Number 45 Year 2023

Journal of New Theory

ISSN: 2149-1402

www.dergipark.org.tr/en/pub/jnt

Journal of New Theory (abbreviated as J. New Theory or JNT) is a mathematical journal focusing on new mathematical theories or their applications to science.

J. New Theory is an international, peer-reviewed, and open-access journal.

JNT was founded on 18 November 2014, and its first issue was published on 27 January 2015.

Language: As of 2023, JNT accepts contributions in American English only.

Editor-in-Chief: Naim Çağman

E-mail: journalofnewtheory@gmail.com

APC: JNT incurs no article processing charges.

Review Process: Blind Peer Review

DOI Numbers: The published papers are assigned DOI numbers.

The policy of Screening for Plagiarism: JNT accepts submissions for pre-review only if their reference-excluded similarity rate is a **maximum of 30%**.

Publication Ethics: The governance structure of Journal New Theory and its acceptance procedures are transparent and designed to ensure the highest quality of published material. JNT adheres to the international standards developed by the Committee on Publication Ethics (COPE).

Journal Boards

Oktay Muhtaroğlu
Department of Mathematics, Tokat Gaziosmanpaşa University, Tokat, Türkiye
oktay.muhtaroglu@gop.edu.tr
Sturm Liouville Theory, Boundary Value Problem, Spectrum Functions, Green's Function, Differential Operator Equations

Muhammad Irfan Ali
Department of Mathematics, COMSATS Institute of Information Technology Attock, Attock, Pakistan
mirfanali13@yahoo.com
Soft Sets, Rough Sets, Fuzzy Sets, Intuitionistic Fuzzy Sets, Pythagorean Fuzzy Sets

Muhammad Riaz
Department of Mathematics, Punjab University, Quaid-e-Azam Campus, Lahore, Pakistan
mriaz.math@pu.edu.pk
Topology, Fuzzy Sets and Systems, Machine Learning, Computational Intelligence, Linear Diophantine Fuzzy Set

Pabitra Kumar Maji
Department of Mathematics, Bidhan Chandra College, Asansol, Burdwan (W), West Bengal, India.
pabitra_maji@yahoo.com
Soft Sets, Fuzzy Soft Sets, Intuitionistic Fuzzy Sets, Fuzzy Sets, Decision Making Problems

Kalyan Mondal
Department of Mathematics, Jadavpur University, Kolkata, West Bengal, India
kalyanmathematic@gmail.com
Neutrosophic Sets, Rough Sets, Decision Making, Similarity Measures, Neutrosophic Soft Topological Space

Sunil Jacob John
Department of Mathematics, National Institute of Technology Calicut, Calicut Kerala, India
sunil@nitc.ac.in
Topology, Fuzzy Mathematics, Rough Sets, Soft Sets, Multisets

Layout Editors

Production Editor

CONTENTS

Journal of New Theory

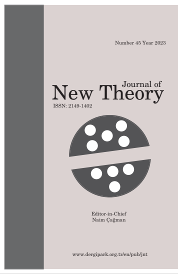
ISSN: 2149-1402

45 (2023) 1-17

Journal of New Theory

https://dergipark.org.tr/en/pub/jnt

Open Access



Analyzing Stability and Data Dependence Notions by a Novel Jungck-Type Iteration Method

Yunus Atalan[1], **Esra Erbaş**[2]

Article Info

Received: 12 Jul 2023

Accepted: 27 Oct 2023

Published: 31 Dec 2023

doi:10.53570/jnt.1326344

Research Article

Abstract — Finding the ideal circumstances for a mapping to have a fixed point is the fundamental goal of fixed point theory. These criteria can also be used for the structure under investigation. One of this theory's most well-known theorems, Banach's fixed point theorem, has been expanded adopting various methods, making it possible to conduct numerous research studies. Thanks to the Jungck-Contraction Theorem, which has been proven through commutative mappings, many fixed point theorems have been obtained using classical fixed point iteration methods and newly defined methods. This study aims to investigate the convergence, stability, convergence rate, and data dependency of the new four-step fixed-point iteration method. Nontrivial examples are also included to support some of the results herein.

Keywords *Jungck-contraction principle, fixed point, iteration method, stability, data dependence*

Mathematics Subject Classification (2020) 47H09, 47H10

1. Introduction

The solutions of some problems in mathematics can be reduced to finding the solution of an equation that can be written as $f(x)-x=0$ for a function f satisfying the appropriate conditions. The points x, which are the solutions of equations of this type, are called the fixed points of the f function. With its extensive range of applications in fields such as differential and integral equations [1], approximation theory and game theory [2], fixed point theory has emerged as a captivating and fundamental subject within nonlinear analysis. Moreover, this theory yields fruitful outcomes across various domains, including optimization [3], physics [4], economics [5], and medicine [6]. Consequently, fixed point theory has remained a dynamic research area, drawing significant attention from researchers in the past fifty years, due to its foundation in analysis and topology, and continues to generate a vibrant body of literature.

Geometrically, the definition of a fixed point means the point on the $y=x$ line. The theorems formulated to establish the existence and uniqueness of a fixed point are commonly referred to as fixed-point theorems. One of the most famous existence and uniqueness theorems is the theorem, which was proved by Banach [7] in 1922 and called the Banach Contraction Principle. While this theorem states that a contraction mapping defined on itself in complete metric spaces will have a unique fixed point, it also offers a method called iteration in order to reach this unique fixed point.

[1]yunusatalan@aksaray.edu.tr (Corresponding Author); [2]esraa_erbas@hotmail.com

[1,2]Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, Aksaray, Türkiye

The main idea in the studies on the iterations mentioned above is to determine under which conditions the sequences obtained from these algorithms, which are formed by using certain mapping classes, converge to the fixed point, the equivalence of the convergence behavior with other methods. Furthermore, testing the convergence speed, analysis of the data dependency, and stability of the iteration methods are considered one of the main targets of these studies.

Since the Picard iteration used in the Banach Contraction Principle cannot converge to the fixed point of non-expansive mappings, this problem has been tried to be overcome by defining new iteration methods. As a result of this approach, many iteration methods have been brought to the literature and studies on the definition of new iterations have continued to maintain their popularity today.

While the iterative sequence converges to the fixed point of a certain mapping class, it may not converge to the fixed point of another mapping class. This problem has revealed the concept of equivalence of convergence for iteration methods, and whether the iteration methods in the literature and the newly defined iteration methods are equivalent in terms of convergence have been examined in various spaces [8, 9]. A large literature has been created as a result of trying to determine which of the two iteration methods, which are shown to be equivalent in terms of convergence, converges to the fixed point of the relevant mapping more rapidly [10, 11].

After showing that the iterative sequence converges to the fixed point of the used mapping, it can be shown that the new sequence to be obtained by using another mapping called the approximation operator for this iteration method is also convergent to the fixed point of the approximate operator. In such a case, the questions of how close the fixed points of both mappings are to each other and how to calculate this distance bring up the concept of data dependency. There are many studies on different kinds of constructs on whether fixed point iteration methods are data dependent [12–15].

Mathematically, the concept of stability can be thought of as the fact that small changes to be applied to the structure studied cannot disrupt the functioning of it. In this context, many studies have been carried out on the stability of fixed-point iteration methods. The approach here is; instead of the sequence to be obtained from the iteration method used, calculation errors, rounding errors, etc., it can be characterized as the convergence of the new sequence to the fixed point of the mapping, although another sequence is obtained for various reasons [16, 17].

Because the mapping used in the Banach Contraction Principle is contraction, researchers have sought to obtain various generalizations of this theorem for different types of mappings [18–20]. One of the notable generalizations of this theorem was made by Jungck [21] in 1976 using commutative mappings.

In this paper, a Jungck-type four-step iteration method is introduced and the convergence and stability of the sequence obtained from this method, which is constructed using a certain type of mapping, under favorable conditions are investigated. Moreover, the convergence behavior of the new iterative sequence is compared with other Jungck-type iterative sequences in the literature. In addition, the concept of data dependence is analyzed and some of the results mentioned here are supported by numerical examples.

2. Preliminaries

Jungck [21] expressed one of the noteworthy generalizations of the Banach Contraction Principle using commutative mappings as follows:

Theorem 2.1. Let $f_1, f_2 : \mathfrak{B} \to \mathfrak{B}$ be two functions satisfy in the following conditions, for all $b_1, b_2 \in \mathfrak{B}$:

i. (f_1, f_2) is a commutative pair of map *ii.* f_2 is continuous

iii. $f_1(\mathfrak{B}) \subsetneq f_2(\mathfrak{B})$

iv. $\wp(f_1b_1, f_1b_2) \leq t\wp(f_2b_1, f_2b_2)$ such that $t \in [0,1]$

in which $\mathfrak{B}$ is complete metric space with respect to metric function $\wp$. In this case f_1 and f_2 have a unique common fixed point $p \in \mathfrak{B}$.

The condition specified by *iv* in this theorem is known as the Jungck Contraction mapping, and when taking f_2 as a unit function, it corresponds to the classical Banach Contraction Principle. Building upon this theorem, Jungck introduced the following iteration method:

Assume that $\mathfrak{B}$ be a Banach space, $\mathcal{C}$ any set, and $S,T:\mathfrak{B}\to\mathcal{C}$ satisfy $T(\mathcal{C}) \subseteq S(\mathcal{C})$.

$$Sx_{n+1} = Tx_n \tag{1}$$

This is referred to as the Jungck iteration method. If $S = I$ and $\mathcal{C} = \mathfrak{B}$ in Equation 1, the classical Picard iteration method [22] is obtained. Many researchers have worked on this method introduced by Jungck and have obtained many fixed point theorems by rewriting the classical iteration methods in Jungck type. Some of the works done with this approach are as follows for $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\mu_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty} \subseteq [0,1]$:

Jungck-SP iteration method [23] is defined as under:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n) Sy_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n) Sz_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n) Sx_n + \gamma_n Tx_n \end{cases} \tag{2}$$

Jungck-CR iteration is defined by [24]:

$$\begin{cases} Su_{n+1} = (1-\alpha_n) Sv_n + \alpha_n Tv_n \\ Sv_n = (1-\beta_n) Tu_n + \beta_n Tw_n \\ Sw_n = (1-\gamma_n) Su_n + \gamma_n Tu_n \end{cases} \tag{3}$$

Furthermore, if $\{\alpha_n\}_{n=0}^{\infty} = 0$ in Equation 3, the following Jungck-type Agarwal iteration method is obtained [25]:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n) Tx_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n) Sx_n + \beta_n Tx_n \end{cases} \tag{4}$$

If $\{\alpha_n\}_{n=0}^{\infty} = 0$ and $\{\beta_n\}_{n=0}^{\infty} = 1$ in Equation 3, the following Jungck-type Sahu iteration method is obtained [25]:

$$\begin{cases} Sx_{n+1} = Ty_n \\ Sy_n = (1-\gamma_n) Sx_n + \gamma_n Tx_n \end{cases} \tag{5}$$

The Jungck-Khan iteration method is defined as follows [26]:

$$\begin{cases} Su_{n+1} = (1-\alpha_n-\beta_n) Su_n + \alpha_n Tv_n + \beta_n Tu_n \\ Sv_n = (1-b_n-c_n) Su_n + b_n Tw_n + c_n Tu_n \\ Sw_n = (1-a_n) Su_n + a_n Tu_n \end{cases} \tag{6}$$

The new four-step iteration method that we have defined inspired by the literature on the iteration methods given above is as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, y_n + \alpha_n T y_n \\ y_n = (1-\beta_n)\, T x_n + \beta_n T z_n \\ z_n = (1-\gamma_n)\, w_n + \gamma_n T w_n \\ w_n = (1-\mu_n)\, x_n + \mu_n T x_n \end{cases} \tag{7}$$

The following iteration method is obtained by rewriting the iteration method given by Equation 7 in Jungck-type:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)\, Sy_n + \alpha_n T y_n \\ Sy_n = (1-\beta_n)\, T x_n + \beta_n T z_n \\ Sz_n = (1-\gamma_n)\, Sw_n + \gamma_n T w_n \\ Sw_n = (1-\mu_n)\, Sx_n + \mu_n T x_n \end{cases} \tag{8}$$

The following statements hold for the Jungck-type iteration methods given above for $n \in \{0, 1, 2...\}$, taking $S = I$ and $\mathcal{C} = \mathfrak{B}$:

Remark 2.2. *i.* The classical SP iteration method [27] can be obtained from the iteration method provided by Equation 2;

ii. The classical CR iteration method [28] can be obtained from the iteration method provided by Equation 3;

iii. The classical Agarwal-S [29] and classical Sahu [30] iteration methods can be obtained from the iteration methods provided by Equation 4 and Equation 5, respectively.

iv. If $\mu_n = 0$ is chosen in the iteration method provided by Equation 7, the classical CR iteration [28] is obtained.

v. If $\mu_n = 0$ is chosen in the iteration method provided by Equation 8, the Jungck-CR iteration method provided by Equation 3 is obtained.

Some auxiliary theorems and definitions have been given to obtain the main results in the following:

Definition 2.3. [24] Suppose that $\mathfrak{B} \neq \emptyset$ and $S, T : \mathfrak{B} \to \mathfrak{B}$ are mappings.

i. $b \in \mathfrak{B}$ is referred to as the common fixed point of T and S if $b = Tb = Sb$

ii. $c \in \mathfrak{B}$ is referred to as the coincidence point of T and S if $c = Tb = Sb$

iii. The pair of maps (S, T) is referred to as commuting if $TSb = STb$ for all $b \in \mathfrak{B}$

iv. The pair of maps (S, T) is referred to as weakly compatible if $TSb = STb$ whenever $Tb = Sb$ for some $b \in \mathfrak{B}$.

Definition 2.4. [31] Let $\left\{\Theta_n^{(i)}\right\}_{n=0}^{\infty}$ be two sequences with $\lim_{n\to\infty} \Theta_n^{(i)} = \Theta_i$, $i \in \{1, 2\}$. Then, it is said that $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{\left\|\Theta_n^{(1)} - \Theta_1\right\|}{\left\|\Theta_n^{(2)} - \Theta_2\right\|} = 0$$

Definition 2.5. [31] Assume that $\left\{\Theta_n^{(i)}\right\}_{n=0}^{\infty}$ and $\left\{\Pi_n^{(i)}\right\}_{n=0}^{\infty}$ are four sequences for $i \in \{1, 2\}$ such that $\Pi_n^{(i)} \geq 0$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} \Theta_n^{(i)} = \Theta^*$, and $\lim_{n\to\infty} \Pi_n^{(i)} = 0$. Suppose that the following error estimates are available:

$$(\forall n \in \mathbb{N}) \quad \left\|\Theta_n^{(i)} - \Theta^*\right\| \leq \Pi_n^{(i)} \quad i \in \{1, 2\}$$

If $\left\{\Pi_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{\Pi_n^{(2)}\right\}_{n=0}^{\infty}$ (in the sense of Definition 2.4), then it is said that $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ converges to Θ^* faster than $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$.

Definition 2.6. [32] Assume that $S,T:\mathcal{C}\to\mathfrak{B}$ are mappings satisfy $T(\mathcal{C})\subseteq S(\mathcal{C})$ and $p=Tb=Sb$. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ attained by $Sx_{n+1}=f(T,x_n)$ converges to p for any $x_0\in\mathcal{C}$. Let $\{Sy_n\}_{n=0}^{\infty}\subsetneq\mathfrak{B}$ be an arbitrary sequence and set $\epsilon_n=d\left(Sy_{n+1},f\left(T,y_n\right)\right),\ n\in\{0,1,2,\dots\}$. Then $f\left(T,x_n\right)$ will be called (S,T)-stable if and only if $\lim\limits_{n\to\infty}\epsilon_n=0$ implies that $\lim\limits_{n\to\infty}Sy_n=p$.

Definition 2.7. [33] Assume that (X,d) is a metric space and the maps $S,T:X\to X$ satisfy the following conditions for all $x,y\in X$:

i. $T(X)\subseteq S(X)$

ii. for non-negative λ and μ satisfying the condition $\lambda+\mu<1$,

$$d\left(Tx,Ty\right)\leq\lambda d\left(Sx,Sy\right)+\mu\left(\frac{d\left(Sx,Tx\right).d\left(Sy,Ty\right)}{1+d\left(Sx,Sy\right)}\right)\tag{9}$$

iii. $S\left(X\right)$ is complete sub-space of X

Then, the mappings S and T have a coincidence point. In addition, if S and T are weakly compatible, these mappings have a unique common fixed point.

Lemma 2.8. [34] Suppose that $\left\{\rho_n^{(k)}\right\}_{n=0}^{\infty}$ are two sequences such that $\rho_n^{(k)}\geq 0$, for each $n\in\mathbb{N}$ and for $k\in\{1,2\}$. Assume that $\lim\limits_{n\to\infty}\rho_n^{(2)}=0$ and $\mu\in(0,1)$. If $\rho_{n+1}^{(1)}\leq\mu\rho_n^{(1)}+\rho_n^{(2)}$, then $\lim\limits_{n\to\infty}\rho_n^{(1)}=0$.

Lemma 2.9. [35] Assume that $\{a_n\}_{n=1}^{\infty}$ is a non negative real sequence and there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ satisfying the following condition:

$$a_{n+1}\leq(1-\mu_n)a_n+\mu_n\eta_n$$

where $\mu_n\in$ (0,1) such that $\sum\limits_{n=1}^{\infty}\mu_n=\infty$ and $\eta_n\geq 0$. Then, the following inequality holds:

$$0\leq\lim_{n\to\infty}\sup a_n\leq\lim_{n\to\infty}\sup\eta_n$$

Definition 2.10. [36] Suppose that $(\mathfrak{B},d)$ is a metric space and $A_1:\mathfrak{B}\to\mathfrak{B}$ is operator with fixed point p and there exist a fixed point iteration method that converges to p. $A_2:\mathfrak{B}\to\mathfrak{B}$ is referred to as approximate operator of A_1 for a suitable $\mu>0$ if $d\left(A_1x,A_2x\right)\leq\mu$, for each $x\in\mathfrak{B}$.

3. Main Results

In this part of the study, the concept of convergence is analyzed using the new iteration method. It is also shown that this result can be obtained independently of the condition applied to the control sequences. In addition, the theorems such as stability, convergence speed, and data dependence are proved.

Theorem 3.1. Assume that X is a Banach space, Y an arbitrary set and $S,T:Y\to X$ satisfy the condition given by Inequality 9 with $p=Tx_p=Sx_p$. Suppose that $S(Y)$ is a complete subset of X such that $T(Y)\subseteq S(Y)$ and $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence given by Equation 8 with $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. If $Y=X$ and S and T are weakly compatible then, p is a unique common fixed point of S and T.

Proof.

By using Equation 8, Inequality 9, and $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty},\{\gamma_n\}_{n=0}^{\infty},\{\mu_n\}_{n=0}^{\infty}\subseteq[0,1]$, in the following inequalities are obtained:

$$\begin{aligned}\|Sx_{n+1}-p\| &= \|(1-\alpha_n)\,Sy_n+\alpha_n Ty_n-p\| \\ &\le (1-\alpha_n)\,\|Sy_n-Tx_p\| + \alpha_n\,\|Ty_n-Tx_p\| \\ &\le (1-\alpha_n)\,\|Sy_n-Sx_p\| \\ &\quad + \alpha_n\left\{\lambda\,\|Sy_n-Sx_p\|+\mu\left(\frac{\|Sy_n-Ty_n\|\,.\,\|Sx_p-Tx_p\|}{1+\|Sy_n-Sx_p\|}\right)\right\} \\ &\le (1-\alpha_n)\,\|Sy_n-Sx_p\| + \lambda\alpha_n\,\|Sy_n-Sx_p\| \\ &= [1-\alpha_n\,(1-\lambda)]\,\|Sy_n-Sx_p\|\end{aligned}$$

and

$$\begin{aligned}\|Sy_n-p\| &= \|(1-\beta_n)\,Tx_n+\beta_n Tz_n-p\| \\ &\le (1-\beta_n)\,\|Tx_n-Tx_p\| + \beta_n\,\|Tz_n-Tx_p\| \\ &\le (1-\beta_n)\left\{\lambda\,\|Sx_n-Sx_p\|+\mu\left(\frac{\|Sx_n-Tx_n\|\,.\,\|Sx_p-Tx_p\|}{1+\|Sx_n-Sx_p\|}\right)\right\} \\ &\quad + \beta_n\left\{\lambda\,\|Sz_n-Sx_p\|+\mu\left(\frac{\|Sz_n-Tz_n\|\,.\,\|Sx_p-Tx_p\|}{1+\|Sz_n-Sx_p\|}\right)\right\} \\ &= \lambda\,(1-\beta_n)\,\|Sx_n-Sx_p\| + \lambda\beta_n\,\|Sz_n-Sx_p\|\end{aligned}$$

Similarly,

$$\|Sz_n-p\| \le [1-\gamma_n\,(1-\lambda)]\,\|Sw_n-Sx_p\|$$

and

$$\|Sw_n-p\| \le [1-\mu_n\,(1-\lambda)]\,\|Sx_n-Sx_p\|$$

If these inequalities are nested and necessary simplifications are made considering that $[1-\gamma_n\,(1-\lambda)] \le 1$ and $[1-\mu_n\,(1-\gamma)] \le 1$, then it is attained that

$$\|Sx_{n+1}-p\| \le \lambda[1-\alpha_n\,(1-\lambda)]\,\|Sx_n-p\| \tag{10}$$

If induction is applied to the last inequality, then

$$\|Sx_{n+1}-p\| \le \lambda^{n+1}\prod_{i=0}^{n}\,[1-\alpha_i(1-\lambda)]\,\|Sx_0-p\| \tag{11}$$

By using $1-x \le e^{-x}$, for all $x \in [0,1]$, it is obtained in the following inequality:

$$\begin{aligned}\|Sx_{n+1}-p\| &\le \lambda^{n+1}\,\|Sx_0-p\|\prod_{i=0}^{n} e^{-(1-\lambda)\alpha_i} \\ &= \lambda^{n+1}\,\|Sx_0-p\|\,e^{-(1-\lambda)\sum\limits_{i=0}^{n}\alpha_i}\end{aligned}$$

If the limit for the last inequality as $n \to \infty$ is taken, it can be observed that $Sx_n \to p$. It will be demonstrated that S and T have a unique common fixed point like p. Suppose the pair (S,T) has another coincidence point, say q. Therefore,

$$\begin{aligned}0 \le \|p-q\| = \|Tx_p-Tx_q\| &\le \lambda\,(\|Sx_p-Sx_q\|)+\mu\left(\frac{\|Sx_p-Tx_p\|\,.\,\|Sx_q-Tx_q\|}{1+\|Sx_p-Sx_q\|}\right) \\ &= \lambda\,\|Sx_p-Sx_q\|\end{aligned}$$

which implies that $p = q$, that is S and T have a unique coincidence point. Since S and T are weakly compatible and $Sx_p = Tx_p = p$, then $TTp = TTx_p = TSx_p = STx_p$ signifies $Tp = Sp$. Thus, Tp is the unique coincidence point of (S,T), then $Tp = p$. As a result, the (S,T) pair of maps have a unique common fixed point. □

In the next theorem, it is proven that the result of Theorem 3.1 can be derived without the $\sum_{n=0}^{\infty} \alpha_n = \infty$ condition:

Theorem 3.2. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Then $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Moreover, if $Y = X$ and S and T are weakly compatible, then p is a unique common fixed point of S and T.

Proof.
Since $[1 - \alpha_n (1 - \lambda)] \leq 1$, from Inequality 10, it is attained the following inequality

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1} \|Sx_0 - p\|$$

Given that $\lambda < 1$ and taking the limit in the last inequality, one can obtain $Sx_n \to p$ as $n \to \infty$. It can be observed from Theorem 3.1 that p is the unique common fixed point of the T and S. □

Theorem 3.3. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Suppose that iterative sequence $\{Sx_n\}_{n=0}^{\infty}$ given by Equation 8 converges to p with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, it is (S,T)-stable.

Proof.
Assume that $\varepsilon_n = \|Sa_{n+1} - f(T, a_n)\|$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Besides, $\{Sa_n\}_{n=0}^{\infty} \subsetneq X$ is any sequence obtained from the following equation:

$$\begin{cases} Sa_{n+1} = (1-\alpha_n) Sb_n + \alpha_n Tb_n \\ Sb_n = (1-\beta_n) Ta_n + \beta_n Tc_n \\ Sc_n = (1-\gamma_n) Sd_n + \gamma_n Td_n \\ Sd_n = (1-\mu_n) Sa_n + \mu_n Ta_n \end{cases} \tag{12}$$

It will be shown that $\lim\limits_{n\to\infty} Sa_n = p$. By using Inequality 9 and Equation 12, the following inequalities are obtained:

$$\begin{aligned} \|Sd_n - p\| &= \|(1-\mu_n) Sa_n + \mu_n Ta_n - p\| \\ &\leq (1-\mu_n) \|Sa_n - p\| + \mu_n \|Ta_n - Tx_p\| \\ &\leq (1-\mu_n) \|Sa_n - p\| + \mu_n \left\{ \lambda \|Sa_n - Sx_p\| + \mu \left(\frac{\|Sa_n - Ta_n\| . \|Sx_p - Tx_p\|}{1 + \|Sa_n - Sx_p\|} \right) \right\} \\ &\leq [1 - \mu_n (1-\lambda)] \|Sa_n - p\| \end{aligned} \tag{13}$$

and

$$\begin{aligned} \|Sc_n - p\| &= \|(1-\gamma_n) Sd_n + \gamma_n Td_n - p\| \\ &\leq (1-\gamma_n) \|Sd_n - p\| + \gamma_n \|Td_n - Tx_p\| \\ &\leq (1-\gamma_n) \|Sd_n - p\| + \gamma_n \left\{ \lambda \|Sd_n - p\| + \mu \left(\frac{\|Sd_n - Td_n\| . \|Sx_p - Tx_p\|}{1 + \|Sd_n - Sx_p\|} \right) \right\} \\ &\leq [1 - \gamma_n (1-\lambda)] \|Sd_n - p\| \end{aligned} \tag{14}$$

Similarly,

$$\|Sb_n - p\| \leq (1-\beta_n)\,\lambda\,\|Sa_n - p\| + \beta_n\lambda\,\|Sc_n - p\| \tag{15}$$

Substituting Inequality 13 in Inequality 14 and Inequality 14 in Inequality 15, and making the necessary simplifications considering that $[1-\mu_n(1-\lambda)] \leq 1$, $[1-\gamma_n(1-\lambda)] \leq 1$, and

$$\|Sb_n - p\| \leq \lambda\,\|Sa_n - p\| \tag{16}$$

In addition,

$$\begin{aligned}
\|Sa_{n+1} - p\| &\leq \|Sa_{n+1} - f(T,a_n)\| + \|f(T,a_n) - p\| \\
&\leq \varepsilon_n + \|Sa_{n+1} - p\| \\
&\leq \varepsilon_n + (1-\alpha_n)\,\|Sb_n - p\| + \alpha_n\,\|Tb_n - p\| \\
&\leq \varepsilon_n + (1-\alpha_n)\,\|Sb_n - p\| + \alpha_n\left\{\lambda\,\|Sb_n - p\| + \mu\left(\frac{\|Sb_n - Tb_n\|\,.\,\|Sx_p - Tx_p\|}{1+\|Sb_n - Sx_p\|}\right)\right\} \\
&= \varepsilon_n + [1-\alpha_n(1-\lambda)]\,\|Sb_n - p\|
\end{aligned} \tag{17}$$

Substituting Inequality 16 in Inequality 17,

$$\|Sa_{n+1} - p\| \leq \varepsilon_n + \lambda[1-\alpha_n(1-\lambda)]\,\|Sa_n - p\|$$

Hence, from Lemma 2.8, it is obtained that $\lim\limits_{n\to\infty} Sa_n = p$.

Conversely, assume that $\lim\limits_{n\to\infty} Sa_n = p$. It will be shown that $\lim\limits_{n\to\infty} \varepsilon_n = 0$:

$$\begin{aligned}
\varepsilon_n &= \|Sa_{n+1} - f(T,a_n)\| \\
&\leq \|Sa_{n+1} - p\| + \|f(T,a_n) - p\| \\
&\leq \|Sa_{n+1} - p\| + (1-\alpha_n)\,\|Sb_n - p\| + \alpha_n\,\|Tb_n - p\|
\end{aligned} \tag{18}$$

By using similar operations in Inequalities 13-17, from Inequality 18,

$$\varepsilon_n \leq \|Sa_{n+1} - p\| + \lambda[1-\alpha_n(1-\lambda)]\,\|Sa_n - p\|$$

If the limit for the above inequality is taken, then it is obtained that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. □

Example 3.4. Assume that $X = \mathbb{R}$ is Banach space, $Y = [0,1]$, and $S, T : Y \to X$ are defined by $Sx = \frac{1}{5}\sin 2x$ and $Tx = \frac{1}{10}\sin^2 x$ respectively. It can be observed that S and T are pairs of maps satisfying Inequality 9 and having unique common fixed point $p = 0$. If the iteration method given by Equation 8 is rewritten for S and T with $\alpha_n = \beta_n = \gamma_n = \mu_n = \frac{1}{n+1}$:

$$\begin{cases}
x_{n+1} = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2y_n + \frac{1}{2(n+1)}\sin^2 y_n\right] \\
y_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{2(n+1)}\right)\sin^2 x_n + \frac{1}{2(n+1)}\sin^2 z_n\right] \\
z_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2w_n + \frac{1}{2(n+1)}\sin^2 w_n\right] \\
w_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2x_n + \frac{1}{2(n+1)}\sin^2 x_n\right]
\end{cases}$$

It can be observed from Theorem 3.1 that the $\{Sx_n\}_{n=0}^{\infty}$ sequence to be obtained from the above equation converges to $p = 0$. If the sequence $\{Sa_n\}_{n=0}^{\infty}$ is chosen as $Sa_n = (\frac{1}{n+5})$, then $\lim\limits_{n\to\infty} |Sx_n - Sa_n| = 0$. Hence, $\{Sa_n\}_{n=0}^{\infty}$ is approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$. If the iteration method given by Equation

12 is rewritten using S and T:

$$\begin{cases} a_{n+1} = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2b_n + \frac{1}{2(n+1)}\sin^2 b_n\right] \\ b_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{2(n+1)}\right)\sin^2 a_n + \frac{1}{2(n+1)}\sin^2 c_n\right] \\ c_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2d_n + \frac{1}{2(n+1)}\sin^2 d_n\right] \\ d_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2a_n + \frac{1}{2(n+1)}\sin^2 a_n\right] \end{cases}$$

From the above equality, it is obtained that

$$a_{n+1} = \frac{1}{2}\sin^{-1}\left[\begin{array}{c} \frac{1}{2}\left(\frac{n}{n+1}\right)^2\sin^2 a_n + \frac{n}{2(n+1)^2}\sin^2\left\{\frac{1}{2}\sin^{-1}\left(\frac{n}{n+1}\right)u_1 + \frac{1}{2(n+1)}\sin^2\left(\frac{1}{2}\sin^{-1}u_1\right)\right\} \\ +\frac{1}{2(n+1)}\sin^2\left\{\frac{1}{2}\sin^{-1}\left\{\frac{n}{2(n+1)}\sin^2 a_n + \frac{1}{2(n+1)}\sin^2\left(\frac{1}{2}\sin^{-1}u_2\right)\right\}\right\} \end{array}\right]$$

in which $u_1 = \left(\frac{n}{n+1}\right)\sin 2a_n + \frac{1}{2(n+1)}\sin^2 a_n$ and $u_2 = \left(\frac{n}{n+1}\right)\mathrm{u}_1 + \frac{1}{2(n+1)}\sin^2\left(\frac{1}{2}\sin^{-1}\mathrm{u}_1\right)$. If $\varepsilon_n = |Sa_{n+1} - f(T, a_n)|$, then $\lim_{n\to\infty}\left|\left(\frac{1}{n+6}\right) - f(T, a_n)\right| = 0$. As a result, $\lim\limits_{n\to\infty}\varepsilon_n = 0$.

Theorem 3.5. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ obtained from the iteration method given by Equation 8 and the sequence $\{Su_n\}_{n=0}^{\infty}$ obtained from the Jungck-CR iteration method given by Equation 3 under the condition $\alpha_1 < \alpha_n \leq 1$, where $x_0 = u_0 \in Y$. In this case, $\{Sx_n\}_{n=0}^{\infty}$ has a better convergence rate with respect to $\{Su_n\}_{n=0}^{\infty}$.

Proof.
From Inequality 11, it is attained that

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1}\prod_{i=0}^{n}[1 - \alpha_i(1-\lambda)]\,\|Sx_0 - p\| \tag{19}$$

In addition, if similar steps are taken as in the proof of Theorem 3.1 for the Jungck-CR iteration method, then

$$\|Su_{n+1} - p\| \leq [1 - \alpha_n(1-\lambda)]\,\|Su_n - p\|$$

If induction is applied to the above inequality, then

$$\|Su_{n+1} - p\| \leq \prod_{i=0}^{n}[1 - \alpha_i(1-\lambda)]\,\|Su_0 - p\| \tag{20}$$

If the assumption $\alpha_1 < \alpha_n \leq 1$ is applied to Inequalities 19 and 20, then

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1}[1 - \alpha_1(1-\lambda)]^{n+1}\,\|Sx_0 - p\|$$

and

$$\|Su_{n+1} - p\| \leq [1 - \alpha_1(1-\lambda)]^{n+1}\,\|Su_0 - p\|$$

Denote

$$a_n = \lambda^{n+1}[1 - \alpha_1(1-\lambda)]^{n+1}$$

and

$$b_n = [1 - \alpha_1(1-\lambda)]^{n+1}$$

Then,

$$\begin{aligned}\psi_n &= \frac{a_n}{b_n} \\ &= \frac{\lambda^{n+1}[1-\alpha_1(1-\lambda)]^{n+1}}{[1-\alpha_1(1-\lambda)]^{n+1}} \\ &= \lambda^{n+1}\end{aligned}$$

Since $\lambda^{n+1} < 1$, it is obtained that $\lim_{n\to\infty} \psi_n = 0$. From Definition 2.5, $\{Sx_n\}_{n=0}^{\infty}$ has a better convergence speed than $\{Su_n\}_{n=0}^{\infty}$. □

The following example shows that iteration method given by Equation 8 has a higher convergence speed under favorable conditions than the other Jungck-type methods presented in this paper:

Example 3.6. Assume that $X = \mathbb{R}$ is Banacah space, $Y = [0.5, 1.5]$, and $S, T : [0.5, 1.5] \to [1, 81]$ are defined by $Sx = 16x^4$ and $Tx = x^8 + 24x^3 - 44x^2 + 35$, respectively. It can be observed that $T1 = S1 = 16$ and $T([0.5, 1.5]) \subseteq S([0.5, 1.5])$, and (S, T) are pairs of maps satisfying Inequality 9 with $\lambda = 0.4$ and $\mu = 0.2$. The convergence of the Jungck-type iteration methods provided by Equations 2-6 and Equation 8 to the $p = T1 = S1 = 16$ with the control sequences $\alpha_n = \beta_n = \gamma_n = \mu_n = a_n = b_n = c_n = \frac{3}{20}$, for the initial condition $x_0 = 0.75$, are shown in Tables 1 and 2. The following conclusions can be obtained from these tables:

- While newly defined iteration method given by Equation 8 reaches the fixed point at the 16th step,
- the Jungck-SP iteration method given by Equation 2 reaches the fixed point at the 72nd step,
- the Jungck-CR iteration method given by Equation 3 reaches the fixed point at the 17th step,
- the Jungck-Agarwal iteration method given by Equation 4 reaches the fixed point at the 17th step,
- the Jungck-Sahu iteration method given by Equation 5 reaches the fixed point at the 17th step, and
- the Jungck-Khan iteration method given by Equation 6 reaches the fixed point at the 101st.

Table 1. Convergence of some iteration methods for the initial point $x_0 = 0.75$

x_n	New Jungck Type	Jungck-CR	Jungck-Agarwal
x_1	0.75	0.75	0.75
x_2	1.05295512496838	1.05414377123399	1.06137648831351
⋮	⋮	⋮	⋮
x_{11}	0.99999999994918	0.99999999994220	0.99999999978115
x_{12}	1.00000000000533	1.00000000000615	1.00000000002691
⋮	⋮	⋮	⋮
x_{15}	0.99999999999999	1.00000000000007	1.00000000000041
x_{16}	1.00000000000000	0.99999999999999	0.99999999999995
x_{17}	⋮	1.00000000000000	1.00000000000000
⋮	⋮	⋮	⋮

Table 2. Convergence of some iteration methods for the initial point $x_0 = 0.75$

x_n	Jungck-Sahu	Jungck-SP	Jungck-Khan
x_1	0.75	0.75	0.75
x_2	1.04466023384367	0.87897378710221	0.85315042838595
⋮	⋮	⋮	⋮
x_{11}	0.99999999993357	0.99820409366943	0.999542288145375
x_{12}	1.00000000000718	0.99883957379660	0.999202265784102
⋮	⋮	⋮	⋮
x_{16}	0.99999999999999	0.99979688882405	0.999951335812014
x_{17}	1.00000000000000	0.99986856789305	0.99998589613924
⋮	⋮	⋮	⋮
x_{72}	⋮	1.00000000000000	⋮
x_{101}	⋮	⋮	1.00000000000000
⋮	⋮	⋮	⋮

Theorem 3.7. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Suppose that $S_1, T_1 : Y \to X$ are the approximation operators of S and T, respectively, satisfying the conditions $T_1x_p = S_1x_p = q$, $\|Tx - T_1x\| \leq \varepsilon_1$, and $\|Sx - S_1x\| \leq \varepsilon_2$, for ε_1 and ε_2 and for each $x \in Y$. Consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ obtained from the iteration method given by Equation 8 with the condition $\frac{1}{2} \leq \alpha_n$. Moreover, suppose that $\{S_1e_n\}_{n=0}^{\infty}$ is any sequence obtained from the following equation:

$$\begin{cases} S_1e_{n+1} = (1-\alpha_n) S_1f_n + \alpha_n T_1 f_n \\ S_1f_n = (1-\beta_n) T_1e_n + \beta_n T_1 g_n \\ S_1g_n = (1-\gamma_n) S_1h_n + \gamma_n T_1 h_n \\ S_1h_n = (1-\mu_n) S_1e_n + \mu_n T_1 e_n \end{cases} \tag{21}$$

If $\{S_1e_n\}_{n=0}^{\infty} \to q$ as $n \to \infty$, then

$$\|p-q\| \leq \frac{7\varepsilon_1 + 11\varepsilon_2}{1-\lambda}$$

Proof.
By using Equation 8 and Inequalities 9 and 21,

$$\begin{aligned} \|Sw_n - S_1h_n\| &= \|(1-\mu_n) Sx_n + \mu_n Tx_n - (1-\mu_n) S_1e_n - \mu_n T_1e_n\| \\ &\leq (1-\mu_n)\|Sx_n - S_1e_n\| + \mu_n \|Tx_n - T_1e_n\| \\ &\leq (1-\mu_n)\|Sx_n - Se_n\| + (1-\mu_n)\|Se_n - S_1e_n\| \\ &\quad + \mu_n\|Tx_n - Te_n\| + \mu_n \|Te_n - T_1e_n\| \\ &\leq (1-\mu_n)\|Sx_n - Se_n\| + (1-\mu_n)\varepsilon_2 \\ &\quad + \mu_n \|Tx_n - Te_n\| + \mu_n\varepsilon_1 \end{aligned} \tag{22}$$

Moreover,

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - Se_n\| + \mu \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$$

Suppose that $D_1 = \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$. Then, it is attained that

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - Se_n\| + \mu D_1 \tag{23}$$

Substituting Inequality 23 in Inequality 22,

$$\|Sw_n - S_1 h_n\| \leq [1 - \mu_n (1 - \lambda)] \|Sx_n - Se_n\| + \mu_n \mu D_1 + (1 - \mu_n) \varepsilon_2 + \mu_n \varepsilon_1 \tag{24}$$

Similarly,

$$\|Sz_n - S_1 g_n\| \leq (1 - \gamma_n) \|Sw_n - Sh_n\| + (1 - \gamma_n) \varepsilon_2 + \gamma_n \|Tw_n - Th_n\| + \gamma_n \varepsilon_1$$

and

$$\|Tw_n - Th_n\| \leq \lambda \|Sw_n - Sh_n\| + \mu \left(\frac{\|Sw_n - Tw_n\| . \|Sh_n - Th_n\|}{1 + \|Sw_n - Sh_n\|} \right)$$

Suppose that $D_2 = \left(\frac{\|Sw_n - Tw_n\| . \|Sh_n - Th_n\|}{1 + \|Sw_n - Sh_n\|} \right)$. Then, it is obtained that

$$\|Sz_n - S_1 g_n\| \leq [1 - \gamma_n (1 - \lambda)] \|Sw_n - Sh_n\| + \gamma_n \mu D_2 + (1 - \gamma_n) \varepsilon_2 + \gamma_n \varepsilon_1 \tag{25}$$

Moreover,

$$\|Sw_n - Sh_n\| \leq \|Sw_n - S_1 h_n\| + \varepsilon_2 \tag{26}$$

Substituting Inequality 26 in Inequality 25,

$$\|Sz_n - S_1 g_n\| \leq [1 - \gamma_n (1 - \lambda)] \|Sw_n - S_1 h_n\| + [1 - \gamma_n (1 - \lambda)]\varepsilon_2 + \gamma_n \mu D_2 + (1 - \gamma_n) \varepsilon_2 + \gamma_n \varepsilon_1 \tag{27}$$

Substituting Inequality 24 in Inequality 27,

$$\begin{aligned} \|Sz_n - S_1 g_n\| \leq{} & [1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)] \|Sx_n - S_1 e_n\| + [1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)]\varepsilon_2 \\ & + [1 - \gamma_n (1 - \lambda)]\mu_n \mu D_1 + [1 - \gamma_n (1 - \lambda)] (1 - \mu_n) \varepsilon_2 + [1 - \gamma_n (1 - \lambda)]\mu_n \varepsilon_1 \\ & + [1 - \gamma_n (1 - \lambda)]\varepsilon_2 + (1 - \gamma_n) \varepsilon_2 + \gamma_n \varepsilon_1 + \gamma_n \mu D_2 \end{aligned}$$

Similarly,

$$\|Sy_n - S_1 f_n\| \leq (1 - \beta_n) \|Tx_n - Te_n\| + (1 - \beta_n) \varepsilon_1 + \beta_n \|Tz_n - Tg_n\| + \beta_n \varepsilon_1 \tag{28}$$

and

$$\|Tz_n - Tg_n\| \leq \lambda \|Sz_n - Sg_n\| + \mu \left(\frac{\|Sz_n - Tz_n\| . \|Sg_n - Tg_n\|}{1 + \|Sz_n - Sg_n\|} \right)$$

Suppose that $D_3 = \left(\frac{\|Sz_n - Tz_n\| . \|Sg_n - Tg_n\|}{1 + \|Sz_n - Sg_n\|} \right)$. Then, it is obtained that

$$\|Tz_n - Tg_n\| \leq \lambda \|Sz_n - S_1 g_n\| + \lambda \varepsilon_2 + \mu D_3$$

Therefore,

$$\begin{aligned} \|Tz_n - Tg_n\| \leq{} & \lambda[1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)] \|Sx_n - S_1 e_n\| \\ & + \lambda[1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)]\varepsilon_2 \\ & + \lambda[1 - \gamma_n (1 - \lambda)]\mu_n \mu D_1 + \lambda[1 - \gamma_n (1 - \lambda)] (1 - \mu_n) \varepsilon_2 \\ & + \lambda[1 - \gamma_n (1 - \lambda)]\mu_n \varepsilon_1 + \lambda[1 - \gamma_n (1 - \lambda)]\varepsilon_2 \\ & + \lambda (1 - \gamma_n) \varepsilon_2 + \gamma_n \lambda \varepsilon_1 + \gamma_n \lambda \mu D_2 + \lambda \varepsilon_2 + \mu D_3 \end{aligned} \tag{29}$$

In addition,

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - Se_n\| + \mu \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$$

Suppose that $D_4 = \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$. Then, it is attained that

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - S_1e_n\| + \lambda\varepsilon_2 + \mu D_4 \tag{30}$$

Substituting Inequalities 29 and 30 in Inequality 28,

$$\begin{aligned} \|Sy_n - S_1f_n\| \leq & (1-\beta_n)\lambda \|Sx_n - S_1e_n\| + (1-\beta_n)\mu D_4 + (1-\beta_n)\lambda\varepsilon_2 + (1-\beta_n)\varepsilon_1 \\ & + \beta_n\lambda[1-\mu_n(1-\lambda)][1-\gamma_n(1-\lambda)]\|Sx_n - S_1e_n\| \\ & + \beta_n\lambda[1-\mu_n(1-\lambda)][1-\gamma_n(1-\lambda)]\varepsilon_2 \\ & + \beta_n\lambda[1-\gamma_n(1-\lambda)]\mu_n\mu D_1 + \beta_n\gamma_n\lambda\mu D_2 + \beta_n\mu D_3 \\ & + \beta_n\lambda[1-\gamma_n(1-\lambda)](1-\mu_n)\varepsilon_2 + \beta_n\lambda[1-\gamma_n(1-\lambda)]\mu_n\varepsilon_1 \\ & + \beta_n\lambda[1-\gamma_n(1-\lambda)]\varepsilon_2 + \beta_n\lambda(1-\gamma_n)\varepsilon_2 + \beta_n\gamma_n\lambda\varepsilon_1 + \beta_n\lambda\varepsilon_2 + \beta_n\varepsilon_1 \end{aligned} \tag{31}$$

Moreover,

$$\begin{aligned} \|Sx_{n+1} - S_1e_{n+1}\| \leq & (1-\alpha_n)\|Sy_n - S_1f_n\| \\ & + \alpha_n \|Ty_n - Tf_n\| + \alpha_n\varepsilon_1 \end{aligned} \tag{32}$$

and

$$\|Ty_n - Tf_n\| \leq \lambda \|Sy_n - Sf_n\| + \mu \left(\frac{\|Sy_n - Ty_n\| . \|Sf_n - Tf_n\|}{1 + \|Sy_n - Sf_n\|} \right)$$

Suppose that $D_5 = \left(\frac{\|Sy_n - Ty_n\| . \|Sf_n - Tf_n\|}{1 + \|Sy_n - Sf_n\|} \right)$. Then,

$$\|Ty_n - Tf_n\| \leq \lambda \|Sy_n - S_1f_n\| + \lambda\varepsilon_2 + \mu D_5 \tag{33}$$

Substituting Inequalities 31 and 33 in Inequality 32,

$$\begin{aligned} \|Sx_{n+1} - S_1e_{n+1}\| \leq & [1-\alpha_n(1-\lambda)](1-\beta_n)\lambda\|Sx_n - S_1e_n\| + [1-\alpha_n(1-\lambda)](1-\beta_n)\lambda\varepsilon_2 \\ & + [1-\alpha_n(1-\lambda)](1-\beta_n)\mu D_4 + [1-\alpha_n(1-\lambda)](1-\beta_n)\varepsilon_1 \\ & + [1-\alpha_n(1-\lambda)]\beta_n[1-\mu_n(1-\lambda)][1-\gamma_n(1-\lambda]\|Sx_n - S_1e_n\| \\ & + [1-\alpha_n(1-\lambda)]\beta_n\lambda[1-\mu_n(1-\lambda)][1-\gamma_n(1-\lambda]\varepsilon_2 \\ & + [1-\alpha_n(1-\lambda)]\beta_n\lambda[1-\gamma_n(1-\lambda)]\mu_n\mu D_1 \\ & + [1-\alpha_n(1-\lambda)]\beta_n\gamma_n\lambda\,\mu D_2 + [1-\alpha_n(1-\lambda)]\beta_n\mu D_3 \\ & + [1-\alpha_n(1-\lambda)]\beta_n\lambda[1-\gamma_n(1-\lambda)](1-\mu_n)\varepsilon_2 \\ & + [1-\alpha_n(1-\lambda)]\beta_n\lambda[1-\gamma_n(1-\lambda)]\mu_n\varepsilon_1 \\ & + [1-\alpha_n(1-\lambda)]\ \beta_n\lambda[1-\gamma_n(1-\lambda)]\varepsilon_2 \\ & + [1-\alpha_n(1-\lambda)]\beta_n\lambda(1-\gamma_n)\varepsilon_2 + [1-\alpha_n(1-\lambda)]\beta_n\lambda\gamma_n\varepsilon_1 \\ & + [1-\alpha_n(1-\lambda)]\beta_n\lambda\varepsilon_2 + [1-\alpha_n(1-\lambda)]\beta_n\varepsilon_1 + \alpha_n\lambda\varepsilon_2 + \alpha_n\mu D_5 + \alpha_n\varepsilon_1 \end{aligned}$$

For the above inequality , if necessary simplifications are made considering that $\frac{1}{2} \leq \alpha_n$ and $\alpha_n, \beta_n, \gamma_n, \mu_n \in [0,1]$ and $\lambda < 1$, then it is attained that

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & \{[1-\alpha_n(1-\lambda)](1-\beta_n)+[1-\alpha_n(1-\lambda)]\beta_n\}\|Sx_n-S_1e_n\| \\ & +[1-\alpha_n(1-\lambda)]D_1+[1-\alpha_n(1-\lambda)]D_2+[1-\alpha_n(1-\lambda)]D_3 \\ & +[1-\alpha_n(1-\lambda)]D_4+\alpha_nD_5 \\ & +\Big\{[1-\alpha_n(1-\lambda)](1-\beta_n)+[1-\alpha_n(1-\lambda)]\beta_n \\ & \quad +[1-\alpha_n(1-\lambda)]+[1-\alpha_n(1-\lambda)]+\alpha_n\Big\}\varepsilon_1 \\ & +\Big\{[1-\alpha_n(1-\lambda)](1-\beta_n)+[1-\alpha_n(1-\lambda)]\beta_n+[1-\alpha_n(1-\lambda)] \\ & \quad +[1-\alpha_n(1-\lambda)]+[1-\alpha_n(1-\lambda)]+[1-\alpha_n(1-\lambda)]+\alpha_n\Big\}\varepsilon_2\end{aligned}$$

and

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & [1-\alpha_n(1-\lambda)]\|Sx_n-S_1e_n\|+[1-\alpha_n(1-\lambda)](D_1+D_2+D_3+D_4) \\ & +\alpha_nD_5+\{3[1-\alpha_n(1-\lambda)]+\alpha_n\}\varepsilon_1+\{5[1-\alpha_n(1-\lambda)]+\alpha_n\}\varepsilon_2\end{aligned}$$

Therefore,

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & [1-\alpha_n(1-\lambda)]\|Sx_n-S_1e_n\| \\ & +2\alpha_n(D_1+D_2+D_3+D_4+D_5) \\ & +7\alpha_n\varepsilon_1+11\alpha_n\varepsilon_2\end{aligned}$$

Hence,

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq & [1-\alpha_n(1-\lambda)]\|Sx_n-S_1e_n\| \\ & +\alpha_n(1-\lambda)\left\{\frac{7\varepsilon_1+11\varepsilon_2+2(D_1+D_2+D_3+D_4+D_5)}{1-\lambda}\right\}\end{aligned} \tag{34}$$

It is clear that $\lim_{n\to\infty}(D_1+D_2+D_3+D_4+D_5)=0$. With this in mind, consider the following equalities:

$$a_n=\|Sx_n-S_1e_n\|$$

$$\mu_n=\alpha_n(1-\lambda)\in(0,1)$$

and

$$\eta_n=\left\{\frac{7\varepsilon_1+11\varepsilon_2+2(D_1+D_2+D_3+D_4+D_5)}{1-\lambda}\right\}$$

It can be observed that Inequality 34 satisfies all the conditions of Lemma 2.9. Hence, it follows by its conclusion that

$$0\leq\limsup_{n\to\infty}\|Sx_n-S_1e_n\|\leq\limsup_{n\to\infty}\left\{\frac{7\varepsilon_1+11\varepsilon_2+2(D_1+D_2+D_3+D_4+D_5)}{1-\lambda}\right\}=\frac{7\varepsilon_1+11\varepsilon_2}{1-\lambda}$$

By using $\{S_1e_n\}_{n=0}^{\infty}\to q$ and $\{Sx_n\}_{n=0}^{\infty}\to p$,

$$\|p-q\|\leq\frac{7\varepsilon_1+11\varepsilon_2}{1-\lambda}$$

□

4. Conclusion

This paper introduces a new four-step fixed-point iteration method, which is rewritten with the help of the Jungck Contraction Principle, and some fixed-point theorems for a general class of mappings are investigated. The results show that the new iteration method converges faster than the other methods presented in this paper. This method is stable and can obtain a data dependence result. Numerical examples are given to concretize the stability and convergence speed analysis. In future work, researchers can rewrite the iteration method provided in this paper by considering the Volterra-Fredholm integral equations as an operator in complex-valued Banach spaces with appropriate conditions and study the solution of these integral equations.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

References

[1] A. Amini-Harandi, H. Emami, *A Fixed Point Theorem for Contraction Type Maps in Partially Ordered Metric Spaces and Application to Ordinary Differential Equations*, Nonlinear Analysis: Theory, Methods and Applications 72 (5) (2010) 2238–2242.

[2] A. Wieczorek, *Applications of Fixed-Point Theorems in Game Theory and Mathematical Economics*, Wisdom Mathematics (28) (1988) 25–34.

[3] L. C. Ceng, Q. Ansari, J. C. Yao, *Some Iterative Methods for Finding Fixed Points and for Solving Constrained Convex Minimization Problems*, Nonlinear Analysis: Theory, Methods and Applications (74) (2011) 5286–5302.

[4] J. Borwein, B. Sims, Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Vol. 49 of *The Douglas–Rachford Algorithm in the Absence of Convexity*, Springer, New York, 2011, Ch. 6, pp. 93-109.

[5] K. C. Border, Fixed Point Theorems with Applications to Economics and Game Theory, Cambridge University Press, Cambridge, 1989.

[6] M. Chen, W. Lu, Q. Chen, K. J. Ruchala, G. H. Olivera, *A Simple Fixed-Point Approach to Invert a Deformation Field*, Medical Physics 35 (1) (2008) 81–88.

[7] S. Banach, *Sur Les Opérations Dans Les Ensembles Abstraits Et Leur Application Aux Equations Intégrales*, Fundamenta Mathematicae 3 (1) (1922) 133–181.

[8] V. Karakaya, K. Doğan, F. Gürsoy, M. Ertürk, *Fixed Point of a New Three-Step Iteration Algorithm under Contractive-like Operators over Normed Spaces*, Abstract and Applied Analysis 2013 (2013) Article ID 560258 9 pages.

