition 2.4. [24] Let Q be a quasilinear space, P be a parameter set, and G be a soft set over (Q, P). Then, G is said to be a soft quasilinear space of Q if $Q(\gamma)$ is a quasilinear subspace of Q, for every $\gamma \in P$.

Remark 2.5. [24] Soft quasi vectors in a soft quasilinear space are represented by $\widetilde{q}$, $\widetilde{w}$, and $\widetilde{z}$, and $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are used to specify soft real numbers.

Definition 2.6. If a soft quasi element $\widetilde{q}$ has an inverse, i.e., $\widetilde{q} - \widetilde{q} = \widetilde{\theta}$ such that $\widetilde{q}(\gamma) - \widetilde{q}(\gamma) = \widetilde{\theta}(\gamma)$, for every $\gamma \in P$, then it is called regular. If a soft quasi element $\widetilde{q}$ has no inverse, then it is called singular.

Definition 2.7. [24] Let $\widetilde{Q}$ be the absolute soft quasilinear space, i.e., $\widetilde{Q}(\gamma) = Q$, for every $\gamma \in P$ and $\mathbb{R}(P)$ denote all soft real numbers. Then, a mapping $\|.\| : SE\left(\widetilde{Q}\right) \longrightarrow \mathbb{R}(P)$ is said to be the soft norm on the soft quasilinear space $\widetilde{Q}$ if $\|.\|$ satisfies the following conditions:

i. $\|\widetilde{q}\| \geq 0$ if $\widetilde{q} \neq \widetilde{\theta}$, for every $\widetilde{q} \in \widetilde{Q}$.

ii. $\|\widetilde{q} + \widetilde{w}\| \leq \|\widetilde{q}\| + \|\widetilde{w}\|$, for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

iii. $\|\widetilde{\alpha} \cdot \widetilde{q}\| = |\widetilde{\alpha}| \cdot \|\widetilde{q}\|$, for every $\widetilde{q} \in \widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$.

iv. If $\widetilde{q} \widetilde{\preceq} \widetilde{w}$, then $\|\widetilde{q}\| \leq \|\widetilde{w}\|$, for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

v. If, for any $\varepsilon > 0$, there exists a quasi vector $\widetilde{q}_\varepsilon \in \widetilde{Q}$ such that $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_\varepsilon$ and $\|\widetilde{q}_\varepsilon\| \leq \varepsilon$, then $\widetilde{q} \widetilde{\preceq} \widetilde{w}$, for any soft quasi vectors $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

A soft quasilinear space $\widetilde{Q}$ with a soft norm $\|.\|$ on $\widetilde{Q}$ is called a soft normed quasilinear space and is indicated by $\left(\widetilde{Q}, \|.\|\right)$ or $\left(\widetilde{Q}, \|.\|, P\right)$.

Lemma 2.8. [26] Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and a soft quasi norm $\|.\|$ satisfy the condition:

$$\{\|\widetilde{q}\|(\gamma) : \widetilde{q}(\gamma) = q, \text{for } q \in Q \text{ and } \gamma \in P\} \text{ is a singleton set.} \tag{1}$$

Then, for every $\gamma \in P$, $\|.\|_\gamma : Q \to \mathbb{R}^+$ defined by $\|q\|_\gamma = \|\widetilde{q}\|(\gamma)$, for every $q \in Q$ and $\widetilde{q} \in \widetilde{Q}$ such that $\widetilde{q}(\gamma) = q$, is a quasi norm on Q.

Let $\widetilde{Q}$ be a soft normed quasilinear space. Then, soft Hausdorff or soft norm metric on $\widetilde{Q}$ is defined by

$$h_Q(\widetilde{q}, \widetilde{w}) = \inf\left\{\widetilde{r} \geq \widetilde{0} : \widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_1^r,\ \widetilde{w} \widetilde{\preceq} \widetilde{q} + \widetilde{q}_2^r, \text{ and } \|\widetilde{q}_i^r\| \leq \widetilde{r}\right\}$$

3. Some New Results Related to Soft Quasi Sequences

Throughout this section, let Q and W be two quasilinear spaces over field $\mathbb{R}$, P be a non-empty parameter set, and $\widetilde{Q}$ and $\widetilde{W}$ be two absolute soft quasilinear spaces, i.e., $\widetilde{Q}(\gamma) = Q$ and $\widetilde{W}(\gamma) = W$, for every $\gamma \in P$, respectively. $SE(\widetilde{Q})$ denotes the set of all the soft quasi vectors of $\widetilde{Q}$. The notations $\widetilde{q}$ and $\widetilde{w}$ demonstrate also soft quasi vectors of $\widetilde{Q}$. Further, we will take $\widetilde{\alpha}(\gamma) = \alpha$, for every soft scalar $\widetilde{\alpha}$ and $\gamma \in P$.

Definition 3.1. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ be a sequence of soft quasi vectors in $\widetilde{Q}$. If $h(\widetilde{q}_n, \widetilde{q}) \to \widetilde{0}$ as $n \to \infty$, then $\{\widetilde{q}_n\}$ is referred to as a convergent soft quasi sequence and converges to soft quasi vector $\widetilde{q} \in \widetilde{Q}$. In other words, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists $N \in \mathbb{N}$ such that the following condition applies for $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^\varepsilon,\ \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^\varepsilon, \text{ and } \|\widetilde{q}_{in}^\varepsilon\| \leq \widetilde{\varepsilon} \tag{2}$$

Example 3.2. Let $\widetilde{Q}$ be a soft normed linear space. In this case, $\widetilde{Q}$ is a soft normed quasilinear space. The partial ordering relation that gives $\widetilde{Q}$ a soft quasilinear space structure is equality. Moreover, if $\widetilde{Q}$ is a soft normed quasilinear space and every soft quasi vector $\widetilde{q}$ in $\widetilde{Q}$ has an inverse, then $\widetilde{Q}$ is called a soft normed linear space and partial order relation on $\widetilde{Q}$ turns into equality relation. Besides, $h_{\widetilde{Q}}(\widetilde{q}, \widetilde{w}) = \|\widetilde{q} - \widetilde{w}\|_{\widetilde{Q}}$.

Theorem 3.3. In a quasilinear soft normed space, the limit of a sequence is unique if it exists.

Proof.
Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. Suppose that $\{\widetilde{q}_n\}$ is a sequence of soft quasi vectors in $\widetilde{Q}$ such that $h(\widetilde{q}_n, \widetilde{q}) \to \widetilde{0}$ and $h(\widetilde{q}_n, \widetilde{w}) \to \widetilde{0}$ as $n \to \infty$ where $\widetilde{q} \neq \widetilde{w}$. If $h(\widetilde{q}_n, \widetilde{q}) \to \widetilde{0}$ as $n \to \infty$, then for every $\widetilde{\varepsilon} > \widetilde{0}$ there exists $N \in \mathbb{N}$ such that the following conditions are satisfied for $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^\varepsilon,\ \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^\varepsilon, \text{ and } \|\widetilde{q}_{in}^\varepsilon\| \leq \frac{\widetilde{\varepsilon}}{2}$$

Further, if $h(\widetilde{q}_n, \widetilde{w}) \to \widetilde{0}$ as $n \to \infty$, then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists $M \in \mathbb{N}$ such that the following conditions are satisfied, for $n > M$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^\varepsilon,\ \widetilde{w} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_{2n}^\varepsilon, \text{ and } \|\widetilde{w}_{in}^\varepsilon\| \leq \frac{\widetilde{\varepsilon}}{2}$$

If we get $K = \max\{N, M\}$, then we get $\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^\varepsilon$, $\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^\varepsilon$, and $\|\widetilde{q}_{in}^\varepsilon\| \leq \frac{\widetilde{\varepsilon}}{2}$ and $\widetilde{q}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^\varepsilon$, $\widetilde{w} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_{2n}^\varepsilon$, and $\|\widetilde{w}_{in}^\varepsilon\| \leq \frac{\widetilde{\varepsilon}}{2}$, for every $n > K$. Since $\widetilde{Q}$ is a soft normed quasilinear space, we obtain

$$\widetilde{q}\widetilde{\preceq}\widetilde{q}_n+\widetilde{q}^{\varepsilon}_{2n}\widetilde{\preceq}\widetilde{w}+\widetilde{w}^{\varepsilon}_{1n}+\widetilde{q}^{\varepsilon}_{2n}$$

and

$$\widetilde{w}\widetilde{\preceq}\widetilde{q}_n+\widetilde{w}^{\varepsilon}_{2n}\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n}+\widetilde{w}^{\varepsilon}_{2n}$$

for every $n>K$. Thus, we find

$$\|\widetilde{q}^{\varepsilon}_{in}+\widetilde{w}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$$

since $\|\widetilde{q}^{\varepsilon}_{in}\|\leq\frac{\widetilde{\varepsilon}}{\widetilde{2}}$ and $\|\widetilde{w}^{\varepsilon}_{in}\|\leq\frac{\widetilde{\varepsilon}}{\widetilde{2}}$, for every $\widetilde{\varepsilon}>\widetilde{0}$. This gives $\widetilde{q}\widetilde{\preceq}\widetilde{w}+\widetilde{w}^{\varepsilon}_{1n}+\widetilde{q}^{\varepsilon}_{2n}$, $\widetilde{w}\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n}+\widetilde{w}^{\varepsilon}_{2n}$, and $\|\widetilde{q}^{\varepsilon}_{in}+\widetilde{w}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$, for every $\widetilde{\varepsilon}>\widetilde{0}$. Since $\widetilde{Q}$ is a soft normed quasilinear space, we get $\widetilde{q}\widetilde{\preceq}\widetilde{w}$ and $\widetilde{w}\widetilde{\preceq}\widetilde{q}$ from Definition 2.7. Thus, we obtain $\widetilde{q}=\widetilde{w}$. This is a contradiction. In this way, the proof is complete. □

Definition 3.4. Let $\left(\widetilde{Q},\|.\|,P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ be a sequence of soft quasi vectors in $\widetilde{Q}$. If $\{h(\widetilde{q}_n,\widetilde{q}_m):m,n\in\mathbb{N}\}$ is a bounded set, i.e., there exists $\widetilde{N}\widetilde{\succeq}\widetilde{0}$ such that $h(\widetilde{q}_n,\widetilde{q}_m)\widetilde{\preceq}\widetilde{N}$, for every $n,m\in\mathbb{N}$, then $\{\widetilde{q}_n\}$ is called a bounded soft quasi sequence in $\widetilde{Q}$.

Theorem 3.5. In a soft normed quasilinear space, every convergent sequence is bounded.

Proof.
Assume that $\{\widetilde{q}_n\}$ is a convergent sequence converging to $\widetilde{q}$ in $\widetilde{Q}$. Then, for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists an $N\in\mathbb{N}$ such that, for all $n>N$, there are soft quasi vectors $\widetilde{q}^{\varepsilon}_{1n},\widetilde{q}^{\varepsilon}_{2n}\in\widetilde{Q}$ satisfying the conditions

$$\widetilde{q}_n\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n},\quad \widetilde{q}\widetilde{\preceq}\widetilde{q}_n+\widetilde{q}^{\varepsilon}_{2n},\quad\text{and}\quad \|\widetilde{q}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$$

This means $h(\widetilde{q}_n,\widetilde{q})\to\widetilde{0}$ as $n\to\infty$. For an arbitrary soft quasi element $\widetilde{q}_0\in\widetilde{Q}$, we can write

$$h(\widetilde{q}_n,\widetilde{q}_0)\widetilde{\preceq}h(\widetilde{q}_n,\widetilde{q})+h(\widetilde{q},\widetilde{q}_0)$$

from properties of Hausdorff metric. Let

$$\widetilde{M}=\max\left\{h(\widetilde{q}_1,\widetilde{q}),h(\widetilde{q}_2,\widetilde{q}),\cdots,\widetilde{1}+h(\widetilde{q},\widetilde{q}_0)\right\}$$

Then, we find $h(\widetilde{q}_n,\widetilde{q}_0)\widetilde{\preceq}\widetilde{M}$, if we take $h(\widetilde{q}_n,\widetilde{q})\widetilde{\preceq}\widetilde{1}$, for every $\widetilde{\varepsilon}>\widetilde{0}$. This gives $\widetilde{q}_n\in\widetilde{S}_{\widetilde{M}}(\widetilde{q}_0)$. □

Every bounded soft quasi-sequence is not necessarily convergent. For example, we take a bounded soft quasi sequence in soft normed quasilinear space $\widetilde{\Omega}_C(\mathbb{R})$ such that $\widetilde{q}_n(\gamma)=\{(-1)^n\}\in\Omega_C(\mathbb{R})$, for $\gamma\in P$. Clearly, this sequence is bounded in $\widetilde{\Omega}_C(\mathbb{R})$. However, $\widetilde{q}_n$ is not convergent in $\widetilde{\Omega}_C(\mathbb{R})$.

Definition 3.6. Let $\left(\widetilde{Q},\|.\|,P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ be a sequence of soft quasi vectors in $\widetilde{Q}$. If, for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists an $M\in\mathbb{N}$ such that $h(\widetilde{q}_n,\widetilde{q}_m)\preceq\widetilde{\varepsilon}$, for every $n,m>M$, then $\{\widetilde{q}_n\}$ is called a soft quasi-Cauchy sequence in $\widetilde{Q}$.

Theorem 3.7. In a soft normed quasilinear space, every convergent sequence is a Cauchy sequence.

Proof.
Assume that $\{\widetilde{q}_n\}$ is convergent to $\widetilde{q}$ in $\widetilde{Q}$. Then, for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists an $N\in\mathbb{N}$ such that, for $n>N$, there are soft quasi vectors $\widetilde{q}^{\varepsilon}_{1n},\widetilde{q}^{\varepsilon}_{2n}\in\widetilde{Q}$ satisfying the conditions

$$\widetilde{q}_n\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n},\quad \widetilde{q}\widetilde{\preceq}\widetilde{q}_n+\widetilde{q}^{\varepsilon}_{2n},\quad\text{and}\quad \|\widetilde{q}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$$

Then, we clearly get $\widetilde{w}^{\varepsilon}_{1n},\widetilde{w}^{\varepsilon}_{2n}\in\widetilde{Q}$ such that $\widetilde{q}_m\widetilde{\preceq}\widetilde{q}+\widetilde{w}^{\varepsilon}_{1n}$, $\widetilde{q}\widetilde{\preceq}\widetilde{q}_m+\widetilde{w}^{\varepsilon}_{2n}$, and $\|\widetilde{w}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$, for all $m>N$. Similar to the proof of Theorem 3.5, if we get $K=\max\{N,M\}$, then

$$\widetilde{q}_n\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n}\widetilde{\preceq}\widetilde{q}_m+\widetilde{w}^{\varepsilon}_{2n}+\widetilde{q}^{\varepsilon}_{1n}$$

and

$$\widetilde{q}_m \widetilde{\preceq} \widetilde{q} + \widetilde{w}^{\varepsilon}_{1n} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{1n}$$

for every $n, m > K$. Moreover, if we take $\widetilde{2} \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}'$, then we have $\|\widetilde{q}^{\varepsilon}_{in} + \widetilde{w}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}'$ since $\|\widetilde{q}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$ and $\|\widetilde{w}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$. This gives that every $\widetilde{\varepsilon}' > \widetilde{0}$, there exists a $K \in \mathbb{N}$ such that $h(\widetilde{q}_n, \widetilde{q}_m) \preceq \widetilde{\varepsilon}$, for every $n, m > K$. □

The converse of Theorem 3.7 is not always correct.

Theorem 3.8. If there is a convergent soft quasi subsequence of a soft quasi-Cauchy sequence in a soft normed quasilinear space, this soft quasi-Cauchy sequence converges to the soft quasi vector at which the soft quasi-subsequence converges.

PROOF.
Let $\{\widetilde{q}_n\}$ be a soft quasi-Cauchy sequence in $\widetilde{Q}$. Then, for every $\widetilde{\epsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for $n, m > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{q}^{\epsilon}_{1n}, \quad \widetilde{q}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}^{\epsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\epsilon}_{in}\| \leq \frac{\widetilde{\epsilon}}{2}$$

We define a convergent soft quasi subsequence of $\{\widetilde{q}_n\}$ with $\{\widetilde{q}_{n_k}\}$ and $\widetilde{q}_{n_k} \to \widetilde{q}$ as $n \to \infty$. Since $\{\widetilde{q}_n\}$ is a soft quasi-Cauchy sequence and $\{\widetilde{q}_{n_k}\}$ is a soft quasi subsequence of $\{\widetilde{q}_n\}$, then

$$\widetilde{q}_{n_m} \widetilde{\preceq} \widetilde{q}_n + \widetilde{k}^{\epsilon}_{1n}, \quad \widetilde{q}_n \widetilde{\preceq} \widetilde{q}_{n_m} + \widetilde{k}^{\epsilon}_{2n}, \quad \text{and} \quad \|\widetilde{k}^{\epsilon}_{in}\| \leq \frac{\widetilde{\epsilon}}{2}$$

Moreover, since $\widetilde{q}_{n_k} \to \widetilde{q}$ as $n \to \infty$, then, for $n, m > N$ and for every $\widetilde{\epsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_{n_m} \widetilde{\preceq} \widetilde{q} + \widetilde{l}^{\epsilon}_{1n}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_{n_m} + \widetilde{l}^{\epsilon}_{2n}, \quad \text{and} \quad \|\widetilde{l}^{\epsilon}_{in}\| \leq \frac{\widetilde{\epsilon}}{2}$$

Then, from the above two inequalities,

$$\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{l}^{\epsilon}_{2n} + \widetilde{k}^{\epsilon}_{1n} \quad \text{and} \quad \widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{l}^{\epsilon}_{1n} + \widetilde{k}^{\epsilon}_{2n}$$

for all $n > N$. Further, we find $\left\|\widetilde{l}^{\epsilon}_{1n} + \widetilde{k}^{\epsilon}_{2n}\right\| \leq \widetilde{\epsilon}$. This gives $\widetilde{q}_n \to \widetilde{q}$ as $n \to \infty$. □

Theorem 3.9. In a soft normed quasilinear space, every Cauchy sequence is a bounded soft quasi-sequence.

PROOF.
Let $\{\widetilde{q}_n\}$ be a soft quasi-Cauchy sequence in $\widetilde{Q}$. Then, there exists $\widetilde{N} \widetilde{\succeq} \widetilde{0}$ such that $h(\widetilde{q}_k, \widetilde{q}_l) \widetilde{\preceq} \widetilde{1}$, for every $k, l > N$. If we take $\widetilde{K}(\gamma) = \max_{1 \leq k, l \leq m} \{h(\widetilde{q}_k, \widetilde{q}_l)(\gamma)\}$, for all $\gamma \in P$, then

$$\begin{aligned} h(\widetilde{q}_k, \widetilde{q}_l)(\gamma) &\preceq h(\widetilde{q}_k, \widetilde{q}_m)(\gamma) + h(\widetilde{q}_m, \widetilde{q}_l)(\gamma) \\ &\preceq \widetilde{K}(\gamma) + \widetilde{1}(\gamma) \\ &= \left(\widetilde{K} + \widetilde{1}\right)(\gamma) \end{aligned}$$

for $1 \leq k \leq m$ and $l \geq m$. Thus, we find $h(\widetilde{q}_k, \widetilde{q}_l) \leq \left(\widetilde{K} + \widetilde{1}\right)$, for every $k, l \in \mathbb{N}$. This gives that $\{\widetilde{q}_n\}$ is a bounded soft quasi sequence in $\widetilde{Q}$. □

Definition 3.10. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and (S, P) be a soft quasi subset in $\widetilde{Q}$ such that $S(\gamma) \neq \emptyset$, for every $\gamma \in P$. If there exists a soft real number $\widetilde{m}$ such that $\|\widetilde{q}\| \leq \widetilde{m}$, for every $\widetilde{q} \in \widetilde{S}$, then the soft quasi subset (S, P) is referred to as bounded in $\widetilde{Q}$.

Example 3.11. Let $B = \{\widetilde{q} : \widetilde{q}(\gamma) \subseteq [0,1], \gamma \in P\}$, a soft quasi subset of soft quasilinear space $\widetilde{\Omega}_C(\mathbb{R})$ with $\|\widetilde{q}\| = \sup \|\widetilde{q}(\gamma)\|_{\Omega_C(\mathbb{R})}$. Then, the soft quasi subset B is bounded since

$$\|\widetilde{q}\| = \sup \|\widetilde{q}(\gamma)\|_{\Omega_C(\mathbb{R})} \leq \sup \|[0,1]\|_{\Omega_C(\mathbb{R})} = 1$$

Definition 3.12. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. If every soft quasi-Cauchy sequences in $\widetilde{Q}$ converges to a soft quasi element in $\widetilde{Q}$, then $\widetilde{Q}$ is called a complete soft normed quasilinear space. Generally, a soft quasilinear Banach space can be described as a complete soft normed quasilinear space.

Theorem 3.13. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. Then, the following statements are valid:

i. If $\widetilde{q}_n \longrightarrow \widetilde{q}$ and $\widetilde{w}_n \longrightarrow \widetilde{w}$, then $\widetilde{q}_n + \widetilde{w}_n \longrightarrow \widetilde{q} + \widetilde{w}$, i.e., according to the Hausdorff metric, the algebraic sum is continuous.

ii. If $\widetilde{q}_n \longrightarrow \widetilde{q}$ and $\widetilde{\gamma}_n \longrightarrow \widetilde{\gamma}$, then $\widetilde{\gamma}_n \cdot \widetilde{q}_n \longrightarrow \widetilde{\gamma} \cdot \widetilde{q}$, i.e., according to the Hausdorff metric, multiplication by soft real numbers is continuous. The sequence $\widetilde{\gamma}_n$ consists of soft scalars.

Proof.
Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space.

i. Suppose that $\widetilde{q}_n \longrightarrow \widetilde{q}$ and $\widetilde{w}_n \longrightarrow \widetilde{w}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}^{\varepsilon}_{1n}, \quad \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{w}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$$

Therefore,

$$\widetilde{q}_n + \widetilde{w}_n \widetilde{\preceq} \widetilde{q} + \widetilde{w} + \widetilde{q}^{\varepsilon}_{1n} + \widetilde{w}^{\varepsilon}_{1n}, \quad \widetilde{q} + \ \widetilde{w} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\varepsilon}_{in} + \widetilde{w}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}'$$

such that $\widetilde{\varepsilon}' = \widetilde{2}\widetilde{\varepsilon}$. Thus, $\widetilde{q}_n + \widetilde{w}_n \longrightarrow \widetilde{q} + \widetilde{w}$.

Similarly, it can be demonstrated that soft real number multiplication is continuous. □

Theorem 3.14. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. The soft quasi-norm is continuous according to the Hausdorff metric.

Proof.
Suppose that $\widetilde{q}_n \longrightarrow \widetilde{q}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$$

Since $\widetilde{Q}$ is a soft normed quasilinear space with $\|.\|$,

$$\|\widetilde{q}_n\| \leq \|\widetilde{q}\| + \|\widetilde{q}^{\varepsilon}_{1n}\| \quad \text{and} \quad \|\widetilde{q}\| \widetilde{\leq} \|\widetilde{q}_n\| + \|\widetilde{q}^{\varepsilon}_{2n}\|$$

This gives $\|\widetilde{q}_n\| \to \|\widetilde{q}\|$ as $n \to \infty$ because $\|\widetilde{q}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$. □

Theorem 3.15. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ and $\{\widetilde{w}_n\}$ be two soft quasi-Cauchy sequences in $\widetilde{Q}$. Then, $\{\widetilde{q}_n + \widetilde{w}_n\}$ is soft quasi-Cauchy sequence in $\widetilde{Q}$.

Proof.
Let $\{\widetilde{q}_n\}$ and $\{\widetilde{w}_n\}$ be two soft quasi-Cauchy sequences in $\widetilde{Q}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exist

$N, M \in \mathbb{N}$ such that, for all $n, m > N$ and $n, m > M$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w}_m + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w}_m \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

If we take $K = \max\{N, M\}$, then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $K \in \mathbb{N}$ such that, for all $n, m > K$,

$$\widetilde{q}_n + \widetilde{w}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{w}_m + \widetilde{q}_{1n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon} \quad \text{and} \quad \widetilde{q}_m + \widetilde{w}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{2n}^{\varepsilon}$$

Moreover, $\|\widetilde{q}_{in}^{\varepsilon} + \widetilde{w}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$ since $\|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$ and $\|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$. This gives $\{\widetilde{q}_n + \widetilde{w}_n\}$ is a soft quasi-Cauchy sequence in $\widetilde{Q}$. □

Theorem 3.16. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. The following statements are provided:

i. Assume that $\widetilde{q}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{w}$. If $\widetilde{q}_n \widetilde{\preceq} \widetilde{w}_n$, for every $n \in \mathbb{N}$, then $\widetilde{q} \widetilde{\preceq} \widetilde{w}$.

ii. Assume that $\widetilde{q}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{q}$. If $\widetilde{q}_n \widetilde{\preceq} \widetilde{m}_n \widetilde{\preceq} \widetilde{w}_n$, for every $n \in \mathbb{N}$, then $\widetilde{m}_n \to \widetilde{q}$.

iii. If $\widetilde{q}_n + \widetilde{w}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{\theta}$, then $\widetilde{q}_n \to \widetilde{q}$.

Proof.
Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space.

i. Let $\widetilde{q}_n \to \widetilde{q}$, $\widetilde{w}_n \to \widetilde{w}$, and $\widetilde{q}_n \widetilde{\preceq} \widetilde{w}_n$, for every $n \in \mathbb{N}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

Moreover, since $\widetilde{q}_n \widetilde{\preceq} \widetilde{w}_n$, for every $n \in \mathbb{N}$,

$$\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon} \widetilde{\preceq} \widetilde{w}_n + \widetilde{q}_{2n}^{\varepsilon} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon}$$

Further,

$$\|\widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

This gives $\widetilde{q} \widetilde{\preceq} \widetilde{w}$ because $\widetilde{Q}$ is a soft normed quasilinear space.

ii. Let $\widetilde{q}_n \to \widetilde{q}$, $\widetilde{w}_n \to \widetilde{q}$, and $\widetilde{q}_n \widetilde{\preceq} \widetilde{m}_n \widetilde{\preceq} \widetilde{w}_n$, for every $n \in \mathbb{N}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

Moreover, since $\widetilde{q}_n \widetilde{\preceq} \widetilde{m}_n$ for every $n \in \mathbb{N}$,

$$\widetilde{q} \widetilde{\preceq} \widetilde{m}_n + \widetilde{q}_{2n}^{\varepsilon}$$

Further, as $\widetilde{m}_n \widetilde{\preceq} \widetilde{w}_n$ for every $n \in \mathbb{N}$,

$$\widetilde{m}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}$$

Besides, because $\|\widetilde{q}_{2n}^{\varepsilon}\| \leq \widetilde{\varepsilon}$ and $\|\widetilde{w}_{1n}^{\varepsilon}\| \leq \widetilde{\varepsilon}$, for every $n \in \mathbb{N}$, $\widetilde{m}_n \to \widetilde{q}$ as $n \to \infty$.

iii. Let $\widetilde{q}_n + \widetilde{w}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{\theta}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n + \widetilde{w}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{\theta} + \widetilde{w}^{\varepsilon}_{1n}, \quad \widetilde{\theta} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{w}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

From these above relations,

$$\widetilde{q}_n + \widetilde{\theta} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{w}^{\varepsilon}_{2n} \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n} + \widetilde{w}^{\varepsilon}_{2n}$$

and

$$\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}^{\varepsilon}_{2n} \widetilde{\preceq} \widetilde{q}_n + \widetilde{\theta}_n + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{1n}$$

Besides, $\|\widetilde{q}^{\varepsilon}_{1n} + \widetilde{w}^{\varepsilon}_{2n}\| \leq \widetilde{\varepsilon}$ and $\|\widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{1n}\| \leq \widetilde{\varepsilon}$ since $\|\widetilde{q}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$ and $\|\widetilde{w}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$. This gives $\widetilde{q}_n \to \widetilde{q}$ as $n \to \infty$. □

4. Some New Results Concerning to Soft Quasi Subspaces of Soft Normed Quasilinear Space

In this section, we provide some results on soft quasilinear subspaces of soft normed quasilinear spaces. Further, we define the regular and singular soft quasi vectors of a soft quasilinear space and exemplify them.

Lemma 4.1. Let $\widetilde{W}$ and $\widetilde{Z}$ be soft closed subspaces of a soft normed quasilinear space $\widetilde{Q}$ satisfying Condition 1 and $\widetilde{W}$ be a closed proper subset of the subspace $\widetilde{Z}$. Then, for $\widetilde{\varepsilon} \widetilde{\geq} \widetilde{0}$, there exists $\widetilde{z} \in \widetilde{Q} \backslash \widetilde{W}$ with $\|\widetilde{z}\| \widetilde{\geq} \widetilde{1}$ such that, for all $\widetilde{w} \in \widetilde{W}$, the inequality $\|\widetilde{z} - \widetilde{w}\| \widetilde{\geq} \widetilde{1} - \widetilde{\varepsilon}$ is satisfied.

Proof.
Suppose that $\widetilde{\varepsilon} \widetilde{\geq} \widetilde{0}$ and $\widetilde{\varepsilon}(\gamma) = \varepsilon_\gamma > 0$, for every parameter $\gamma \in P$. Since $\widetilde{Q}$ satisfies Condition 1, $\widetilde{Z}(\gamma) = Z_\gamma$ is a closed subspace of the normed quasilinear space Q such that $\widetilde{Q}(\gamma) = Q$. From Reisz's Lemma for normed quasilinear space provided in [4], there exists $\widetilde{z}(\gamma) \in Q \backslash Z_\gamma$ with $\|\widetilde{z}(\gamma)\|_\gamma \geq 1$ such that for all $\widetilde{w}(\gamma) \in W_\gamma$ the inequality

$$\|\widetilde{z}(\gamma) - \widetilde{w}(\gamma)\|_\gamma \geq 1 - \varepsilon_\gamma$$

is satisfied. This gives that, for $\widetilde{\varepsilon} \widetilde{\geq} \widetilde{0}$, there exists $\widetilde{z} \in \widetilde{Q} \backslash \widetilde{W}$ with $\|\widetilde{z}\| \widetilde{\geq} \widetilde{1}$ such that, for all $\widetilde{w} \in \widetilde{W}$, the inequality $\|\widetilde{z} - \widetilde{w}\| \widetilde{\geq} \widetilde{1} - \widetilde{\varepsilon}$ is satisfied. □

Lemma 4.2. Let $\widetilde{Q}$ be a soft quasilinear space. Then, in the soft quasilinear space $\widetilde{Q}$, the soft element $\widetilde{\theta}$ is minimal in $\widetilde{Q}$, i.e., $\widetilde{q} = \widetilde{\theta}$ if $\widetilde{q} \widetilde{\preceq} \widetilde{\theta}$.

Proof.
Assume that $\widetilde{q}$ is a soft quasi vector in $\widetilde{Q}$ and $\widetilde{q} \widetilde{\preceq} \widetilde{\theta}$. Since $(\widetilde{-1})\widetilde{q} \widetilde{\preceq} (\widetilde{-1})\widetilde{q}$, for soft scalar $\widetilde{-1}$, and $\widetilde{Q}$ is a soft quasilinear space, then

$$\widetilde{q} + (\widetilde{-1})\widetilde{q} \widetilde{\preceq} \widetilde{\theta} + (\widetilde{-1})\widetilde{q} = (\widetilde{-1})\widetilde{q}$$

Further,

$$\widetilde{\theta} = \left(\widetilde{1} + (\widetilde{-1})\right)\widetilde{q} = \widetilde{q} + (\widetilde{-1})\widetilde{q} \widetilde{\preceq} \widetilde{\theta} + (\widetilde{-1})\widetilde{q} = (\widetilde{-1})\widetilde{q}$$

from properties of soft quasilinear space. Thus, $(\widetilde{-1})\widetilde{\theta} \widetilde{\preceq} (\widetilde{-1})\left((\widetilde{-1})\widetilde{q}\right) = \widetilde{q}$. Moreover, $(\widetilde{-1})\widetilde{\theta} = \widetilde{\theta}$. Therefore, $\widetilde{\theta} \widetilde{\preceq} \widetilde{q}$. This gives $\widetilde{q} = \widetilde{\theta}$. Consequently, the soft element $\widetilde{\theta}$ is minimal in soft quasilinear space $\widetilde{Q}$. □

Definition 4.3. Let $\widetilde{Q}$ be a soft quasilinear space. A soft quasi vector $\widetilde{q}^{'} \in \widetilde{Q}$ is named an inverse of a soft quasi vector $\widetilde{q} \in \widetilde{Q}$ if $\widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$. The inverse of a soft quasi vector is unique if there exists.

Lemma 4.4. If any soft quasi vector in the soft quasilinear space $\widetilde{Q}$ has an inverse soft quasi vector in $\widetilde{Q}$, then the partial order relation in $\widetilde{Q}$ is achieved through equality. As a result, the distributive property is valid. Therefore, $\widetilde{Q}$ is a soft linear space.

