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Clinical research on humans,

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Retrospective studies by the law on the protection of personal data

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Merve GÖRE Betül AVCI

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On double summability methods $\left|\mathscr{A}_f\right|_k$ and $|C,0,0|_s$

Fadime Gökçe[1]

Keywords
Double series, Factorable matrix, Absolute Cesàro summability

Abstract — Recently, for single series, the necessary and sufficient conditions for $|C,0| \Rightarrow \left|A_f\right|_k$ and vise versa, and $\left|A_f\right| \Rightarrow |C,0|_k$ and vise versa have been established, where $1 < k < \infty$ and A is a factorable matrix. The present study extends these results to double summability, and also provides some new results.

Subject Classification (2020): 40D25, 40F05.

1. Introduction

The summability theory has an important role in applied mathematics, engineering sciences, and analysis essentially in functional analysis, approximation theory, calculus, quantum mechanics, probability theory, Fourier analysis. The main purpose of the theory is to assign a limit value for divergent series or sequences by using a transformation which is given by the most general linear mappings of infinite matrices. The reason why matrices are used for a general linear operator is that a linear operator from a sequence space to another one can be given by an infinite matrix. In this regard, the literature in the field of summability theory continues to develop not only on the generation of sequence spaces through the matrix domain of a particular matrices such as Hölder, Euler, Cesàro, Hausdorff, Nörlund and weighted mean matrices and on the investigation of their topological, algebraic structures and matrix transformations but also on examinations about new series spaces derived by several absolute summability methods from a different perspective (see, [1–9]). Besides of all these, recently, a many of new article using by double series are also placed in literature. For instance, in [10], the characterizations of the equivalence $|C,0,0|_k \Leftrightarrow \left|R,p_n,q_n\right|_k$ for doubly sequences given by Sarıgöl and the necessary and sufficient conditions for the equivalence of absolute weighted mean summability methods of doubly infinite series are given in [11],(see also [12–16]). The main purpose of this paper is to extend certain theorems given by Hazar and Gökçe in [7] to double infinite series a different approach.

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2. Preliminary

Let $\sum x_v$ be an infinite series of complex numbers with partial sums s_n and σ_n^α denote the $n-$th term of the Cesàro $(C;\alpha)$-transform of $s=(s_n)$. If (see [1])

$$\sum_{n=1}^{\infty} n^{k-1}|x_n|^k < \infty$$

then, it is said that the series $\sum x_v$ is summable $|C,0|_k, k \geq 1$.

Let $A_f=(a_{nv})$ be a factorable matrix i.e., the lower triangular with entries

$$a_{nv}=\begin{cases} \widehat{a}_n a_v, & 0 \leq v \leq n \\ 0, & v>n \end{cases} \tag{2.1}$$

where $(\widehat{a}_n)$ and (a_n) are any sequences of real numbers. The series $\sum x_v$ is said to be summable $\left|A_f\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\widehat{a}_n \sum_{v=1}^{n} a_v x_v\right|^k < \infty \tag{2.2}$$

[9].

Let Λ and Γ be two methods of summability. Γ is said to include Λ if every series summable by the method Λ is also summable by the method Γ and it is written $\Lambda \Rightarrow \Gamma$. Also, Λ and Γ are said to be equivalent if each methods includes the other and it is written $\Lambda \Leftrightarrow \Gamma$.

Through the whole paper k^* denotes the conjugate index of k, i.e., $\frac{1}{k}+\frac{1}{k^*}=1$. The following theorems are given by Hazar and Gökçe [7] for single series:

Theorem 2.1. [7] Let $1<k<\infty$ and A be a factorable matrix given by (2.1) such that $\widehat{a}_v$, $a_v \neq 0$ for all v. Then, $\left|A_f\right|_k |\Rightarrow |C,0|$ if and only if

$$\sum_{v=1}^{\infty} \frac{1}{v}\left\{\frac{1}{|\widehat{a}_v|}\left(\frac{1}{|a_v|}+\frac{1}{|a_{v+1}|}\right)\right\}^{k^*} < \infty$$

Theorem 2.2. [7] Let $1<k<\infty$ and A be a factorable matrix. Then, $|C,0|_k \Rightarrow \left|A_f\right|$ if and only if

$$\sum_{v=1}^{\infty} \frac{1}{v}\left(|a_v| \sum_{n=v}^{\infty} |\widehat{a}_n|\right)^{k^*} < \infty$$

Theorem 2.3. [7] Let $1 \leq k<\infty$ and A be a factorable matrix. Then, $|C,0| \Rightarrow \left|A_f\right|_k$ if and only if

$$\sum_{n=v}^{\infty} n^{k-1}|\widehat{a}_n a_v|^k = O(1) \text{ as } v \to \infty$$

Theorem 2.4. [7] Let $1 \leq k<\infty$ and A be a factorable matrix such that $\widehat{a}_v$, $a_v \neq 0$ for all v. Then, $\left|A_f\right| \Rightarrow |C,0|_k$ if and only if

$$\frac{v^{k-1}}{|\widehat{a}_v|^k}\left(\frac{1}{|a_v|^k}+\frac{1}{|a_{v+1}|^k}\right) = O(1) \text{ as } v \to \infty$$