[9] M. Özdemir, S. Akbulut, *On the Equivalance of Some Fixed Point Iterations*, Kyungpook Mathematical Journal 46 (2) (2006) 211–217.

[10] V. Karakaya, Y. Atalan, K. Doğan, N. Bouzara, *Some Fixed Point Results for a New Three Steps Iteration Process in Banach Spaces*, Fixed Point Theory 18 (2) (2017) 625–640.

[11] Y. Atalan, V. Karakaya, *Investigation of Some Fixed Point Theorems in Hyperbolic Spaces for a Three Step Iteration Process*, Korean Journal of Mathematics 27 (4) (2019) 929–947.

[12] V. Karakaya, F. Gürsoy, K. Doğan, M. Ertürk, *Data Dependence Results for Multistep and CR Iterative Schemes in the Class of Contractive-like Operators*, Abstract and Applied Analysis 2013 (2013) Article ID 381980 7 pages.

[13] S. Maldar, Y. Atalan, K. Doğan, *Comparison Rate of Convergence and Data Dependence for a New Iteration Method*, Tbilisi Mathematical Journal 13 (4) (2020) 65–79.

[14] Y. Atalan, *On Numerical Approach to the Rate of Convergence and Data Dependence Results for a New Iterative Scheme*, Konuralp Journal of Mathematics 7 (1) (2019) 97–106.

[15] S. Maldar, Y. Atalan, *Common Fixed Point Theorems for Complex-Valued Mappings with Applications*, Korean Journal of Mathematics 30 (2) (2022) 205–229.

[16] Y. Atalan, V. Karakaya, *Obtaining New Fixed Point Theorems Using Generalized Banach-Contraction Principle*, Erciyes University Journal of the Institute of Science and Technology 35 (3) (2019) 34–45.

[17] K. Doğan, F. Gürsoy, V. Karakaya, S. H. Khan, *Some New Results on Convergence, Stability and Data Dependence in N-normed Spaces*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 69 (1) (2020) 112–122.

[18] L. J. Ciric, *A Generalization of Banach's Contraction Principle*, Proceedings of American Mathematical Society 45 (2) (1974) 267–273.

[19] M. Edelstein, *An Extension of Banach's Contraction Principle*, Proceedings of the American Mathematical Society 12 (1) (1961) 7–10.

[20] S. B. Presic, *Sur Une Classe D' Inequations Aux Differences Finite Et. Sur La Convergence De Certaines Suites*, Publications De I'institut Mathématique 5 (25) (1965) 75–78.

[21] G. Jungck, *Commuting Mappings and Fixed Points*, American Mathematical Monthly 83 (4) (1976) 261–263.

[22] E. Picard, *Memoire Sur La Theorie Des Equations Aux Derivees Partielles Et La Methode Des Approximations Successives*, Journal de Mathématiques Pures et Appliquées 6 (1890) 145–210.

[23] R. Chugh, V. Kumar, *Strong Convergence and Stability Results for Jungck-SP Iterative Scheme*, International Journal of Computer Applications 36 (12) (2011) 40–46.

[24] N. Hussain, V. Kumar, M. A. Kutbi, *On Rate of Convergence of Jungck-type Iterative Schemes*, Abstract and Applied Analysis 2013 (2013) Article ID 132626 15 pages.

[25] R. Chugh, S. Kumar, *On the Stability and Strong Convergence for Jungck-Agarwal et al. Iteration Procedure*, International Journal of Computer Applications 64 (7) (2013) 39–44.

[26] A. R. Khan, V. Kumar, N. Hussain, *Analytical and Numerical Treatment of Jungck-Type Iterative Schemes*, Applied Mathematics and Computation 231 (2014) 521–535.

[27] W. Pheungrattana, S. Suantai, *On the Rate of Convergence of Mann, Ishikawa, Noor and SP Iterations for Continuous on an Arbitrary Interval*, Journal of Computational and Applied Mathematics 235 (9) (2011) 3006–3014.

[28] R. Chugh, V. Kumar, S. Kumar, *Strong Convergence of a New Three Step Iterative Scheme in Banach Spaces*, American Journal of Computational Mathematics 2 (4) (2012) 345–357.

[29] R. P. Agarwal, D. O. Regan, D. R. Sahu, *Iterative Construction of Fixed Points of Nearly Asymptotically Nonexpansive Mappings*, Journal of Nonlinear and Convex Analysis 8 (1) (2007) 61–79.

[30] D. R. Sahu, A. Petruşel, *Strong Convergence of Iterative Methods by Strictly Pseudocontractive Mappings in Banach Spaces*, Nonlinear Analysis: Theory, Methods and Applications 74 (17) (2011) 6012–6023.

[31] V. Berinde, *Picard Iteration Converges Faster Than Mann Iteration for a Class of Quasicontractive Operators*, Fixed Point Theory and Applications 2004 (2004) Article Number 716359 9 pages.

[32] S. L. Singh, C. Bhatnagar, S. N. Mishra, *Stability of Jungck-Type Iterative Procedures*, International Journal of Mathematics and Mathematical Sciences 2005 (2005) Article ID 386375 9 pages.

[33] M. Kumar, P. Kumar, S. Kumar, *Common Fixed Point Theorems in Complex Valued Metric Spaces*, Journal of Analysis and Number Theory 2014 (2014) Article ID 587825 7 pages.

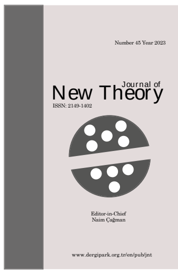
[34] V. Berinde, *On a Family of First Order Difference Inequalities Used in the Iterative Approximation of Fixed Points*, Creative Mathematics and Informatics 18 (2) (2009) 110–122.

[35] S. M. Şoltuz, T. Grosan, *Data Dependence for Ishikawa Iteration when Dealing with Contractive Like Operators*, Fixed Point Theory and Applications 2008 (2008) Article Number 242916 7 pages.

[36] V. Berinde, Iterative Approximation of Fixed Points, Springer-Verlag, Berlin, 2007.

Journal of
New Theory
ISSN: 2149-1402

45 (2023) 18-29
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Approximate Solutions of the Fractional Clannish Random Walker's Parabolic Equation with the Residual Power Series Method

Sevil Çulha Ünal[1]

Article Info
Received: 17 Aug 2023
Accepted: 18 Dec 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1344706
Research Article

Abstract — One of the prominent nonlinear partial differential equations in mathematical physics is the Clannish Random Walker's Parabolic (CRWP) equation. This study uses Residual Power Series Method (RPSM) to solve the time fractional CRWP equation. In this equation, the fractional derivatives are considered in Caputo's sense. The effectiveness of RPSM is illustrated with graphical results. The series solutions are utilized to represent the approximate solutions. Besides, the approximate solutions found by the suggested method ensure good accuracy when compared with the exact solution. Moreover, RPSM efficiently analyzes complex problems that emerge in the related mathematical and scientific fields.

Keywords *Fractional partial differential equation, Caputo derivative, Clannish Random Walker's Parabolic equation, residual power series method, approximate solution*

Mathematics Subject Classification (2020) 26A33, 35C10

1. Introduction

The Clannish Random Walker's Parabolic (CRWP) equation in the form

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 0$$

is a mathematical model of physical problems appearing in various scientific fields such as mathematical biology and physics. This equation describes the behavior of two types that carry out a concurrent one-dimensional random walk defined by the condensation of the clannishness of members as the density of another increases. In the literature, various methods, such as the improved tanh function method [1], homotopy perturbation method [2], Jacobi elliptic function method [3], unified rational expansion method [3], and a direct rational exponential scheme [4], have been used to solve the CRWP equation.

Fractional calculus is a quickly developing branch of mathematics with various applications in numerous chemistry, physics, biology, and engineering fields such as thermodynamics, viscoelasticity, electricity, aerodynamics, fluid dynamics, control theory, turbulence, signal processing, and others [5-10]. Thus, finding exact and approximate solutions to fractional differential equations is important in scientific studies. An important one of these fractional differential equations is the time fractional CRWP equation.

Recently, many methods, such as the adapted (G'/G)-expansion scheme [11,12], the $(G'/G\,, 1/G)$-expansion method [12,13], the Kudryashov method [14], the improved $\tan(Q(\xi)/2)$-expansion method [15], the generalized homotopy analysis method [16], the modified Kudryashov method [17], the extended $\exp(-\varphi(\xi)/2)$- expansion method [18], the modified extended auxiliary mapping method [19], the modified

[1]sevilunal@sdu.edu.tr (Corresponding Author)
[1]Department of Avionics, School of Civil Aviation, Süleyman Demirel University, Isparta, Türkiye

F-expansion method [19, 20], the modified (G'/G^2)-expansion method [20], the power series method [21], the natural decomposition method [22], the energy inequality method [23], and the modified trial equation method [24], have been used to find solutions to the fractional CRWP equation. Residual Power Series Method (RPSM) has not yet been investigated to solve the time fractional CRWP equation in the literature. Thus, the main focus of this paper is to utilize RPSM to calculate the approximate solutions of the time fractional CRWP equation

$$D_t^{\mu}u(x,t) - u_x(x,t) + 2u(x,t)u_x(x,t) + u_{xx}(x,t) = 0, \quad 0 < \mu \leq 1 \tag{1}$$

where D_t^{μ} is the fractional derivative operator in the Caputo sense. Abu Arqub [25] suggested RPSM as a useful method for obtaining coefficients of the power series solution in 2013. RPSM has numerous benefits for solving partial differential equations compared to other methods [26]. RPSM provides an easy and effective power series solution for various equations without linearization, discretization, or perturbation. This method does not need a recursion relationship and does not require comparing the coefficients of the corresponding terms. The suggested method yields the solutions as a convergence series. With this method, infinite series solutions can be gained by iterated operations. Besides, RPSM is unaffected by rounding errors in computation and does not require a lot of computer memory and time. Moreover, there is no need for any transformation with this method. Furthermore, RPSM can be implemented directly into the present equation by choosing an initial guess approximation. In literature, RPSM has been used to find power series solutions for different problems, such as those provided in [27-44].

The organization of the study is as follows: Section 2 provides some definitions and theorems for the Caputo derivative and the fractional power series. Section 3 presents RPSM for the approximate solutions of nonlinear fractional differential equations. Section 4 applies the proposed method for the fractional CRWP equation solutions and exhibits the suggested method's effectiveness with table and graphics. Finally, the last section contains the concluding remarks.

2. Preliminaries

Many fractional derivative definitions, such as Riemann-Liouville, Caputo, Grunwald-Letnikov, Marchaud, Weyl, and Hadamard fractional derivatives, have been used in scientific studies. In this section, the Caputo derivative is considered because the initial conditions of the fractional partial differential equations with the Caputo derivative have the common form of the integer order partial differential equations, and the derivative of the constant is zero.

Definition 2.1. [45] The time-fractional derivative in Caputo sense is described as

$$D_t^{\mu}u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t (t-\tau)^{m-1-\mu}\frac{\partial^m u(x,\tau)}{\partial \tau^m}d\tau, & m-1<\mu<m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & m=\mu\in\mathbb{N} \end{cases}$$

Definition 2.2. [46] The fractional power series about t_0 is defined as

$$\sum_{m=0}^{\infty} c_m(t-t_0)^{m\mu} = c_0 + c_1(t-t_0)^{\mu} + c_2(t-t_0)^{2\mu} + \cdots, \quad 0 \leq m-1 < \mu \leq m \quad \text{and} \quad t \geq t_0$$

Here, c_m are constants, and t is a variable.

Theorem 2.1. [46] Suppose that h is a fractional power series representation about t_0 of the manner

$$h(t) = \sum_{m=0}^{\infty} c_m (t - t_0)^{m\mu}, \quad 0 \leq m - 1 < \mu \leq m \quad \text{and} \quad t_0 \leq t < t_0 + R$$

When $D^{m\mu} h(t)$ are continuous on $(t_0, t_0 + R)$, then coefficients c_m are given as

$$c_m = \frac{D^{m\mu} h(t_0)}{\Gamma(1 + m\mu)}, \quad m \in \{0,1,2, \cdots\}$$

where R is the radius of convergence and $D^{m\mu} = \underbrace{D^{\mu} D^{\mu} \cdots D^{\mu}}_{m\ times}$.

Theorem 2.2. [46] Suppose that $u(x,t)$ has a multivariate fractional power series representation at t_0 of the form

$$u(x,t) = \sum_{m=0}^{\infty} h_m(x)(t - t_0)^{m\mu}, \quad x \in I, \quad 0 \leq m - 1 < \mu \leq m, \quad \text{and} \quad t_0 \leq t < t_0 + R$$

If $D_t^{m\mu} u(x,t)$ are continuous on $I \times (t_0, t_0 + R)$, then $h_m(x)$ are given as

$$h_m(x) = \frac{D_t^{m\mu} u(x, t_0)}{\Gamma(1 + m\mu)}, \quad m \in \{0,1,2, \cdots\}$$

Here, $D_t^{m\mu} = \frac{\partial^{m\mu}}{\partial t^{m\mu}} = \frac{\partial^{\mu}}{\partial t^{\mu}} \frac{\partial^{\mu}}{\partial t^{\mu}} \cdots \frac{\partial^{\mu}}{\partial t^{\mu}}$ and $R = \min_{c \in I} R_c$ that R_c is the radius of convergence of the fractional power series

$$\sum_{m=0}^{\infty} h_m(c)(t - t_0)^{m\mu}$$

3. General Structure of RPSM

In this section, to find the approximate solutions of nonlinear fractional differential equations with the suggested method, we investigate the following general nonlinear fractional differential equation with the initial condition

$$D_t^{\mu} u(x,t) = R(u) + N(u), \quad 0 < \mu \leq 1, \quad t > 0, \quad \text{and} \quad u(x,0) = h(x) \tag{2}$$

where $R(u)$ is the linear term and $N(u)$ is the nonlinear term. Here, D_t^{μ} is the fractional derivative operator in the Caputo sense. The proposed method suggests the solution for Equation 2 as a fractional power series,

$$u(x,t) = \sum_{m=0}^{\infty} h_m(x) \frac{t^{m\mu}}{\Gamma(1 + m\mu)}, \quad x \in I, \quad 0 < \mu \leq 1, \quad \text{and} \quad 0 \leq t < R$$

Then, the $u_k(x,t)$ is given as

$$u_k(x,t) = \sum_{m=0}^{k} h_m(x) \frac{t^{m\mu}}{\Gamma(1 + m\mu)}, \quad x \in I, \quad 0 < \mu \leq 1, \quad \text{and} \quad 0 \leq t < R \tag{3}$$

The 0-th RPSM approximate solution of $u(x,t)$ is expressed as

$$u_0 = h_0(x) = u(x,0) = h(x)$$

Equation 3 can be given as

$$u_k(x,t) = h(x) + \sum_{m=1}^{k} h_m(x)\frac{t^{m\mu}}{\Gamma(1+m\mu)}, \quad x \in I, \quad 0 < \mu \le 1, \quad 0 \le t < R, \quad \text{and} \quad k \in \{1,2,\cdots\} \tag{4}$$

The residual function for Equation 2 is stated by

$$\text{Res}_u(x,t) = D_t^{\mu}u(x,t) - R(u) - N(u)$$

Hence, $\text{Res}_{u,k}$ is expressed as

$$\text{Res}_{u,k}(x,t) = D_t^{\mu}u_k(x,t) - R(u_k) - N(u_k) \tag{5}$$

As can be seen in [25,26, 47-49], it is obvious that $\text{Res}_u(x,t) = 0$ and $\lim_{k\to\infty} \text{Res}_{u,k}(x,t) = \text{Res}_u(x,t)$, for $t \ge 0$ and $x \in I$. Since the fractional derivative of a constant function is zero in the Caputo sense, we express $D_t^{m\mu}\text{Res}_u(x,t) = 0$. Besides, the fractional derivatives of $\text{Res}_u(x,t)$ and $\text{Res}_{u,k}(x,t)$ are matching at $t = 0$ for $m \in \{0,1,\cdots,k\}$; that is $D_t^{m\mu}\text{Res}_u(x,0) = D_t^{m\mu}\text{Res}_{u,k}(x,0) = 0$, $m \in \{0,1,\cdots,k\}$.

To gain the coefficients $h_m(x)$ with $m \in \{1,2,\cdots,k\}$ in Equation 4, we substitute the $u_k(x,t)$ in Equation 5 and calculate the $D_t^{(k-1)\mu}$ of $\text{Res}_{u,k}(x,t)$ for $k \in \{1,2,\cdots\}$ at $t = 0$. Then, we solve the following algebraic equation

$$D_t^{(k-1)\mu}\text{Res}_{u,k}(x,0) = 0, \quad 0 < \mu \le 1, \quad 0 \le t < R, \quad t = 0, \quad \text{and} \quad k \in \{1,2,\cdots\} \tag{6}$$

4. Implementation of RPSM for the Solution of the Fractional CRWP Equation

In this section, the suggested method is used to determine the RPSM solutions for Equation 1 subject to the initial condition

$$u(x,0) = \frac{1}{2} + \frac{1}{1+\cosh x - \sinh x} \tag{7}$$

Here, $u(x,t) = \frac{1}{2} + \frac{1}{1+\cosh(x-t)-\sinh(x-t)}$ is the exact solution of Equation 1 for $\mu = 1$ [14]. We express the residual function of Equation 1 as

$$\text{Res}_u(x,t) = D_t^{\mu}u(x,t) - \frac{\partial}{\partial x}u(x,t) + 2u(x,t)\frac{\partial}{\partial x}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t)$$

Hence, $\text{Res}_{u,k}(x,t)$ is given as

$$\text{Res}_{u,k}(x,t) = D_t^{\mu}u_k(x,t) - \frac{\partial}{\partial x}u_k(x,t) + 2u_k(x,t)\frac{\partial}{\partial x}u_k(x,t) + \frac{\partial^2}{\partial x^2}u_k(x,t) \tag{8}$$

We investigate $k = 1$ in this equation to determine the $h_1(x)$ and gain

$$\text{Res}_{u,1}(x,t) = D_t^{\mu}u_1(x,t) - \frac{\partial}{\partial x}u_1(x,t) + 2u_1(x,t)\frac{\partial}{\partial x}u_1(x,t) + \frac{\partial^2}{\partial x^2}u_1(x,t)$$

From Equation 4 at $k = 1$,

$$u_1(x,t) = h(x) + h_1(x)\frac{t^{\mu}}{\Gamma(1+\mu)}$$

Therefore,

$$\mathrm{Res}_{u,1}(x,t) = h_1(x) - \left(h'(x) + h_1'(x)\frac{t^{\mu}}{\Gamma(1+\mu)}\right) + 2\left(h(x) + h_1(x)\frac{t^{\mu}}{\Gamma(1+\mu)}\right)\left(h'(x) + h_1'(x)\frac{t^{\mu}}{\Gamma(1+\mu)}\right)$$
$$+h''(x) + h_1''(x)\frac{t^{\mu}}{\Gamma(1+\mu)}$$

We gain $\mathrm{Res}_{u,1}(x,0) = 0$ from Equation 6. Hence,

$$h_1(x) = \frac{1}{-2(1+\cosh x)}$$

Therefore,

$$u_1(x,t) = \frac{1}{2} + \frac{1}{1+\cosh x - \sinh x} - \frac{1}{2(1+\cosh x)}\frac{t^{\mu}}{\Gamma(1+\mu)}$$

To determine $h_2(x)$, we investigate $k = 2$ in Equation 8 and gain

$$\mathrm{Res}_{u,2}(x,t) = D_t^{\mu}u_2(x,t) - \frac{\partial}{\partial x}u_2(x,t) + 2u_2(x,t)\frac{\partial}{\partial x}u_2(x,t) + \frac{\partial^2}{\partial x^2}u_2(x,t)$$

From Equation 4 at $k = 2$,

$$u_2(x,t) = h(x) + h_1(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)}$$

Thus,

$$\mathrm{Res}_{u,2}(x,t) = h_1(x) + h_2(x)\frac{t^{\mu}}{\Gamma(1+\mu)} - \left(h'(x) + h_1'(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2'(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)}\right)$$
$$+2\left(h(x) + h_1(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)}\right)\left(h'(x) + h_1'(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2'(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)}\right)$$
$$+h''(x) + h_1''(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2''(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)}$$

We gain $D_t^{\mu}\mathrm{Res}_{u,2}(x,0) = 0$ from Equation 6. Thus,

$$h_2(x) = -2\mathrm{csch}^3 x \sinh^4\left(\frac{x}{2}\right)$$

Hence,

$$u_2(x,t) = \frac{1}{2} + \frac{1}{1+\cosh x - \sinh x} - \frac{1}{2(1+\cosh x)}\frac{t^{\mu}}{\Gamma(1+\mu)} - 2\mathrm{csch}^3 x \sinh^4\left(\frac{x}{2}\right)\frac{t^{2\mu}}{\Gamma(1+2\mu)}$$

To find $h_3(x)$, we investigate $k = 3$ in Equation 8 and gain

$$\mathrm{Res}_{u,3}(x,t) = D_t^{\mu}u_3(x,t) - \frac{\partial}{\partial x}u_3(x,t) + 2u_3(x,t)\frac{\partial}{\partial x}u_3(x,t) + \frac{\partial^2}{\partial x^2}u_3(x,t)$$

From Equation 4 at $k = 3$,

$$u_3(x,t) = h(x) + h_1(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)} + h_3(x)\frac{t^{3\mu}}{\Gamma(1+3\mu)}$$

Hence,

$$\mathrm{Res}_{u,3}(x,t) = h_1(x) + h_2(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_3(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)} - \left(h'(x) + h_1'(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2'(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)} + h_3'(x)\frac{t^{3\mu}}{\Gamma(1+3\mu)}\right)$$

$$+2\left(h(x) + h_1(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)} + h_3(x)\frac{t^{3\mu}}{\Gamma(1+3\mu)}\right)\left(h'(x) + h_1'(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2'(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)} + h_3'(x)\frac{t^{3\mu}}{\Gamma(1+3\mu)}\right)$$

$$+h''(x) + h_1''(x)\frac{t^{\mu}}{\Gamma(1+\mu)} + h_2''(x)\frac{t^{2\mu}}{\Gamma(1+2\mu)} + h_3''(x)\frac{t^{3\mu}}{\Gamma(1+3\mu)}$$

We gain $D_t^{2\mu}\mathrm{Res}_{u,3}(x,0) = 0$ from Equation 6. Thus,

$$h_3(x) = -\frac{1}{8}(-2 + \cosh x)\operatorname{sech}^4\left(\frac{x}{2}\right)$$

Therefore,

$$u_3(x,t) = \frac{1}{2} + \frac{1}{1 + \cosh x - \sinh x} - \frac{1}{2(1+\cosh x)}\frac{t^{\mu}}{\Gamma(1+\mu)} - 2\operatorname{csch}^3 x \sinh^4\left(\frac{x}{2}\right)\frac{t^{2\mu}}{\Gamma(1+2\mu)}$$

$$-\frac{1}{8}(\cosh x - 2)\operatorname{sech}^4\left(\frac{x}{2}\right)\frac{t^{3\mu}}{\Gamma(1+3\mu)}$$

Utilizing the same operation for $k = 4$,

$$h_4(x) = -\frac{1}{16}\operatorname{sech}^5\left(\frac{x}{2}\right)\left(-11\sinh\left(\frac{x}{2}\right) + \sinh\left(\frac{3x}{2}\right)\right)$$

and

$$u_4(x,t) = \frac{1}{2} + \frac{1}{1 + \cosh x - \sinh x} - \frac{1}{2(1+\cosh x)}\frac{t^{\mu}}{\Gamma(1+\mu)} - 2\operatorname{csch}^3 x \sinh^4\left(\frac{x}{2}\right)\frac{t^{2\mu}}{\Gamma(1+2\mu)}$$

$$-\frac{1}{8}(\cosh x - 2x)\operatorname{sech}^4\left(\frac{x}{2}\right)\frac{t^{3\mu}}{\Gamma(1+3\mu)} - \frac{1}{16}\operatorname{sech}^5\left(\frac{x}{2}\right)\left(-11\sinh\left(\frac{x}{2}\right) + \sinh\left(\frac{3x}{2}\right)\right)\frac{t^{4\mu}}{\Gamma(1+4\mu)}$$

The solution $u_4(x,t)$ is obtained for $\mu = 0.25$, $\mu = 0.50$, and $\mu = 1$ with the different values of x and t in Table 1. Besides, $u_4(x,t)$ is compared numerically with the exact solution for $\mu = 1$ in this table. Table 1 indicates that the absolute error increases as the value t increases. When compared with the generalized homotopy analysis method [16] and the natural decomposition method [22], it is seen that more numerical results are presented with the proposed method for the different values of x and t in this table. The comparison of the approximate solution and the exact solution is illustrated for $0 \leq x \leq 1$ and $t = 0.1$ by the natural decomposition method. However, this comparison is illustrated for $-20 \leq x \leq 20$ and $0 \leq t \leq 1$ by the suggested method. Moreover, the comparison of the approximate and exact solutions is demonstrated only with the help of figures by the generalized homotopy analysis method.

Table 1. Comparing the $u_4(x,t)$ solution with the exact solution

x	t	$\mu = 0.25$	$\mu = 0.50$	$\mu = 1$		
		$u_4(x,t)$	$u_4(x,t)$	$u_4(x,t)$	Exact solution	Absolute error
-20	0	0.500000002061	0.500000002061	0.500000002061	0.500000002061	0
	0.2	0.500000001322	0.500000001336	0.500000001688	0.500000001688	5.21805×10^{-15}
	0.4	0.50000000142	0.500000001187	0.500000001382	0.500000001382	1.64757×10^{-13}
	0.6	0.500000001569	0.500000001147	0.500000001132	0.500000001131	1.21259×10^{-12}
	0.8	0.500000001743	0.50000000118	0.500000000931	0.500000000926	4.9557×10^{-12}
	1	0.500000001931	0.500000001277	0.500000000773	0.500000000758	1.46766×10^{-11}

Table 1. (Continued) Comparing the $u_4(x,t)$ solution with the exact solution

x	t	$\mu=0.25$	$\mu=0.50$	$\mu=1$		
		$u_4(x,t)$	$u_4(x,t)$	$u_4(x,t)$	Exact solution	Absolute error
-5	0	0.506692850924	0.506692850924	0.506692850924	0.506692850924	0
	0.2	0.504227945443	0.504341132518	0.50548631283	0.505486298899	1.39308×10^{-8}
	0.4	0.504461951935	0.503840255916	0.504496707853	0.504496273161	4.34692×10^{-7}
	0.6	0.504857371886	0.503672525135	0.503687458916	0.503684239899	3.21902×10^{-6}
	0.8	0.505329920677	0.503726292195	0.503031646234	0.503018416325	1.32299×10^{-5}
	1	0.50585023446	0.50396675685	0.502512007312	0.502472623157	3.93842×10^{-5}
5	0	1.49330714908	1.49330714908	1.49330714908	1.49330714908	0
	0.2	1.48180816187	1.48809037128	1.4918374435	1.49183742885	1.46499×10^{-8}
	0.4	1.47688585161	1.48424157072	1.49004867877	1.49004819813	4.8064×10^{-7}
	0.6	1.47276294279	1.48024305141	1.48787530595	1.48787156502	3.74094×10^{-6}
	0.8	1.46904740969	1.47598316537	1.4852421188	1.48522596831	1.61505×10^{-5}
	1	1.46559068753	1.47142711235	1.48206425377	1.48201379004	5.04637×10^{-5}
20	0	1.5	1.5	1.5	1.5	0
	0.2	1.49999999636	1.49999999837	1.49999999954	1.5	4.56339×10^{-10}
	0.4	1.49999999477	1.49999999715	1.49999999899	1.5	1.01354×10^{-9}
	0.6	1.49999999343	1.49999999587	1.49999999831	1.5	1.69303×10^{-9}
	0.8	1.49999999222	1.4999999945	1.49999999748	1.5	2.51955×10^{-9}
	1	1.4999999911	1.49999999303	1.49999999648	1.4999999851	1.138×10^{-8}

In Figure 1, the comparison between the exact solution and the $u_4(x,t)$ is demonstrated for $-20\le x\le 20$ and $0\le t\le 1$ at $\mu=1$. When equal parameters are chosen, it is clear that the $u_4(x,t)$ solution has almost the same shape as the exact solution in Figure 1.

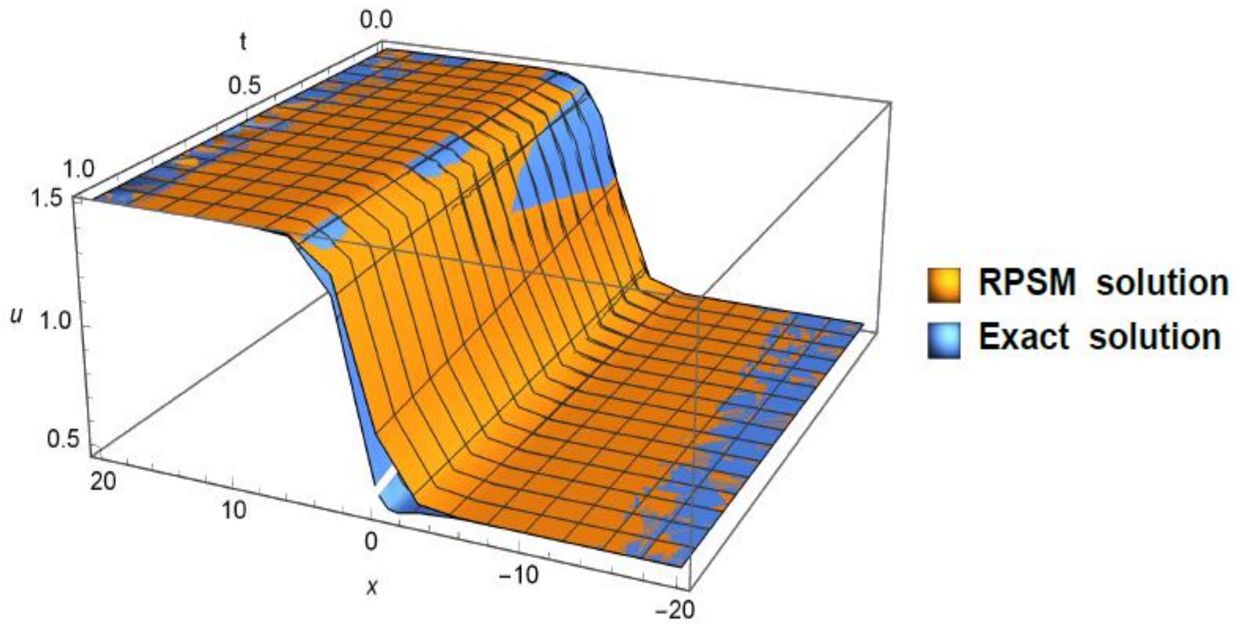


Figure 1. The graphic of the exact solution and $u_4(x,t)$

In Figure 2, the $u_4(x,t)$ is demonstrated for $-10 \le x \le 10$ and $0 \le t \le 5$ when $\mu = 0.1$, $\mu = 0.4$, $\mu = 0.7$, $\mu = 1$. In Figure 3, the same solution is illustrated for $-10 \le x \le 10$ and $t = 4$ with the different values of μ. The solution at $\mu = 0.1$ is demonstrated with the blue line, the solution at $\mu = 0.4$ is demonstrated with the orange line, the solution at $\mu = 0.7$ is demonstrated with the green line, and the solution at $\mu = 1$ is demonstrated with the red line in Figure 3. Cleary observed from Figure 3 that a solitary wave occurs as the values of α increase. All graphics are demonstrated with the aid of Mathematica.

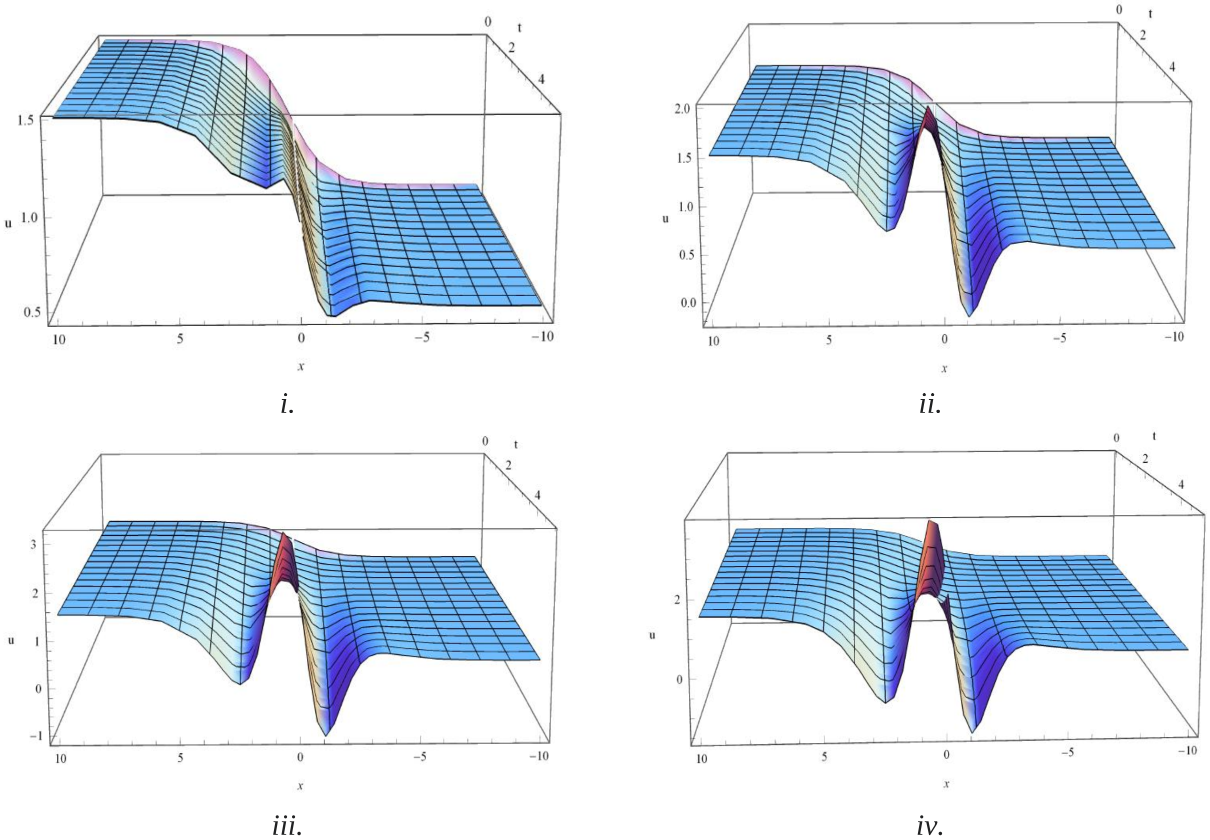


Figure 2. 3D graphics of the $u_4(x,t)$: (*i*) for $\mu = 0.1$, (*ii*) for $\mu = 0.4$, (*iii*) for $\mu = 0.7$, and (*iv*) for $\mu = 1$

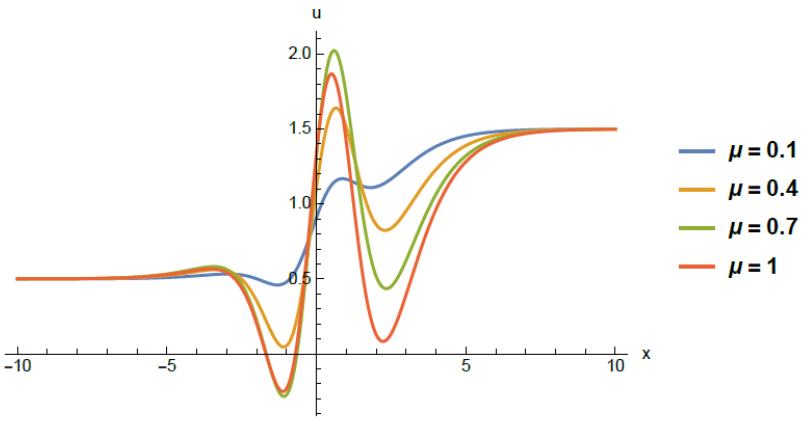


Figure 3. 2D graphic of the $u_4(x,4)$ for the different values of μ

5. Conclusion

In this paper, RPSM is utilized to obtain the approximate solutions of Equation 1. Numerical results are introduced with the different values of μ, x, and t. The proposed method reaches a higher level of accuracy when these results are investigated. It is seen that the approximate solutions are found to have nearly the same shape as the exact solution when equal parameters are chosen. These solutions are also illustrated in 2D and 3D graphics as proof of visualization. The suggested method does not require a lot of calculation work and time. This method can obtain infinite series solutions using only a few iterations. Moreover, RPSM is highly efficient for the fractional CRWP equation. Furthermore, there is no need for perturbation, linearization, discretization, or transformation when utilizing the proposed method. For future studies, RPSM can be used as an alternative to gain the approximate solutions of different types of partial and fractional differential equations encountered in physics, mathematics, and engineering.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

References

[1] Y. Uğurlu, D. Kaya, *Analytic Method for Solitary Solutions of Some Partial Differential Equations*, Physics Letters A 370 (3-4) (2007) 251–259.

[2] Y. Uğurlu, İ. E. İnan, B. Kilic, *Analytic Solutions of Some Partial Differential Equations by Using Homotopy Perturbation Equation*, World Applied Sciences Journal 12 (11) (2011) 2135–2139.

[3] B. Kiliç, *Exact Solutions for Nonlinear Evolution Equations with Jacobi Elliptic Function Rational Expansion Method*, World Applied Sciences Journal 23 (12) (2013) 81–88.

[4] M. S. Khatun, M. F. Hoque, M. A. Rahman, *Multisoliton Solutions, Completely Elastic Collisions and Non-elastic Fusion Phenomena of Two PDEs*, Pramana-Journal of Physics 88 (2017) Article Number 86 9 pages.

[5] B. Ahmad, J. J. Nieto, *Existence of Solutions for Nonlocal Boundary Value Problems of Higher-order Nonlinear Fractional Differential Equations*, Hindawi Abstract and Applied Analysis (2009) Article ID 494720 9 pages.

[6] Y. Wang, S. Liang, Q. Wang, *Existence Results for Fractional Differential Equations with Integral and Multi-point Boundary Conditions*, Boundary Value Problems 2018 (4) (2018) 11 pages.

[7] M. Şenol, A. Ata, *Approximate Solution of Time-fractional KdV Equations by Residual Power Series Method*, Journal of Balıkesir University Institute of Science and Technology 20 (1) (2018) 430–439.

[8] G. Akram, M. Sadaf, M. Abbas, I. Zainab, S. R. Gillani, *Efficient Techniques for Traveling Wave Solutions of Time-fractional Zakharov-Kuznetsov Equation*, Mathematics and Computers in Simulation 193 (2022) 607–622.

[9] M. M. A. Qurashi, Z. Korpinar, D. Baleanu, M. Inc, *A New Iterative Algorithm on the Time-fractional Fisher Equation: Residual Power Series Method*, Advances in Mechanical Engineering 9 (9) (2017) 8 pages.

[10] Z. Körpınar, *The Residual Power Series Method for Solving Fractional Klein-Gordon Equation*, Sakarya University Journal of Science 21 (3) (2017) 285–293.

[11] M. N. Alam, I. Talib, B. Bazighifan, D. N. Chalishajar, B. Almarri, *An Analytical Technique Implemented in the Fractional Clannish Random Walker's Parabolic Equation with Nonlinear Physical Phenomena*, Mathematics 9 (8) (2021) 801 10 pages.

[12] O. Guner, A. Bekir, Ö. Ünsal, *Two Reliable Methods for Solving the Time Fractional Clannish Random Walker's Parabolic Equation*, Optik 127 (20) (2016) 9571–9577.

[13] A. A. Al-Shawba, F. A. Abdullah, A. Azmi, M. A. Akbar, *An Extension of Double* $(G'/G\,,1/G)$-*expansion Method for Conformable Fractional Differential Equations*, Hindawi Complexity (2020) Article ID 7967328 13 pages.

[14] H. Bulut, B. Kılıç, *Exact Solutions for Some Fractional Nonlinear Partial Differential Equations via Kudryashov Method*, Physical Sciences 8 (11) (2013) 24–31.

[15] S. Ampun, S. Sungnul, S. Koonprasert, *New Exact Solutions for the Time fractional Clannish Random Walker's Parabolic Equation by the Improved* $(\tan(\Phi(\xi)/2))$-*expansion Method*, in: C, Likasiri, T. Muakthonglang, P. Phetpradab, T. Suksamran, P. Rojanakul, K. Sangkhanan (Eds.), The 22nd Annual Meeting in Mathematics, Thailand, 2017, 13 pages.

[16] E. Atilgan, O. Taşbozan, A. Kurt, S. T. Mohyud-Din, *Approximate Analytical Solutions of Conformable Time Fractional Clannish Random Walker's Parabolic (CRWP) Equation and Modified Benjamin-Bona-Mahony (BBM) Equation*, Universal Journal of Mathematics and Applications 3 (2) (2020) 61–68.

[17] A. Korkmaz, *Explicit Exact Solutions to Some One-Dimensional Conformable Time Fractional Equations*, Waves in Random and Complex Media 29 (1) (2019) 124–137.

[18] D. Kumar, S. C. Ray, *Application of Extended Exp(-$\varphi(\xi)$)-expansion Method to the Nonlinear Conformable Time-fractional Partial Differential Equations*, International Journal of Physical Research 7 (2) (2019) 81–93.

[19] A. R. Seadawy, A. Ali, M. H. Raddadi, *Exact and Solitary Wave Solutions of Conformable Time Fractional Clannish Random Walker's Parabolic and Ablowitz-Kaup-Newell-Segur Equations via Modified Mathematical Methods*, Results in Physics 26 (2021) Article ID 104374 10 pages.

[20] I. Siddique, K. B. Mehdi, M. A. Akbar, H. A. E-W. Khalifa, A. Zafar, *Diverse Exact Soliton Solutions of the Time Fractional Clannish Random Walker's Parabolic Equation via Dual Novel Techniques*, Hindawi Journal of Function Spaces 2022 (2022) Article ID 1680560 10 pages.

[21] P. Wang, W. Shan, Y. Wang, Q. Li, *Lie Symmetry Analysis, Explicit Solutions and Conservation Laws of the Time Fractional Clannish Random Walker's Parabolic Equation*, Modern Physics Letters B 35 (4) (2021) 2150074 16 pages.

[22] A. Almuneef, A. E. Hagag, *Approximate Solution of the Fractional Differential Equation via the Natural Decomposition Method*, Revista Internacional de Métodos Numércos para Cálculo y Diseño en Ingeniería 39 (4) (2023) 15 pages.

[23] O. Taki-Eddine, B. Abdefatah, *A Priori Estimates for Weak Solution for a Time-fractional Nonlinear Reaction-diffusion Equations with an Integral Condition*, Chaos, Solitons & Fractals 103 (2017) 79–89.

[24] M. Odabaşı, E. Mısırlı, *On the solutions of the Nonlinear Fractional Differential Equations via the Modified Trial Equation Method*, Mathematical Methods in the Applied Sciences 41 (2018) 904–911.

[25] A. Arqub, Series Solution of Fuzzy Differential Equations Under Strongly Generalized Differentiability, Journal of Advanced Research in Applied Mathematics 5 (1) (2013) 31–52.

[26] A. El-Ajou, O. A. Arqub, S. Momani, *Approximate Analytical Solution of the Nonlinear Fractional KdV-Burgers Equation: A New Iterative Algorithm*, Journal of Computational Physics 293 (2015) 81–95.

[27] M. Şenol, M. Alquran, H. D. Kasmaei, *On the Comparison of Perturbation-iteration Algorithm and Residual Power Series Method to Solve Fractional Zakharov-Kuznetsov Equation*, Results in Physics 9 (2018) 321–327.

[28] S. Kumar, A. Kumar, D. Baleanu, *Two Analytical Methods for Time-fractional Nonlinear Coupled Boussinesq-Burger's Equations Arise in Propagation of Shallow Water Waves*, Nonlinear Dynamics 85 (2016) 699–715.

[29] M. Alquran, *Analytical Solutions of Fractional Foam Drainage Equation by Residual Power Series Method*, Mathematical Sciences 8 (2014) 153–160.

[30] D. G. Prakasha, P. Veeresha, H. M. Baskonus, *Residual Power Series Method for Fractional Swift-Hohenberg Equation*, Fractal and Fractional 3 (1) (2019) 9 16 pages.

[31] A. Kumar, S. Kumar, M. Singh, *Residual Power Series Method for Fractional Sharma-Tasso-Olever Equation*, Communications in Numerical Analysis 2016 (1) (2016) Article ID cna-00235 10 pages.

[32] R. M. Jena, S. Chakraverty, *Residual Power Series Method for Solving Time-fractional Model of Vibration Equation of Large Membranes*, Journal of Applied and Computational Mechanics 5 (4) (2019) 603–615.

[33] K. K. Jaber, R. S. Ahmad, *Analytical Solution of the Time Fractional Navier-Stokes Equation*, Ain Shams Engineering Journal 9 (4) (2018) 1917–1927.

[34] A. Arafa, G. Elmahdy, *Application of Residual Power Series Method to Fractional Coupled Physical Equations Arising in Fluids Flow*, Hindawi International Journal of Differential Equations, 2018 (2018) Article ID 7692849 10 pages.

[35] M. H. Darassi, Y. A. Hour, *Residual Power Series Technique for Solving Fokker-Planck Equation*, Italian Journal of Pure and Applied Mathematics (44) (2020) 319–332.

[36] M. Inc, Z. S. Korpinar, M. M. A. Qurashi, D. Baleanu, *A New Method for Approximate Solutions of Some Nonlinear Equations: Residual Power Series Method*, Advances in Mechanical Engineering 8 (4) (2016) 1–7.

[37] H. M. Jaradat, S. Al-Shar'a, Q. J. A. Khan, M. Alquran, K. Al-Khaled, *Analytical Solution of Time-fractional Drinfeld-Sokolov-Wilson System Using Residual Power Series Method*, IAENG International Journal of Applied Mathematics 46 (1) (2016) 64–70.

[38] Z. Korpinar, M. Inc, E. Hınçal, D. Baleanu, *Residual Power Series Algorithm for Fractional Cancer Tumor Models*, Alexandria Engineering Journal 59 (3) (2020) 1405–1412.

[39] B. A. Mahmood, M. A. Yousif, *A Residual Power Series Technique for Solving Boussinesq-Burgers Equations*, Cogent Mathematics 4 (2017) Article ID 1279398 11 pages.

[40] B. A. Mahmood, M. A. Yousif, *A Novel Analytical Solution for the Modified Kawahara Equation Using the Residual Power Series Method*, Nonlinear Dynamics 89 (2017) 1233–1238.

[41] T. R. R. Rao, *Application of Residual Power Series Method to Time Fractional Gas Dynamics Equations*, in: S. Sivasankaran, S. Eswaramoorthi (Eds.), International Conference on Applied and Computational Mathematics, Tamilnadu, 2018, 5 pages.

[42] H. Tariq, G. Akram, *Residual Power Series Method for Solving Time-Space-Fractional Benney-Lin Equation Arising in Falling Film Problems*, Journal of Applied Mathematics and Computing 55 (2017) 683–708.

[43] H. Tariq, G. Akram, *New Traveling Wave Exact and Approximate Solutions for the Nonlinear Cahn-Allen Equation: Evolution of a Nonconserved Quantity*, Nonlinear Dynamics 88 (2017) 581–594.

[44] F. Tchier, M. Inc, Z. S. Korpinar, D. Baleanu, *Solutions of the Time Fractional Reaction-diffusion Equations with Residual Power Series Method*, Advances in Mechanical Engineering 8 (10) (2016) 10 pages.

[45] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[46] A. El-Ajou, O. A. Arqub, Z. A. Zhour, S. Momani, *New Results on Fractional Power Series: Theories and Applications*, Entropy 15 (12) (2013) 5305–5323.

[47] O. A. Arqub, Z. Abo-Hammour, R. Al-Badarneh, S. Momani, *A Reliable Analytical Method for Solving Higher-Order Initial Value Problems*, Hindawi Discrete Dynamics in Nature and Society (2013) Article ID 673829 12 pages.

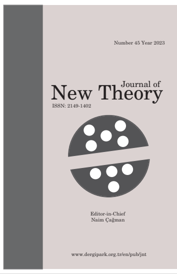
[48] O. A. Arqub, A. El-Ajou, Z. A. Zhour, S. Momani, *Multiple Solutions of Nonlinear Boundary Value Problems of Fractional Order: A New Analytic Iterative Technique*, Entropy 16 (2014) 471–493.

[49] O. A. Arqub, A. El-Ajou, A. S. Bataineh, I. Hashim, *A Representation of the Exact Solution of Generalized Lane-Emden Equations Using a New Analytical Method*, Hindawi Abstract and Applied Analysis (2013) Article ID 378593 10 pages.

New Theory Journal of

ISSN: 2149-1402

45 (2023) 30-45
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



A Unified Approach to Computing the Zeros of Orthogonal Polynomials

Ridha Moussa[1], **James Tipton**[2]

Article Info
Received: 26 Aug 2023
Accepted: 21 Nov 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1350502
Research Article

Abstract — We present a unified approach to calculating the zeros of the classical orthogonal polynomials and discuss the electrostatic interpretation and its connection to the energy minimization problem. This approach works for the generalized Bessel polynomials, including the normalized reversed variant, as well as the Vieté–Pell and Vieté–Pell–Lucas polynomials. We briefly discuss the electrostatic interpretation for each aforesaid case and some recent advances. We provide zeros and error estimates for various cases of the Jacobi, Hermite, and Laguerre polynomials and offer a brief discussion of how the method was implemented symbolically and numerically with Maple. In conclusion, we provide possible avenues for future research.

Keywords *Orthogonal polynomials, zeros, electrostatic interpretation*

Mathematics Subject Classification (2020) 33C45, 42C05

1. Introduction

The Jacobi, Hermite, and Laguerre polynomials are called classical orthogonal polynomials. They have served as objects of study as early as the 19th century and found applications in physics and approximation and number theory. For example, the Jacobi polynomials contain the Legendre polynomials as a special case, the coefficients in the expansion of the gravitational potential associated to a point mass [1]. The last two chapters of Szegö's classic text [2] focus on applications to interpolation and mechanical quadrature. More recently, the theory of orthogonal polynomials was used to present a formulation of quantum mechanics [3]. The classical orthogonal polynomials may be characterized as solutions to a Sturm–Liouville type equation of the form:

$$Q(x)y'' + L(x)y' + \lambda y = 0$$

In the case of the Jacobi polynomials, $Q(x) = 1-x^2$, $L(x) = \beta-\alpha-(\alpha+\beta+2)x$, and $\lambda = n(n+\alpha+\beta+1)$. For the Hermite polynomials, $Q(x) = 1$, $L(x) = -2x$, and $\lambda = 2n$. For the generalized Laguerre polynomials, $Q(x) = x$, $L(x) = (\alpha + 1 - x)$, and $\lambda = n$. In each case, the corresponding polynomial solutions satisfy an orthogonality condition of the form

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)W(x)dx = 0, \quad m \neq n$$

[1]rmoussa@nsu.edu; [2]jetipton@nsu.edu (Corresponding Author)
[1,2]Department of Mathematics, College of Science, Engineering, and Technology, Norfolk State University, Norfolk, United States

where for the Jacobi polynomials,

$$W(x) = \begin{cases} (1-x)^{\alpha}(1+x)^{\beta}, & -1 \leq x \leq 1 \\ 0, & |x| > 1 \end{cases}$$

for the Hermite polynomials,

$$W(x) = e^{-x^2}$$

and for the generalized Laguerre polynomials,

$$W(x) = \begin{cases} x^{\alpha}e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Some topics of recent interest in the area of orthogonal polynomials include generalized Bessel polynomials [4], Vieté–Pell–Lucas polynomials [5, 6], and quasi-orthogonal polynomials [7]. Dunster et al. [4] study the reverse generalized Bessel polynomials by combining a qualitative analysis involving Liouville–Green Stokes lines and anti-Stokes lines with a fixed point method to calculate their zeros. Tasci and Yalcin [6] present some fundamental properties of Vieté–Pell and Vieté–Pell–Lucas polynomials, such as their characteristic equations, Binet formulas, and generating functions. More recently, Kuloğlu et al. [5] study a generalization of these polynomials, incomplete generalized Vieté–Pell and Vieté–Pell–Lucas polynomials, presenting recurrence relations and their generating functions. The well-known electrostatic interpretation is a central reason for continued interest in the zeros of classical and nonclassical orthogonal polynomials. Ismail [8, 9] extensively researches this topic, including a recent study on the general theory of quasi-orthogonal polynomials, which included an investigation into an electrostatic equilibrium problem [7]. Another point of interest is that these zeros have been used in quadrature rules [10–12].

The above treatment can be provided to both the (normalized reversed) generalized Bessel polynomials and the Vieté–Pell and Vieté–Pell–Lucas polynomials. Take $Q(x) = x^2$, $L(x) = \alpha x + \beta$, and $\lambda = -n(n+\alpha-1)$, for the generalized Bessel polynomials, while the normalized reverse Bessel polynomials satisfy $Q(x) = x$, $L(x) = -(2n - 2 + a + 2x)$, and $\lambda = 2n$ (for more details, see [4, 13]). To provide Vieté–Pell and Vieté–Pell–Lucas polynomials a similar treatment, one can exploit their relationship to the Chebyshev polynomials to find that $Q(x) = 4-x^2$, $L(x) = -3x$, and $\lambda = n(n+1)$ for the Vieté–Pell polynomials and $Q(x) = 4 - x^2$, $L(x) = -x$, and $\lambda = n^2$ for the Vieté–Pell–Lucas polynomials.

We present a unified method to calculate the zeros of a class of orthogonal polynomials, including the classical orthogonal polynomials and generalized Bessel polynomials. We discuss the electrostatic interpretation for several cases and the connection to the energy minimization problem. The method in question differs from that used by Dunster et al. [4] and is more akin to an approach developed by Pasquini [14–16] and more recently [17]. In Section 2, we present the details of the method. In Section 3, we discuss the electrostatic interpretation in the context of the energy minimization problem. We briefly outline how to implement the method symbolically and numerically in Section 4. In Section 5, we provide some examples. The paper concludes with possible avenues for future investigation.

2. Method

Given a polynomial $y = c_n \prod_{i=1}^{n}(x - x_i)$, where $c_n, x_i \in \mathbb{R}$, $c_n \neq 0$, and the x_i are distinct,

$$\frac{y'}{y} = \sum_{i=1}^{n} \frac{1}{x - x_i} \tag{1}$$

$$\frac{y''}{y} = \sum_{i=1}^{n}\sum_{j\in J_i}\frac{1}{(x-x_i)(x-x_j)} = 2\sum_{i<j}\frac{1}{(x-x_i)(x-x_j)} \tag{2}$$

and

$$\frac{ax^2+bx+c}{(x-x_i)(x-x_j)} = a + \frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)} + \frac{ax_j^2+b_j+c}{(x_j-x_i)(x-x_j)} \tag{3}$$

where J_i consists of all integers in $[1,n]$ except i. Identities 1 and 2 follow from the product rule. Identity 3 follows from partial fraction decomposition.