Proof.
The proof is similar to the quasilinear spaces if take as $\widetilde{q}(\gamma) = q$ and $\widetilde{q}^{'}(\gamma) = q^{'}$, for all parameter γ. □

Definition 4.5. In a soft quasilinear space $\widetilde{Q}$, a soft quasi vector with an inverse is called a regular soft quasi vector, and a soft quasi vector without an inverse is called a singular soft quasi vector. The set of all the regular and singular soft quasi vectors of $\widetilde{Q}$ is denoted by $\widetilde{Q}_r$ and $\widetilde{Q}_s$, respectively.

Here, the subspace of all the regular soft quasi vectors of the soft quasilinear space $\widetilde{Q}$ is called the soft regular subspace of $\widetilde{Q}$. Similarly, The subspace of all the singular soft quasi vectors of the soft quasilinear space $\widetilde{Q}$ is called the soft singular subspace of $\widetilde{Q}$.

Definition 4.6. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{W} \subseteq \widetilde{Q}$. If $\widetilde{W}$ is a soft quasilinear space with the same operations in $\widetilde{Q}$ and the same partial order relation in $\widetilde{Q}$, then $\widetilde{W}$ is called a soft sub-quasilinear space of $\widetilde{Q}$.

Theorem 4.7. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{W} \subseteq \widetilde{Q}$. Then, $\widetilde{W}$ is a soft sub-quasilinear space of $\widetilde{Q}$ if and only if $\widetilde{\alpha}\widetilde{w}_1 + \widetilde{\beta}\widetilde{w}_2 \in \widetilde{W}$, for every soft quasi vector $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta}$.

Proof.
The theorem can be proved in a similar way to that of soft linear spaces. □

Example 4.8. Consider the absolute soft quasi set generated by $\Omega_C(\mathbb{R})$ and defined by $\widetilde{\Omega}_C(\mathbb{R})$, i.e., $\widetilde{\Omega}_C(\mathbb{R})\ (\gamma) = \Omega_C(\mathbb{R})$, for every $\gamma \in P$. Let

$$\widetilde{W} = \{\widetilde{w} : \widetilde{w}(\gamma) = [a, b],\ a, b \in \mathbb{R},\ a < b,\ \text{and } \gamma \in P\} \cup \left\{\widetilde{0}\right\}$$

Clearly, $\widetilde{W}$ consists of all the soft quasi vectors in which image is a singular element of $\Omega_C(\mathbb{R})$ under the parameter γ. Since

$$\left(\widetilde{\alpha}\widetilde{w}_1 + \widetilde{\beta}\widetilde{w}_2\right)(\gamma) = \widetilde{\alpha}(\gamma)\,\widetilde{w}_1(\gamma) + \widetilde{\beta}(\gamma)\,\widetilde{w}_2(\gamma) = \alpha\widetilde{w}_1(\gamma) + \beta\widetilde{w}_2(\gamma) \in \Omega_C(\mathbb{R})$$

for every soft quasi vectors $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta}$, then $\widetilde{W}$ is a soft subquasilinear space of $\widetilde{\Omega}_C(\mathbb{R})$. Moreover, by Definition 4.5, we get $\widetilde{W}$ is a soft singular subspace of $\widetilde{\Omega}_C(\mathbb{R})$. For another soft quasi set

$$\widetilde{M} = \{\widetilde{m} : \widetilde{m}(\gamma) = \{m\} \in \mathbb{R}, \forall \gamma \in P\}$$

$\widetilde{M}$ is a soft subspace of $\widetilde{\Omega}_C(\mathbb{R})$ since

$$\left(\widetilde{\alpha}\widetilde{m}_1 + \widetilde{\beta}\widetilde{m}_2\right)(\gamma) = \widetilde{\alpha}(\gamma)\,\widetilde{m}_1(\gamma) + \widetilde{\beta}(\gamma)\,\widetilde{m}_2(\gamma) = \alpha m_1 + \beta m_2 \in \mathbb{R}$$

for every soft quasi vectors $\widetilde{m}_1, \widetilde{m}_2 \in \widetilde{M}$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta}$. Further, every soft quasi vector $\widetilde{m} \in \widetilde{M}$ has an inverse because $\widetilde{\Omega}_C(\mathbb{R})$ is an absolute soft quasilinear space. Thus, $\widetilde{M}$ is a soft regular subspace of $\widetilde{\Omega}_C(\mathbb{R})$.

Theorem 4.9. Every regular soft quasi vector in a soft quasilinear space $\widetilde{Q}$ is minimal.

Proof.
Let $\widetilde{q} \in \widetilde{Q}_r$ be an arbitrary soft quasi vector and $\widetilde{w} \widetilde{\preceq} \widetilde{q}$, for any $\widetilde{w} \in \widetilde{Q}$. Then,

$$\widetilde{w} + \widetilde{q}^{'} \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$$

since $\widetilde{q}$ is a soft quasi-regular vector in $\widetilde{Q}$. From Lemma 4.2, $\widetilde{w} + \widetilde{q}^{'} = \widetilde{\theta}$. Thus, $\widetilde{w} = \widetilde{q}$ because the inverse of a soft quasi vector is unique if there exists. Hence, an arbitrary soft quasi vector $\widetilde{q}$ in $\widetilde{Q}$ is minimal. □

Theorem 4.10. Let $\widetilde{Q}$ be a soft normed quasilinear space. Then, the soft quasi set $\widetilde{Q}_r$ is a closed subspace of $\widetilde{Q}$.

Proof.
Let $\widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}(P)$. It is necessary to prove that $\widetilde{\alpha}\widetilde{q} + \widetilde{\beta}\widetilde{w} \in \widetilde{Q}_r$ to show that $\widetilde{Q}_r$ is a subspace of $\widetilde{Q}$. As $\widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$, there exist $\widetilde{q}^{'}, \widetilde{w}^{'} \in \widetilde{Q}$ such that $\widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$ and $\widetilde{w} + \widetilde{w}^{'} = \widetilde{\theta}$. Since $\widetilde{Q}$ is a soft normed quasilinear space,

$$\widetilde{\alpha}\widetilde{q} + \widetilde{\beta}\widetilde{w} + \widetilde{\alpha}\widetilde{q}^{'} + \widetilde{\beta}\widetilde{w}^{'} = \widetilde{\alpha}\left(\widetilde{q} + \widetilde{q}^{'}\right) + \widetilde{\beta}\left(\widetilde{w} + \widetilde{w}^{'}\right) = \widetilde{\theta}$$

This gives $\widetilde{\alpha}\widetilde{q} + \widetilde{\beta}\widetilde{w} \in \widetilde{Q}_r$.

The soft quasi sequence $\{\widetilde{q}_n\}$ in $\widetilde{Q}_r$ converges to $\widetilde{q} \in \widetilde{Q}$, i.e., $\widetilde{q}_n \to \widetilde{q} \in \widetilde{Q}$ as $n \to \infty$. Since $\widetilde{Q}$ is a soft normed quasilinear space, $-\widetilde{q}_n \to -\widetilde{q}$ as $n \to \infty$. Hence, $\widetilde{q}_n - \widetilde{q}_n \to \widetilde{q} - \widetilde{q}$. Since $\widetilde{q}_n \in \widetilde{Q}_r$, $\widetilde{q}_n - \widetilde{q}_n = \widetilde{\theta}$ and then $\widetilde{q} - \widetilde{q} = \widetilde{\theta}$. This gives $\widetilde{q} \in \widetilde{Q}_r$. Therefore, $\widetilde{Q}_r$ is a closed subspace of $\widetilde{Q}$. □

Theorem 4.11. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. If $\widetilde{q} + \widetilde{w} \in \widetilde{Q}_r$, then $\widetilde{q} \in \widetilde{Q}_r$ and $\widetilde{w} \in \widetilde{Q}_r$.

Proof.
Assume that $\widetilde{q} + \widetilde{w} \in \widetilde{Q}_r$ and $\widetilde{q}$ are not soft quasi-regular vectors of $\widetilde{Q}$. Then, there exists a soft quasi vector $\widetilde{m} \in \widetilde{Q}_r$ such that $(\widetilde{q} + \widetilde{w}) + \widetilde{m} = \widetilde{\theta}$. Thus, $\widetilde{q} + (\widetilde{w} + \widetilde{m}) = \widetilde{\theta}$ because $\widetilde{Q}$ is a soft quasilinear space. This implies that the soft quasi vector $\widetilde{q}$ has an inverse soft quasi vector $\widetilde{w} + \widetilde{m}$. However, this contradicts the assumption $\widetilde{q} \notin \widetilde{Q}_r$. Because, if $\widetilde{q} \in \widetilde{Q}_r$, then $\widetilde{q}$ has an inverse soft quasi vector $\widetilde{q}^{'} \in \widetilde{Q}_r$ such that $\widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$. As a result, the assumption is not correct and thus $\widetilde{q} \in \widetilde{Q}_r$. In a similar way, it can be observed that $\widetilde{w}$ is a soft quasi-regular vector of $\widetilde{Q}$. □

Theorem 4.12. Let $\widetilde{Q}$ be a soft quasilinear space. If $\widetilde{q} \in \widetilde{Q}_r$ and $\widetilde{w} \in \widetilde{Q}_s$, then $\widetilde{q} + \widetilde{w} \in \widetilde{Q}_r$.

Proof.
The proof is similar to the proof of Teorem 4.11. □

As in quasilinear spaces, soft quasilinear spaces have a soft quasi-singular vector containing each soft quasi-regular vector.

5. Conclusion

This study provided some results on the convergence and boundedness of a soft quasi sequence in a soft quasilinear space. Further, it investigated some properties of regular and singular subspaces of a soft quasilinear space. In future works, some algebraic properties of soft quasilinear spaces, such as basis, dimensions, and properness, can be studied depending on the descriptions of soft quasilinear spaces. Moreover, whether the class of soft fuzzy sets has a soft quasilinear space structure can be investigated. Applying the soft quasi concept to them is worth studying.

Author Contributions

All the authors contributed equally to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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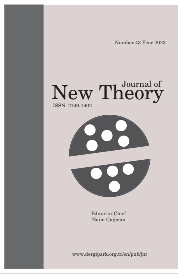

Homoderivations in Prime Rings

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Abstract — The study consists of two parts. The first part shows that if $h_1(x)h_2(y) = h_3(x)h_4(y)$, for all $x, y \in R$, then $h_1 = h_3$ and $h_2 = h_4$. Here, h_1, h_2, h_3, and h_4 are zero-power valued non-zero homoderivations of a prime ring R. Moreover, this study provide an explanation related to h_1 and h_2 satisfying the condition $ah_1 + h_2b = 0$. The second part shows that $L \subseteq Z$ if one of the following conditions is satisfied: $i.\, h(L) = (0)$, $ii.\, h(L) \subseteq Z$, $iii.\, h(xy) = xy$, for all $x, y \in L$, $iv.\, h(xy) = yx$, for all $x, y \in L$, or $v.\, h([x, y]) = 0$, and for all $x, y \in L$. Here, R is a prime ring with a characteristic other than 2, h is a homoderivation of R, and L is a non-zero square closed Lie ideal of R.

Keywords *Prime rings, Lie ideals, homoderivations*

Mathematics Subject Classification (2020) 16N60, 16W25

1. Introduction

Throughout this article, unless otherwise specified, R denotes an associative prime ring, i.e., for all $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$, with the maximal left ring of quotients $Q = Q_{ml}(R)$. It is well known that R is a subring of Q, Q is a prime ring, and the center C of Q is a field and called the extended centroid of R [1]. Z denotes the center of R, and the notation $\mathrm{Char}(R)$ represents the characteristic of R. For all $a, b \in R$, let $[a, b] := ab - ba$, the Lie commutator of a and b. For a subset A of R, $C_R(A)$ means the centralizer of A and defined by $C_R(A) = \{x \in R \mid [x, a] = 0, \text{ for all } a \in A\}$. If L is an additive subgroup of R and $[x, r] \in L$, for all $x \in L$ and $r \in R$, then L is referred to as a Lie ideal of R. If L is a Lie ideal of R and $x^2 \in L$, for all $x \in L$, then L is called a square closed Lie ideal. Since $(x + y)^2 \in L$ and $[x, y] \in L$, for all $x, y \in L$, then $2xy \in L$. Let $\varnothing \neq S \subseteq R$. A mapping $f : R \to R$ is called zero-power valued on S, if $f(S) \subseteq S$, and, for all $s \in S$, there exists a positive integer $n(s) > 1$ such that $f^{n(s)}(s) = 0$. An additive map $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Especially, I_a, defined by $I_a(x) := [a, x]$, for all $x \in R$, is an inner derivation induced by an element $a \in R$.

In [2], El Sofy Aly has introduced a new mapping created by combining the concepts of homomorphisms and derivations on rings. An additive mapping $h : R \to R$ is called a homoderivation if

$$h(xy) = h(x)h(y) + h(x)y + xh(y)$$

for all $x, y \in R$. The only additive mapping, both a derivation and a homoderivation on a prime ring, is the zero map. Some examples of homoderivations are as follows:

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Example 1.1. [2] Let R be a ring and f be an endomorphism of R. Then, the mapping $h : R \to R$ defined by $h(x) = f(x) - x$ is a homoderivation of R.

Example 1.2. [2] Let R be a ring. Then, the additive mapping $h : R \to R$ defined by $h(x) = -x$ is a homoderivation of R.

Example 1.3. [2] Let $R = \mathbb{Z}\left(\sqrt{2}\right)$, a ring of all the real numbers of the form $m + n\sqrt{2}$ such that $m, n \in \mathbb{Z}$, the set of all the integers, under the usual addition and multiplication of real numbers. Then, the map $d : R \to R$ defined by $d(m + n\sqrt{2}) = -2n\sqrt{2}$ is a homoderivation of R.

In 2016, Melaibari et al. [3] have proved the commutativity of a prime ring R admitting a non-zero homoderivation h that satisfies any one of the conditions: *i.* $[x, y] = [h(x), h(y)]$, for all $x, y \in U$, a non-zero ideal of R, *ii.* $h([x, y]) = 0$, for all $x, y \in U$, a non-zero ideal of R, or *iii.* $h([x, y]) \in Z$, for all $x, y \in R$. Alharfie et al. [4] have shown that the commutativity of a prime ring R if any of the following conditions is satisfied: for all $x, y \in I$, *i.* $xh(y) \pm xy \in Z(R)$, *ii.* $xh(y) \pm yx \in Z(R)$, or *iii.* $xh(y) \pm [x, y] \in Z(R)$. Here, I is a non-zero left ideal of R, and h is a homoderivation of R. In 2019, Al Harfien et al. [5] and Rehman et al. [6] have studied the commutativity of a semiprime (prime) ring admitting a homoderivation satisfying some identities on a ring. Researchers [7–14] have executed many noteworthy works concerning various properties of homoderivations during the last decades.

In Theorem 1.4, Bresar [16] has indicated that derivations d, f, g, and h of a prime ring R satisfying the condition $d(x)g(y) = h(x)f(y)$, for all $x, y \in R$, are $C-$dependent. In other words, g and f and h and d are $C-$dependent. In Teorem 1.5, the author has indicated that derivations g and h of a prime ring R satisfying the condition $ag(x) + h(x)b = 0$, for all $x, y \in R$, are $C-$dependent. That is, g and I_b and h and I_a are $C-$dependent. Motivated by the results of Bresar, we create Section 3 of this study. In the section, we research the results of Bresar by homoderivations. We show that homoderivations h_1, h_2, h_3, and h_4 of a prime ring R satisfying the condition $h_1(x)h_2(y) = h_3(x)h_4(y)$, for all $x, y \in R$, are $1-$dependent such that $1 \in C$. That is, $h_1 = h_3$ and $h_2 = h_4$ where $h_{1|_Z} \neq 0$ or $h_{2|_Z} \neq 0$ such that $h_{1|_Z}, h_{2|_Z} : Z \to \mathbb{R}$ are two mapping defined by $h_{1|_Z}(x) := h_1(x)$ and $h_{2|_Z}(x) := h_2(x)$, respectively. In addition, we prove that $a = -b \in Z$, for homoderivations h_1 and h_2 of a prime ring R satisfying the condition $ah_1(x) + h_2(x)b = 0$, for all $x \in R$.

Theorem 1.4. [16] Let R be a prime ring, and d, f, g, and h be derivations of R. Suppose that $d(x)g(y) = h(x)f(y)$, for all $x, y \in R$. If $d \neq 0$ and $f \neq 0$, then there exists a $\lambda \in C$ such that $g(x) = \lambda f(x)$ and $h(x) = \lambda d(x)$, for all $x \in R$

Theorem 1.5. [16] Let R be a prime ring, and g and h be derivations of R. Suppose that there exist $a, b \in R$ such that $ag(x) + h(x)b = 0$, for all $x \in R$. If $a \notin Z$ and $b \notin Z$, then there exists a $\lambda \in C$ such that $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$, for all $x \in R$. Moreover, if $g \neq 0$, then $ab \in Z$.

The purpose of Section 3 is to prove the following two results:

• Let R be a prime ring and h_1, h_2, h_3, and h_4 be zero-power valued non-zero homoderivations on R. Suppose that $h_1(x)h_2(y) = h_3(x)h_4(y)$, for all $x, y \in R$. If $h_{1|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$. Moreover, $h_{1|_Z} = 0$ if and only if (iff) $h_{3|_Z} = 0$. Similarly, If $h_{2|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$. Moreover, $h_{2|_Z} = 0$ iff $h_{4|_Z} = 0$.

• Let R be a prime ring and h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that there are $a, b \in R$ such that $ah_1(x) + h_2(x)b = 0$, for all $x \in R$. Then, $a = -b \in Z$ or $h_{1|_Z} = h_{2|_Z} = 0$.

In Lemma 5 and 6 provided in [15], Bergen et al. have showed that a Lie ideal U of a prime ring R such that $\mathrm{Char}(R) \neq 2$ with derivation d satisfying the condition $d(U) = 0$ or $d(U) \subseteq Z$ is central.

One of our motivations for Section 4 is this result. In this paper, we investigate the hypothesis of this result using homoderivations and provide similar results. Another purpose of Section 4 is to generalize some of the well-known results above using square closed Lie ideals of a prime ring.

The purpose of Section 4 is to prove $L \subseteq Z$ if one of the following conditions is satisfied:

i. $h(L) = (0)$,

ii. $h(L) \subseteq Z$,

iii. $h(xy) = xy$, for all $x, y \in L$,

iv. $h(xy) = yx$, for all $x, y \in L$, or

v. $h([x, y]) = 0$, for all $x, y \in L$

Here, R is a prime ring with a $\text{Char}(R) \neq 2$, h is a homoderivation of R and L is a non-zero square closed Lie ideal of R:

Section 2 of the present study provides some properties on commutativity of prime rings. Section 3 investigates the identity $ah_1(x) + h_2(x)b = 0$ on prime rings such that h_1 and h_2 are two homoderivations on R. Section 4 studies commutativity of a prime ring by square closed Lie ideals and homoderivations. Final section discusses the need for further research.

2. Preliminary

This section uses the following basic identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$, for any $x, y, z \in R$.

Theorem 2.1. [17] Let R be a prime ring whose characteristic is not 2 and d_1 and d_2 derivations of R such that the iterate $d_1 d_2$ is also a derivation, then at least one of d_1 and d_2 is zero.

Lemma 2.2. [15] Let R be a prime ring whose characteristic is not 2. If $U \not\subseteq Z$ is a Lie ideal of R, then $C_R(U) = Z$.

Lemma 2.3. [15] Let R be a prime ring whose characteristic is not 2. If $U \not\subseteq Z$ is a Lie ideal of R and $aUb = 0$, then $a = 0$ or $b = 0$.

Lemma 2.4. [18] If a prime ring R contains a commutative non-zero right ideal I, then R is commutative.

Lemma 2.5. [18] Let b and ab be in the center of a prime ring R. If b is not zero, then $a \in Z$.

Lemma 2.6. [3] Let R be a ring and h be a zero-power valued homoderivation on R. Then, h preserves Z.

Lemma 2.7. Let R be a prime ring. If h is a zero-power valued non-zero homoderivation on R such that $h(x) \in Z$, for all $x \in R$, then R is commutative or

$$h_{|Z} = 0 \quad \text{and} \quad h(xz) = h(x)z \ \ (h(zx) = zh(x)), \text{ for all } x \in R \text{ and } z \in Z$$

Proof.
Let R be a prime ring and h be a zero-power valued non-zero homoderivation on R such that $h(R) \subseteq Z$. By hypothesis, $h(x_1 x_2) \in Z$, for all $x_1, x_2 \in R$. Since Z is a subring of R and h is homoderivation of R, then

$$h(x_1)x_2 + x_1 h(x_2) \in Z \tag{1}$$

Replacing x_2 by $x_2 z$ such that $z \in Z$, then the following expression is obtained by Expression 1,

$$x_1(h(x_2) + x_2)h(z) \in Z \tag{2}$$

Since h is zero-power valued on R, there exists an integer $n(x_2) > 1$ such that $h^{n(x_2)}(x_2) = 0$, for all $x_2 \in R$. Replacing x_2 by $x_2 - h(x_2) + h^2(x_2) + \cdots + (-1)^{n(x_2)-1} h^{n(x_2)-1}(x_2)$ in Expression 2, for all $x_1, x_2 \in R$ and $z \in Z$,

$$x_1 x_2 h(z) \in Z$$

In view of Lemma 2.5, we have $x_1 x_2 \in Z$ or $h(z) = 0$, for all $x_1, x_2 \in R$ and $z \in Z$. Here, there are two cases:

Case 1: If $x_1 x_2 \in Z$, for all $x_1, x_2 \in R$, then $(x_1 x_2) x_3 \in Z$, for all $x_3 \in R$. Hence, $[(x_1 x_2) x_3, x_4] = 0$, for all $x_4 \in R$. That is, $[(x_1 x_2), x_4] x_3 + x_1 x_2 [x_3, x_4] = 0$ and thus

$$x_1 x_2 [x_3, x_4] = 0, \text{ for all } x_1, x_2, x_3, x_4 \in R$$

It follows from the fact that R is a prime ring that R is commutative.

Case 2: If $h(z) = 0$, for all $z \in Z$, then $h_{|_Z} = 0$. In this case, for all $x_1 \in R$ and $z \in Z$,

$$h(x_1 z) = h(x_1) z \quad (h(z x_1) = z h(x_1))$$

is obtained. □

3. The Identity $ah_1(x) + h_2(x)b = 0$

In this section, unless stated otherwise, let R be a prime ring.

Theorem 3.1. Let h_1, h_2, h_3, and h_4 be zero-power valued non-zero homoderivations on R. Suppose that

$$h_1(x_1) h_2(x_2) = h_3(x_1) h_4(x_2), \text{ for all } x_1, x_2 \in R \tag{3}$$

i. If $h_{1|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$.

ii. $h_{1|_Z} = 0$ iff $h_{3|_Z} = 0$

iii. If $h_{2|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$.

iv. $h_{2|_Z} = 0$ iff $h_{4|_Z} = 0$

Proof.
Let h_1, h_2, h_3, and h_4 be zero-power valued non-zero homoderivations on R. Suppose that

$$h_1(x_1) h_2(x_2) = h_3(x_1) h_4(x_2), \text{ for all } x_1, x_2 \in R$$

i. Let $h_{1|_Z} \neq 0$. There is at least $0 \neq z \in Z$ such that $h_1(z) \neq 0$. By Lemma 2.6, it is clear that $h_1(z) \in Z$. In Expression 3, by replacing x_1 by $x_1 z$, for $x_1 \in R$,

$$h_1(x_1 z) h_2(x_2) = h_3(x_1 z) h_4(x_2)$$

Thus,

$$h_1(x_1) h_1(z) h_2(x_2) + h_1(x_1) z h_2(x_2) + x_1 h_1(z) h_2(x_2) = h_3(x_1) h_3(z) h_4(x_2) + h_3(x_1) z h_4(x_2) + x_1 h_3(z) h_4(x_2)$$

From the last equation,for all $x_1, x_2 \in R$, the equation

$$h_1(x_1) h_1(z) h_2(x_2) = h_3(x_1) h_3(z) h_4(x_2)$$

is obtained. In view of hypothesis, for all $x_2 \in R$,

$$h_1(z) h_2(x_2) = h_3(z) h_4(x_2)$$

Using the last equation in the equation $h_1(x_1) h_1(z) h_2(x_2) = h_3(x_1) h_3(z) h_4(x_2)$,

$$(h_1(x_1) - h_3(x_1)) h_1(z) h_2(x_2) = 0, \text{ for all } x_1, x_2 \in R$$

The primeness of R and $0 \neq h_1(z) \in Z$ imply that

$$(h_1(x_1) - h_3(x_1))h_2(x_2) = 0, \text{ for all } x_1, x_2 \in R \tag{4}$$

In Expression 4, replacing x_2 by x_2x_3 such that $x_3 \in R$ and using Expression 4,

$$(h_1(x_1) - h_3(x_1))x_2h_2(x_3) = 0$$

for all $x_1, x_2, x_3 \in R$. Since R is a prime ring and h_2 is a non-zero homoderivation of R, then $h_1(x_1) = h_3(x_1)$, for all $x_1 \in R$. In that case, by hypothesis, $h_1(x_1)h_2(x_2) = h_1(x_1)h_4(x_2)$ for all $x_1, x_2 \in R$. That is,

$$h_1(x_1)(h_2(x_2) - h_4(x_2)) = 0, \text{ for all } x_1, x_2 \in R \tag{5}$$

In Expression 5, replacing x_1 by x_1x_3, $x_3 \in R$, and using Expression 5,

$$h_1(x_1)x_3(h_2(x_2) - h_4(x_2)) = 0, \text{ for all } x_1, x_2, x_3 \in R$$

Since R is a prime ring and h_1 is a non-zero homoderivation of R, then $h_2(x_2) = h_4(x_2)$, for all $x_2 \in R$.

ii. ($\Rightarrow$): Let $h_{1|_Z} = 0$. In Expression 3, replacing x_1 by $z \in Z$ for and using $h_1(z) = 0$,

$$h_3(z)h_4(x_2) = 0, \text{ for all } x_2 \in R$$

In this equation, replacing x_2 by x_3x_2 for $x_3 \in R$ and using the hypothesis,

$$h_3(z)x_3h_4(x_2) = 0, \text{ for all } x_2, x_3 \in R$$

The primeness of R implies $h_{3|_Z} = 0$. Thus, if $h_{1|_Z} = 0$, then $h_{3|_Z} = 0$.

($\Leftarrow$): Let $h_{3|_Z} = 0$. With similar steps above, $h_{1|_Z} = 0$ is obtained. Hence, if $h_{3|_Z} = 0$, then $h_{1|_Z} = 0$.

The proofs of *iii.* and *iv.* are similar to *i.* and *ii.*, respectively. □

Example 3.2. Let $\Re$ be a ring with the unit and no zero divisors. For the subring

$$\wp = \{r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} : r_{11}, r_{12}, r_{22} \in \Re\}$$

of $M_2(\Re)$, the ring of 2×2 matrices over $\Re$, it is easy to validate that $\wp$ is not a prime ring. Here,

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Moreover, $Z_\wp = \{ze_{11} + ze_{22} : z \in Z_\Re\}$ is the center of ring $\wp$. Let

$$\begin{array}{rccl} h_1 : & \wp & \to & \wp \\ & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \to & -r_{11}e_{11} - r_{12}e_{12} \end{array}$$

and

$$\begin{array}{rccl} h_2 : & \wp & \to & \wp \\ & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \to & -r_{12}e_{12} - r_{22}e_{22} \end{array}$$

Then, it is easy to check that h_1 and h_2 are homoderivations of $\wp$. Let $\Im = \wp \times \wp$. It is easy to validate that $\Im$ is not a prime ring. Besides,

$$Z_\Im = \{(z_{11}e_{11} + z_{22}e_{22}, \alpha_{11}e_{11} + \alpha_{22}e_{22}) : z_{11}, z_{22}, \alpha_{11}, \alpha_{22} \in Z_\Re\}$$

is the center of ring $\Im$. Let $X = (r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}, s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22}) \in \Im$ and $Y = (x_{11}e_{11} + x_{12}e_{12} + x_{22}e_{22}, y_{11}e_{11} + y_{12}e_{12} + y_{22}e_{22}) \in \Im$. Define the maps $H_1, H_2, H_3, H_4 : \Im \to \Im$ as follows:

$$\begin{array}{rcl} H_1 : \Im & \to & \Im \\ X & \to & (h_1(r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}), 0_\Im) = (-r_{11}e_{11} - r_{12}e_{12}, 0_\Im) \end{array}$$

$$\begin{array}{rcl} H_2 & : & \Im \rightarrow \Im \\ & & X \rightarrow (0_\Im, h_1(s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22})) = (0_\Im, -s_{11}e_{11} - s_{12}e_{12}) \end{array}$$

$$\begin{array}{rcl} H_3 & : & \Im \rightarrow \Im \\ & & X \rightarrow (h_2(r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}), 0_\Im) = (-r_{12}e_{12} - r_{22}e_{22}, 0_\Im) \end{array}$$

and

$$\begin{array}{rcl} H_4 & : & \Im \rightarrow \Im \\ & & X \rightarrow (0_\Im, h_2(s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22})) = (0_\Im, -s_{12}e_{12} - s_{22}e_{22}) \end{array}$$

Then, it is easy to check that H_1, H_2, H_3, and H_4 are homoderivations of $\Im$. For any two elements $X, Y \in \Im$,

$$H_1(X)H_2(Y) = H_3(X)H_4(Y)$$

However, neither

$$H_1 = H_3 \quad \text{and } H_2 = H_4$$

nor

$$H_{1|_{Z_\Im}} = 0, \quad H_{2|_{Z_\Im}} = 0, \quad H_{3|_{Z_\Im}} = 0, \quad \text{and} \quad H_{4|_{Z_\Im}} = 0$$

Hence, this example shows that it is crucial that the considered ring is a prime ring and the selected homoderivations are zero-power valued, as stated in Theorem 3.1.

Note 3.3. From Theorem 3.1, it can be observed that the statements "If $h_{3|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$" and "If $h_{4|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$" are valid.

From Theorem 3.1, the following corollaries are obtained.

Corollary 3.4. Let h_1 and h_2 be zero-power valued non-zero homoderivations on R satisfying the condition

$$h_1(x)h_1(y) = h_2(x)h_2(y), \text{ for all } x, y \in R$$

Then, $h_1 = h_2$ or $h_{1|_Z} = h_{2|_Z} = 0$.

Corollary 3.5. Let h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that

$$h_1(x)h_2(y) = h_2(x)h_1(y), \text{ for all } x, y \in R$$

Then, $h_1 = h_2$ or $h_{1|_Z} = h_{2|_Z} = 0$.

Theorem 3.6. Let h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that there are $a, b \in R$ such that

$$ah_1(x) + h_2(x)b = 0, \text{ for all } x \in R \tag{6}$$

Then, $a = -b \in Z$ or $h_{1|_Z} = h_{2|_Z} = 0$.