Lemma 2.1. From the above setting,

$$(\mu x+\nu)\frac{y'}{y} = \mu n + \sum_{i=1}^{n}\frac{\nu+\mu x_i}{x-x_i}$$

and

$$(ax^2+bx+c)\frac{y''}{y} = a(n^2-n) + 2\sum_{i\neq j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)}$$

Proof.
The first identity follows directly from Identity 1 and some long division. For the second identity, combine Identities 2 and 3 and get the equality

$$(ax^2+bx+c)\frac{y''}{y} = 2\sum_{i<j}\left[a + \frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)} + \frac{ax_j^2+bx_j+c}{(x_j-x_i)(x-x_j)}\right]$$

There are $\frac{n^2-n}{2}$ terms in the above summation. Thus, $2\sum\limits_{i<j} a = a(n^2-n)$. Observe that

$$\begin{aligned}\sum_{i<j}\left[\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)} + \frac{ax_j^2+bx_j+c}{(x_j-x_i)(x-x_j)}\right] &= \sum_{i<j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)} + \sum_{i<j}\frac{ax_j^2+bx_j+c}{(x_j-x_i)(x-x_j)}\\ &= \sum_{i<j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)} + \sum_{j<i}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)}\\ &= \sum_{i\neq j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)}\end{aligned}$$

where the second to last equality follows from index swapping on the second summation. Putting the above calculations together yields the desired result. □

Proposition 2.2. Suppose y is a degree n polynomial solution to the differential equation

$$(ax^2+bx+c)y'' + (\mu x+\nu)y' + \kappa y = 0$$

If the zeros of y, $x_1, \cdots, x_n$ are distinct, then for each integer $k\in[1,n]$,

$$2\sum_{j\in J_k}\frac{ax_k^2+bx_k+c}{x_k-x_j} + \nu + \mu x_k = 0$$

Proof.
Divide by y and apply Lemma 1,

$$a(n^2-n)+2\sum_{i\neq j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)}+\mu n+\sum_{i=1}^{n}\frac{\nu+\mu x_i}{x-x_i}+\kappa=0 \Leftrightarrow 2\sum_{i\neq j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)}+\sum_{i=1}^{n}\frac{\nu+\mu x_i}{x-x_i}+M=0$$

$$\Leftrightarrow 2(x-x_k)\sum_{i\neq j}\frac{ax_i^2+bx_i+c}{(x_i-x_j)(x-x_i)}$$

$$+(x-x_k)\sum_{i=1}^{n}\frac{\nu+\mu x_i}{x-x_i}+(x-x_k)M=0$$

where $M=\kappa+a(n^2-n)+\mu n$ and k is some integer in $[1,n]$. As x approaches x_k, all terms will approach zero except those for $i=k$. Taking this limit gives the desired result. □

2.1. Jacobi Polynomials

For $\alpha,\beta>-1$, the degree n Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ solves the differential equation

$$(1-x^2)y''+(\beta-\alpha-(\alpha+\beta+2)x)y'+n(n+\alpha+\beta+1)y=0$$

Denote the n distinct zeros of $P_n^{(\alpha,\beta)}(x)$ by $x_1,\cdots,x_n$. Let $a=-1$, $b=0$, $c=1$, $\mu=-(\alpha+\beta+2)$, and $\nu=\beta-\alpha$. By Proposition 2.2, we see that the zeros must satisfy

$$2\sum_{j\in J_k}\frac{-x_k^2+1}{x_k-x_j}+\beta-\alpha-(\alpha+\beta+2)x_k=0 \Leftrightarrow \frac{\frac{1}{2}(\alpha+1)}{x_k-1}+\frac{\frac{1}{2}(\beta+1)}{x_k+1}+\sum_{j\in J_k}\frac{1}{x_k-x_j}=0 \tag{4}$$

The following theorem, which we have adapted from the Marsden and Hoffman classic [18], expresses the well-known fact that a continuous real-valued function over a compact set must attain an absolute maximum:

Theorem 2.3. [18] Suppose $A\subset\mathbb{R}^n$ and let $f:A\to\mathbb{R}$ be continuous. If $K\subset A$ is compact, then f is bounded on K. Furthermore, there exists an $x_0\in K$ such that $f(x_0)=\sup f(A)$.

In what follows, consider the real-valued function

$$f(\vec{x})=\prod_{k=1}^{n}\left[(1-x_k)^{(\alpha+1)/2}(1+x_k)^{(\beta+1)/2}\right]\prod_{i<j}(x_j-x_i)$$

defined over the set $D_n=\{\vec{x}\in\mathbb{R}^n:-1<x_1<x_2<\cdots<x_n<1\}$. Note that f is smooth over D_n and continuous on $\overline{D_n}$. Note that f vanishes on the boundary of D_n but is positive over D_n. Since f must attain an absolute maximum in $\overline{D_n}$, the previous observations show that this maximum occurs in D_n and must be a critical point.

Lemma 2.4. A point $\vec{x}\in D_n$ is a critical point of f if and only if Expression 4 holds for $k\in\{1,2,3,\cdots,n\}$.

PROOF.
Consider instead

$$\ln(f)=\sum_{k=1}^{n}\left[\frac{\alpha+1}{2}\ln(1-x_k)+\frac{\beta+1}{2}\ln(1+x_k)\right]+\sum_{i<j}\ln(x_j-x_i)$$

we have that

$$\frac{\partial\ln(f)}{\partial x_k}=\frac{f_{x_k}}{f}=\frac{\frac{1}{2}(\alpha+1)}{x_k-1}+\frac{\frac{1}{2}(\beta+1)}{x_k+1}+\sum_{j\in J_k}\frac{1}{x_k-x_j}$$

demonstrating the claim. □

Lemma 2.5. The function $\ln(f)$ has only one critical point in D_n.

PROOF.
The claim holds if we can show that $\ln(f)$ is concave in D_n. This, in turn, will follow if we can show that the Hessian of $\ln(f)$ is diagonally dominant and negative definite. To that extent, observe that

$$\frac{\partial^2 \ln(f)}{\partial x_k^2} = \frac{-\frac{1}{2}(\alpha+1)}{(x_k-1)^2} - \frac{\frac{1}{2}(\beta+1)}{(x_k+1)^2} - \sum_{j\in J_k} \frac{1}{(x_k-x_j)^2}$$

and for $i \neq j$ that

$$\frac{\partial^2 \ln(f)}{\partial x_i x_j} = \frac{1}{(x_i-x_j)^2}$$

The Hessian is thus diagonally dominant and negative definite. □

Example 2.6. To see the above results in action, set $\alpha = \beta = 0$, giving the Legendre differential equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

The general solution is

$$y = k_1 P_n(x) + k_2 Q_n(x)$$

where

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

is the n-th Legendre polynomial and

$$Q_n = \begin{cases} \frac{1}{2}\log\frac{1+x}{1-x}, & \text{if } n=0 \\ P_1(x)Q_0(x) - 1, & \text{if } n=1 \\ \frac{2n-1}{n}Q_{n-1}(x) - \frac{n-1}{n}Q_{n-2}(x), & \text{if } n \geq 2 \end{cases}$$

is the n-th Legendre function of the second kind [2]. For $n = 2$, the second Legendre polynomial $P_2(x) = \frac{3x^2-1}{2}$ solves the following differential equation:

$$(1-x^2)y'' - 2xy' + 6y = 0$$

The corresponding real-valued function on D_2 is

$$f(x_1, x_2) = (x_2 - x_1)\sqrt{(1-x_1^2)(1-x_2^2)}$$

which attains a global maximum at $x_1 = -1/\sqrt{3}$ and $x_2 = 1/\sqrt{3}$. It is clear that $P_2(x_1) = P_2(x_2) = 0$.

2.2. Hermite Polynomials

The degree n Hermite polynomial $H_n(x)$ solves the differential equation $y'' - 2xy' + 2ny = 0$. Denote the n distinct zeros of $H_n(x)$ by $x_1, \cdots, x_n$. Let $a = b = \nu = 0$, $c = 1$, $\mu = -2$, and $\kappa = 2n$. By Proposition 2.2, we observe that the zeros must satisfy

$$2\sum_{j\in J_k} \frac{1}{x_k - x_j} - 2x_k = 0 \Leftrightarrow \sum_{j\in J_k} \frac{1}{x_k - x_j} - x_k = 0 \tag{5}$$

In what follows, consider the real-valued function

$$f(\vec{x}) = \prod_{i<j}\Big[x_j - x_i\Big] e^{-\frac{1}{2}\sum_{k=1}^{n} x_k^2}$$

defined over the set $D_n = \{\vec{x} \in \mathbb{R}^n : -\infty < x_1 < x_2 < \cdots < x_n < \infty\}$. Note that f is smooth, positive and bounded over D_n but approaches 0 on the boundary. Thus, f must have a critical point in D_n.

Lemma 2.7. A point $\vec{x} \in D_n$ is a critical point of f if and only if Expression 5 holds for $k \in \{1, 2, 3, \cdots, n\}$.

Proof.
Consider instead

$$\ln(f) = \sum_{i<j} \ln(x_j - x_i) - \frac{1}{2}\sum_{k=1}^{n} x_k^2$$

we have that

$$\frac{\partial \ln(f)}{\partial x_k} = \frac{f_{x_k}}{f} = \sum_{j\in J_k} \frac{1}{x_k - x_j} - x_k$$

demonstrating the claim. □

Lemma 2.8. The function $\ln(f)$ has only one critical point in D_n.

Proof.
The claim holds if we can show that $\ln(f)$ is concave in D_n. This, in turn, will follow if we can show that the Hessian of $\ln(f)$ is diagonally dominant and negative definite. To that extent, observe that

$$\frac{\partial^2 \ln(f)}{\partial x_k^2} = -\sum_{j\in J_k} \frac{1}{(x_k - x_j)^2} - 1$$

and for $i \neq j$ that

$$\frac{\partial^2 \ln(f)}{\partial x_i x_j} = \frac{1}{(x_i - x_j)^2}$$

The Hessian is thus diagonally dominant and negative definite. □

2.3. Laguerre Polynomials

The degree n generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ solves the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0$$

Denote the n distinct zeros of $L_n^{(\alpha)}(x)$ by $x_1, \cdots, x_n$. Let $a = c = 0$, $b = 1$, $\mu = -1$, $\nu = \alpha + 1$, and $\kappa = n$. By Proposition 2.2, we see that the zeros must satisfy

$$2\sum_{j\in J_k} \frac{x_k}{x_k - x_j} + \alpha + 1 - x_k = 0 \Leftrightarrow \sum_{j\in J_k} \frac{1}{x_k - x_j} + \frac{\frac{1}{2}(\alpha+1)}{x_k} - \frac{1}{2} = 0 \tag{6}$$

In what follows, consider the real-valued function

$$f(\vec{x}) = \prod_{i<j} \left[x_j - x_i\right] \prod_{k=1}^{n} \left[x_k^{(\alpha+1)/2}\right] e^{-\frac{1}{2}\sum_{k=1}^{n} x_k}$$

defined over the set $D_n = \{\vec{x} \in \mathbb{R}^n : 0 < x_1 < x_2 < \cdots < x_n < \infty\}$. Note that f is smooth, positive and bounded over D_n but approaches 0 on the boundary. Thus, f must have a critical point in D_n.

Lemma 2.9. A point $\vec{x} \in D_n$ is a critical point of f if and only if Expression 6 holds for $k \in \{1, 2, 3, \cdots, n\}$.

Proof.
Consider instead

$$\ln(f) = \sum_{i<j} \ln(x_j - x_i) + \sum_{k=1}^{n} \left[\frac{\alpha+1}{2} \ln x_k\right] - \frac{1}{2}\sum_{k=1}^{n} x_k$$

we have that

$$\frac{\partial \ln(f)}{\partial x_k} = \frac{f_{x_k}}{f} = \sum_{j\in J_k} \frac{1}{x_k - x_j} + \frac{\frac{1}{2}(\alpha+1)}{x_k} - \frac{1}{2}$$

demonstrating the claim. □

Lemma 2.10. The function $\ln(f)$ has only one critical point in D_n.

Proof.
The claim holds if we can show that $\ln(f)$ is concave in D_n. This, in turn, will follow if we can show that the Hessian of $\ln(f)$ is diagonally dominant and negative definite. To that extent, observe that

$$\frac{\partial^2 \ln(f)}{\partial x_k^2} = -\sum_{j\in J_k} \frac{1}{(x_k - x_j)^2} - \frac{\frac{1}{2}(\alpha+1)}{x_k^2} < 0$$

and for $i \neq j$ that

$$\frac{\partial^2 \ln(f)}{\partial x_i x_j} = \frac{1}{(x_i - x_j)^2}$$

The Hessian is thus diagonally dominant and negative definite. □

2.4. Normalized Reversed Generalized Bessel Polynomials

The normalized reversed generalized Bessel polynomials (RGBP)

$$\hat{\theta}_n(z;a) = \theta_n\left(\frac{(2n+a-2)z}{2};a\right)$$

satisfy the differential equation

$$\frac{2z}{2n+a-2}\hat{\theta}_n'' - (2z+2)\hat{\theta}_n' + 2n\hat{\theta}_n = 0$$

Applying Proposition 2.2,

$$\sum_{j\in J_k} \frac{1}{z_k - z_j} + \frac{M_{n,a}}{z_k} + M_{n,a} = 0 \tag{7}$$

where $M_{n,a} = \dfrac{2-2n-a}{2}$, which correspond to the critical points of the function

$$f(\vec{z}) = \prod_{i<j}(z_j - z_i)\prod_{i=1}^{n}\left(z_i^{M_{n,a}}\right)e^{M_{n,a}\sum z_i}$$

with domain $D_n = \{\vec{z} : z_i \neq z_j \text{ if } i \neq j\}$.

Lemma 2.11. A point $\vec{z} \in D_n$ is a critical point of f if and only if Equation 7 holds for $k \in \{1, 2, 3, \cdots, n\}$.

Proof.
Consider instead

$$\ln(f) = \sum_{i<j}\ln(z_j - z_i) + M_{n,a}\sum_{i=1}^{n}\ln z_i - M_{n,a}\sum_{i=1}^{n} z_k$$

we have that

$$\frac{\partial \ln(f)}{\partial z_k} = \frac{f_{z_k}}{f} = \sum_{j\in J_k} \frac{1}{z_k - z_j} + \frac{M_{n,a}}{z_k} - M_{n,a}$$

demonstrating the claim. □

Lemma 2.12. The function $\ln(f)$ has only one critical point in D_n.

Proof.
The claim holds if we can show that $\ln(f)$ is concave in D_n. This, in turn, will follow if we can show that the Hessian of $\ln(f)$ is diagonally dominant and negative definite. To that extent, observe that

$$\frac{\partial^2 \ln(f)}{\partial z_k^2} = -\sum_{j\in J_k} \frac{1}{(z_k - z_j)^2} - \frac{M_{n,a}}{z_k^2}$$

and for $i \neq j$ that

$$\frac{\partial^2 \ln(f)}{\partial z_i z_j} = \frac{1}{(z_i - z_j)^2}$$

The Hessian is thus diagonally dominant and negative definite. □

2.5. Generalized Bessel Polynomials

In the case of generalized Beesel Polynomials (GBP), which satisfies the differential equation $x^2y'' + (\alpha x + \beta)y' + n(n + \alpha - 1)y = 0$, we can take $a = 1$, $b = c = 0$, $\mu = \alpha$, $\nu = \beta$, and $\kappa = n(n + \alpha - 1)$, to get that the zeros of the nth GBP satisfy

$$\sum_{j\in J_k} \frac{1}{x_k - x_j} + \frac{\frac{1}{2}\beta}{x_k^2} + \frac{\frac{1}{2}\alpha}{x_k} = 0 \tag{8}$$

These correspond to the critical points of the function

$$f(\vec{x}) = \prod_{i<j} (x_j - x_i) \prod_{i=1}^{n} \left(x_i^{\alpha/2}\right) e^{-\frac{1}{2}\beta \sum 1/x_i}$$

with domain $D_n = \{\vec{x} : x_i \neq x_j \text{ if } i \neq j\}$.

Lemma 2.13. A point $\vec{x} \in D_n$ is a critical point of f if and only if Equation 8 holds for $k \in \{1, 2, 3, \cdots, n\}$.

Proof.
Consider instead

$$\ln(f) = \sum_{i<j} \ln(x_j - x_i) + \frac{\alpha}{2}\sum_{i=1}^{n} \ln x_i - \frac{\beta}{2}\sum_{i=1}^{n} \frac{1}{x_i}$$

we have that

$$\frac{\partial \ln(f)}{\partial x_k} = \frac{f_{x_k}}{f} = \sum_{j\in J_k} \frac{1}{x_k - x_j} + \frac{\frac{1}{2}\alpha}{x_k} + \frac{\frac{1}{2}\beta}{x_k^2}$$

demonstrating the claim. □

Lemma 2.14. The function $\ln(f)$ has only one critical point in D_n.

Proof.
The claim holds if we can show that $\ln(f)$ is concave in D_n. This, in turn, will follow if we can show that the Hessian of $\ln(f)$ is diagonally dominant and negative definite. To that extent, observe that

$$\frac{\partial^2 \ln(f)}{\partial x_k^2} = -\sum_{j\in J_k} \frac{1}{(x_k - x_j)^2} - \frac{\frac{1}{2}\alpha}{x_k^2} - \frac{\frac{1}{2}\beta}{x_k^2}$$

and for $i \neq j$ that

$$\frac{\partial^2 \ln(f)}{\partial x_i x_j} = -\sum_{j\in J_k} \frac{1}{(x_i - x_j)^2}$$

The Hessian is thus diagonally dominant and negative definite. □

2.6. Vieté–Pell and Vieté–Pell–Lucas Polynomials

Vieté–Pell polynomials satisfy the differential equation $(4-x^2)y''-3xy'+n(n+1)y=0$ and Vieté–Pell–Lucas polynomials satisfy the differential equation $(4-x^2)y''-xy'+n^2y=0$ where n is the degree of the polynomial. Applying Proposition 2.2 in each case, we find that the zeros satisfy

$$\sum_{j\in J_k}\frac{1}{x_k-x_j}+\frac{\frac{3}{4}}{x_k+2}+\frac{\frac{3}{4}}{x_k-2}=0 \tag{9}$$

and

$$\sum_{j\in J_k}\frac{1}{x_k-x_j}+\frac{\frac{1}{2}}{x_k+2}+\frac{\frac{1}{2}}{x_k-2}=0 \tag{10}$$

respectively. Consider the functions

$$f(\vec{x})=\prod_{k=1}^{n}\left[(2-x_k)^{\frac{3}{4}}(2+x_k)^{\frac{3}{4}}\right]\prod_{i<j}(x_j-x_i)$$

and

$$g(\vec{x})=\prod_{k=1}^{n}\left[(2-x_k)^{\frac{1}{2}}(2+x_k)^{\frac{1}{2}}\right]\prod_{i<j}(x_j-x_i)$$

Proceeding as in the Jacobi case, one finds that:

Lemma 2.15. A point $\vec{x}\in D_n$ is a critical point of f (resp. g) if and only if Equation 9 (resp. Equation 10) holds for $k\in\{1,2,\cdots,n\}$.

PROOF.
Consider instead

$$\ln(f)=\sum_{i<j}\ln(x_j-x_i)+\frac{3}{4}\sum_{i=1}^{n}\ln(2-x_i)+\frac{3}{4}\sum_{i=1}^{n}\ln(2+x_i)$$

we have that

$$\frac{\partial\ln(f)}{\partial x_k}=\frac{f_{x_k}}{f}=\sum_{j\in J_k}\frac{1}{x_k-x_j}+\frac{\frac{3}{4}}{x_k-2}+\frac{\frac{3}{4}}{x_k+2}$$

demonstrating the claim for f. For g, replace $\frac{3}{4}$ with $\frac{1}{2}$. □

Lemma 2.16. The functions $\ln(f)$ and $\ln(g)$ have only one critical point in D_n.

PROOF.
The claim holds if we can show that both $\ln(f)$ and $\ln(g)$ are concave in D_n. This, in turn, will follow if we can show that their Hessians are diagonally dominant and negative definite. To that extent, observe that

$$\frac{\partial^2\ln(f)}{\partial x_k^2}=-\sum_{j\in J_k}\frac{1}{(x_k-x_j)^2}-\frac{\frac{3}{4}}{(x_k-2)^2}-\frac{\frac{3}{4}}{(x_k+2)^2}$$

and for $i\neq j$ that

$$\frac{\partial^2\ln(f)}{\partial x_i x_j}=-\sum_{j\in J_k}\frac{1}{(x_i-x_j)^2}$$

For g, replace $\frac{3}{4}$ with $\frac{1}{2}$. In either case, the Hessian is thus diagonally dominant and negative definite.
□

3. Electrostatic Interpretation and the Connection to the Energy Minimization Problem

As detailed by Szego in [2], the zeros of the classical orthogonal polynomials may be interpreted as the equilibrium position of an electrostatic problem. Stieltjes derived this connection in the case of the Jacobi polynomials in 1885. In this case, the problem is to find the position of $n \geq 2$ unit masses in the interval $[-1, 1]$ given two fixed positive masses $\frac{\alpha+1}{2}$ and $\frac{\beta+1}{2}$ at -1 and 1, respectively, for which electrostatic equilibrium is attained. The problem is solved by locating the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ [2]. Stieltjes provided a similar interpretation to the other classical orthogonal polynomials. The unit "masses" lie in the interval $(0, \infty)$ for the Laguerre polynomials with the restriction that the arithmetic mean of the unit charges is uniformly bounded and $(-\infty, \infty)$ for the Hermite polynomials with the restriction that the square arithmetic mean of the unit charges is uniformly bounded [2, 19].

A similar electrostatic interpretation may be presented for Vieté–Pell and Vieté–Pell–Lucas polynomials; the unit masses lie in the interval $[-2, 2]$, where we have positive mass $\frac{3}{4}$ at both -2 and 2 in Vieté–Pell case and positive mass $\frac{1}{2}$ at both -2 and 2 in Vieté–Pell–Lucas case. To the best of our knowledge, an electrostatic interpretation for the normalized RGBP and the GBP remains open. Interest in this connection has been steadily growing; see Marcellán, Martínez-Finkelshtein and Martínez-González's excellent survey [19] for details. As noted in [19], this is due in part to advances in the theory of logarithmic potentials as well as special functions from other areas of study, such as physics, combinatorics and number theory. Marcellán et al. [19] consider the following natural questions:

i. Can the electrostatic interpretation be generalized to other families of polynomials?

ii. Is it necessary to consider the global minimum of the energy? What about other equilibria?

In regards to the first question, it is noted in [19] that Ismail [8, 9] has provided an electrostatic model for general orthogonal polynomials, in which the external field is given as the sum of a long-range and short-range potential. For example, in [8], an explicit formula is given for the total energy of the model at the equilibrium position, and this energy is shown to be minimum. In the case of Freud weights, the total energy is shown to be asymptotic to $\frac{-n^2}{\alpha} \ln n$.

The authors [19] consider a more general case where the weight function satisfies the Pearson equation, particularly with the weight function corresponding to the Freud-type polynomials. It is noted that, in this case, the zeros of the Freud-type polynomials provide a critical configuration for the total energy. Still, it is an open problem whether the zeros are in a stable equilibrium. Regarding the second question, it is posited whether other types of equilibria are preserved in this case. The authors [19] present a max-min characterization of the zeros of the Jacobi polynomials, which is amenable to complex zeros of the family when the parameters fall out of the "classical" bounds. Loosely speaking, the characterization shows that of all possible compact continua from -1 to 1 (within the complex plane), the energy (minimized over n points for a given compact continua) is maximized over all compact continua when the n points are the zeros of the Jacobi polynomial.

More recently, regarding the first question above, Ismail and Wang developed an electrostatic interpretation of quasi-orthogonal polynomials in [7]. The main result is analogous to one given in [8]. In brief, it says that the equilibrium position of n unit charges in the presence of a given external field is uniquely attained at the zeros of the associated quasi-orthogonal polynomials.

4. Implementation

The above method was implemented using an amalgamation of symbolic and numerical approaches in Maple 2018. As an illustration, we present the steps taken to calculate the zeros of the Laguerre polynomial $L_9^{(0)}(x)$.

Step 1. We implement the initial guess procedure using the asymptotic formula in the Digital Library of Mathematical Functions section 18.16. [20]. The input for the initial guess procedure is α and n corresponding to the desired Laguerre polynomial $L_n^{(\alpha)}(x)$.

Step 2. We define the nonlinear system Expression 6, corresponding to the Laguerre polynomials.

Step 3. We calculate the Jacobian matrix using the built-in Maple function "Jacobian".

Step 4. For instructive purposes, we perform one iteration of Newton's method before writing a loop to iterate it ten times. We evaluate $L_9^{(0)}(x)$ at the approximated zeros as a quick check for accuracy. Maple produces the zeros after each iteration.

5. Illustrative Examples

In the following Tables 1-7, zeros approximations are listed for a variety of classical orthogonal polynomials of a specified degree n. The Jacobi column corresponds to the general Jacobi polynomial with $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{8}$. The Chebyshev column refers to the Chebyshev polynomials of the 1st kind, which correspond to Jacobi polynomials with $\alpha = \beta = \frac{-1}{2}$. The Gegenbauer column corresponds to Jacobi polynomials with $\alpha = \beta = \frac{1}{4}$. The Legendre column corresponds to Jacobi polynomials with $\alpha = \beta = 0$. The Laguerre column corresponds to the classical Laguerre polynomials. The General Laguerre column corresponds to Laguerre polynomials with $\alpha = 1$.

These results are obtained by using a straightforward implementation of Newton's method in the following way: Let n be a fixed natural number and consider the vector $\vec{x} = (x_1, x_2, \cdots, x_n)$ which contains the zeros of the orthogonal polynomial of degree n and $\vec{f} = (f_1, f_2, \cdots, f_n)$ be a vector-valued function. With this notation, we can write the system of equations as $\vec{f}(\vec{x}) = \vec{0}$. The nonlinear equation above is represented by Expression 4 in the case of the Jacobi polynomials, by Expression 5 in the case of the generalized Laguerre polynomials and by Expression 6 in the case of the Hermite polynomials. As for the initial guess, we relied on formulas given in Section 18.16 of [20].

Since the exact roots are known for the Chebyshev case, one may calculate the exact error. Thus, the same can be said for Vieté–Pell and Vieté–Pell–Lucas polynomials. Using the infinity norm we have for $n = 20$ the exact error is 6.749×10^{-17}, while for $n = 25$ the exact error is 6.297×10^{-17}. We provide error estimates in each case using the infinity norm.

Table 1. Error estimates for $n = 20$

Polynomial	Error Estimate
Legendre	$1.6064700823479085388 \times 10^{-16}$
General Jacobi $\alpha = 1/4$, $\beta = 1/8$	$2.0443258006786251481 \times 10^{-16}$
Gegenbauer	$2.4276213934271014550 \times 10^{-16}$
Chebyshev 1st Kind	$2.775557561562891350 \times 10^{-17}$
Classical Laguerre	$7.0122389569333584353 \times 10^{-15}$
General Laguerre $\alpha = 1$	$1.0850726264919494635 \times 10^{-14}$
Hermite	$1.2572574676652352260 \times 10^{-16}$

Table 2. Error estimates for $n = 25$

Polynomial	**Error Estimate**
Legendre	$2.1554887097997079110 \times 10^{-16}$
General Jacobi $\alpha = 1/4$, $\beta = 1/8$	$1.1129640277756032144 \times 10^{-16}$
Gegenbauer	$1.4290762156055028002 \times 10^{-16}$
Chebyshev 1st Kind	$1.3834655062070259971 \times 10^{-16}$
Classical Laguerre	$8.9260826473499668326 \times 10^{-15}$
General Laguerre $\alpha = 1$	$2.4825341532472729961 \times 10^{-16}$
Hermite	$4.7043788112778503159 \times 10^{-16}$

Table 3. Newton's Method results for $n = 20$ and 30 iterations

Jacobi	Chebyshev	Gegenbauer
-0.992143445584654	-0.996917333733128	-0.991034230192877
-0.962098494639669	-0.972369920397677	-0.959770495283156
-0.90991914333223	-0.923879532511287	-0.906555627647643
-0.83679724371729	-0.852640164354092	-0.832601034386276
-0.744414638606914	-0.760405965600031	-0.739597864903566
-0.634897399553407	-0.649448048330184	-0.629673706991205
-0.510766182525352	-0.522498564715949	-0.505343420884813
-0.374878073128636	-0.382683432365090	-0.369451505240359
-0.230360787671044	-0.233445363855905	-0.22510699141448
-0.080540669675107	-0.0784590957278449	-0.0756123031135758
0.071133877871622	0.078459095727845	0.0756123031135758
0.221171767113119	0.233445363855905	0.22510699141448
0.366119581638305	0.382683432365090	0.369451505240359
0.502641066039214	0.522498564715949	0.505343420884813
0.627593920566186	0.649448048330184	0.629673706991205
0.738102136937797	0.760405965600031	0.739597864903566
0.831622222573934	0.852640164354092	0.832601034386276
0.906001841773546	0.923879532511287	0.906555627647643
0.959529848266796	0.972369920397677	0.959770495283156
0.99098031100982	0.996917333733128	0.991034230192877

Table 4. Newton's Method results for $n = 20$ and 30 iterations

Legendre	Laguerre	General Laguerre
-0.993128599185095	0.0705398896919887	0.174906752386615
-0.963971927277914	0.372126818001611	0.587303080638269
-0.912234428251326	0.916582102483273	1.23822510183424
-0.839116971822219	1.70730653102834	2.13139626007693
-0.746331906460151	2.74919925530943	3.27213313351699
-0.636053680726515	4.04892531385089	4.66749446588836
-0.510867001950827	5.61517497086162	6.32653619767384
-0.37370608871542	7.45901745367106	8.26067095201373
-0.227785851141645	9.5943928695811	10.4841673812082
-0.0765265211334974	12.0388025469643	13.0148487721526
0.0765265211334973	14.8142934426307	15.8750870127848
0.227785851141645	17.9488955205194	19.0932519076063
0.373706088715419	21.478788240285	22.7058938881731
0.510867001950827	25.4517027931869	26.7611702293794
0.636053680726515	29.9325546317006	31.3245161370075
0.746331906460151	35.013434240479	36.4887033461491
0.839116971822219	40.8330570567286	42.3934227457745
0.912234428251326	47.6199940473465	49.2688138498685
0.963971927277914	55.8107957500639	57.5544209713148
0.993128599185095	66.5244165256157	68.3770378145523

Table 5. Newton's Method results for $n = 25$ and 30 iterations

Jacobi	Chebyshev	Gegenbauer
-0.994901665878463	-0.998026728428272	-0.994174685362604
-0.975360959985654	-0.982287250728689	-0.973813483540093
-0.941256322689963	-0.951056516295154	-0.938979875687483
-0.893091307988287	-0.90482705246602	-0.890187770804335
-0.831584665110590	-0.844327925502015	-0.828161987824607
-0.757655035013272	-0.770513242775789	-0.75382448992158
-0.672406769138576	-0.684547105928689	-0.668280361715944
-0.577113343604359	-0.587785252292473	-0.57280131807384
-0.473198311079934	-0.481753674101715	-0.468806780981076
-0.362214026547642	-0.368124552684678	-0.357842771895352
-0.245818453819013	-0.248689887164855	-0.241558925568652
-0.125750396162197	-0.125333233564304	-0.121683964806954
-0.0038035200079399	8.36062906219094E-18	2.87922513006768E-17
0.118200440621912	0.125333233564304	0.121683964806954
0.238438898854630	0.248689887164855	0.241558925568652
0.355115642426439	0.368124552684678	0.357842771895352
0.466487667212620	0.481753674101715	0.468806780981076
0.570891216112889	0.587785252292473	0.57280131807384
0.666766634609609	0.684547105928689	0.668280361715944
0.752681672462637	0.770513242775789	0.75382448992158
0.827352885709386	0.844327925502015	0.828161987824607
0.889664827092574	0.904827052466020	0.890187770804335
0.938686772318027	0.951056516295154	0.938979875687483
0.973686941970036	0.982287250728689	0.973813483540093
0.994146438181037	0.998026728428272	0.994174685362604

Table 6. Newton's Method results for $n = 25$ and 30 iterations

Legendre	Laguerre	General Laguerre
-0.995556969790498	0.0567047754527055	0.141236726258096
-0.976663921459518	0.299010898586989	0.473974537884425
-0.942974571228974	0.735909555435016	0.998383405621479
-0.894991997878275	1.36918311603519	1.71638168719236
-0.833442628760834	2.20132605372147	2.63069311458477
-0.759259263037358	3.23567580355804	3.7448777262027
-0.673566368473468	4.47649661507383	5.06340831233858
-0.577662930241223	5.92908376270045	6.59177560687321
-0.473002731445715	7.59989930995675	8.33662635980513
-0.361172305809388	9.49674922093243	10.3059430256137
-0.243866883720988	11.6290149117788	12.5092780113164
-0.12286469261071	14.0079579765451	14.9580612826525
-3.94351965660777E-18	16.6471255972888	17.6660089928416
0.12286469261071	19.5628980114691	20.6496747456588
0.243866883720988	22.775241986835	23.9292078044927
0.361172305809388	26.3087723909689	27.5294209021358
0.473002731445715	30.1942911633161	31.481337894211
0.577662930241223	34.471097571922	35.8245167628475
0.673566368473468	39.1906088039374	40.61069001566
0.759259263037358	44.422349336162	45.9097868582297
0.833442628760834	50.2645749938335	51.8206158754045
0.894991997878275	56.8649671739402	58.4916748142772
0.942974571228974	64.4666706159541	66.1674493598106
0.976663921459518	73.5342347921002	75.315081358106
0.995556969790498	85.260155562496	87.1338948199813

Table 7. Newton's Method results for $n = 12$ and 30 iterations

Hermite
0.440147298645308
0.881982756213821
1.32728070207308
1.77800112433715
2.23642013026728
2.70532023717303
3.1882949244251
3.69028287699836
4.21860944438656
4.78532036735222
5.41363635528003
6.16427243405245

6. Conclusion

We have presented a unified approach for calculating the zeros of the classical orthogonal polynomials and provided examples involving the Jacobi polynomials, including Chebyshev and Gengebauer, the General Laguerre polynomials, including Legendre and Laguerre and the Hermite polynomials. We are working on a similar approach that works for more general classes of polynomials, the Heine–Stieltjes polynomials. The difficulty lies in choosing a decent guess for the zeros of the given Heine–Stieltjes polynomial. We have had some success using the electrostatic interpretation for the initial guess, but more work is needed. Other future studies include expanding the family of orthogonal polynomials to which this method applies, expanding the electrostatic interpretation to other families of polynomials, such as the generalized Bessel polynomials, and exploring connections between orthogonal polynomials and Lucas polynomial identities, such as was done in [21].

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

References

[1] A. M. Legendre, *Recherches sur L'attraction des Sphéroïdes Homogènes*, Universitätsbibliothek Johann Christian Senckenberg 1785 (1785) 411–434.

[2] G. Szegö, Orthogonal Polynomials, 4th Edition, American Mathematical Society, Rhode Island, 1975.

[3] A. Alhaidari, *Representation of the Quantum Mechanical Wavefunction by Orthogonal Polynomials in the Energy and Physical Parameters*, Communications in Theoretical Physics 72 (1) (2019) 015104 15 pages.

[4] T. M. Dunster, A. Gil, D. Ruiz-Antolín, J. Segura, *Computation of the Reverse Generalized Bessel Polynomials and Their Zeros*, Computational and Mathematical Methods 3 (6) (2021) e1198 12 pages.

[5] B. Kuloğlu, E. Özkan, A. G. Shannon, *Incomplete Generalized Vieta–Pell and Vieta–Pell–Lucas Polynomials*, Notes on Number Theory and Discrete Mathematics 27 (4) (2021) 245–256.

[6] D. Tasci, F. Yalcin, *Vieta-Pell and Vieta-Pell-Lucas Polynomials*, Advances in Difference Equations 2013 (2013) Article Number 224 8 pages.

[7] M. E. H. Ismail, X.-S. Wang, *On Quasi-Orthogonal Polynomials: Their Differential Equations, Discriminants and Electrostatics*, Journal of Mathematical Analysis and Applications 474 (2) (2019) 1178–1197.

[8] M. E. H. Ismail, *An Electrostatics Model for Zeros of General Orthogonal Polynomials*, Pacific Journal of Mathematics 193 (2) (2000) 355–369.

[9] M. E. H. Ismail, *More on Electrostatic Models for Zeros of Orthagonal Polynomials*, Numerical Functional Analysis and Optimization 21 (1) (2007) 191–204.

[10] A. N. Lowan, N. Davids, A. Levenson., *Table of the Zeros of the Legendre Polynomials of Order 1-16 and the Weight Coefficients for Gauss' Mechanical Quadrature Formula*, Bulletin of the American Mathematical Society 48 (10) (1942) 739–743.

[11] R. E. Greenwood, J. J. Miller, *Zeros of the Hermite Polynomials and Weights for Gauss' Mechanical Quadrature Formula*, Bulletin of the American Mathematical Society 54 (1948) 765–769.

[12] H. E. Salzer, R. Zucker, *Table of the Zeros and Weight Factors of the First Fifteen Laguerre Polynomials*, Bulletin of the American Mathematical Society 55 (10) (1949) 1004–1012.

[13] H. L. Krall, O. Frink, *A New Class of Orthogonal Polynomials: The Bessel Polynomials*, Transactions of the American Mathematical Society 65 (1) (1949) 100–115.

[14] L. Pasquini, *Polynomial Solutions to Second Order Linear Homogeneous Ordinary Differential Equations. Properties and Approximation*, Calcolo 26 (1989) 167–183.

[15] L. Pasquini, *On the Computation of the Zeros of the Bessel Polynomials*, in: R. V. M. Zahar (Ed.), Approximation and Computation: A Festschrift in Honor of Walter Gautschi, Vol. 119 of *ISNM International Series of Numerical Mathematics*, Birkhäuser, Boston, 1994, pp. 511–534.

[16] L. Pasquini, *Accurate Computation of the Zeros of the Generalized Bessel Polynomials*, Numerische Mathematik 86 (3) (2000) 507–538.

[17] S. Steinerberger, *Electrostatic Interpretation of Zeros of Orthogonal Polynomials*, Proceedings of the American Mathematical Society 146 (12) (2018) 5323–5331.

[18] J. E. Marsden, M. J. Hoffman, Elementary Classical Analysis, 2nd Edition, W. H. Freeman, San Francisco, 1993.

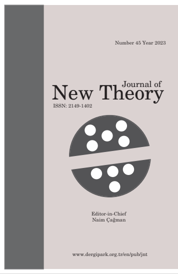
[19] F. Marcellán, A. Martínez-Finkelshtein, P. Martínez-González, *Electrostatic Models for Zeros of Polynomials: Old, New, and Some Open Problems*, Journal of Computational and Applied Mathematics 207 (2) (2007) 258–272.

[20] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, M. A. McClain, *NIST Digital Library of Mathematical Functions* (2010), http://dlmf.nist.gov/, Accessed 20 Nov 2023 to Release 1.1.0 of 2020-12-15.

[21] W. M. Abd-Elhameed, A. Napoli, *Some Novel Formulas of Lucas Polynomials via Different Approaches*, Symmetry 15 (1) (2023) 185 19 pages.

Journal of
New Theory
ISSN: 2149-1402

45 (2023) 46-56
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



On Non-Archimedean $\mathcal{L}$-Fuzzy Vector Metric Spaces

Şehla Eminoğlu[1]

Article Info
Received: 29 Aug 2023
Accepted: 21 Nov 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1351848
Research Article

Abstract — This paper contributes to the broader studies of fuzzy vector metric spaces and fuzzy metric spaces based on order structures beyond the unit interval. It defines the notions of the left (right) order convergence and continuity in non-Arcimedean $\mathcal{L}$-fuzzy vector metric spaces. The notation $\mathcal{M}_E(a, b, s)$ means the nearness between a and b according to any positive vector s. This study exemplifies definitions and reaches some well-known results. Moreover, it proposes the concept of $\mathcal{L}$-fuzzy vector metric diameter and studies some of its basic properties. Further, the present paper proves the Cantor intersection theorem and the Baire category theorem via these concepts. Finally, this study discusses the need for further research.

Keywords *Non-Archimedean $\mathcal{L}$-fuzzy vector metrics, left and right order convergence, $\mathcal{L}$-fuzzy vector diameter, Riesz spaces*

Mathematics Subject Classification (2020) 46A40, 47H10

1. Introduction

In the field of engineering design, it is often the case that there is no clear solution or design, which often leads to fuzziness, and Zadeh [1] proposed a rule to address such issues in engineering and design. Goguen [2] expanded Zadeh's study with a fresh viewpoint, considering the ordered structures beyond the unit interval. It is typically necessary for a partially ordered set (poset) to be at least a complete lattice with distributive law to query what the maximum and minimum values of a fuzzy set are. A detailed study about these concepts can be found in [2, 3].

Moreover, Menger [4] presented probabilistic metric spaces and associated ideas. The notion was then greatly improved by Schweizer and Sklar [5, 6]. Subsequently, Kramosil and Michálek in [7] provided an equivalent definition for the term probabilistic metric in the form of fuzzy metric spaces, which George and Veeramani [8] later adapted to provide a Hausdorff topology. The degree of nearness between two elements a and b of a set X concerning the real number s is the subject of the notion of fuzzy metric. The reality of X having a vector space structure is a common occurrence (for more details, see [9–11]). Alternatively, the distance in a Riesz space can be defined as a vector; more details can be found in [12–15].

In this study, we consider the parameter s as a vector based on $\mathcal{L}$-fuzzy sets given by Goguen and the fuzzy metric space provided by Kramosil and Michálek. In this case, the order structure must be added to the concept of left-hand continuity. Thus, we define left (right) order continuity to construct

[1]sehla.eminoglu@ostimteknik.edu.tr (Corresponding Author)
[1]Department of Industrial Engineering, Faculty of Engineering, Ostim Technical University, Ankara, Türkiye

$\mathcal{L}$-fuzzy vector metric spaces and non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. Then, we obtain some new results and provide Cantor's intersection theorem and Baire's theorem in non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces.

2. Preliminaries

This section provides some basic notions to be needed in the next section. The concept of an $\mathcal{L}$-fuzzy set was introduced by Goguen [2], who generalized the notion of a fuzzy set nicely introduced by Zadeh. Goguen defined an $\mathcal{L}$-fuzzy set as a function that maps elements of a universe of discourse to elements of a complete lattice $\mathcal{L}$, where each lattice element represents the degree of membership of the corresponding universe element in the fuzzy set. He defined $\mathcal{L}$-fuzzy set in the following manner.

Definition 2.1. [2] Let $X \neq \emptyset$ and $\mathcal{L} = (L, \leqslant_L)$ be a complete lattice with distributive law. Then, an $\mathcal{L}$-fuzzy set $\mathcal{A}$ is a function such that $\mathcal{A} : X \to L$ and $\mathcal{A}(a)$, for each $a \in X$, means the degree of a in L.

Definition 2.2. [10] Let $X \neq \emptyset$. Then, an intuitionistic $\mathcal{L}$-fuzzy set $\mathcal{A}_{\xi,\vartheta}$ is an object on X such that $\mathcal{A}_{\xi,\vartheta} = \{(\xi_{\mathcal{A}}(a), \vartheta_{\mathcal{A}}(a)) : a \in X\}$, where the notations $\xi_{\mathcal{A}}(a)$ and $\vartheta_{\mathcal{A}}(a)$ represent the membership and non-membership degrees of a, respectively, and satisfy the condition $\xi_{\mathcal{A}}(a) + \vartheta_{\mathcal{A}}(a) \leqslant_L 1_{\mathcal{L}}$.

Goguen [2] and Sadati et al. [10] provided the definitions of t-norm, decreasing negation function, and involutive negation as follows:

Definition 2.3. [2,10] A t-norm on $\mathcal{L}$ is a function $\mathcal{T} : L^2 \to L$ holding following properties, for all $k, l, m, n \in L$, where $\inf L = 0_{\mathcal{L}}$ and $\sup L = 1_{\mathcal{L}}$.

i. $\mathcal{T}(k, 1_{\mathcal{L}}) = k$ (boundary condition)

ii. $\mathcal{T}(k, l) = \mathcal{T}(l, k)$ (commutativity)

iii. $\mathcal{T}(k, \mathcal{T}(l, m)) = \mathcal{T}(\mathcal{T}(k, l), m)$ (associativity)

iv. $k \leqslant_L m$ and $l \leqslant_L n \Rightarrow \mathcal{T}(k, l) \leqslant_L \mathcal{T}(m, n)$ (monotonicity)

Definition 2.4. [2,10] Let $\mathcal{L} = (L, \leqslant_L)$ be a complete lattice. Then, $\mathcal{N} : L \to L$ is a decreasing negation function on $\mathcal{L}$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. Furthermore, $\mathcal{N}$ is called an involutive negation if $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$.

Aliprantis, in his books Infinite Dimensional Analysis [12] and Positive Operators [13], discussed the concept of ordered vector space in the following fashion.

Definition 2.5. [12,13] Let E be a real vector space. If E has an order relation $\leq$, which is compatible with the algebraic structure of E in terms of the following two axioms:

i. if $s \leq u$, then $s + w \leq u + w$, for all $w \in E$

ii. if $s \leq u$, then $\gamma s \leq \gamma u$, for all $\gamma \in \mathbb{R}^+$

then E is called an ordered vector space.

For any two vectors $s, u \in E$, the notation $s \leq u$ can be represented by $u \geq s$ in another way. If $\theta \leq s$ where θ represents the zero vector of E, then the vector s is called positive. The set of all the positive vectors of E is denoted by $E_+ := \{s \in E : \theta \leq s\}$.

Aliprantis et al. [12, 13] also proposed the concept of Riesz spaces and some related concepts in the following form.

Definition 2.6. [12,13] Let E be an ordered vector space. For all $s, u \in E$, if E has the supremum and the infimum of the set $\{s, u\}$, then E is called a Riesz space or a vector lattice. The notations used for $\sup\{s, u\}$ and $\inf\{s, u\}$ are as follows:

$$s \vee u = \sup\{s, u\} \quad \text{and} \quad s \wedge u = \inf\{s, u\}$$

An example of a Riesz space is the space of real-valued continuous functions on a set X, considering the pointwise ordering, defined as follows: $f_1 \leq f_2$ in E if and only if $f_1(a) \leq f_2(a)$, for all $a \in X$. The lattice operation in any function space E can be defined as

$$[f_1 \vee f_2](a) = \max\{f_1(a), f_2(a)\} \quad \text{and} \quad [f_1 \wedge f_2](a) = \min\{f_1(a), f_2(a)\}$$

for each pair $f_1, f_2 \in E$ and for all $a \in X$.

We will denote Riesz spaces with the letter E in the rest of this study.

Theorem 2.7. [12,13] For all $s, u, w \in E$, the following properties hold:

i. $s \vee u = -[(-s) \wedge (-u)]$ and $s \wedge u = -[(-s) \vee (-u)]$

ii. $s + u = (s \wedge u) + (s \vee u)$

iii. $s + (u \vee w) = (s + u) \vee (s + w)$ and $s + (u \wedge w) = (s + u) \wedge (s + w)$

iv. $\gamma(s \vee u) = (\gamma s) \vee (\gamma u)$ and $\gamma(s \wedge u) = (\gamma s) \wedge (\gamma u)$, for all $\gamma \geqslant 0$

For any vector $s \in E$, the positive part, negative part, and absolute value of s are denoted by s^+, s^-, and $|s|$, respectively, and defined as follows:

$$s^+ := s \vee \theta, \quad s^- := (-s) \vee \theta, \quad \text{and} \quad |s| = s \vee (-s)$$

Theorem 2.8. [12,13] For any vector $s \in E$, the following properties hold:

i. $s = s^+ - s^-$

ii. $|s| = s^+ + s^-$

iii. $s^+ \wedge s^- = \theta$

A sequence $(s_n) \subseteq E$ is decreasing, denoted by $s_n \downarrow$, if and only if $n \geqslant m$ implies $s_n \leq s_m$. In addition the notation $s_n \downarrow s$ means $s_n \downarrow$ and $\inf\{s_n\} = s$. Similarly, a sequence $(s_n) \subseteq E$ is increasing, represented by $s_n \uparrow$, if and only if $n \leqslant m$ implies $s_n \leq s_m$. In addition the notation $s_n \uparrow s$ means $s_n \uparrow$ and $\sup\{s_n\} = s$.

Aliprantis et al. [12,13] set forth the concepts of ordered convergence and lattice norm in the following way.

Definition 2.9. [12,13] Let $(s_n) \subseteq E$ be a sequence and $s \in E$ be a vector. Then, (s_n) is called order convergent to s, denoted by $s_n \xrightarrow{o} s$, if there exists another sequence (u_n) satisfying $|s_n - s| \leq u_n \downarrow \theta$.

Definition 2.10. [3,13] Let s and u be some vectors of E and $\|\cdot\|$ be a defined norm on E. If $|s| \leq |u|$ implies $\|s\| \leq \|u\|$, then $\|\cdot\|$ is called a lattice norm. In addition, a Riesz space equipped with this norm is called a normed Riesz space.

The notion of vector metric spaces, where the distance function takes values in Riesz spaces, was first mentioned in [14].

Definition 2.11. [14] Let $X \neq \emptyset$, E be a Riesz space, and $d_E : X \times X \to E$ be a function. Then, (X, d_E) is called a vector metric space if the function d_E satisfies the following properties, for all $a, b, c \in X$:

i. $\theta \leq d_E(a, b)$

ii. $d_E(a,b) = \theta$ if and only if $a = b$

iii. $d_E(a,b) = d_E(b,a)$

iv. $d_E(a,c) \leq d_E(a,b) + d_E(b,c)$

Since the set of real numbers $\mathbb{R}$ is a Riesz space with the usual ordering, it is obvious that every metric space is a vector metric space.

Example 2.12. [14] Every Riesz space E is a vector metric space with the function $d_E : E \times E \to E$ defined by $d_E(a,b) = |a - b|$. This vector metric is called the absolute valued vector metric on E.

To set up the definition of non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces, we benefit from the definition of fuzzy metric space suggested by Kramosil and Michálek [7].

Definition 2.13. [7] Let $X \neq \emptyset$, M be a fuzzy set on $X \times X \times [0,\infty)$, and $\mathcal{T}$ be a continuous t-norm. Then, the triple $(X, M, *)$ is a fuzzy metric space as Kramosil and Michálek describe, if for all $a, b, c \in X$ and $0 < s, u$, the following properties hold:

i. $M(a,b,0) = 0$

ii. $M(a,b,s) = 1$ if and only if $a = b$

iii. $M(a,b,s) = M(b,a,s)$

iv. $\mathcal{T}(M(a,b,s), M(b,c,u)) \leqslant M(a,c,s+u)$

v. $M(a,b,.) : [0,\infty) \to [0,1]$ is left-continuous

Here, the notation $M(a,b,s)$ denotes the nearness degree between a and b according to s.

3. Main Results

We define the concepts of left and right-order convergence and continuity. Thanks to these concepts, new ideas on $\mathcal{L}$-fuzzy vector metric space will be built.

Definition 3.1. Let $(s_n) \subseteq E$ be a sequence and $s \in E$ be a vector. Then,

i. (s_n) is called left-order convergent to some vector s, denoted by $s_n \xrightarrow{o^-} s$, if there exists another sequence (u_n) satisfying $(s_n - s)^- \leq u_n \downarrow \theta$.

ii. (s_n) is called right-order convergent to some vector s, denoted by $s_n \xrightarrow{o^+} s$, if there exists another sequence (u_n) satisfying $(s_n - s)^+ \leq u_n \downarrow \theta$.

Definition 3.2. Let $X \neq \emptyset$, $\mathcal{M}_E$ be an $\mathcal{L}$-fuzzy set on $X \times X \times E^+$, and $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}$. Then, the triple $(X, \mathcal{M}_E, \mathcal{T})$ is an $\mathcal{L}$-fuzzy vector metric space if for all $a, b, c \in X$ and $s, u \in E_+$, the following properties hold:

i. $\mathcal{M}_E(a,b,\theta) = 0_{\mathcal{L}}$

ii. $\mathcal{M}_E(a,b,s) = 1_{\mathcal{L}}$ if and only if $a = b$

iii. $\mathcal{M}_E(a,b,s) = \mathcal{M}_E(b,a,s)$

iv. $\mathcal{T}(\mathcal{M}_E(a,b,s), \mathcal{M}_E(b,c,u)) \leqslant_L \mathcal{M}_E(a,c,s+u)$

v. $\mathcal{M}_E(a,b,.) : E_+ \to L$ is left-order-continuous

If the condition *vi* below is used instead of the condition *iv*, then the triple $(X, \mathcal{M}_E, \mathcal{T})$ is said to be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

vi. $\mathcal{T}(\mathcal{M}_E(a,b,s), \mathcal{M}_E(b,c,u)) \leqslant_L \mathcal{M}_E(a,c,s \vee u)$

It can be observed that every non-Archimedean $\mathcal{L}$-fuzzy vector metric space is an $\mathcal{L}$-fuzzy vector metric space because the triangular inequality *vi* implies *iv*. Moreover, if $s \wedge u = 0$, then every $\mathcal{L}$-fuzzy vector metric space becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Lemma 3.3. In a non-Archimedean $\mathcal{L}$-fuzzy vector metric space, the function $\mathcal{M}_E(a, b, .)$ is non-decreasing, for all $a, b \in X$.

Lemma 3.4. In a non-Archimedean $\mathcal{L}$-fuzzy vector metric space, the following statements hold:

i. If $s_n \overset{o}{\to} s$ and $s_n \overset{o}{\to} u$, then $\mathcal{M}_E(a, b, s) = \mathcal{M}_E(a, b, u)$

ii. If $s_n \overset{o}{\to} s$ and $u \leqslant s_n$ hold for $n \in \mathbb{N}$, then $\mathcal{M}_E(a, b, u) \leqslant_L \mathcal{M}_E(a, b, s)$

iii. If $s_n \downarrow$ and $s_n \overset{o}{\to} s$, which means both $s_n \overset{o^+}{\to} s$ and $s_n \downarrow s$, then for all $n \in \mathbb{N}$, $\mathcal{M}_E(x, y, s) \leqslant_L \mathcal{M}_E(x, y, s_n)$

iv. If $s_n \uparrow$ and $s_n \overset{o}{\to} s$, which means both $s_n \overset{o^-}{\to} s$ and $s_n \uparrow s$, then for all $n \in \mathbb{N}$, $\mathcal{M}_E(a, b, s_n) \leqslant_L \mathcal{M}_E(a, b, s)$

v. If $s_n \overset{o}{\to} s$ and $u_n \overset{o}{\to} u$, then $\lim\limits_{n\to\infty} \mathcal{M}_E(a, b, ks_n + ru_n) = \mathcal{M}_E(a, b, ks + ru)$, for all $n \in \mathbb{N}$ and $k, r \in \mathbb{R}$

Corollary 3.5. By the definition $s^+ := s \vee \theta$, if $s, u \in E_+$, then $\mathcal{M}_E(a, c, s \vee u) = \mathcal{M}_E(a, c, s^+ \vee u^+)$.