Proof.
Let h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that there are $a, b \in R$ such that

$$ah_1(x) + h_2(x)b = 0, \text{ for all } x \in R$$

If $a = b = 0$, then the proof is clear. From now on, $a \neq 0$ and $b \neq 0$. Suppose that $h_{1|_Z} = 0$. In Expression 6, replacing x by z for $z \in Z$,

$$ah_1(z) + h_2(z)b = 0$$

Since $h_{1|_Z} = 0$, then $h_2(z)b = 0$. This means that $h_2(z) = 0$, for all $z \in Z$, by the primeness of R. With the same arguments above, it can be shown that if $h_{2|_Z} = 0$, then $h_{1|_Z} = 0$. Assume that

$h_{1|_Z} \neq 0$. In light of Lemma 2.6 and $h_{1|_Z} \neq 0$, there is at least $0 \neq z_1 \in Z$ such that $0 \neq h_1(z_1) \in Z$ and $h_2(z_1) \in Z$. Replacing x by xz_1 in Expression 6,

$$0 = ah_1(x)h_1(z_1) + ah_1(x)z_1 + axh_1(z_1) + h_2(x)h_2(z_1)b + h_2(x)z_1b + xh_2(z_1)b$$

Using $z_1, h_1(z_1), h_2(z_1) \in Z$ and Expression 6 in the last equation,

$$(a(h_1(x) + x) - (h_2(x) + x)a)h_1(z_1) = 0$$

Since R is a prime ring and $h_1(z_1) \neq 0$, for all $x \in R$,

$$a(h_1(x) + x) - (h_2(x) + x)a = 0 \tag{7}$$

Since $h_{2|_Z} \neq 0$, there is at least $0 \neq z_2 \in Z$ such that $0 \neq h_2(z_2) \in Z$. In Expression 7, replacing x by z_2,

$$ah_1(z_2) + h_2(z_2)(-a) = 0$$

Combining the last equations and Expression 6,

$$h_2(z_2)(b + a) = 0$$

The primeness of R and $h_2(z_2) \neq 0$ implies $a = -b$. In that case, for any $x \in R$,

$$ah_1(x) - h_2(x)a = 0 \tag{8}$$

In Expression 8, replacing x by xz_1,

$$0 = ah_1(x)h_1(z_1) + axh_1(z_1) - h_2(x)h_2(z_1)a - xh_2(z_1)a$$

According to the last equation and Expression 8,

$$ah_1(x)h_1(z_1) + axh_1(z_1) - ah_1(x)h_1(z_1) - xah_1(z_1) = 0$$

This implies $[a, x]\, h_1(z_1) = 0$, for all $x \in R$. The primeness of R and $h_1(z_1) \neq 0$ implies $a \in Z$. $\square$

Example 3.7. Consider the ring $\wp$ provided in Example 3.2. Let

$$\begin{array}{rccl} h_1 & : & \wp & \to \wp \\ & & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \to -r_{11}e_{11} - r_{12}e_{12} - r_{22_{e_{22}}} \end{array}$$

and

$$\begin{array}{rccl} h_2 & : & \wp & \to \wp \\ & & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \to -r_{11}e_{12} - r_{12}e_{12} \end{array}$$

Then, it is easy to check that h_1 and h_2 are homoderivations of $\wp$. Let $\alpha = -1_{\Re}e_{11}$ and $\beta = 1_{\Re}e_{11} + 1_{\Re}e_{22}$ be fixed elements. For any element $X = r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} \in \wp$,

$$\alpha h_1(X) + h_2(X)\beta = 0_{\wp}$$

However, neither $\alpha = -\beta$ nor $h_{1|_{Z_\wp}} = h_{2|_{Z_\wp}} = 0$. Hence, this examples show that it is crucial that the considered ring is a prime ring and the selected homoderivations are zero-power valued, as stated in Theorem 3.6.

4. Central Lie Ideals of Prime Rings with Homoderivations

In this section, unless stated otherwise, R is a prime ring with $\text{Char}(R) \neq 2$.

Lemma 4.1. Let L be a non-zero Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) = 0$, for all $x \in L$. Then, $L \subseteq Z$.

Proof.
Let L be a non-zero Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) = 0$, for all $x \in L$. Since h is a homoderivations of R,

$$h([x_1, r_1]) = [h(x_1), h(r_1)] + [h(x_1), r_1] + [x_1, h(r_1)]\,, \text{ for all } x_1 \in L, r_1 \in R$$

By hypothesis, $[x_1, h(r_1)] = 0$, for all $x_1 \in L$ and $r_1 \in R$. By taking $r_1 = r_1 x_2$, for any $x_2 \in L$, in the last equation,

$$h(r_1)\,[x_1, x_2] = 0, \text{ for all } x_1, x_2 \in L, r_1 \in R \tag{9}$$

In Expression 9, replacing r_1 by $r_1 r_2$, $r_2 \in R$,

$$h(r_1) r_2\,[x_1, x_2] = 0$$

Hence, $[x_1, x_2] = 0$, for all $x_1, x_2 \in L$, by the primeness of R. By replacing x_2 by $[x_2, r_1]$ in the last equation,

$$[x_1, [x_2, r_1]] = 0, \text{ for all } x_1, x_2 \in L, r_1 \in R \tag{10}$$

Consider two inner derivations of R, $I_{x_1} : R \to R$ and $I_{x_2} : R \to R$ defined by $I_{x_1}(s) = [x_1, s]$ and $I_{x_2}(s) = [x_2, s]$, respectively. Thus, $I_{x_1} I_{x_2}(r_1) = 0$, for all $r_1 \in R$, by Expression 10. In view of Theorem 2.1, $I_{x_1} = 0$ or $I_{x_2} = 0$. That is, $x_1 \in Z$ or $x_2 \in Z$. This prove that $L \subseteq Z$. □

Lemma 4.2. Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) \in Z$, for all $x \in L$. Then, $L \subseteq Z$.

Proof.
Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) \in Z$, for all $x \in L$. By hypothesis for all $x_1 \in L$ and $r_1 \in R$,

$$[h(x_1), r_1] = 0 \tag{11}$$

In Expression 11, by replacing x_1 by x_1^2, $[h(x_1^2), r_1] = 0$. From the last equation, since $h(x_1) \in Z$ and using $\mathrm{Char}(R) \neq 2$,

$$h(x_1)\,[x_1, r_1] = 0 \tag{12}$$

In Expression 12, substituting $r_1 r_2$ instead of r_1, $r_2 \in R$,

$$h(x_1) r_1\,[x_1, r_2] = 0$$

The primeness of R implies $h(x_1) = 0$ or $x_1 \in Z$, for all $x_1 \in L$. Define

$$A = \{x \in L : h(x) = 0\}$$

and

$$B = \{x \in L : x \in Z\}$$

Note that both are additive subgroups of L, and their union equals L. Thus, either $A = L$ or $B = L$. Suppose first that $A = L$. Then, $h(L) = 0$. In view of Lemma 4.1, $L \subseteq Z$. In other case, $x_1 \in Z$, for all $x_1 \in L$. That is $L \subseteq Z$. □

The following example shows that the above result is not true in the types of some other rings. In the example, it is emphasized that the hypothesis primeness of the result provided above is all-important.

Example 4.3. Let R_1 be a non-commutative ring with the unit, no zero divisors, and $\mathrm{Char}(R_1) \neq 2$, and R_2 be a non-commutative ring with the unit, no zero divisors, and $\mathrm{Char}(R_2) \neq 2$. For a fixed $(1_{R_1}, 0_{R_2}), (0_{R_1}, 1_{R_2}) \neq (0_{R_1}, 0_{R_2}) \in R^* = R_1 \times R_2$, it holds that $(1_{R_1}, 0_{R_2})\, R^*\, (0_{R_1}, 1_{R_2}) = (0_{R_1}, 0_{R_2})$. Thus, R^* is not a prime ring. Let $L = Z_{R_1} \times R_2$ such that Z_{R_1} is a the center of R_1. It is easy to verify that L is a subgroup of R^*. For $(z, s_1) \in L$ and $(r, s_2) \in R^*$,

$$[(z, s_1), (r, s_2)] = (zr - rz, s_1 s_2 - s_2 s_1) \overset{z \in Z_{R_1}}{=} (0_{R_1}, s_1 s_2 - s_2 s_1) \in L$$

and

$$(z, s_1)(z, s_1) = \left(z^2, s_1 s_2\right) \in L$$

Thus, L is a square closed Lie ideal of R^* and $L \nsubseteq Z_{R^*}$. Let

$$\begin{array}{rccc} h & : & R^* & \to & R^* \\ & & (r, s) & \to & (-r, 0_{R_2}) \end{array}$$

Then, it is easy to check that h is a homoderivation of R^*. For any element $(z, s_1) \in L$, $h(z, s_1) \in Z_{R^*}$. However, L is not a central square closed Lie ideal of R^*.

Theorem 4.4. Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h(xy) = xy \text{ (or } h(xy) = yx\text{), for all } x, y \in L$$

Then, $L \subseteq Z$.

Proof.
Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h(xy) = xy, \text{ for all } x, y \in L$$

Suppose that $L \nsubseteq Z$. Since h is homoderivation of R, for all $x_1, x_2, x_3 \in L$,

$$\begin{aligned} x_1 2(x_2 x_3) = h(x_1 2(x_2 x_3)) &= 2h(x_1(x_2 x_3)) \\ &= 2\left(h(x_1)h(x_2 x_3) + h(x_1)x_2 x_3 + x_1 h(x_2 x_3)\right) \\ &= 2\left(h(x_1)x_2 x_3 + h(x_1)x_2 x_3 + x_1 x_2 x_3\right) \end{aligned}$$

This implies $4h(x_1)x_2 x_3 = 0$. Since $\text{Char}(R) \neq 2$,

$$h(x_1)x_2 x_3 = 0, \text{ for all } x_1, x_2, x_3 \in L \tag{13}$$

In Expression 13, replacing x_2 by $2x_4 x_2$ such that $x_4 \in L$ and using $\text{Char}(R) \neq 2$,

$$h(x_1)x_4 x_2 x_3 = 0 \tag{14}$$

Multiplying Expression 13 by x_4 from the left,

$$x_4 h(x_1)x_2 x_3 = 0 \tag{15}$$

Combining Expression 14 and Expression 15,

$$[h(x_1), x_4]\, x_2 x_3 = 0$$

for all $x_1, x_2, x_3, x_4 \in L$. In view of Lemma 2.3 and $L \neq (0)$, for all $x_1, x_4 \in L$,

$$[h(x_1), x_4] = 0$$

We have proved $h(L) \subseteq C_R(L)$. In this case, $h(L) \subseteq Z$ by Lemma 2.2. In view of Lemma 4.2, $L \subseteq Z$. This is a contradiction. That proves that $L \subseteq Z$.

For the condition $h(xy) = yx$, for all $x, y \in L$, the proof is similar. □

Since every ideal is a square closed Lie ideal, an ideal can be considered instead of a square closed Lie ideal in Theorem 4.4. Thus, Corollary 4.5 is obtained by Lemma 2.4.

Corollary 4.5. Let R be a prime ring with $\text{Char}(R) \neq 2$, I be a non-zero ideal of R, and h be a non-zero homoderivation of R. If one of the following conditions is satisfied, for all $x, y \in I$,

i. $h(xy) = xy$

ii. $h(xy) = yx$

then R is commutative.

Here, it can be observed that Corollary 4.5 without hypothesis "zero-power valued homoderivation on the ideal" is a more general version of Theorem 3 provided in [4].

Theorem 4.6. Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h([x,y]) = 0, \text{ for all } x, y \in L \tag{16}$$

Then, $L \subseteq Z$.

Proof.
Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h([x,y]) = 0, \text{ for all } x, y \in L$$

Suppose that $L \not\subseteq Z$. Let $x_1, x_2 \in L$. By taking $x = 2x_2x_1$ and $y = x_2$ in Expression 16 and using $\text{Char}(R) \neq 2$,

$$h(x_2)\,[x_1, x_2] = 0 \tag{17}$$

and then replacing x_1 with $2x_1x_3$ such that $x_3 \in L$ in Expression 17 and using $\text{Char}(R) \neq 2$,

$$h(x_2)x_1\,[x_3, x_2] = 0 \tag{18}$$

Let $x_4 \in L$. In Expression 18, replacing x_1 by $2x_4x_1$ and using $\text{Char}(R) \neq 2$,

$$h(x_2)x_4x_1\,[x_3, x_2] = 0 \tag{19}$$

Multiplying Expression 18 by x_4 from the left,

$$x_4h(x_2)x_1\,[x_3, x_2] = 0 \tag{20}$$

By comparing Expression 19 and Expression 20,

$$[h(x_2), x_4]\,x_1\,[x_3, x_2] = 0, \text{ for all } x_1, x_2, x_3, x_4 \in L$$

In view of Lemma 2.3,

$$[h(x_2), x_4] = 0 \quad \text{or} \quad [x_3, x_2] = 0$$

for all $x_2, x_3, x_4 \in L$. This proves that $h(x_2) \in C_R(L)$ or $[x_3, x_2] = 0$, for all $x_2, x_3 \in L$. Define

$$A = \{x \in L : h(x) \in C_R(L)\}$$

and

$$B = \{x \in L : [y, x] = 0, \text{ for all } y \in L\}$$

Note that both are additive subgroups of L and their union equals L. Thus either $A = L$ or $B = L$. Suppose first that $A = L$. Then, $h(x_2) \in C_R(L)$, for all $x_2 \in L$. Moreover, by Lemma 2.2, $h(x_2) \in Z$, for all $x_2 \in L$. In view of Lemma 4.2, $L \subseteq Z$, a contradiction. Suppose that $B = L$. Then, $[x_3, x_2] = 0$, for all $x_2, x_3 \in L$. Let $r \in R$ and fix $x_2, x_3 \in L$. By replacing x_2 by $[x_2, r]$ in $[x_3, x_2] = 0$,

$$[x_3, [x_2, r]] = 0$$

Using similar techniques after Expression 10, $L \subseteq Z$, a contradiction. That proves that $L \subseteq Z$. □

5. Conclusion

In this paper, Section 3 discussed algebraic identities including homoderivations on a prime ring. Section 4 also investigated algebraic identities involving homoderivations on a square closed Lie ideal of a prime ring. It proved that a square closed Lie ideal, satisfying the identities discussed in the section, is contained in the center of a prime ring. The obtained results extended several well-known results in the literature. In future studies, the hypotheses in this study can be studied using a semiprime ring and an ideal of a semiprime ring.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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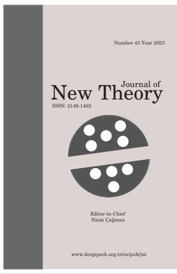

Cofinitely Goldie*-Supplemented Modules

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Abstract — One of the generalizations of supplemented modules is the Goldie*-supplemented module, defined by Birkenmeier et al. using β^* relation. In this work, we deal with the concept of the cofinitely Goldie*-supplemented modules as a version of Goldie*-supplemented module. A left R-module M is called a cofinitely Goldie*-supplemented module if there is a supplement submodule S of M with $C\beta^*S$, for each cofinite submodule C of M. Evidently, Goldie*-supplemented are cofinitely Goldie*-supplemented. Further, if M is cofinitely Goldie*-supplemented, then M/C is cofinitely Goldie*-supplemented, for any submodule C of M. If A and B are cofinitely Goldie*-supplemented with $M = A \oplus B$, then M is cofinitely Goldie*-supplemented. Additionally, we investigate some properties of the cofinitely Goldie*-supplemented module and compare this module with supplemented and Goldie*-supplemented modules.

Keywords *Cofinitely supplemented module, Goldie*-supplemented module, cofinitely Goldie*-supplemented module*

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1. Introduction

Cofinitely supplemented modules were introduced by Alizade et al. [1] and Smith [2]. Following these works, various generalizations of cofinitely supplemented modules, such as totally cofinitely supplemented [3], cofinitely weak supplemented [4], an H-cofinitely supplemented [5, 6] and cofinitely weak rad-supplemented [7] were studied. The Goldie*-supplemented modules were introduced and characterized in [8, 9]. A left module M is called a Goldie*-supplemented module (or concisely, $\mathcal{G}^*$s module) if there is a supplement submodule S of M with $C\beta^*S$, for each submodule C of M. Furthermore, the authors [8, 9] stated that Goldie*-supplemented modules ($\mathcal{G}^*$s) are located between amply supplemented and supplemented. Afterward, a new equivalence relation β^{**} was defined, inspired by β^* relation, and the properties of the equivalence relation β^{**} were analyzed in [10]. The relation β^{**} has helped to describe two concepts, namely Goldie-rad-supplemented and amply (weakly) Goldie-rad-supplemented modules. After presenting the relation β^{**}, Talebi et al. [10] characterized Goldie-rad-supplemented modules as a perspective of H-supplemented modules. This module corresponds to rad-H-supplemented modules. Meanwhile, another version of the Goldie-rad-supplemented modules, called amply (weakly) Goldie-rad-supplemented modules, were developed based on the relation β^{**} [11]. It was shown that an amply (weakly) Goldie-rad-supplemented module is a (weakly) Goldie-rad-supplemented [11]. Inspired by these works, we concentrate on cofinitely Goldie*-supplemented modules as a generalization of $\mathcal{G}^*$s modules. A module M is called a cofinitely Goldie*-supplemented

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module (or concisely, c$\mathcal{G}^*$s module) if there is a supplement submodule S of M with $C\beta^*S$, for each cofinite submodule C of M, equivalently, $C+S/C$ is small in M/C, and $C+S/S$ is small in M/S. This definition is closely related to the concept of H-cofinitely supplemented. A module M is called H-cofinitely supplemented if, for each cofinite submodule C of M, there exists a direct summand D of M such that $C+D/C$ is small in M/C, and $C+D/D$ is small in M/D. Clearly, H-cofinitely supplemented is c$\mathcal{G}^*$s. We provide an example to show that the converse implication does not hold. However, if M is refinable, then H-cofinitely supplemented and c$\mathcal{G}^*$s coincide. Therefore, c$\mathcal{G}^*$s modules are situated between H-cofinitely supplemented and cofinitely weak supplemented. Moreover, we observe that if M is c$\mathcal{G}^*$s, then M/C is c$\mathcal{G}^*$s, for any submodule C of M. In addition, we provide that the cofinite direct summand of c$\mathcal{G}^*$s is c$\mathcal{G}^*$s. We investigate the relations between c$\mathcal{G}^*$s, $\mathcal{G}^*$s, and cofinitely supplemented modules under some restrictions.

Section 2 of the handled study presents some basic definitions and properties. Section 3 studies cofinitely Goldie*-supplemented modules. Final section discusses the need for further research.

2. Preliminaries

This section provides some essential definitions to be needed for the following sections. Throughout this paper, let M be an unital left module over an associative unital ring R and $Rad(M)$ be a Jacobson radical of M.

Definition 2.1. [12] Let A be a submodule of M. If $A + B \neq M$, for every proper submodule B of M, A is called superfluous (or small) in M and denoted by $A \ll M$.

Lemma 2.2. [13] Let A, B be submodules of M.

i. If $A \subseteq B \subseteq M$, then $B \ll M$ if and only if $A \ll M$ and $B/A \ll M/A$.

ii. If $A \subseteq B \subseteq M$ and $A \ll B$, then $A \ll M$. Moreover, if B is a direct summand in M and $A \ll M$, then $A \ll B$.

iii. For $A \ll M$, if $f : M \to N$, then $f(A) \ll N$. If f is a small epimorphism, the converse is also true.

Definition 2.3. [13] A submodule A of M is called a (weak) supplement of B in M if $A + B = M$ and $A \cap B \ll A$ ($A \cap B \ll M$), for some submodule B of M. If every submodule of M has a (weak) supplement in M, then M is (weak) supplemented.

It is clear that the supplemented module is weak supplemented.

Lemma 2.4. [14] If $f : M \to N$ is a small epimorphism with a small kernel, and A is a supplement of B in M, then $f(A)$ is a supplement of $f(B)$ in N.

Definition 2.5. [13] A submodule C of M is called a cofinite submodule in M if M/C is finitely generated. A module M is said to be cofinitely weak supplemented (briefly, cws) if every cofinite submodule of M has a weak supplement in M.

Definition 2.6. [13] If every cofinite submodule of M has a supplement in M, M is called a cofinitely supplemented module (briefly, cs).

Indeed, if M is supplemented module, then M is cofinitely supplemented, and cofinitely weak supplemented. For the converse, finitely generated property is needed. Namely, finitely generated cofinitely supplemented is supplemented.

Proposition 2.7. [4] An arbitrary sum of cws-modules is a cws-module.

Theorem 2.8. [4] Let M be an R-module such that $Rad(A) = A \cap Rad(M)$, for every finitely generated submodule A of M. Then, M is cws if and only if M is cs.

Theorem 2.9. [4] Let M be a module with a small radical. Then, the following statements are equivalent:

i. M is a cws-module.

ii. $M/Rad(M)$ is a cws-module.

iii. Every cofinite submodule of $M/Rad(M)$ is a direct summand.

Definition 2.10. [13] Let $M = X + Y$, for submodules X and Y of M. Then, M is called a refinable module if there is a direct summand A of M so that $A \subseteq X$ and $M = A + Y$.

Definition 2.11. [13] Any submodule A of M has ample supplements in M if $A + B = M$, for every submodule B of M, there is a supplement A' of A with $A' \subseteq B$. Then, M is called an amply supplemented if all submodules have ample supplements in M.

Evidently, if M is an amply supplemented module, then M is supplemented. Supplemented modules over a non-local Dedekind domain provided in [2] are amply supplemented. Additionally, if R is semiperfect ring, then every finitely generated left R-module is amply supplemented.

Definition 2.12. [8] Let A and B be submodules of M. Then, $A\beta^*B$ if $A + B/B$ is small in M/B, and $A + B/A$ is small in M/A.

In [8], it is shown that β^* is an equivalence relation, and if A is small in M, then $0\beta^*A$.

Definition 2.13. [8] If there is a supplement submodule B of M with $A\beta^*B$, for each submodule A of M, then M is called a Goldie*-supplemented module ($\mathcal{G}^*$s).

Every linearly compact and semisimple module is $\mathcal{G}^*$s. Moreover, if M is amply supplemented, then M is $\mathcal{G}^*$s. In addition, if M is $\mathcal{G}^*$s, then M is supplemented [8].

Theorem 2.14. [8] Let A, B be submodules of M such that $A\beta^*B$. Then, A has a (weak) supplement C in M if and only if C is a (weak) supplement for B in M.

Corollary 2.15. [8] Let A, B be submodules of M such that $A \subseteq B$, and A has a weak supplement C in M. Then, $A\beta^*B$ if and only if $B \cap C \ll M$.

Proposition 2.16. [8] Let $f : M \to N$ be an epimorphism.

i. If A and B are two submodules of M such that $A\beta^*B$, then $f(A)\beta^*f(B)$.

ii. If A and B are two submodules of N such that $A\beta^*B$, then $f^{-1}(A)\beta^*f^{-1}(B)$.

Corollary 2.17. [8] Let A, B, and C be submodules of M such that $C \ll M$. Then, $A\beta^*B$ if and only if $A\beta^*(B + C)$.

Definition 2.18. [5] A module M is called an H-cofinitely supplemented if, for each cofinite submodule C of M, there exists a direct summand D of M such that $C + D/C$ is small in M/C, and $C + D/D$ is small in M/D. It is obvious that H-cofinitely supplemented is c$\mathcal{G}^*$s.

Definition 2.19. [15] A ring R is called a left V-ring if every simple left R-module is injective.

Theorem 2.20. [15] For any ring R, the following are equivalent:

i. R is a left V-ring.

ii. Any left ideal A of R is an intersection of maximal left ideals.

iii. For any left R-module M, $Rad(M) = 0$.

3. Cofinitely Goldie*-Supplemented Modules

Definition 3.1. A module M is called a cofinitely Goldie*-supplemented (c$\mathcal{G}$*s) if there is a supplement submodule S of M with $C\beta^*S$, for each cofinite submodule C of M. It is obvious that every $\mathcal{G}$*s is c$\mathcal{G}$*s.

Example 3.2. Every semisimple and local module is c$\mathcal{G}$*s. Let M be a semisimple. In other words, M is $\mathcal{G}$*s. Therefore, M is c$\mathcal{G}$*s. Let us take a submodule C as a cofinite in M. Because M is local, C is small in M, that is, $C\beta^*0$. Thereby, M is c$\mathcal{G}$*s.

Proposition 3.3. Every c$\mathcal{G}$*s module is cws.

Proof.
To prove this, consider the cofinite submodule C of M. Then, from the hypothesis, we get $C\beta^*S$ where $M = S + K$ and $K \cap S \ll S$, for some submodule K of M, that is, S is a supplement in M. Besides, $K \cap S$ is also small in M from Lemma 2.2. Thus, S has a weak supplement K by Definition 2.3. Moreover, from Theorem 2.14, C has a weak supplement K in M. Consequently, M is cws. □

Proposition 3.4. If M is a refinable cws-module, then M is c$\mathcal{G}$*s.

Proof.
Assume that C is cofinite in M. Then, C has a weak supplement S in M as M is cws. In other words, $M = C + S$ and $C \cap S$ is small in M. Using the refinable property, we observe that there exists a direct summand A of M, such that $A \subseteq C$ and $M = A + S$. Thus, $A \cap S \subseteq C \cap S \ll M$ implies from Lemma 2.2 *i* that $A \cap S \ll M$. Thus, A has a weak supplement S in M. Hence, $A\beta^*C$ from Corollary 2.15. □

Theorem 3.5. Let M be a module and consider the following conditions:

i. M is amply supplemented.

ii. M is $\mathcal{G}$*s.

iii. M is c$\mathcal{G}$*s.

Then, $i \Rightarrow ii$ and $ii \Rightarrow iii$. Moreover, if M is finitely generated, then $iii \Rightarrow ii$, and if R is a non-local domain, then $ii \Rightarrow i$.

Proof.
$i \Rightarrow ii$ Clear.

$ii \Rightarrow iii$ Clear.

$iii \Rightarrow ii$ Let M be a c$\mathcal{G}$*s module. If M is finitely generated, then every submodule of M is cofinite. Hence, M is $\mathcal{G}$*s.

$ii \Rightarrow i$ M is supplemented since every $\mathcal{G}$*s is supplemented. Hence, M is amply supplemented because R is a non-local domain. □

The following example shows that every H-cofinitely supplemented module need not be c$\mathcal{G}$*s.

Example 3.6. [5] Let $R = F[[x, y]]$ be the ring of formal power series over a field F in the indeterminates x and y. Then, R is a commutative noetherian local domain with maximal ideal $J = Rx + Ry$. Therefore, the ring R is semiperfect, and the ideal J is finitely generated. Since R is a domain, J_R is a uniform module. Thus, J_R is not a direct sum of cylic modules. Then, J_R is not H-cofinitely supplemented. Since R is semiperfect, J_R is amply supplemented. Hence, J_R is c$\mathcal{G}$*s by Theorem 3.5.

The relationships between c$\mathcal{G}$*s and cs modules under some conditions are as follows:

Proposition 3.7. If M is c$\mathcal{G}$*s with zero radical, then M is cs.

Proof.
Let C be a cofinite submodule of M. From the hypothesis, there exists a supplement submodule S of M such that $C\beta^*S$. We observe that $M = S + K$, and $K \cap S$ is small in S, for some submodule K of M. When the radical is zero, $K \cap S = 0$. This means $M = S \oplus K$. In particular, K is also a supplement of C in M because of Theorem 2.14. Therefore, M is cs. □

Proposition 3.8. If M is refinable c$\mathcal{G}$*s, then M is cs.

Proof.
Take a cofinite submodule C of M. As M is c$\mathcal{G}$*s, $C\beta^*S$ where S is a supplement submodule of M. Therefore, $M = S + S'$, and $S' \cap S$ is small in S, for submodule S' of M. According to Lemma 2.2, $S' \cap S$ is small in M. More precisely, S and S' are weak supplements of each other. In addition, based on Theorem 2.14, we realize that C also has a weak supplement S' in M. Then, we mean $M = C + S'$ and $C \cap S'$ is small in M. The refinable property admits a direct summand A of M so that $A \subseteq C$ and $M = S' + A$. Taking a submodule A' of M, we write as $M = A \oplus A'$. In these circumstances, A' is a supplement of A. By the modular property, we see that $C = A + (C \cap S')$. Moreover, $A \cap S'$ is small in M. Here, we emphasize that A is a weak supplement of S' in M. Corollary 2.15 shows that $C\beta^*A$. We conclude from Theorem 2.14 that A' is a supplement of C in M. □

Proposition 3.9. Let M be c$\mathcal{G}$*s with $Rad(A) = A \cap Rad(M)$, for finitely generated submodule A of M. Therefore, M is cs.

Proof.
Based on Proposition 3.3, we have that M is cws. We provide from Theorem 2.8 that M is cs. □

Proposition 3.10. If M is c$\mathcal{G}$*s, then M/A is c$\mathcal{G}$*s, for every small submodule A of M.

Proof.
Take a submodule C of M containing A, and let C/A be a cofinite submodule in M/A. Then, C is a cofinite submodule in M, as $(M/A)/(C/A) \cong M/C$ is finitely generated. From the hypothesis, $C\beta^*S$ with a supplement S in M. If $g : M \to M/A$ is a canonical epimorphism, following Proposition 2.16, we get $g(C)\beta^*g(S)$, that is, $(C/A)\beta^*(S + A/A)$. Taking into account Lemma 2.4, we have that $S + A/A$ is a supplement in M/A. As a consequence, M/A is c$\mathcal{G}$*s. □

Proposition 3.11. If M/A is refinable c$\mathcal{G}$*s with $A \ll M$, M is c$\mathcal{G}$*s.

Proof.
If C is a cofinite submodule in M, then $C + A/A$ is a cofinite in M/A. Since M/A is c$\mathcal{G}$*s,

$$(C + A/A)\beta^*(S + A/A)$$

where $S+A/A$ is a supplement in M/A. Observe that $M/A = (S+A/A)+(B/A)$ and $(S+A/A)\cap(B/A)$ is small in $S+A/A$, for submodule B of M containing A, equivalently, $M = S+B$, $(S\cap B)+A/A$ is small in $S + A/A$. Furthermore, $(S \cap B) + A/A$ is small in M/A. If $f : M \to M/A$ is a small epimorphism, we obtain $f^{-1}(C + A/A)\beta^*f^{-1}(S + A/A)$ from Proposition 2.16, that is, $(C + A)\beta^*(S + A)$. We can see from Corollary 2.17 that $C\beta^*S$. By Lemma 2.2, $S \cap B$ is small in M. Since $M = S + B$, S has a weak supplement B in M. In fact, following Theorem 2.14, we get $M = C + B$, and $C \cap B$ is small in M. Since M is refinable, $M = C' \oplus C''$ for some submodules C' and C'' of M with $C' \subseteq C$, and $M = C' + B$. If C' is contained in C, by Lemma 2.2, $C' \cap B$ is also small in M. This implies that C' has a weak supplement B in M. Using Corollary 2.15, we have $C\beta^*C'$. Finally, M is c$\mathcal{G}$*s. □

Proposition 3.12. Let M be a c$\mathcal{G}$*s with a small radical. Then, every cofinite submodule of $M/Rad(M)$ is a direct summand.

Proof.
We deduce from Proposition 3.3 that M is cws. Then, Theorem 2.9 shows the result. $\square$

Proposition 3.13. Let M be refinable c$\mathcal{G}$*s, and C be a cofinite direct summand of M. Thus, C is c$\mathcal{G}$*s.

Proof.
Assume that $M = C \oplus B$, for some submodule B of M. Here, B is finitely generated. Consider a cofinite submodule A of C. Then, C/A is finitely generated. Further, A is a cofinite in M because $M/A = (C \oplus B)/A$. Since M is c$\mathcal{G}$*s, there exists a supplement S in M such that $A\beta^*S$. Thus, for submodule S' of M, $M = S + S'$, and $S \cap S'$ is small in S. Note that $S \cap S'$ is small in M from Lemma 2.2. Moreover, S has a weak supplement S' in M. Following Theorem 2.14, $M = A + S'$ and $A \cap S'$ is small in M. Because M is refinable, then $M = X \oplus X'$, for some submodules X and X' of M with $X \subseteq A$ and $M = X + S'$. Since X is contained in A, then $X \cap S' \subseteq A \cap S'$, and $A \cap S' \ll M$ implies that $X \cap S' \ll M$ from Lemma 2.2. Hence, S' is a weak supplement of X in M. Applying Corollary 2.15, we get $X\beta^*A$. From the modular law, $C = X \oplus (C \cap X')$. Obviously, X is a supplement submodule in C. $\square$

Proposition 3.14. Let M be refinable. If $M = A \oplus B$ where A and B are c$\mathcal{G}$*s, then M is c$\mathcal{G}$*s.

Proof.
A and B are cws by Proposition 3.3. Furthermore, M is cws by Proposition 2.7. Thus, M is c$\mathcal{G}$*s because of Proposition 3.4. $\square$

Proposition 3.15. Let C be a cofinite submodule in M such that $C = S + A$, for some supplement submodule S and small submodule A of M. Then, M is c$\mathcal{G}$*s.

Proof.
Because β^* is an equivalence relation, $C\beta^*C$. Thus, $C\beta^*(S + A)$. By Corollary 2.17, $C\beta^*S$. $\square$

In addition,the converse of Proposition 3.15 under refinable conditions is as follows:

Proposition 3.16. If M is refinable and c$\mathcal{G}$*s, then $C = S + A$, for every cofinite submodule C of M, such that S is a supplement in M and A is small in M.

Proof.
From the hypothesis, there is a supplement S in M such that $C\beta^*S$. In this situation, $M = S + S'$ and $S' \cap S \ll S$, for some submodule S' of M. In other words, S' has a weak supplement S in M as $S' \cap S \ll M$ by Lemma 2.2 *ii*. According to Theorem 2.14, we can write as $M = C + S'$ and $S' \cap C$ is small in M. As M is refinable, for the direct summand submodule C' of M, $C' \subseteq C$, and $M = C' + S'$. From modularity, $C = C' + (S' \cap C)$. $\square$

Proposition 3.17. Let M be c$\mathcal{G}$*s module over a commutative V-ring and C be a cofinite submodule in M. Then, C is a direct summand in M.

Proof.
From the assumption, $C\beta^*S$, for supplement submodule S of M. Thus, $M = S + S'$ and $S' \cap S \ll S$, for some submodule S' of M, and based on Lemma 2.2, we have $S' \cap S \ll M$. Moreover, from Theorem 2.14, $M = S' + C$ and $S' \cap C \ll M$. Then, $S' \cap C \subseteq Rad(M) = 0$ by Theorem 2.20. Consequently, $S' \cap C = 0$ and thus $M = S' \oplus C$. $\square$

Corollary 3.18. If M is c$\mathcal{G}$*s over a commutative V-ring, then M is cs.

Theorem 3.19. If M is a torsion module and R is a Dedekind domain, then $M/Rad(M)$ is c$\mathcal{G}^*$s.

Proof.
From assumption, $M/Rad(M)$ is semisimple. Hence, $M/Rad(M)$ is $\mathcal{G}^*$s. Therefore, $M/Rad(M)$ is c$\mathcal{G}^*$s. □

4. Conclusion

In this study, we discussed some results of cofinitely Goldie*-supplemented modules using β^* relation. We proved that any factor module of cofinitely Goldie*-supplemented is cofinitely Goldie*-supplemented. In addition, the finite sum of cofinitely Goldie*-supplemented is cofinitely Goldie*-supplemented. For future studies, modules for which every submodule is cofinitely Goldie*-supplemented may be an interesting subject. Moreover, one can investigate the rings whose modules are cofinitely Goldie*-supplemented.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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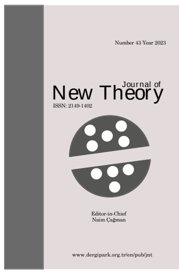

Chebyshev Collocation Method for the Fractional Fredholm Integro-Differential Equations

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Abstract — In this study, Chebyshev polynomials have been applied to construct an approximation method to attain the solutions of the linear fractional Fredholm integro-differential equations (IDEs). By this approximation method, the fractional IDE has been transformed into a linear algebraic equations system with the aid of the collocation points. In the method, the conformable fractional derivatives of the Chebyshev polynomials have been calculated in terms of the Chebyshev polynomials. Using the results of these calculations, the matrix relation for the conformable fractional derivatives of Chebyshev polynomials was attained for the first time in the literature. After that, the matrix forms have been replaced with the corresponding terms in the given fractional integro-differential equation, and the collocation points have been used to have a linear algebraic system. Furthermore, some numerical examples have been presented to demonstrate the preciseness of the method. It is inferable from these examples that the solutions have been obtained as the exact solutions or approximate solutions with minimum errors.

Keywords *Conformable fractional derivative, Chebyshev polynomials, numerical solutions*

Mathematics Subject Classification (2020) 26A33, 33C45

1. Introduction

The theory of fractional derivatives plays an impressive role in the field of the study of applied mathematics to analyze innumerable problems through the diverse areas of engineering and science, such as bioengineering, mathematical physics, astrophysics, hydrology, control theory, biophysics, statistical mechanics, thermodynamics, cosmology, and finance [1]. As much as the theory of fractional derivatives has drawn considerable attention among scientists, especially mathematicians, investigating the solution methods for the fractional linear and nonlinear IDEs has been the focus point continually in the last decades [2, 3]. The methods utilized to obtain the solutions of the Fredholm IDEs, fractional in the Caputo sense with the aid of the Chebyshev polynomials are given as the Chebyshev wavelet method of the second kind [4, 5] and least squares method [6, 7]. Besides, Chebyshev wavelet methods of the second kind [8-10] and the fourth kind [11] have been applied to attain the solutions of the fractional integro-differential equations of the Fredholm-Volterra type in the sense of the Caputo differentiation operator.

Moreover, investigating the exact and numerical solutions of the fractional integro-differential equations in the conformable sense is a fresh and strange field of investigation among applied mathematicians. Preliminarily, Bayram et al. [12] have applied the Sinc-collocation method, and Daşcıoğlu et al. [13] have used a collocation method based upon the Laguerre polynomials to attain the solutions of the linear fractional IDEs in the conformable sense. This method mentioned in [13] is an improvement of the method that had been used for

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the solutions of the linear Caputo fractional IDEs of the Volterra type [14] and Caputo fractional linear IDEs of the Fredholm type [15]. However, for the conformable fractional Fredholm IDEs, there has not been a method in the literature in the sense of Chebyshev polynomials. To this respect, in that study, a method predicated on the Chebyshev polynomials of the first kind is announced to obtain the numerical (in some cases exact) solutions of the linear conformable fractional integro-differential equation of the Fredholm type having the fractionality in the differential part as

$$\sum_{i=0}^{m} p_i(x)D^{\alpha_i}y(x) = g(x) + \lambda \int_{-1}^{1} K(x,t)y(t)dt, \quad -1 \leq x \leq 1 \tag{1}$$

with the initial conditions

$$y(0) = c_0 \tag{2}$$

where $l \in N$, $\lambda \in R$, $0 < \alpha_i \leq 1$, $K(x,t)$, p_i, and g are given (known) functions, $y(x)$ stands for the unknown function to be found, and $D^{\alpha_i}y(x)$ represents the fractional derivative in the conformable sense of the unknown function $y(x)$.

In the present paper, Section 2 provides the basic definitions and their properties. Section 3 constitutes the fundamental matrix relations for each term in the fractional integro-differential equation provided in Equation 1. Section 4 presents a well-functional collocation method based on the Chebyshev polynomials. Section 5 resolves some numerical examples and exhibits their results to affirm the preciseness and effectiveness of the introduced method. Finally, the last section discusses the need for further research.

2. Preliminaries

This section provides some basic notions to be needed in the following sections.

Definition 2.1. [16] The conformable fractional derivative of a function f of the α-th order is described as

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \quad \alpha \in (0,1)$$

where $f: [0, \infty) \to \mathbb{R}$. Here, if the function f is differentiable of the order α in the conformable sense in some open interval $(0, \alpha)$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then $\lim_{t \to 0^+} f^{(\alpha)}(t) = f^{(\alpha)}(0)$.

Since we have become familiar with the definition of the conformable fractional derivative, it is obvious that the notion of the conformable fractional derivative is the most analogous to the classical definition of the usual derivative. By the theorem below, we recognize the similarity between the conformable fractional derivative and the ordinary derivative:

Theorem 2.2. [16] Suppose that $\alpha \in (0,1]$ and the functions f and g are differentiable of the order α in the conformable sense at the point $t > 0$. Therefore, the following statements are satisfied.

i. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$

ii. $D^{\alpha}(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$

iii. $D^{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$

iv. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$

v. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$

vi. If f is differentiable, then $D^{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}$

Proposition 2.3. [16] The obtained expression for the fractional derivative of the power function x^k in the conformable sense for $k \in \{0,1,2,\cdots\}$

$$D^\alpha x^k = \begin{cases} 0, & k < 1 \\ k x^{k-\alpha}, & k \geq 1 \end{cases}$$

The following theorem introduces the chain rule for the conformable fractional derivative:

Theorem 2.4. [17] Assume $f, g : (0,\infty) \to \mathbb{R}$ be the differentiable functions of the order α in the conformable sense where $0 < \alpha \leq 1$. Suppose $h(t) = f\big(g(t)\big)$. Then, the composite function $h(t)$ is differentiable of the order α in the conformable sense and, for all t with $t \neq 0$ and $g(t) \neq 0$,

$$D^\alpha(h)(t) = D^\alpha(f)\big(g(t)\big).D^\alpha(g)(t).g(t)^{\alpha-1}$$

For $t = 0$, we can use the following limit

$$D^\alpha(h)(0) = \lim_{t\to 0} D^\alpha(f)\big(g(t)\big).D^\alpha(g)(t).g(t)^{\alpha-1}$$

The fundamental goal of this research is to introduce a useful approximation method that will provide an approximate solution (in some cases an exact solution) of the fractional Fredholm integro-differential equation in Problem 1 under the Condition 2 in the type

$$y(x) \cong y_N(x) = \sum_{i=0}^{N} a_i T_i(x) \tag{3}$$

where the upper limit of the sum $N \geq 1$ is any selected positive integer, the term T_i stand for the Chebyshev polynomials of the first kind of the order i, and the coefficients a_i are unknown and to be determined. Afterward, we provide the definition of the Chebyshev polynomials:

Definition 2.5. [18] The Chebyshev polynomial of degree n of the first kind is a polynomial in variable x is denoted by $T_n(x)$ and defined as

$$T_n(x) = \cos n\theta, \quad \cos\theta = x, \quad -1 \leq x \leq 1$$

Moreover, these well-known Chebyshev polynomials satisfy the following recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \in \{2,3,\cdots\}$$

together with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$ recursively generates all the polynomials $\{T_n(x)\}$ efficiently.

Furthermore, the following properties present the relation between the Chebyshev polynomials and the power function:

Proposition 2.6. [18] The Chebyshev polynomials are provided in terms of the powers of x as

$$T_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[(-1)^k \sum_{j=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j}\binom{j}{k} \right] x^{n-2k}$$

or

$$T_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}$$

where $\left\lfloor \frac{n}{2} \right\rfloor$ denotes the integer part of $\frac{n}{2}$.

Proposition 2.7. [18] The famous Chebyshev series in the Chebyshev polynomials of the first kind of the power function x^n has been stated as

$$x^n = 2^{1-n} \sum_{i=0'}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} T_{n-2i}(x), \quad n \in \{0,1,2,\cdots\}$$

where the dashed sigma stands for that the ith term in the sum is to be halved if n is even and $i = \frac{n}{2}$; in other words, the term in $T_0(x)$, if there is one, is to be halved.

3. Elementary Matrix Formulas

In this part of the paper, we transform Equation 1 by formulating the matrix forms of the unknown function and the fractional derivative of that function in a conformable sense. First, we can formulate the approximate solution in Equation 3 as the product of the Chebyshev matrix $\mathrm{T}(x)$ and the coefficient matrix A by

$$y_N(x) = \mathrm{T}(x)\mathrm{A} \tag{4}$$

where the matrices are as follows:

$$\mathrm{T}(x) = [T_0(x) \quad T_1(x) \quad \cdots \quad T_N(x)] \text{ and } \mathrm{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$$

For that purpose, we prove a theorem that states the relation between the conformable fractional derivative of the Chebyshev polynomials and the Chebyshev polynomials of the first kind:

Theorem 3.1. Suppose that $T_i(x)$ denotes the ith order Chebyshev polynomial of the first kind. Then, the fractional derivative of the Chebyshev polynomial $T_i(x)$ in the conformable sense in terms of the Chebyshev polynomials of the first kind are constructed as:

$$D^\alpha T_0(x) = 0 \tag{5}$$

and otherwise

$$D^\alpha T_n(x) = x^{1-\alpha} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{j}{k} \binom{n}{2j} (n-2k) 2^{2k-n+2} \sum_{i=0'}^{\lfloor \frac{n-2k-1}{2} \rfloor} \binom{n-2k-1}{i} T_{n-2k-2i-1}(x) \tag{6}$$

where $\lfloor n \rfloor$ denotes the integer part of n and the dashed sigma ($\Sigma_{\prime}$) stands for that the ith term in the sum is to be halved if $n-2k-1$ is even and $i = \frac{n-2k-1}{2}$.

Proof. We will originate with the expression of the Chebyshev polynomials in terms of the powers of x, and α-differentiate these polynomials as

$$D^\alpha T_n(x) = D^\alpha \left\{ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[(-1)^k \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{k} \right] x^{n-2k} \right\}$$

Since the conformable fractional derivative is linear, we have the equality

$$D^\alpha T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[(-1)^k \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{k} \right] D^\alpha \left(x^{n-2k} \right)$$

Utilizing the conformable fractional derivative of the power function x^k, for $k \in \{0,1,2,\cdots\}$,

$$D^\alpha x^k = \begin{cases} 0, & k < 1 \\ kx^{k-\alpha}, & k \geq 1 \end{cases}$$

we obtain $D^\alpha T_0(x) = 0$ and

$$D^\alpha T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{k} (n-2k) x^{n-2k-\alpha}, \quad n \in \{1,2,\cdots\}$$

At that point, we will take the term $x^{1-\alpha}$ out of the series since it is independent of the indices of the sums

$$D^\alpha T_n(x) = x^{1-\alpha} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{k} (n-2k) x^{n-2k-1}, \quad n \in \{1,2,\cdots\}$$

and utilizing the Chebyshev series of x^n mentioned with Property 2

$$x^n = 2^{1-n} \sum_{i=0'}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} T_{n-2i}(x), \quad n \in \{0,1,2,\cdots\}$$

where the dashed sigma stands for that the ith term in the sum is to be halved if n is even and $i = \frac{n}{2}$; in other words, the term in $T_0(x)$, if there is one, is to be halved; we get the statement of the formulas given by Equations 5 and 6, and the proof of Theorem 3.1 is accomplished. □

Theorem 3.2. Suppose that $\mathrm{T}(x)$ is a row matrix with $(N+1)$ columns and is called as Chebyshev matrix, and $D^\alpha \mathrm{T}(x)$ stands for the conformable fractional derivative of α-th order of the Chebyshev matrix $\mathrm{T}(x)$. Then, the matrix relation for the conformable fractional derivative of $\mathrm{T}(x)$ is attained as

$$D^\alpha \mathrm{T}(x) = 2x^{1-\alpha} \mathrm{T}(x) \mathrm{M} \tag{7}$$

where the $(N+1)$ dimensional square matrix M is characterized by odd N as

$$\mathrm{M} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and for even N

$$\mathrm{M} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

PROOF.

The explicit forms of the Chebyshev matrix $\mathrm{T}(x)$ and $D^{\alpha}\mathrm{T}(x)$ are

$$\mathrm{T}(x) = [T_0(x) \quad T_1(x) \quad \cdots \quad T_N(x)]$$

and

$$D^{\alpha}\mathrm{T}(x) = [D^{\alpha}T_0(x) \quad D^{\alpha}T_1(x) \quad \cdots \quad D^{\alpha}T_N(x)]$$

The statement of Theorem 3.1. is utilized to obtain the relation between the matrices above. Using Equations 5 and 6, the terms in $D^{\alpha}\mathrm{T}(x)$ can be expressed explicitly, for $n \in \{0,1,\dots,N\}$, as formulated below:

For $n = 0$,

$$D^{\alpha}T_0(x) = 0$$

For $n = 1$,

$$\begin{aligned} D^{\alpha}T_1(x) &= x^{1-\alpha}\sum_{k=0}^{\lfloor \frac{1}{2} \rfloor}\sum_{j=k}^{\lfloor \frac{1}{2} \rfloor}(-1)^k\binom{1}{2j}\binom{j}{k}(1-2k)2^{2k-1+2}\sum_{i=0'}^{\lfloor \frac{1-2k-1}{2} \rfloor}\binom{1-2k-1}{i}T_{1-2k-2i-1}(x) \\ &= x^{1-\alpha}T_0(x) \\ &= 2x^{1-\alpha}\left[\frac{1}{2}T_0(x)\right] \end{aligned}$$

For $n = 2$,

$$\begin{aligned} D^{\alpha}T_2(x) &= x^{1-\alpha}\sum_{k=0}^{\lfloor 1 \rfloor}\sum_{j=k}^{\lfloor 1 \rfloor}(-1)^k\binom{2}{2j}\binom{j}{k}(2-2k)2^{2k-2+2}\sum_{i=0'}^{\lfloor \frac{2-2k-1}{2} \rfloor}\binom{2-2k-1}{i}T_{2-2k-2i-1}(x) \\ &= 4x^{1-\alpha}T_1(x) \\ &= 2x^{1-\alpha}[2T_1(x)] \end{aligned}$$

For $j = N$ and odd N,

$$\begin{aligned} D^{\alpha}T_N(x) &= x^{1-\alpha}\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor}\sum_{j=k}^{\lfloor \frac{N}{2} \rfloor}(-1)^k\binom{N}{2j}\binom{j}{k}(N-2k)2^{2k-N+2}\sum_{i=0'}^{\lfloor \frac{N-2k-1}{2} \rfloor}\binom{N-2k-1}{i}T_{N-2k-2i-1}(x) \\ &= 2x^{1-\alpha}\left[\frac{N}{2}T_0(x) + NT_2(x)+\ldots+NT_{N-1}(x)\right] \end{aligned}$$

and for even N

$$\begin{aligned} D^{\alpha}T_N(x) &= x^{1-\alpha}\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor}\sum_{j=k}^{\lfloor \frac{N}{2} \rfloor}(-1)^k\binom{N}{2j}\binom{j}{k}(N-2k)2^{2k-N+2}\sum_{i=0'}^{\lfloor \frac{N-2k-1}{2} \rfloor}\binom{N-2k-1}{i}T_{N-2k-2i-1}(x) \\ &= 2x^{1-\alpha}[NT_1(x) + NT_3(x)+\ldots+NT_N(x)] \end{aligned}$$

It can be observed that the relation between the fractional derivative of the Chebyshev matrix in the conformable sense $D^{\alpha}\mathrm{T}(x)$ and the Chebyshev matrix $\mathrm{T}(x)$ is in the form as stated in Equation 7. This proves the theorem. □

After that, by applying Equations 4 and 7, the left-hand side of Equation 1 could be expressed as

$$D^{\alpha}y(x) \cong D^{\alpha}\mathrm{T}(x)\mathrm{A} = x^{1-\alpha}\mathrm{T}(x)\mathrm{MA} \tag{8}$$

It is remarked that this matrix is the same as the matrix provided by Sezer et al. [19] and Akyüz [20] for the usual first-order derivative. Thus, it is obvious that there is a correlation between the methods for the conformable fractional derivative and the usual derivative.

Finally, the corresponding matrix relation of the conditions in Equation 2 is formulated as

$$y(0) = \mathrm{T}(0)\mathrm{A} = c_0 \tag{9}$$

At this stage, the condition matrix $\mathrm{T}(0)$ is referred to as U where the matrix U is a row matrix with $(N+1)$ columns. Thus, Equation 9 transforms into $\mathrm{UA} = c_0$.

4. Solution Method

In this section, we maintain the approximate solution method, which can be specified as a collocation method since we use the collocation points at the end to solve the matrix equation. In other words, we determine the unknown coefficients a_i in Equation 3 to attain the solution of Equations 1 and 2 by a collocation method.

Before all, we interchange the formulated matrix forms given with Equations 4 and 8 into Equation 1, and thus we attain the matrix equation of the fractional integro-differential equation

$$\sum_{i=0}^{l} p_i(x)x^{1-\alpha_i}2\mathrm{T}(x)\mathrm{MA} = g(x) + \lambda\int_{-1}^{1} K(x,t)\mathrm{T}(t)\mathrm{A}dt \tag{10}$$

Secondly, we substitute the chosen collocation points $x_s > 0$, for $s \in \{0,1,\dots,N\}$, into the matrix Equation 10, we get a linear system of the $N+1$ equations

$$\left\{\sum_{i=0}^{l} p_i(x_s){x_s}^{1-\alpha_i}2\mathrm{T}(x_s)\mathrm{MA} - \lambda f(x_s)\right\}\mathrm{A} = g(x_s) \tag{11}$$

where $f(x_s) = \int_{-1}^{1} K(x_s,t)\mathrm{T}(t)dt$. This linear system can be expressed in compact forms:

$$\left\{\sum_{i=0}^{l} 2\mathrm{P}_i\mathrm{X}_{\alpha_i}\mathrm{LTM} - \lambda\mathrm{F}\right\}\mathrm{A} = \mathrm{G} \tag{12}$$

where

$$\mathrm{X}_{\alpha_i} = \begin{bmatrix} x_0^{1-\alpha_i} & 0 & \cdots & 0 \\ 0 & x_1^{1-\alpha_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_N^{1-\alpha_i} \end{bmatrix}, \quad \mathrm{T} = \begin{bmatrix} \mathrm{T}(x_0) \\ \mathrm{T}(x_1) \\ \vdots \\ \mathrm{T}(x_N) \end{bmatrix}, \quad \mathrm{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

$$\mathrm{P}_i = \begin{bmatrix} p_i(x_0) & 0 & \cdots & 0 \\ 0 & p_i(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_i(x_N) \end{bmatrix}, \quad \text{and} \quad \mathrm{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

After that, when we denote the formulation in parenthesis of Equation 12 by W, the main matrix equation for Equation 1 is abbreviated into the equation $\mathrm{WA} = \mathrm{G}$ representing a system of $N+1$ linear algebraic equations with $N+1$ undetermined Chebyshev coefficients a_i's, for $i \in \{0,1,\dots,N\}$.

Eventually, we solve the obtained linear algebraic system to calculate the unknown coefficients. For that purpose, there are several ways to solve this system, but we primarily use it to replace or to stack up the n rows of the augmented matrix $[\mathrm{W};\mathrm{G}]$ with the rows of the augmented matrix $[\mathrm{U};c_0]$. We choose the best way to get the most accurate solutions for each problem. Therefore, since the unknown Chebyshev coefficients are discovered by resolving this system, we end up with the solution of Equation 1 under Condition 2.

5. Numerical Examples

In that part of the paper, we use the presented method in the previous section for two different examples. The collocation points that are used to transform the equations have been formalized as

$$x_s = \frac{\left[1 - \cos\left(\frac{(s+1)\pi}{N+1}\right)\right]}{2}, \quad s \in \{0,1,\dots N\}$$

for these two examples. All the numerical calculations have been executed with the program Mathcad 15.

Example 5.1. The fractional Fredholm IDE in the form of Equation 1

$$D^{\frac{1}{2}}y(x) = y(x) + 2x^{1.5} - x^2 - \frac{2}{3} + \int_{-1}^{1} y(t)dt$$

subject to initial condition $y(0) = 0$ in the form of Equation 2.

It can easily be confirmed that the exact solution to the above problem is the polynomial solution of degree two, $y(x) = x^2$. Implementing the methodology explained in Section 4, the expected fundamental matrix equation of the problem and its conditions can be presented as $\mathrm{P}_0 = \mathrm{I}$, I is the identity matrix, $\lambda = 1$,

$$\left\{2\mathrm{X}_{\frac{1}{2}}\mathrm{TM} - \mathrm{T} - \mathrm{F}\right\}\mathrm{A} = \mathrm{G}, \text{ and } \mathrm{UA} = 0$$

When we select $N = 2$, the formula gives us the points $x_0 = 0.25$, $x_1 = 0.75$, and $x_2 = 1$ as the collocation points. Then, the matrices mentioned above are

$$\mathrm{X}_{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{T} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{7}{8} \\ 1 & \frac{3}{4} & \frac{1}{8} \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} 2 & 0 & -\frac{2}{3} \\ 2 & 0 & -\frac{2}{3} \\ 2 & 0 & -\frac{2}{3} \end{bmatrix},$$

$$\mathrm{G} = \begin{bmatrix} -\frac{23}{48} \\ \frac{3\sqrt{3}}{4} - \frac{59}{48} \\ \frac{1}{3} \end{bmatrix}, \quad \text{and} \quad \mathrm{U} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

As a result of the solution of the above system, the unknown coefficients in Equation 3, for $N = 2$, can be calculated as $a_0 = \frac{1}{2}$, $a_1 = 0$, and $a_2 = \frac{1}{2}$. In the final step, we substitute these coefficients into approximate Equation 3. Then, we obtain the exact solution.

Example 5.2. The Fredholm fractional IDE in the form of Equation 1

$$D^{\frac{1}{3}}y(x) = x^{\frac{2}{3}}y(x) - 2e^x + \int_{-1}^{1} e^{x-t}y(t)dt\,, 0 \leq x \leq 1$$

subject to $y(0) = 1$ having the exponential function e^x as the exact solution.

The exact solution could not be attained by the introduced method in Section 4 since this problem does not have a polynomial solution. Therefore, we attain the approximate solutions with some insignificant errors. The absolute maximum errors between the approximate solution obtained by the proposed method and the exponential function e^x, the exact solution to the given problem is stated in Table 1. In Table 1, the maximum absolute errors are calculated by interchanging the row in the last place of the evaluated augmented matrix $[\mathrm{W};\mathrm{G}]$ with the augmented matrix $[\mathrm{U};1]$, for the values $N \in \{2,4,6,8,10\}$ and the values $N \in \{14,16\}$; by stacking up the rows of the computed augmented matrices for this problem.

Table 1. The maximum errors of Example 2 for different N values

$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 10$	$N = 14$	$N = 16$
0.36	1.9×10^{-2}	6.0×10^{-5}	1.2×10^{-7}	1.3×10^{-10}	6.4×10^{-14}	5.4×10^{-14}

6. Conclusion

This paper uses Chebyshev polynomials to construct an approximation method to attain the solutions of the linear fractional Fredholm integro-differential equations (IDEs). By this approximation method, the fractional IDE has been transformed into a linear algebraic equations system with the aid of the collocation points. There are numerous methods for obtaining the solutions of the fractional IDEs in the Caputo differential operator sense. However, investigating the solutions of the fractional IDEs in the conformable differential operator sense is a new field of study among mathematicians. Therefore, the relation for the matrix of the conformable fractional derivative of the Chebyshev polynomials is attained for the first time in fractional calculus literature. The fractional IDE has been turned into an algebraic equations system using suitable collocation points and the obtained matrix relations. The proposed approximation method's simplicity and efficiency have been strengthened by the results of the numerical examples provided in the previous section. For future research, the fractional derivatives of the Chebyshev polynomials and the related matrix relations can be obtained for the different types of fractional derivatives, such as the Caputo fractional derivative and fractional beta derivative.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

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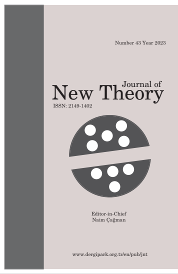

Construction of Analytical Solutions to the Conformable New (3+1)-Dimensional Shallow Water Wave Equation

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Abstract — This study investigates the new (3+1)-dimensional shallow water wave equation. To do so, the definitions of conformable derivatives and their descriptions are given. Using the Riccati equation and modified Kudryashov methods, exact solutions to this problem are discovered. The gathered data's contour plot surfaces and related 3D and 2D surfaces emphasize the result's physical nature. To monitor the problem's physical activity, exact and complete solutions are necessary. The results demonstrate the potential applicability of additional nonlinear physical models from mathematical physics and under-investigation in real-world settings. In order to solve fractional differential equations, it may prove helpful to use these methods in various situations.

Keywords *Riccati equation method, modified Kudryashov method, new* $(3+1)$*-dimensional shallow water wave (SWW) equation, conformable derivative*

Mathematics Subject Classification (2020) 35R11, 35C07

1. Introduction

Determining a created model's analytical solution enables the physical phenomena expressed by that model to be understood and interpreted. As a result of modeling problems encountered in applied science fields, nonlinear fractional differential equations are usually found. The fractional differential equation has applications in biology, chemistry, medicine, pharmacy, psychology, economics, statistics, and natural sciences, especially engineering and physics. Examples of these application areas are the movements of fluids, earthquake vibration movements, shallow water waves, and propagation movements of acoustic sound vibrations. Research on this and other topics continues rapidly, and thus new fields of applications are also used.

Hence, it is crucial to solve nonlinear fractional differential equations analytically. Thanks to the developed analytical methods, exact solutions of fractional differential equations have been found. Thereby, it has become easier to understand and interpret physical phenomena.

Many different types of fractional derivative operators have been described in the literature. Some of them are Caputo derivative [1], Riemann-Liouville derivative [2], Caputo-Fabrizio derivative [3], modified Riemann Liouville derivative [4], Atangana-Baleanu derivative [5], and conformable derivative [6]. With the help of these derivative operators, various techniques have been developed that provide

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exact solutions of partial differential equations with nonlinear fractional derivatives. Moreover, these equations are used with the help of fractional derivatives with various analytical methods to get better results. Among these techniques are Generalized (G'/G)-expansion Method [7], Riccati Equation Method [8,9], Exp-function Method [10], Generalized Riccati Equation Mapping Method [11], Jacobi Elliptic Function Method [12], Improved F-expansion Method [13], tanh-sech Method [14], Hirota Bilinear Method [15], Inverse Scattering Method [16], and Extended tanh Method [17], etc.

This study presents analytical solutions to the conformable form of the following new (3+1)-dimensional shallow water wave equation (SWW) [18]

$$\alpha_1((u_x u_t)_x + u_{xxxt}) + \alpha_2((u_x u_y)_x + u_{xxxy}) + \alpha_3 u_{yt} + \alpha_4 u_{xx} + \alpha_5 u_{xy} + \alpha_6 u_{xt} + \alpha_7 u_{yy} + \alpha_8 u_{zz} = 0 \quad (1)$$

We conducted a study by applying analytical solution methods using the conformable derivative of this new equation.

The following is the layout of the paper. The preliminaries appear in Section 2. The Riccati Equation and Modified Kudryashov Methods are presented in Section 3. In Section 4, the solutions to the considered equation are provided. Finally, the paper includes a discussion in Section 5.

2. Preliminaries

This section provides a basic definition of the conformable derivative and some of its properties.

Definition 2.1. Let $\gamma : [0,\infty) \to \mathbb{R}$ be a function, $t > 0$, and $\omega \in (0,1)$. Then, ω^{th} order conformable fractional derivative of the γ function is defined by

$$\mathscr{D}_t^{\omega}(\gamma)(t) = \lim_{\chi \to 0} \frac{\gamma(t + \chi t^{1-\omega}) - \gamma(t)}{\chi}$$

Lemma 2.2. [19–21] For $\omega \in (0,1)$ and $t > 0$, let γ_1 and γ_2 be ω^{th} order conformable fractional differentiable functions. Then,

i. $\mathscr{D}_t^{\omega}(t^{\Omega_1}) = \Omega_1 t^{\Omega_1 - \omega}$, $\Omega_1 \in \mathbb{R}$

ii. $\mathscr{D}_t^{\omega}(\Omega_1\gamma_1 + \Omega_2\gamma_2) = \Omega_1\mathscr{D}_t^{\omega}(\gamma_1) + \Omega_2\mathscr{D}_t^{\omega}(\gamma_2)$, $\Omega_1, \Omega_2 \in \mathbb{R}$

iii. $\mathscr{D}_t^{\omega}(\frac{\gamma_1}{\gamma_2}) = \frac{\gamma_2.\mathscr{D}_t^{\omega}(\gamma_1) - \gamma_1 \mathscr{T}_t^{\omega}(\gamma_2)}{\gamma_2^2}$

iv. $\mathscr{D}_t^{\omega}(\gamma_1.\gamma_2) = \gamma_1.\mathscr{D}_t^{\omega}(\gamma_2) + \gamma_2.\mathscr{D}_t^{\omega}(\gamma_1)$

v. If w is differentiable, then $\mathscr{D}_t^{\omega}(\gamma_1)(t) = \frac{t^{1-\omega} d\gamma_1(t)}{dt}$

vi. $\mathscr{D}_t^{\omega}(\mathscr{K}) = 0$ such that $\mathscr{K} \in \mathbb{R}$

3. The Procedures of the Analytical Methods

In this section, the procedures of the Riccati equation and modified Kudryashov methods are presented. A general form of a partial differential equation (PDE) is as follows:

$$\mathscr{B}(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \cdots) = 0 \quad (2)$$

If a special wave transform is defined as, for $h \neq 0$,

$$u(x, \cdots, t) = u(\xi), \quad \xi = kx + \cdots + h\frac{t^{\omega}}{\omega} \quad (3)$$

then a nonlinear ordinary differential equation (ODE) of the form is obtained:

$$\mathscr{F}(u(\xi), u'(\xi), u''(\xi), \cdots) = 0 \quad (4)$$

3.1. Riccati Equation Method

The technique is based on the following equation:

$$\varphi'(\xi) = \sigma + \varphi(\xi)^2 \tag{5}$$

Suppose that the following is the general form of a nonlinear conformable PDE

$$\mathscr{Q}(f, \mathscr{D}_t^{\omega}, \mathscr{D}_x f, \mathscr{D}_y f, \mathscr{D}_x^2 f, \mathscr{D}_y^2 f, \cdots) = 0$$

In this case, the derivative operator denoted by $\mathscr{D}_t^{\omega}$ appears in any order. Equation 3 provides the definition of conformable transformations. Making use of the chain rule, $k, \cdots ., h$ constants represent arbitrary values that will be determined later. Equation 2 is transformed into a nonlinear ODE as shown below. Thus, Equation 4 should have the following solution:

$$u(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi),\ a_N \neq 0 \tag{6}$$

Then, N is calculated by using the balancing rule in Equation 4, where $\varphi(\xi)$ is solution to the Riccati equation. A list of solutions satisfying Equation 5 is provided below.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi), & \sigma < 0 \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi), & \sigma < 0 \\ \sqrt{\sigma}\tan(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\frac{1}{\xi+\theta},\ \theta \text{ is a constant}, & \sigma = 0 \end{cases} \tag{7}$$

By inserting all of the values found in the terms in Equation 7, we can determine the exact solutions of Equation 2. A polynomial of $\varphi(\xi)$ is produced by combining all of the results. A system of nonlinear algebraic equations for $a_i, \cdots, h, (i \in \{0, 1, \cdots, N\})$ is produced when the coefficients of $\varphi(\xi)$ disappear. These nonlinear algebraic equations solutions are calculated, and the results are used to determine the values of $a_i, \cdots, h, (i \in \{0, 1, \cdots, N\})$.