Theorem 3.6. Let $\emptyset \neq A \subseteq E_+$ and $s \in E_+$. If $\inf A$ exists, then the infimum of the set $(s \vee A)$ exists and

$$\mathcal{T}(\mathcal{M}_E(a, b, s), \mathcal{M}_E(b, c, \inf A)) \leqslant_L \mathcal{M}_E(a, c, s \vee \inf A) = \mathcal{M}_E(a, c, \inf(s \vee A))$$

Proof.
Assume that $\inf A$ exists. Let $u = \inf A$, then $s \vee u \leq s \vee w$, for all $w \in A$ and $s \in E$, which means that $s \vee u$ is a lower bound of the set $s \vee A$ and $\mathcal{M}_E(a, b, s \vee u) \leqslant_L \mathcal{M}_E(a, b, s \vee w)$ holds. Let r be another lower bound. To show that $s \vee u$ is the greatest lower bound of $s \vee A$, we must show $r \leq s \vee u$. Besides, $w + s = (s \wedge w) + (s \vee w)$, for all $w \in E$. From the properties in Theorem 2.7,

$$w = (s \wedge w) + (s \vee w) - s \geq (s \wedge w) + r - s \geq (s \wedge u) + r - s$$

Because $\inf A = u$, it follows that $u \geq (s \wedge u) + r - s$. This implies $u \geq (u + s) - (s \vee u) + r - s$. Thus, $s \vee u \geq r$ is obtained. It means that $s \vee u$ is the greatest lower bound. Then, $\inf(s \vee A)$ exists and $\inf(s \vee A) = s \vee \inf A$. Consequently,

$$\mathcal{T}(\mathcal{M}_E(a, b, s), \mathcal{M}_E(b, c, \inf A)) \leqslant_L \mathcal{M}_E(a, c, s \vee \inf A) = \mathcal{M}_E(a, c, \inf(s \vee A))$$

□

Example 3.7. Let $(X, \mathcal{M}_E, \mathcal{T})$ be an $\mathcal{L}$-fuzzy vector space with (s_n) and (u_n) in $C[0, 1] = \{h \mid h : [0, 1] \to \mathbb{R}$ is a continuous function$\}$ define as follows:

$$s_n = \begin{cases} 0 & , \quad x \in [0, \frac{1}{n+1}] \\ \frac{(n+1)x-1}{n} & , \quad x \in (\frac{1}{n+1}, 1] \end{cases}$$

$$u_n = \begin{cases} -(n+1)x + 1 & , \quad x \in [0, \frac{1}{n+1}] \\ 0 & , \quad x \in (\frac{1}{n+1}, 1] \end{cases}$$

Since $s_n \uparrow 1_{\mathcal{L}} = \mathbb{1}$ and $u_n \downarrow 0_{\mathcal{L}} = \theta$, then $s_n \wedge u_n = \theta$ holds, where $\mathbb{1}(x) = 1$ and $\theta(x) = 0$ are constant functions in $C[0, 1]$. Hence, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.8. Let the pair (X, d_E) be a bounded vector metric space such that $d_E(a, b) < k$, for all $a, b \in X$ and $k \in E$. In addition, let $g : E_+ \to (\|k\|, +\infty)$ be an increasing continuous function.

Define $\mathcal{T}(l,t) = \sup\{l+t-1_\mathcal{L}, 0_\mathcal{L}\}$ and the function $\mathcal{M}_E$ by

$$\mathcal{M}_E(a,b,s) = 1_\mathcal{L} - \frac{d_E(a,b)}{g(s)}$$

In this case, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.9. For $\mathcal{T}(k,l) = \inf\{k,l\}$, define the function $\mathcal{M}_E$ by

$$\mathcal{M}_E(a,b,s) = \begin{cases} 1, & a = b \\ \varphi(s), & a \neq b \end{cases}$$

where $\varphi : E_+ \to [0_\mathcal{L}, 1_\mathcal{L})$ is an increasing continuous function. In this case, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.10. Let the pair (X, d_E) be a vector metric space and E be a normed Riesz space. For all $a, b \in X$ and $s \in E_+$ and for $\mathcal{T}(k,l) = \inf\{k,l\}$, define the function $\mathcal{M}_E$ by

$$\mathcal{M}_E(a,b,s) = \frac{\|s\|}{\|s\| + \|d_E(a,b)\|}$$

Particularly, $\mathcal{M}_E$ is called the standard $\mathcal{L}$-fuzzy vector metric induced by the vector metric d_E. Then, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Moreover, this example is used successfully in color image processing in [9,11] as a real-life application. For this, let F_i and F_j be two image pixels. In this case, the spatial closeness between F_i and F_j is calculated with

$$\mathcal{S}(F_i, F_j, s) = \frac{s}{s + \|d_E(F_i, F_j)\|}$$

where $s \in \mathbb{R}^+$ is a parameter adjusting the sensitivity of S.

Definition 3.11. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space. In this case, $\mathcal{B}_E(a,r,s)$ and $\mathcal{B}_E[a,r,s]$, for $s \in E_+$, with center $a \in X$ and radius $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ are defined as follows:

$$\mathcal{B}_E(a,r,s) = \{b \in X : \mathcal{M}_E(a,b,s) >_L \mathcal{N}(r)\}$$

and

$$\mathcal{B}_E[a,r,s] = \{b \in X : \mathcal{M}_E(a,b,s) \geqslant_L \mathcal{N}(r)\}$$

Corollary 3.12. A subset $\Omega \subseteq X$ is said to be open if for $a \in \Omega$, there exist an $s \in E_+$ and a radius $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that $\mathcal{B}_E(a,r,s) \subset \Omega$. Then, every open ball is an open set. Futhermore, $\tau_{\mathcal{M}_E} = \{\Omega \subseteq X : \Omega \text{ is open}\}$ is a topology on X.

Definition 3.13. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

i. Let $\emptyset \neq \Omega \subseteq X$. For every $a, b \in \Omega$ and $s \in E_+$, if there exists an $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that $\mathcal{M}_E(a,b,s) \geqslant_L \mathcal{N}(r)$, then Ω is bounded. Moreover, for all $n \in \mathbb{N}$, $(a_n) \subseteq X$ is called bounded if there exists an $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that $(a_n) \subseteq \mathcal{B}_E[a,r,s]$.

ii. For every $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, $(a_n) \subseteq X$ is convergent to $a \in X$ if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s) >_L \mathcal{N}(\varepsilon)$, for all $n \geqslant n_0$ and denoted by

$$\lim_{n\to\infty} \mathcal{M}_E(a_n, a, s) = 1_\mathcal{L} \text{ or } a_n \overset{\mathcal{M}_E}{\to} a$$

iii. For each $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, $(a_n) \subseteq X$ is a Cauchy sequence in X if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a_m, s) >_L \mathcal{N}(\varepsilon)$, for all $n, m \geqslant n_0$.

iv. $(X, \mathcal{M}_E, \mathcal{T})$ is complete if and only if every Cauchy sequence in X is convergent.

v. Let $\Omega \subseteq X$. Then, Ω is said to be closed if $(a_n) \subseteq \Omega$ and $a_n \stackrel{\mathcal{M}_E}{\to} a$ imply $a \in \Omega$.

In the following example, we provide a nonconvergent sequence in a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.14. Let $X = (a_n) \cup \{1\}$ for $(a_n) \subseteq \mathbb{R}^+$ with $a_n \uparrow 1$. Define $\mathcal{M}_E(a_n, a_n, s) = 1_\mathcal{L}$, $\mathcal{M}_E(1, 1, s) = 1_\mathcal{L}$, and

$$\mathcal{M}_E(a_n, 1, s) = \begin{cases} \inf\{a_n, s\} & , \quad \theta < s < \mathbb{1} \\ a_n & , \quad s > \mathbb{1} \end{cases}$$

for all n and $s \in E^+$. Then, $(X, \mathcal{M}_E, \mathcal{T})$ is a non-Archimedean $\mathcal{L}$-fuzzy vector metric space where $\mathcal{T}(k, l) = \inf\{k, l\}$. Since $\lim_{n\to\infty} \mathcal{M}_E(a_n, 1, \frac{\mathbb{1}}{3}) = \frac{\mathbb{1}}{3}$, (a_n) is not a convergent sequence in this space.

Proposition 3.15. Let $(X, \mathcal{M}_{E_1}, \mathcal{T})$ and $(Y, \mathcal{M}_{E_2}, \mathcal{T})$ be two non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. If

$$\mathcal{M}_E((a_1, b_1), (a_2, b_2), s) = \mathcal{T}(\mathcal{M}_{E_1}(a_1, a_2, s), \mathcal{M}_{E_2}(b_1, b_2, s))$$

for $(a_1, b_1), (a_2, b_2) \in X \times Y$ and for all $s \in E_+$, then $\mathcal{M}_E$ is a non-Archimedean $\mathcal{L}$-fuzzy vector metric on $X \times Y$.

Note 3.16. For the rest of this study, $\mathcal{T}$ stands for a continuous t-norm on L such that for any $s \in E_+$ and $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, there exists an element $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ satisfying the condition $\mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \geqslant_L \mathcal{N}(\varepsilon)$.

Theorem 3.17. Let $\mathcal{M}_E$ be defined as in Proposition 3.15 and $(a_n) \subseteq X$ and $(b_n) \subseteq Y$ be two sequences. If $a_n \stackrel{\mathcal{M}_{E_1}}{\to} a$ in X and $b_n \stackrel{\mathcal{M}_{E_2}}{\to} b$ in Y, then $(a_n, b_n) \stackrel{\mathcal{M}_E}{\to} (a, b)$ in $X \times Y$.

Proof.
Let $a_n \stackrel{\mathcal{M}_{E_1}}{\to} a$ in X and $b_n \stackrel{\mathcal{M}_{E_2}}{\to} b$ in Y. Then, according to Definition 3.13 (*ii*) there exist $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that $\mathcal{M}_{E_1}(a_n, a, s) >_L \mathcal{N}(r)$, for all $n \geqslant n_1$ and $\mathcal{M}_{E_2}(b_n, b, s) >_L \mathcal{N}(r)$, for all $n \geqslant n_2$. If $n_0 = \max\{n_1, n_2\}$, then

$$\begin{aligned} \mathcal{M}_E((a_n, b_n), (a, b), s) &= \mathcal{T}(\mathcal{M}_{E_1}(a_n, a, s), \mathcal{M}_{E_2}(b_n, b, s)) \\ &>_L \mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \\ &\geqslant_L \mathcal{N}(\varepsilon) \end{aligned}$$

is obtained. Thus, the proof is completed. □

Theorem 3.18. Suppose $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Arcimedean $\mathcal{L}$-fuzzy vector metric space and $(a_n) \subseteq X$ be a convergent sequence. Then, the following properties hold:

i. (a_n) is bounded and its limit is unique.

ii. (a_n) is a Cauchy sequence.

iii. Any subsequence (a_{n_k}) of (a_n) converges to the same limit.

Proof.
Suppose $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Arcimedean $\mathcal{L}$-fuzzy vector metric space and $(a_n) \subseteq X$ be a convergent sequence.

i. Let $a_n \stackrel{\mathcal{M}_E}{\to} a$. Then, for each $\varepsilon, \eta \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, there exists $n_1 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) \geqslant_L \mathcal{N}(\varepsilon)$, for all $n \geqslant n_1$ and $a_0 \in X$ such that $\mathcal{M}_E(a_0, a, s/2) \geqslant_L \mathcal{N}(\eta)$. For some $\lambda \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, suppose

$$\min\{\mathcal{M}_E(a_n, a, s/2) : n_1 > n\} = \mathcal{N}(\lambda)$$

Then, an $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ can be found such that

$$\min\{\mathcal{T}(\mathcal{N}(\eta), \mathcal{N}(\lambda)), \mathcal{T}(\mathcal{N}(\eta), \mathcal{N}(\varepsilon))\} = \mathcal{N}(r)$$

Thereby, for all $n \in \mathbb{N}^+$

$$\mathcal{M}_E(a_0, a_n, s) \geqslant_L \mathcal{T}(\mathcal{M}_E(a_0, a, s/2), \mathcal{M}_E(a_n, a, s/2)) \geqslant_L \mathcal{N}(r)$$

is obtained. As a result, $(a_n) \subseteq \mathcal{B}_E[a_0, r, s]$, which means (a_n) is bounded. To illustrate the uniqueness of the limit, suppose the sequence (a_n) has two different limits a and b. Let $\varepsilon = \mathcal{N}(\mathcal{M}_E(a, b, s))$, for any $s \in E_+$. Since (a_n) is convergent, then there exist $n_1, n_2 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) \geqslant_L \mathcal{N}(\lambda)$ and $\mathcal{M}_E(a_n, b, s/2) \geqslant_L \mathcal{N}(\lambda)$, for all $n \geqslant n_1, n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then, for $n \geqslant n_0$,

$$\begin{aligned}\mathcal{M}_E(a, b, s) &\geqslant_L \mathcal{T}(\mathcal{M}_E(a_n, a, s/2), \mathcal{M}_E(a_n, b, s/2)) \\ &>_L \mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \\ &\geqslant_L \mathcal{N}(\varepsilon)\end{aligned}$$

which means a contraction. Hence, the limit of the convergent sequence is unique.

ii. Let $s \in E_+$ and $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$. Because of the convergent of the sequence (a_n), there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) > \mathcal{N}(\lambda)$, for all $n \geqslant n_0$. Then, for all $m \geqslant n_0$,

$$\begin{aligned}\mathcal{M}_E(a_n, a_m, s) &\geqslant_L \mathcal{T}(\mathcal{M}_E(a_n, a, s/2), \mathcal{M}_E(a, a_m, s/2)) \\ &>_L \mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \\ &\geqslant_L \mathcal{N}(\varepsilon)\end{aligned}$$

Thus, every convergent sequence is a Cauchy sequence.

iii. Let $a_n \overset{\mathcal{M}_E}{\to} a$ and $(a_{n_i}) \subseteq (a_n)$. Thus, for all $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) > \mathcal{N}(\varepsilon)$, for all $n \geqslant n_0$. If $i \geqslant n_0$, then $n_0 \leqslant i \leqslant n_i$ and thus $\mathcal{M}_E(a_{n_i}, a, s) > \mathcal{N}(\varepsilon)$.

□

Definition 3.19. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space and $\Omega \subseteq X$. Then, the $\mathcal{L}$-fuzzy vector metric diameter $\mathcal{D}_E(\Omega)$ is defined as follows:

$$\mathcal{D}_E(\Omega) = \sup_{s \in E_+} \{\inf \mathcal{M}_E(a, b, s) : a, b \in \Omega\}$$

If $\mathcal{D}_E(\Omega) = 1_L$, then Ω is said to be bounded.

Remark 3.20. If Ω is a singleton set, then $\mathcal{D}_E(\Omega) = 1_L$. However, unlike crisp sets, the converse may not always be true. For example, for the standard non-Archimedean $\mathcal{L}$-fuzzy vector metric defined in Example 3.10 as follows

$$\mathcal{M}_E(a, b, s) = \frac{\|s\|}{\|s\| + \|d_E(a, b)\|}$$

and for $\Omega = \{a_0, b_0\} \subset X$,

$$\mathcal{D}_E(\Omega) = \sup_{s \in E_+} \frac{\|s\|}{\|s\| + \|d_E(a_0, b_0)\|} = 1_L$$

is obtained.

Theorem 3.21. For $\mathcal{D}_E(\Omega)$, the following statements hold:

i. Let $\Omega \subseteq \Psi$. Then, $\mathcal{D}_E(\Psi) \leqslant_L \mathcal{D}_E(\Omega)$

ii. $\mathcal{D}_E(\Omega) \leqslant_L \mathcal{M}_E(a, b, s)$, for any $a, b \in \Omega$

iii. Let $\Omega = \{a, b\}$. Then, $\mathcal{D}_E(\Omega) = \mathcal{M}_E(a, b, s)$

iv. Let $\Omega \cap \Psi \neq \emptyset$. Then, $\mathcal{T}(\mathcal{D}_E(\Omega), \mathcal{D}_E(\Psi)) \leqslant_L \mathcal{D}_E(\Omega \cup \Psi)$

Definition 3.22. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space. For $\emptyset \neq (\Omega_n) \subseteq X$ if

$$\lim_{n\to\infty} \mathcal{D}_E(\Omega_n) = 1_L$$

then it is said to be Ω has appearing $\mathcal{L}$-fuzzy vector metric diameter. Moreover, for all $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, a number $n_0 \in \mathbb{N}$ can be found such that $\mathcal{M}_E(a, b, s) >_L \mathcal{N}(r)$, for all $a, b \in \Omega_{n_0}$.

Theorem 3.23 (Theorem of Cantor Intersection)**.** Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space. Let $\emptyset \neq \Omega_n$ be closed and decreasing sequence of subsets of X. Suppose that $\lim\limits_{n\to\infty} \mathcal{D}_E(\Omega_n) = 1_L$. Then, X is complete if and only if the intersection of the sequence is a singleton.

Proof.
Let X be complete. For each $n \in \mathbb{N}$ by considering a point $a_n \in \Omega_n$, a sequence (a_n) can be formed. If $m \geqslant n$ is chosen, $\Omega_m \subseteq \Omega_n$ is obtained such that all the points $\{a_m : m \geqslant n\}$ of the sequence belong to the set Ω_n. According to Theorem 3.21, $\mathcal{D}_E(\Omega_n) \leqslant_L \mathcal{M}_E(a_m, a_n, s)$, for $s \in E_+$ and for all $m \geqslant n$. Since the sequence (Ω_n) has an appearing $\mathcal{L}$-fuzzy vector diameter, $\lim\limits_{n,m\to\infty} \mathcal{M}_E(a_m, a_n, s) = 1_L$. Thus, (a_n) is a Cauchy sequence. Since X is complete, there is a point $a \in X$ such that $\lim\limits_{n\to\infty} \mathcal{M}_E(a_n, a, s) = 1_L$. If a set Ω_{n_0} is taken and formed the sequence $(a_n) \subset \Omega_{n_0}$, for $n \geqslant n_0$, then $\lim\limits_{n\to\infty} \mathcal{M}_E(a_n, a, s) = 1_L$. Moreover, $a \in \Omega_{n_0}$ because Ω_{n_0} is closed. As a result, it follows that a belongs to all the members of the sequence (Ω_n). Hence, $a \in \bigcap\limits_{n=1}^{\infty} \Omega_n$ is obtained. Considering another point $a' \in \bigcap\limits_{n=1}^{\infty} \Omega_n$, $\mathcal{D}_E(\Omega_n) \leqslant_L \mathcal{M}_E(a, a', s)$, for all $s \in E^+$. Since the sequence (Ω_n) has an appearing $\mathcal{L}$-fuzzy vector diameter, $\mathcal{M}_E(a, a', s) = 1_L$. As a result, it follows that $\bigcap\limits_{n=1}^{\infty} \Omega_n = \{a\}$ because of $a = a'$.

Conversely, considering a Cauchy sequence $(a_n) \subseteq X$ and closed nonempty subset $\Omega_n = \overline{\{a_m : m \geqslant n\}}$ of X, then $\lim\limits_{n\to\infty} \mathcal{D}_E(\Omega_n) = 1_L$ because the sequence (Ω_n) is decreasing and (a_n) is a Cauchy sequence. According to the assumption of the theorem, there is only a single point a such that $\bigcap\limits_{n=1}^{\infty} \Omega_n = \{a\}$. Then, because of the definition of $\mathcal{L}$-fuzzy vector diameter there is a natural number n_0 such that $\mathcal{D}_E(\Omega_{n_0}) >_L \mathcal{N}(\varepsilon)$, for each $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$. Moreover, since $a \in \Omega_{n_0}$, $\mathcal{M}(a_n, a, s) >_L \mathcal{N}(\varepsilon)$, for all $n \geqslant n_0$. It means that $a_n \overset{\mathcal{M}_E}{\to} a$. Consequently, the non-Archimedean $\mathcal{L}$-fuzzy vector metric space X is a complete space. □

Theorem 3.24 (Baire Category Theorem)**.** Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space and let $(\Omega_n) \subset X$ be a countable collection of open and dense subsets. Then, the intersection of (Ω_n) is also dense in X.

Proof.
For proof, it is necessary that

$$\mathcal{B}_E(a, r, s) \cap \left(\bigcap_{n=1}^{\infty} \Omega_n \right) \neq \emptyset$$

is satisfied for all $a \in X$, $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$. For Ω_1, $\mathcal{B}_E(a, r, s) \cap \Omega_1$ is open and nonempty because $\Omega_1 \subset X$ is dense. Considering the element $a_1 \in \mathcal{B}_E(a, r, s) \cap \Omega_1$, then there exist $r_1 \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s_1 \in E_+$ such that $\mathcal{B}_E[a_1, r_1, s_1] \subset \mathcal{B}_E(a, r, s) \cap \Omega_1$. Let $\mathcal{B}_{E_1} = \mathcal{B}_E(a_1, r_1, s_1)$. $\mathcal{B}_{E_1} \cap \Omega_2$ is open and nonempty because $\Omega_2 \subset X$ is dense. Considering the element $a_2 \in \mathcal{B}_{E_1} \cap \Omega_2$, then there exist $r_2 \in (0_L, 1_L/2)$ and $s_2 \in E_+$ such that $\mathcal{B}_E[a_2, r_2, s_2] \subset \mathcal{B}_{E_1} \cap \Omega_2$. Let $\mathcal{B}_{E_2} = \mathcal{B}_E(a_2, r_2, s_2)$.

If continued inductively, two sequences $(a_n) \subseteq X$ and $(r_n) \subseteq \mathbb{R}$ are obtained such that

$$\mathcal{B}_E[a_{n+1}, r_{n+1}, s_{n+1}] \subset \mathcal{B}_{E_n} \cap \Omega_{n+1} \subset \mathcal{B}_E[a_n, r_n, s_n] \quad \text{and} \quad r_n \in (0_L, 1_L/n)$$

for all $n \in \mathbb{N}$. According to Theorem 3.23, $\bigcap\limits_{n=1}^{\infty} \mathcal{B}_E[a_n, r_n, s_n]$ has only one element. As a result, from

$$\bigcap_{n=1}^{\infty} \mathcal{B}_E[a_n, r_n, s_n] \subset \mathcal{B}_E(a, r, s) \cap \left(\bigcap_{n=1}^{\infty} \Omega_n \right)$$

we reach the conclusion $\mathcal{B}_E(a, r, s) \cap \left(\bigcap\limits_{n=1}^{\infty} \Omega_n \right) \neq \emptyset$. This completes the proof. □

4. Conclusion

In conclusion, this article contributes to the field of fuzzy metric spaces by defining left and right-order convergence and continuity within the framework of non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. Left and right-order continuity concepts are used to construct $\mathcal{L}$-fuzzy vector metric spaces and non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. Furthermore, some non-trivial examples are built, and as an implication, the findings are used to prove Cantor's intersection theorem and Baire's theorem. In the next stages, as a continuation of this study, examples of these spaces can be multiplied, and fixed point theorems can be studied.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (3) (1965) 338–353.

[2] J. Goguen, *$\mathcal{L}$-Fuzzy Sets*, Journal of Mathematical Analysis and Applications 18 (1) (1967) 145–174.

[3] G. D. Birkhoff, Lattice Theory, 3rd Edition, American Mathematical Society, New York, 1973.

[4] K. Menger, *Statistical Metrics*, Proceedings of the National Academy of Sciences 28 (12) (1942) 535–537.

[5] B. Schweizer, A. Sklar, *Statistical Metric Spaces*, Pacific Journal of Mathematics 10 (1) (1960) 313–334.

[6] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Dover Publications, New York, 2011.

[7] I. Kramosil, J. Michálek, *Fuzzy Metrics and Statistical Metric Spaces*, Kybernetica 11 (5) (1975) 336–344.

[8] A. George, P. Veeramani, *On Some Results in Fuzzy Metric Spaces*, Fuzzy Sets and Systems 64 (3) (1994) 395–399.

[9] V. Gregori, S. Morillas, A. Sapena, *Examples of Fuzzy Metrics and Applications*, Fuzzy Sets and Systems 170 (1) (2011) 95–111.

[10] R. Saadati, A. Razani, H. Adibi, *A Common Fixed Point Theorem in $\mathcal{L}$-Fuzzy Metric Spaces*, Chaos, Solitons & Fractals 33 (2) (2007) 358–363.

[11] S. Morillas, V. Gregori, G. Peris-Fajarnés, P. Latorre, *A Fast Impulsive Noise Color Image Filter Using Fuzzy Metrics*, Real-Time Imaging 11 (5-6) (2005) 417–428.

[12] C. D. Aliprantis, K. C. Border, Infinite Dimensional Analysis, Springer-Verlag Berlin, Heidelberg, 1999.

[13] C. D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, Dordrecht, 2006.

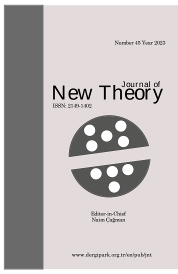
[14] C. Çevik, I. Altun, *Vector Metric Spaces and Some Properties*, Topological Methods in Nonlinear Analysis 34 (2) (2009) 375–382.

[15] Ş. Eminoğlu, C. Çevik, *Fuzzy Vector Metric Spaces and Some Results*, Journal of Nonlinear Sciences and Applications 10 (2017) 3429–3436.

New Theory Journal of

ISSN: 2149-1402

45 (2023) 57-72
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



On Finite and Non-Finite Bayesian Mixture Models

Rasaki Olawale Olanrewaju[1], Sodiq Adejare Olanrewaju[2], Adedeji Adigun Oyinloye[3], Wasiu Adesoji Adepoju[4]

Article Info
Received: 21 Sep 2023
Accepted: 23 Nov 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1358754
Research Article

Abstract — In this paper, a Bayesian paradigm of a mixture model with finite and non-finite components is expounded for a generic prior and likelihood that can be of any distributional random noise. The mixture model consists of stylized properties-proportional allocation, sample size allocation, and latent (unobserved) variable for similar probabilistic generalization. The Expectation-Maximization (EM) algorithm technique of parameter estimation was adopted to estimate the stated stylized parameters. The Markov Chain Monte Carlo (MCMC) and Metropolis–Hastings sampler algorithms were adopted as an alternative to the EM algorithm when it is not analytically feasible, that is, when the unobserved variable cannot be replaced by imposed expectations (means) and when there is need for correction of exploration of posterior distribution by means of acceptance ratio quantity, respectively. Label switching for exchangeability of posterior distribution via truncated or alternating prior distributional form was imposed on the posterior distribution for robust tailoring inference through Maximum a Posterior (MAP) index. In conclusion, it was deduced via simulation study that the number of components grows large for all permutations to be considered for subsample permutations.

Keywords *Bayesian paradigm, expectation-maximization, MCMC, proportional allocation, metropolis–hastings*

Mathematics Subject Classification (2020) 62F15, 62H30

1. Introduction

The essence of Bayesian methods, inference, and statistics in time-series analysis, mixture models, econometric, machine learning, and data science had received great attention since the propounded of Bayes' theorem by Thomas Bayes in the early 1980s [1]. Its great advantage is via its simplicity and bedrock of simplifying Bayes' theorem in a discretized or continuous form. However, it provides subjective pre-judgment and pre-knowledge (prior information) about the data that is to be incorporated into likelihoods, in order to aid sequential learning, decision-making, and prediction [2]. In other words, an inescapable requirement of Bayesian methods is to rightfully specify prior distributions for all involved parameters in the model that are usually regarded as unknown quantities. However, there have been debates and controversies over the choice of priors that truly advocate for likelihood(s) of interest [3]. Although, different types of prior have been proposed, types like, conjugate and non-conjugate priors; horseshoe prior, improper priors (otherwise refer to

[1]olanrewaju_rasaq@yahoo.com (Corresponding Author); [2]sodiqadejare19@gmail; [3]adedeji.oyinloye@bouesti.edu.ng; [4]adepojuwasiu@gmail.com
[1]Department of Business Analytics and Value Network, Africa Business School, Mohammed VI Polytechnic University, Rabat, Morocco
[2]Department of Statistics, Faculty of Science, University of Ibadan, Ibadan, Nigeria
[3]Department of Mathematical Sciences, College of Sciences, School of Pure and Applied Sciences, Bamidele Olumilua University of Education, Science and Technology, Ikere-Ekiti, Nigeria
[4]Department of Mathematics Education, Faculty of Education, University of Ibadan, Ibadan, Nigeria

as σ-finite measure), Zellner's *G*-prior, *G*-Priors, non-informative *G*-Priors, Jeffreys' prior, but there has not been a clear distinction as regards the ideal one [4]. However, prior distributions to be considered depend on how they inclusively or exclusively contained in the mixture model, the model to be considered, their involvements in the likelihood coefficients; and their bearing resulting inference via sensitivity analysis. Consequently, it is not all the time that conjugate priors defined for certain likelihoods do give posterior forms of the likelihoods. In addition, the deduction that some non-informative priors usually accompany undefined posteriors irrespective of the sample size is a clear indicator of the complexity of Bayesian inference for some models [5].

According to [6], prior distributions are being specified based on principles, relying on asymptotes, approximations, algorithms' flexibilities, and ignorance about the parameters. This makes it feasible for emergence of any inferential probabilistic prior with its corresponding likelihood to yield closed form solution and limiting distribution for the embedded parameters. Contrary to the adoption of priors' principles, [7] proposed Approximate Bayesian Computation–Population Monte Carlo (ABC–PMC) algorithm as an alternative technique for finite mixture model inferential. [7] adopted a kernel function as a substitute for prior distribution and explicitly highlighted how the problem of label switching can be solved with the use of the adopted kernel. In extension, [8] adopted Bayes factor to find required number of K-components that will be associated to a finite mixture model. The adopted Bayes factor ratio was incorporated in parametric family of finite mixtures and that of nonparametric via `strongly identifiable' Dirichlet Process Mixture (DPM) model and inferred that scalable evidence estimation technique for non-conjugate Dirichlet Process mixtures will be needed to derive the parametric and nonparametric processes. Prediction is one of the key factors that distinguished Bayesian paradigm, because of its ability to take into account all involved parameters and integrate them into posterior distribution in order to iteratively estimate their reliable and inferential solutions [9].

The continual usage of Bayesian methods in mixture models and econometrics in general, is because of the repercussion in flexibility and efficient algorithms used in conducting inferential inference through estimating unknown quantities. However, various powerful Bayesian computational techniques have been designed to estimate posterior solutions analytically and intractably for dimensional models. Among the techniques designed are Markov Chain Monte Carlo (MCMC) methods, Gibbs sampler and the Metropolis-Hastings sampler, Arnason–Schwarz Gibbs Sampler; and Stan implemented Hamiltonian Monte Carlo called "Stan" algorithms. The MCMC method is a veritable revolution and procedure for implementing a class of computational algorithms that can be easily applied to almost every model. The idea behind MCMC method is to generate analytically intractable posterior solutions via Markov Chain that converge to a chain of selections from the posterior distribution [10,11]. Once one of the chain drawings is available or successful, predictive inference will also be achieved. There are various ways of designing Markov Chain depending on the structure of the problem, once the chain exists, Gibbs sampler and the Metropolis-Hastings sampler; Arnason–Schwarz Gibbs Sampler; or Stan implemented Hamiltonian Monte Carlo can then be employed. Adding of additional auxiliary variable(s) (data augmentation) by the sampler (MCMC sampler) usually facilitate the implementation and analysis to be conducted on the augmented domain on not only the unknown quantities (model parameters), but also on unobserved variables (latent variables) and missing observational structure [12, 13]. This accessible reference and ability to incorporate additional data augmentation by MCMC sampler makes Bayesian paradigm to be the feasible method for mixture models (mixture models that involve incorporation of latent variable for regime switching, sample size allocation, and proportional allocation (mixing weights)).

This article covers a wide range of mixture models for simple probability distribution to be made more complex and less informative by a mechanism that combines several known or unknown same distribution. This composition is what is called mixture model or mixture of distributions. Inference made about the known and unknown quantities (parameters) of the ingredients of the mixture model and proportional allocations (mixing weights) is what is usually referred to as mixture estimation. In relation to machine learning, the repossession of the source distribution of each observation from the mixture of distributions is usually termed as classification (that is, distinguishing unsupervised classification from supervised classification). This technique requires advanced and sophisticated computational tools since the composition of the posterior distribution might not be easily computed. However, this article covers theoretical cases, as well as simulation studies for generic finite and non-finite Bayesian mixture models for common likelihoods with their prior distributions for known and unknown number of components, proportional allocation, and allotted sample size for specific approximation of Expectation-Maximization (EM) parameter estimation. The EM parameter estimation technique will be alternatively updated via MCMC, Gibbs sampler and the Metropolis-Hastings algorithms. The problem of identifying exchangeable posterior distribution will be treated via label switching, with its associated total allocation sample size being carve-out via Monte Carlo approximation with the use of Maximum a Posteriori (MAP) estimator.

2. Method

[14-16] propounded that a finite mingle of mixture model of similar probabilistic distribution to be a generalization of,

$$h_i(y) = \lambda_1 g_1(y) + \lambda_2 g_2(y) + \cdots + \lambda_k g_k(y) \tag{1}$$

$$h_i(y) = \sum_{i=1}^{k} \lambda_i \, g_i(y) \tag{2}$$

That is, a mixture model of same distribution is nothing but a convex combination, such that, $h_i(y)$ is the complete function of the mixture generalization model, with $g_i(y)(i \in \{1,2,3,\cdots,k\})$ being the known and inferential probabilistic distribution for each allocation (λ_k) with their corresponding unknown proportions on sample points $(y_1,\cdots,y_n)$ on g_i components, such that, $\lambda_i \geq 0 \ni \sum_{i=1}^{k} \lambda_i \approx 1$, for $i \in \{1,2,3,\cdots,k\}$, for a drawn or selected sample size of size (n).

However, in a parametric setting, where $g_i(y)(i \in \{1,2,3,\cdots,k\})$ can take any distributional form, forms like Gaussian, exponential, Beta, or student-*t* distribution with unknown coefficients (parameters, say ϑ_i), Equation 2 can then be rewritten as:

$$h(y) = \sum_{i=1}^{k} \lambda_i \, g_i(y \,|\, \vartheta_i) \tag{3}$$

With λ_i as the mixing weights (or proportional allocation) and ϑ_i the component coefficients for $(i \in \{1,2,3,\cdots,k\})$. The idea of mixing allocations from a parametric point of view makes it possible to associate component coefficients with missing data structure (unobserved or latent variable), while in a subjective prejudgment manner (Bayesian paradigm), they are known to be related observations. This noticeable assertion might not be germane in a computational setting that involves likelihood function or construction of prior distribution, pertinent in the interpretation of posterior results.

One of the reasons (motivations) for constructing mixtures of same distributions is to usefully extend "standard" distributions statistically in an approach that envisions observations as several unobserved (latent) sub-populations (strata). Conditioning it on the setting, the inferential goal is to associate selected samples (n_i) or drawings from mixtures of finite or non-finite, but with components of the same distribution to reassemble selected groups (usually refer to as cluster) by estimating the unobserved component, say "s", to provide estimators for unknown coefficients for several groups, or to estimate the number of k -groups.

2.1. Procedure for Generic Mixture Likelihoods and Posteriors Drawings

Assuming an Independent and Identically Distributed (IID) sample drawings of $(y_1, \cdots, y_n)$ was drawn from the mixture of distributions with, proportional allocations, and component parameters of Equation 3. Then, the likelihood is such that,

$$\ell(\vartheta, \gamma|y) = \prod_{j=1}^{n} \sum_{i=1}^{k} \lambda_i g(y_j|\vartheta_i) \tag{4}$$

Equation 4 is the likelihood that contains k^n-terms of $\lambda_i g(y_j|\vartheta_i)$. The computational acumen of Equation 4 depends on the feasibility of the order $O(nk)$ of Equation 3 that can cater for the analytical solution of either the Bayes estimators or Maximum Likelihood (ML) estimators.

Let $T(\vartheta, \lambda)$ be the prior distribution of any form for the likelihood distribution, then the posterior distribution of $(\vartheta, \lambda|y)$ can be added-up for a multiplicative constant, say,

$$T(\vartheta, \lambda|y) \infty \left(\prod_{j=1}^{n} \sum_{i=1}^{k} \lambda_i g\left(y_j \middle| \vartheta_i\right) \right) T(\vartheta, \lambda) \tag{5}$$

For $T(\vartheta, \lambda|y)$ that can be computed for guess values of the parameters in $T(\vartheta, \lambda)$ at a contrivance order of $O(nk)$. The derivation of $T(\vartheta, \lambda|y)$ posterior outputs and expectations (means) of the mixture distribution coefficients of interest can only be achieved in an exponential time order of $O(nk)$. Incorporating the latent (unobserved or missing variable) intuition into the mixture posterior distribution of Equation 5 for each y_i in association to the latent variable "s" indicated a Markov chain component of distribution that can be generated. This makes it to be seen as hierarchical structure associated with the mixture model to be:

$$s_i|\lambda \sim M_k(\lambda_1, \cdots, \lambda_k)$$

where M_k denotes the Multinomial distribution for $y_i|\, s_i, \vartheta \sim g\left(y|\vartheta_{s_i}\right)$.

The complete data, that is, (y_i, s_i) is the complete likelihood corresponding to the unobserved variable, s_i is

$$\ell(\vartheta, \lambda|y, s) = \prod_{j=1}^{n} \lambda_{s_j} g(y_j|\vartheta_{s_j}) \tag{6}$$

Then, the posterior distribution that added-up for a multiplicative constant is given as

$$T(\vartheta, \lambda|y, s) \infty \left(\prod_{j=1}^{n} \lambda_{s_j} g(y_j|\vartheta_{s_j}) \right) T(\vartheta, \lambda) \tag{7}$$

where $s = \{s_1, s_2, s_3, \cdots, s_n\}$ for k^n-terms of $\Xi = \{1,2,3, \cdots, k\}^n$ possible values of the specified vector of "s". Having ascertained proportional allocation for each mixture distribution to be (λ_k). In a similar manner, sample size allocation can also be ascertained via partitioning (decomposition) to R via $\mathrm{R} = \bigcup_{i=1}^{r} \Xi_i$ for a given allocation size vector of $(n_1, n_2, n_3, \cdots, n_k)$, where, $\sum_{i=1}^{k} n_k$, the number of observations allotted to each component, then partition sets can be worked-out as:

$$\Xi_j = \left\{ s: \sum_{i=1}^{n} I_{s_i=1}, \cdots, \sum_{i=1}^{n} I_{s_k=k} \right\} \tag{8}$$

Ξ_j consist of all proportional allocations with their corresponding or given allocation sizes $(n_1, n_2, n_3, \cdots, n_k)$, such that, the partitioning sets with $j = (n_1, n_2, n_3, \cdots, n_k)j$ can be conceptualized as a lexicographical ordering of $(n_1, n_2, n_3, \cdots, n_k)$'s.

$$j = 1, \quad (n_1, n_2, n_3, \cdots, n_k) = (n, 0, \cdots, 0)$$

$$j = 2, \quad (n_1, n_2, n_3, \cdots, n_k) = (n - 1, 1, \cdots, 0)$$

$$j = 3, \quad (n_1, n_2, n_3, \cdots, n_k) = (n - 1, 0, 1, \cdots, 0)$$

and

$$j = 4, \quad (n_1, n_2, n_3, \cdots, n_k) = (n - 1, 1, 0, 1, \cdots, 0)$$

So that the posterior distribution of (θ, λ) can be rewritten in a close form as:

$$T(\theta, \lambda|y) = \sum_{j=1}^{r} \sum_{s \in \Xi_r} \eta(s) T(\theta, \lambda|y, s) = \sum_{s \in \Xi} T(\theta, \lambda|y, s) \tag{9}$$

Such that $\eta(s)$ is the marginal posterior likelihood of the allotted "s" conditioned on sample points' domain of "y". This can be derived after integrating out "ϑ" and "λ", such that, the close form of the Bayes estimator of (ϑ, λ) is

$$E^T[\vartheta, \lambda|y] = \sum_{j=1}^{r} \sum_{s \in \Xi_r} \eta(s)\, E^T[\vartheta, \lambda|y] \tag{10}$$

Decomposing Equation 9, for an inferential point of view. It connotes that the posterior distribution takes into account each possible partition of "s" in the dataset and allocate a posterior likelihood of $\eta(s)$ to these partitions, as well construct a posterior distribution for the embedded coefficients conditioned on the allocations.

Employing the Expectation-Maximization (EM) algorithm procedure for completion of parameter estimation mechanism that involves latent (unobserved) variable. The "E" stands for expectation and "M" connotes maximization steps that involve convergence of local maximum of likelihood.

Iteratively in time variant "t", the E-step computational function corresponds to

$$Q\{(\vartheta^{(t)}, \lambda^{(t)}), (\vartheta, \lambda)\} = E_{(\vartheta^{(t)}, \lambda^{(t)})}[\log \ell\,(\vartheta, \lambda|y, s)|y] \tag{11}$$

$\log \ell\,(\vartheta, \lambda|y, s)$is regarded as the likelihood of the joint distribution of "y" and "s", such that, imposed means of the coefficients to be calculated under the conditional distribution of "s" given "y" for the value of $(\vartheta^{(t)}, \lambda^{(t)})$. The second case, which is the M-step, is the maximization of $Q\{(\vartheta^{(t)}, \lambda^{(t)}), (\vartheta, \lambda)\}$ in (ϑ, λ) with convergence solution of $(\vartheta^{(t+1)}, \lambda^{(t+1)})$ (see [17,18]).

2.2. MCMC Solutions as an Alternate for Expectation-Maximization (EM) Algorithm

In situation where the first step (that is, the E-step) of the EM-algorithm is not analytically feasible, that is, when the unobserved variable "s" cannot be replaced by imposed expectations (means) for the joint distribution of Equation 5, then the full conditional distribution of "s" given "y" will be evaluated as,

$$T(s|y, \vartheta, \lambda) \infty \prod_{j=1}^{n} \lambda_{s_j} g\left(y_i|\vartheta_{s_i}\right) \tag{12}$$

Equation 12 can be computed for guess value of $T(\vartheta,\lambda)$ at a contrivance order of $O(n)$ for standard distributions of $g(\cdot\,|\vartheta)$.

It is to be noted that if $T(\vartheta,\lambda)$ is a conjugate prior, then its full conditional posterior can also be worked-out via Gibbs sampler. Assuming "ϑ" and "λ" are independent a priori, then conditioning "s", the vectors "y" and "λ" are independent; that is,

$$T(\lambda|s,y)\infty T(\lambda)g(s\,|\,\lambda)g(y\,|\,s)\infty T(\lambda)g(s\,|\,\lambda)\infty)T(\lambda|s) \tag{13}$$

It is to be noted that "ϑ" is independent posterior of the form "λ" given "s" and "y", with density $T(\vartheta|s,y)$ for successive simulation of "s" and (λ,ϑ) conditional on one another, as well on the data points (y).

This implies that Gibbs sampler will be ideal under the umbrella of latent variable (unobserved variable) by simulation under data augmentation. The simulation of $\vartheta j's$ depends solely on sampling density of $g(\cdot\,|\vartheta)$ coupled with the prior, $T(\vartheta,\lambda)$.

The marginal distribution of s_i's is nothing but a multinomial distribution of $M_k(\lambda_1,\cdots,\lambda_k)$, which allow a conjugate prior on "λ", such that, $\lambda=(\lambda_1,\cdots,\lambda_k)$, where "$\lambda$" follows a Dirichet distribution, that is, $\lambda\sim\wp(\delta_1,\cdots,\delta_k)$ with density

$$\frac{\Gamma(\delta_1+\cdots+\delta_k)}{\Gamma(\delta_1)\cdots\Gamma(\delta_k)}\lambda_1^{\delta_1}\cdots\lambda_k^{\delta_k}$$

on k-real number line $\Re^k$,

$$\wp=\left\{(\lambda_1,\cdots,\lambda_k)\in[0,1]^k;\sum_{i=1}^{k}\lambda_i=1\right\} \tag{14}$$

Its own sample size allocation can be denoted as

$$n_i=\sum_{\iota=1}^{n}I_{s_\iota=i}\,(1\le i\le k)$$

for posterior distribution of "λ" given "s" that is,

$$\lambda|s\sim\wp(n_1+\delta_1,\cdots,n_k+\delta_k) \tag{15}$$

2.2.1. Algorithm for the Gibbs Sampler

Guess an initialization (that is starting values) for "λ" and "ϑ": That is, choose $\lambda^{(0)}$ and $\vartheta^{(0)}$ arbitrarily. Note that $0\le\lambda^{(0)}\le 1$.

Iteration $t(t\ge 1)$:

i. For $i\in\{1,2,3,\cdots,n\}$, generate $s_i^{(t)}\ni P(s_i=j|\lambda,\vartheta)\infty\lambda_j^{(t-1)}g\left(y_i|\vartheta_j^{(t-1)}\right)$

ii. Generate $\lambda^{(t)}$ according to $\left(\lambda|s^{(t)}\right)$

iii. Generate $\vartheta^{(t)}$ according to $\left(\vartheta|s^{(t)},y\right)$

The simulation of ϑ_j's exists only for conjugate prior. The intricacy in the simulation of the ϑ_j's depends on the sampling density $g(\cdot\,|\vartheta)$ as well as the prior distribution of $T(\vartheta,\lambda)$.

2.3. Metropolis–Hastings Algorithm as an Alternate to Expectation-Maximization (EM) and MCMC Algorithms

Knowing that the likelihood of mixture model is usually in a closed-form manner when computation is in $O(nk)$ order and time variant "t" and the posterior distribution is thus up to a multiplicative constant. One can alternatively switch to Metropolis–Hastings algorithm, as long as a new quantity "Υ" provides a correct exploration for the posterior distribution with acceptance ratio,

$$\gamma(y \mid \vartheta', \lambda') = \frac{T(\vartheta', \lambda'|y)}{T(\vartheta, \lambda|y)} \frac{\Upsilon(\vartheta, \lambda|\vartheta', \lambda')}{\Upsilon(\vartheta', \lambda'|\vartheta, \lambda)} \wedge 1 \tag{16}$$

computed in $O(nk)$ time.

2.3.1. Algorithm for the Metropolis–Hastings

Guess an initialization (that is starting values) for $y^{(0)}$: That is, choose $y^{(0)}$ arbitrarily.

Iteration $t (t \geq 1)$:

i. Given $y^{(t-1)}$, generate $\vartheta' \sim \lambda'\left(y^{(t-1)}, y\right)$

ii. Given $y^{(t-1)}$, generate $\lambda' \sim \vartheta'\left(y^{(t-1)}, y\right)$

iii. Compute $\gamma\left(y^{(t-1)} \mid \vartheta', \lambda'\right) = \min\left(\frac{T(\vartheta', \lambda'|y)}{T(\vartheta, \lambda|y)} \frac{\Upsilon(\vartheta, \lambda|\vartheta', \lambda')}{\Upsilon(\vartheta', \lambda'|\vartheta, \lambda)} \wedge 1\right)$

iv. With probability $\gamma\left(y^{(t-1)} \mid \vartheta', \lambda'\right)$, accept ϑ' and set $y^{(t)} = \vartheta'$; or accept λ' and set $y^{(t)} = \lambda'$ otherwise reject ϑ' and set $y^{(t)} = y^{(t-1)}$ or reject λ' and set $y^{(t)} = y^{(t-1)}$

The distribution of ϑ', λ' is called the instrumental distribution for acceptance–rejection method. ϑ', λ' and T are proportionality constants in the calculation of "γ". The merit of this approach in comparison to Gibbs sampler is that it does not necessarily need the usage of conditional distributions of "T". The Metropolis-Hastings algorithm proposed that the distribution of ϑ' to provide correct exploration of the posterior surface, since the acceptance ratio

$$\frac{T(\vartheta', \lambda'|y)}{T(\vartheta, \lambda|y)} \frac{\Upsilon(\vartheta, \lambda|\vartheta', \lambda')}{\Upsilon(\vartheta', \lambda'|\vartheta, \lambda)} \wedge 1$$

2.4. Label Switching for Exchangeability of Posterior Distribution

In scenarios, where either robust alternative prior distribution is needed for more tailoring inference, label switching is what is termed to be required. The ability to identify exchangeable posterior distribution answers the problem of imposing identifiability restriction to estimate the unknown quantities (parameters). A typical example is by defining components via ordering means, allocation sample size, or proportional allocation in a mixture model. From Bayesian perspective, this is nothing but truncating the source or first used prior distribution from $T(\vartheta, \lambda)$ to

$$T(\vartheta, \lambda) I_{\vartheta_1 \leq \cdots \leq \vartheta_k} \tag{17}$$

The resolution to label switching problem is to shun imposition of the restriction mingling that consist arbitrarily drawing of the $k!$ (k factorial).

2.5. Monte Carlo Approximation for Estimating Maximum a Posteriori (MAP)

Given a MCMC of total allocation sample of size of say "N", We might be interested in finding the Monte Carlo approximation of the MAP estimator by taking $\vartheta^{(i*)}, \lambda^{(i*)}$, such that,

$$i^* = \arg\max_{i=1,\cdots,N} T\{(\lambda, \vartheta)^{(i)} \mid y\} \tag{18}$$

The approximate MAP estimate would act as pivot that yields good approximation for the mode and when reordering iterations with respect to the mode. It is to be noted that Equation 18 is for simulation value that produces maximal posterior density.

In case where the reordering is based on Euclidean distance in the parameter space domain of ϑ, one can employ the distance in the domain of the allotted proportions. Assuming Ψ_k is a k −permutation set and $r \in \Psi_k$, minimizing "r" in an entropy Euclidean distance by adding the relative entropies between $P(s_j = t|\vartheta^{(i*)}, \lambda^{(i*)})'s$ and $P(s_j = t|\, r\{\vartheta^{(i)}, \lambda^{(i)}\})$ such that,

$$f(i,r) = \sum_{j=1}^{n}\sum_{t=1}^{k} P(s_j = t|\vartheta^{(i*)}, \lambda^{(i*)}) \log\left[\frac{P(s_j = t|\vartheta^{(i*)}, \lambda^{(i*)})}{P(s_j = t|\, r[\vartheta^{(i)}, \lambda^{(i)}])}\right] \tag{19}$$

See the permutations of selection of reordering the MCMC output algorithm below.

Algorithm for Pivotal Reordering

For iteration of $i \in \{1,2,3,\cdots,N\}$

i. Compute $r_i = \arg\min_{\nu' \in \Psi_k} f(i,r)$

ii. Set $(\vartheta^{(i)}, \lambda^{(i)}) = r_i(\vartheta^{(i)}, \lambda^{(i)})$

Therefore, after the reordering steps from Equation 16 to 18, the Monte Carlo estimate of the posterior expectation can be written as $E^T[\vartheta_j|y] = \sum_{i=1}^{N} \frac{\vartheta_j^{(i)}}{N}$, where $E^T[\vartheta_j|y]$ (or its approximation) can be compared with Ψ_k in order to check for convergence.

2.6. Mixtures with an Unknown Number of Components

It is to be noted that the number of homogeneous components (k-components) connotes the degree of approximation, and cannot be fixed in advance, except one ascertained the number of components (proportional allocation) via visualization or other detection techniques. Even from the classical approach perspective, the number of homogeneous clustering (usually via the mean) within the population of interest is usually not ascertained and first-hand inference is usually employed to determine the number of components. For example, in financial stock where the number of different patterns of studied stock evolution that may be unknown to analyst (unknown homogeneous components) (for more details, see [19,20]).

In this type of computational resolution, the number of models is infinite and requires special type of MCMC exploration with variability inference. Mixture models with unknown number of proportional allocations are usually referred to as variable dimensional models that require special simulation technique called reversible jump and collection of 2^k-sub-models. It usually requires high degree of formalization, sensitive calibration, and approximated marginal likelihoods in this kind of special case of mixture model. The enumeration of mixture model with unknown components depends on sampling approximation via the marginal likelihood of whole range of potentials models as

$$f_J(y|\varsigma_J) = \prod_{i=1}^{n}\sum_{j=1}^{J} \lambda_j\, f(y_i|\vartheta_j) \tag{20}$$

such that $\varsigma_J = (\vartheta, \lambda) = (\vartheta_1, \cdots, \vartheta_J, \lambda_1, \cdots, \lambda_J)$. $f_J(y|\varsigma_J)$ evolve round the marginal likelihood integral via the sampling approximation,

$$c_J(y) = \int f_J(y|\,\varsigma_J)T_J(\varsigma_J)\partial\varsigma_J \tag{21}$$

"J" connotes the model index, that is, the infinitesimal case of the components in a different representation that starts from another arbitrary density say d_J, then

$$\Gamma = \int d_J(\varsigma_J)\partial\varsigma_J = \int \frac{d_J(\varsigma_J)}{f_J(y|\varsigma_J)T_J(\varsigma_J)} f_J(y|\varsigma_J)T_J(\varsigma_J)\partial\varsigma_J \tag{22}$$

$$\Gamma = c_J(y)\int \frac{d_J(\varsigma_J)}{f_J(y|\varsigma_J)T_J(\varsigma_J)} T_J(\varsigma_J)\partial\varsigma_J \tag{23}$$

This connotes that the estimate of $c_J(y)$ is

$$c_J(\hat{y}) = \frac{1}{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{d_J(\varsigma_J^{(t)})}{f_J(y|\varsigma_J^{(t)})T_J\left(\varsigma_J^{(t)}\right)}\right]} \tag{24}$$

where $\varsigma_J^{(t)}$ are products of the MCMC sampler pointed at $T_J\left(\varsigma_J^{(t)}\right)$.