3.2. Modified Kudryashov Method

The general form of the solution to Equation 4 is as follows:

$$u(\xi) = \sum_{r=0}^{N} B_r \varphi^r(\xi), \quad B_N \neq 0 \tag{8}$$

$\varphi(\xi)$ meets the ODE in Equation 8.

$$\varphi'(\xi) = \log(\vartheta)\varphi(\xi)(\varphi(\xi) - 1) \tag{9}$$

The solution to this equation is given by

$$\varphi(\xi) = \frac{1}{d\vartheta^{\xi} + 1}, \quad \vartheta > 0, \quad \vartheta \neq 0, \quad d \text{ is a constant} \tag{10}$$

The homogeneous balancing concept is used to compute N at Equation 4. We may compute a polynomial of $\varphi^r(\xi)$ by putting Equation 8 into Equation 4 without ignoring Equation 9. Setting all of the coefficients of $\varphi^r(\xi)$ to zero results in a series of algebraic equations in $k, \cdots, h$ and B_r [22]. Finally, the soliton-type solutions of the given model are attained.

4. Solutions to the New (3+1)-Dimensional SWW Equation

Take into account below equation, the conformable form of Equation 1:

$$\alpha_1((u_x\mathscr{D}_t^\omega u)_x+\mathscr{D}_t^\omega u_{xxx})+\alpha_2((u_xu_y)_x+u_{xxxy})+\alpha_3\mathscr{D}_t^\omega u_y+\alpha_4u_{xx}+\alpha_5u_{xy}+\alpha_6\mathscr{D}_t^\omega u_x+\alpha_7u_{yy}+\alpha_8u_{zz}=0$$

Having the transformation $u(x,y,z,t)=u(\xi)$, for $\xi=kx+wy+sz+h\frac{t^\omega}{\omega}$, and integrating yields

$$0=\alpha_1hk^3u^{(3)}(\xi)+\alpha_1hk^2u'(\xi)^2+\alpha_6hku'(\xi)+\alpha_3hwu'(\xi)+\alpha_2k^3wu^{(3)}(\xi)+\alpha_4k^2u'(\xi)$$
$$+\alpha_2k^2wu'(\xi)^2+\alpha_5kwu'(\xi)+\alpha_8s^2u'(\xi)+\alpha_7w^2u'(\xi)$$

Balancing $u^{(3)}=N+3$, $(u')^2=2(N+1)$ gives $N=1$. If it is replaced in Equation 6 and Equation 8, the outcomes are as follows:

4.1. Riccati Equation-based Analytical Solutions

For $N=1$, the series of sums that result from substituting Equation 6 appears,

$$u=a_0+a_1\varphi,\quad a_1\neq 0$$

In combination with Equation 5, the following algebraic system is created

$$0=2a_1\alpha_1hk^3\sigma^2+a_1^2\alpha_1hk^2\sigma^2+a_1\alpha_6hk\sigma+a_1\alpha_3h\sigma w+2a_1\alpha_2k^3\sigma^2w+a_1\alpha_4k^2\sigma+a_1^2\alpha_2k^2\sigma^2w$$
$$+a_1\alpha_5k\sigma w+a_1\alpha_8s^2\sigma+a_1\alpha_7\sigma w^2$$
$$0=8a_1\alpha_1hk^3\sigma+2a_1^2\alpha_1hk^2\sigma+a_1\alpha_6hk+a_1\alpha_3hw+8a_1\alpha_2k^3\sigma w+a_1\alpha_4k^2+2a_1^2\alpha_2k^2\sigma w+a_1\alpha_5kw$$
$$+a_1\alpha_8s^2+a_1\alpha_7w^2$$
$$0=6a_1\alpha_1hk^3+a_1^2\alpha_1hk^2+6a_1\alpha_2k^3w+a_1^2\alpha_2k^2w$$

Here, we obtain one case and one set of solutions for a_0, a_1, and h.

Case 1.

$$a_1=-6k,\quad \text{and}\quad h=\frac{-4\alpha_2k^3\sigma w+\alpha_4k^2+\alpha_5kw+\alpha_8s^2+\alpha_7w^2}{4\alpha_1k^3\sigma-\alpha_6k-\alpha_3w}$$

Set 1.

For $\sigma<0$,

$$u_1=a_0+6k\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(\frac{t^\omega\left(-4\alpha_2k^3\sigma w+\alpha_4k^2+\alpha_5kw+\alpha_8s^2+\alpha_7w^2\right)}{\omega\left(4\alpha_1k^3\sigma-\alpha_6k-\alpha_3w\right)}+kx+sz+wy\right)\right) \quad (11)$$

$$u_2=a_0+6k\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(\frac{t^\omega\left(-4\alpha_2k^3\sigma w+\alpha_4k^2+\alpha_5kw+\alpha_8s^2+\alpha_7w^2\right)}{\omega\left(4\alpha_1k^3\sigma-\alpha_6k-\alpha_3w\right)}+kx+sz+wy\right)\right)$$

for $\sigma>0$,

$$u_3=a_0-6k\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(\frac{t^\omega\left(-4\alpha_2k^3\sigma w+\alpha_4k^2+\alpha_5kw+\alpha_8s^2+\alpha_7w^2\right)}{\omega\left(4\alpha_1k^3\sigma-\alpha_6k-\alpha_3w\right)}+kx+sz+wy\right)\right)$$

$$u_4=a_0+6k\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(\frac{t^\omega\left(-4\alpha_2k^3\sigma w+\alpha_4k^2+\alpha_5kw+\alpha_8s^2+\alpha_7w^2\right)}{\omega\left(4\alpha_1k^3\sigma-\alpha_6k-\alpha_3w\right)}+kx+sz+wy\right)\right) \quad (12)$$

and for $\sigma=0$,

$$u_5=a_0+\frac{6k}{\theta+\frac{t^\omega(\alpha_4k^2+\alpha_5kw+\alpha_8s^2+\alpha_7w^2)}{\omega(\alpha_3(-w)-\alpha_6k)}+kx+sz+wy}$$

4.2. The Modified Kudryashov Method-based Analytical Solutions

For $N = 1$, the series of sums that result from substituting Equation 8 appears,

$$u = B_0 + B_1\varphi(\xi), \quad B_1 \neq 0 \tag{13}$$

The below system is attained with combining Equation 9,

$$0 = 7\alpha_1 B_1 hk^3 \log^3(a) + \alpha_1 B_1^2 hk^2 \log^2(a) + \alpha_6 B_1 hk \log(a) + \alpha_3 B_1 hw \log(a) + 7\alpha_2 B_1 k^3 w \log^3(a)$$
$$+\alpha_4 B_1 k^2 \log(a) + \alpha_2 B_1^2 k^2 w \log^2(a) + \alpha_5 B_1 kw \log(a) + \alpha_8 B_1 s^2 \log(a) + \alpha_7 B_1 w^2 \log(a)$$
$$0 = \alpha_1 B_1(-h)k^3 \log^3(a) - \alpha_6 B_1 hk \log(a) - \alpha_3 B_1 hw \log(a) - \alpha_2 B_1 k^3 w \log^3(a) - \alpha_4 B_1 k^2 \log(a)$$
$$-\alpha_5 B_1 kw \log(a) - \alpha_8 B_1 s^2 \log(a) - \alpha_7 B_1 w^2 \log(a)$$
$$0 = -12\alpha_1 B_1 hk^3 \log^3(a) - 2\alpha_1 B_1^2 hk^2 \log^2(a) - 12\alpha_2 B_1 k^3 w \log^3(a) - 2\alpha_2 B_1^2 k^2 w \log^2(a)$$
$$0 = 6\alpha_1 B_1 hk^3 \log^3(a) + \alpha_1 B_1^2 hk^2 \log^2(a) + 6\alpha_2 B_1 k^3 w \log^3(a) + \alpha_2 B_1^2 k^2 w \log^2(a)$$

Here, we obtain one case and one set of solutions for B_0, B_1 and h.

Case 2.

$$B_1 = -6k\log(a) \quad \text{and} \quad h = -\frac{\alpha_2 k^3 w \log^2(a) + \alpha_4 k^2 + \alpha_5 kw + \alpha_8 s^2 + \alpha_7 w^2}{\alpha_1 k^3 \log^2(a) + \alpha_6 k + \alpha_3 w}$$

Using Equation 10 and these values with Equation 13, the solutions are obtained as follows:

Set 2.

$$u_5 = B_0 - \frac{6k\log(a)}{da^{-\frac{t^{\omega}\left(\alpha_2 k^3 w \log^2(a) + \alpha_4 k^2 + \alpha_5 kw + \alpha_8 s^2 + \alpha_7 w^2\right)}{\omega\left(\alpha_1 k^3 \log^2(a) + \alpha_6 k + \alpha_3 w\right)} + kx + sz + wy} + 1} \tag{14}$$

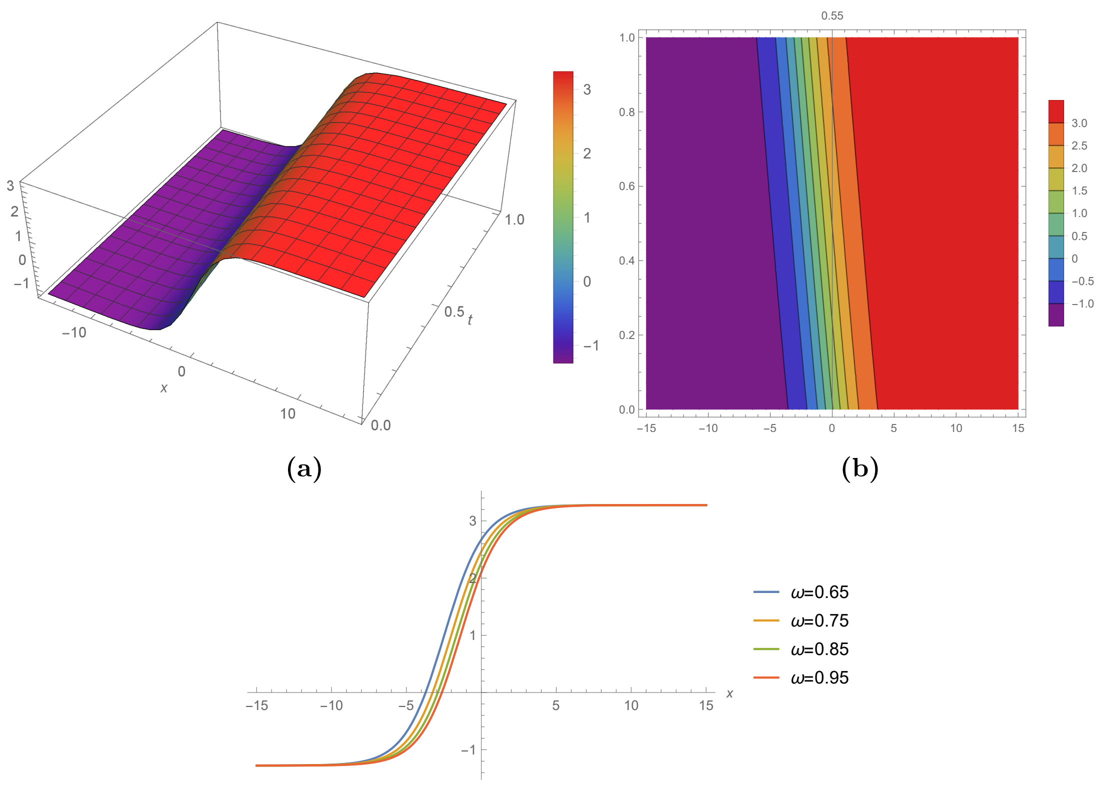

Fig. 1. The plot of Equation 11 for (a) 3D, (b) contour, and (c) 2D plot

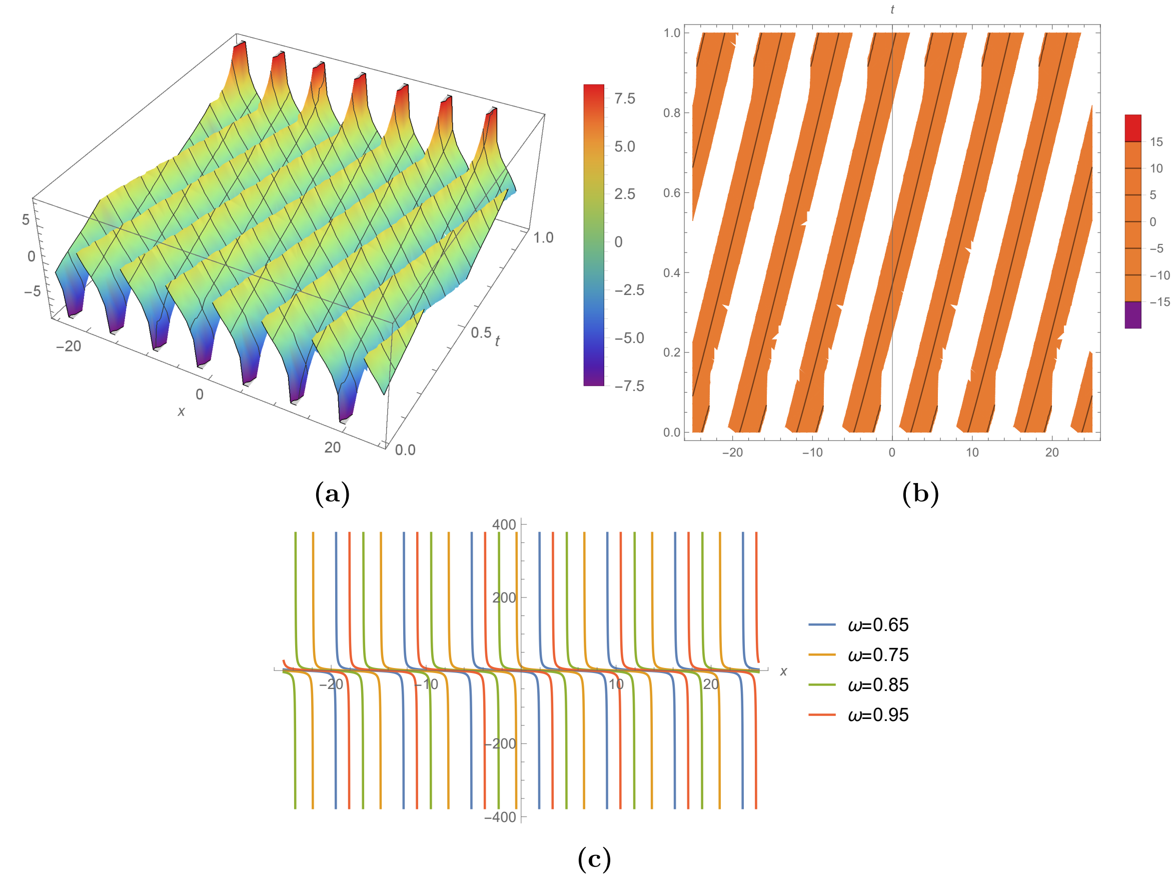

Fig. 2. The plot of Equation 12 for (a) 3D, (b) contour, and (c) 2D plot

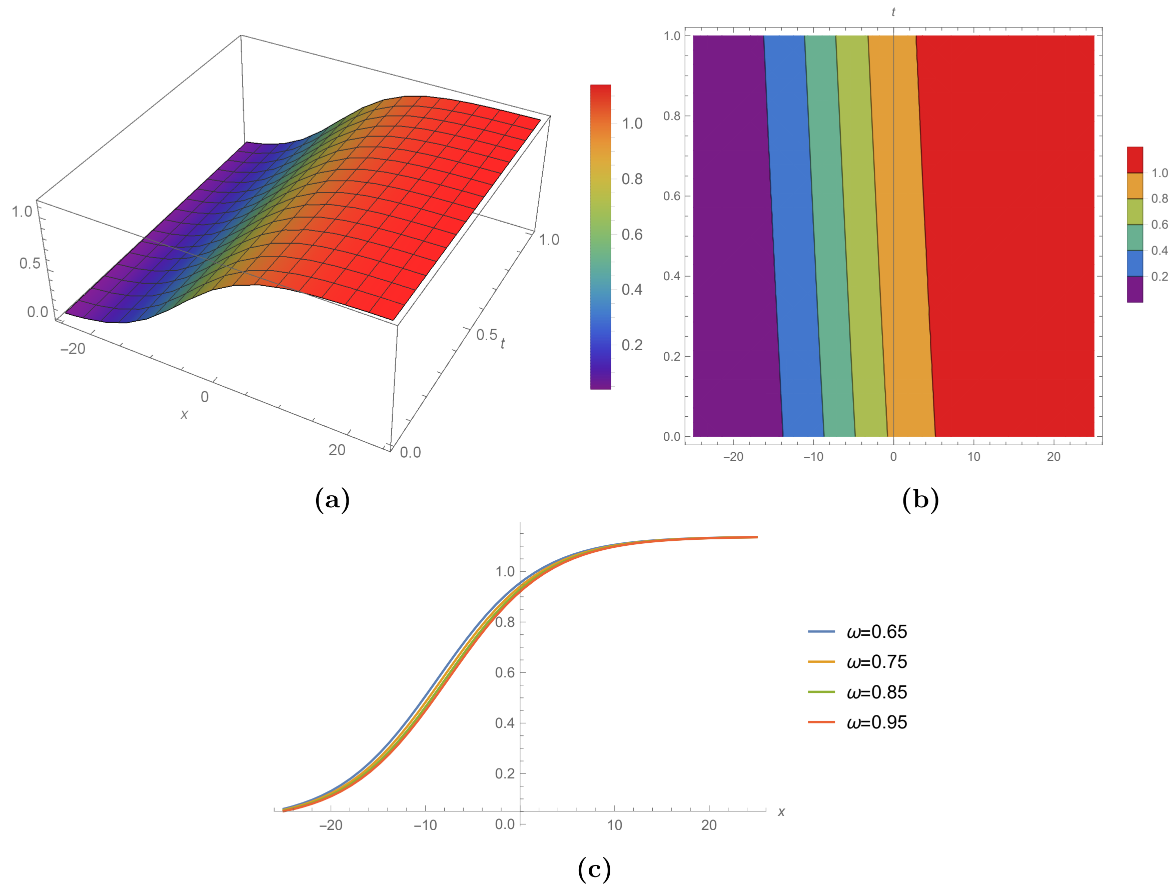

Fig. 3. The plot of Equation 14 for (a) 3D, (b) contour, and (c) 2D plot

For Figures 1-3, the following numerical values are employed:

i. In Figure 1, for (a) and (b), $a_0 = 1, k = -0.6, w = 0.1, s = 0.05, y = 0.1, z = 0.5, \alpha_1 = 0.4, \alpha_2 = -0.2, \alpha_3 = 0.6, \alpha_4 = -0.4, \alpha_5 = 0.99, \alpha_6 = 0.1, \alpha_7 = -0.7, \alpha_8 = 0.8, \sigma = -0.4$, and $\omega = 0.95$ and for (c), $t = 0.55$.

ii. In Figure 2, for (a) and (b), $a_0 = 0.1, k = 0.22, w = 0.5, s = 0.5, y = 0.1, z = 0.5, \alpha_1 = 0.95, \alpha_2 = 0.85, \alpha_3 = 0.65, \alpha_4 = 0.4, \alpha_5 = 0.99, \alpha_6 = 0.1, \alpha_7 = 0.9, \alpha_8 = 0.8, \sigma = 4$, and $\omega = 0.95$ and for (c), $t = 0.55$.

iii. In Figure 3, for (a) and (b), $B_0 = 0.01, a = 0.625, k = 0.4, w = -0.02, s = 0.55, y = 0.5, z = -0.09, d = 0.36, \alpha_1 = 0.25, \alpha_2 = 0.55, \alpha_3 = 0.45, \alpha_4 = -0.65, \alpha_5 = 0.35, \alpha_6 = 0.75, \alpha_7 = 0.15, \alpha_8 = -0.55$, and $\omega = 0.95$ and for (c), $t = 0.95$.

The presented methods have several novel solutions revealed by the graphical representations and may be applied to other kinds of equations.

5. Conclusion

This work investigated the new $(3 + 1)$-dimensional shallow water wave equation with conformable derivative's soliton characteristics using the Riccati equation and modified Kudryashov methods. Then, to visualize some of the solutions with the proper values, 3D, contour, and 2D graphics are presented. Graphical representations and analytical solutions have been provided to show these techniques' accuracy. Furthermore, the physical characteristics of these solutions are distinctive and significant, and they have been researched in the literature. Consequently, future studies may use the proposed approaches to handle and solve various additional fractional differential equations.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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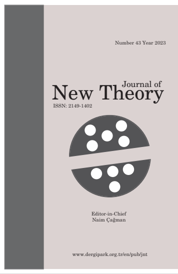

Exact Solutions of Some Basic Cardiovascular Models by Kashuri Fundo Transform

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Abstract — Differential equations refer to the mathematical modeling of phenomena in various applied fields, such as engineering, physics, chemistry, astronomy, biology, psychology, finance, and economics. The solutions of these models can be more complicated than those of algebraic equations. Therefore, it is convenient to use integral transformations to attain the solutions of these models. In this study, we find exact solutions to two cardiovascular models through an integral transformation, namely the Kashuri Fundo transform. It can be observed that the considered transform is a practical, reliable, and easy-to-use method for obtaining solutions to differential equations.

Keywords *Kashuri Fundo transform, inverse Kashuri Fundo transform, cardiovascular models*

Mathematics Subject Classification (2020) 34A30, 44A15

1. Introduction

Ordinary differential equations are essential in describing the rates of change of quantities in diverse scientific disciplines, including physics, chemistry, biology, engineering, and economics. These equations provide a concise mathematical framework for modeling dynamic systems, where variables depend on a single independent variable, such as time. By formulating ordinary differential equations, scientists can represent complex real-world problems as mathematical equations, facilitating their analysis and prediction. Ordinary differential equations enable researchers to investigate the behavior of systems over time, making them invaluable in studying dynamic processes and phenomena. For this reason, differential equations are used to analyze many problems in many fields of applied sciences [1, 2].

Ordinary differential equations play a pivotal role in biology, providing a powerful mathematical tool for understanding and analyzing complex biological systems [3]. These equations are of paramount importance in biology due to the dynamic nature of biological processes, where variables such as concentrations, populations, and reaction rates change over time. Ordinary differential equations allow researchers to investigate the dynamics of biological systems, predict their behavior under different conditions, and gain insights into fundamental biological principles. They are instrumental in studying population dynamics, the spread of diseases, gene regulation, cellular signaling, and many other biological phenomena [4]. Ordinary differential equations provide a powerful mathematical framework for unraveling the intricacies of biological systems, allowing scientists to deepen their understanding of life processes and contribute to advancements in biological research and healthcare.

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Finding a field of application in many fields has made it important to reach the solutions of this type of equations. The solution of differential equations can be more complicated than that of algebraic equations. Therefore, researchers have sought ways to convert differential equations into algebraic equations. One of the solution methods that emerged as a result of this search is integral transforms that convert differential equations into algebraic equations. These transforms give very effective results in solving a wide variety of problems in many different fields. Its application to various problems has led to the diversification of integral transforms [2]. In this study, we consider a type of integral transform, namely Kashuri Fundo transform [5].

The Kashuri Fundo transform is a powerful mathematical tool that has gained significant attention in the field of differential equations. The Kashuri Fundo integral transform offers a systematic approach to transform differential equations into algebraic equations, making it easier to solve them and obtain analytical or numerical solutions [5]. By employing this transform, researchers can simplify the mathematical representation of differential equations, which often leads to more tractable equations and allows for the application of established techniques for solving algebraic equations. The importance of the Kashuri Fundo transform lies in its potential to overcome the challenges associated with solving differential equations analytically or numerically, offering a promising alternative approach for obtaining solutions to a wide range of differential equations encountered in various scientific and engineering fields. Its utilization can enhance the efficiency and accuracy of solving differential equations, ultimately advancing our understanding and prediction of dynamic systems in applied science and engineering disciplines. Kashuri Fundo transform was introduced to the literature by Kashuri and Fundo with the statement that various properties can be found easily due to its deep connection with Laplace transform [2]. In later processes, many researchers, including Kashuri and Fundo, worked on different applications [6–16] of this transform. Helmi et al. [17], Singh [18], Dhange [19], and Güngör [20] investigated various applications of Kashuri Fundo transformation. Later, Peker et al. [21–26] applied this transform to the models, namely steady heat transfer, decay, some chemical reaction, one-dimensional Bratu's problem, Michaelis-Menten's biochemical reaction model, population growth and mixing problem to demonstrate the competence of the Kashuri Fundo transform in reaching solutions of ordinary differential equations.

Extracting analytic or approximate solutions for differential equations by using new mathematical methods always attract the attention of the researchers due to the academic curiosity and practical applications. Motivation for having a technique more effective, more applicable, and easier to use induces the possibility of analyzing the utility of other non-conventional solution techniques or methods.

In this study, we aim to present that the Kashuri Fundo transform is a technique that facilitates the solution of differential equations through two different cardiovascular models, one of which is glucose concentration in the blood during continuous intravenous glucose injection. The other is pressure in the aorta.

2. Preliminaries

This section provides some of basic definitions and properties related to Kashuri Fundo transform.

Definition 2.1. [5] Let F be a function set defined by

$$F = \left\{ f(t) \mid \exists M, k_1, k_2 > 0 \ni |f(t)| \leqslant M e^{\frac{|t|}{k_i^2}}, \text{ if } t \in (-1)^i \times [0, \infty) \right\}$$

where M is a constant and k_1 and k_2 are finite constants or infinite.

Definition 2.2. [5] Kashuri Fundo transform, defined on the set F and denoted by the operator $\mathscr{K}(.)$, is defined by

$$\mathscr{K}[f(t)](v) = A(v) = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} f(t)\ dt, \quad t \geqslant 0 \quad \text{and} \quad -k_1 < v < k_2$$

which can be stated as well by

$$\mathscr{K}[f(t)](v) = A(v) = v\int_0^\infty e^{-t} f(v^2 t)\ dt$$

Inverse Kashuri Fundo transform is denoted by $\mathscr{K}^{-1}[A(v)] = f(t)$, $t \geq 0$.

Definition 2.3. [5] A function $f(t)$ is said to be of exponential order $\frac{1}{k^2}$, if there are positive constants T and M such that $|f(t)| \leq Me^{\frac{-t}{k^2}}$, for all $t \geq T$.

Theorem 2.4. [5] [Sufficient Conditions for Existence] If $f(t)$ is piecewise continuous on $[0, \infty)$ and has exponential order $\frac{1}{k^2}$, then $\mathscr{K}[f(t)](v)$ exists, for $|v| < k$.

Theorem 2.5. [5] [Linearity Property] Let $f(t)$ and $g(t)$ be functions whose Kashuri Fundo transforms exist and c_1 and c_2 be constants. Then,

$$\mathscr{K}[(c_1 f + c_2 g)(t)](v) = c_1\mathscr{K}[f(t)](v) + c_2\mathscr{K}[g(t)](v)$$

Theorem 2.6. [5] [Derivatives of a Function $f(t)$] Let $A(v)$ be a Kashuri Fundo transform of $f(t)$. Then,

$$\mathscr{K}[f^{(n)}(t)](v) = \frac{A(v)}{v^{2n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2(n-k)-1}} \tag{1}$$

Table 1 presents transformations of some special functions.

Table 1. Kashuri Fundo and Laplace transforms of some special functions [2, 5, 13]

$f(t)$	$\mathscr{K}[f(t)] = A(v)$	$\mathscr{L}[f(t)] = F(s)$
1	v	$\frac{1}{s}$
t	v^3	$\frac{1}{s^2}$
t^n, $n \in \mathbb{Z}$	$n!v^{2n+1}$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{v}{1-av^2}$	$\frac{1}{s-a}$
$\sin(at)$	$\frac{av^3}{1+a^2v^4}$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{v}{1+a^2v^4}$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{av^3}{1-a^2v^4}$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{v}{1-a^2v^4}$	$\frac{s}{s^2-a^2}$
t^α, $\alpha \in \mathbb{R}^+$	$\Gamma(1+\alpha)v^{2\alpha+1}$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
$\sum_{k=0}^{n} a_k t^k$	$\sum_{k=0}^{n} k!a_k v^{2k+1}$	$\sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}$

Table 2 presents inverse transformations of some special functions.

Table 2. Inverse Kashuri Fundo transform of some special functions [5, 13]

$A(v)$	$\mathscr{K}^{-1}[A(v)] = f(t)$
v	1
v^3	t
$n!v^{2n+1}$	$t^n,\ n \in \mathbb{Z}$
$\frac{v}{1-av^2}$	e^{at}
$\frac{av^3}{1+a^2v^4}$	$\sin(at)$
$\frac{v}{1+a^2v^4}$	$\cos(at)$
$\frac{av^3}{1-a^2v^4}$	$\sinh(at)$
$\frac{v}{1-a^2v^4}$	$\cosh(at)$
$\Gamma(1+\alpha)v^{2\alpha+1}$	$t^\alpha,\ \alpha \in \mathbb{R}^+$
$\sum_{k=0}^{n} k!a_k v^{2k+1}$	$\sum_{k=0}^{n} a_k t^k$

In order to better understand the application of Kashuri Fundo transform to ordinary differential equations, two simple numerical examples are provided below.

Example 2.7. [27] Consider differential equation

$$\frac{dy}{dt} - 16y = 2 \tag{2}$$

with the initial condition

$$y(0) = -4$$

Applying the Kashuri Fundo transform bilaterally to both sides of Equation 2,

$$\mathscr{K}\left[\frac{dy}{dt}\right] - \mathscr{K}[16y] = \mathscr{K}[2]$$

If we write the equivalent in Equation 1 instead of the expression $\mathscr{K}\left[\frac{dy}{dt}\right]$ and arrange the expression $\mathscr{K}[2]$ according to Table 1 using the linearity property of the transform,

$$\frac{A(v)}{v^2} - \frac{y(0)}{v} - 16A(v) = 2v \tag{3}$$

where $A(v) = \mathscr{K}[y(t)]$. Substituting the initial condition in Equation 3 and rearranging the equation,

$$A(v) = \frac{2v^3 - 4v}{1 - 16v^2} \tag{4}$$

We use Table 2 to apply the inverse Kashuri Fundo transform to Equation 4. For this, if we rearrange Equation 4,

$$A(v) = -\frac{1}{8}v - \frac{31}{8}\frac{v}{1 - 16v^2} \tag{5}$$

If we apply the inverse Kashuri Fundo transform to both sides of Equation 5 using Table 2, we find the solution of the given differential equation as

$$y(t) = -\frac{1}{8} - \frac{31}{8}e^{16t}$$

The 3D graph of this solution is in Figure 1. The graphics in this study is drawn using Python.

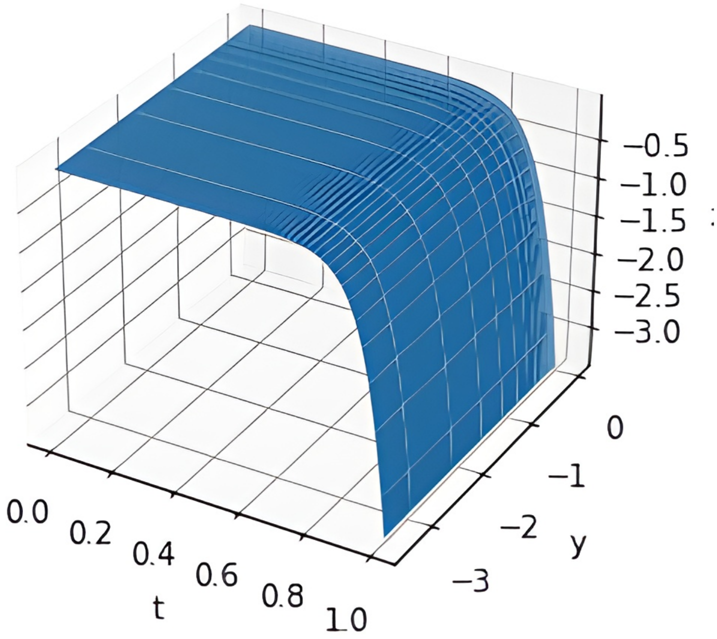

Figure 1. Graph of $y(t) = -\frac{1}{8} - \frac{31}{8}e^{16t}$

Example 2.8. [28] Consider differential equation

$$\frac{dy}{dt} - y = e^{3t} \tag{6}$$

with the initial condition

$$y(0) = 2$$

Applying the Kashuri Fundo transform bilaterally to both sides of Equation 6,

$$\mathscr{K}\left[\frac{dy}{dt}\right] - \mathscr{K}[y] = \mathscr{K}[e^{3t}]$$

If we write the equivalent in Equation 1 instead of the expression $\mathscr{K}\left[\frac{dy}{dt}\right]$ and arrange the expression $\mathscr{K}[e^{3t}]$ according to Table 1 using the linearity property of the transform,

$$\frac{A(v)}{v^2} - \frac{y(0)}{v} - A(v) = \frac{v}{1-3v^2} \tag{7}$$

where $A(v) = \mathscr{K}[y(t)]$. Substituting the initial condition in Equation 7 and rearranging the equation,

$$A(v) = \frac{2v - 5v^3}{(1-v^2)(1-3v^2)} \tag{8}$$

We use Table 2 to apply the inverse Kashuri Fundo transform to Equation 8. For this, if we rearrange the right-hand side of Equation 8,

$$\frac{2v - 5v^3}{(1-v^2)(1-3v^2)} = \frac{Av}{1-v^2} + \frac{Bv}{1-3v^2} \tag{9}$$

If this equation is solved, then

$$A = \frac{3}{2} \quad \text{and} \quad B = \frac{1}{2}$$

If we write these values in their places in Equation 9 and use the equation here in Equation 8,

$$A(v) = \frac{3}{2}\frac{v}{1-v^2} + \frac{1}{2}\frac{v}{1-3v^2} \tag{10}$$

If we apply the inverse Kashuri Fundo transform to both sides of Equation 10 using Table 2, we find the solution of the given differential equation as

$$y(t) = \frac{1}{2}(3e^t + e^{3t})$$

The 3D graph of this solution is in Figure 2.