3. Simulation Studies

A simulated dataset of 1000 observations from independent Gaussian randomly selected observations with true mean values $(\mu_1, \mu_2, \lambda_1) = (2.3,0,0.7)$ were considered, such that, Dirichet variates were used as prior, we consider a two-component Gaussian mixture model of

$$\lambda_1 N(\mu_1, 1) + \lambda_2 N(\mu_2, 1) \ni \lambda_2 = (1 - \lambda_1) \tag{25}$$

In this scenario, the Gaussian coefficients (parameters) are identifiable. This connotes that μ_1 and μ_2 cannot be bewildered for each other, such that, λ_1 is not equal to 0.5. If λ_1 is equal to 0.5, it implies that $\lambda[N(\mu_1, 1) + N(\mu_2, 1)]$. The log-likelihood surfaces of Figure 1 below give the image representation of Equation 25. Two modes were exhibited and expounded, such that the upper chamber with larger mode is noted to be closer to the neighborhood of the true values of the average coefficients simulated. The mode with the lower chamber possessed an inverse separation of the dataset of the two clusters. For better understanding of the lower chamber mode, if a limit of $\lambda_1 = \lambda_2 = 0.5$ is set, it means that there is high likelihood that the two equivalent modes will be approximately equal, that is $(\mu_1, \mu_2) = (\mu_2, \mu_1)$. If λ_1 will be different from 0.5, the lower chamber mode becomes smaller and smaller in comparison with the larger chamber. It is to be noted that the starting guessing points in both cases of μ_1 and μ_2 are saddled points between the two modes.

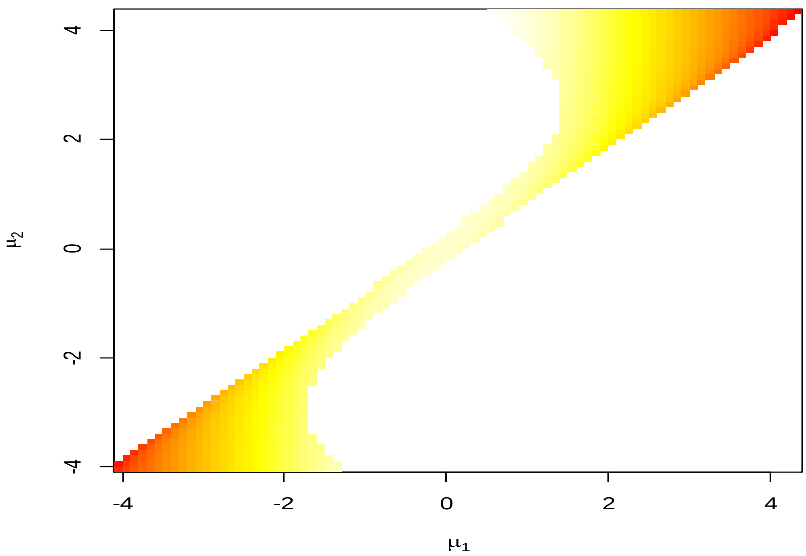


Figure 1. Log-likelihood of the mixture of distribution in Equation 25

A special case of Equation 25 is when two different independent Gaussian priors of $\mu_1 \sim \mathbb{N}(0,4)$ and $\mu_2 \sim \mathbb{N}(2,4)$ are considered from the simulated dataset, then the posterior allocation weight vector of "s" is given as

$$\bar{y}(s) = \frac{1}{1000}\sum_{i=1}^{1000} I_{s_i=1} y_i \text{ and } \bar{y}(s) = \frac{1}{1000 - n_1}\sum_{i=1}^{1000} I_{s_i=2} y_i$$

Its variance is equal to

$$\hat{\sigma}_1^2(s) = \sum_{i=1}^{1000} I_{s_i=1}\left(y_i - \bar{y}_1(s)\right)^2 \text{ and } \hat{\sigma}_2^2(s) = \sum_{i=1}^{1000} I_{s_i=2}\left(y_i - \bar{y}_2(s)\right)^2$$

The log-likelihood of the posterior distribution was carved-out in Figure 2 below (contour plot function that exhibits an additional mode on the likelihood surface). It simply connotes that each of the two partitions of "s" of the simulated dataset, such that 0.0002 and 0.0006 allocates a posterior probability to each of the partition, and afterwards construct a posterior distribution for the conditional coefficients on μ_1 and μ_2. The conditional posterior distribution of the s_i's given (μ_1, μ_2), for $i \in \{1,2,3,\cdots,n\}$

$$P(s_i = 1|\mu_1, y_i) \propto \lambda \exp(-0.5(y_i - \mu_1)^2)$$

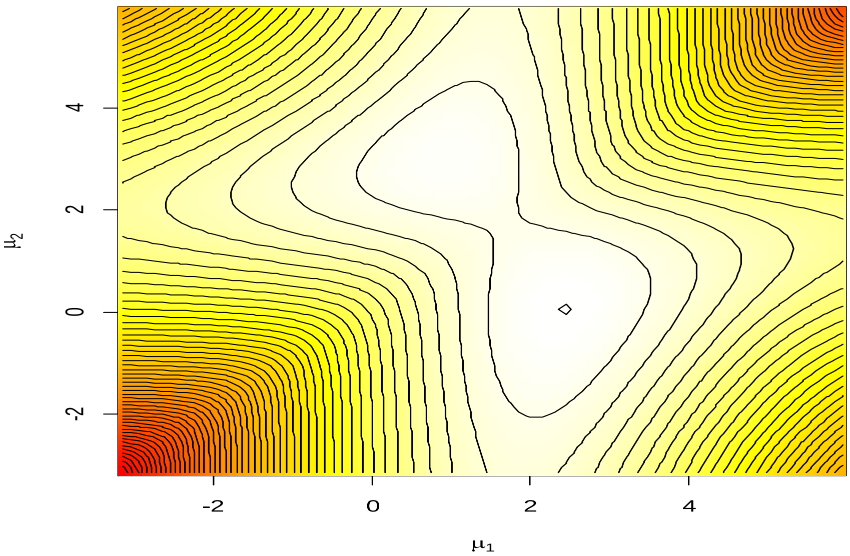


Figure 2. Contour plot of two different independent Gaussian priors of $\mu_1 \sim \mathbb{N}(0,4)$;$\mu_2 \sim \mathbb{N}(2,4)$

Considering a more general case of a mixture of two Gaussian distributions with their parameters unknown, such that, $\lambda_1 N(\mu_1, \sigma_1^2) + (1-\lambda_1)N(\mu_2, \sigma_2^2)$ and for the conjugate prior distribution ($j \in \{1,2\}$). The same starting guessing points in both cases of μ_1 and μ_2 were the saddled points between the modes. Gaussian random walk of scaled unity was adopted because of its smaller magnitude require to paddle more iterations for proper modal region to be reached. For the posterior associated with Equation 25, the Gaussian random walk proposal is $\hat{\mu}_1 \sim N\left(\mu_1^{(t-1)}, \gamma^2\right)$ and $\hat{\mu}_2 \sim N\left(\mu_2^{(t-1)}, \gamma^2\right)$ which leads to the acceptance probability of

$$r = \min\left\{1, \frac{T\left(\hat{\mu}_1, \hat{\mu}_2 | y\right)}{\left(\mu_1^{(t-1)}, \mu_2^{(t-1)} | y\right)}\right\}$$

"γ" was chosen to achieve a reasonable acceptance rate. However, Metropolis–Hastings algorithm checkmated the drawback of Gibbs sampler of insignificantly smaller index that can trap in phenomenon as the scale. This corresponds to $0.25N(2.35,1) + 0.75N(0.02,1)$ of the likelihood surface of Figure 3. The Gibbs sampler was based on 10,000 iterations in agreement with the likelihood surface. It was deduced that the Gibbs sampler ended-up in trapping the lower mode.

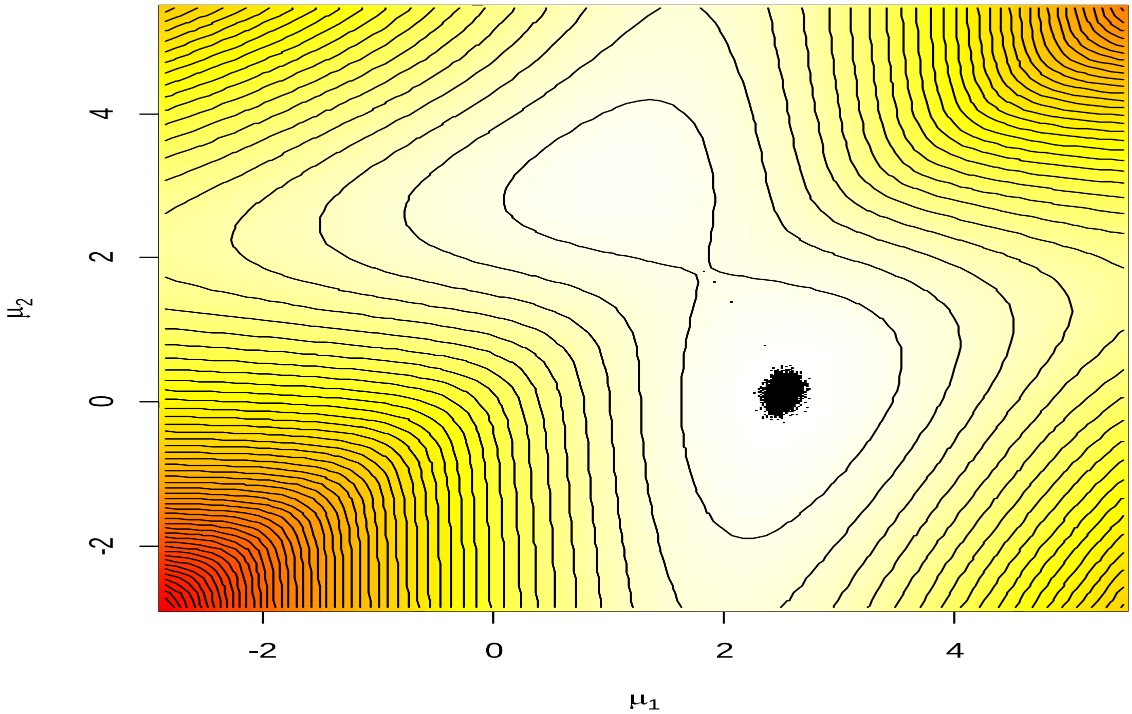


Figure 3. Log-likelihood surface and the corresponding Gibbs sampler for the model, based on 10,000 iterations

It is to be noted that the starting point of the Gibbs sampler in Figure 3 is ($\mu_1 = 0.005$ and $\mu_2 = 0.005$). It clearly indicates that the unconstrained random walk of Metropolis-Hastings remains justifiable for constrained parameters, but not efficient when the Markov Chain moves closer to the boundary of the parameter domain of Figure 4. It also needs to be noted that the parameter domain moves slowly by conditioning the proposed values to be incompatible with the constraints, thus leading to the rejection of the Metropolis-Hastings acceptance ratio. For label switching under invariant permutation indices of components, the Gaussian mixture of $0.25N(2.35,1) + 0.75N(0.002,1)$ and $0.75N(0.002,1) + 0.25N(2.35,1)$ are similar. This does not tantamount to $0.75N(0.002,1)$ distribution that can be called the first component of the mixture model. However, the component parameters ϑ_i are not identifiable marginal, such that, $\vartheta_1 = 0.002$ maybe 2.35 as well. In this case, the quantities (ϑ_1, λ_1) and (ϑ_2, λ_2) are exchangeable.

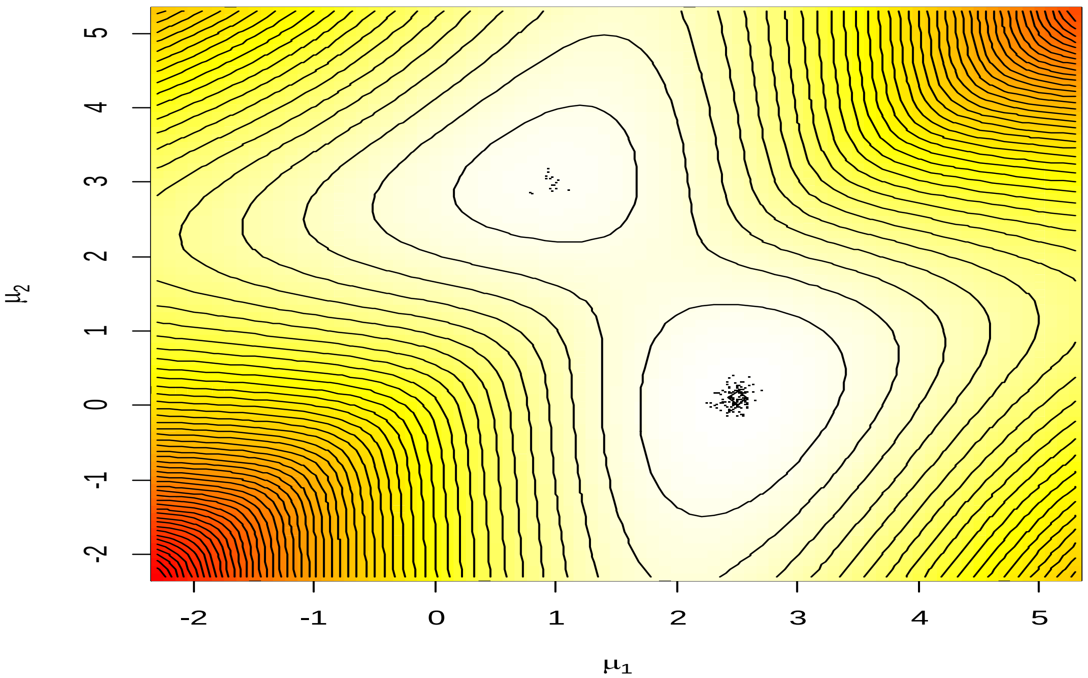


Figure 4. 10,000-iteration outcome of the random walk metropolis-hastings sample on the log-likelihood surface with guessing starting point of (0.5,0.4)

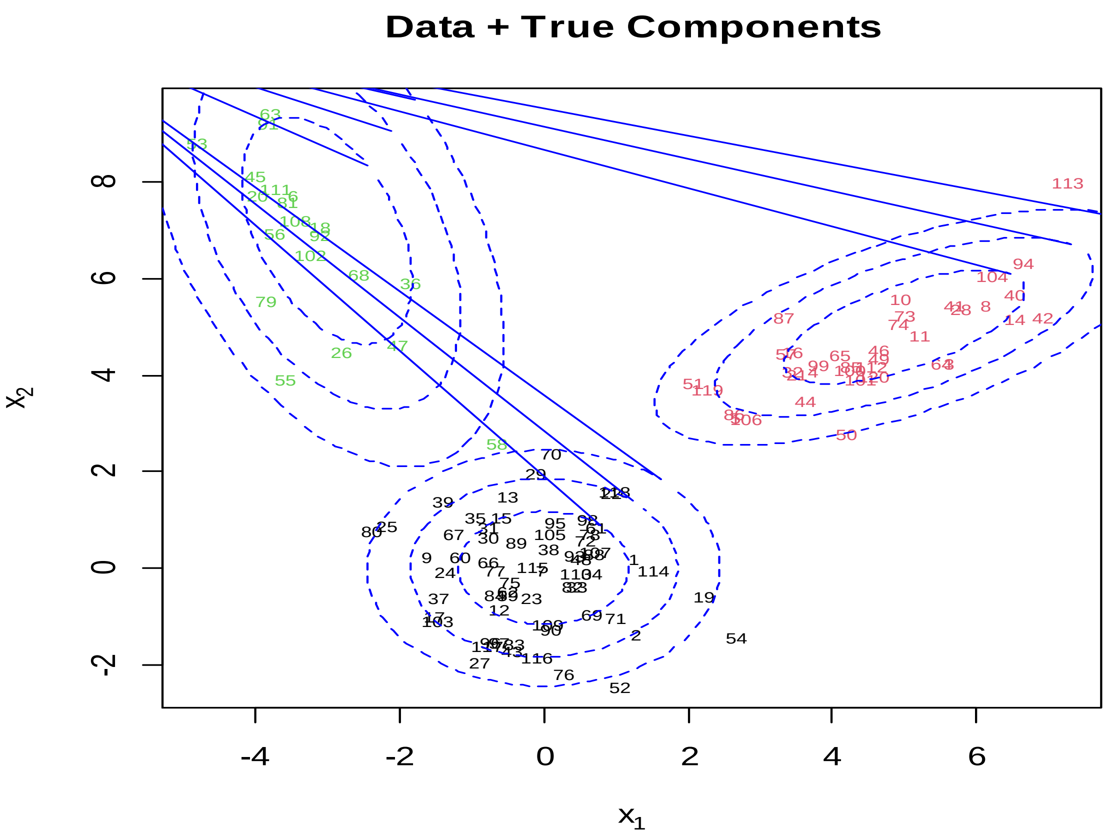


Figure 5. Contour plot of the three-component multivariate mixture model of data + true components

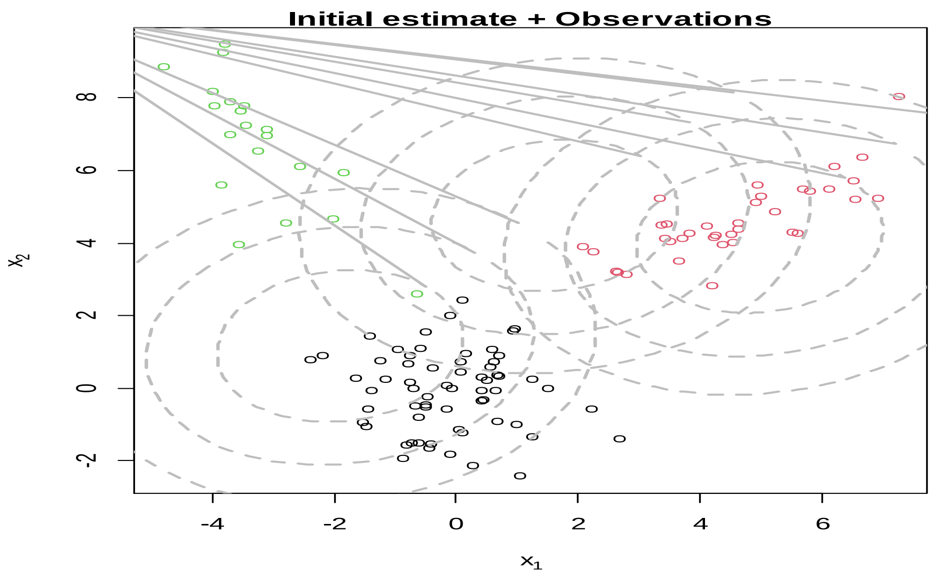


Figure 6. Contour plot of the three-component multivariate mixture model of Initial Estimate + Observations

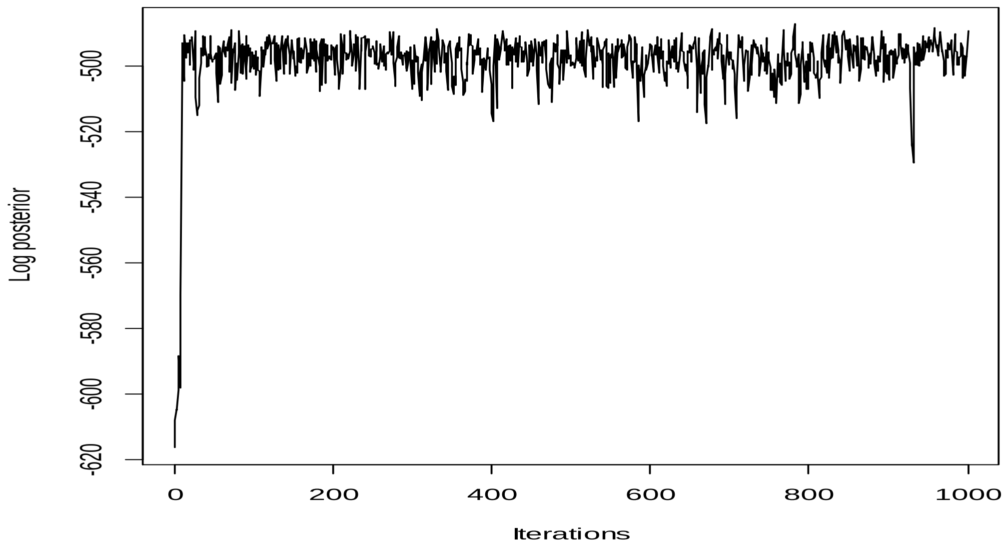


Figure 7. Plot the log-posterior distribution for various samples

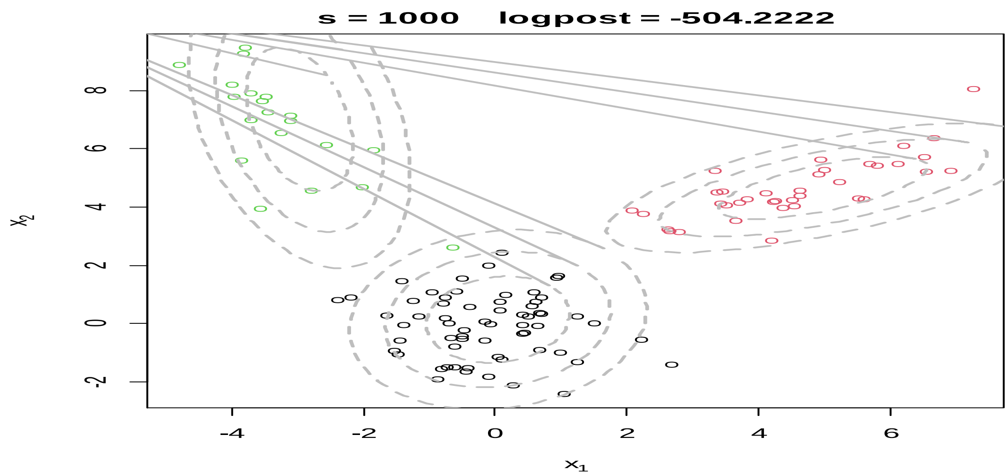


Figure 8. Plot the density estimate for the last iteration of the MCMC

3.1. Discussion of Simulation Results

MCMC algorithm of mixture models of three-component of 2-variates Gaussian components was tested to produce Figures 5 to 8, respectively. The true weights for the three associated components are 0.5, 0.3, and 0.2 respectively. Their true means are (0,0), (5,5), and (-3,7) for the first, second and third components respectively for the simulated data of sample of size 120, such that their true sigma (variance) are (1,0), (2,0.9), and (1,-0.9) respectively. The starting guess value of the weights for each component was assigned to by equal weight of repetition (1,3)/3 via iteration of the sampler. From Figure 5, this is the cluster of three contour plots of the simulated data. The first contour of cluster is denser with clustered of numbers in black color, where the second contour with red character number is lesser than the first one. The third contour at the left corner possessed a scanty cluster of numbers in green color. However, there are three numbers that can be regarded as point outliers: 54, 58, and 133 for the first, second and third contour respectively. The three-compartmental multivariate contour plot of Figure 6 is similar to that of Figure 5 and Figure 8 but it was notably contaminated with outliers, positively abnormal values to indicate that there is possibility of absolving a robust noisy prior-posterior distribution for proper capture. In comparison, the logarithm of the posterior was estimated to be -504.222, such that, the outliers possessed by the posterior density of Figure 7 carved-out six (6) point outliers in contrast to by the true data.

4. Conclusion

This article studies the generic procedure of mixture models with similar inferential probabilistic distribution in a Bayesian setting was proposed. Mixture of similar distributions via Bayesian paradigm was expounded in a finite and non-finite setting, such that the proportional allocation, sample size allocation, and mixing weights for the posterior distribution were carved-out for *k*-components. The parameter estimation of the generic posterior distribution of proportional allocation, sample size allocation, and mixing components coupled with the embedded latent (unobserved) variable was carried-out via the EM algorithm. Metropolis–Hastings and MCMC algorithms were alternately employed in place of the EM algorithm under some conditions. Monte Carlo approximation technique was employed to estimate MAP, such that label switching for exchangeability of posterior distribution was carried-out under different prior for known and unknown components with finite and non-finite mixture. In conclusion, it was deduced that the number of components grows large for all permutations to be considered for subsample of permutations simulated. Further study can be extended to generic procedure of mixture models with different inferential probabilistic distributions from both a parametric, non-parametric and Bayesian point of view. In addition, in scenarios where different prior-likelihood distributions are merged or convoluted, label switching for exchangeability of posterior distribution for finite and non-finite mixture needs to be further studied. In extension, the finite and non-finite mixture models via Bayesian paradigm can be extended to time-varying processes like networking autoregressive and mixture autoregressive processes.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] A. Gelman, J. Carlin, H. Stern, D. Dunson, A. Vehtari, D. Rubin, Bayesian Data Analysis, 3rd Edition, Chapman and Hall, New York, 2013.

[2] G. Wioletta, *The Advantages of Bayesian Methods over Classical Methods in the Context of Credible Intervals*, Information Systems in Management 4 (1) (2015) 53–63.

[3] C. Charlton, J. Rasbash, W. J. Browne, M. Healy, B. Cameron, *MLwiN. In: Centre for Multilevel Modeling* (2020), https://www.bristol.ac.uk/cmm/, Accessed 20 Sep 2023.

[4] J. E. Johndrow, A. Smith, N. Pillai, N. Dunson, *MCMC for Imbalanced Categorical Data*, Journal of the American Statistical Association 114 (527) (2019) 1394–1403.

[5] R. O. Olanrewaju, S. A. Olanrewaju, L. A. Nafiu, *Multinomial Naive Bayes Classifier: Bayesian versus Non-parametric Classifier Approach*, European Journal of Statistics 2 (8) (2022) 1–14.

[6] R. O. Olanrewaju, *Bayesian Approach: An Alternative to Periodogram and Time Axes Estimation for Known and Unknown White Noise*, International Journal of Mathematical Sciences and Computing 2 (5) (2018) 22–33.

[7] U. Simola, J. Cisewski-Kehe, L. R. Wolpert, *Approximate Bayesian Computation for Finite Mixture Models*, Journal of Statistical Computation and Simulation 91 (6) (2021) 1155–1174.

[8] A. Hairault, C. P. Robert, J. Rousseau, *Evidence Estimation in Finite and Infinite Mixture Models and Applications* (2022) 43 pages, https://arxiv.org/abs/2205.05416.

[9] A. R. Hassan, R. O. Olanrewaju, Q. C. Chukwudum, S. A. Olanrewaju, S. E. Fadugba, *Comparison Study of Generative and Discriminative Models for Classification of Classifiers*, International Journal of Mathematics and Computer Simulation 16 (12) (2022) 76–87.

[10] M. Betancourt, *A Conceptual Introduction to Hamiltonian Monte Carlo* (2017) 60 pages, https://arxiv.org/abs/1701.02434.

[11] J. F. Ojo, R. O. Olanrewaju, S. A. Folorunsho, *Bayesian Logistic Regression Using Gaussian Naıve Bayes (GNB)*, Journal of Medical and Applied Biosciences 9 (2) (2017) 1–18.

[12] R. O. Olanrewwaju, L. O. Adekola, E. Oseni, S. A. Phillips, A. A. Oyinloye, *Disintegration of Price Ordered Probit Model: An Application to Prices of Cereal Crops in Nigeria*, African Journal of Applied Statistics 7 (1) (2020) 781–804.

[13] S. Virolainen, *A Mixture Autoregressive Model Based on Gaussian and Student-t-Soft Distributions*, Studies in Nonlinear Dynamics & Econometrics 26 (4) (2022) 559–580.

[14] R. O. Olanrewaju, A. G. Waititu, L. A. Nafiu, *Bull and Bear Dynamics of the Nigeria Stock Returns Transitory via Mingled Autoregressive Random Processes*, Open Journal of Statistics 11 (2021) 870–885.

[15] R. O. Olanrewaju, A. G. Waititu, L. A. Nafiu, *On the Estimation of k-Regimes Switching of Mixture Autoregressive Model via Weibull Distributional Random Noise*, International Journal of Probability and Statistics 10 (1) (2021) 1–8.

[16] J. F. Ojo, R. O. Olanrewaju, *On Mixture Auto-Regressive (MAR) Using Naira-Dollar Exchange Rates*, Journal of Nigeria Association Mathematical Physics 38 (12) (2016) 155-165.

[17] R. O. Olanrewaju, S. A. Olanrewaju, *An Alternative Mean Variance Portfolio Theoretical Framework: Nigeria Banks' Market Shares Analysis*, Global Journal of Business, Economics, and Management 11 (3) (2021) 220–234.

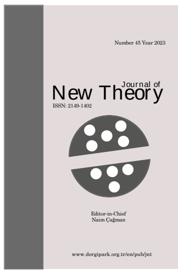
[18] R. O. Olanrewaju, *On the Application of Generalized Beta-G Family of Distributions to Prices of Cereals*, Journal of Mathematical Finance 11 (4) (2021) 670–685.

[19] R. O. Olanrewaju, M. A. Jallow, S. A. Olanrewaju, *An Analysis of the Atlantic Ocean Random Cosine and Sine Alternate Wavy ARIMA Functions*, International Journal of Intelligent Systems and Applications 14 (5) (2022) 22–34.

[20] J. F. Olanrewaju, R. O. Olanrewaju, S. A. Folorunso, *Performance of all Nigeria Banks' Shares using Student-t Mixture Autoregressive Model*, Journal of Engineering and Applied 9 (1) (2017) 69–82.

New Theory Journal of
ISSN: 2149-1402

45 (2023) 73-82
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Altered Numbers of Fibonacci Number Squared

Fikri Köken[1], Emre Kankal[2]

Article Info
Received: 29 Sep 2023
Accepted: 30 Nov 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1368751
Research Article

Abstract — We investigate two types of altered Fibonacci numbers obtained by adding or subtracting a specific value $\{a\}$ from the square of the n^{th} Fibonacci numbers $G_{F_{(n)}}^{(2)}(a)$ and $H_{F_{(n)}}^{(2)}(a)$. These numbers are significant as they are related to the consecutive products of the Fibonacci numbers. As a result, we establish consecutive sum-subtraction relations of altered Fibonacci numbers and their Binet-like formulas. Moreover, we explore greatest common divisor (GCD) sequences of r-successive terms of altered Fibonacci numbers represented by $\left\{G_{F(n),r}^{(2)}(a)\right\}$ and $\left\{H_{F(n),r}^{(2)}(a)\right\}$ such that $r \in \{1,2,3\}$ and $a \in \{1,4\}$. The sequences are based on the GCD properties of consecutive terms of the Fibonacci numbers and structured as periodic or Fibonacci sequences.

Keywords *Altered Fibonacci number, greatest common divisor (GCD) sequence, Fibonacci sequence*

Mathematics Subject Classification (2020) 11B39, 11B50

1. Introduction

It is known [1] that the Fibonacci sequence is defined recursively as

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with initial values $F_0 = 0$ and $F_1 = 1$ (A000045 in OEIS). As a similar, L_n is the n^{th} term in the Lucas sequence (A000032) and defined by

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2 \quad \text{and} \quad L_1 = 1$$

Their characteristic equation is $x^2 = x + 1$ and its roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Hence, the Binet formulas for the Fibonacci F_n and Lucas L_n numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

Binet formulas can be used to prove certain properties of the Fibonacci and Lucas numbers. For instance, for negative subscripts the n^{th} Fibonacci number can be established as $F_{-n} = (-1)^{n+1} F_n$, for all $n \geq 1$, or two useful identities can be confirmed the Cassini identity and the d'Ocagne identity [1-3], respectively,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

and

[1]kokenfikri@gmail.com (Corresponding Author); [2]kangalemre56@gmail.com
[1]Department of Computer Engineering, Seydişehir Ahmet Cengiz Faculty of Engineering, Necmettin Erbakan University, Konya, Türkiye
[2]Department of Mathematics, Institute of Science, Necmettin Erbakan University, Konya, Türkiye

$$F_mF_{n+1} = F_nF_{m+1} + (-1)^nF_{m-n}, \quad m > n \geq 1$$

Additionally, the formulas sum and subtraction for the Fibonacci numbers squared are

$$F^2_{m+n+1} + F^2_{m-n} = F_{2m+1}F_{2n+1} \tag{1}$$

and

$$F^2_{m+n} - F^2_{m-n} = F_{2m}F_{2n} \tag{2}$$

Many sum properties [1-3] can be provided as examples of sequences derived from the Fibonacci numbers. The sum of the Fibonacci numbers is $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ (A000071 in OEIS [2]), and the sum of even-indices Fibonacci numbers is $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$ (A027941 in [2]). These findings have been scrutinized as the altered Fibonacci sequences [4]. The sum of odd-indices Fibonacci numbers is $\sum_{i=1}^{n} F_{2i-1} = F_nL_n$ (A001906 in [2]). The sum of the Fibonacci numbers squared between F_1 and F_n is $\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$ (A180662, The golden rectangle numbers in [2]).

In the literature, numerous researchers [4-8] have developed novel sequences utilizing Fibonacci numbers and analyzed some of their basic properties. Dudley et al. [4] studied two altered Fibonacci sequences $\{G_n\} = \{F_n + (-1)^n\}$ and $\{H_n\} = \{F_n - (-1)^n\}$, concerned with number sequences A000071 and A027941, using the equations given by Theorem 1 in [4]

$$F_{4k} + 1 = F_{2k-1}L_{2k+1} \qquad F_{4k} - 1 = F_{2k+1}L_{2k-1} \tag{3}$$

$$F_{4k+1} + 1 = F_{2k+1}L_{2k} \qquad F_{4k+1} - 1 = F_{2k}L_{2k+1} \tag{4}$$

$$F_{4k+2} + 1 = F_{2k+2}L_{2k} \qquad F_{4k+2} - 1 = F_{2k}L_{2k+2} \tag{5}$$

$$F_{4k+3} + 1 = F_{2k+1}L_{2k+2} \qquad F_{4k+3} - 1 = F_{2k+2}L_{2k+1} \tag{6}$$

Some of those are easily obtained according to whether n is odd or even in the Cassini identity. Moreover, $\{(G_n, G_{n+1})\}_{n\geq 0}$ and $\{(H_n, H_{n+1})\}_{n\geq 0}$ sequences are defined by using the greatest common divisor (GCD) of the numbers G_n and H_n considering Equations 3-6 are multiplication cases. These sequences produce Fibonacci subsequences, such as $(G_{4k}, G_{4k+1}) = L_{2k+1}$, $(G_{4k+2}, G_{4k+3}) = F_{2k+2}$, $(H_{4k}, H_{4k+1}) = F_{2k+1}$, and $(H_{4k+2}, H_{4k+3}) = L_{2k+2}$ [4]. Hernandez and Luca [5] proved the existence of an integer c in the form of an infinite number $(F_n + a, F_m + b) > e^{(cm)}$ of any positive integer $n < m$, according to various n and m for the positive integers a and b. Chen [6] defined a sequence $\{F_n + a\}_{n\geq 0}$ such that $a \in \mathbb{Z}$, called a shifted Fibonacci sequence, and established a sequence $\{f_n(a)\}_{n\geq 0} = \{(F_n + a, F_{n+1} + a)\}_{n\geq 0}$, referred to as a GCD sequence of the shifted Fibonacci sequence. He showed that some successive terms of the altered and shifted sequences have different behavior, such as $f_{4n-1}(1) = F_{2n-1}$, $f_{4n+1}(1) = L_{2n}$, $f_{4n-1}(-1) = L_{2n-1}$, and $f_{4n+1}(-1) = F_{2n}$. The author showed that $\{f_n(a)\}$ is bounded from above if $a \neq \pm 1$. In [7], in addition to the properties of $\{f_n(a)\}$, Spilker showed that for two integers a and n if $m = a^4 - 1$ is not 0 and $f_n(a)$ divides $a^2 + (-1)^n$, then $f_n(a)$ is simply periodic such that a period p is defined by $F_p \equiv 0 \pmod m$. Koken [8] defined the altered sequences $\{L_n^+\}_{n>0}$ and $\{L_n^-\}_{n>0}$ such that $L_{4k}^+ = 5F_{2k+1}F_{2k-1}$, $L_{4k+1}^+ = 5F_{2k+1}F_{2k}$, $L_{4k+2}^+ = L_{2k+2}L_{2k}$, and $L_{4k+3}^+ = L_{2k+2}L_{2k+1}$ and $L_{4k}^- = L_{2k+1}L_{2k-1}$, $L_{4k+1}^- = L_{2k+1}L_{2k}$, $L_{4k+2}^- = 5F_{2k+2}F_{2k}$, and $L_{4k+3}^- = 5F_{2k+2}F_{2k+1}$. Furthermore, he presented the numbers $L_{4k,1}^+ = 5F_{2k+1}, k \geq 1$, $L_{4k-2,1}^+ = L_{2k}, k \geq 1$, $L_{4k,1}^- = L_{2k+1}$ and $L_{4k+2,1}^- = 5F_{2k+2}$ where $L_{n,r}^\pm = \left(L_n^\pm, L_{n+r}^\pm\right)$ denotes *r*-successive GCD numbers. Besides, the GCD numbers $L_{n,r}^+$ and $L_{n,r}^-$ are obtained by $r \in \{2,3,4\}$. For over 50 years, many authors [9-14] have studied to determine all such numbers of the forms w^2, w^3, $w^2 \pm 1$, and $w^3 \pm 1$ in the Fibonacci sequences. Marques

[15] has considered the Fibonacci variant of the Brocard-Ramanujan equation and claimed that the Diophantine equation

$$F_n F_{n+1} \cdots F_{n+k-2} F_{n+k-1} + 1 = F_m^2 \tag{7}$$

has no solution according to the positive integer values k, m, and n. However, according to equations $F_2 F_4 + 1 = F_1 F_4 + 1 = F_3^2$ and $F_2 F_6 + 1 = F_1 F_6 + 1 = F_4^2$, it can be observed that the Fibonacci Brocard-Ramanujan version in Equality 7 has solutions. Szalay [16] obtains the solutions of the equations by accepting a correct version of the result of Marques [15] more general than the Fibonacci Brocard-Ramanujan equation in Equality 7. Pongsriiam [17] has continued to search for the solutions of the Diophantine equations:

$$F_{n_1} F_{n_2} \cdots F_{n_{k-1}} F_{n_k} \pm 1 = F_m^2 \quad \text{and} \quad L_{n_1} L_{n_2} \cdots L_{n_{k-1}} L_{n_k} \pm 1 = F_m^2$$

such that $0 \leq n_1 < n_2 < \cdots < n_{k-1} < n_k$, $m \geq 0$, and $k \geq 1$.

Inspired by previous research on altered Fibonacci numbers and the Brocard-Ramanujan equation, this study aims to explore their applications and altered sequences of Fibonacci numbers squared. This investigation is continued by the question of whether it is possible to define altered Fibonacci sequences, specifically those of the form $\{F_n^2 \pm a\}$. Unlike [18,19], related to the sum of sequences of k-consecutive Fibonacci numbers, the paper considers the results of altered Fibonacci numbers squared through the following sums:

$$\sum_{j=1}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 1 \quad \text{or} \quad \sum_{j=2}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 2$$

and

$$\sum_{j=1}^{2n-1} F_j F_{j+1} = F_{2n}^2$$

Koken [20] investigate two types altered Lucas numbers $G_{L(n)}^{(2)}(a)$ and $H_{L(n)}^{(2)}(a)$. Since these numbers form as the consecutive products of the Fibonacci numbers, they give the GCD sequences of r–successive terms of altered Lucas numbers denoted $\left\{G_{L(n),r}^{(2)}(a)\right\}$ and $\left\{H_{L(n),r}^{(2)}(a)\right\}$ such that $r \in \{1,2\}$ and $a \in \{1,9\}$. We show that these sequences are periodic or Fibonacci sequences.

This present paper is organized as follows: Section 2 provides brief definitions and properties. Section 3 defines two altered sequences and investigates some of their properties. This includes analyzing the sum and difference, Binet formula, and closed forms for the numbers $G_{F(n)}^{(2)}(a) = F_n^2 + (-1)^n a$ and $H_{F(n)}^{(2)}(a) = F_n^2 - (-1)^n a$. Section 4 establishes two types of r-successive altered Fibonacci GCD sequences, referred to as $G_{F(n),r}^{(2)}(a)$ and $H_{F(n),r}^{(2)}(a)$, and investigates these sequences according to the cases $r \in \{1,2,3\}$ for the values $G_{F(n)}^{(2)}(a)$ and $H_{F(n)}^{(2)}(a)$ such that $a \in \{1,4\}$.

2. Preliminaries

This section defines two types of altered numbers derived by using a value $\{a\}$ from the n^{th} Fibonacci number squared. It works on taking values $\{\pm 1\}$ instead of $\{a\}$.

Definition 2.1. The n^{th} altered Fibonacci numbers denoted by $G_{F_{(n)}^{(2)}}(a)$ and $H_{F_{(n)}^{(2)}}(a)$ are defined as

$$G_{F(n)}^{(2)}(a) = F_n^2 + (-1)^n a \tag{8}$$

and

$$H^{(2)}_{F(n)}(a) = F_n^2 - (-1)^n a \tag{9}$$

where F_n be the n^{th} Fibonacci number and $a \in \mathbb{Z}$.

For example, particular values $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$ numbers are provided in Table 1, and they follow $G_{F_{(n)}}^{(2)}(1) = H_{F_{(n)}}^{(2)}(-1)$ and $H_{F_{(n)}}^{(2)}(1) = G_{F_{(n)}}^{(2)}(-1)$.

Table 1. First few terms of $G^{(2)}_{F(n)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G^{(2)}_{F(n)}(1)$	1	0	2	3	10	24	65	168	442	1155	3026	7920	20737
$H_{F_{(n)}}^{(2)}(1)$	-1	2	0	5	8	26	63	170	440	1157	3024	7922	20735

Table 1 shows that $G_{F_{(3n)}}^{(2)}(1)$ and $H_{F_{(3n)}}^{(2)}(1)$ are odd, and the others are even, any increasing sequences with special values except the first few values. The general terms of the sequences $\left\{G_{F_{(n)}}^{(2)}(1)\right\}$ and $\left\{H_{F_{(n)}}^{(2)}(1)\right\}$ can be given as follows:

Theorem 2.2. Let $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$ denote the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}}^{(2)}(1) = F_{n+1}F_{n-1} \tag{10}$$

and

$$H_{F_{(n)}}^{(2)}(1) = F_{n+2}F_{n-2} \tag{11}$$

Proof.

If $m = k+1$ and $n = k$ in Equation 1, then $G_{F_{(2k)}}^{(2)}(1) = F_{2k-1}F_{2k+1}$, for $a = 1$ and $n = 2k$ in Equation 8. In addition, if $m = k+1$ and $n = k$ in Equation (2), then $G_{F_{(2k+1)}}^{(2)}(1) = F_{2k+2}F_{2k}$, for $a = 1$ and $n = 2k + 1$ in Equation 8. The number $G_{F_{(n)}}^{(2)}(1)$ is observed from these equations for $n = 2k$ and $n = 2k+1$.

If $m = k+2$ and $n = k$ in Equation 1, then $H_{F_{(2k+1)}}^{(2)}(1) = F_{2k+3}F_{2k-1}$, for $a = 1$ and $n = 2k+1$ in Equation 9. For $m = k+2$ and $n = k$ in Equation 2, $H_{F_{(2k)}}^{(2)}(1) = F_{2k+2}F_{2k-2}$ when $n = 2k$ in Equation 9. The number $H_{F_{(n)}}^{(2)}(1)$ is observed from these equations for $n = 2k$ and $n = 2k+1$. □

We have conducted research on several addition and subtraction identities of numbers $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$.

Theorem 2.3. Let $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$ be the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}}^{(2)}(1) + G_{F_{(n+1)}}^{(2)}(1) = H_{F_{(n)}}^{(2)}(1) + H_{F_{(n+1)}}^{(2)}(1) = F_{2n+1} \tag{12}$$

$$G_{F_{(n+1)}}^{(2)}(1) - G_{F_{(n-1)}}^{(2)}(1) = H_{F_{(n+1)}}^{(2)}(1) - H_{F_{(n-1)}}^{(2)}(1) = F_{2n} \tag{13}$$

$$2G_{F_{(n+1)}}^{(2)}(1) + G_{F_{(n)}}^{(2)}(1) - G_{F_{(n-1)}}^{(2)}(1) = F_{2n+2} \tag{14}$$

$$2H_{F_{(n+1)}^{(2)}}(1) + H_{F_{(n)}^{(2)}}(1) - H_{F_{(n-1)}^{(2)}}(1) = F_{n+1}L_{n+1} \tag{15}$$

PROOF.

From Equations 10 and 11 and the identities $F_n^2 + F_{n+1}^2 = F_{2n+1}$ and $F_nL_n = F_{2n}$,

$$H_{F_{(n)}^{(2)}}(1) + H_{F_{(n+1)}^{(2)}}(1) = F_{n+2}F_{n-2} + (F_{n+2} + F_{n+1})F_{n-1} = F_{2n+1}$$

and

$$G_{F(n+1)}^{(2)}(1) - G_{F(n-1)}^{(2)}(1) = F_n(F_{n+1} + F_n - F_{n-2}) = F_nL_n$$

The others in Equations 12 and 13 are obtained similarly. If Equations 12 and 13 are summed side-to-side collection, then Equations 14 and 15 are obtained. □

2.1. Altered Fibonacci Sequences $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$

This subsection generalizes the value $\{a\}$ in Equations 8 and 9 as the square of t^{th} Fibonacci numbers such that $t \in \mathbb{Z}$.

Theorem 2.4. Let $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ denote the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is odd} \tag{16}$$

and

$$H_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is even} \tag{17}$$

where F_t^2 is the square of the t^{th} Fibonacci numbers.

PROOF.

Let t is odd. If $m = k + (t+1)/2$ and $n = k - (t-1)/2$ are taken in Equation 1, for $a = F_t^2$ and $n = 2k$ in Equation 8, then $G_{F_{(2k)}^{(2)}}(F_t^2) = F_{(2k)+t}F_{(2k)-t}$. Moreover, if values of $m = k + (t+1)/2$ and $n = k - (t-1)/2$ are considered in Equation 2, according to $a = F_t^2$ and $n = 2k+1$ in Equation 8, then $G_{F_{(2k+1)}^{(2)}}(F_t^2) = F_{(2k+1)+t}F_{(2k+1)-t}$.

Similarly, let t is even. If $m = k + t/2$ and $n = k - t/2$ in Equations 1 and 2, then the desired result is obtained. □

As a result, the sum of two successive altered Fibonacci numbers equals the Fibonacci number, and no alike Fibonacci recurrence relation is provided. However, a Binet-like formula for the numbers $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ can be obtained by using the Fibonacci Binet formula.

Theorem 2.5. Let $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ be the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}^{(2)}}(F_t^2) = \frac{(\alpha^{2n} + \beta^{2n}) + (-1)^n(\alpha^{2t} + \beta^{2t})}{5}, \qquad t \text{ is odd} \tag{18}$$

and

$$H_{F_{(n)}^{(2)}}(F_t^2) = \frac{(\alpha^{2n} + \beta^{2n}) - (-1)^n(\alpha^{2t} + \beta^{2t})}{5}, \qquad t \text{ is even} \tag{19}$$

PROOF.

Let t is odd. If we substitute the Fibonacci Binet formula in Equation 16. Then,

$$G_{F_{(n)}^{(2)}}(F_t^2) = \frac{(\alpha^{n+t} - \beta^{n+t})(\alpha^{n-t} - \beta^{n-t})}{(\alpha - \beta)^2}$$

By using $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$, the desired expression is obtained. The other appeared as an application of the Fibonacci Binet formula in Equation 17. □

As a result of Equations 18 and 19, Binet-like formulas for the numbers $G_{F_{(n)}^{(2)}}(1)$ and $H_{F_{(n)}^{(2)}}(1)$ are

$$G_{F_{(n)}^{(2)}}(1) = \frac{(\alpha^{2n} + \beta^{2n}) + (-1)^n 3}{5} = \frac{L_{2n} + (-1)^n 3}{5}$$

and

$$H_{F_{(n)}^{(2)}}(1) = \frac{(\alpha^{2n} + \beta^{2n}) - (-1)^n 7}{5} = \frac{L_{2n} - (-1)^n 7}{5}$$

More details about the sequences $a(n) = F_n F_{n+2}$ and $b(n) = F_n F_{n+4}$ can be found in (A059929) and (A192883). We study the special terms of the altered Fibonacci numbers $G_{F_{(n)}^{(2)}}(F_t^2) = H_{F_{(n)}^{(2)}}(-F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2) = G_{F_{(n)}^{(2)}}(-F_t^2)$. The altered number $G_{F_{(n)}^{(2)}}(4) = F_{n+3}F_{n-3}$ is the case $t = 3$ in Equation 16. Furthermore, the sequence $x(n) = F_{n+3}F_{n-3}$ has been studied in the literature (A292612) with its different applications. The altered number $H_{F_{(n)}^{(2)}}(9) = F_{n+4}F_{n-4}$ is the case $t = 4$ in Equation 17. In addition, the sequence $b(n) = F_{n+4}F_{n-4}$ has been studied in the literature (A292612) with its different applications. However, $H_{F_{(n)}^{(2)}}(4)$ and $G_{F_{(n)}^{(2)}}(9)$ could not be generalized as the product of Fibonacci or Lucas numbers.

3. Altered Fibonacci GCD Sequences $G_{F_{(n),r}^{(2)}}(a)$ and $H_{F_{(n),r}^{(2)}}(a)$

A GCD of two Fibonacci numbers is a Fibonacci number, such as $(F_m, F_n) = F_{(m,n)}$ and $(F_m, F_n) = (F_n, F_r)$, for all $m = qn + r$ such that $m, n, r, q \in \mathbb{N}$. Thus, two successive Fibonacci numbers are relatively prime, i.e., $(F_n, F_{n+1}) = 1$ and $\left(F_{qn-1}, F_n\right) = (F_n, F_{n+2}) = 1$ [1-3]. This section investigates properties related to GCD of two numbers whose indices differ r from the altered sequences $\left\{G_{F_{(n)}^{(2)}}(a)\right\}$ and $\left\{H_{F_{(n)}^{(2)}}(a)\right\}$.

Definition 3.1. Let $G_{F_{(n)}^{(2)}}(a)$ and $H_{F_{(n)}^{(2)}}(a)$ be the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n),r}^{(2)}}(a) = \left(G_{F_{(n)}^{(2)}}(a), G_{F_{(n+r)}^{(2)}}(a)\right)$$

and

$$H_{F_{(n),r}^{(2)}}(a) = \left(H_{F_{(n)}^{(2)}}(a), H_{F_{(n+r)}^{(2)}}(a)\right)$$

The sequences $\left\{G_{F_{(n),r}^{(2)}}(a)\right\}$ and $\left\{H_{F_{(n),r}^{(2)}}(a)\right\}$ formed by these numbers are called the r-successive altered Fibonacci GCD sequences.

Table 2 shows $\left\{G_{F_{(n),1}^{(2)}}(1)\right\}$ and $\left\{H_{F_{(n),1}^{(2)}}(1)\right\}$ are not increasing or decreasing but can be periodic sequences.

Table 2. 1-successive altered Fibonacci GCD numbers $G_{F(n),1}^{(2)}(1)$ and $H_{F(n),1}^{(2)}(1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$G_{F(n),1}^{(2)}(1)$	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1
$H_{F(n),1}^{(2)}(1)$	1	2	5	1	2	1	1	10	1	1	2	1	5	2	1	1

The following theorem investigates whether 1-successive altered Fibonacci GCD sequences take special values in certain periods.

Theorem 3.2. Let $G_{F(n),1}^{(2)}(1)$ and $H_{F(n),1}^{(2)}(1)$ be the n^{th} 1-successive altered Fibonacci GCD numbers. Then,

$$G_{F(n),1}^{(2)}(1) = \begin{cases} 2, & n \equiv 1 \pmod 3 \\ 1, & \text{otherwise} \end{cases}$$

and

$$H_{F(n),1}^{(2)}(1) = \begin{cases} 10, & n \equiv 7 \pmod{15} \\ 5, & n \equiv 2,12 \pmod{15} \\ 2, & n \equiv 1,4,10,13 \pmod{15} \\ 1, & \text{otherwise} \end{cases}$$

Proof.

According to Equation 10, $G_{F(n),1}^{(2)}(1) = (F_{n+1}F_{n-1}, F_nF_{n+2})$. Since $(F_{n+1}, F_n) = (F_{n+1}, F_{n+2}) = (F_{n-1}, F_n) = 1$, then $G_{F(n),1}^{(2)}(1) = (F_{n-1}, F_{n+2})$. Therefore, let $(F_{n-1}, F_{n+2}) = d$. By using $(F_x, F_y) = F_{(x,y-x)}$, $(F_{n-1}, F_{n+2}) = F_{(n-1,3)} = F_3, n \equiv 1 \pmod 3$. Otherwise, $(F_{n-1}, F_{n+2}) = F_1$.

According to Equation 11, $H_{F(n),1}^{(2)}(1) = (F_{n+2}F_{n-2}, F_{n-1}F_{n+3})$. Since $(F_{n+2}, F_{n+3}) = (F_{n-2}, F_{n-1}) = 1$, then $H_{F(n),1}^{(2)}(1) = (F_{n-2}, F_{n+3})(F_{n+2}, F_{n-1})$. Thus, if $(F_{n-2}, F_{n+3}) = F_{(n-2,5)} = F_5$, $n \equiv 2 \pmod 5$ and $(F_{n+2}, F_{n-1}) = F_{(n-1,3)} = F_3$, $n \equiv 1 \pmod 3$, then we can obtain desired results by using the Chinese remainder theorem. ◻

Table 3 manifests that the 2-successive altered Fibonacci GCD sequence $\left\{G_{F(n),2}^{(2)}(1)\right\}$, for $n \geq 2$, takes values according to a specific increasing sequence, and the sequence $\left\{H_{F(n),2}^{(2)}(1)\right\}$ is seen periodic.

Table 3. 2-successive altered Fibonacci GCD numbers $G_{F(n),2}^{(2)}(1)$ and $H_{F(n),2}^{(2)}(1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$G_{F(n),2}^{(2)}(1)$	1	3	2	3	5	24	13	21	34	165	89	144	233	1131	610	987
$H_{F(n),2}^{(2)}(1)$	1	1	8	1	1	2	1	1	8	1	1	2	1	1	8	1

Some properties of the aforesaid sequences are as follows:

Theorem 3.3. Let $G_{F(n),2}^{(2)}(1)$ and $H_{F(n),2}^{(2)}(1)$ be the n^{th} 2-successive altered Fibonacci GCD numbers. Then,

$$G_{F(n),2}^{(2)}(1) = \begin{cases} 3F_{n+1}, & n \equiv 1 \pmod 4 \\ F_{n+1}, & \text{otherwise} \end{cases}$$

and

$$H_{F(n),2}^{(2)}(1) = \begin{cases} 8, & n \equiv 2 \pmod 6 \\ 2, & n \equiv 5 \pmod 6 \\ 1, & \text{otherwise} \end{cases}$$

PROOF.