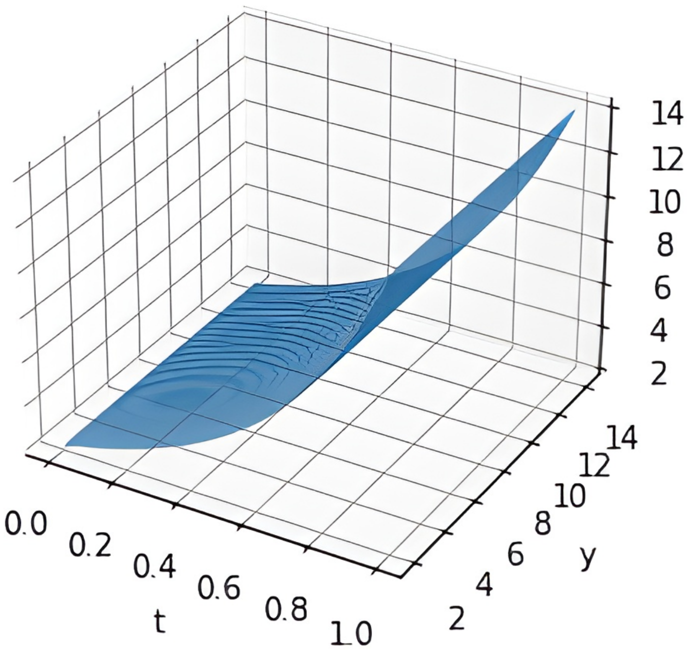

Figure 2. Graph of $y(t) = \frac{1}{2}(3e^t + e^{3t})$

3. Applications of Kashuri Fundo Transform to Cardiovascular Models

This section presents two applications of Kashuri Fundo transform to two cardiovascular models.

Application 3.1. (Glucose concentration in the blood) During continuous intravenous glucose injection, the concentration of glucose in the blood is $C(t)$ exceeding the baseline value at the start of the infusion. The function $C(t)$ satisfies the initial value problem [2]

$$\frac{dC(t)}{dt} + kC(t) = \frac{\alpha}{V}, \quad t > 0 \tag{11}$$

and

$$C(0) = 0$$

where k is the constant velocity of elimination, α is the rate of infusion, and V is the volume in which glucose is distributed. We will find the concentration of glucose in the blood by using the Kashuri Fundo transform method.

Applying the transform bilaterally to the given Equation 11,

$$\mathscr{K}\left[\frac{dC(t)}{dt}\right] + k\,\mathscr{K}[C(t)] = \mathscr{K}\left[\frac{\alpha}{V}\right] \tag{12}$$

Let $\mathscr{K}[C(t)] = A(v)$. If we rearrange the Equation 12 by using the Equation 1 with the initial condition and the transforms in Table 1,

$$\frac{A(v)}{v^2} - \frac{C(0)}{v} + kA(v) = \frac{\alpha}{V}v$$

and thus

$$A(v) = \frac{\alpha}{V}\left(\frac{v^3}{1+kv^2}\right) \tag{13}$$

Having rearranged the Equation 13,

$$A(v) = \frac{\alpha}{V}\left(v - \frac{v}{1+kv^2}\right) \tag{14}$$

Applying the inverse Kashuri Fundo transform to Equation 14,

$$\mathscr{K}^{-1}[A(v)] = \mathscr{K}^{-1}\left[\frac{\alpha}{V}\left(v - \frac{v}{1+kv^2}\right)\right] \tag{15}$$

If we rearrange Equation 15 using the linearity property of the inverse Kashuri Fundo transform,

$$\mathscr{K}^{-1}[A(v)] = \frac{\alpha}{V}\left(\mathscr{K}^{-1}[v] - \mathscr{K}^{-1}\left[\frac{v}{1+kv^2}\right]\right) \tag{16}$$

According to Table 2, the equivalents of the expressions in Equation 16 are

$$\mathscr{K}^{-1}[A(v)] = C(t), \quad \mathscr{K}^{-1}[v] = 1, \quad \text{and} \quad \mathscr{K}^{-1}\left[\frac{v}{1+kv^2}\right] = e^{-kt}$$

Finally, substitute these expressions in Equation 16, we find the concentration of glucose in the blood as

$$C(t) = \frac{\alpha}{kV}(1 - e^{-kt})$$

Application 3.2. (Pressure in the aorta) The blood is pumped into the aorta by the contraction of the heart. The pressure $p(t)$ in the aorta satisfies the initial value problem [2]

$$\frac{dp(t)}{dt} + \frac{c}{k}p(t) = cA\sin\omega t \tag{17}$$

and

$$p(0) = p_0$$

where c, k, A, and p_0 are constants. We will obtain the pressure in the aorta by using the Kashuri Fundo transform method.

Applying the transform bilaterally to the given Equation 17,

$$\mathscr{K}\left[\frac{dp(t)}{dt}\right] + \frac{c}{k}\mathscr{K}[p(t)] = cA\mathscr{K}[\sin\omega t] \tag{18}$$

Let $\mathscr{K}[p(t)] = A(v)$. If we rearrange the Equation 18 by using the Equation 1 with the initial condition and the transforms in Table 1,

$$\frac{A(v)}{v^2} - \frac{p_0}{v} + \frac{c}{k}A(v) = cA\frac{\omega v^3}{1+\omega^2 v^4}$$

and thus

$$A(v) = p_0\frac{v}{1+\frac{c}{k}v^2} + cA\left(\frac{k\omega v^5}{(k+cv^2)(1+\omega^2 v^4)}\right) \tag{19}$$

Regrouping the Equation 19,

$$A(v) = p_0\frac{v}{1+\frac{c}{k}v^2} + cA\left(\frac{kc}{\omega^2k^2+c^2}\frac{v^3}{1+\omega^2v^4} - \frac{\omega k^2}{\omega^2k^2+c^2}\frac{v}{1+\omega^2v^4} + \frac{\omega k^2}{\omega^2k^2+c^2}\frac{v}{1+\frac{c}{k}v^2}\right) \tag{20}$$

Applying the inverse Kashuri Fundo transform to Equation 20,

$$\mathscr{K}^{-1}[A(v)] = \mathscr{K}^{-1}\left[p_0\frac{v}{1+\frac{c}{k}v^2} + cA\left(\frac{kc}{\omega^2k^2+c^2}\frac{v^3}{1+\omega^2v^4} - \frac{\omega k^2}{\omega^2k^2+c^2}\frac{v}{1+\omega^2v^4} + \frac{\omega k^2}{\omega^2k^2+c^2}\frac{v}{1+\frac{c}{k}v^2}\right)\right] \tag{21}$$

If we rearrange Equation 21 using the linearity property of the inverse Kashuri Fundo transform,

$$\begin{aligned}\mathscr{K}^{-1}[A(v)] = cA\Bigg(&\tfrac{kc}{\omega^2k^2+c^2}\mathscr{K}^{-1}\left[\tfrac{v^3}{1+\omega^2v^4}\right] - \tfrac{\omega k^2}{\omega^2k^2+c^2}\mathscr{K}^{-1}\left[\tfrac{v}{1+\omega^2v^4}\right] + \tfrac{\omega k^2}{\omega^2k^2+c^2}\mathscr{K}^{-1}\left[\tfrac{v}{1+\frac{c}{k}v^2}\right]\Bigg) \\ &+p_0\mathscr{K}^{-1}\left[\tfrac{v}{1+\frac{c}{k}v^2}\right]\end{aligned} \tag{22}$$

According to Table 2, the equivalents of the expressions in Equation 22 are

$$\mathscr{K}^{-1}[A(v)] = p(t), \quad \mathscr{K}^{-1}\left[\frac{v}{1+\frac{c}{k}v^2}\right] = e^{-\frac{c}{k}t},$$

$$\mathscr{K}^{-1}\left[\frac{v^3}{1+\omega^2v^4}\right] = \sin\omega t, \quad \text{and} \quad \mathscr{K}^{-1}\left[\frac{v}{1+\omega^2v^4}\right] = \cos\omega t.$$

Finally, substitute these expressions in Equation 22, we obtain the pressure in the aorta as

$$p(t) = p_0e^{-\frac{c}{k}t} + cA\frac{\omega k^2}{\omega^2k^2+c^2}\left(\frac{c}{\omega k}\sin\omega t - \cos\omega t + e^{-\frac{c}{k}t}\right)$$

4. Conclusion

Differential equations appear in the modeling of many phenomena in applied sciences. Using these equations in modeling makes understanding and interpreting the phenomenon underlying these events more accessible. The fact that it is used in modeling many important events makes it essential to solving these equations. On the other hand, differential equations are more difficult to solve than algebraic equations. Thus, using integral transformations is very helpful in solving these equations. In this study, we applied the Kashuri Fundo transform, one of the integral transforms closely related to the Laplace transform, with a few cardiovascular models. As seen from these applications, the considered transform is suitable for applicability, reliability, effectiveness, and ease of use.

In future studies, researchers can study by considering systems of differential equations as well as fractional differential equations emerging in applied fields.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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A Note on Equivalence of G-Cone Metric Spaces and G-Metric Spaces

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Abstract — This paper contains the equivalence between tvs-G cone metric and G-metric using a scalarization function ζ_p, defined over a locally convex Hausdorff topological vector space. This function ensures that most studies on the existence and uniqueness of fixed-point theorems on G-metric space and tvs-G cone metric spaces are equivalent. We prove the equivalence between the vector-valued version and scalar-valued version of the fixed-point theorems of those spaces. Moreover, we present that if a real Banach space is considered instead of a locally convex Hausdorff space, then the theorems of this article extend some results of G-cone metric spaces and ensure the correspondence between any G-cone metric space and the G-metric space.

Keywords *G-metric space, G-cone metric space, tvs-G cone metric space*

Mathematics Subject Classification (2020) 47H10, 54H25

1. Introduction

In 1906, French mathematician Frechet introduced metric space by generalizing the concept of the Euclidean distance function. Later, Hausdorff, in 1914, formalized the definition of metric space by a set of axioms inherited from the basic properties of Euclidean distance. After that, several generalized metric spaces, such as 2-metric [1], b-metric [2, 3], strong b-metric [4], D-metric [5], G-metric [6], S-metric [7], cone-metric [8], parametric S-metric [9], generalized parametric metric [10], and fuzzy metric [11] have been familiarized. Since 1922, when Banach proved the celebrated Banach fixed point theorem in complete metric spaces, several researchers have tried to generalize it. Sometimes this generalization is by changing the contraction condition or reforming it to some generalized metric spaces. Based on the types of self-mappings, such as contractive or expansive, single-valued or multivalued, fixed point theories have been developed on those spaces.

In 2006, G-metric space, one of those generalized metric spaces, was brought to light by Mustafa and Sims [6] to overcome elementary imperfection in the structure of D-metric spaces, defined by Dhage [5]. Immediately after, Guang and Xian [8] introduced the idea of cone metric in 2007, where they replaced the set of non-negative real numbers with an ordered real Banach space. Following them, Beg et al. [12] extended the concept of G-metric and cone metric to G-cone metric space in 2010. Consequently, many study on fixed point theory have been done in G-cone metric spaces.

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This article aims to investigate the relation between the vector-valued and scalar-valued versions of fixed point theorems of generalized cone-metric spaces and G-metric spaces. We show that there is a relationship between G-metric and G-cone metric with the help of scalarization function ζ_p, defined on a locally convex Hausdorff topological vector space.

This article is presented as follows: Section 2 consists of some definitions and results used in the main result Section. Section 3 establishes the relation between G-metric and G-cone metric utilizing the tvs-G cone metric space definition. The final section contains some fixed point results ensuring their equivalence in general G-metric spaces and tvs-G cone metric spaces and discusses the need for further research.

2. Preliminaries

This section contains some definitions and results related to the main results of this study.

Let E be a topological vector space (tvs in short), θ be zero vector, and P be a nonempty convex, i.e., $P+P \subseteq P$ and $\mu P \subseteq P$, for $\mu \geq 0$, and pointed, i.e., $P \cap (-P) = \{\theta\}$, cone in E. For the cone $P \in E$, $\preceq$ is a partial ordering with respect to P given by $x \preceq \mu \Leftrightarrow \mu - x \in P$. $x \prec \mu$ stands for $x \preceq \mu$ but $x \neq \mu$ and $x \prec\prec \mu$ stands for $\mu - x \in \text{int}(P)$ where $\text{int}(P)$ denotes the interior of P.

Throughout the article, Y is a real locally convex Hausdorff TVS and P is closed, proper, and convex pointed cone with the non-empty interior, $p \in \text{int}(P)$, and $\preceq$ is a partial ordering with respect to P defined as above.

Consider the nonlinear scalarization function $\zeta_p : Y \to \mathbb{R}$ is defined by

$$\zeta_p(x) = \inf\{s \in \mathbb{R} : x \in ps - P\}, \quad \text{for all } x \in Y$$

Lemma 2.1. [13–17] For each $s \in \mathbb{R}$, $p \in \text{int}(P)$, and $x, x_1, x_2 \in Y$, the following conditions are satisfied:

i. $\zeta_p(x) \leq s \Leftrightarrow x \in ps - P$

ii. $\zeta_p(x) > s \Leftrightarrow x \notin ps - P$

iii. $\zeta_p(x) \geq s \Leftrightarrow x \notin ps - \text{int}(P)$

iv. $\zeta_p(x) < s \Leftrightarrow x \in ps - \text{int}(P)$

v. $\zeta_p(.)$ is continuous and positively homogeneous function on Y

vi. $x_2 \preceq x_1$ implies $\zeta_p(x_2) \leq \zeta_p(x_1)$

vii. $\zeta_p(x_1 + x_2) \leq \zeta_p(x_1) + \zeta_p(x_2)$.

Note 2.2. [14] Clearly $\zeta_p(\theta) = 0$. Moreover, the converse statement of *vi.* in Lemma 2.1 is not true necessarily. For example, let $E = \mathbb{R}^2$, $P = \{(x, y) \in X : x, y \geq 0\}$, and $p = (1, 1)$. Then, P is a closed, convex, proper, and pointed cone in Y with $\text{int}(P) \neq \phi$. For $r = 1$, it can be observed that $x_1 = (8, -15) \notin rp - \text{int}(P)$ and $x_2 = (0, 0) \in rp - \text{int}(P)$. By applying *iii.* and *iv.* of Lemma 2.1, we have $\zeta_p(x_1) < 1 \leq \zeta_p(x_2)$ while $x_1 \notin x_2 + P$.

Definition 2.3. [6] Let $\Im$ be a nonempty set and $\mathcal{G} : \Im \times \Im \times \Im \to [0, \infty)$ be a mapping that satisfies the following conditions:

(G1) $\mathcal{G}(x, \zeta, z) = 0$ if $x = \zeta = z$

(G2) $0 < \mathcal{G}(x, x, \zeta)$ if $x \neq \zeta$

(G3) $\mathcal{G}(x,x,\zeta) \leq \mathcal{G}(x,\zeta,z)$ if $\zeta \neq z$

(G4) $\mathcal{G}(x,\zeta,z) = \mathcal{G}(x,z,\zeta) = \mathcal{G}(\zeta,x,z) = \cdots$ (Symmetric in all three variables)

(G5) $\mathcal{G}(x,\zeta,z) \leq \mathcal{G}(x,\mu,\mu) + \mathcal{G}(\mu,\zeta,z)$

for all $x,\zeta,z,\mu \in \Im$. Then, $(\Im,\mathcal{G})$ is called a G-metric space.

Definition 2.4. [6] Let $(\Im,\mathcal{G}_1)$ and $(\Im,\mathcal{G}_2)$ be two G-metric spaces. A function $\mathcal{F} : (\Im,\mathcal{G}_1) \to (\Im,\mathcal{G}_2)$ is said to be

i. G-continuous at a point $\eta \in X$ if, for given $\alpha > 0$, there exists $\beta > 0$ such that, for all $\eta, b, z \in \Im$, $\mathcal{G}_2(\mathcal{F}\eta, \mathcal{F}b, \mathcal{F}z) < \alpha$ if $\mathcal{G}_1(\eta,b,z) < \beta$.

ii. G-sequentially continuous at a point $\eta \in X$ if $\{\eta_n\}$ is G-converges to η implies $\{\mathcal{F}(\eta_n)\}$ is G-converges to $\mathcal{F}(\eta)$.

Theorem 2.5. [6] Let $(\Im,\mathcal{G}_1)$ and $(\Im,\mathcal{G}_2)$ be two G-metric spaces. A function $\mathcal{F} : (\Im,\mathcal{G}_1) \to (\Im,\mathcal{G}_2)$ is G-continuous at a point $\eta \in \Im$ if and only if (iff) $\mathcal{F}$ is G-sequentially continuous at η.

Definition 2.6. [8] Let $\Im$ be a nonempty set, E be a real Banach space, P be a cone in E, and $\preceq$ is a partial ordering in E with respect to P. A mapping $M : \Im \times \Im \to E$ is called a cone metric on $\Im$ if it satisfies the following properties:

(M1) $M(y,\alpha) \succ \theta$, for all $y,\alpha \in \Im$, and $M(y,\alpha) = \theta$ iff $y = \alpha$

(M2) $M(y,\alpha) = M(\alpha,y)$, for all $y,\alpha \in \Im$

(M3) $M(y,\alpha) \preceq M(y,\eta) + M(\eta,\alpha)$, for all $y,\alpha,\eta \in \Im$

Moreover, the pair $(\Im, M)$ is called a cone metric space.

Definition 2.7. [12] Let $\Im$ be a nonempty set, E be a real Banach space, P be a cone in E, and $\preceq$ is a partial ordering in E with respect to P. A mapping $\mathcal{G} : \Im \times \Im \times \Im \to E$ satisfying, for all $x,\zeta,z,\mu \in \Im$,

i. $\mathcal{G}(x,\zeta,z) = \theta$ if $x = \zeta = z$

ii. $\theta \prec \mathcal{G}(x,x,\zeta)$ if $x \neq \zeta$

iii. $\mathcal{G}(x,x,\zeta) \preceq \mathcal{G}(x,\zeta,z)$ if $\zeta \neq z$

iv. $\mathcal{G}(x,\zeta,z) = \mathcal{G}(x,z,\zeta) = \mathcal{G}(\zeta,x,z) = \cdots$ (Symmetric in all three variables)

v. $\mathcal{G}(x,\zeta,z) \preceq \mathcal{G}(x,\mu,\mu) + \mathcal{G}(\mu,\zeta,z)$

is called a G-cone metric and $(\Im,\mathcal{G})$ is called a G-cone metric space.

Definition 2.8. [18] Let $\Im$ be a nonempty set, Y be a real Hausdorff tvs, and $\preceq$ is a partial ordering in Y with respect to a cone P. A vector-valued mapping $\mathcal{T} : \Im \times \Im \to \Im$ is called a tvs-cone metric if it satisfies

($\mathcal{T}$1) $\mathcal{T}(x,\eta) \succ \theta$, for all $x,\eta \in \Im$, and $\mathcal{T}(x,\eta) = \theta$ iff $x = \eta$

($\mathcal{T}$2) $\mathcal{T}(x,\eta) = \mathcal{T}(\eta,y)$, for all $x,\eta \in \Im$

($\mathcal{T}$3) $\mathcal{T}(x,\eta) \preceq \mathcal{T}(x,\alpha) + \mathcal{T}(\alpha,\eta)$, for all $x,\eta,\alpha \in \Im$

Moreover, the pair $(\Im,\mathcal{T})$ is called a tvs-cone metric space.

Definition 2.9. [19] Let $\Im$ be a nonempty set, Y be a tvs, $\preceq$ be a partial ordering in Y with respect to cone P, and $\mathcal{G} : \Im \times \Im \times \Im \to Y$ be a mapping satisfying the following conditions:

(G1) $\mathcal{G}(x,\zeta,z)=\theta$ if $x=\zeta=z$

(G2) $\theta \prec \mathcal{G}(x,x,\zeta)$ if $x\neq\zeta$, for all $x,\zeta\in\Im$

(G3) $\mathcal{G}(x,x,\zeta)\preceq\mathcal{G}(x,\zeta,z)$ if $\zeta\neq z$

(G4) $\mathcal{G}(x,\zeta,z)=\mathcal{G}(x,z,\zeta)=\mathcal{G}(\zeta,x,z)=\cdots$ (Symmetric in all three variables)

(G5) $\mathcal{G}(x,\zeta,z)\preceq\mathcal{G}(x,\mu,\mu)+\mathcal{G}(\mu,\zeta,z)$, for all $x,\zeta,z,\mu\in\Im$

Then, G is called a tvs-G cone metric, and the pair $(\Im,\mathcal{G})$ is called a tvs-G-cone metric space.

Definition 2.10. [19] Let $(\Im,\mathcal{G})$ be a tvs-G-cone metric space and $\{x_n\}$ be a sequence in $\Im$. Then,

i. $\{x_n\}$ is said to be tvs-G-cone convergent to x if, for all $\alpha\in Y$ with $0\prec\prec\alpha$, there is a $K\in\mathbb{N}$ such that $\mathcal{G}(x_m,x_n,x)\prec\prec\alpha$, for all $m,n\geq K$ and we write $\lim\limits_{n\to\infty}x_n=x$.

ii. $\{x_n\}$ is said to be a tvs-G-cone Cauchy if, for all $\alpha\in Y$ with $0\prec\prec\alpha$, there is a $K\in\mathbb{N}$ such that $G(x_m,x_n,x_k)\prec\prec c$, for all $m,n,k\geq K$.

iii. $\Im$ is called tvs-G-cone complete if every Cauchy sequence in $\Im$ converges to some element in $\Im$.

3. Main Result

In the following, we consider Y as a real locally convex Hausdorff tvs, P as a closed, proper, and convex pointed cone in Y with non-empty interior, $p\in\text{int}(P)$, and $\preceq$ as a partial ordering with respect to P defined as above.

Definition 3.1. A tvs-G-cone metric space $(\Im,\mathcal{G})$ is said to be symmetric if, for all $\alpha,y\in\Im$, $\mathcal{G}(\alpha,y,y)=\mathcal{G}(y,\alpha,\alpha)$.

Note 3.2. In particular, if we take E as a real Banach space, then the definition of tvs-G-cone metrics is reduced to G cone metrics of Beg et al. [12]. Hence, for examples of symmetric and non-symmetric tvs-G-cone metric spaces, please see Examples 2.4 and 2.5 of Beg et al. [12].

Definition 3.3. Let $(\Im,\mathcal{G}_1)$ and $(\Im,\mathcal{G}_2)$ be two tvs-G-cone metric spaces and $\mathcal{F}:(\Im,\mathcal{G}_1)\to(\Im,\mathcal{G}_2)$ be a function. Then, $\mathcal{F}$ is

i. tvs-G-cone continuous at a point $\xi\in\Im$ if, for any $\alpha\succ\succ\theta$, there exists $\beta\succ\succ\theta$ such that, for all $\xi,y,c\in\Im$, $\mathcal{G}_2(\mathcal{F}\xi,\mathcal{F}y,\mathcal{F}c)\prec\prec\alpha$ if $\mathcal{G}_1(\xi,y,c)\prec\prec\beta$.

ii. tvs-G-cone sequentially continuous at a point $\xi\in\Im$ if $\{\xi_n\}$ is tvs-G-cone converges to ξ implies $\{\mathcal{F}(\xi_n)\}$ is tvs-G-cone converges to $\mathcal{F}(\xi)$.

In the following theorem, we show that there is a relationship between a tvs-G cone metric space and a G-metric space.

Theorem 3.4. Let $\Im$ be a nonempty set and $(\Im,\mathcal{G})$ be a tvs-G cone metric space. Define a mapping $M_{\mathcal{G}}:\Im\times\Im\times\Im\to\mathbb{R}_{\geq 0}$ by $M_{\mathcal{G}}=\zeta_p\ o\ \mathcal{G}$ where $p\in\text{int}(P)$ in Y. Then, $M_{\mathcal{G}}$ is a G-metric on $\Im$.

Proof.
Let $\Im$ be a nonempty set and $(\Im,\mathcal{G})$ be a tvs-G cone metric space. Define a mapping $M_{\mathcal{G}}:\Im\times\Im\times\Im\to\mathbb{R}_{\geq 0}$ by $M_{\mathcal{G}}=\zeta_p\ o\ \mathcal{G}$ where $p\in\text{int}(P)$ in Y.

i. If $x=\mu=z$, then $\mathcal{G}(x,\mu,z)=\theta$. Therefore, $M_{\mathcal{G}}(x,\mu,z)=\zeta_p(\mathcal{G}(x,\mu,z))=0$, since $\zeta_p(\theta)=0$. Thus, (G1) holds.

ii. If $\alpha\neq y$, then $\mathcal{G}(\alpha,\alpha,y)\succ\theta$. Thus,

$$M_{\mathcal{G}}(x,y,z)=(\zeta_p o\mathcal{G})(x,x,y)=\zeta_p(\mathcal{G}(x,x,y))\geq\zeta_p(\theta)=0$$

Hence, (G2) holds for $M_{\mathcal{G}}$.

iii. Since $\mathcal{G}(\alpha,\alpha,y) \preceq \mathcal{G}(\alpha,y,c)$, for $\alpha, y \neq c \in \Im$, thus $\zeta_p(\mathcal{G}(\alpha,\alpha,y)) \leq \zeta_p(\mathcal{G}(\alpha,y,c))$ and hence $M_{\mathcal{G}}(\alpha,\alpha,y) \leq M_{\mathcal{G}}(\alpha,y,c)$, for $\alpha, y \neq c \in \Im$. Therefore, $M_{\mathcal{G}}$ satisfies the condition (G3).

iv. (G4) is valid for $M_{\mathcal{G}}$, since $\mathcal{G}$ is symmetric in all three variables implies $M_{\mathcal{G}}$ is.

v. For all $\alpha, y, a, \mu \in \Im$, we have $\mathcal{G}(\alpha,y,a) \preceq \mathcal{G}(\alpha,\mu,\mu) + \mathcal{G}(\mu,y,a)$ which implies

$$\zeta_p(\mathcal{G}(\alpha,y,a)) \leq \zeta_p(\mathcal{G}(\alpha,\mu,\mu) + \mathcal{G}(\mu,y,a)) \leq \zeta_p(\mathcal{G}(\alpha,\mu,\mu)) + \zeta_p(\mathcal{G}(\mu,y,a))$$

Therefore, $M_{\mathcal{G}}(\alpha,y,a) \leq M_{\mathcal{G}}(\alpha,\mu,\mu) + M_{\mathcal{G}}(\mu,y,a)$, for all $\alpha, y, a, \mu \in \Im$ and thus (G5) is valid.

Hence, $M_{\mathcal{G}}$ is a G-metric on $\Im$ and the pair $(\Im, M_{\mathcal{G}})$ is a G-metric space. □

Corollary 3.5. If $\mathcal{G}$ is a G-cone metric on $\Im$ in the sense of Beg et al. [12], then $M_{\mathcal{G}} = \zeta_p \ o \ \mathcal{G}$ is a G-metric on $\Im$.

Proof.
In the above theorem, in particular, if we take Y as a real Banach space, then the result can be concluded from the above. □

Next theorem establishes the relation between the notions of convergence of sequences in tvs-G cone metric spaces and G-metric spaces.

Theorem 3.6. Suppose that $\mathcal{G}$ is a tvs-G cone metric and $M_{\mathcal{G}}$ is a G-metric on $\Im$ where $M_{\mathcal{G}}$ is defined in Theorem 3.4. Then,

i. A sequence $\{\eta_n\}$ converges in $(\Im, \mathcal{G})$ iff $\{\eta_n\}$ converges in $(\Im, M_{\mathcal{G}})$.

ii. A sequence $\{\eta_n\}$ is a Cauchy sequence in $(\Im, \mathcal{G})$ iff $\{\eta_n\}$ is Cauchy in $(\Im, M_{\mathcal{G}})$.

iii. $(\Im, \mathcal{G})$ is complete iff $(\Im, M_{\mathcal{G}})$ is complete.

Proof.
i. ($\Rightarrow$): Assume that $\{\eta_n\}$ converges to η in $(\Im, \mathcal{G})$. Let $\epsilon > 0$ be arbitrary. For any $p \succ\succ \theta$ in Y, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\begin{aligned} \mathcal{G}(\eta_n,\eta_n,\eta) \prec\prec p\epsilon \quad &\Longrightarrow \quad \mathcal{G}(\eta_n,\eta_n,\eta) \in p\epsilon - \text{int}(P) \\ &\Longrightarrow \quad \zeta_p(\mathcal{G}(\eta_n,\eta_n,\eta)) < \epsilon \\ &\Longrightarrow \quad (\zeta_p o \mathcal{G})(\eta_n,\eta_n,\eta)) < \epsilon \end{aligned}$$

That is, $M_{\mathcal{G}}(\eta_n,\eta_n,\eta) < \epsilon$, for all $m, n \geq N$, which implies $\{\eta_n\}$ converges to η in $(\Im, M_{\mathcal{G}})$.

($\Leftarrow$): Assume that a sequence $\{\eta_n\}$ converges to η in $(\Im, M_{\mathcal{G}})$. Let $c \succ\succ \theta$ in Y be arbitrary. Take $p \in intP$ and $\epsilon > 0$ be such that $p\epsilon \prec\prec c$. Since $\{\eta_n\}$ converges to η in $(\Im, M_{\mathcal{G}})$, thus there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

$$\begin{aligned} M_{\mathcal{G}}(\eta_n,\eta_n,\eta) < \epsilon \quad &\Longrightarrow \quad (\zeta_p o \mathcal{G})(\eta_n,\eta_n,\eta) < \epsilon \\ &\Longrightarrow \quad \zeta_p(\mathcal{G}(\eta_n,\eta_n,\eta)) < \epsilon \\ &\Longrightarrow \quad \mathcal{G}(\eta_n,\eta_n,\eta) \in p\epsilon - \text{int}(P) \end{aligned}$$

which implies $\mathcal{G}(\eta_n,\eta_n,\eta) \prec\prec p\epsilon \prec\prec c$, for all $m, n \geq N$, and thus $\{\eta_n\}$ converges to η in $(\Im, \mathcal{G})$.

Proof of *ii* can be derived in a similar way of *i*, and *iii* is immediate consequence of *i* and *ii*. □

Theorem 3.7. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two tvs-G-cone metrics on $\Im$ and M_{G_1}, respectively, and M_{G_2} is the induced G-metrics on $\Im$, as defined in Theorem 3.4. Then, a function $\mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2)$ is tvs-G-cone continuous iff $\mathcal{T} : (\Im, M_{\mathcal{G}_1}) \to (\Im, M_{\mathcal{G}_2})$ is G-continuous.

PROOF.
(⇐): Assume that $\mathcal{T}$ is G-continuous. Let $c \succ\succ \theta$ in Y. Take $p \in \text{int}(P)$ and $\epsilon > 0$ be such that $p\epsilon \prec\prec c$. Then, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$M_{\mathcal{G}_2}(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } M_{\mathcal{G}_1}(a, b, c) < \delta \tag{1}$$

For $p \in \text{int}(P)$ and $\delta > 0$, there exists an $e \succ\succ \theta$ be such that $p\delta \prec\prec e$. From Relation 1, it follows that

$$\begin{aligned}(\zeta_p o \mathcal{G}_2)(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } (\zeta_p o \mathcal{G}_1)(a, b, c) < \delta &\implies \zeta_p(\mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c)) < \epsilon \text{ if } \zeta_p(\mathcal{G}_1(a, b, c)) < \delta \\ &\implies \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \in p\epsilon - \text{int}(P) \text{ if } G_1(a, b, c) \in p\delta - \text{int}(P) \\ &\implies \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \prec\prec p\epsilon \prec\prec c \text{ if } \mathcal{G}_1(a, b, c) \prec\prec p\delta \prec\prec e\end{aligned}$$

Therefore, $\mathcal{T}$ is tvs-G-cone continuous.

(⇒): Assume that $\mathcal{T}$ is tvs-G-cone continuous. Let $\epsilon > 0$. Then, for any $p \in \text{int}(P)$, there exists a $\delta > 0$ such that

$$\begin{aligned}\mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \prec\prec p\epsilon \text{ if } \mathcal{G}_1(a, b, c) \prec\prec p\delta &\implies \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \in p\epsilon - \text{int}(P) \text{ if } \mathcal{G}_1(a, b, c) \in p\delta - \text{int}(P) \\ &\implies \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \in p\epsilon - \text{int}(P) \text{ if } \mathcal{G}_1(a, b, c) \in p\delta - \text{int}(P) \\ &\implies \zeta_p(\mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c)) < \epsilon \text{ if } \zeta_p(\mathcal{G}_1(a, b, c)) < \delta \\ &\implies (\zeta_p \ o \ \mathcal{G}_2)(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } (\zeta_p \ o \ \mathcal{G}_1)(a, b, c) < \delta \\ &\implies M_{\mathcal{G}_2}(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } M_{\mathcal{G}_1}(a, b, c) < \delta.\end{aligned}$$

Hence, $\mathcal{T}$ is G-continuous. □

Theorem 3.8. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two tvs-G-cone metrics on $\Im$ and $\mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2)$ be a function. Then, $\mathcal{T}$ is tvs-G-cone continuous on $\Im$ iff $\mathcal{T}$ is tvs-G-cone sequentially continuous on $\Im$.

PROOF.
For $p \in \text{int}(P)$ in Y, the mapping $M_{\mathcal{G}_i} = \zeta_p o \mathcal{G}_i$ such that $i \in \{1, 2\}$ are the induced G-metrics on $\Im$. Then,

$$\begin{aligned}\mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2) \text{ is tvs-G-cone continuous on } \Im &\iff \mathcal{T} : (\Im, M_{\mathcal{G}_1}) \to (\Im, M_{\mathcal{G}_2}) \text{ is G-cone continuous on } \Im \\ &\iff \mathcal{T} : (\Im, M_{\mathcal{G}_1}) \to (\Im, M_{\mathcal{G}_2}) \text{ is G-cone sequentially continuous on } \Im \\ &\iff \mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2) \text{ is tvs-G-cone sequentially continuous on } \Im\end{aligned}$$

□

Theorem 3.9. Let $\mathcal{G}$ be a tvs-G-cone metric space on $\Im$. Then, a mapping $p : \Im \times \Im \to Y$ defined by $p(x, \xi) = \mathcal{G}(x, x, \xi) + \mathcal{G}(x, \xi, \xi)$, for all $x, \xi \in \Im$, is a tvs-cone-metric on $\Im$.

PROOF.
Let $\mathcal{G}$ be a tvs-G-cone metric space on $\Im$. Define a mapping $p : \Im \times \Im \to Y$ by $p(x, \xi) = \mathcal{G}(x, x, \xi) + \mathcal{G}(x, \xi, \xi)$, for all $x, \xi \in \Im$.

i. Clearly $p(x, \xi) \succ \theta$, for all $x, \xi \in \Im$ and

$$\begin{aligned}p(x, \xi) = \theta &\iff \mathcal{G}(x, x, \xi) + \mathcal{G}(x, \xi, \xi) = \theta \\ &\iff \mathcal{G}(x, x, \xi) = \theta \text{ and } \mathcal{G}(x, \xi, \xi) = \theta \\ &\iff x = \xi\end{aligned}$$

Therefore, $(\mathcal{T}1)$ holds.

ii. $(\mathcal{T}2)$ holds trivially since $\mathcal{G}$ is symmetric in its all three variables.

iii. $\mathcal{G}$ satisfies the inequality $\mathcal{G}(x,\xi,z) \preceq \mathcal{G}(x,a,a)+\mathcal{G}(a,\xi,z)$, for all $x,\xi,z,a \in \Im$. Therefore, for all $x,\xi,a \in \Im$,

$$\begin{aligned} p(x,\xi) = \mathcal{G}(x,x,\xi)+\mathcal{G}(x,\xi,\xi) &= \mathcal{G}(\xi,x,x)+\mathcal{G}(x,\xi,\xi), \quad (\text{since } \mathcal{G} \text{ is symmetric}) \\ &\preceq \mathcal{G}(\xi,a,a)+\mathcal{G}(a,x,x)+\mathcal{G}(x,a,a)+\mathcal{G}(a,\xi,\xi) \\ &= \mathcal{G}(x,x,a)+\mathcal{G}(x,a,a)+\mathcal{G}(\xi,a,a)+\mathcal{G}(a,\xi,\xi) \\ &= p(x,a)+p(a,\xi) \end{aligned}$$

Thus, p satisfies the condition $(\mathcal{T}3)$.

This shows that p is a tvs-cone-metric on $\Im$. □

Afterward, we show that fixed point theorems on tvs-G-cone metric spaces can be presented via G-metric spaces with the help of the scalarization function ζ_p.

Theorem 3.10. Suppose that $\mathcal{G}$ is a complete tvs-G cone metric and $M_{\mathcal{G}}$ be the induced G-metric on $\Im$. If $\mathcal{T}:\Im\to\Im$ is a mapping satisfying either

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)+n\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z) \tag{2}$$

or

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,x)+n\mathcal{G}(\xi,\xi,\mathcal{T}\xi)+r\mathcal{G}(z,z,\mathcal{T}z) \tag{3}$$

for all $x,\xi,z \in \Im$ where $0<l+m+n+r<1$, then $\mathcal{T}$ has a unique fixed point.

Proof.
For any $p \in \text{int}(P)$ in Y, we consider the function ζ_p. Then, by Theorem 3.4, $M_{\mathcal{G}}=\zeta_p o\mathcal{G}$ is a G-metric on $\Im$. Since $(\Im,\mathcal{G})$ is tvs-G-cone complete, then $(\Im,M_{\mathcal{G}})$ is also G-complete by the Theorem 3.6. Let $\mathcal{T}$ satisfies Condition 2. Then, for all $x,\xi,z \in \Im$, Lemma 2.1 implies if

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)+n\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)$$

then

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \zeta_p(l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)+n\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z))$$

Thus,

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq l\zeta_p(\mathcal{G}(x,\xi,z))+m\zeta_p(\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x))+n\zeta_p(\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi))+r\zeta_p(\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z))$$

Hence,

$$(\zeta_p o\mathcal{G})(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \leq l(\zeta_p o\mathcal{G})(x,\xi,z)+m(\zeta_p o\mathcal{G})(x,\mathcal{T}x,\mathcal{T}x)+n(\zeta_p o\mathcal{G})(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r(\zeta_p o\mathcal{G})(z,\mathcal{T}z,\mathcal{T}z)$$

Therefore,

$$M_{\mathcal{G}}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \leq lM_{\mathcal{G}}(x,\xi,z)+mM_{\mathcal{G}}(x,\mathcal{T}x,\mathcal{T}x)+nM_{\mathcal{G}}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+rM_{\mathcal{G}}(z,\mathcal{T}z,\mathcal{T}z)$$

This shows that $\mathcal{T}$ satisfies Condition 2.1 of Theorem 2.1 [20]. Since $(\Im,M_{\mathcal{G}})$ is a complete G-metric space, the existence and uniqueness of fixed point $\mathcal{T}$ follows from the Theorem 2.1 [20] in G-metric spaces. Consequently, $\mathcal{T}$ has a unique fixed point in $(\Im,\mathcal{G})$. Similarly, we can draw the conclusion if $\mathcal{T}$ satisfies Condition 3. □

Note 3.11. In particular, when we take $Y=E$, a real Banach space, the above theorem reduces to the theorem of Beg et al. [12].

Theorem 3.12. Suppose that $(\Im,\mathcal{G})$ is a complete tvs-G cone metric space and $M_{\mathcal{G}}$ is the induced

G-metric on $\Im$. If $\mathcal{T}$ is a self mapping on $\Im$ satisfying either of the following conditions

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \preceq \kappa\{\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x)\} \tag{4}$$

or

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \preceq \kappa\{\mathcal{G}(x,x,\mathcal{T}\xi)+\mathcal{G}(\xi,\xi,\mathcal{T}x)\} \tag{5}$$

for all $x,\xi \in \Im$ where $0<\kappa\leq \frac{1}{2}$, then $\mathcal{T}$ has a unique fixed point in $\Im$ and $\mathcal{T}$ is tvs-G-cone continuous on $\Im$.

Proof.
For any $p\in \operatorname{int}(P)$ in Y, we consider the function ζ_p. Then, by Theorem 3.4, $M_{\mathcal{G}}=\zeta_p o\mathcal{G}$ is a G-metric on $\Im$. If, for all $x,\xi\in\Im$, $\mathcal{T}$ satisfies Condition 4, then Lemma 2.1 gives

$$\begin{aligned}\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \preceq \kappa\{\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x)\} &\Longrightarrow \zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \zeta_p(\kappa\{\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x)\})\\ &\Longrightarrow \zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \kappa\zeta_p(\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x))\\ &\Longrightarrow \zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \kappa\{\zeta_p(\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi))+\zeta_p(\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x))\}\\ &\Longrightarrow (\zeta_p o\mathcal{G})(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \leq \kappa\{(\zeta_p o\mathcal{G})(x,\mathcal{T}\xi,\mathcal{T}\xi)+(\zeta_p o\mathcal{G})(\xi,\mathcal{T}x,\mathcal{T}x)\}\\ &\Longrightarrow M_{\mathcal{G}}(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \kappa\{M_{\mathcal{G}}(x,\mathcal{T}\xi,\mathcal{T}\xi)+M_{\mathcal{G}}(\xi,\mathcal{T}x,\mathcal{T}x)\}\end{aligned}$$

This shows that $\mathcal{T}$ satisfies Condition 2.49 [20] in G-metric space $(\Im,M_{\mathcal{G}})$. Since $(\Im,\mathcal{G})$ is tvs-G-cone complete, by the Theorem 3.6, $(\Im,M_{\mathcal{G}})$ is G-complete. Therefore, by the Theorem 2.8, $\mathcal{T}$ has a unique fixed point in $(\Im,M_{\mathcal{G}})$ and $\mathcal{T}$ is G-continuous. Hence, $\mathcal{T}$ has a unique fixed point in $(\Im,\mathcal{G})$ and $\mathcal{T}$ is tvs-G-cone continuous by the Theorem 3.7. If $\mathcal{T}$ satisfies the Condition 5, then the conclusion can be drawn in a similar way. □

Theorem 3.13. Let $\mathcal{G}$ be a complete tvs-G cone metric and $M_{\mathcal{G}}$ be the induced G-metric on $\Im$. If $\mathcal{T}$ is a self mapping on $\Im$ satisfying either of the conditions

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq \kappa\max\{\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x),\ \mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ \mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)\} \tag{6}$$

or

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq \kappa\max\{\mathcal{G}(x,x,\mathcal{T}x),\ \mathcal{G}(\xi,\xi,\mathcal{T}\xi),\ \mathcal{G}(z,z,\mathcal{T}z)\} \tag{7}$$

for all $x,\xi,z\in\Im$ where $0<\kappa\leq 1$, then $\mathcal{T}$ has a unique fixed point in $\Im$ and $\mathcal{T}$ is tvs-G-cone continuous on $\Im$.

Proof.
For any $p\in \operatorname{int}(P)$ in Y, we consider the function ζ_p. Then, by Theorem 3.4, $M_{\mathcal{G}}=\zeta_p o\mathcal{G}$ is a G-metric on $\Im$. Since $\mathcal{G}$ is a tvs-G-cone complete metric on $\Im$, thus $(\Im,M_{\mathcal{G}})$ is also G-complete. If, for all $x,\xi,z\in\Im$, $\mathcal{T}$ satisfies Condition 6, then applying Lemma 2.1, if

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq \kappa\max\{\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x),\ \mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ \mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)$$

then

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \zeta_p(\kappa\max\{\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x),\ \mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ \mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)\}$$

Thus,

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \kappa\max\{\zeta_p(\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)),\ \zeta_p(\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)),\ \zeta_p(\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z))\}$$

Hence,

$$(\zeta_p o\mathcal{G})(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \leq \kappa\max\{(\zeta_p o\mathcal{G})(x,\mathcal{T}x,\mathcal{T}x),\ (\zeta_p o\mathcal{G})(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ (\zeta_p o\mathcal{G})(z,\mathcal{T}z,\mathcal{T}z)\}$$

Therefore,

$$M_{\mathcal{G}}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \kappa \max\{M_{\mathcal{G}}(x,\mathcal{T}x,\mathcal{T}x),\ M_{\mathcal{G}}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ M_{\mathcal{G}}(z,\mathcal{T}z,\mathcal{T}z)\}$$

This shows that $\mathcal{T}$ satisfies Condition 2.19 [20] in the complete G-metric space $(\Im, M_{\mathcal{G}})$. Thus, Theorem 2.3 [20] ensures that $\mathcal{T}$ has a unique fixed point in $(\Im, M_{\mathcal{G}})$. Therefore, $\mathcal{T}$ has a unique fixed point in $(\Im,\mathcal{G})$. Moreover, Theorem 2.3 [20] shows that $\mathcal{T}$ is continuous in $(\Im, M_{\mathcal{G}})$. Since, by the Theorem 3.7, continuity of $\mathcal{T}$ in $(\Im, M_{\mathcal{G}})$ implies the continuity of $\mathcal{T}$ in $(\Im,\mathcal{G})$, thus $\mathcal{T}$ is tvs-G-cone continuous. We can prove the theorem similarly, if $\mathcal{T}$ satisfies Condition 7 in $(\Im,\mathcal{G})$. □

4. Conclusion

In this paper, we investigated the relationship between the vector-valued version and scalar-valued version of fixed point theorems of generalized cone-metric spaces and G-metric spaces. We showed a correspondence between G-metric and tvs-G cone metric with the help of a scalarization function defined on a locally convex Hausdorff topological vector space. If we take a real Banach space E instead of locally convex Hausdorff space X and P is the cone in E as defined in [8]. Then, all the results for X hold for G-cone metric spaces. Hence, these theorems extended some results of G-cone metric spaces and proved a correspondence between any G-cone metric space and the G-metric space. The remarkable point is that all of these are possible only because of the non-empty interior of P. Like Theorems 3.10 and 3.12, the equivalence between the non-negative scalar-valued version and vector-valued versions of these fixed point theorems can be proved easily.

Shortly, new generalized metric spaces are expected to be introduced, and studies on fixed point theory are expected to continue. We hope that the results of this paper will be helpful to researchers in this field for further research. Researchers may study the equivalence between the vector-valued and scalar-valued versions of fixed point results in new generalized metric spaces, getting inspired by the relations provided herein between the tvs-G cone metric spaces and G-metric spaces.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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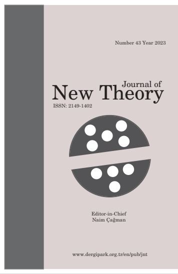

On the Orbit Problem of Free Lie Algebras

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Abstract — By operationalizing F_n as a free Lie Algebra of finite rank n, this work considers the orbit problem for F_n. The orbit problem is the following: given an element $u \in F_n$ and a finitely generated subalgebra H of F_n, does H meet the orbit of u under the automorphism group $AutF_n$ of F_n? It is proven that the orbit problem is decidable for finite rank n, $n \geqslant 2$. Furthermore, we solve a particular instance of the problem – i.e., whether H contains a primitive element of F_n. In addition, some applications are provided. Finally, the paper inquires the need for further research.

Keywords *Orbit, automorphism, free Lie algebras*

Mathematics Subject Classification (2020) 17B01, 17B40

1. Introduction

The orbit problem is one of the most studied algorithmic problems in algebra. The problem generally concerns a subalgebra H of an algebra F, the orbit of an element u of F under the action of a subgroup G of Aut$F$, and it is checked whether or not the subalgebra contains the orbit of a given element. Indeed, the orbit problem has been extensively studied in various algebraic structures, including groups, Lie algebras, and associative algebras. Computational group theory, in particular, has been a prominent field where the orbit problem has been investigated. Whitehead's [1] work in computational group theory proved the decidability of the orbit problem for free groups. This means that there exists an algorithm that can effectively determine whether the orbit of a given element under the action of a subgroup of the automorphism group lies within a subgroup of a free group. In [2, 3], the authors established similar results regarding the orbit problem of finitely generated subgroups. The problem was also studied for a cyclic subgroup of the automorphism group of a free group, e.g., [4, 5]. Furthermore, in [6], Kozen focused on the decidability of the orbit problem for infinite algebras. In 2011, Bahturin and Olshanskii [7] investigated if the subalgebra membership problem is decidable for free Lie algebras. The membership problem for free Lie algebras asks whether a given element belongs to a subalgebra of a free Lie algebra. This problem's decidability would imply a systematic and algorithmic approach to determine whether a given element belongs to a subalgebra. The results of this study determined that the subalgebra membership problem for free Lie algebras is, in fact, undecidable. This means that there is no general algorithm that can solve this problem for all cases. Consequently, the subalgebra membership problem for free Lie algebras remains an open and challenging research topic in algebraic computation. It is worth noting that even though the

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subalgebra membership problem is undecidable for free Lie algebras, specific cases may exist where the problem can be solved. In the context of the present paper, the orbit problem is a particular case of the membership problem. In the case of free Lie algebras, the orbit problem considers whether an automorphic image of a given Lie element is contained in a finitely generated subalgebra, while the membership problem asks whether a given element belongs to a free Lie algebra or a given finitely generated subalgebra.

This paper considers the orbit problem for finitely generated free Lie algebras. The technique used to solve the problem is inspired by the results of a similar problem in groups [3]. We give algorithms if an automorphic image of a given Lie element u is contained by a given finitely generated subalgebra H of a free Lie algebra F_n with finite rank n such that $n \geq 2$. Moreover, we prove that it is decidable whether or not a primitive element is contained by a given finitely generated subalgebra H.

2. Preliminaries

Let F_n be a free Lie algebra generated by $X = \{x_1, x_2, \cdots, x_n\}$ over a field K of characteristic 0. Denote by $U(F_n)$, the universal enveloping algebra of F_n, i.e., the free associative algebra with the same generating set X over the field K. There is the augmentation homomorphism $\varepsilon : U(F_n) \to K$ defined by $\varepsilon(x_i) = 0,\ i \in \{1, 2, \cdots, n\}$. Fox derivations [8,9]

$$\frac{\partial}{\partial x_i} : U(F_n) \to U(F_n), \quad i \in \{1, 2, \cdots, n\}$$

satisfy the following conditions for each $a, b \in K$ and $u, v \in U(F_n)$,

i. $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, (Kronecker delta)

ii. $\frac{\partial}{\partial x_i}(au + bv) = a\frac{\partial}{\partial x_i}(u) + b\frac{\partial}{\partial x_i}(v)$

iii. $\frac{\partial}{\partial x_i}(uv) = u\frac{\partial}{\partial x_i}(v) + \varepsilon(v)\frac{\partial}{\partial x_i}(u)$

such that $\frac{\partial}{\partial x_i}(a) = 0$, for any $a \in K$. A primitive element in F_n is an element belonging to a free generating set of F_n. Given an arbitrary element u in F_n, the rank of u, denoted by rank(u), is defined as the least number of free generators from X on which the image of u under any automorphism of F_n can depend. This definition is in line with the work of [9]. We introduce the left $U(F_n)$-module M_u generated by the elements $\frac{\partial u}{\partial x_i}$, for $i \in \{1, \cdots, n\}$. The algebra $U(F_n)$ as a left $U(F_n)$-module is a free cyclic module. It is known that any left ideal of a free associative algebra is a free module of unique rank [10]. We denote the rank of the module M_u as rank(M_u).

Lemma 2.1. [11] Let $u \in F_n$ and $\varphi \in \text{Aut}F_n$. Then, rank$(M_{\varphi(u)})$ =rank(M_u) =rank(u).

Lemma 2.2. [11] Let H be a subalgebra generated by $\{x_1, x_2, \cdots, x_r\}$, $1 \leq r < n$ and $u \in F_n$. If rank$(M_u) \leq r$, then there is an automorphism φ of F_n such that $\varphi(u) \in H$.

For an element u of F_n, we write $u = u(x_1, \cdots, x_k)$ if u depends on the generators $x_1, \cdots, x_k$. We use bracket notation $[u, v]$ to denote the Lie product of elements u and v of F_n. Lie monomials of F_n are defined in the usual way as non-zero Lie products of elements of X. The degree of a monomial is the length of this product. We call an element u of F_n is homogeneous if it is a linear combination of the monomials with the same degree. A subalgebra of F_n generated by a set Y is denoted by $\langle Y \rangle$.

Definition 2.3. [12] We define elementary transformations of F_n by one of the following transformations applied to X

i. A non-singular linear transformation is applied to X

ii. An element x of X is replaced by $x + u(x_1 \cdots, x_k)$ where u is an expression in the elements

$x_1, \cdots, x_k$ of $X\backslash\{x\}$

In [12], Cohn proved that every automorphism of a finitely generated free Lie algebra is a composition of elementary transformations.

Proposition 2.4. [12] Every automorphism φ of F_2 belongs to the general linear group $GL_2(K)$ and is defined by

$$\begin{aligned} \varphi \quad : \quad & x_1 \to \alpha x_1 + \beta x_2 \\ & x_2 \to \gamma x_1 + \delta x_2 \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in K$ and $\alpha\delta - \beta\gamma \neq 0$.

Proposition 2.5. [13] An endomorphism $\varphi : F_2 \to F_2$ defined as

$$\begin{aligned} \varphi \quad : \quad & x_1 \to \alpha x_1 + \beta x_2 \\ & x_2 \to \gamma x_1 + \delta x_2 \end{aligned}$$

is an automorphism if and only if

$$[\varphi(x_1), \varphi(x_2)] = k[x_1, x_2]$$

where $\alpha, \beta, \gamma, \delta \in K$, $k = \alpha\delta - \beta\gamma \neq 0$.

Thus, we can decide whether a given pair of elements of F_2 generates this algebra with this criterion.

3. The Orbit Problem

In this section, we discuss the decidability of the orbit problem and the existence of primitive elements in a finitely generated subalgebra H of a free Lie algebra F_n with finite rank $n \geq 2$. Decidability of the orbit problem means that there exists an algorithm or a systematic procedure that can determine whether the orbit of a given element under the action of a given subgroup of automorphisms belongs to a subalgebra. Firstly, we prove that for the case of rank 2, it is possible to decide whether the orbit of a given element $u \in F_2$ under the action of $\text{Aut}F_2$ is in H. This result is significant because it establishes a decision algorithm for a specific case of the orbit problem. In addition, we show that for rank 2, it is also possible to decide whether or not H contains a primitive element. Furthermore, we extend the results to larger ranks and provide algorithms to solve the orbit problem and determine the existence of primitive elements in H, for $n > 2$.

3.1. Case of Rank $n = 2$

Theorem 3.1. Given $u \in F_2$ and a finitely generated subalgebra H of F_2, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_2$.

Proof.
Let F_2 be a free Lie algebra generated by $\{x_1, x_2\}$ and H be a finitely generated subalgebra of F_2. Given $\varphi \in \text{Aut}F_2$ defined by $\varphi(x_1) = a$ and $\varphi(x_2) = b$. Since φ is an automorphism, the set $\{a, b\}$ freely generates F_2. For any element $u = u(x_1, x_2) \in F_2$,

$$\varphi(u(x_1, x_2)) = u(\varphi(x_1), \varphi(x_2)) = u(a, b)$$

Thus, if $u \in F_2$, then $\varphi(u) \in H$ if and only if $u(a, b) \in H$ for some free generating set $\{a, b\}$ of F_2. By Proposition 2.5,

$$[a, b] = [\varphi(x_1), \varphi(x_2)] = \lambda[x_1, x_2]$$

where $\lambda \in K\backslash\{0\}$. Hence, we obtain that there exists an automorphism φ such that $\varphi(u) \in H$ if and only if the following system admits a solution

$$[a, b] = \lambda[x_1, x_2]$$

and

$$u(a, b) = h$$

where $h \in H$, $\lambda \in K\backslash\{0\}$, and a and b are free generators of F_2. This completes the proof. □

Example 3.2. Given a subalgebra H generated by the subset $\{[x_1, x_2], x_2\}$ of F_2. Consider the element $u(x_1, x_2) = x_1 + [[x_1, x_2], x_1]$ of F_2. We find a solution to the system

$$[a, b] = \lambda[x_1, x_2]$$

and

$$u(a, b) = h$$

where $h \in H$, $\lambda \in K\backslash\{0\}$, and a and b are free generators of F_2. It implies

$$u(a, b) = a + [[a, b], a] = \alpha x_2 + \beta[[x_1, x_2], x_2] \tag{1}$$

where $\alpha, \beta \in K\backslash\{0\}$. By grading, $a = \alpha x_2$, and replacing a in Equation 1, $b = -\dfrac{\beta}{\alpha^2}x_1$ can be obtained. Then, by the equation

$$[a, b] = -\frac{\beta}{\alpha}[x_1, x_2]$$

a and b are free generators. Hence, by Theorem 3.1, there exists an automorphism φ such that $\varphi(u) \in H$.

Corollary 3.3. Let H be a subalgebra of F_2. It is decidable whether or not H contains a primitive element.

We consider a tuple element of F_2 rather than a single element in the following theorem.

Theorem 3.4. Let $u_1, u_2, \cdots, u_k \in F_2$ and $H_1, H_2, \cdots, H_k, H$, and K be subalgebras of F_2. The following problems are decidable

i. whether $\varphi(u_1) \in H_1, \cdots, \varphi(u_k) \in H_k$, for some $\varphi \in \text{Aut}F_2$

ii. whether $\varphi(K) \subseteq H$, for some $\varphi \in \text{Aut}F_2$

Proof.
Let $u_1, u_2, \cdots, u_k \in F_2$ and $H_1, H_2, \cdots, H_k, H$, and K be subalgebras of F_2.

i. We prove this statement as Theorem 3.1, by reduction to a system of equations. Let F_2 be a free Lie algebra generated by $\{x_1, x_2\}$. We consider the system

$$[a, b] = \alpha[x_1, x_2]$$

and

$$u_i(a, b) = h_i, \quad i \in \{1, 2, \cdots, k\}$$

where $h_1, \cdots, h_k \in H$, $\alpha \in K\backslash\{0\}$, and a and b are free generators of F_2. Clearly, if this system admits a solution, then $\varphi(u_i) \in H_i$, $i \in \{1, 2, \cdots, k\}$.

ii. This statement is a particular case of *i*, when $\{u_1, u_2, \cdots, u_k\}$ is a generating set of K and $H_1 = H_2 = \cdots = H_k = H$. □

3.2. Case of Rank $n > 2$

Theorem 3.5. Let u be a homogeneous element of F_n and H be a subalgebra of F_n. If H is a free factor of F_n or $\text{rank}H = 1$, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$.

PROOF.
Assume that H is a free factor of F_n, i.e., $F_n = H * G$ where $\text{rank}H = r$, $1 < r < n$, and G is a subalgebra of F_n. Let $u \in F_n$ and M_u be the left $U(F_n)-$module generated by $\frac{\partial u}{\partial x_i}$, $i \in \{1, \cdots, n\}$. By Lemma 2.1,

$$\text{rank}u = \text{rank}M_u = \text{rank}kM_{\varphi(u)}$$

for some $\varphi \in \text{Aut}F_n$. By [9], we can compute a minimum rank element v in the automorphic orbit

$$\text{Orb}(u) = \{\psi(u) : \psi \in \text{Aut}F_n\}$$

of u. If $\text{rank}v = r$, it is easily verified that $\phi(v) \in H$ for some automorphism ϕ of F_n by Lemma 2.2. Thus, $\phi(v) = \varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$. Assume that $H = \langle y \rangle$, for an element y of F_n. Given $u = \alpha u_1$ and $y = \beta y_1$ where $\alpha, \beta \in K\backslash\{0\}$ and $u_1, y_1 \in F_n$. If $\varphi(u) \in H$, then

$$\varphi(u) = \alpha\varphi(u_1) = \gamma y = \gamma\beta y_1$$

where $\gamma \in K$. It implies $\alpha = \gamma\beta$ and $\varphi(u_1) = y_1$. Therefore, we obtain $\varphi(u) \in H$ if and only if $\varphi(u_1) = y_1$, i.e., u_1 and y_1 are in each other's automorphic orbit if and only if $\varphi(u) \in H$. $\square$

We require the following technical result.

Theorem 3.6. Let $u \in F_n$. $A = \{x_1, x_2, \cdots, x_{n-1}, u\}$ is a free generating set of F_n if and only if $u = \alpha x_n + f(x_1, \cdots, x_{n-1})$ where $\alpha \in K\backslash\{0\}$ and $f(x_1, \cdots, x_{n-1})$ is an element of F_n depends on the free generators $x_1, \cdots, x_{n-1}$.

PROOF.
If A is a free generating set then the Jacobian matrix $J(A)$ is invertible over $U(F_n)$ by [14]. The Jacobian matrix

$$J(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \end{pmatrix}$$

can be reduced to

$$J(A)^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial u}{\partial x_n} \end{pmatrix}$$

by applying elementary transformations to its rows. Clearly, $J(A)$ is invertible if and only if $J(A)^*$ is invertible. Therefore, $\frac{\partial u}{\partial x_n}$ is an invertible element of $U(F_n)$. Since the only invertible elements of $U(F_n)$ are the elements of the field K, $\frac{\partial u}{\partial x_n}$ belongs to K. Thus, for a nonzero element $\alpha \in K$, $\frac{\partial u}{\partial x_n} = \alpha$ and the element u is of the form

$$\alpha x_n + f(x_1, \cdots, x_{n-1})$$

Conversely, if

$$u \in Kx_n + \langle x_1, \cdots, x_{n-1}\rangle$$

then $J(A)$ is invertible. Hence, A is a free generating set. $\square$

Proposition 3.7. Let $\{v_1, \cdots, v_{n-1}\}$ be a primitive subset of F_n. Then, there exists a set

$$A = \{w \in F_n \mid \{v_1, \cdots, v_{n-1}, w\} \text{ is a free generating set of } F_n\}$$

PROOF.
Let $\{v_1, \cdots, v_{n-1}\}$ be a primitive subset in F_n and φ be an automorphism of F_n defined by

$$\varphi : x_i \to v_i$$
$$x_n \to z$$

where $1 \le i \le n-1$ and $z \in F_n$. Then, $\{x_1, \cdots, x_{n-1},\ \varphi^{-1}(z)\}$ is a free generating set of F_n. This shows that

$$\varphi^{-1}(z) \in Kx_n + \langle x_1, \cdots, x_{n-1}\rangle$$

by Theorem 3.6. Thus, $z \in K\varphi(x_n) + \langle v_1, \cdots, v_{n-1}\rangle$, and we obtain a free generating set $\{v_1, \cdots, v_{n-1}, z\}$. Hence, we obtain a set A such that

$$A = K\varphi(x_n) + \langle v_1, \cdots, v_{n-1}\rangle = Kz + \langle v_1, \cdots, v_{n-1}\rangle, \quad z \in F_n$$

$\square$

Theorem 3.8. Given $u \in F_n$ and a subalgebra H of F_n. If $\operatorname{rank} H = n-1$, then it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \operatorname{Aut} F_n$.

PROOF.
Let H be a subalgebra of F_n generated by the set $\{v_1, \cdots, v_{n-1}\}$ freely and φ be an automorphism of F_n. Assume that $\varphi(x_i) = v_i$, for $1 \le i \le n-1$. Consider the set

$$A = \{w \in F_n \mid \{v_1, \cdots, v_{n-1}, w\} \text{ is a free generating set of } F_n\}$$

It implies $w = \varphi(x_n)$. By [15],

$$F_n/\langle x_n\rangle \cong \langle x_1, \cdots, x_{n-1}\rangle$$

and $\langle x_1, \cdots, x_{n-1}\rangle$ is a free Lie algebra. For $u \in F_n$,

$$u + \langle x_n\rangle \in \langle x_1, \cdots, x_{n-1}\rangle$$

Thus, it is obtained

$$u = \sum \alpha_s[\cdots[x_n, x_{i_1}], \cdots], x_{i_s}] + f(x_1, \cdots, x_{n-1})$$

where $x_{i_1}, x_{i_2}, \cdots, x_{i_s} \in \{x_1, \cdots, x_{n-1}\}$. We compute

$$\begin{aligned}\varphi(u) &= \sum \alpha_s[\cdots[\varphi(x_n), \varphi(x_{i_1})]\cdots], \varphi(x_{i_s})] + \varphi(f(x_1, \cdots, x_{n-1}))\\ &= \sum \alpha_s[\cdots[w, v_{i_1}]\cdots]v_{i_s}] + f(v_1, \cdots, v_{n-1})\end{aligned}$$

Therefore, we decide whether there exists some $w \in A$ such that $\varphi(u) \in H$. This is equivalent to deciding whether the equation

$$y = \sum \alpha_s[\cdots[w, v_{i_1}]\cdots]v_{i_s}] + f(v_1, \cdots, v_{n-1})$$

on the variables w and y has a solution in F_n with $w \in A$ and $y \in H$. $\square$

Corollary 3.9. Given $u \in F_n$ and a subalgebra H of F_n. Then, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_n$.