According to Equation 10, $G_{F_{(n),2}}^{(2)}(1) = \left(G_{F_{(n)}}^{(2)}(1), G_{F_{(n+2)}}^{(2)}(1)\right) = F_{n+1}(F_{n-1}, F_{n+3})$. Therefore, $(F_{n-1}, F_{n+3}) = F_{(n-1,4)} = F_4$, $n \equiv 1 \pmod 4$ by using $(F_x, F_y) = F_{(x,y-x)}$. Otherwise, it is seen that $(F_{n-1}, F_{n+3}) = F_{(n-1,4)} = F_2$ or F_1.

According to Equation 11, $H_{F(n),2}^{(2)}(1) = (F_{n+2}F_{n-2}, F_nF_{n+4})$. Because of $(F_{n+2}, F_n) = (F_{n+2}, F_{n+4}) = (F_{n-2}, F_n) = 1$, we study on $H_{F(n),2}^{(2)}(1) = (F_{n-2}, F_{n+4})$. Thus, $H_{F(n),2}^{(2)}(1) = F_{(n-2,6)} = F_6$, $n \equiv 2 \pmod 6$. Otherwise, the others are $H_{F(n),2}^{(2)}(1) = F_{(n-2,6)} = F_3$, $n \equiv 5 \pmod 6$; $H_{F(n),2}^{(2)}(1) = F_2, n \equiv 0,4 \pmod 6$; or $H_{F(n),2}^{(2)}(1) = F_1, n \equiv 1,3 \pmod 6$. □

Theorem 3.4. Let $G_{F_{(n),2}}^{(2)}(1)$ be the n^{th} 2-successive altered Fibonacci GCD number. Then,

$$G_{F_{(n),2}}^{(2)}(1) + G_{F_{(n+1),2}}^{(2)}(1) = \begin{cases} F_{n+1} + L_{n+2}, & n \equiv 1 \pmod 4 \\ L_{n+3}, & n \equiv 0 \pmod 4 \\ F_{n+3}, & \text{otherwise} \end{cases}$$

PROOF.

According to $G_{F_{(n),2}}^{(2)}(1)$ in Theorem 3.3,

$$G_{F_{(n),2}}^{(2)}(1) + G_{F_{(n+1),2}}^{(2)}(1) = \begin{cases} F_{n+2} + 3F_{n+1}, & n \equiv 1 \pmod 4 \\ 3F_{n+2} + F_{n+1}, & n \equiv 0 \pmod 4 \\ F_{n+1} + F_{n+2}, & \text{otherwise} \end{cases}$$

$$= \begin{cases} F_{n+3} + 2F_{n+1}, & n \equiv 1 \pmod 4 \\ F_{n+2} + F_{n+4}, & n \equiv 0 \pmod 4 \\ F_{n+3}, & \text{otherwise} \end{cases}$$

by the identity $F_{n+1} + F_{n-1} = L_n$. □

This study continues according to the particular values of the numbers $G_{F_{(n)}}^{(2)}(4) = F_{n+3}F_{n-3}$ and $H_{F_{(n)}}^{(2)}(9) = F_{n+4}F_{n-4}$ provided in Table 4.

Table 4. Altered Lucas numbers $G_{F(n)}^{(2)}(4)$ and $H_{F_{(n)}}^{(2)}(9)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G_{F(n)}^{(2)}(4)$	4	-3	5	0	13	21	68	165	445	1152	3029	7917	20740
$H_{F_{(n)}}^{(2)}(9)$	-9	10	-8	13	0	34	55	178	432	1165	3016	7930	20727

By utilizing properties divisibility and GCD of Fibonacci numbers, GCD sequences $G_{F_{(n),r}}^{(2)}(4)$, $r \in \{1,2,3\}$, of the sequences $G_{F(n)}^{(2)}(4)$ presented in Table 4 are observed periodic.

$$G_{F_{(n),1}}^{(2)}(4) = (F_{n+3}F_{n-3}, F_{n+4}F_{n-2}) = \begin{cases} F_5F_7, & n \equiv 17 \pmod{3\,5} \\ F_7, & n \equiv 3,10,24,31 \pmod{3\,5} \\ F_5, & n \equiv 2,7,12,22,27,32 \pmod{3\,5} \\ 1, & \text{otherwise} \end{cases}$$

$$G_{F_{(n),2}}^{(2)}(4) = (F_{n+3}F_{n-3}, F_{n+5}F_{n-1}) = \begin{cases} F_8, & n \equiv 3 \pmod 8 \\ F_4, & n \equiv 1,5,7 \pmod 8 \\ 1, & \text{otherwise} \end{cases}$$

and

$$G_{F_{(n),3}^{(2)}}(4) = (F_{n+3}F_{n-3}, F_{n+6}F_n) = \begin{cases} F_3F_9, & n \equiv 3 \pmod 8 \\ F_3^2, & n \equiv 1,5,7 \pmod 8 \\ 1, & \text{otherwise} \end{cases}$$

We haven't got a closed-form expression for the numbers $G_{F_{(n)}^{(2)}}(9) = H_{F_{(n)}^{(2)}}(-9)$ and $G_{F_{(n)}^{(2)}}(-4) = H_{F_{(n)}^{(2)}}(4)$. Thus, the properties of the GCD sequences $G_{F_{(n),r}^{(2)}}(9)$ and $H_{F_{(n),r}^{(2)}}(4)$, $r \in \{1,2,3\}$, have been investigated by using MAPLE up to $n < 100$. It is seen that all sequences are bounded and periodic sequences.

4. Conclusion

In this study, we derived two types of altered numbers of the Fibonacci numbers squared, defined as $G_{F_{(n)}^{(2)}}(a) = F_n^2 + (-1)^n a$ and $H_{F_{(n)}^{(2)}}(a) = F_n^2 - (-1)^n a$, for $a \in \mathbb{Z}$. We observed that the numbers $G_{F_{(n)}^{(2)}}(1)$ and $H_{F_{(n)}^{(2)}}(1)$ correspond to an extraordinary multiplication of the Fibonacci numbers. Furthermore, their generalizations $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ exhibit the same unique Fibonacci multiplication as follows:

$$G_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is odd}$$

and

$$H_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is even}$$

Therefore, we researched r-successive altered Fibonacci GCD sequences $\left\{G_{F_{(n),r}^{(2)}}(a)\right\}$ and $\left\{H_{F_{(n),r}^{(2)}}(a)\right\}$, where $a \in \{-1,1\}$ and $r \in \{1,2\}$. We could refer that the sequences $\left\{G_{F_{(n),2}^{(2)}}(1)\right\}$ and $\left\{H_{F_{(n),4}^{(2)}}(1)\right\}$ are Fibonacci subsequences. The other GCD sequences are periodic and bounded. In future studies, other properties of the sequences $\left\{G_{F_{(n),r}^{(2)}}(F_t^2)\right\}$ and $\left\{H_{F_{(n),r}^{(2)}}(F_t^2)\right\}$ and their r-successive GCD sequences are worth studying. Besides, matrix and graph applications may be handled.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] T. Koshy, Fibonacci and Lucas Numbers with Applications, John Wiley and Sons, New York, 2001.

[2] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (1964), https://oeis.org/, Accessed 20 Sep 2023.

[3] T. Koshy, Elementary Number Theory with Applications, 2nd Edition, Academic Press, California, 2007.

[4] U. Dudley, B. Tucker, *Greatest Common Divisors in Altered Fibonacci Sequences*, Fibonacci Quarterly 9 (1971) 89–91.

[5] S. Hernandez, F. Luca, *Common Factors of Shifted Fibonacci Numbers*, Periodica Mathematica Hungarica 47 (2003) 95–110.

[6] J. Spilker, *The GCD of the Shifted Fibonacci Sequence*, in: J. Sander, J. Steuding, R. Steuding (Eds.), From Arithmetic to Zeta-Functions: Number Theory in Memory of Wolfgang Schwarz, Springer, Cham, 2016, pp. 473–483.

[7] Chen, K.W, *Greatest Common Divisors in Shifted Fibonacci Sequences*, Journal of Integer Sequences 14 (11) (2011) 4–7.

[8] F. Koken, *The GCD Sequences of the Altered Lucas Sequences*, Annales Mathematicae Silesianae 34 (2) (2020) 222–240.

[9] N. Robbins, *Fibonacci and Lucas numbers of the Forms* $w^2 - 1$, $w^3 \pm 1$, Fibonacci Quarterly 19 (4) (1981) 369–373.

[10] J. H. E. Cohn, *Square Fibonacci Numbers*, The Fibonacci Quarterly 2 (2) (1964) 109–113.

[11] H. London, R. Finkelstein, *On Fibonacci and Lucas Numbers Which are Perfect Powers*, The Fibonacci Quarterly 7 (5) (1969) 476–481.

[12] R. Finkelstein, *On Lucas Numbers Which are One More Than a Square*, Fibonacci Quarterly 14 (1) (1973) 340–342.

[13] H. C. Williams, *On Fibonacci Numbers of the Form* $k^2 + 1$, The Fibonacci Quarterly 13 (2) (1975) 213–214.

[14] J. C. Lagarias, D. P. Weisser, *Fibonacci and Lucas Cubes*, The Fibonacci Quarterly 19 (1) (1981) 39–43.

[15] D. Marques, *The Fibonacci Version of the Brocard–Ramanujan Diophantine Equation*, Portugaliae Mathematica 68 (2) (2011) 185–189.

[16] L. Szalay, *Diophantine Equations with Binary Recurrences Associated to the Brocard–Ramanujan Problem*, Portugaliae Mathematica 69 (3) (2012) 213–220.

[17] P. Pongsriiam, *Fibonacci and Lucas Numbers Associated with Brocard-Ramanujan Equation*, Communications of the Korean Mathematical Society 32 (3) (2017) 511–522.

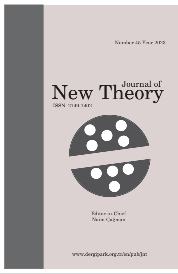
[18] Z. Cerin, *On Factors of Sums of Consecutive Fibonacci and Lucas Numbers*, Annales Mathematicae et Informaticae 41 (2013) 19–25.

[19] A. Tekcan, A. Ozkoc, B. Gezer, O. Bizim, *Some Relations Involving the Sums of Fibonacci Numbers*, Proceedings of the Jangjeon Mathematical Society 11 (1) (2008) 1–12.

[20] F. Koken, E. Kankal, *Altered Numbers of Lucas Number Squared*, Journal of Scientific Reports A 54 (2023) 62–75.

Journal of
New Theory
ISSN: 2149-1402

45 (2023) 83-94
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Screen Semi-Invariant Lightlike Hypersurfaces on Hermite-Like Manifolds

Ömer Aksu[1] , **Mehmet Gülbahar**[2]

Article Info
Received: 26 Oct 2023
Accepted: 25 Dec 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1381549
Research Article

Abstract — Hermite-like manifolds, which admit two different, almost complex structures, can be considered a general concept of Hermitian manifolds. Factoring in the effects of these two complex structures on the radical, screen, and transversal spaces, a new classification of lightlike hypersurfaces of Hermite-like manifolds is proposed in the present paper. Moreover, an example of screen semi-invariant lightlike hypersurfaces of Hermite-like manifolds is provided. Besides, some results on these hypersurfaces admitting a statistical structure are obtained. Further, screen semi-invariant lightlike hypersurfaces are investigated on Kaehler-like statistical manifolds. In addition, several characteristics of totally geodesic, mixed geodesic, and totally umbilical screen lightlike hypersurfaces are obtained. Finally, the need for further research is discussed.

Keywords *Hermite-like manifolds, complex structures, lightlike hypersurfaces*

Mathematics Subject Classification (2020) 53C40, 53C50

1. Introduction

Firstly, the concept of Hermite-like manifolds was given by Takano [1, 2]. A different feature of these manifolds that differs from Hermitian manifolds is that even the simplest examples are not found in Euclidean spaces but are found in non-Euclidean spaces. A pseudo-Riemannian manifold $(\widetilde{H}, \widetilde{h})$ with two different almost complex structures J and $J^\star$ providing

$$\widetilde{h}(JZ_1, Z_2) = -\widetilde{h}(Z_1, J^\star Z_2) \tag{1}$$

for any $Z_1, Z_2, \in \Gamma(T\widetilde{H})$ is entitled a Hermite-like manifold. For any Hermite-like manifold, we possess

$$\widetilde{h}(JZ_1, J^\star Z_2) = \widetilde{h}(Z_1, Z_2) \tag{2}$$

If we indite $J = J^\star$ in Equations 1 and 2, then a Hermite-like manifold becomes an almost Hermitian manifold.

Various authors have investigated non-degenerate submanifolds of Hermite-like manifolds [3–5]. Moreover, the authors have researched Riemannian submersions admitting Hermite-like manifolds [6–11]. However, no studies on degenerate submanifolds of Hermite-like manifolds have been published thus far.

In addition to the above facts, lightlike geometry has interesting results thanks to the different ge-

[1]omeraksu@harran.edu.tr; [2]mehmetgulbahar@harran.edu.tr (Corresponding Author)
[1,2]Department of Mathematics, Faculty of Arts and Sciences, Harran University, Şanlıurfa, Türkiye

ometric properties of radical, screen, and transversal distributions. Considering the effects of J and $J^\star$ on the radical, screen, and transversal spaces, new classifications of lightlike hypersurfaces can be identified. With this perspective, we familiarize the impression of screen semi-invariant lightlike hypersurfaces of Hermite-like manifolds and Hermite-like statistical manifolds in this paper.

Section 2 of the handled study presents some basic notions to be used in the following sections. Section 3 provides lightlike hypersurfaces of Hermite-like manifolds. Section 4 analyzes screen semi-invariant lightlike hypersurfaces of Kaehler-like statistical manifolds $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$.

2. Preliminaries

This section provides some basic properties to be needed in the following sections. For any lightlike hypersurface (H, h) of a pseudo-Riemannian manifold, we invite the radical space at each point $p \in H$ by

$$Rad\ T_pH = \{\xi \in T_pH : h_p(\xi, Z) = 0, \forall Z \in T_pH\}$$

Here, h is the induced degenerate metric from $\widetilde{h}$. The complementary non-degenerate vector bundle of $Rad\ T_pH$ is indicated by $S(TH)$ and we indite

$$TH = Rad\ TH \oplus_{orth} S(TH)$$

There exists a lightlike transversal bundle $ltr\ TH = span\{N\}$ such that we possess

$$\widetilde{h}(Z, N) = \widetilde{h}(N, N) = 0,\ \widetilde{h}(\xi, N) = 1 \tag{3}$$

for any $Z \in \Gamma(S(TH))$. Therefore, the tangent bundle $T\widetilde{H}$ of $\widetilde{H}$ is decomposed as follows:

$$T\widetilde{H} = TH \oplus ltr\ TH = \left\{TH^\perp \oplus ltr(TH)\right\} \oplus_{orth} S(TH) \tag{4}$$

where $\oplus$ indicates the direct sum, not orthogonal. Let $\widetilde{D}^0$ be the Riemannian connection of $(\widetilde{H}, \widetilde{h})$. The Gauss and Weingarten formulas for (H, h) are represented by

$$\begin{aligned} \widetilde{D}^0_{Z_1} Z_2 &= D^0_{Z_1} Z_2 + B^0(Z_1, Z_2)N \\ \widetilde{D}^0_{Z_1} N &= -A^0_N Z_1 + \tau^0(Z_1)N \end{aligned} \tag{5}$$

for any $Z_1, Z_2 \in \Gamma(TH)$. Here, D^0 is the induced connection, B^0 is the second fundamental form, A^0_N is the shape operator, and τ^0 is a $1-$form on $\Gamma(TH)$. We note that D^0 is not a Riemannian connection [12].

If $B^0 = 0$, then a lightlike hypersurface $(H, h, S(TH))$ is called totally geodesic. If there exists a function λ on H satisfying

$$B^0(Z_1, Z_2) = \lambda h(Z_1, Z_2)$$

then $(H, h, S(TH))$ is entitled totally umbilical [13].

Let $(\widetilde{H}, \widetilde{h}, \widetilde{D})$ be a statistical manifold. Then,

$$Z_3\widetilde{h}(Z_1, Z_2) = \widetilde{h}(\widetilde{D}_{Z_3} Z_1, Z_2) + \widetilde{h}(Z_1, \widetilde{D}^\star_{Z_3} Z_2) \tag{6}$$

and

$$\widetilde{D}^0_{Z_1} Z_2 = \frac{1}{2}(\widetilde{D}_{Z_1} Z_2 + \widetilde{D}^\star_{Z_1} Z_2) \tag{7}$$

The connection $\widetilde{D}^\star$ is entitled the dual of $\widetilde{D}$ [14]. Indicate the Riemannian curvature tensors with regard to connections $\widetilde{D}$ and $\widetilde{D}^{\star}$ by $\widetilde{R}$ and $\widetilde{R}^{\star}$, respectively. In this regard,

$$\widetilde{h}(\widetilde{R}^{\star}(Z_1, Z_2)Z_3, Z_4) = -\widetilde{h}(Z_3, \widetilde{R}(Z_1, Z_2)Z_4) \tag{8}$$

for any $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T\widetilde{H})$ [15]. Equation 8 implies that $\widetilde{R}$ and $\widetilde{R}^{\star}$ are not symmetric.

Let $(H, h, S(TH))$ be a lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, \widetilde{D})$. The Gauss and Weingarten type formulas with regard to $(\widetilde{D}, \widetilde{D}^{\star})$ are formulated by

$$\widetilde{D}_{Z_1} Z_2 = D_{Z_1} Z_2 + B(Z_1, Z_2)N \tag{9}$$

$$\widetilde{D}_{Z_1} N = -A^{\star}_N Z_1 + \tau^{\star}(Z_1)N \tag{10}$$

and

$$\widetilde{D}^{\star}_{Z_1} Z_2 = D^{\star}_{Z_1} Z_2 + B^{\star}(Z_1, Z_2)N \tag{11}$$

$$\widetilde{D}^{\star}_{Z_1} N = -A_N Z_1 + \tau(Z_1)N \tag{12}$$

where $D_{Z_1} Z_2$, $D^{\star}_{Z_1} Z_2$, $A_N Z_1$, and $A^{\star}_N Z_1$ are included in $\Gamma(TH)$ and D and $D^{\star}$ are the induced connections on H.

Suppose that P is the projection mapping from $\Gamma(TH)$ onto $\Gamma(S(TH))$. In this regard,

$$D_{Z_1} PZ_2 = \widetilde{D}_{Z_1} PZ_2 + C(Z_1, PZ_2)\xi \tag{13}$$

and

$$D_{Z_1}\xi = -\widetilde{A}_{\xi} Z_1 - \tau(Z_1)\xi \tag{14}$$

where $\widetilde{D}_{Z_1} PZ_2$ and $\widetilde{A}_{\xi} Z_1$ are included in $\Gamma(S(TH))$. Then,

$$B(Z_1, Z_2) = \widetilde{h}(\widetilde{D}_{Z_1} Z_2, \xi), \quad \tau^{\star}(Z_1) = \widetilde{h}(\widetilde{D}_{Z_1} N, \xi) \tag{15}$$

and

$$B^{\star}(Z_1, Z_2) = \widetilde{h}(\widetilde{D}^{\star}_{Z_1} Z_2, \xi), \quad \tau(Z_1) = \widetilde{h}(\widetilde{D}^{\star}_{Z_1} N, \xi) \tag{16}$$

Similarly, in view of Equations 13 and 14, we indite

$$D^{\star}_{Z_1} PZ_2 \;\; = \;\; \widetilde{D}^{\star}_{Z_1} PZ_2 + C^{\star}(Z_1, PZ_2)\xi \tag{17}$$

and

$$D^{\star}_{Z_1}\xi \;\; = \;\; -\widetilde{A}^{\star}_{\xi} Z_1 - \tau^{\star}(Z_1)\xi \tag{18}$$

where $\widetilde{D}^{\star}_{Z_1} PZ_2$ and $\widetilde{A}^{\star}_{\xi} Z_1$ are included in $\Gamma(S(TH))$ [16]. Using Equations 11-18, the following relations are provided:

$$B(Z_1, Z_2) = h(\widetilde{A}^{\star}_{\xi} Z_1, Z_2) + B^{\star}(Z_1, \xi)\widetilde{h}(Z_2, N) \tag{19}$$

and

$$B^{\star}(Z_1, Z_2) = h(\widetilde{A}_{\xi} Z_1, Z_2) + B(Z_1, \xi)\widetilde{h}(Z_2, N) \tag{20}$$

In view of Equations 19 and 20,

$$B(Z_1, \xi) + B^{\star}(Z_1, \xi) = 0, \quad h(A_N Z_1 + A^{\star}_N Z_1, Z_2) = 0, \quad \text{and} \quad C(Z_1, PZ_2) = h(A_N Z_1, PZ_2) \tag{21}$$

As a result of Equation 21, we obtain that B and $B^{\star}$ do not vanish on the radical space [17, 18].

A lightlike hypersurface of a statistical manifold is entitled *i.* totally geodesic with regard to $\widetilde{D}$ if $B = 0$,

ii. totally geodesic with regard to $\widetilde{D}^\star$ if $B^\star = 0$,

iii. totally tangential umbilical about $\widetilde{D}$ if there exists a smooth function k such that $B(Z_1, Z_2) = kh(Z_1, Z_2)$,

and

iv. totally tangential umbilical with respect to $\widetilde{D}^\star$ if there exists a smooth function $k^\star$ such that $B^\star(Z_1, Z_2) = k^\star h(Z_1, Z_2)$ [17].

3. Lightlike Hypersurfaces of Hermite-like Manifolds

This section presents lightlike hypersurfaces of Hermite-like manifolds.

Definition 3.1. [2] A Hermite-like manifold is called a Hermite-like statistical manifold if there is a linear connection $\widetilde{D}$ providing Equations 6 and 7. A Hermite-like statistical manifold is specified by $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$.

Definition 3.2. [2] A Hermite-like statistical manifold $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$ is entitled a Kaehler-like statistical manifold if $\widetilde{D}J = 0$. For each Kaehler-like statistical manifold $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$, $\widetilde{D}^\star J^\star = 0$.

We define semi-invariant lightlike hypersurfaces inspiring [19–24] as follows:

Definition 3.3. A lightlike hypersurface $(H, h, S(TH))$ is called screen semi-invariant if $J(Rad\, TH)$ and $J(ltr\ TH)$ are included in $S(TH)$.

In view of Equation 1, if $(H, h, S(TH))$ is a screen semi-invariant lightlike hypersurface, then $J^\star(Rad\, TH)$ and $J^\star(ltr\ TH)$ are included in $S(TH)$.

Example 3.4. Let $(\widetilde{H}, \widetilde{h})$ be a 6−dimensional pseudo-Riemannian manifold with a pseudo-Riemannian metric $\widetilde{h}$ provided by

$$\widetilde{h} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Define almost complex structures

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$J^\star = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Then, Equation 1 is satisfied. Therefore, $(\widetilde{H}, \widetilde{h}, J)$ is an example of Hermite-like manifold.

Let $\{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6\}$ be the standard frame field on $\Gamma(TH)$). Denote $\widetilde{\nabla}_{\partial_i}\partial_j = \sum_{k=1}^{8} \Gamma_{ij}^k \partial_k$ and $\widetilde{\nabla}^\star_{\partial_i}\partial_j = \sum_{k=1}^{8} \Gamma_{ij}^{\star k} \partial_k$, for $i, j \in \{1, \dots, 8\}$. From Equation 6, $\Gamma_{ij}^k \, \widetilde{h}(\partial_k, \partial_k) + \Gamma_{ik}^{\star j} \, \widetilde{h}(\partial_j, \partial_j) = 0$. Considering this fact, $\widetilde{\nabla}_{\partial_1}\partial_1 = \partial_1$, $\widetilde{\nabla}_{\partial_1}\partial_2 = \partial_2$, $\widetilde{\nabla}^\star_{\partial_1}\partial_1 = -\partial_1$, and $\widetilde{\nabla}^\star_{\partial_1}\partial_2 = -\partial_2$ and the other terms of $\widetilde{\nabla}$ and $\widetilde{\nabla}^\star$ vanish. Then, $(\widetilde{H}, \widetilde{h}, J)$ becomes a Kaehler-like statistical manifold.

Regard as a hypersurface of $(\widetilde{H}, \widetilde{h}, J)$ described by

$$H = \left\{ (z_i)_{i \in \{1,2,3,4,5,6\}} : z_1 = z_3 \right\}$$

In this case, the induced metric h becomes

$$h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By a straightforward computation,

$$Rad\ TH = \operatorname{span}\{\xi = \partial_1 + \partial_3\}$$

$$ltr\ TH = \operatorname{span}\left\{ N = -\frac{1}{2}(\partial_1 + \partial_3) \right\}$$

and

$$S(TH) = \operatorname{span}\{e_1 = \partial_1, e_2 = \partial_4, e_3 = \partial_5, e_4 = \partial_6\}$$

Therefore,

$$J\xi = e_2 + e_4, \quad JN = \frac{1}{2}(e_2 + e_4), \quad J^\star\xi = -(e_2 + e_4), \quad \text{and} \quad J^\star N = \frac{1}{2}(e_2 - e_4)$$

which indicate that $(H, h, S(TH))$ is screen semi-invariant.

Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J)$. In this regard,

$$JN = \alpha, \quad J^\star N = \alpha^\star, \quad J\xi = \beta, \quad \text{and} \quad J^\star\xi = \beta^\star \tag{22}$$

where α, $\alpha^\star$, β, and $\beta^\star$ are included in $\Gamma(S(TH))$. For each $Z \in \Gamma(TH)$,

$$JZ = \psi Z + w^\star(Z)\xi + \eta^\star(Z)N \tag{23}$$

and

$$J^\star Z = \psi^\star Z + w(Z)\xi + \eta(Z)N \tag{24}$$

where ψ and $\psi^\star$ are projections from $\Gamma(TH)$ onto $\Gamma(S(TH))$ and w, $w^\star$, η, and $\eta^\star$ are $1-$forms described by

$$w(Z) = h(Z, \alpha), \quad w^\star(Z) = h(Z, \alpha^\star)$$

and

$$\eta(Z) = h(Z, \beta), \quad \eta^\star(Z) = h(Z, \beta^\star)$$

for all $Z \in \Gamma(TH)$.

Proposition 3.5. Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J)$.

Then, the following equations hold:

$$\eta^\star(\psi Z) = 0 \;\text{ and }\; \eta(\psi^\star Z) = 0$$

for all $Z \in \Gamma(TH)$. In particular,

$$w^\star(\psi Z) = 0 \;\text{ and }\; w(\psi^\star Z) = 0$$

for all $Z \in \Gamma(S(TH))$.

PROOF.
Using Equations 22-24,

$$-Z = J^2 Z = J(\psi Z) + w^\star(Z)J\xi + \eta^\star(Z)JN$$

and

$$-Z = \psi^2 Z + w^\star(\psi Z)\xi + \eta^\star(\psi Z)N + w^\star(Z)\beta + \eta^\star(Z)\alpha \tag{25}$$

Investigating the tangential and transversal sides of Equation 25, $\eta^\star(\psi Z) = 0$.

If Z is included in $\Gamma(S(TH))$, then $w^\star(\psi Z) = 0$. Applying $(J^\star)^2 = -\mathbf{I}_{n+2}$ and a similar technique as in the proof of Equation 25,

$$-Z = (\psi^\star)^2 Z + w(\psi^\star Z)\xi + \eta(\psi^\star Z)N + w(Z)\beta^\star + \eta(Z)\alpha^\star \tag{26}$$

which indicates $\eta(\psi^\star Z) = 0$, for all $Z \in \Gamma(TH)$. If $Z \in \Gamma(S(TH))$, then $w(\varphi^\star Z) = 0$ from Equation 26. □

Using Equations 25 and 26, the following results are obtained.

Proposition 3.6. For any screen semi-invariant lightlike hypersurface $(M, g, S(TH))$ of $(\widetilde{H}, \widetilde{h}, J)$, the following relations occur, for all $Z \in \Gamma(TH)$,

$$\psi^2 Z = -PZ - w^\star(Z)\beta - \eta^\star(Z)\alpha$$

$$(\psi^\star)^2 Z = -PZ - w(Z)\beta^\star - \eta(Z)\alpha^\star$$

and

$$w^\star(\psi Z) = w(\psi^\star Z)$$

Proposition 3.7. For any screen semi-invariant lightlike hypersurface $(M, g, S(TH))$, the following relations occur, for all $Z_1, Z_2 \in \Gamma(TH)$,

$$\widetilde{h}(\psi Z_1, Z_2) + \eta^\star(Z_2)\widetilde{h}(Z_2, N) = \widetilde{h}(Z_1, \psi^\star Z_2) + \eta(Z_2)\widetilde{h}(Z_1, N)$$

and

$$\widetilde{h}(\psi Z_1, \psi^\star Z_2) = -\widetilde{h}(Z_1, Z_2) - w^\star(Z_1)\eta(Z_2) - \eta^\star(Z_1)w(Z_2)$$

In particular, the relation

$$h(\psi Z_1, Z_2) = h(Z_1, \psi^\star Z_2)$$

is valid, for all $Z_1, Z_2 \in \Gamma(S(TH))$.

The proof is obvious by utilizing Equations 23 and 24 in Equation 1.

4. Screen Semi-Invariant Lightlike Hypersurface of Kaehler-like Statistical Manifolds

This section analyzes screen semi-invariant lightlike hypersurfaces of Kaehler-like statistical manifolds $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$.

Proposition 4.1. Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$. Then, the following relations occur, for all $Z \in \Gamma(TH)$,

$$\widetilde{D}_Z\alpha - \eta^\star(D_Z\alpha)\alpha = \psi A_N Z \tag{27}$$

and

$$\eta^\star(D_Z\alpha) = -\tau^\star(Z) \tag{28}$$

Proof.
Considering $(\widetilde{H}, \widetilde{h}, J)$ is a Kaehler-like statistical manifold,

$$-\widetilde{D}_Z N = \widetilde{D}_Z J\alpha = J\widetilde{D}_Z\alpha \tag{29}$$

Using Equations 10 and 29,

$$J\widetilde{D}_Z\alpha = A_N Z - \tau^\star(Z)N \tag{30}$$

From Equation 9 in Equation 30,

$$\psi D_Z\alpha + w^\star(D_Z\alpha)\xi + \eta^\star(D_Z\alpha)N + B(Z,\alpha)\alpha = A_N Z - \tau^\star(Z)N \tag{31}$$

Because $\psi\alpha = 0$ and investigating the tangential and transversal sides of Equation 31, Equation 28 is obtained and

$$\psi D_Z\alpha + w^\star(D_Z\alpha)\xi + B(Z,\alpha)\alpha = A_N Z \tag{32}$$

From Equation 32,

$$\psi^2 D_Z\alpha + w^\star(D_Z\alpha)\psi\xi = \psi A_N Z \tag{33}$$

Using Equations 22, 25, and 33, Equation 27 is obtained. □

Definition 4.2. Let $(\widetilde{H}, \widetilde{h})$ be pseudo-Riemannian manifold and $\widetilde{D}^{\widetilde{H}}$ indicate a linear connection on $(\widetilde{H}, \widetilde{h})$. A vector field v on $\widetilde{H}$ is entitled torse-forming with regard to $\widetilde{D}^{\widetilde{H}}$ if the following circumstance is provided, for each $Z \in \Gamma(\widetilde{H})$,

$$\widetilde{D}^{\widetilde{H}}_Z v = \gamma Z + \varphi(Z)v$$

where φ is a linear form and γ is a function [25]. A torse-forming vector field is entitled

i. torqued if $\varphi(v) = 0$,

ii. concircular if $\varphi = 0$,

iii. concurrent if $\gamma = 1$ and $\varphi = 0$,

and

iv. reccurrent if $\gamma = 0$.

In view of Proposition 4.1 and Definition 4.2, the following are obtained.

Corollary 4.3. If $(H, h, S(TH))$ is totally geodesic with regard to $\widetilde{D}$, then there is no less than one vector field lying on $\Gamma(S(TH))$, reccurrent with regard to D.

Corollary 4.4. If α is a torse-forming vector field with regard to D, then $(H, h, S(TH))$ can not be totally geodesic with regard to $\widetilde{D}$.

PROOF.
Assume that α is torse-forming with regard to D. If we indite Equation 5 in Equation 27, then

$$PA_N Z = \alpha\psi Z \tag{34}$$

for each vector field Z, orthogonal to α and β. From Equation 34, if H is totally geodesic with regard to $\widetilde{D}$, then $\psi Z = 0$. This contradicts the fact that $(H, h, S(TH))$ is screen semi-invariant. □

Corollary 4.5. If α is parallel with regard to D (or $\widetilde{D}$), then the shape operator takes the following format:

$$A_N Z = w^\star(A_N Z)\beta + \eta^\star(A_N Z)\alpha$$

Corollary 4.6. There does not exist any totally umbilical semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$ admitting a parallel vector field $JN = \alpha$ with regard to D (or $\widetilde{D}$).

Contemplate the following distributions:

$$\mathbb{D}_1 = span\{\beta, \beta^\star\} \text{ and } \mathbb{D}_2 = span\{\alpha, \alpha^\star\}$$

Hence, there exists a $(n-4)-$dimensional pseudo-Riemannian distribution $\mathbb{D}$ in $S(TH)$ such that

$$S(TH) = \mathbb{D} \oplus_{orth} \{\mathbb{D}_1 \oplus \mathbb{D}_2\}$$

Therefore, from Equations 3 and 4,

$$TH = \mathbb{D} \oplus_{orth} \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \oplus_{orth} Rad(TH)$$

and

$$T\widetilde{H} = \mathbb{D} \oplus_{orth} \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \oplus_{orth} \{Rad\ TH \oplus ltr\ TH\}$$

From the above verities, $\mathbb{D}$ is invariant with respect to J and $J^\star$. Suppose that

$$\widetilde{\mathbb{D}} = \mathbb{D} \oplus_{orth} Rad\ TH \oplus_{orth} J(Rad\ TH) \oplus_{orth} J^\star(Rad\ TH)$$

Hence, $\widetilde{\mathbb{D}}$ is invariant with respect to J and $J^\star$.

Theorem 4.7. Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$. Then, the following assertions are equivalent:

i) $\widetilde{\mathbb{D}}$ is integrable with regard to D.

ii) The equality

$$B(Z_1, tZ_2) = B(Z_2, tZ_1)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$.

iii) The equality

$$h(\widetilde{A}^\star_\xi Z_1, tZ_2) - h(\widetilde{A}^\star_\xi Z_2, tZ_1) = B(Z_1, \xi)\widetilde{h}(tZ_2, N) - B(Z_2, \xi)\widetilde{h}(tZ_1, N)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$, where $tZ_1 = \psi Z_1 + w^\star(Z_1)\xi$.

PROOF.
Since $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$ is a Kaehler-like statistical manifold, it is obvious that

$$\widetilde{D}_{Z_1} JZ_2 = J\widetilde{D}_{Z_1} Z_2 \tag{35}$$

for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$. If Z_2 is perpendicular to α and $\alpha^\star$, then

$$\widetilde{D}_{Z_1} J Z_2 = \widetilde{D}_{Z_1}(\psi Z_2 + w^\star(Z_2)\xi) \tag{36}$$

From Equations 9 and 36,

$$\widetilde{D}_{Z_1} J Z_2 = \widetilde{D}_{Z_1}\psi Z_2 + B(Z_1, \psi Z_2)N + Z_1\left[w^\star(Z_2)\right]\xi + w^\star(Z_2)D_{Z_1}\xi + w^\star(Z_2)B(Z_1,\xi)N \tag{37}$$

Combining Equations 9 and 23,

$$J\widetilde{D}_{Z_1}Z_2 = \psi D_{Z_1}Z_2 + w^\star(D_{Z_1}Z_2)\xi + \eta^\star(D_{Z_1}Z_2)N + B(Z_1,Z_2)\alpha \tag{38}$$

Taking into account of Equations 35, 37, and 38,

$$\begin{aligned}\widetilde{D}_{Z_1}\psi Z_2 + B(Z_1,\psi Z_2)N + Z_1\left[w^\star(Z_2)\right]\xi + w^\star(Z_2)D_{Z_1}\xi + w^\star(Z_2)B(Z_1,\xi)N = \psi D_{Z_1}Z_2 + w^\star(D_{Z_1}Z_2)\xi \\ +\eta^\star(D_{Z_1}Z_2)N + B(Z_1,Z_2)\alpha\end{aligned} \tag{39}$$

Altering the position of Z_1 and Z_2 in Equation 39,

$$\begin{aligned}\widetilde{D}_{Z_2}\psi Z_1 + B(Z_2,\psi Z_1)N + Y\left[w^\star(Z_1)\right]\xi + w^\star(Z_1)D_{Z_2}\xi + w^\star(Z_1)B(Z_2,\xi)N = \psi D_{Z_2}Z_1 + w^\star(D_{Z_2}Z_1)\xi \\ +\eta^\star(D_{Z_2}Z_1)N + B(Z_2,Z_1)\alpha\end{aligned} \tag{40}$$

If we subtract Equations 39 and 40 side to side,

$$\eta^\star(D_{Z_1}Z_2) - \eta^\star(D_{Z_2}Z_1) = B(Z_1,\psi Z_2) - B(Z_2,\psi Z_1) + w^\star(Z_2)B(Z_1,\xi) - w^\star(Z_1)B(Z_2,\xi)$$

which shows

$$B(Z_1, tZ_2) - B(Z_2, tZ_1) = \eta^\star([Z_1, Z_2]) \tag{41}$$

Taking into consideration of Equation 41, $B(Z_1, tZ_2) = B(Z_2, tZ_1)$ is provided for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$ if and only if $[Z_1, Z_2] \in \Gamma(\widetilde{\mathbb{D}})$. Hence, $(i) \Leftrightarrow (ii)$. From Equations 19 and 41, $(ii) \Leftrightarrow (iii)$.

□

From Theorem 4.7, the following results are obtained.

Corollary 4.8. If $(H, h, S(TH))$ is totally geodesic with regard to D, then $\widetilde{\mathbb{D}}$ is integrable with regard to D.

Corollary 4.9. If $(H, h, S(TH))$ is totally umbilical with regard to D, then $\widetilde{\mathbb{D}}$ is not integrable with regard to D.

An analogous to Theorem 4.7 is as follows:

Theorem 4.10. For any screen semi-invariant lightlike hypersurface $(H, h, S(TH))$, the following assertions are equivalent:

$i)$ $\widetilde{\mathbb{D}}$ is integrable with regard to $D^\star$.

$ii)$ The equality

$$B^\star(Z_1, t^\star Z_2) = B^\star(Z_2, t^\star Z_1)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$.

$iii)$ The equality

$$h(\widetilde{A}_\xi Z_1, t^\star Z_2) - h(A^\star_\xi Z_2, t^\star Z_1) = B^\star(Z_1,\xi)\widetilde{h}(t^\star Z_2, N) - B^\star(Z_2,\xi)\widetilde{h}(t^\star Z_1, N)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$, where $t^\star Z_1 = \psi^\star Z_1 + w(Z_1)\xi$.

Theorem 4.11. $(H, h, S(TH))$ is mixed geodesic with regard to $\widetilde{D}$ if and only if $A^\star_N Z$ is included in $\mathbb{D}_1^\perp$, for all $Z \in \Gamma(\widetilde{\mathbb{D}})$.

Proof.
Assume that $(H, h, S(TH))$ is mixed geodesic with regard to $\widetilde{D}$. In view of Equations 9 and 10,

$$\widetilde{D}_Z\alpha = D_Z\alpha + B(Z,\alpha)N \tag{42}$$

and

$$J\widetilde{D}_Z N = -tA_N^{\star}Z - \eta^{\star}(A_N^{\star}Z)N + \tau^{\star}(Z)\alpha \tag{43}$$

for all $Z \in \Gamma(\widetilde{\mathbb{D}})$. Since $(\widetilde{H}, \widetilde{h}, J)$ is a Kaehler-like statistical manifold, we derive the following relation using Equations 42 and 43:

$$0 = B(Z,\alpha) = -\eta^{\star}(A_N^{\star}Z) = -\widetilde{g}(JA_N^{\star}Z, \xi)$$

Hence, $h(A_N^{\star}Z, \beta^{\star}) = 0$. With similar arguments,

$$\widetilde{D}_Z\alpha^{\star} = D_Z\alpha^{\star} + B(Z,\alpha^{\star})N \tag{44}$$

and

$$J^{\star}\widetilde{D}_Z N = -h^{\star}A_N^{\star}Z - \eta(A_N^{\star}Z)N + \tau^{\star}(Z)\alpha^{\star} \tag{45}$$

From Equations 44 and 45,

$$0 = B(Z,\alpha^{\star}) = -\eta(A_N^{\star}Z) = -\widetilde{g}(J^{\star}A_N^{\star}Z, \xi)$$

Hence, $h(A_N^{\star}Z, \beta) = 0$. Therefore, $A_N^{\star}Z$ is included in $\mathbb{D}_1^{\perp}$ for all $Z \in \Gamma(\widetilde{\mathbb{D}})$. The proof of converse is clear. □

With a similar method of Theorem 4.11, the following result is obtained.

Theorem 4.12. $(H, h, S(TH))$ is mixed geodesic with regard to $\widetilde{D}^{\star}$ if and only if $A_N Z$ is included in $\mathbb{D}_1^{\perp}$, for all $Z \in \Gamma(\widetilde{\mathbb{D}})$.

5. Conclusion

This study investigates the geometry of screen semi-invariant lightlike hypersurfaces, where almost complex structures J and includes $J^{\star}$ in the screen distribution. With this perspective, new types of lightlike hypersurfaces can be introduced. For example, the cases where almost complex structures J and $J^{\star}$ are invariant or anti-invariant in the radical space or invariant and anti-invariant on the screen space can be examined. Thus, the problem of the existence of new kinds of lightlike hypersurfaces for almost Hermite-like manifolds and Kaehler-like statistical manifolds arises in the future.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Acknowledgement

This work was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK), Project number: 122F326.

References

[1] K. Takano, *Statistical Manifolds with Almost Contact Structures and Its Statistical Submersions*, Journal of Geometry 85 (1-2) (2006) 171–187.

[2] K. Takano, *Statistical Manifolds with Almost Complex Structures*, in: H. Shimada (Ed.), Proceedings of the 45th Symposium on Finsler Geometry, Tokyo, 2011, pp. 54–57.

[3] M. Aquib, *Some Inequalities for Statistical Submanifolds of Quaternion Kaehler-like Statistical Space Forms*, International Journal of Geometric Methods in Modern Physics 16 (08) (2019) 1950129 16 pages.

[4] M. Aquib, M. H. Shahid, *Generalized Normalized δ-Casorati Curvature for Statistical Submanifolds in Quaternion Kaehler-like Statistical Space Forms*, Journal of Geometry 109 (1) (2018) Article Number 13 13 pages.

[5] H. Aytimur, M. Kon, A. Mihai, C. Özgür, K. Takano, *Chen Inequalities for Statistical Submanifolds of Kähler-like Statistical Manifolds*, Mathematics 7 (12) (2019) Article ID 1202 19 pages.

[6] H. Aytimur, C. Özgür, *On Cosymplectic-like Statistical Submersions*, Mediterranean Journal of Mathematics 16 (2019) 1–14.

[7] S. Kazan, K. Takano, *Anti-invariant Holomorphic Statistical Submersions*, Results in Mathematics 78 (4) (2023) Article Number 128 18 pages.

[8] S. Kazan, *Anti-Invariant $\xi^{\perp}$-Cosymplectic-like Statistical Submersions*, Thermal Science 26 (4) (2022) 2991–3001.

[9] A. N. Siddiqui, S. Uddin, M. H. Shahid, *B.-Y. Chen's Inequality for Kähler-like Statistical Submersions*, International Electronic Journal of Geometry 15 (2) (2022) 277–286.

[10] M. D. Siddiqi, A. N. Siddiqui, F. Mofarreh, H. Aytimur, *A Study of Kenmotsu-like Statistical Submersions*, Symmetry 14 (8) (2022) Article ID 1681 13 pages.

[11] A. D. Vilcu, G. E. Vilcu, *Statistical Manifolds with Almost Quaternionic Structures and Quaternionic Kähler-like Statistical Submersions*, Entropy 17 (9) (2015) 6213–6228.

[12] K. L. Duggal, B. Şahin, Differential Geometry of Lightlike Submanifolds, Birkhäuser, Basel, 2011.

[13] K. L. Duggal, D. H. Jin, *Totally Umbilical Lightlike Submanifolds*, Kodai Mathematical Journal 26 (2003) 49–68.

[14] S. Amari, Differential-Geometrical Methods in Statistics, Springer, New York, 1985.

[15] H. Furuhata, *Hypersurfaces in Statistical Manifolds*, Differential Geometry and its Applications 27 (3) (2009) 420–429.

[16] O. Bahadır, *On Lightlike Geometry of Indefinite Sasakian Statistical Manifolds*, AIMS Mathematics 6 (11) (2021) 12845–12862.

[17] O. Bahadır, M. M. Tripathi, *Geometry of Lightlike Hypersurfaces of a Statistical Manifold*, WSEAS Transactions on Mathematics 22 (2023) 466–474.

[18] O. Bahadır, A. N. Siddiqui, M. Gülbahar, A. H. Alkhaldi, *Main Curvatures Identities on Lightlike Hypersurfaces of Statistical Manifolds and Their Characterizations*, Mathematics 10 (13) (2022) Article ID 2290 18 pages.

[19] B. E. Acet, S. Yüksel Perktaş, E. Kılıç, *On Lightlike Geometry of Para-Sasakian Manifolds*, The Scientific World Journal 2014 (2014) Article ID 696231 12 pages.

[20] B. E. Acet, *Lightlike Hypersurfaces of Metallic Semi-Riemannian Manifolds*, International Journal of Geometric Methods in Modern Physics 15 (12) (2018) Article ID 1850201 16 pages.

[21] E. Kılıç, B. Şahin, S. Keleş, *Screen Semi Invariant Lightlike Submanifolds of Semi-Riemannian Product Manifolds*, International Electronic Journal of Geometry 4 (2) (2011) 120–135.

[22] S: Y. Perktaş, E. Kılıç, B. E. Acet, *Lightlike Hypersurfaces of a Para-Sasakian Space Form*, Gulf Journal of Mathematics 2 (2) (2014) 7–18.

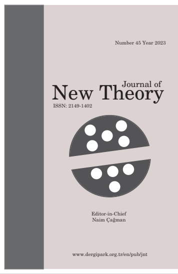
[23] N. Ö. Poyraz, E. Yaşar, *Lightlike Hypersurfaces of a Golden Semi-Riemannian Manifold*, Mediterranean Journal of Mathematics 14 (2017) 1–20.

[24] N. Ö. Poyraz, *Screen Semi-Invariant Lightlike Submanifolds of Golden Semi-Riemannian Manifolds*, International Electronic Journal of Geometry 14 (1) (2021) 207–216.

[25] K. Yano, *On the Torse-Forming Directions in Riemannian Spaces*, Proceedings of the Imperial Academy Tokyo 20 (6) (1944) 340–345.

Journal of
New Theory
ISSN: 2149-1402

45 (2023) 95-104
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Crossed Corner and Reduced Simplicial Commutative Algebras

Özgün Gürmen Alansal[1]

Article Info
Received: 15 Nov 2023
Accepted: 15 Dec 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1391397
Research Article

Abstract — In this paper, we describe the crossed corner of commutative algebras and present the relation between the category of crossed corners of commutative algebras and the category of reduced simplicial commutative algebras with Moore complex of length 2. We provide a passage from crossed corners to bisimplicial algebras. In this construction, we utilize the Artin-Mazur codiagonal functor from reduced bisimplicial algebras to simplicial algebras and the hypercrossed complex pairings in the Moore complex of a simplicial algebra. Using the coskeleton functor from the category of k-truncated simplicial algebras to the category simplicial algebras with Moore complex of length k, we see that the length of Moore complex of the reduced simplicial algebra obtained from a crossed corner is 2.

Keywords *Crossed modules, simplicial algebras, crossed corner*

Mathematics Subject Classification (2020) 18N50, 55U10

1. Introduction

Whitehead [1] introduced the concept of crossed modules of groups as an algebraic model of connected homotopy 2-types of topological spaces. As a 2-dimensional crossed module, or a crossed module of crossed modules, the notion of crossed square has been introduced by Guin-Waléry and Loday [2]. Another 2-dimensional crossed modules of groups is the quadratic module was introduced by Baues as an algebraic model for 3-types in [3]. The commutative algebra and the Lie algebra versions of quadratic modules were introduced by Arvasi and Ulualan [4] and Ulualan and Uslu [5], respectively. The quasi quadratic modules over Lie algebras has been studied in [6]. For further work about the 2-dimensional crossed modules, see [7].

Alp [8] has defined crossed corners of groups, closely associated with crossed squares, and studied relationships between them. The commutative algebra analogue of crossed modules has been studied by Porter [9]. Moreover, the commutative, associative, and Lie algebra versions of crossed squares has been defined by Ellis [10], as higher dimensional versions of crossed modules of algebras. The equivalence between simplicial algebras and these crossed structures was proven in [4, 10–12]. In this paper, our first aim is to achieve the definition of a crossed corner over commutative algebras. We investigate the close relationship between the categories of crossed corners of commutative algebras and reduced simplicial algebras with Moore complex of length 2 in terms of Peiffer pairings in the Moore complex. Throughout this paper, an algebra action of $r \in R$ on $s \in S$ will be denoted by $r \cdot s$ or $s \cdot r$. Since all algebras in this work are commutative algebras, we can write $r \cdot s = s \cdot r$. Recall from [13]

[1]ozgun.gurmen@dpu.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Arts and Sciences, Kütahya Dumlupınar University, Kütahya, Türkiye

that a crossed module of algebras is a homomorphism of R-algebras $\partial : S \to R$ with the algebra action of R on S such that the following axioms are satisfied: $CM1$. $\partial(s \cdot r) = \partial(s)r$, $\partial(r \cdot s) = r\partial(s)$, and $CM2$. $\partial(s) \cdot s' = ss' = s \cdot \partial(s')$, for all $r \in R$, and $s, s' \in S$. It is well known that a crossed module is equivalent to a simplicial algebra with Moore complex of length 1. For the connection between crossed modules of Lie algebras and simplicial Lie algebras and for the Lie-Rinehart version of this connection, see [14, 15].

A crossed corner can be regarded as a 2-dimensional crossed module. By giving the definition of a crossed corner of commutative algebras, we will prove that the category of crossed corners is equivalent to the category of reduced simplicial commutative algebras with Moore complex of length 2. In this equivalence, we will define a passage from the crossed corners to reduced bisimplicial algebras and Artin-Mazur codioganal functor from bisimplicial algebras to simplicial algebras. In this construction, we see that the length of this reduced simplicial algebras is 2.

2. Crossed Corner of Commutative Algebras

Suppose that k is a fixed commutative ring. All of the k-algebras studied in this work are assumed to be commutative and associative. We will denote the category of commutative algebras by Alg_k. In this section, we provide the commutative algebra version of a crossed corner of groups, presented by Alp [8, 16, 17].

Definition 2.1. A crossed corner of algebras is a diagram of commutative algebras

$$\begin{array}{ccc} K_1 & \xrightarrow{\partial} & K_2 \\ {\scriptstyle \partial'}\downarrow & & \\ K_3 & & \end{array}$$

together with algebra actions of K_2 on K_1 and K_3 on K_1 and homomorphisms $\partial : K_1 \to K_2$ and $\partial' : K_1 \to K_3$ of algebras with a map $h : K_2 \otimes K_3 \to K_1$ satisfying the following axioms:

CC1. ∂ and ∂' are crossed modules of algebras

CC2. $h((k_2 + k_2{}') \otimes k_3) = h(k_2 \otimes k_3) + h(k_2{}' \otimes k_3)$ and $h(k_2 \otimes (k_3 + k_3{}')) = h(k_2 \otimes k_3) + h(k_2 \otimes k_3{}')$

CC3. $h(\partial(k_1) \otimes k_3) = k_3 \cdot k_1$ and $h(k_2 \otimes \partial'(k_1)) = k_2 \cdot k_1$

CC4. $(k_2 \cdot k_3) \cdot k_1 = (k_2k_3) \cdot k_1$ and $(k_3 \cdot k_2) \cdot k_1 = (k_3k_2) \cdot k_1$

where the actions

$$k_3 \cdot k_2 = \partial' h(k_2 \otimes k_3)$$

and

$$k_2 \cdot k_3 = \partial h(k_2 \otimes k_3)$$

These two actions are commutative algebra actions, for all $k_1 \in K_1$, $k_2, k_2' \in K_2$, $k_3, k_3' \in K_3$.

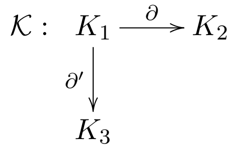
Example 2.2. Let I_1 and I_2 be two ideals of a k-algebra I. The following diagram of inclusions

$$\begin{array}{ccc} I_1 \cap I_2 & \xrightarrow{\partial} & I_1 \\ {\scriptstyle \partial'}\downarrow & & \\ I_2 & & \end{array}$$

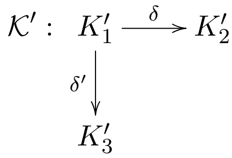
together with the actions of I_1, I_2 on $I_1 \cap I_2$ given by multiplication and the function $h : I_1 \otimes I_2 \to I_1 \cap I_2$, $h(i_1 \otimes i_2) = i_1 i_2$ is a crossed corner. It can be observed that this is a crossed corner of commutative algebras.

2.1. Morphisms of Crossed Corners

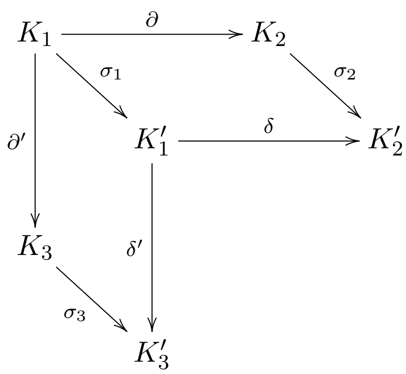
In this section, we define the morphism between two crossed corners. Let



and



be crossed corners together with maps $h : K_2 \otimes K_3 \to K_1$ and $h' : K'_2 \otimes K'_3 \to K'_1$. The morphism $\sigma = (\sigma_1, \sigma_2, \sigma_3) : \mathcal{K} \to \mathcal{K}'$ is provided by the following commutative diagram



where $\delta'\sigma_1 = \sigma_3\partial'$ and $\delta\sigma_1 = \sigma_2\partial$ and for $k_2 \in K_2, k_3 \in K_3$

$$\sigma_1 h(k_2 \otimes k_3) = h'(\sigma_2(k_2) \otimes \sigma_3(k_3))$$

Furthermore, for $k_2 \in K_2, k_1 \in K_1$,

$$\sigma_1(k_2 \cdot k_1) = \sigma_2(k_2) \cdot \sigma_1(k_1)$$

and for $k_3 \in K_3$

$$\sigma_1(k_3 \cdot k_1) = \sigma_3(k_3) \cdot \sigma_1(k_1)$$

and where $\sigma_1, \sigma_2, \sigma_3$ are k-algebra homomorphism.