PROOF.
Let $K_i = \langle x_1, \cdots, x_i \rangle$, $i \in \{1, \cdots, n\}$. It is known that K_{i-1} is a subalgebra of K_i and by [15]

$$K_i/\langle x_i \rangle \cong K_{i-1}, \quad i \in \{2, \cdots, n\}$$

Therefore, for an element u of K_n, we have $u + \langle x_n \rangle \in K_{n-1}$. By the same way $u + \langle x_n \rangle + \langle x_{n-1} \rangle \in K_{n-2}$ and with consecutive applications $u + \langle x_n \rangle + \cdots + \langle x_{n-r} \rangle \in K_r$ are obtained. Hence,

$$u = \sum \alpha_{n_s}[\cdots[x_n, y_{n_1}]\cdots], y_{n_s}] + \cdots + \sum \alpha_{(n-r)_s}[\cdots[x_{n-r}, y_{(n-r)_1}], \cdots], y_{(n-r)_s}] + f(x_1, \cdots, x_r)$$

where $y_{j_1}, \cdots, y_{j_s} \in \{x_1, \cdots, x_{j-1}\}$ and $j = n-r, \cdots, n$. Let H be a subalgebra of F_n freely generated by a set $\{v_1, \cdots, v_r\}$, $r < n$, and φ be an automorphism of F_n. Assume that $\varphi(x_i) = v_i$, for $1 \leq i \leq r$, and $\varphi(x_i) = w_i$, for $r+1 \leq i \leq n$. Hence, we compute

$$\varphi(u) = \sum \alpha_{n_s}[\cdots[w_n, v_{n_1}]\cdots]v_{n_s}] + \cdots + \sum \alpha_{(n-r)_s}[\cdots[w_{n-r}, v_{(n-r)_1}]\cdots], v_{(n-r)_s}] + f(v_1, \cdots, v_r)$$

Therefore, we decide whether the equation

$$y = \sum \alpha_{n_s}[\cdots[w_n, v_{n_1}], \cdots], v_{n_s}] + \cdots + \sum \alpha_{(n-r)_s}[\cdots[w_{n-r}, v_{(n-r)_1}], \cdots], v_{(n-r)_s}] + f(v_1, \cdots, v_r)$$

has a solution on the variables $w_{n-r}, \cdots w_n$ of F_n and $y \in H$. □

Corollary 3.10. Let H be a subalgebra of F_n. Then, it is decidable whether or not H contains a primitive element.

Theorem 3.11. Let $u_1, u_2, \cdots, u_m \in F_n$ and H and G be subalgebras of F_n. The following problems are decidable

i. whether, $\varphi(u_1), \cdots, \varphi(u_m) \in H$, for some $\varphi \in \mathrm{Aut}F_n$

ii. whether, $\varphi(G) \subseteq H$, for some $\varphi \in \mathrm{Aut}F_n$

PROOF.
Let $u_1, u_2, \cdots, u_m \in F_n$ and H and G be subalgebras of F_n.

i. Let

$$y_i = \sum \alpha^{(i)}_{n_s}[\cdots[w_n, v_{n_1}]\cdots], v_{n_s}] + \cdots + \sum \alpha^{(i)}_{(n-r)_s}[\cdots[w_{n-r}, v_{(n-r)_1}]\cdots], v_{(n-r)_s}] + f_i(v_1, \cdots, v_r)$$

such that $i \in \{1, \cdots, m\}$. If this equation on the variables $w_n, \cdots w_{n-r}, y_1, \cdots, y_m$ has a solution in F_n, then $\varphi(u_i) = y_i \in H$ by Corollary 3.9.

ii. This statement is a particular case of *i*, when $\{u_1, u_2, \cdots, u_k\}$ is a generating set of G. Then, it is decidable whether or not $\varphi(u_i) \in H$, for some $\varphi \in \mathrm{Aut}F_n$. Hence, $\varphi(G) \subseteq H$. □

Example 3.12. Let $H = \langle x_1, x_2 \rangle$ be a subalgebra of F_n. Given $u = [x_3, x_2] \in F_n$. It is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_n$. By [8],

$$\frac{\partial u}{\partial x_2} = x_3, \quad \frac{\partial u}{\partial x_3} = -x_2, \quad \text{and} \quad \mathrm{rank}u = 2$$

Since M_u is left $U(F_n)-$module generated by $\frac{\partial u}{\partial x_2}$ and $\frac{\partial u}{\partial x_3}$, $\mathrm{rank}M_u = 2$. Given an automorphism φ of F_n defined by

$$\begin{aligned} \varphi \ : \ & x_i \to x_i \\ & x_2 \to x_2 + x_1 \\ & x_3 \to x_3 - x_1 \end{aligned}$$

where $i \notin \{2,3\}$. Then,

$$\varphi(u) = [x_3 - x_1, x_2 + x_1] = [x_3, x_2] + [x_3, x_1] - [x_1, x_2]$$

We calculate

$$\frac{\partial \varphi(u)}{\partial x_1} = x_3 + x_2, \quad \frac{\partial \varphi(u)}{\partial x_2} = x_3 - x_1, \quad \text{and} \quad \frac{\partial \varphi(u)}{\partial x_3} = -x_2 - x_1$$

Since

$$\frac{\partial \varphi(u)}{\partial x_3} = -\frac{\partial \varphi(u)}{\partial x_1} + \frac{\partial \varphi(u)}{\partial x_2}$$

then

$$\text{rank}u = \text{rank}M_u = \text{rank}M_{\varphi(u)} = 2$$

By [11], $\varphi(u)$ belongs to a subalgebra which has rank 2. It seems that $\varphi(u)$ involves the generator x_3, therefore, $\varphi(u) \notin H = \langle x_1, x_2 \rangle$. However, it is verified that $\sigma(\varphi(u)) \in H$ for some automorphism σ of F_n by Lemma 2.2. Therefore,

$$\sigma(\varphi(u)) = \sigma([x_3 - x_1, x_2 + x_1]) = [\sigma(x_3) - \sigma(x_1), \sigma(x_2) + \sigma(x_1)] \in H$$

for some automorphism σ of F_n. Hence, solving the equation

$$[\sigma(x_3) - \sigma(x_1), \sigma(x_2) + \sigma(x_1)] = [x_1, x_2]$$

for an appropriate automorphism σ,

$$\sigma(x_3) - \sigma(x_1) = x_1$$

and

$$\sigma(x_2) + \sigma(x_1) = x_2$$

Choose $\sigma(x_1) = x_3$ and $\sigma(x_i) = x_i, i \notin \{1,2,3\}$. Then, $\sigma(x_3) = x_3 + x_1$ and $\sigma(x_2) = x_3 + x_2$. Hence, for the automorphism σ of F_n, we obtain $\sigma(\varphi(u)) \in H$.

Example 3.13. Let $H = \langle x_1 + [x_2, x_3], x_2, x_3, x_4 \rangle$ be a subalgebra of F_n and $u = [x_1, x_2] + [x_3, x_4] \in F_n$. It is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$. Given an automorphism φ of F_n defined by

$$\begin{aligned} \varphi : x_1 &\to x_1 + [x_2, x_3] \\ x_i &\to x_i \end{aligned}$$

such that $i \neq 1$. Therefore,

$$\begin{aligned} \varphi(u) &= [x_1, x_2] + [[x_2, x_3], x_2] + [x_3, x_4] \\ &= [x_1 + [x_2, x_3], x_2] + [x_3, x_4] \end{aligned}$$

Clearly, $\varphi(u)$ belongs to a subalgebra generated by $\{x_1 + [x_2, x_3], x_2, x_3, x_4\}$.

4. Conclusion

In this study, the orbit problem for free Lie algebras of finite rank n such that $n \geqslant 2$ is solved. In this context, we prove that for a given element u and a subalgebra H of F_n, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$. In addition, we get the decidability of the problem for given primitive elements of free Lie algebras of finite rank. Furthermore, in future research, the decidability of the orbit problem for relatively free Lie algebras can be investigated.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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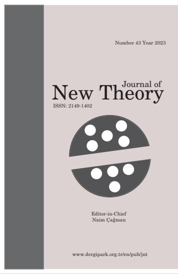

Magnetized Strange Quark Models in Lyra Theory

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Abstract — In this study, the behavior of magnetized strange quark matter (MSQM) distribution in Lyra theory was investigated for homogeneous anisotropic Bianchi III, locally rotationally symmetric (LRS) Bianchi I, and Kantowski-Sachs universe models. We have used the equations of state, anisotropy, and linearly varying deceleration parameters to obtain the exact solutions of field equations in Lyra theory. When switching from the anisotropic universe model to the isotropic universe model, the magnetic field was not observed in the LRS Bianchi I universe. Besides, the graphs of the dynamic quantities obtained for each universe model were analyzed in detail. Finally, we inquire whether further research should be conducted.

Keywords *Magnetized strange quark matter, Lyra theory, LRS Bianchi I, Kantowski-Sachs, Bianchi III*

Mathematics Subject Classification (2020) 83C05,83C15

1. Introduction

Recent experiments and observations show that the universe is accelerating and expanding [1,2]. What caused this accelerating expansion is still unknown. However, scientists have some assumptions about the causes of the accelerating expansion of the universe. The strongest of these assumptions can be counted as dark matter - dark energy. Einstein published General Relativity theory in 1916. General Relativity theory is one of the most important theories explaining the relationship between matter and space-time geometry. General Relativity Theory tries to explain the universe's structure on a large scale. However, General Relativity Theory falls short of explaining the universe's accelerating expansion. Edwin Hubble proved with observations that the universe is accelerating and expanding. After this proof, other theories that could be alternatives to General Relativity Theory were put forward. Among these alternative theories are Lyra, Brans-Dicke, $f(T)$, $f(G)$, and $f(R,T)$, etc. These alternative theories are reduced to the General Relative theory in special cases. Lyra theory, one of these alternative theories, was put forward in 1951 [3]. Lyra is a modified theory created by adding the term containing the scalar field to the left side of the field equations in theory.

There are many articles in the literature on both Lyra theory and magnetized strange quark matter dispersion. Some of these can be summarized as follows. Katore and Kapse [4] have investigated magnetized dark energy model behaviors in Lyra theory for axially symmetric space-time. Mishra et al. [5] have researched 5D Kaluza-Klein universe with magnetized anisotropic fluid matter distribution in Lyra theory. Katore and Hatkar [6] have studied magnetized anisotropic dark energy for Kaluza-Klein universe model in the context of Lyra manifold. Anisotropic dark energy and massive scalar

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field for Bianchi VI_0 metric was investigated by Ram and Verma [7] within the framework of Lyra theory. The holographic Ricci dark energy universe model was analyzed by Das and Bharali [8] within the framework of Lyra theory in high-dimensional metric. The perfect fluid matter distribution was studied by Raushan et al. [9] within the framework of Lyra theory for a homogeneous isotropic Friedmann-Robertson-Walker (FRW) universe. Naidu et al. [10] have investigated massive scalar field and perfect fluid for Bianchi I universe model in Lyra manifold. Aktaş and Aygün [11] have researched magnetized strange quark matter (MSQM) distribution for FRW universe model in $f(R,T)$ theory. Aygün et al. [12] have investigated scalar field solutions both Lyra geometry and Riemannian geometry for Marder space-time. Kalkan and Aktaş [13] have studied MSQM for 5D Kaluza-Klein metric in $f(R,T)$ theory. The behavior of MSQM was investigated by Kalkan et al. [14] within the framework of $f(R,T)$ theory in inhomogeneous anisotropic space-time. In addition, the physical properties of MSQM for the Bianchi VI_0 metric $f(R,T)$ were examined in theory by Kalkan and Aktaş [15]. Güdekli et al. [16] have researched strange stars for Krori Barua space-time in $f(T,\tau)$ theory. Tsallis dark energy universe model was explored by Khan et al. [17] in the Saez–Ballester theory of gravity for the locally rotationally symmetric (LRS) Bianchi V metric. Can and Güdekli [18] have analyzed for conservative and non-conservative $f(R,T)$ models. Abebe et al. [19] have studied viscous fluid matter distribution for Bianchi V universe model. The role of the jerk parameter in $f(R,T)$ gravitation theory were analyzed by Tiwari et al. [20].

Our motivation in this study is to investigate the space-time geometry of magnetized strange quark matter in Lyra theory, one of the alternative gravitational theories, for Bianchi III, LRS Bianchi I, and Kantowski-Sachs metrics.

This article is organized as follows: In Section 2, the field equations in Lyra theory, the general form of Bianchi III, LRS Bianchi I, and Kantowski-Sachs metrics in spherical coordinates, and the energy-momentum tensor of MSQM are provided. In Section 3, solutions are obtained for each metric using the deceleration parameter, the anisotropy parameter, and the equation of state for the MSQM distribution. In Section 4, the solutions obtained for each metric are analyzed in detail both mathematically and physically, and their graphs are drawn. The final section discusses the need for further research.

2. Field Equations in Lyra Theory

The field equations in Lyra theory can be written as follows [3, 21]:

$$R_{ik} - \frac{1}{2}g_{ik}R + \frac{3}{2}\left(\phi_i\phi_k - \frac{1}{2}g_{ik}\phi_j\phi^j\right) = T_{ik} \tag{1}$$

Here, R_{ik} is Ricci tensor, R is Ricci scalar, g_{ik} is metric tensor, T_{ik} is energy momentum tensor, and ϕ_i is the displacement vector, defined by

$$\phi_i = (0,0,0,\beta(t)) = \delta_i^4 \cdot \beta(t) \tag{2}$$

where $i \in \{1,2,3,4\}$. The general form of homogeneous anisotropic Bianchi III, LRS Bianchi I, and Kantowski-Sachs metric in spherical coordinates (r,θ,Φ,t) is as follows:

$$ds^2 = -dt^2 + A^2dr^2 + B^2(d\theta^2 + K_l^2(\theta)d\Phi^2) \tag{3}$$

where the metric coefficients A and B are functions of t. Moreover, $K_l^2(\theta)$ is a function defined as follows [22]:

$$K_l^2(\theta) = \begin{cases} \sinh^2\theta, & \text{if } l=-1 \text{ Bianchi III model} \\ \theta^2, & \text{if } l=0 \text{ LRS Bianchi I model} \\ \sin^2\theta, & \text{if } l=1 \text{ Kantowski - Sachs model} \end{cases}$$

The energy-momentum tensor for magnetized quark matter distribution is

$$T_{ik} = (\rho + p)\, u_i u_k + p g_{ik} + (2u_i u_k + g_{ik})\, \frac{h^2}{2} - h_i h_k \tag{4}$$

where p, ρ, and h^2 denote pressure, energy density, and magnetic field, respectively [23,24]. Moreover, u_i and h_i denote the 4-velocity and magnetic field vector, respectively. Besides, h_i and u^i have the relations $h_i u^i = 0$ and $u_i u^i = -1$. Due to the condition $h_i u^i = 0$, the magnetic field is selected in the radial direction.

Kinematic quantities for the given metric; spatial volume (V), Hubble parameter (H), expansion scalar (θ), shear scalar (σ^2), deceleration parameter (q), and mean anisotropy parameter are defined as follows:

$$V = a^3 = AB^2 \tag{5}$$

$$H = \frac{\dot{a}}{a} = \frac{\dot{A}}{3A} + \frac{2\dot{B}}{3B} \tag{6}$$

$$\theta = \frac{\dot{A}}{A} + \frac{2\dot{B}}{B} \tag{7}$$

$$\sigma^2 = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2 \tag{8}$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = \frac{-3AB(\ddot{A}B + 2A\ddot{B}) + 2(A\dot{B} - \dot{A}B)^2}{(A\dot{B} + \dot{A}B)^2} \tag{9}$$

and

$$AP = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i}{H} - 1\right)^2 = \frac{6\left(\dot{A}B - A\dot{B}\right)^2}{\left(2\dot{A}B + A\dot{B}\right)^2} \tag{10}$$

Here, the dot represents the derivative with respect to time and H_i is component of Hubble parameter such that $H_1 = \frac{\dot{A}}{A}$ and $H_2 = H_3 = \frac{\dot{B}}{B}$.

3. Magnetized Strange Quark Matter Solutions for Bianchi III, LRS Bianchi I, and Kantowski-Sachs Metrics

From Equations 1, 3, and 4, we obtain the field equations in Lyra theory as follows:

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}}{B} - \frac{K_l''}{K_l\, B^2} + \frac{3}{4}\beta^2 = -p + \frac{1}{2}h^2 \tag{11}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} - \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -p - \frac{1}{2}h^2 \tag{12}$$

$$\frac{2\dot{A}\dot{B}}{B} + \frac{\dot{B}^2}{B} - \frac{K_l''}{K_l\, B^2} - \frac{3}{4}\beta^2 = \rho + \frac{1}{2}h^2 \tag{13}$$

where $K_l'' = \frac{d^2 K_l}{d\theta^2}$. As can be seen from Equations 11-13, we have three equations with six unknowns A, B, p, ρ, β^2, and h^2. We need three additional equations such as anisotropy parameter, deceleration parameter, and equation of state to solve the system of equations exactly.

Firstly, we can take the deceleration parameter as an additional equation. Deceleration parameter is known as one of the important parameters showing whether the universe is accelerating or not. In many studies, the deceleration parameter was taken as constant. However, in studies in recent years, the deceleration parameter is taken depending on time. One of the deceleration parameters taken depending on time, especially the one in linear form, has become prominent in recent years.

The deceleration parameter in linear form was proposed by Akarsu and Dereli [25]. The deceleration parameter in linear form is

$$q = -kt + m - 1 \tag{14}$$

where k and m are constants. From the solution of this equation, the metric potential A is obtained as follows:

$$A = c_1 \frac{e^{\left(\frac{\tanh^{-1}\left(\frac{kt-m}{\sqrt{m^2+6c_2k}}\right)}{\sqrt{m^2+6c_2k}}\right)^6}}{B^2} \tag{15}$$

such that c_1 and c_2 are integral constants. Without loss of generality, we can take $c_1 = 1$ and $c_2 = 0$. In this situation, we get the metric potential A as

$$A = \frac{\left(\frac{kt}{kt-2m}\right)^{\frac{3}{m}}}{B^2} \tag{16}$$

Secondly, we can use the anisotropy parameter as an additional equation. The anisotropy parameter is a parameter that gives information about the isotropy of the universe. It can take values between 0 and 1. If the anisotropy parameter is zero, then the universe is said to be isotropic. The anisotropy parameter is defined as follows:

$$\frac{\sigma}{\theta} = \xi \tag{17}$$

where ξ is constant and $0 \leq \xi \leq 1$. From Equations 7, 8, and 17, we get metric potential B

$$B = c_3 \left(\frac{t}{kt-2m}\right)^{\frac{\sqrt{3}\xi+1}{m}} \tag{18}$$

where c_3 is integral constant. From Equations 16 and 18, we have

$$A = \frac{(-1)^{\frac{3}{m}}}{c_3^2} \left(\frac{kt-2m}{t}\right)^{\frac{2\sqrt{3}\xi-1}{m}} \tag{19}$$

Finally, we can use the equation of state for strange quark matter as an additional equation. The equation of state for strange quark matter is defined as follows:

$$p = \frac{\rho - 4B_c}{3} \tag{20}$$

where B_c is a bag constant [26]. If Equations 18 and 19 are substituted in Equations 11-13, then the pressure

$$p = -\frac{2\left(\sqrt{3}\,\xi - 2\right)(kt-m+3)}{(kt-2m)^2 t^2} - \frac{K_l''}{2c_3^2 K_l}\left(\frac{kt-2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} - 2B_c \tag{21}$$

the energy density,

$$\rho = -\frac{6\left(\sqrt{3}\,\xi - 2\right)(kt-m+3)}{(kt-2m)^2 t^2} - \frac{3K_l''}{2c_3^2 K_l}\left(\frac{kt-2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} - 2B_c \tag{22}$$

the magnetic field,

$$h^2 = \frac{12\xi\,(kt-m+3)\sqrt{3}}{t^2\,(kt-2m)^2} - \frac{K_l''}{c_3^2 K_l}\left(\frac{kt-2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} \tag{23}$$

and the displacement vector component β^2

$$\beta^2 = \frac{4(4kt - 9\xi^2 - 4m + 8)}{t^2(kt-2m)^2} - \frac{4K_l''}{3c_3^2 K_l}\left(\frac{kt-2m}{kt}\right)^{\frac{2-3\xi}{m}} + \frac{8}{3}B_c \tag{24}$$

As can be seen from Equations 21-24, pressure, energy density, magnetic field, and displacement vector depend on $K_l(\theta)$. According to the states of $K_l(\theta)$, we obtain the solutions in Bianchi III, LRS Bianchi I, and Kantowski-Sachs universe models as follows:

i. If $K_l(\theta) = \sinh\theta$, then solutions are obtained in the Bianchi III universe model for magnetized strange quark matter distribution:

Pressure

$$p = -\frac{2\left(\sqrt{3}\,\xi - 2\right)(kt - m + 3)}{(kt - 2m)^2 t^2} - \frac{1}{2c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} - 2B_c \tag{25}$$

energy density

$$\rho = -\frac{6\left(\sqrt{3}\,\xi - 2\right)(kt - m + 3)}{(kt - 2m)^2 t^2} - \frac{3}{2c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} - 2B_c \tag{26}$$

magnetic field

$$h^2 = \frac{12\xi\,(kt - m + 3)\sqrt{3}}{t^2\,(kt - 2m)^2} - \frac{1}{c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} \tag{27}$$

and displacement vector component

$$\beta^2 = \frac{4(4kt - 9\xi^2 - 4m + 8)}{t^2(kt - 2m)^2} - \frac{4}{3c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2-3\xi}{m}} + \frac{8}{3}B_c \tag{28}$$

ii. If $K_l(\theta) = \theta$, then solutions are obtained in the LRS Bianchi I universe model for magnetized strange quark matter distribution:

Pressure

$$p = -\frac{2\left(\sqrt{3}\,\xi - 2\right)(kt - m + 3)}{(kt - 2m)^2 t^2} - 2B_c \tag{29}$$

energy density

$$\rho = -\frac{6\left(\sqrt{3}\,\xi - 2\right)(kt - m + 3)}{(kt - 2m)^2 t^2} - 2B_c \tag{30}$$

magnetic field

$$h^2 = \frac{12\xi\,(kt - m + 3)\sqrt{3}}{t^2\,(kt - 2m)^2} \tag{31}$$

and displacement vector component

$$\beta^2 = \frac{4(4kt - 9\xi^2 - 4m + 8)}{t^2(kt - 2m)^2} + \frac{8}{3}B_c \tag{32}$$

iii. If $K_l(\theta) = \sin\theta$, then solutions are obtained in the Kantowski-Sachs universe model for magnetized strange quark matter distribution:

Pressure

$$p = -\frac{2\left(\sqrt{3}\,\xi - 2\right)(kt - m + 3)}{(kt - 2m)^2 t^2} + \frac{1}{2c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} - 2B_c \tag{33}$$

energy density

$$\rho = -\frac{6\left(\sqrt{3}\,\xi - 2\right)(kt - m + 3)}{(kt - 2m)^2 t^2} + \frac{3}{2c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} - 2B_c \tag{34}$$

magnetic field

$$h^2 = \frac{12\xi\left(kt - m + 3\right)\sqrt{3}}{t^2\left(kt - 2m\right)^2} + \frac{1}{c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2+2\sqrt{3}\xi}{m}} \tag{35}$$

and displacement vector component

$$\beta^2 = \frac{4(4kt - 9\xi^2 - 4m + 8)}{t^2(kt - 2m)^2} + \frac{4}{3c_3^2}\left(\frac{kt - 2m}{kt}\right)^{\frac{2-3\xi}{m}} + \frac{8}{3}B_c \tag{36}$$

4. Results and Discussions

From Equations 6-8, 10, 18, and 19, some of the kinematic quantities are obtained as follows:

Hubble parameter

$$H = \frac{2}{t(2m - kt)}$$

expansion scalar

$$\theta = \frac{6}{t(2m - kt)}$$

shear scalar

$$\sigma^2 = \frac{36\xi^2}{t^2\left(2m - kt\right)^2}$$

and mean anisotropy parameter

$$AP = 18\xi^2$$

As can be seen from Equations 29-36, there are singularities at points $t = 0$ and $t = \frac{2m}{k}$, for all three universe models (Bianchi III, LRS Bianchi I, and Kantowski-Sachs). In order to be valid these solutions, it must be $t \neq 0$ and $t \neq \frac{2m}{k}$. At these points, kinematic quantities have singularities. Moreover, c_3, k, and m must be non zero. For $t \to 0$, Hubble parameter, expansion scalar, and shear scalar approach infinity, while they approach zero, for $t \to \infty$. The metric potentials A and B increase with time. Figure of pressure and energy density are presented in Figures 1 and 2. As can be observed from Figures 1 and 2, pressure and energy density decrease with time.

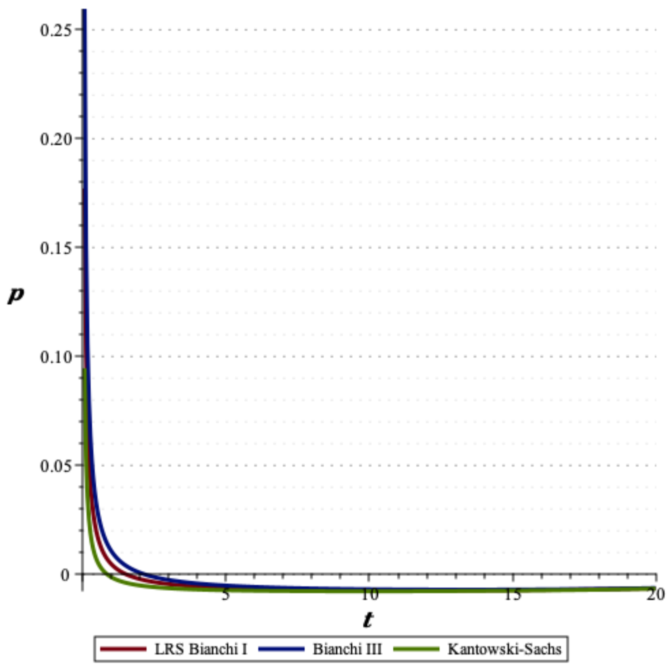

Figure 1. Pressure-time variation

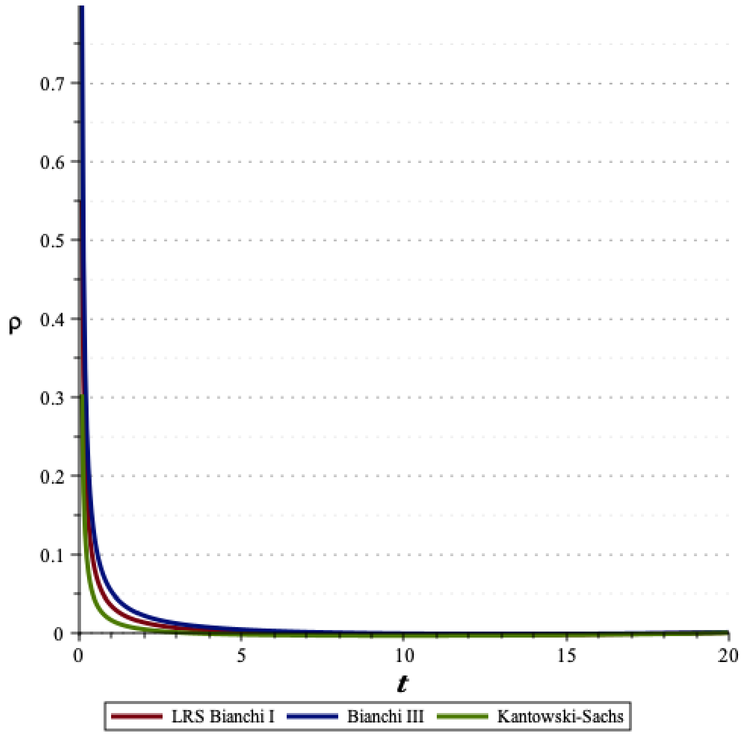

Figure 2. Energy density-time variation

The graphs of variation of magnetic field and displacement vector component with respect to time are provided in Figures 3 and 4. When Figures 3 and 4 are investigated, it is observed that the magnetic field and displacement vector component also decrease with time.

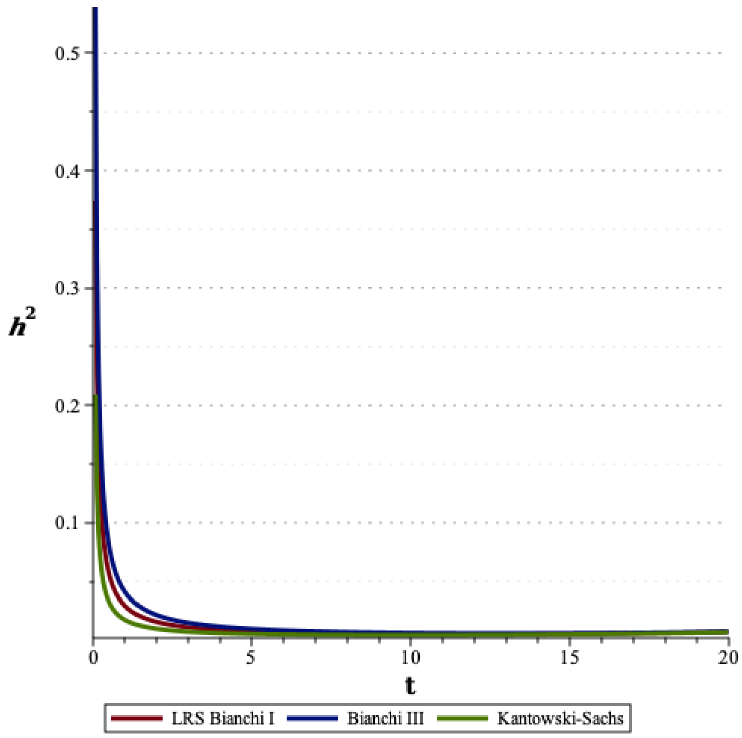

Figure 3. Magnetic field-time variation

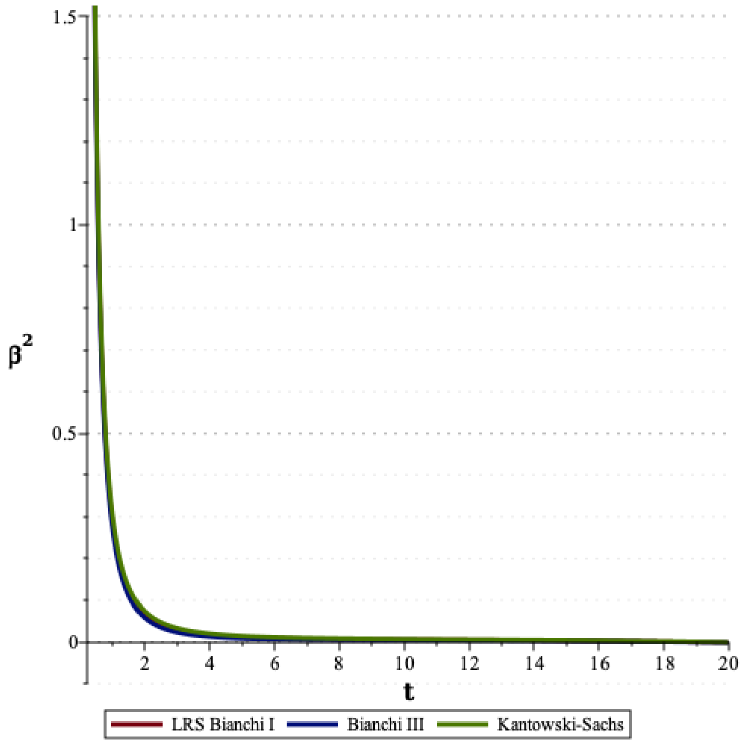

Figure 4. Displacement vector component-time variation

As can be observed from Equation 31 obtained in case of LRS Bianchi I metric, the magnetic field becomes zero when $\xi = 0$. $\xi = 0$ indicates that the universe model is isotropic. If we switch from the homogeneous anisotropic universe model to the homogeneous isotropic universe model, the magnetic field disappears. This shows that the source of the magnetic field may be the anisotropy of the universe. In other words, the anisotropy of the universe plays an important role in the formation of the magnetic field.

5. Conclusion

In this article, the behavior of MSQM distribution in homogeneous anisotropic Bianchi III, LRS Bianchi I, and Kantowski-Sachs metrics was investigated within the framework of Lyra manifolds. While investigating the solutions, the time-dependent linear deceleration parameter and anisotropy parameter were used. In future studies, investigating the space-time geometry of the MSQM distribution using other alternative gravity theories, such as $f(G)$, $f(Q)$, and $f(Q,T)$, or taking different deceleration parameters is worth studying.

Author Contributions

All the authors contributed equally to this work. This paper is derived from the first author's master's thesis supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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