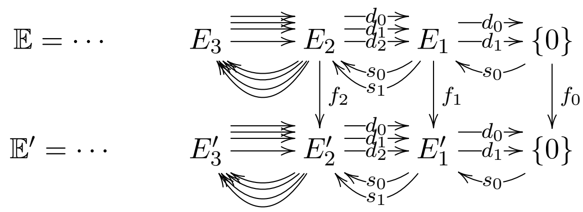
Thus, we can define the category of crossed corners of algebras denoting it as CC.

3. (Bi)simplicial Algebras

Recall from [12] that a simplicial algebra $\mathbb{E}$ consists of k-algebras E_n, for $n \in Z^+ \cup \{0\}$, together with the homomorphisms $d_i^n : E_n \to E_{n-1}$, $0 \le i \le n$, and $s_j^n : E_n \to E_{n+1}$, $0 \le j \le n$, called faces and degeneracies, respectively, satisfying the usual simplicial identities given in [4]. As an alternative description of a simplicial algebra, we can say that a simplicial algebra $\mathbb{E}$ can be regarded as a functor from the opposite category of finite ordinals $\Delta^{op}[n]$, for $n \in Z^+ \cup \{0\}$ to the category of algebras. That is, $\mathbb{E}$ is simplicial object in the category of commutative algebras. For each $k \ge 0$, it is obtained a subcategory $\Delta[n]_{\le k}$ of $\Delta[n]$ whose objects are $[j] = \{0 < 1 < \cdots < j\}$ of $\Delta[n]$ with $j \le k$. Then, for each $k \ge 0$, we can obtain a k-truncated simplicial algebra by defining the functor $\mathbb{E} : \Delta[n]_{\le k} \to Alg$. Let $\mathbb{E}$ be a simplicial algebra. Then, its Moore complex (NE, ∂) is a chain complex defined on each level by $NE_n = \bigcap\limits_{i=0}^{n-1} \mathrm{Ker} d_i^n = \mathrm{Ker} d_0^n \cap \mathrm{Ker} d_1^n \cap \ldots \cap \mathrm{Ker} d_{n-1}^n$ with the boundary morphism $\partial_n : NE_n \to NE_{n-1}$ restricted to NE_n of the morphism $d_n^n : E_n \to E_{n-1}$. Thus, we can illustrate the Moore (chain) complex by

$$(NE, \partial) : \cdots \xrightarrow{\partial_3} NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0$$

If $NE_n = \{0\}$, for $n \geq k+1$, then the Moore complex NE is of length k. We will denote the category of simplicial algebras with Moore complex of length k by $SimpAlg_{\leq k}$. If the first component E_0 of a simplicial algebra $\mathbb{E}$ is zero, that is $E_0 = \{0\}$, then $\mathbb{E}$ is called a reduced simplicial algebra. We denote the category of reduced simplicial algebras with Moore complex of length k by $ReSimpAlg_{\leq k}$. A morphism between reduced simplicial algebras in this category is given by the following diagram



in which $f : \mathbb{E} \to \mathbb{E}'$ consists of k-algebra homomorphisms $f_i : E_i \to E'_i$, $i \in \mathbb{Z}^+ \cup \{0\}$, commuting with all the face and degeneracy operators.

Arvasi and Porter [12] have defined the functions $C_{\alpha,\beta}$ in the Moore complex of a simplicial algebra $\mathbb{E}$. We recall these functions to use them in the connection between reduced simplicial algebras and crossed corners. We only use these functions in dimension 3. These functions are

$$C_{(0),(2,1)}(x,z) = (s_2 s_1(x))(-s_0(z) + s_1(z) + s_2(z))$$

and

$$C_{(2,0),(1)}(x,z) = (s_2 s_0(x) - s_2 s_1(x))(s_1(z) - s_2(z))$$

For the images of these functions under the boundary map ∂_3, see [12].

Now consider the product category $\Delta[n] \times \Delta[n]$ whose objects are the pairs $([p],[q])$ and whose morphisms between objects are the pairs of non-decreasing maps. Then, the functor $\mathbb{E}_{.,.}$ from $(\Delta \times \Delta)^{op}$ to Alg can be regarded as a bisimplicial algebra. Thus, we can give the definition of a bisimplicial algebra equivalently as follows. For each object (p,q) of $(\Delta \times \Delta)^{op}$, there is an k-algebra $E_{p,q}$ and for each morphism between the pairs (p,q), there are homomorphisms of algebras

$$\begin{array}{lll} d_i^h : E_{p,q} \to E_{p-1,q}; & s_i^h : E_{p,q} \to E_{p+1,q}, & p \geq i \geq 0 \\ d_j^v : E_{p,q} \to E_{p,q-1}; & s_j^v : E_{p,q} \to E_{p,q+1}, & q \geq j \geq 0 \end{array}$$

such that morphisms d_j^v, s_j^v commute with d_i^h, s_i^h. Furthermore, these morphisms satisfy the usual simplicial identities. The Moore bicomplex of a bisimplicial algebra $\mathbb{E}_{.,.}$ is given by

$$NE_{n,m} = \bigcap_{(i,j)=(0,0)}^{(n-1,m-1)} \mathrm{Ker} d_i^h \cap \mathrm{Ker} d_j^v$$

with the boundary homomorphisms $\partial_i^h : NE_{n,m} \to NE_{n-1,m}$ and $\partial_j^v : NE_{n,m} \to NE_{n,m-1}$ obtained by the restriction to d_i^h and d_j^v, respectively. Thus, we can show pictorially a Moore bicomplex of a bisimplicial algebra by the following diagram

$$\begin{array}{ccccc} & & \downarrow & & \downarrow \\ \cdots & \overset{\vdots}{\longrightarrow} & NE_{1,2} & \overset{\vdots}{\longrightarrow} & NE_{0,2} \\ & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & NE_{1,1} & \longrightarrow & NE_{0,1} \\ & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & NE_{1,0} & \longrightarrow & NE_{0,0} \end{array}$$

If $E_{0,0}$ is a zero in a bisimplicial algebra $\mathbb{E}_{.,.}$, then it is called a reduced bisimplicial algebra. If $NE_{p,q} = \{0\}$, for $p+q \geq k+1$, then the Moore bicomplex is of length k. For 2-dimensional version of $C_{\alpha,\beta}$ functions for bisimplicial algebras, see [18].

4. From Reduced Simplicial Algebras to Crossed Corners

In this section, we investigate the relation between the categories of crossed corners and reduced simplicial algebras. Suppose that $\mathbb{E}$ is a reduced simplicial algebra with $E_0 = \{0\}$. We will construct a crossed corner of commutative algebras as

$$\begin{array}{ccc} K_1 & \xrightarrow{\partial} & K_2 \\ {\scriptstyle \partial'}\downarrow & & \\ K_3 & & \end{array}$$

with the h-map $h : K_2 \otimes K_3 \to K_1$.

Suppose $K_2 = NE_1 = \mathrm{Ker} d_0^1$ and $K_3 = NE_1^* = \mathrm{Ker} d_1^1$. Let $K_1 = \overline{NE_2} = NE_2/_{\partial_3(NE_3 \cap I_3)}$, where I_3 is the ideal of E_3 generated by the degeneracy elements given in [12]. Then, the action of K_2 on K_1 is given by $k_2 \in K_2$ and $\overline{k_1} = k_1 + \partial_3(NE_3 \cap I_3) \in K_1$, $k_2 \cdot \overline{k_1} = \overline{s_1(k_2)k_1}$, and $\overline{k_1} \cdot k_2 = \overline{k_1 s_1(k_2)}$. The action of $k_3 \in K_3$ on K_1 is given by $k_3 \cdot \overline{k_1} = \overline{s_1(k_3)k_1} = \overline{k_1 s_1(k_3)} = \overline{k_1} \cdot k_3$. The homomorphism $\partial : K_1 \to K_2$ is given by the restriction of $d_2^2 : E_2 \to E_1$ on $\mathrm{Ker} d_0^1$ and similarly $\partial' : K_1 \to K_3$ is given by the restriction of d_2^2 on $\mathrm{Ker} d_1^1$. Then, we obtain the following diagram:

$$\begin{array}{ccc} NE_2/_{\partial_3(NE_3 \cap I_3)} & \xrightarrow{\partial_2} & NE_1 \\ {\scriptstyle \partial_2^*}\downarrow & & \\ NE_1^* & & \end{array}$$

where $x \in NE_1 = \mathrm{Ker} d_0^1$ and $y \in NE_1^* = \mathrm{Ker} d_1^1$ and h map is provided by

$$\begin{array}{rcl} h : NE_1 \otimes NE_1^* & \longrightarrow & NE_2/_{\partial_3(NE_3 \cap I_3)} \\ (x \otimes y) & \longmapsto & \overline{s_1(x)s_1(y) - s_0(x)s_1(y)} = (s_1(x) - s_0(x))s_1(y) + \partial_3(NE_3 \cap I_3) \end{array}$$

We will show that all axioms of crossed corner are verified.

CC1. ∂_2 and ∂_2^* are crossed modules. Because, there are actions of NE_1^* on $\overline{NE_2} = NE_2/_{\partial_3(NE_3 \cap I_3)}$ and NE_1 via s_1 and NE_1 acts on $NE_2/_{\partial_3(NE_3 \cap I_3)}$and NE_1^* via s_1. For $x \in NE_1$ and $\overline{y} = y + \partial_3 NE_3 \in \overline{NE_2}$,

$$\partial_2(x \cdot \overline{y}) = \partial_2(\overline{x \cdot y}) = \partial_2(\overline{s_1 x y}) = x\partial_2(\overline{y})$$

and for $\overline{y}, \overline{y'} \in \overline{NE_2}$,

$$\partial_2(\overline{y}) \cdot \overline{y'} = s_1 d_2 y.y' + \partial_3(NE_3 \cap I_3)$$

We know from the $C_{\alpha,\beta}$ functions from [12] that

$$yy' - s_1 d_2 y \cdot y' = d_2(s_1 y s_1 y' - s_0 y s_1 y') \in \partial_3(NE_3 \cap I_3)$$

Thus,

$$\partial_2(\overline{y}) \cdot \overline{y'} = \overline{yy'}$$

and then ∂_2 is a crossed module of algebras. Similarly ∂_2^* is a crossed module of algebras.

CC2. For $x_1, x_2 \in NE_1$ and $y \in NE_1^*$, it must be $h((x_1 + x_2) \otimes y) = h(x_1 \otimes y) + h(x_2 \otimes y)$.

$$\begin{aligned} h((x_1 + x_2) \otimes y) &= s_1(x_1 + x_2)s_1(y) - s_0(x_1 + x_2)s_1(y) + \partial_3(NE_3 \cap I_3) \\ &= (s_1(x_1) + s_1(x_2))s_1(y) - (s_0(x_1) + s_0(x_2))s_1(y) + \partial_3(NE_3 \cap I_3) \\ &= (s_1(x_1)s_1(y) + s_1(x_2)s_1(y)) - (s_0(x_1)s_1(y) + s_0(x_2)s_1(y)) + \partial_3(NE_3 \cap I_3) \\ &= (s_1(x_1)s_1(y) - s_0(x_1)s_1(y)) + (s_1(x_2)s_1(y) - s_0(x_2)s_1(y)) + \partial_3(NE_3 \cap I_3) \\ &= h(x_1 \otimes y) + h(x_2 \otimes y) \end{aligned}$$

Similarly, for $x \in NE_1$ and $y_1, y_2 \in NE_1^*$, $h(x \otimes (y_1 + y_2)) = h(x \otimes y_1) + h(x \otimes y_2)$ is satisfied.

CC3. For $\overline{z} = z + \partial_3(NE_3 \cap I_3) \in NE_2/_{\partial_3(NE_3 \cap I_3)}$, $x \in NE_1, y \in NE_1^*$ it must be $h(\partial_2(\overline{z}) \otimes y) = y \cdot \overline{z}$ and $h(x \otimes \partial_2^*(\overline{z})) = x \cdot \overline{z}$. We will use the image of the $C_{\alpha,\beta}$ pairings in the Moore complex of a simplicial commutative algebra. For the image of these elements, see [12].

Firstly, $h(\partial_2(\overline{z}) \otimes y) = s_1 d_2(z)s_1(y) - s_0 d_2(z)s_1(y) + \partial_3(NE_3 \cap I_3)$. For $\alpha = (0)$ and $\beta = (2,1)$, from [12],

$$\begin{aligned} d_3(C_{(0),(2,1)}(y,z)) &= d_3[(s_2 s_1(y))(-s_0(z) + s_1(z) + s_2(z))] \\ &= d_3(s_2 s_1(y))(-d_3 s_0(z) + d_3 s_1(z) + d_3 s_2(z)) \\ &= s_1(y)(-s_0 d_2(z) + s_1 d_2(z) + z) \qquad (\because d_3 s_2 = id, d_3 s_0 = s_0 d_2, d_3 s_1 = s_1 d_2) \\ &= s_1 d_2(z)s_1(y) - s_0 d_2(z)s_1(y) + s_1(y)z \in \partial_3(NE_3 \cap I_3) \end{aligned}$$

Thus,

$$\begin{aligned} h(\partial_2(\overline{z}) \otimes y) &= s_1(y)\overline{z} \qquad (\text{mod } \partial_3(NE_3 \cap I_3)) \\ &= y \cdot \overline{z} \end{aligned}$$

Similarly, $h(x \otimes d_2^*(\overline{z})) = s_1(x)s_1 d_2(z) - s_0(x)s_1 d_2(z) + \partial_3(NE_3 \cap I_3)$. For $\alpha = (2,0)$ and $\beta = (1)$, from [12],

$$\begin{aligned} d_3(C_{(2,0),(1)}(x,z)) &= d_3[(s_2 s_0(x) - s_2 s_1(x))(s_1(z) - s_2(z))] \\ &= d_3 s_2 s_0(x) d_3 s_1(z) - d_3 s_2 s_0(x) d_3 s_2(z) - d_3 s_2 s_1(x) d_3 s_1(z) + d_3 s_2 s_1(x) d_3 s_2(z) \\ &= s_0(x)s_1 d_2(z) - s_0(x)z - s_1(x)s_1 d_2(z) + s_1(x)z \in \partial_3(NE_3 \cap I_3) \end{aligned}$$

Then,

$$\begin{aligned} h(x \otimes \partial_2^*(\overline{z})) &= s_1(x)\overline{z} - s_0(x)\overline{z} \qquad (\text{mod } \partial_3(NE_3 \cap I_3)) \\ &= s_1(x)\overline{z} \qquad (\because \partial_1(x) = 0) \\ &= x \cdot \overline{z} \end{aligned}$$

CC4. We show $(x \cdot y) \cdot \overline{z} = (xy) \cdot \overline{z}$, for $x \in NE_1, y \in NE_1^*$, and $\overline{z} \in NE_2/_{\partial_3(NE_3 \cap I_3)}$:

$$
\begin{aligned}
(x \cdot y) \cdot \overline{z} &= (\partial_2 h(x \otimes y)) \cdot \overline{z} \qquad (\because x \cdot y = \partial_2 h(x \otimes y)) \\
&= d_2(s_1(x)s_1(y) - s_0(x)s_1(y)) \cdot \overline{z} \\
&= (d_2s_1(x)d_2s_1(y) - d_2s_0(x)d_2s_1(y)) \cdot \overline{z} \\
&= (xy - d_2s_0(x)y) \cdot \overline{z} \\
&= (xy - s_0d_1(x)y) \cdot \overline{z} \\
&= (xy) \cdot \overline{z} \qquad (\because\ \partial_1(x) = 0)
\end{aligned}
$$

Similarly, the axiom $(y \cdot x) \cdot \overline{z} = (yx) \cdot \overline{z}$ is satisfied.

Thus, we obtained a crossed corner of a reduced simplicial algebra. If the length of the Moore complex of given reduced simplicial algebra $\mathbb{E}$ is 2, then $NE_3 = \{0\}$ and thus $\partial_3(NE_3 \cap I_3) = \{0\}$. Therefore, the equivalence between cosets becomes equality. Thus, we have defined a functor from the category of reduced simplicial algebras to the category of crossed corners,

$$N : ReSimpAlg_{\leq 2} \to CC$$

5. From Crossed Corners to Reduced Simplicial Algebras

In this section, we will construct a reduced simplicial algebra with Moore complex of length ≤ 2 from a crossed corner

$$
\begin{array}{ccc}
K_1 & \xrightarrow{\partial} & K_2 \\
{\scriptstyle \partial'}\downarrow & & \\
K_3 & &
\end{array}
$$

together with the h-map $h : K_2 \otimes K_3 \to K_1$. We can consider this crossed corner as a crossed square

$$
\begin{array}{ccc}
K_1 & \xrightarrow{\partial} & K_2 \\
{\scriptstyle \partial'}\downarrow & & \downarrow{\scriptstyle \zeta} \\
K_3 & \xrightarrow[\zeta']{} & K_0 = \{0\}
\end{array}
$$

with the h-map $h : K_2 \otimes K_3 \to K_1$. Since $\zeta : K_2 \to \{0\}$ is the zero morphism, then we obtain a diagonal simplicial algebra

$$\cdots K_2 \ltimes (K_2 \ltimes \{0\}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow} K_2 \ltimes \{0\} \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow} \{0\}$$

and then we can say that this is a reduced simplicial algebra. In this structure, the face and degeneracy maps are given by

$$d_0^1(k_2, 0) = d_1^1(k_2, 0) = 0 \qquad s_0^0(0) = (0, 0)$$

and

$$
\begin{aligned}
d_0^1(k_2, k_2', 0) &= (k_2k_2', 0) \\
d_1^1(k_2, k_2', 0) &= (k_2, 0) \\
d_2^1(k_2, k_2', 0) &= (k_2', 0) \\
s_0^1(k_2, 0) &= (0, k_2, 0) \\
s_1^1(k_2, 0) &= (k_2, 0, 0)
\end{aligned}
$$

Since $K_2 \times \{0\} \cong K_2$, we can write it as

$$\cdots K_2 \ltimes K_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} \; K_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \; \{0\}$$

Similarly, since $\zeta' : K_3 \to \{0\}$ is a crossed module, we obtain a reduced simplicial algebra as

$$\cdots K_3 \ltimes K_3 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} \; K_3 \; \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \; \{0\}$$

Using the actions of K_2 on K_1 and of K_3 on K_1, we have a reduced bisimplicial algebra.

$$\begin{array}{ccccc}
\cdots & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & (K_1 \ltimes K_3) \ltimes ((K_1 \ltimes K_3) \ltimes K_2) & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & K_2 \ltimes K_2 \\
\uparrow\uparrow\downarrow\downarrow\downarrow & & \uparrow\uparrow\downarrow\downarrow\downarrow & & \uparrow\uparrow\downarrow\downarrow\downarrow \\
((K_1 \ltimes K_3) \ltimes K_2) \ltimes (K_2 \ltimes K_2) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & (K_1 \ltimes K_3) \ltimes K_2 & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & K_2 \\
\uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow \\
K_3 \ltimes K_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & K_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & \{0\}
\end{array}$$

We will use the way from bisimplicial algebras to simplicial algebras with the help of the functor defined by Artin Mazur [19]. The subset of the algebra $E_{1,0} \times E_{0,1} = (K_2 \ltimes \{0\}) \times (K_3 \ltimes \{0\})$ is

$$E_1 = \{((k_2, 0), (k_3, 0)) | d_0^v(k_2, 0) = d_1^h(k_3, 0) = 0\}$$

where $\{0\} \cong E_0$. The isomorphism between E_1 and $E_{1,0} \times E_{0,1}$ can be defined by

$$\begin{array}{rcl}
\eta : E_1 & \longrightarrow & K_3 \ltimes K_2 \ltimes \{0\} \\
((k_2, 0), (k_3, 0)) & \longmapsto & (k_3, k_2, 0)
\end{array}$$

Thus, we can write $E_1 \cong K_3 \ltimes K_2 \ltimes \{0\}$. Then, we have the structural homomorphisms between E_0 and E_1 obviously

$$d_0(k_3, k_2, 0) = 0$$

and

$$d_1(k_3, k_2, 0) = 0$$

Hence, we obtain $\{E_1, E_0\}$ as a reduced 1-truncated simplicial algebra together with these zero homomorphisms. Moreover, the elements of the subalgebra

$$E_{2,0} \times E_{1,1} \times E_{0,2} = (K_2 \ltimes (K_2 \ltimes \{0\})) \times ((K_1 \ltimes K_3) \ltimes (K_2 \ltimes \{0\})) \times (K_3 \ltimes (K_3 \ltimes \{0\}))$$

can be written by

$$((k_2', k_2'', 0), ((k_1, k_3), (k_2, 0)), (k_3', k_3'', 0))$$

Moreover, the vertical face maps are

$$d_0^v(k_2', k_2'', 0) = d_1^h(k_1, k_3, k_2, 0)$$

and

$$d_1^v(k_1, k_3, k_2, 0) = d_2^h(k_3', k_3'', 0)$$

Then, $k_2'' = k_2$ and $\partial'(k_1)k_3 = k_3'$. Thus, it can be written by

$$((k_2', k_2'', 0), ((k_1, \partial'(k_1)k_3'), (k_2, 0)), (\partial'(k_1)k_3, k_3'', 0))$$

as elements of E_2. We see that the map

$$\begin{array}{rccl}
\eta' : & E_2 & \longrightarrow & (K_1 \ltimes (K_3 \ltimes K_2)) \ltimes (K_3 \ltimes (K_2 \ltimes \{0\})) \\
& ((k_2', k_2'', 0), ((k_1, k_3), (k_2'', 0)), (\partial'(k_1)k_3, k_3'', 0)) & \longmapsto & ((k_1, (k_3, k_2'')), (k_3'', (k_2, 0)))
\end{array}$$

is an isomorphism. Consequently, using the Artin-Mazur codiagonal functor, we obtain the following reduced simplicial algebra.

$$\mathbb{E} : (K_1 \ltimes (K_3 \ltimes K_2)) \ltimes (K_3 \ltimes (K_2 \ltimes \{0\})) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} (K_3 \ltimes (K_2 \ltimes \{0\})) \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \{0\}$$

The faces and degeneracies maps are defined as follows:

$$\begin{aligned} d_0^1(k_3, k_2, 0) &= 0 \\ d_1^1(k_3, k_2, 0) &= 0 \\ s_0^0(0) &= (0, 0, 0) \end{aligned}$$

and

$$\begin{aligned} d_0^2((k_1, (k_3, k_2^{''})), (k_3^{''}, (k_2^{'}, 0))) &= (k_3^{''}, \partial(k_1)k_2^{''}, 0) \\ d_1^2((k_1, (k_3, k_2^{''})), (k_3^{''}, (k_2^{'}, 0))) &= (k_3^{''}, \partial^{'}(k_1)k_3, k_2^{''}k_2^{'}, 0) \\ d_2^2((k_1, (k_3, k_2^{''})), (k_3^{''}, (k_2^{'}, 0))) &= (k_3, k_2^{'}, 0) \\ s_0^1(k_3, k_2, 0) &= ((0, (0, k_2)), (k_3, (0, 0))) \\ s_1^1(k_3, k_2, 0) &= ((0, (k_3, 0)), (0, (k_2, 0))) \end{aligned}$$

We can get a 2-truncated reduced simplicial algebra. Using the coskeleton functor from k-truncated simplicial algebras to simplicial algebras with Moore complex of length k given in [12], we can see that $\mathbb{E}$ is a reduced simplicial algebra with Moore complex of length 2. Therefore, we obtained the following functor

$$\Delta : CC \to ReSimpAlg_{\leq 2}$$

We can provide the following result:

Theorem 5.1. The category of reduced simplicial algebras with Moore complex of length 2 is equivalent to that of crossed corners of commutative algebras.

6. Conclusion

In this paper, the commutative algebra analog of crossed corners has been introduced. We have obtained that the category of reduced simplicial algebras with Moore complex of length 2 is equivalent to that of crossed corners of commutative algebras. We know a categorical equivalence exists between braided crossed modules, reduced quadratic modules, and reduced simplicial groups with Moore complex of length 2. This result establishes the equivalence between crossed corners and braided crossed modules or reduced quadratic modules. Furthermore, this idea can be extended to the Lie algebra case for further research.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bulletin of the American Mathematical Society 55 (1949) 453–496.

[2] D. Guin-Waléry, J-L. Loday, *Obsruction á l'excision en K-theories Algébrique*, in: E. M. Friedlander, M. R. Stein (Eds.), Algebraic K-Theory Evanston 1980, Vol. 854 of *Lectute Notes Mathematics*, Springer, Berlin, 1981, pp. 179–216.

[3] H. J. Baues, Combinatorial Homotopy and 4-Dimensional Complexes, Walter de Gruyter, Berlin, 1991.

[4] Z. Arvasi, E. Ulualan, *Quadratic and 2-Crossed Modules of Algebras*, Algebra Colloquium 14 (2007) 669–686.

[5] E. Ulualan, E. Uslu, *Quadratic Modules for Lie Algebras*, Hacettepe Journal of Mathematics and Statistics 40 (3) (2011) 409–419.

[6] E. Özel, U. E. Arslan, *On Quasi Quadratic Modules of Lie Algebras*, Journal of New Theory (41) (2022) 62–69.

[7] E. Soylu Yılmaz, K. Yılmaz, *On Relations among Quadratic Modules*, Mathematical Methods in the Applied Sciences 45 (18) (2022) 12231–12244.

[8] M. Alp, *Characterization of Crossed Corner*, Algebras, Groups and Geometries 16 (2) (1999) 173–182.

[9] T. Porter, *Homology of Commutative Algebras and an Invariant of Simis and Vasconceles*, Journal of Algebra 99 (1986) 458–465.

[10] G. J. Ellis, *Higher Dimensional Crossed Modules of Algebras*, Journal of Pure and Applied Algebra 52 (1988) 277–282.

[11] Z. Arvasi, *Crossed Squares and 2-Crossed Modules of Commutative Algebras*, Theory and Applications of Categories 3 (7) (1997) 160–181.

[12] Z. Arvasi, T. Porter, *Higher Dimensional Peiffer Elements in Simplicial Commutative Algebras*, Theory and Applications of Categories 3 (1) (1997) 1–23.

[13] N. M. Shammu, *Algebraic and Categorical Structure of Categories of Crossed Modules of Algebras*, Doctoral Dissertation North Carolina Wilmington University (1992) Bangor.

[14] I. Akça, Z. Arvasi, *Simplicial and Crossed Lie Algebras*, Homology, Homotopy and Applications 4 (1) (2002) 43–57.

[15] A. Aytekin, *Categorical Structures of Lie-Rinehart Crossed Module*, Turkish Journal of Mathematics 43 (1) (2019) 511–522.

[16] M. Alp, *Applications of Crossed Corner*, Algebras, Groups and Geometries 16 (2) (1999) 337–344.

[17] M. Alp, A. Bekir, E. Ulualan, *Relation Between Crossed Square and Crossed Corner*, Journal of Science and Technology of Dumlupınar University (002) (2001) 89–96.

[18] Ö. Gürmen Alansal, E. Ulualan, *Bisimplicial Commutative Algebras and Crossed Squares*, Fundamental Journal of Mathematics and Applications 6 (2023) 177–187.

[19] M. Artin, B. Mazur, *On the Van Kampen Theorem*, Topology 5 (1966) 179–189.

Journal of New Theory

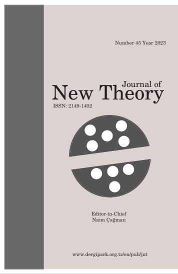
ISSN: 2149-1402

45 (2023) 105-119

Journal of New Theory

https://dergipark.org.tr/en/pub/jnt

Open Access



An Application of Nonstandard Finite Difference Method to a Model Describing Diabetes Mellitus and Its Complications

İlkem Turhan Çetinkaya[1]

Article Info
Received: 15 Nov 2023
Accepted: 27 Dec 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1391403
Research Article

Abstract — In this study, a mathematical model describing diabetes mellitus and its complications in a population is considered. Since standard numerical methods can lead to numerical instabilities, it aims to solve the problem using a nonstandard method. Among the nonstandard methods, nonstandard finite difference (NSFD) schemes that satisfy dynamical consistency are preferred to make the model discrete. Both continuous and discrete models are analyzed to show the stability of the model at the equilibrium points. The Schur-Cohn criterion is used to perform stability analysis at the equilibrium point of the discretized model. Thus, asymptotically stability of the model is presented. Moreover, the advantages of the NSFD method are emphasized by comparing the stability for different step sizes with classical methods, such as Euler and Runge-Kutta. It has been observed that the NSFD method is convergence for larger step sizes. In addition, the numerical results obtained by NSFD schemes are compared with the Runge–Kutta–Fehlberg (RKF45) method in graphical forms. The accuracy of the NSFD method is observed.

Keywords *Diabetes mellitus, nonstandard finite difference scheme, stability analysis*

Mathematics Subject Classification (2020) 34A30, 65L07

1. Introduction

Many biological problems can be modeled by using differential equations. As is known, diabetes has become a very common disease recently. Many important studies about diabetes have been performed. In epidemic models, stability analysis has an important role. Some of the studies about diabetes can be summarized as follows:

Boutayeb et al. [1] propose a mathematical model of diabetes to present a better quality of line for humans. The numerical solution and the stability analysis for the linear model in which the unknowns are numbers of diabetics with and without complications are presented. Akinsola and Oluyo [2–4] obtain the numerical and analytical solution of the model of complications and control of diabetes mellitus in their studies with different methods. Moreover, the linear diabetes mellitus model is considered by AlShurbaji et al. [5]. The numerical comparison of the solution of a system of linear differential equations by numerical methods such as Euler, Heun, Runge-Kutta, and Adams-Moulton is presented. Stability analysis is given. Furthermore, Vanitha and Porchelvi [6] consider the linear mathematical model of diabetes mellitus. A numerical solution by the Euler-Cauchy method is

[1]ilkem.turhan@dpu.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Arts and Sciences, Kütahya Dumlupınar University, Kütahya, Türkiye

presented. Besides, Boutayeb et al. [7] present a nonlinear mathematical model of diabetes mellitus by applying appropriate parameters. Stability analysis and numerical experiments are presented. de Oliveira et al. [8] consider the model proposed in [7] and global asymptotic stability is studied. The stability is verified through numerical simulations. Boutayeb et al. [9] present the dynamics of a population of healthy people, pre-diabetics, and diabetics with and without complications. Optimal control theory is used. Permatasari et al. [10] considered the model developed by [9]. Global stability and controllability for linear and nonlinear systems are presented, respectively. Widyaningsih et al. [11] consider the nonlinear diabetes mellitus model in terms of lifestyle and genetic factors. Fourth-order Runge-Kutta method is applied to predict the number of deaths due to diabetes, recently. Aye [12] presents stability analysis of a linear model describing diabetes mellitus and its complications. The stability is tested by using the Bellman and Coke theorem. Aye et al. [13] solve a similar model using the Homotopy Perturbation Method. Aye [14] investigates the effect of control on the same model.

It is known that stability analysis of mathematical models plays an important role in the disciplines of applied mathematics. Since, in real-life problems, the points are discrete, discretizing the models is very important in stability analysis. Likewise, in solving such problems, standard numerical methods can lead to numerical instabilities. Hence, nonstandard methods are important. Among the discrete methods, the Nonstandard Finite Difference (NSFD) method developed by Mickens [15–20] is very effective and easy to apply. Moreover, it provides convergence results in even bigger step sizes. The detailed literature survey about NSFD schemes is presented in the studies of Patidar [21, 22]. There are many studies about NSFD schemes in many disciplines of applied mathematics. Some of the recent studies about NSFD schemes and stability analysis can be listed as follows:

Adekanye and Washington [23] consider a mathematical model presented by the collapse of the Tacoma Narrows Bridge. Two NSFD schemes are constructed for the vertical and torsional models. Graphics present vertical and torsional motions. An application of NSFD schemes to a model of the Ebola virus in Africa is presented in [24] by Anguelov et al. Epidemic fractional models about susceptible-infected (SI) and susceptible-infected-recovered (SIR) are proposed by Arenas et al. in [25]. NSFD schemes are applied, and some comparisons with classical methods are given. Baleanu et al. [26] analyze a novel fractional chaotic system for integer and fractional order cases. Stability analysis is presented for both cases. Numerical simulations are presented with the help of NSFD schemes. Dang and Hoang [27] construct NSFD schemes for two metapopulation models. Stability analysis and other properties, such as positivity, boundedness, and monotone convergence, are presented. Numerical calculations are given to support the theoretical study. Dang and Hoang [28], and Kocabıyık et al. [29] approximate a computer virus model with the NSFD method. Ozdogan and Ongun [30] solve a mathematical model describing the Michaelis-Menten harvesting rate with the help of NSFD schemes. NSFD discretization of a distributed order smoking model is presented to determine the effects of smoking on humans by Kocabiyik and Ongun [31]. A comparison of two different NSFD schemes for the fractional order Hantavirus model is given in the study of Ongun and Arslan [32]. A predator-prey model is constructed by NSFD schemes by Shabbir et al. in [33]. Stability analysis and other properties such as positivity, boundedness, and persistence of solutions are investigated. Vaz and Torres [34] proposed an NSFD scheme for the Susceptible–Infected–Chronic–AIDS (SICA) model. Elementary and global stability are studied. A linear mathematical model of pharmacokinetics is considered by Egbelowo et al. in [35]. The Standard Finite Difference method, NSFD method, and analytical solution are presented. More recent studies about the stability analysis of the mathematical models are presented in [36–39].

In this study, a system of linear ordinary differential equations led from diabetes mellitus and its complications given in [13] is considered. The second section defines the mathematical model and its parameters and variables. The stability of the continuous model is given. The third section is devoted to discretizing the model by the NSFD method. The fourth section includes the stability analysis of the discrete model. The fifth section is the numerical simulation section. Finally, the last section is the conclusion section.

2. The Continuous Model Describing Diabetes Mellitus and Its Complications

This section presents the definition of a mathematical model of diabetes mellitus and its complications provided in [13]. The model consists of a system of linear ordinary differential equations and is defined as

$$\begin{aligned}
\frac{dH}{dt} &= \beta\theta - (\mu+\tau)H + \sigma S \\
\frac{dS}{dt} &= \beta(1-\theta) + \tau H - (\mu+\alpha+\sigma)S \\
\frac{dD}{dt} &= \alpha S - (\mu+\lambda)D + \omega T \\
\frac{dC}{dt} &= \lambda D - (\mu+\delta+\gamma)C \\
\frac{dT}{dt} &= \gamma C - (\mu+\omega)T
\end{aligned} \tag{1}$$

with the initial conditions $H(0) = H_0$, $S(0) = S_0$, $D(0) = D_0$, $C(0) = C_0$, and $T(0) = T_0$, where the variables $H(t)$, $S(t)$, $D(t)$, $C(t)$, and $T(t)$ denote to the healthy, susceptible, diabetics without complication, diabetics with complication and diabetics with complications receiving a cure, respectively. The parameters β, θ, μ, τ, σ, α, λ, ω, δ, and γ denote rate of birth, rate of children born healthy, rate of natural mortality death, the rate at which healthy individual become susceptible, the rate at which susceptible individual become healthy, probability rate of incidence of diabetes, rate of a diabetic person developing complications, rate at which diabetic with complications after cured return diabetic without complications, rate of mortality due to complications and rate at which diabetic with complications are cured.

Hereinafter, the asymptotic stability of the continuous model described by Equation 1 will be presented. Thus, we first give some basic preliminaries. For a general autonomous vector field

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{2}$$

the linearized system can be defined as

$$\frac{dy}{dt} = J(E)y$$

where E and $J(E)$ denotes the equilibrium point of the Equation 2 and Jacobian matrix of the Equation 2 at the equilibrium point E, respectively.

Theorem 2.1. [40] Suppose all the $J(E)$ have negative reel parts. Then, the equilibrium solution of the nonlinear vector field defined by Equation 2 is asymptotically stable.

The equilibrium point of Equation 1 is obtained as $E^* = (H^*, S^*, D^*, C^*, T^*)$, where

$$H^* = \frac{\beta(\sigma + \theta\mu + \theta\alpha)}{\chi}$$

$$S^* = \frac{-\beta(-\mu - \tau + \theta\mu)}{\chi}$$

$$D^* = \frac{-\alpha\beta(\mu+\omega)(-\mu-\tau+\theta\mu)(\mu+\delta+\gamma)}{\varpi}$$
$$C^* = \frac{-\alpha\beta\lambda(\mu+\omega)(-\mu-\tau+\theta\mu)}{\varpi}$$
$$T^* = \frac{-\alpha\beta\lambda\gamma(-\mu-\tau+\theta\mu)}{\varpi}$$
$$\chi = \mu^2 + (\sigma+\tau+\alpha)\mu + \alpha\tau$$

and

$$\varpi = (\mu^3 + (\omega+\delta+\lambda+\gamma)\mu^2 + (\omega(\gamma+\delta+\lambda) + \lambda(\delta+\gamma))\mu + \omega\lambda\delta)\chi$$

The Jacobian matrix of the continuous model at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$ is determined as

$$J(H^*, S^*, D^*, C^*, T^*) = \begin{pmatrix} -\tau-\mu & \sigma & 0 & 0 & 0 \\ \tau & -\mu-\alpha-\sigma & 0 & 0 & 0 \\ 0 & \alpha & -\mu-\lambda & 0 & \omega \\ 0 & 0 & \lambda & -\mu-\delta-\lambda & 0 \\ 0 & 0 & 0 & \gamma & -\mu-\omega \end{pmatrix}$$

Thus, considering Theorem 2.1, the continuous system defined by Equation 1 is asymptotically stable if all the eigenvalues of $J(H^*, S^*, D^*, C^*, T^*)$ have negative reel parts. A detailed analysis of the asymptotic stability of the continuous model will be given in Section 5.

3. Discretization of the Model by NSFD Schemes

In this section, the model defined by Equation 1 is discretized by using NSFD schemes, an effective method. Some advantages of the proposed method are that it removes the numerical instabilities obtained by standard finite difference procedures, gives more approximate results compared to classical methods such as Runge-Kutta and Euler methods, and is converged for bigger step sizes compared to classical methods.

The rules for constructing NSFD schemes and determination of denominator function can be summarized as follows [16]:

i. To avoid numerical instabilities, the order of discrete derivatives should be equal to the derivatives in the differential equations.

ii. The discretization of first-order derivatives is usually in the following general form:

$$\frac{dx}{dt} \to \frac{x_{n+1} - \psi(h)x_n}{\phi(h)}$$

where $\psi(h)$ and $\phi(h)$ are called numerator and denominator functions, respectively.

iii. Nonlinear terms should be replaced by nonlocal discrete terms such as $x^2 \to x_{k+1}x_k$ and $x^2 \to x_k^2$.

iv. Additional conditions for the differential equations should be satisfied for difference equations.

In the view of the procedure given above, the model is discretized by using the following steps to satisfy the positivity conditions:

In the first equation of Equation 1, the replacements $H(t) \to H(n+1)$ and $S(t) \to S(n)$ are used. Similarly, in the second equation of Equation 1, the replacements $H(t) \to H(n)$ and $S(t) \to S(n+1)$; in the third equation of Equation 1, the replacements $S(t) \to S(n)$, $D(t) \to D(n+1)$, and $T(t) \to T(n)$; in the fourth equation of Equation 1, the replacements $D(t) \to D(n)$ and $C(t) \to C(n+1)$; and finally, in the last equation of Equation 1, the replacements $C(t) \to C(n)$ and $T(t) \to T(n+1)$ are implemented. Thus, the following discrete model is obtained:

$$\begin{aligned}
H(n+1) &= \frac{H(n) + (\beta\theta + \sigma S(n))\phi_1}{1 + (\mu + \tau)\phi_1} \\
S(n+1) &= \frac{S(n) + (\beta(1-\theta) + \tau H(n))\phi_2}{1 + (\mu + \alpha + \sigma)\phi_2} \\
D(n+1) &= \frac{D(n) + (\alpha S(n) + \omega T(n))\phi_3}{1 + (\mu + \lambda)\phi_3} \\
C(n+1) &= \frac{C(n) + \lambda\phi_4 D(n)}{1 + (\mu + \delta + \gamma)\phi_4} \\
T(n+1) &= \frac{T(n) + \gamma\phi_5 C(n)}{1 + (\mu + \omega)\phi_5}
\end{aligned} \tag{3}$$

where $\phi_i,\ i = \overline{1,5}$ are denominator functions and determined as

$$\phi_1 = \frac{e^{h(\mu+\tau)} - 1}{\mu + \tau}$$

$$\phi_2 = \frac{e^{h(\mu+\alpha+\sigma)} - 1}{\mu + \alpha + \sigma}$$

$$\phi_3 = \frac{e^{h(\mu+\lambda)} - 1}{\mu + \lambda}$$

$$\phi_4 = \frac{e^{h(\mu+\delta+\gamma)} - 1}{\mu + \delta + \gamma}$$

and

$$\phi_5 = \frac{e^{h(\mu+\omega)} - 1}{\mu + \omega}$$

4. Stability Analysis of the Discretized Model

In this section, the stability analysis of the discretized model is performed. Some theorems and lemmas about the stability and properties such as positivity, and permanence of the solutions of the discrete system given by Equation 3 are presented.

Lemma 4.1. All solutions of discrete system given in Equation 3 are positive with positive initial conditions and positive parameters $\beta, \theta, \mu, \tau, \sigma, \alpha, \lambda, \omega, \delta, \gamma$, and h under the assumption of

$$\frac{S(n)}{\phi_2} > -(\beta(1-\theta) + \tau H(n))$$

Proof.
Assume that the parameters $\beta, \theta, \mu, \tau, \sigma, \alpha, \lambda, \omega, \delta, \gamma$, and h are positive. Moreover, assume that the initial conditions $H(0) = H_0,\ S(0) = S_0,\ D(0) = D_0,\ C(0) = C_0$, and $T(0) = T_0$ are positive. Then, it is obvious that the denominator functions are all positive, i.e.,

$$\phi_1 = \frac{e^{h(\mu+\tau)} - 1}{\mu + \tau} > 0, \qquad \phi_2 = \frac{e^{h(\mu+\alpha+\sigma)} - 1}{\mu + \alpha + \sigma} > 0,$$

$$\phi_3 = \frac{e^{h(\mu+\lambda)} - 1}{\mu + \lambda} > 0, \qquad \phi_4 = \frac{e^{h(\mu+\delta+\gamma)} - 1}{\mu + \delta + \gamma} > 0,$$

and

$$\phi_5 = \frac{e^{h(\mu+\omega)} - 1}{\mu+\omega} > 0$$

Therefore, for the positive parameters, it is obvious that

$$H(n+1) = \frac{H(n) + (\beta\theta + \sigma S(n))\phi_1}{1 + (\mu+\tau)\phi_1} > 0$$

$$D(n+1) = \frac{D(n) + (\alpha S(n) + \omega T(n))\phi_3}{1 + (\mu+\lambda)\phi_3} > 0$$

$$C(n+1) = \frac{C(n) + \lambda\phi_4 D(n)}{1 + (\mu+\delta+\gamma)\phi_4} > 0$$

and

$$T(n+1) = \frac{T(n) + \gamma\phi_5 C(n)}{1 + (\mu+\omega)\phi_5} > 0$$

As well, assuming $\frac{S(n)}{\phi_2} > -(\beta(1-\theta) + \tau H(n))$, it can be concluded that

$$S(n+1) = \frac{S(n) + (\beta(1-\theta) + \tau H(n))\phi_2}{1 + (\mu+\alpha+\sigma)\phi_2}$$

Thus, the discrete system is positive for all the positive parameters and initial conditions. □

Locally asymptotic stability of the model can be analyzed by obtaining the eigenvalues of the Jacobian matrix at equilibrium points. Local asymptotic stability of the system depends on the eigenvalues of the Jacobian matrix at the equilibrium points.

Theorem 4.2 (Schur-Cohn Criterion). [41] Consider the characteristic polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{4}$$

where $a_1, a_2, \cdots, a_n$ are constants. The zeros of the characteristic polynomial defined by Equation 4 lie inside the unit disk if and only if the following conditions hold:

i. $p(1) > 0$

ii. $(-1)^n p(-1) = 1 - a_1 + a_2 - \cdots + (-1)^n a_n > 0$

iii. The matrices

$$B_{n-1}^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-4} & \cdots & 1 & 0 \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_n & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_n & \cdots & a_4 & a_3 \\ a_n & a_{n-1} & \cdots & a_3 & a_2 \end{pmatrix}$$

are positive innerwise.

Hence, one can conclude that if the Schur-Cohn criterion is satisfied, then the discrete system is asymptotically stable. Note that the equilibrium point of the discrete system given by Equation 3 is the same as the continuous model. Therefore, the Jacobian matrix at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$ can be written as

$$J^* = J(H^*, S^*, D^*, C^*, T^*) = \begin{pmatrix} j_1 & \phi_1\sigma j_1 & 0 & 0 & 0 \\ \phi_2\tau j_2 & j_2 & 0 & 0 & 0 \\ 0 & \phi_3\alpha j_3 & j_3 & 0 & \phi_3\omega j_3 \\ 0 & 0 & \phi_4\lambda j_4 & j_4 & 0 \\ 0 & 0 & 0 & \phi_5\gamma j_5 & j_5 \end{pmatrix}$$

where

$$j_1 = \frac{1}{1+\phi_1(\mu+\tau)}$$

$$j_2 = \frac{1}{1+\phi_2(\mu+\alpha+\sigma)}$$

$$j_3 = \frac{1}{1+\phi_3(\mu+\lambda)}$$

$$j_4 = \frac{1}{1+\phi_4(\mu+\delta+\gamma)}$$

and

$$j_5 = \frac{1}{1+\phi_5(\mu+\omega)}$$

The characteristic equation is as follows:

$$p(\lambda) = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 \tag{5}$$

where the coefficients of Equation 5 are determined as

$$\begin{aligned} a_1 &= -(j_1+j_2+j_3+j_4+j_5) \\ a_2 &= (j_3+j_4)j_5 + j_3j_4 + (j_1+j_2)(j_3+j_4+j_5) - (1-\tau\sigma\phi_1\phi_2)j_1j_2 \\ a_3 &= -(1+\gamma\lambda\omega\phi_3\phi_4\phi_5)j_5j_4j_3 + (\tau\sigma\phi_1\phi_2-1)j_2j_1(j_5+j_4+j_3) - (j_2+j_1)[j_5(j_4+j_3)+j_3j_4] \\ a_4 &= (j_2+j_1)(1+\phi_3\phi_4\phi_5\lambda\omega\gamma) + j_2j_1(1-\sigma\tau\phi_1\phi_2)[j_5(j_4+j_3)+j_4j_3] \\ a_5 &= j_5j_4j_3j_2j_1(\sigma\tau\phi_1\phi_2-1)(1+\phi_3\phi_4\phi_5\lambda\omega\gamma) \end{aligned} \tag{6}$$

To analyze the stability of the model at the equilibrium point, the following theorem for the discrete system given by Equation 3 is presented.

Theorem 4.3. The discrete system in Equation 3 is locally asymptotically stable at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$ if the following conditions are satisfied:

i. $[-1+\phi_3\phi_4\phi_5\gamma\omega\lambda j_3j_4j_5 + j_5 + (j_5-1)(j_4j_3-j_4-j_3)][\phi_1\phi_2\tau\sigma j_1j_2 + (1-j_1)(j_2-1)] > 0$

ii. $[1+\phi_3\phi_4\phi_5\gamma\omega\lambda j_3j_4j_5 + (1+j_3)(j_4+j_5(1+j_4)) + j_3][-\phi_1\phi_2\tau\sigma j_1j_2 + (j_2+1)(j_1+1)] > 0$

iii. $1 - a_1a_5 + a_4 - a_5^2 > 0$

$$[1+(a_1a_2a_5 - a_1a_5(1-a_4))(1+a_4) + a_4a_5(2a_1+a_3) - a_4^2(a_4+a_2+1) + (a_5+a_3+a_1)(a_5^3-a_3)$$
$$-a_5^2(a_4+2+a_2+a_1^2+a_2^2) - a_5a_1^2(a_1+a_3) + a_4(1+a_1(a_3+a_1)) + a_2(1+a_5(2a_3-a_5a_4))] > 0$$

$$[1+a_4a_5(2a_1-3a_3) + a_4^2(a_4-1-a_2) + a_3(a_5-a_3+a_1(1+a_4)) - a_1^2(a_4+a_5(a_3-a_1+a_5))$$
$$+a_1a_5(1-3a_2+a_4(a_2-a_4)) + a_5^2(a_5(a_5-a_3-a_1) + a_2(1-a_2) + a_4(1+a_2) - 2(1+a_1a_3))$$
$$-a_4(1-2a_2) - a_2(1-2a_2a_3a_5)] > 0$$

where a_1, a_2, a_3, a_4, and a_5 are denoted by Equation 6.

PROOF.
Considering Theorem 4.2, it is obvious that if the conditions *i-iii* in Theorem 4.3 are satisfied, then the discrete system given by Equation 3 is locally asymptotically stable at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$. □

5. Numerical Results

This section presents the stability analysis for the parameters given in [13]. The parameters are taken into consideration as

$$\begin{aligned} &\beta = 0.038, \quad \theta = 0.923, \quad \mu = 0.118, \quad \tau = 0.04, \quad \sigma = 0.08, \\ &\alpha = 0.02, \quad \lambda = 0.05, \quad \omega = 0.08, \quad \delta = 0.02, \quad \text{and} \quad \gamma = 0.08 \end{aligned} \tag{7}$$

Under the given parameters above, the stability of the continuous model and discrete model will be analyzed in the view of Theorems 2.1 and 4.2.

5.1. Stability Analysis of the Continuous Model

The characteristic polynomial of the Jacobian matrix at the equilibrium point

$$E^* = (0.2522152093, 0.0597000384, 0.007435253915, 0.001705333467, 0.000689023623)$$

is determined as

$$J(E^*) = \begin{pmatrix} -0.158 & 0.08 & 0 & 0 & 0 \\ 0.04 & -0.218 & 0 & 0 & 0 \\ 0 & 0.02 & -0.168 & 0 & 0.08 \\ 0 & 0 & 0.05 & -0.218 & 0 \\ 0 & 0 & 0 & 0.08 & -0.198 \end{pmatrix}$$

The eigenvalues of $J(E^*)$ is as follows:

$$\lambda_1 = -0.123042327249903$$

$$\lambda_2 = -0.123968757625671$$

$$\lambda_3 = -0.252031242374328$$

and

$$\lambda_{4,5} = -0.230478836375048 \mp 0.0566939252859813i$$

Since all the eigenvalues of $J(E^*)$ have negative reel parts, according to Theorem 2.1, the continuous model defined by Equation 1 is asymptotically stable at the equilibrium point E^*.

5.2. Stability Analysis of the Discrete Model

In addition to the parameters given in Equation 7, the step size is chosen as $h = 0.01$. The Jacobian matrix at the equilibrium point

$$E^* = (0.2522152093, 0.0597000384, 0.007435253915, 0.001705333467, 0.000689023623)$$

is determined as

$$J(E^*) = \begin{pmatrix} 0.9984212475 & 0.0007993683 & 0 & 0 & 0 \\ 0.0003995643 & 0.9978223744 & 0 & 0 & 0 \\ 0 & 0.00019983209 & 0.9983214104 & 0 & 0.0007993283 \\ 0 & 0 & 0.0004994553 & 0.9978223744 & 0 \\ 0 & 0 & 0 & 0.0007992085 & 0.9980219589 \end{pmatrix} \tag{8}$$

The characteristic polynomial of the Jacobian matrix defined by Equation 8 is as

$$p(\lambda) = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5$$

where the constants of characteristic polynomials are

$$a_1 = -4.990409365, \quad a_2 = 9.9616737805, \quad a_3 = -9.9425650792, \quad a_4 = 4.9617462802,$$

and

$$a_5 = -0.99044561567$$

In the view of Theorem 4.2, since

i. $p(1) = 1 + a_1 + a_2 + a_3 + a_4 + a_5 = 0.215 \times 10^{-13} > 0$

ii. $-p(-1) = 1 - a_1 + a_2 - a_3 + a_4 - a_5 = 31.84684012 > 0$

iii. The inners of the matrices

$$B_4^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \pm a_5 \\ a_1 & 1 & \pm a_5 & \pm a_4 \\ a_2 & a_1 \pm a_5 & 1 \pm a_4 & \pm a_3 \\ a_3 \pm a_5 & a_2 \pm a_4 & a_1 \pm a_3 & 1 \pm a_2 \end{pmatrix}$$

are the matrices $B_4^{\pm}$ itself and the matrice

$$IB^{\pm} = \begin{pmatrix} 1 & \pm a_5 \\ a_1 \pm a_5 & 1 \pm a_4 \end{pmatrix}$$

Since the determinants of the inners of the matrice $B_4^{\pm}$

$$\det(B_4^{+}) = 0.213 \times 10^{-12}$$

$$\det(B_4^{-}) = 0.614 \times 10^{-24}$$

$$\det(IB_4^{+}) = 0.038034685$$

and

$$\det(IB_4^{-}) = 0.278 \times 10^{-6}$$

are positive, the matrices $B_4^{\pm}$ are positive innerwise. Thus, since all the conditions of the Schur-Cohn criterion are satisfied, the discrete system given in Equation 3 is locally asymptotically stable for the estimated parameters.

A numerical solution obtained by NSFD schemes is presented in the figures to support the stability of the discrete model. Moreover, the Runge-Kutta-Fehlberg (RKF45) method is applied for the estimated parameters. Therefore, the accuracy of the results obtained by the NSFD method is shown.

The estimated parameters defined in Equation 7, the step size $h = 0.01$, and the positive initial condition

$$H(0) = 198195839, \quad S(0) = 101535728, \quad D(0) = 940000, \quad C(0) = 3760000,$$

and

$$T(0) = 1193250$$

are used during the calculations.

Figures 1-5 present the numerical comparison of the results obtained by the NSFD method with the RKF45 method. It can be observed from Figures 1-5 that the results approach the equilibrium point

$$E^* = (0.2522152093, 0.0597000384, 0.007435253915, 0.001705333467, 0.000689023623)$$

The compatibility of the results can also be observed in Figures 1-5.

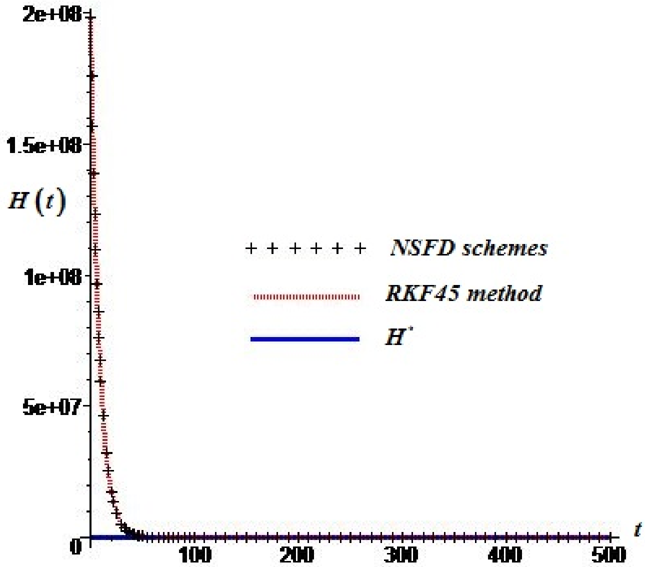


Figure 1. Variation of healthy class $H(t)$

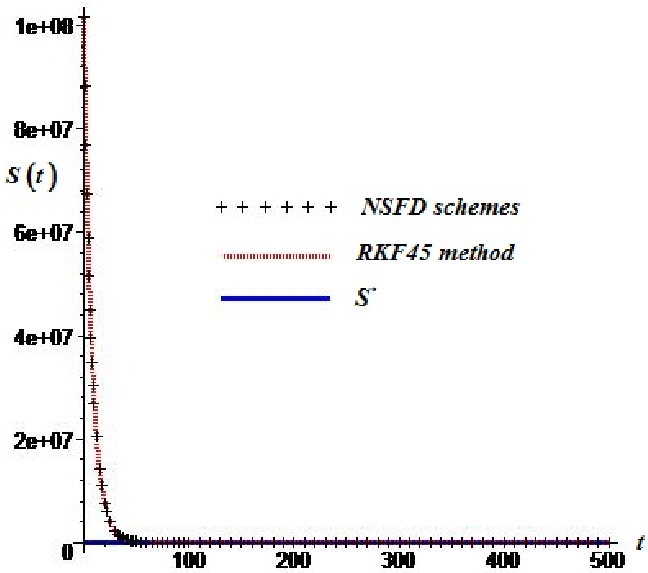


Figure 2. Variation of susceptible class $S(t)$

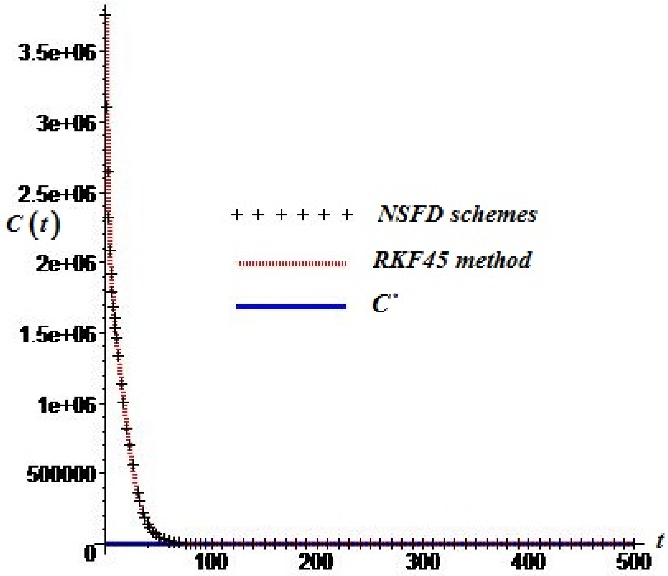


Figure 3. Variation of diabetics without complication $C(t)$

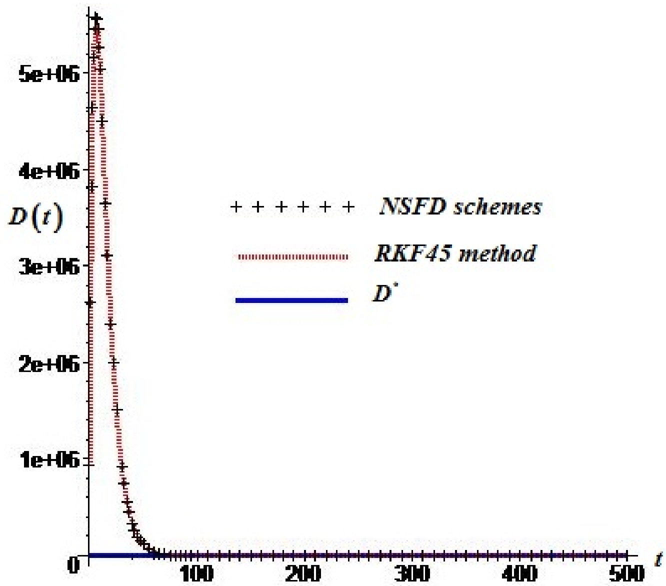


Figure 4. Variation of diabetics with complication $D(t)$

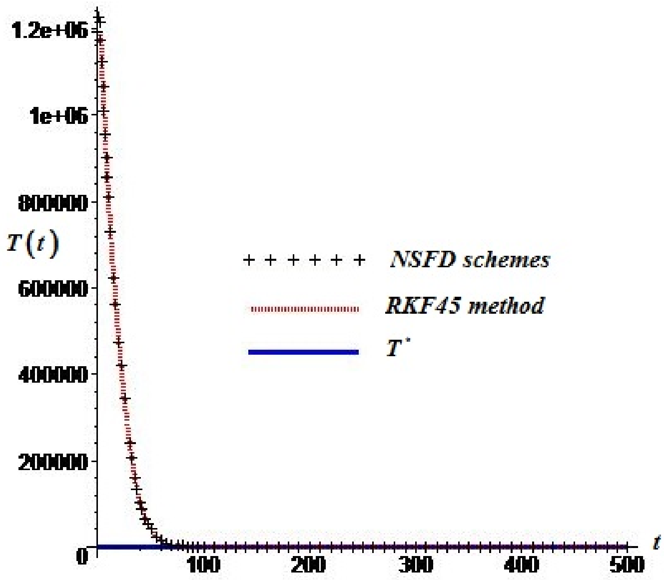


Figure 5. Variation of diabetics with complications receiving a treatment $T(t)$

NSFD method is a very effective method for the bigger step size. Table 1 compares the convergence of the methods. One can see the effectiveness of NSFD schemes from Table 1.

Table 1. Stability of equilibrium point E^* under application of different methods for different step size h

h	**Euler Method**	**Fourth Order Runge-Kutta Method**	**NSFD Schemes**
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.5	Convergence	Convergence	Convergence
1	Convergence	Convergence	Convergence
5	Convergence	Convergence	Convergence
7	Divergence	Convergence	Convergence
10	Divergence	Divergence	Convergence
25	Divergence	Divergence	Convergence
50	Divergence	Divergence	Convergence
100	Divergence	Divergence	Convergence

6. Conclusion

This paper presents the stability analysis of a mathematical model describing diabetes mellitus and its complications. The main aims of the study are to analyze the stability of the model and show the advantages of the NSFD method. Thus, the stability of the continuous model is analyzed, and it is concluded that the model is asymptotically stable. Moreover, the continuous model is discretized with the help of the NSFD method. Considering the Schur-Cohn criterion, it is concluded that the discrete model is asymptotically stable, too. The accuracy of the NSFD scheme is supported by comparing the numerical results with the RKF45 method. The compatibility of the numerical results can be seen through graphics. One of the advantages of the NSFD method is to be convergence for the bigger step sizes. The efficiency of the NSFD method for the bigger step size is presented in tabular form.

In future studies, the NSFD schemes for the linear and nonlinear models can be constructed, and their stability analysis can be performed using a similar technique. Moreover, since the NSFD method can be applied to the fractional order differential equations, fractional diabetes models can be solved by the NSFD method. In addition, stability analysis can be given.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] A. Boutayeb, E. H. Twizell, K. Achouayb, A. Chetouani, *A Mathematical Model for the Burden of Diabetes and Its Complications*, Biomedical Engineering Online 3 (2004) 1–8.

[2] V. O. Akinsola, T. O. Oluyo, *Analytic Solution of Mathematical Model of the Complications and Control of Diabetes Mellitus Using Fundamental Matrix Method*, Journal of Interdisciplinary Mathematics 23 (4) (2020) 877–884.

[3] V. O. Akinsola, T. O. Oluyo, *A Note on the Divergence of the Numerical Solution of the Mathematical Model for the Burden of Diabetes and Its Complications Using Euler Method*, International Journal of Mathematics and Computer Applications Research 5 (3) (2015) 93–100.

[4] V. O. Akinsola, T. O. Oluyo, *Mathematical Analysis with Numerical Solutions of the Mathematical Model for the Complications and Control of Diabetes Mellitus*, Journal of Statistics and Management Systems 22 (5) (2019) 845–869.

[5] M. AlShurbaji, L.A. Kader, H. Hannan, M. Mortula, G. A. Husseini, *Comprehensive Study of a Diabetes Mellitus Mathematical Model Using Numerical Methods with Stability and Parametric Analysis*, International Journal of Environmental Research and Public Health 20 (2) (2023) 939 23 pages.

[6] R. Vanitha, R. Porchelvi, *A Linear Population Model for Diabetes Mellitus*, Bulletin of Pure and Applied Sciences-Mathematics and Statistics 36 (2) (2017) 311–315.

[7] A. Boutayeb, A. Chetouani, A. Achouyab, E. H. Twizell, *A Nonlinear Population Model of Diabetes Mellitus*, Journal of Applied Mathematics and Computing 21 (2006) 127–139.

[8] S. R. de Oliveira, S. Raha, D. Pal, *Global Asymptotic Stability of a Nonlinear Population Model of Diabetes Mellitus*, in: S. Pinelas, T. Caraballo, P. Kloeden, J. Graef (Eds.), Differential and Difference Equations with Applications: ICDDEA 2017, Springer Proceedings in Mathematics and Statistics, Vol 230. Springer, Cham, 2018, pp. 351–357.

[9] W. Boutayeb, M. E. N. Lamlili, A. Boutayeb, M. Derouich, *The Dynamics of a Population of Healthy People, Pre-diabetics and Diabetics with and without Complications with Optimal Control*, in: A. El Oualkadi, F. Choubani, A. El Moussati (Eds.), Proceedings of the Mediterranean Conference on Information and Communication Technologies, Vol. 380 of *Lecture Notes in Electrical Engineering*, Springer, Cham, 2016, pp. 463–471.

[10] A. H. Permatasari, R. H. Tjahjana, T. Udjiani. *Global Stability for Linear System and Controllability for Nonlinear System in the Dynamics Model of Diabetics Population*, Journal of Physics: Conference Series, Vol. 1025 (1), IOP Publishing, 2018.

[11] P. Widyaningsih, R. C. Affan, D. R. S. Saputro, *A Mathematical Model for the Epidemiology of Diabetes Mellitus with Lifestyle and Genetic Factors*, Journal of Physics: Conference series, Vol. 1028 (1) IOP Publishing, 2018.

[12] P. O. Aye, *Stability Analysis of Mathematical Model for the Dynamics of Diabetes Mellitus and Its Complications in a Population*, Data Analytics and Applied Mathematics 3 (1) (2022) 20–27.

[13] P. O. Aye, K. A. Adeyemo, A. S. Oke, A. E. Omotoye, *Analysis of Mathematical Model for the Dynamics of Diabetes Mellitus and Its Complications*, Applied Mathematics and Computational Intelligence, 10 (1) (2021) 57–77.

[14] P. O. Aye, *Mathematical Analysis of the Effect of Control on the Dynamics of Diabetes Mellitus and Its Complications*, Earthline Journal of Mathematical Sciences 6 (2) (2021) 375–395.

[15] R. E. Mickens, Difference Equations Theory and Applications, Chapman and Hall, Atlanta, 1990.

[16] R. E. Mickens, Advances in the Applications of Nonstandard Finite Difference Schemes, Wiley-Interscience, Singapore, 2005.

[17] R. E. Mickens, Nonstandard Finite Difference Models of Differential Equations, World Scientific Publishing Company, New Jersey, 1993.

[18] R. E. Mickens, *Nonstandard Finite Difference Schemes for Differential Equations*, Journal of Difference Equations and Applications 8 (9) (2002) 823–847.

[19] R. E. Mickens, *Calculation of Denominator Functions for Nonstandard Finite Difference Schemes for Differential Equations Satisfying a Positivity Condition*, Numerical Methods for Partial Differential Equations 23 (3) (2006) 672–691.

[20] R. E. Mickens, *Exact Solutions to a Fnite-Difference Model of a Nonlinear Reaction-Advection Equation: Implications for Numerical Analysis*, Numerical Methods for Partial Differential Equations 5 (4) (1989) 313–325.

[21] K. C. Patidar, *On the Use of Nonstandard Finite Difference Methods*, Journal of Difference Equations and Applications 11 (8) (2005) 735–758.

[22] K. C. Patidar, *Nonstandard Finite Difference Methods: Recent Trends and Further Developments*, Journal of Difference Equations and Applications 22 (6) (2016) 817–849.

[23] O. Adekanye, T. Washington, *Nonstandard Finite Difference Scheme for a Tacoma Narrows Bridge Model*, Applied Mathematical Modelling 62 (2018) 223–236.

[24] R. Anguelov, T. Berge, M. Chapwanya, J. K. Djoko, P. Kama, J. S. Lubuma, Y. Terefe, *Nonstandard Finite Difference Method Revisited and Application to the Ebola Virus Disease Transmission Dynamics*, Journal of Difference Equations and Applications 26 (6) (2020) 818–854.

[25] A. J. Arenas, G. Gonzalez-Parra, B. M. Chen-Charpentier, *Construction of Nonstandard Finite Difference Schemes for the SI and SIR Epidemic Models of Fractional Order*, Mathematics and Computers in Simulation 121 (2016) 48–63.

[26] D. Baleanu, S. Zibaei, M. Namjoo, A. Jajarmi, *A Nonstandard Finite Difference Scheme for the Modeling and Nonidentical Synchronization of a Novel Fractional Chaotic System*, Advances in Difference Equations 2021 (1) (2021) Article Number 308 19 pages.

[27] Q. A. Dang, M. T. Hoang, *Lyapunov Direct Method for Investigating Stability of Nonstandard Finite Difference Schemes for Metapopulation models*, Journal of Difference Equations and Applications 24 (1) (2018) 15–47.

[28] Q. A. Dang, M. T. Hoang, *Numerical Dynamics of Nonstandard Finite Difference Schemes for a Computer Virus Propagation Model*, International Journal of Dynamics and Control 8 (3) (2020) 772–778.

[29] M. Kocabıyık, N. Özdoğan, M. Y. Ongun, *Nonstandard Finite Difference Scheme for a Computer Virus Model*, Journal of Innovative Science and Engineering 4 (2) (2020) 96–108.

[30] N. Özdoğan, M. Y. Ongun, *Dynamical Behaviours of a Discretized Model with Michaelis-Menten Harvesting Rate*, Journal of Universal Mathematics 5 (2) (2022) 159–176.

[31] M. Kocabıyık, M. Y. Ongun, *Construction a Distributed Order Smoking Model and Its Nonstandard Finite Difference Discretization*, AIMS Mathematics 7 (3) (2021) 4636—4654.

[32] M. Y. Ongun, D. Arslan, *Explicit and Implicit Schemes for Fractional–order Hantavirus Model*, Iranian Journal of Numerical Analysis and Optimization 8 (2) (2018) 75–94.

[33] M. S. Shabbir, Q. Din, M. Safeer, M. A. Khan, K. Ahmad, *A Dynamically Consistent Nonstandard Finite Difference Scheme for a Predator–prey Model*, Advances in Difference Equations 2019 (1) (2019) 1–17.

[34] S. Vaz, D. F. Torres, *A Dynamically-consistent Nonstandard Finite Difference Scheme for the SICA Model*, Mathematical Biosciences and Engineering 18 (4) (2021) 4552–4571.

[35] O. Egbelowo, C. Harley, B. Jacobs, *Nonstandard Finite Difference Method Applied to a Linear Pharmacokinetics Model*, Bioengineering 4 (2) (2017) 40 21 pages.

[36] I. U. Khan, S. Mustafa, A. Shokri, S. Li, A. Akgül, A. Bariq, *The Stability Analysis of a Nonlinear Mathematical Model for Typhoid Fever Disease*, Scientific Reports 13 (1) (2023) Article Number 15284 15 pages.

[37] F. Özköse, M. Yavuz, *Investigation of Interactions between COVID-19 and Diabetes with Hereditary Traits Using Real Data: A Case Study in Turkey*, Computers in Biology and Medicine 141 (2022) 105044 22 pages.

[38] A. Atangana, S. İğret Araz, *Mathematical Model of COVID-19 Spread in Turkey and South Africa: Theory, Methods, and Applications*, Advances in Difference Equations 2020 (1) (2020) 1–89.

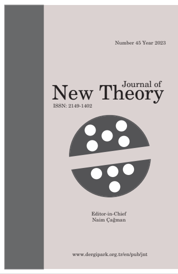
[39] Ö. A. Gümüs, Q. Cui, G. M. Selvam, A. Vianny, *Global Stability and Bifurcation Analysis of a Discrete Time SIR Epidemic Model*, Miskolc Mathematical Notes 23 (1) (2022) 193–210.

[40] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York, 2003.

[41] S. Elaydi, An Introduction to Difference Equations, Springer, New York, 1999.

Journal of
New Theory
ISSN: 2149-1402

45 (2023) 120-130
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Spacelike Ac-Slant Curves with Non-Null Principal Normal in Minkowski 3-Space

Hasan Altınbaş[1]

Article Info
Received: 6 Dec 2023
Accepted: 28 Dec 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1401001
Research Article

Abstract — In this paper, we define a spacelike ac-slant curve whose scalar product of its acceleration vector and a unit non-null fixed direction is a constant in Minkowski 3-space. Furthermore, we give a characterization depending on the curvatures of the spacelike ac-slant curve. After that, we get the relationship between a spacelike ac-slant curve and several distinct types of curves, such as spacelike Lorentzian spherical curves, spacelike helices, spacelike slant helices, and spacelike Salkowski curves, enhancing our understanding of its geometric properties in Minkowski 3-space. Finally, we used Mathematica, a symbolic computation software, to support the notions of an ac-slant curve with attractive images.

Keywords *Acceleration, helices, slant helices, Minkowski 3-spaces*

Mathematics Subject Classification (2020) 53A04, 53A35

1. Introduction

The motion of the object has a route that looks like a curve in space. The position of the object at time t is represented by the position vector of the curve at parameter t in the space. The first, second, and third derivatives of the curve are represented by the object's velocity, acceleration, and jerk vectors at any time t, respectively.

In kinematics and classical mechanics, which deal with the motion of bodies, the physical vector quantities are significant. The magnitude of velocity is known as speed. The rate at which velocity changes is called acceleration. The direction of acceleration is determined by the total force applied to the object. Newton's Second Law was first articulated in the seventeenth century by the English mathematician and scientist Sir Isaac Newton, who also described the magnitude of acceleration. Additionally, the jerk is the acceleration's rate of change [1–3].

In Euclidean 3-space, a regular curve α is said to be a helix if the tangent vector of α makes the fixed angle ϕ with a fixed direction which is the axis of helix where $\phi \in (0,\pi) \setminus \frac{\pi}{2}$. Moreover, the ratio τ/κ is a constant if and only if it is a general helix [4, 5]. A regular curve α is called a slant helix if its principal normal vector of α makes the fixed angle ϕ with a fixed direction which is the axis where ϕ is a constant [6]. If a regular curve α has nonconstant torsion τ but constant curvature κ, then α is called a Salkowski curve [7].

In Minkowski 3-space, A curve is called a helix (resp. slant helix) if the scalar product of its tangent

[1]hasan.altinbas@ahievran.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Arts and Sciences, Kırşehir Ahi Evran University, Kırşehir, Türkiye

(resp. principal normal) vector and fixed direction is constant [8, 9]. Furthermore, Ali [10] modified the definition of spacelike Salkowski curves with spacelike or timelike principle normal in this space with an explicit parametrization. These special curves are studied in different ambient spaces by some authors [11–17].

The plan of this paper is as follows: In section 2, we review the fundamental theory of curves in Minkowski 3-space. In section 3, we define a spacelike ac-slant curve whose scalar product of its acceleration vector and a non-null fixed direction is a constant. First, we provide a characterization based on the torsion and curvature of a spacelike ac-slant curve. Later, we get to the conclusion that when the ac-slant curve is a helix, either the acceleration vector is orthogonal to its axis or the magnitude of its velocity vector is a linear function. Later on, a unit speed curve with constant magnitude acceleration is an ac-slant curve if and only if it is a slant helix. Moreover, a unit speed curve is only a spacelike ac-slant curve if and only if it is a Salkowski curve when the magnitude of the acceleration is equal to one (i.e. $\kappa = 1$).

2. Preliminaries

In this section, we provide basic facts for Minkowski 3-space. For more detail and background, see [8, 18, 19].

Let $\mathbb{E}_1^3 = \left(\mathbb{R}^3\left(t, x, y\right), g\right)$ be a Minkowski 3-space where $g = -dt^2 + dx^2 + dy^2$ denotes the standard metric and (t, x, y) is the connanical coordinates in 3-dimensional real vector space $\mathbb{R}^3$. A vector u in $\mathbb{E}_1^3$ is called spacelike if $g\left(u, u\right) > 0$ or $u = 0$, timelike if $g\left(u, u\right) < 0$, and null if $g\left(u, u\right) = 0$ and $u \neq 0$, respectively. Moreover, the norm of u is defined by $\|u\| = \sqrt{|g(u, u)|}$. Furthermore, u is a unit vector if $g\left(u, u\right) = \pm 1$.

A curve $\alpha(t)$ is called spacelike, timelike, or null if velocity vector $v = \alpha'(t)$ of $\alpha(t)$ are spacelike, timelike, or null in $\mathbb{E}_1^3$ for each parameter t, respectively. Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve α in $\mathbb{E}_1^3$. Then, T, N, and B are the tangent, the principal normal, and the binormal vector fields, respectively. Besides, Frenet-Serret formulae are provided as follows:

$$T' = \varepsilon\nu\kappa N, \quad N' = -\nu\kappa T - \varepsilon\nu\tau B, \quad B' = -\varepsilon\nu\tau N \tag{1}$$

where κ and τ are curvatures of the curve α and

g	T	N	B
T	1	0	0
N	0	ε	0
B	0	0	$-\varepsilon$

such that $\varepsilon = \pm 1$.

In here, $\nu = g\left(\alpha'\left(t_0\right), \alpha'\left(t_0\right)\right)$ is called speed of α at $t_0 \in I$. Moreover, if $\nu = 1$, for all $t \in I$, then α is a unit speed curve. Lorentzian unit sphere is $S_1^2 = \left\{x \in \mathbb{E}_1^3 : g\left(x, x\right) = 1\right\}$. A curve that lies on the Lorentzian unit sphere is called a Lorentzian spherical curve.

From a physical point of view, the motion of particle P along the curve α at time t is correspond to the position vector of α. Then, it is widely known that the first, second, and third derivatives of α concerning time determine the velocity vector $v(t)$, acceleration vector $\mathbf{a}(t)$, and jerk vector $j(t)$, respectively. These vectors are determined by Equations 1 as follows:

$$\begin{aligned} v &= \alpha' = \nu T \\ \mathbf{a} &= \alpha'' = \nu' T + \varepsilon \nu^2 \kappa N \\ j &= \alpha''' = \left(\nu'' - \varepsilon \nu^3 \kappa^2\right) T + \left(3\varepsilon \nu' \nu \kappa + \varepsilon \nu^2 \kappa'\right) N - \nu^3 \kappa \tau B \end{aligned} \tag{2}$$

3. Spacelike Ac-Slant Curves with Non-Null Principal Normal

This section provides new curves, called spacelike ac-slant curves in Minkowski 3-space. Additionally, we characterize these curves.

Definition 3.1. A spacelike curve α is called a spacelike ac-slant curve whose inner product of a unit non-null fixed direction u, called axis of spacelike ac-slant curve, and acceleration vector $\mathbf{a}$ of the curve is constant, i.e., $g(\mathbf{a}, u) = c$, in Minkowski 3−space.

Remark 3.2. Let α be spacelike curve in $\mathbb{E}_1^3$. Then, α is a spacelike ac-slant helix if and only if the jerk vector of α is orthogonal to its axis u, i.e., $g(j, u) = 0$.

Theorem 3.3. Let α be a spacelike curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ in Minkowski 3−space. Then, α is a spacelike ac-slant curve if and only if

$$\eta_1^2 + \varepsilon \eta_2^2 - \varepsilon \eta_3^2 = \epsilon \tag{3}$$

such that

$$\eta_1 = \frac{\frac{1}{\nu}\left(\varepsilon \frac{\tau}{\kappa} - f\right) - \left(\frac{1}{\varepsilon \nu \tau}\left(\frac{1}{\nu^2 \kappa}\right)'\right)' + \left(\frac{1}{\varepsilon \nu \tau}\left(\frac{\nu'}{\nu^3 \kappa}\right)\right)'}{f' + \frac{\nu'}{\nu}\left(\varepsilon \frac{\tau}{\kappa} - f\right)} \tag{4}$$

$$\eta_2 = \frac{1}{\nu^2 \kappa} - \frac{\nu'}{\nu^2 \kappa}\eta_1 \tag{5}$$

and

$$\eta_3 = f\eta_1 + \frac{1}{\varepsilon \nu \tau}\left(\left(\frac{1}{\nu^2 \kappa}\right)' - \frac{\nu'}{\nu^3 \kappa}\right) \tag{6}$$

where c is a nonzero constant, $\epsilon = \pm 1$, and

$$f = \frac{\kappa}{\tau} - \frac{1}{\varepsilon \nu \tau}\left(\left(\frac{\nu'}{\nu^2 \kappa}\right)' - \frac{(\nu')^2}{\nu^3 \kappa}\right)$$

Proof.

Assume that α is a spacelike ac-slant curve with timelike or spacelike axis u. By Definition 3.1, there exist a constant $c = g(\mathbf{a}, u)$ and differentiable functions λ_i such that

$$u = \lambda_1 T + \lambda_2 N + \lambda_3 B \tag{7}$$

By using Equation 2 and 7,

$$\lambda_2 = \frac{c}{\nu^2 \kappa} - \frac{\nu'}{\nu^2 \kappa}\lambda_1 \tag{8}$$

After differentiating of Equation 7 and using Equation 8,

$$\lambda_1' - \frac{c}{\nu} + \frac{\nu'}{\nu}\lambda_1 = 0 \tag{9}$$

$$\varepsilon \nu \kappa \lambda_1 + c\left(\frac{1}{\nu^2 \kappa}\right)' - \left(\frac{\nu'}{\nu^2 \kappa}\right)'\lambda_1 - \frac{\nu'}{\nu^2 \kappa}\lambda_1' - \varepsilon \nu \tau \lambda_3 = 0 \tag{10}$$

and

$$\lambda_3' - \varepsilon \frac{c\tau}{\nu \kappa} + \varepsilon \frac{\nu' \tau}{\nu \kappa}\lambda_1 = 0 \tag{11}$$

By substituting Equation 9 in Equation 10,

$$f\lambda_1 + \frac{c}{\varepsilon\nu\tau}\left(\left(\frac{1}{\nu^2\kappa}\right)' - \frac{\nu'}{\nu^3\kappa}\right) - \lambda_3 = 0 \tag{12}$$

where

$$f = \frac{\kappa}{\tau} - \frac{1}{\varepsilon\nu\tau}\left(\left(\frac{\nu'}{\nu^2\kappa}\right)' - \frac{(\nu')^2}{\nu^3\kappa}\right)$$

After differentiating of Equation 12, by using Equations 9 and 11,

$$\lambda_1 = \frac{\frac{c}{\nu}\left(\varepsilon\frac{\tau}{\kappa} - f\right) - \left(\frac{c}{\varepsilon\nu\tau}\left(\frac{1}{\nu^2\kappa}\right)'\right)' + \left(\frac{c}{\varepsilon\nu\tau}\left(\frac{\nu'}{\nu^3\kappa}\right)\right)'}{f' + \frac{\nu'}{\nu}\left(\varepsilon\frac{\tau}{\kappa} - f\right)} \tag{13}$$

Clearly, from Equation 12,

$$\lambda_3 = f\lambda_1 + \frac{c}{\varepsilon\nu\tau}\left(\left(\frac{1}{\nu^2\kappa}\right)' - \frac{\nu'}{\nu^3\kappa}\right) \tag{14}$$

Hence, by using Equations 8, 13, and 14, it is clear that there exist differentiable functions $\eta_i = \frac{1}{c}\lambda_i$ which is satisfying Equation 3, for $i \in \{1, 2, 3\}$

Conversely, let α be a spacelike curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$. Assume that there exists a unit non-null fixed direction u provided by Equation 7 where differentiable functions λ_i are presented by Equations 4-6. Then, it is observed that the scalar product of acceleration vector $\mathbf{a}$ is given by Equation 2 of α, and u is equal to a nonzero constant c. Thus, α is an ac-slant curve with the axis u. □

Thus, we conclude the following Corollaries from Theorem 3.3.

Corollary 3.4. Let α be a unit speed spacelike non helix curve with curvatures κ and τ in $\mathbb{E}_1^3$. Then, α is a spacelike ac-slant curve if and only if

$$\left(\frac{\left(\frac{\tau}{\kappa}\right)m}{\left(\frac{\tau}{\kappa}\right)'}\right)^2 + \varepsilon\frac{1}{\kappa^2} - \varepsilon\left(\frac{1}{\varepsilon\tau}\left(\frac{1}{\kappa}\right)' + \frac{m}{\left(\frac{\tau}{\kappa}\right)'}\right)^2 = \epsilon$$

where $m = 1 - \varepsilon\left(\frac{\tau}{\kappa}\right)^2 + \varepsilon\frac{\tau}{\kappa}\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'$.

Corollary 3.5. Let α be a unit speed spacelike ac-slant curve with curvature $\kappa = 1$ in $\mathbb{E}_1^3$. Then,

$$\tau(t) = \pm\frac{\sqrt{\frac{c^2}{c^2-\epsilon\varepsilon}}\,t}{\sqrt{1 + \varepsilon\frac{c^2}{c^2-\epsilon\varepsilon}t^2}}$$

where c is a nonzero constant.

Example 3.6. The curve

$$\alpha(t) = \left(\frac{1}{4}(t+2)^2, \frac{1}{4}(t+2)^2\sin t, \frac{1}{4}(t+2)^2\cos t\right)$$

is a spacelike curve in $\mathbb{E}_1^3$ since $g(\alpha'(t), \alpha'(t)) = \frac{1}{16}(t+2)^4 > 0$, for $t \in \mathbb{R}$. Moreover, the curve α lies on the surface $y^2 + z^2 = x^2$, and its acceleration vector is

$$\mathbf{a}(t) = \left(\frac{1}{2}, \frac{\sin t}{2} + (t+2)\cos t - \frac{1}{4}(t+2)^2\sin t, \frac{\cos t}{2} - (t+2)\sin t - \frac{1}{4}(t+2)^2\cos t\right)$$

in $\mathbb{E}_1^3$. Furthermore, α is a spacelike ac-slant curve with the timelike axis $u = (1, 0, 0)$ such that $c = -\frac{1}{2}$ (see Figure 1).

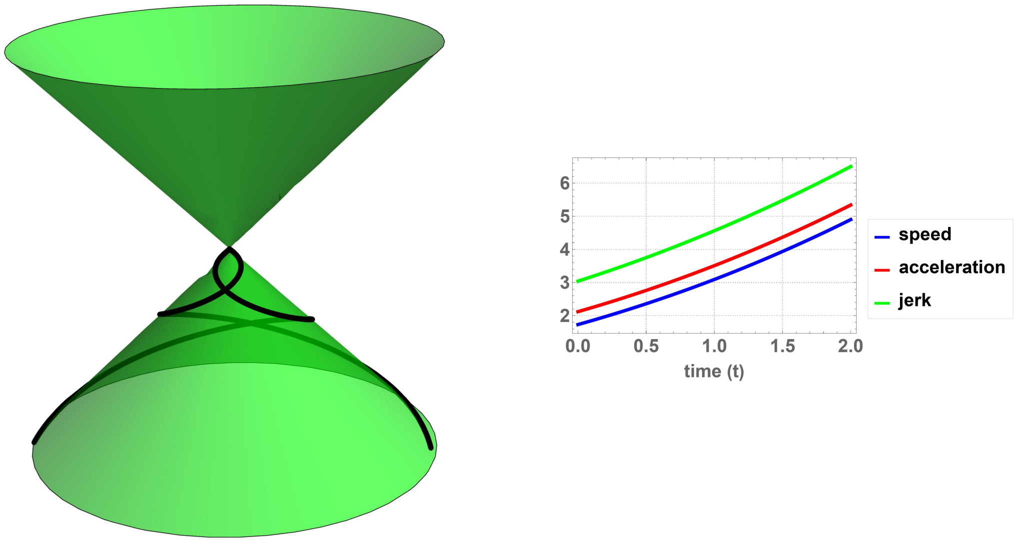


Figure 1. A spacelike ac-slant curve lying on the surface $y^2 + z^2 = x^2$

Example 3.7. The curve

$$\alpha(t) = \left(\log(\cos t), \frac{t^2}{2\sqrt{3}}, \log\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right) - \log\left(\cos\frac{t}{2} - \sin\frac{t}{2}\right)\right)$$

is a spacelike curve in $\mathbb{E}_1^3$ since $g\left(\alpha'\left(t\right), \alpha'\left(t\right)\right) = \frac{1}{3}\left(t^2 + 3\right) > 0$, for $t \in \mathbb{R}$. Moreover, the acceleration vector of α is

$$\mathbf{a}(t) = \left(-\sec^2 t, \frac{1}{\sqrt{3}}, \tan t \sec t\right)$$

in $\mathbb{E}_1^3$. Furthermore, α is a spacelike ac-slant curve with the spacelike axis $u = (0, 1, 0)$ such that $c = \frac{1}{\sqrt{3}}$ (see Figure 2).

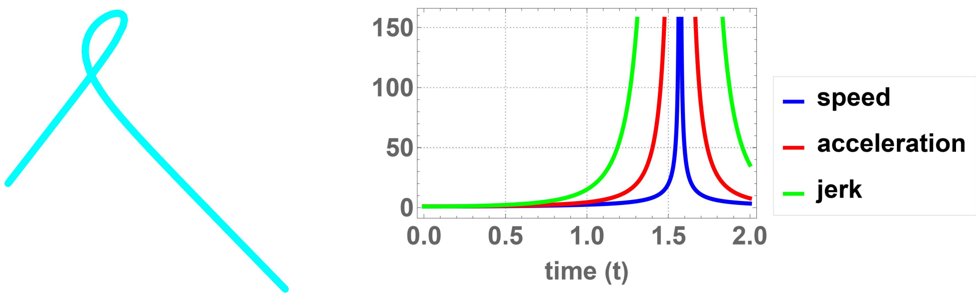


Figure 2. A spacelike ac-slant curve

Lemma 3.8. Let ν be a constant function and $c = 0$. Then, $\boldsymbol{\gamma}$ is a spacelike ac-slant curve if and only if

$$\lambda_1^2 - \varepsilon\lambda_3^2 = \epsilon \quad \text{and} \quad \lambda_2 = 0 \tag{15}$$

where λ_1 and λ_3 are nonzero constants.

PROOF.

Assume that α is a spacelike ac-slant curve with timelike or spacelike axis u. By Definition 3.1, there exist a constant $c = g\left(\mathbf{a}, u\right)$ and differentiable functions λ_i such that

$$u = \lambda_1 T + \lambda_2 N + \lambda_3 B \tag{16}$$

By using assumption ν is a constant function and $c=0$, we obtain $\lambda_2=0$ by using Equation 2. After differentiating of Equation 16 and using the fact of $\lambda_2=0$, we have following differential equations

$$\lambda_1'=0$$

$$\varepsilon\nu\kappa\lambda_1-\varepsilon\nu\tau\lambda_3=0$$

and

$$\lambda_3'=0$$

It can be observed that Equation 15 is satisfied since u is a unit fixed direction.

Conversely, let α be a spacelike curve with Frenet apparatus $\{T,N,B,\kappa,\tau\}$. Suppose that there exists a unit fixed direction u presented by Equation 16 where differentiable functions λ_i are provided by Equation 15. Then, it is clear that $g(\mathbf{a},u)=0$ where ν is constant function. □

Remark 3.9. It can be observed that u is not exist if $c=0$ and ν is a nonconstant function.

In the light of [20], we can provide following Corollary.

Corollary 3.10. Let α be a spacelike Lorentzian spherical curve with radius $r\in\mathbb{R}^+$ in Minkowski 3−space. Then, following equations are satisfied:

$$\left(\frac{1}{\kappa}\right)^2-\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=\varepsilon r^2$$

and

$$\frac{\tau}{\kappa}=\frac{1}{\nu}\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)'$$

Theorem 3.11. Let α be a unit speed spacelike spherical curve with radius $r\in\mathbb{R}^+$, not a helix in $\mathbb{S}_1^2$. Then, α is a spacelike ac-slant curve if and only if

$$\left(\frac{1}{\left(\frac{\tau}{\kappa}\right)'}\right)^2\left(\frac{\tau^2}{\kappa^2}-\varepsilon\right)-2\frac{\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'}{\left(\frac{\tau}{\kappa}\right)'}=\frac{1-r^2}{c^2} \tag{17}$$

where c is a nonzero constant.

Proof.
Let α be a unit speed spacelike spherical curve in $\mathbb{S}_1^2$. Then, Corollary 3.10 is satisfied. Assume that α is a spacelike ac-slant curve with a non-zero constant $c=g(\mathbf{a},u)$. Then, by using Corollary 3.10 join with Equations 4-6, there exist differentiable functions λ_i such that

$$u=\lambda_1T+\lambda_2N+\lambda_3B$$

where

$$\lambda_1=c\frac{\frac{\tau}{\kappa}}{\left(\frac{\tau}{\kappa}\right)'}$$

$$\lambda_2=c\frac{1}{\kappa}$$

and

$$\lambda_3=c\left(\frac{1}{\varepsilon\tau}\left(\frac{1}{\kappa}\right)'+\frac{1}{\left(\frac{\tau}{\kappa}\right)'}\right)$$

Using Corollary 3.10, since u is a unit fixed direction, we obtain Equation 17. Conversely, the proof is clear. □

Moreover, we get the following characterization for ac-slant curves from Lemma 3.8.

Remark 3.12. The curve α is a spacelike ac-slant curve with the non-null axis u satisfying Equation 15 if and only if α is a helix with the same axis u.

The proof is straightforward from Equation 15.

Theorem 3.13. Let α be a spacelike helix with axis u in $\mathbb{E}_1^3$. Then, α is a spacelike ac-slant curve with the axis u if and only if the magnitude of the velocity, i.e., $|v| = |\alpha'| = \nu$, is a linear function concerning parameter of α.

Proof.
Let α be a spacelike curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ in $\mathbb{E}_1^3$. We assume that α is a spacelike helix with axis u. Then, there exists a constant c_1 such that

$$g(T, u) = c_1 \tag{18}$$

After differentiating of Equation 18 and using Equation 1,

$$g(N, u) = 0 \tag{19}$$

Suppose that α is a spacelike ac-slant curve with the same axis u which is provided by Equation 16. Then, $\lambda_1 = c_1$ by using Equations 18 and 19. By using Equation 8,

$$\lambda_2 = \frac{1}{\nu^2\kappa}(c - \nu' c_1) = 0$$

Therefore, ν' is a constant.

Conversely, suppose that α is a spacelike helix with axis u in $\mathbb{E}_1^3$ and ν is a linear function with respect to the parameter of α. Then, there exists a constant c_1 such that $u = c_1 T + \sqrt{1 - \varepsilon c_1^2} B$. Thus, by using Equation 2, $g(\mathbf{a}, u) = \nu' c_1 = const$. Hence, α is a spacelike ac-slant curve with the axis u. □

Example 3.14. The curve

$$\alpha(t) = \left(t\cosh t - \sinh t, t\sinh t - \cosh t, \frac{t^2}{2}\right)$$

is a spacelike curve in $\mathbb{E}_1^3$ since $g(\alpha'(t), \alpha'(t)) = 2t^2 > 0$, for $t \in \mathbb{R}$. Moreover, the curve α lies on the surface $y^2 - x^2 + 2z = 1$, and its acceleration vector is

$$\mathbf{a}(t) = (\sinh t + t\cosh t, t\sinh t + \cosh t, 1)$$

in $\mathbb{E}_1^3$. Furthermore, α is an ac-slant curve with the spacelike axis $u = (0, 0, 1)$ such that $c = 1$ (see Figure 3).

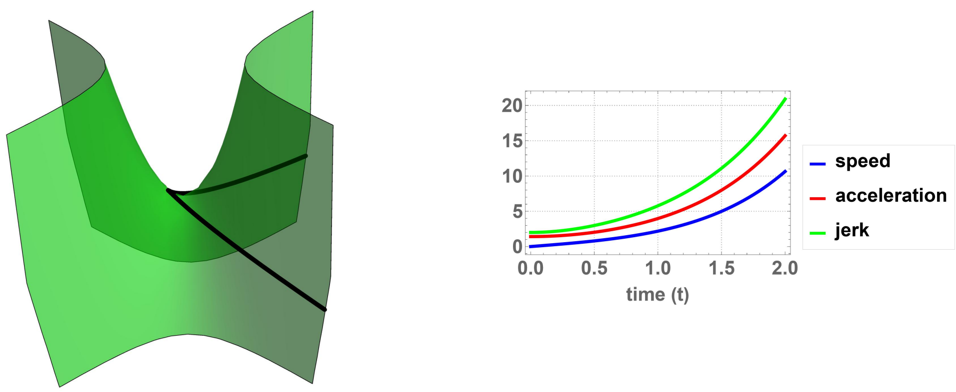


Figure 3. A spacelike ac-slant helix lying on the surface $y^2 - x^2 + 2z = 1$

Moreover, tangent vector field of $\alpha(t)$ is

$$T = \left(\frac{\sinh t}{\sqrt{2}}, \frac{\cosh t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Then, $\alpha(t)$ is a spacelike helix since $g(T, u) = \frac{1}{\sqrt{2}}$ and $\frac{\tau}{\kappa} = 1$. Furthermore, Theorem 3.13 is satisfied, i.e., $|v(t)| = \sqrt{2}t$ is a lineer function.

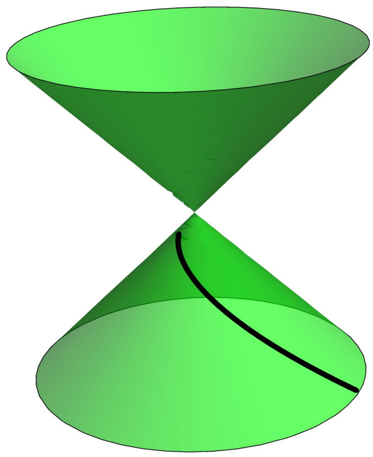
Example 3.15. The curve

$$\alpha(t) = \left(t^2, t^2 \sin\left(\log\left(t^2\right)\right), t^2 \cos\left(\log\left(t^2\right)\right)\right)$$

is a spacelike curve in $\mathbb{E}_1^3$ since $g(\alpha'(t), \alpha'(t)) = 4t^2 > 0$, for $t \in \mathbb{R}$. Moreover, the curve α lies on the surface $y^2 + z^2 = x^2$, and its acceleration vector is

$$\mathbf{a}(t) = (2, 6\cos(2\log(t)) - 2\sin(2\log(t)), -2(3\sin(2\log(t)) + \cos(2\log(t))))$$

in $\mathbb{E}_1^3$. Furthermore, α is an ac-slant curve with the timelike axis $u = (1, 0, 0)$ such that $c = -2$ (see Figure 4).



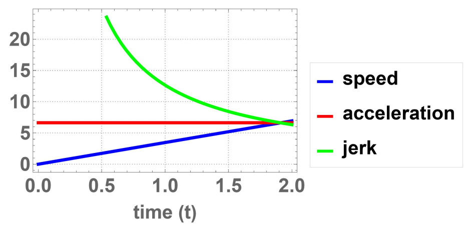


Figure 4. A spacelike ac-slant helix lying on the surface $y^2 - z^2 = x^2$

Moreover, tangent vector field of $\alpha(t)$ is

$$T = (1, \sin(2\log(t)) + \cos(2\log(t)), \cos(2\log(t)) - \sin(2\log(t)))$$

Then, $\alpha(t)$ is a spacelike helix since $g(T, u) = -1$ and $\frac{\tau}{\kappa} = \frac{1}{\sqrt{2}}$. Furthermore, Theorem 3.13 is satisfied, i.e., $|v(t)| = 2t$ is a lineer function.

Corollary 3.16. Let α be a unit speed spacelike curve with a constant magnitude of acceleration, i.e., with constant curvature, in $\mathbb{E}_1^3$. Then, α is a spacelike ac-slant curve if and only if α is a slant helix.

Proof.
Suppose that α is a unit speed spacelike curve with nonzero constant curvature $\kappa = \kappa_0$. Then, $\mathbf{a} = \varepsilon\kappa_0 N$. Hence, the proof is clear. □

Example 3.17. The curve

$$\alpha(t) = \left(t^2, \sqrt{t^4 + 1}\cos t, \sqrt{t^4 + 1}\sin t\right)$$

spacelike curve in $\mathbb{E}_1^3$ since $g(\alpha'(t), \alpha'(t)) > 0$, for $t \in (1, \infty)$. Moreover, the curve α lies on the

Lorentzian sphere $y^2+z^2-x^2=1$, and its acceleration vector is

$$\mathbf{a}(t)=\begin{pmatrix} 2, \\ \dfrac{-4\left(t^4+1\right)t^3\sin t-\left(t^8-2t^6+2t^4-6t^2+1\right)\cos t}{\left(t^4+1\right)^{3/2}}, \\ \dfrac{4\left(t^7+t^3\right)\cos t-\left(t^8-2t^6+2t^4-6t^2+1\right)\sin t}{\left(t^4+1\right)^{3/2}} \end{pmatrix}$$

in $\mathbb{E}_1^3$. Furthermore, α is an ac-slant curve with the timelike axis $u=(1,0,0)$ such that $c=-2$ (see Figure 5).

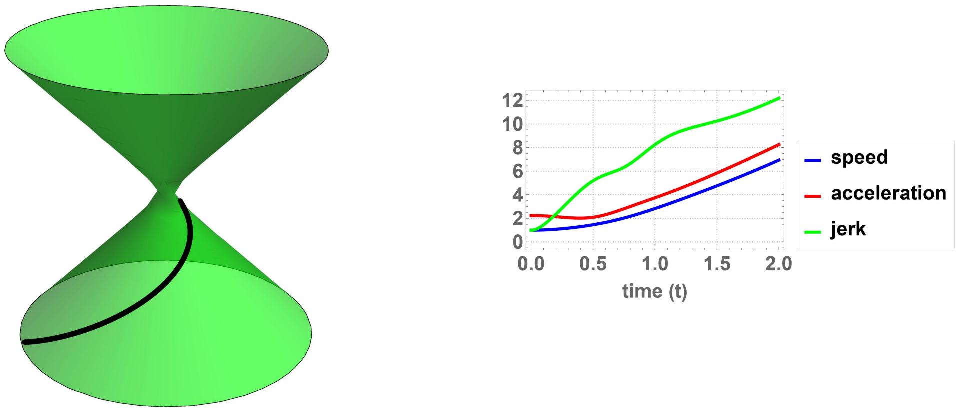


Figure 5. A spacelike Lorentzian spherical ac-slant helix lies on the surface $y^2+z^2-x^2=1$

Moreover, Corollary 3.10 is satisfied since α is spacelike Lorentzian sphere.

In the light of [10], we can provide the following lemma.

Lemma 3.18. Let α be a unit speed spacelike curve with non-null principal normal vector field with $\kappa=1$. Its normal vector N makes a constant hyperbolic angle ϕ with a fixed straight line in $\mathbb{E}_1^3$ if and only if $\tau(s)=\pm\frac{s}{\sqrt{\varepsilon(s^2-\tanh^2\phi)}}$.

Corollary 3.19. Let α be a unit speed spacelike curve with non-null principal normal vector with $\kappa=1$. Then, α is a spacelike ac-slant curve if and only if α is a Salkowski curve.

Proof.
Assume that α is a unit speed spacelike ac-slant curve with $\kappa=1$. Then, α is a Salkowski curve by Corollary 3.16 and Lemma 3.18. Conversely, α is a unit speed spacelike Salkowski curve with $\kappa=1$. Then, α is a spacelike ac-slant curve by Lemma 3.18 and Corollary 3.4. □

Example 3.20. The curve

$$\alpha(t)=\begin{pmatrix} t^2-\dfrac{3}{8}, \\ -\dfrac{4t\left(\sqrt{3-2\sqrt{3}t}\sqrt{3-4t^2}+3t\sqrt{2\sqrt{3}t+3}\right)+3\sqrt{2\sqrt{3}t+3}}{15\sqrt{2}}, \\ \dfrac{t^2 12\sqrt{3-2\sqrt{3}t}-4t\sqrt{2\sqrt{3}t+3}\sqrt{3-4t^2}+3\sqrt{3-2\sqrt{3}t}}{15\sqrt{2}} \end{pmatrix}$$

is a unit speed spacelike curve in $\mathbb{E}_1^3$ since $g\left(\alpha'(t),\alpha'(t)\right)=1>0$, for $t\in\mathbb{R}$. Moreover, the curve α

lies on the surface

$$\left(\frac{x+\frac{9}{8}}{\frac{\sqrt{15}}{2\sqrt{2}}}\right)^2 - \frac{y^2}{2} - \frac{z^2}{2} = \frac{6}{25}$$

and α is a spacelike ac-slant curve with timelike axis $u = (1,0,0)$ since $g(\mathbf{a}, u) = -2$ (see Figure 6).

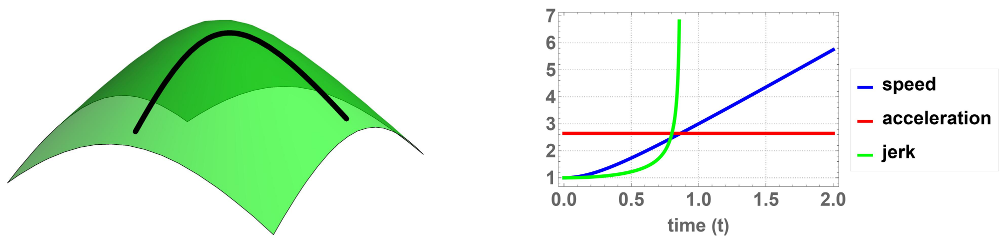


Figure 6. A spacelike ac-slant helix, Salkowski and slant helix lying on the surface $\left(\frac{x+\frac{9}{8}}{\frac{\sqrt{15}}{2\sqrt{2}}}\right)^2 - \frac{y^2}{2} - \frac{z^2}{2} = \frac{6}{25}$

Moreover, $g(N, u) = 2$. Thus, α is also a spacelike slant helix. Furthermore, since $\kappa = 1$, Corollary 3.16 is satisfied. Hence, α is a Salkowski curve. Further, Corollary 3.5 is satisfied and $\tau(t) = -\frac{2t}{\sqrt{3-4t^2}}$

4. Conclusion

Acceleration helps us understand the motion state of an object and aids in controlling that motion. Moreover, acceleration is a fundamental parameter for comprehending object interactions and explaining physical events. This comprehensive study contributes to the theoretical foundation of spacelike ac-slant curves and demonstrates their connections to well-known curves in Minkowski 3-space. We believe further investigation of spacelike ac-slant curves applies to other spaces.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] S. H. Schot, *Jerk: The Time Rate of Change of Acceleration*, American Journal of Physics 46 (11) (1978) 1090–1094.

[2] H. Bondi, R. J. Seeger, *Relativity and Common Sense: A New Approach to Einstein*, American Journal of Physics 34 (4) (1966) 372–372.

[3] H. Crew, The Principles of Mechanics, BiblioBazaar, Boston, 2008.

[4] M. A. Lancret, *Memoire Sur les Courbes á Double Courbure*, Memoires Presentes a Institut (1806) 416–454.

[5] D. J. Struik, Lectures on Classical Differential Geometry, Courier Corporation, London, 1961.

[6] S. Izumiya and N. Takeuchi, *New Special Curves and Developable Surfaces*, Turkish Journal of Mathematics 28 (2) (2004) 153–164.

[7] E. Salkowski, *Zur Transformation Von Raumkurven*, Mathematische Annalen 66 (4) (1909) 517–557.

[8] R. López, *Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space*, International Electronic Journal of Geometry 7 (1) (2014) 44–107.

[9] A. T. Ali, R. López, *Slant Helices in Minkowski Space* E_1^3, Journal of the Korean Mathematical Society 48 (1) (2011) 159–167.

[10] A. T. Ali, *Spacelike Salkowski and Anti-Salkowski Curves with a Spacelike Principal Normal in Minkowski 3-Space*, International Journal of Open Problems in Computer Science and Mathematics 2 (3) (2009) 451–460.

[11] E. Nesovic, U. Ozturk, E. B. Koc Ozturk, *On Non-Null Relatively Normal-Slant Helices in Minkowski 3-Space*, Filomat 36 (6) (2022) 2051–2062.

[12] S. Kızıltug, S. Kaya, O. Tarakcı, *The Slant Helices According to Type-2 Bishop Frame in Euclidean 3-Space*, International Journal of Pure and Applied Mathematics 2 (2013) 211-222.

[13] B. Bukcu, M. K. Karacan, *The Slant Helices According to Bishop Frame of the Spacelike Curve in Lorentzian Space*, Journal of Interdisciplinary Mathematics 12 (5) (2009) 691-700.

[14] H. Altinbas, M. Mak, B. Altunkaya, L. Kula, *Mappings That Transform Helices From Euclidean Space to Minkowski Space*, Hacettepe Journal of Mathematics and Statistics 51 (5) (2022) 1333–1347.

[15] K. Ilarslan, *Spacelike Normal Curves in Minkowski Space* E_1^3, Turkish Journal of Mathematics 29 (1) (2005) 53–63.

[16] M. Babaarslan, Y. Yayli, *On Helices and Bertrand Curves in Euclidean 3-Space*, Mathematical and Computational Applications 18 (1) (2013) 1–11.

[17] L. Kula, Y. Yayli, *On Slant Helix and Its Spherical Indicatrix*, Applied Mathematics and Computation 169 (1) (2005) 600–607.

[18] M. P. Do Carmo, Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition, Courier Dover Publications, New York, 2016.

[19] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[20] M. Petrovic-Torgasev, E. Sucurovic, *Some Characterizations of Lorentzian Spherical Spacelike Curves with the Timelike and the Null Principal Normal*, Mathematica Moravica (4) (2000) 83–92.