For any double sequence (x_{ij}), define

$$\Delta_{11} x_{ij} = x_{ij} - x_{i-1,j} - x_{i,j-1} + x_{i-1,j-1};\ i,j \geq 1$$

Let $\sum\sum x_{ij}$ be a double infinite series with partial sums s_{nm}. By $t_{nm}^{\alpha\beta}$, we denote the double Cesàro means (C,α,β) of the double sequence (s_{nm}), that is

$$t_{nm}^{\alpha\beta} = \frac{1}{A_n^{\alpha} A_m^{\beta}} \sum_{i=0}^{n} \sum_{j=0}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} s_{ij} \tag{2.3}$$

where

$$A_0^{\alpha} = 1, A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, A_{-n}^{\alpha} = 0 \text{ for } n \geq 1$$

The series $\sum\sum x_{ij}$ is said to be summable $\left|C;\alpha;\beta\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nm)^{k-1} \left|\Delta_{11} t_{nm}^{\alpha\beta}\right|^k < \infty \tag{2.4}$$

[12, 13]. In the special case for $\beta = 0$ and $\alpha = 0$ the summability method $\left|C;\alpha;\beta\right|_k$, reduces to $|C;0;0|_k$.

A double infinite matrix is called factorable if there exist sequences $a_n^{(1)}, \hat{a}_n^{(1)}, a_n^{(2)}, \hat{a}_n^{(2)}$ such that

$$a_{nmij} = \begin{cases} a_i^{(1)} \hat{a}_n^{(1)} a_j^{(2)} \hat{a}_m^{(2)}, & 0 \leq i \leq n, 0 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}$$

where $a_n^{(1)}, \hat{a}_n^{(1)}, a_n^{(2)}, \hat{a}_n^{(2)}$ are any sequences of real numbers.

We say that the series $\sum\sum x_{ij}$ is summable $\left|\mathscr{A}_f\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nm)^{k-1} \left| \hat{a}_n^{(1)} \hat{a}_m^{(2)} \sum_{i=1}^{n} a_i^{(1)} \sum_{j=1}^{m} a_j^{(2)} x_{ij} \right|^k < \infty \tag{2.5}$$

where $\mathscr{A}$ is factorable matrix.

Let us consider the space

$$\mathscr{L}_k = \left\{ x = (x_{ij}) \in \Omega : \sum_{i,j=0}^{\infty} \left|x_{ij}\right|^k < \infty \right\}, 1 \leq k < \infty$$

which is the set of double sequences corresponding to the well known space ℓ_k of single sequences [16], where Ω is the set of all double sequences of complex numbers. Also, in the case $k = 1$ the space reduces to $\mathscr{L}$, studied by Zeltser [17]. On the other hand, $\mathscr{L}_k$ is the Banach space [16] according to its natural norm

$$\|x\|_{\mathscr{L}_k} = \left(\sum_{i,j=0}^{\infty} \left|x_{ij}\right|^k \right)^{1/k}, 1 \leq k < \infty$$

and, for $k = \infty$, $\mathscr{L}_\infty$ is the space of all bounded double sequences, which is a Banach space with the norm $\|x\|_{\mathscr{L}_\infty} = \sup_{i,j} \left|x_{ij}\right|$.

The following lemmas play significant role in our paper:

Lemma 2.5. [14] Let $k \geq 1$ and $A = (a_{mnrs})$ be a four dimensional infinite matrix of complex numbers. Then, $A \in (\mathscr{L}, \mathscr{L}_k)$ if and only if

$$\sum_{m,n=0}^{\infty} |a_{mnrs}|^k = O(1) \text{ as } r,s \to \infty \tag{2.6}$$

Lemma 2.6. [14] Let $1 \leq k < \infty$ and $A = \left(a_{mnij}\right)$ be an four dimensional infinite matrix of complex numbers. Define $W_k(A)$ and $w_k(A)$ by

$$W_k(A) = \sum_{r,s=0}^{\infty} \left(\sum_{m,n=0}^{\infty} |a_{mnrs}| \right)^k \tag{2.7}$$

and

$$w_k(A) = \sup_{MXN} \sum_{r,s=0}^{\infty} \left| \sum_{(m,n)\in MXN} a_{mnrs} \right|^k \tag{2.8}$$

where M and N are finite subsets of natural numbers. Then, the following statements are equivalent:

i. $W_{k^*}(A) < \infty$

ii. $A \in (\mathscr{L}_k, \mathscr{L})$

iii. $A^t \in (\mathscr{L}_\infty, \mathscr{L}_{k^*})$

iv. $w_{k^*}(A) < \infty$

Lemma 2.7. [11] Let $k > 0$, (p_n) and (q_n) are positive sequences such that $P_n = p_0 + p_1 + \ldots + p_n \to \infty, n \to \infty$, $(P_{-1} = p_{-1} = 0)$. Then, there exists two strictly positive constans M and N, depending only on k, such that

$$\frac{M}{P_{i-1}Q_{j-1}} \leq \sum_{n=i}^{\infty} \sum_{m=j}^{\infty} \frac{p_n q_m}{P_n P_{n-1}^k Q_m Q_{m-1}^k} \leq \frac{N}{P_{i-1}Q_{j-1}}$$

for all $i, j \leq 1$, where M, N are independent of $(p_m), (q_n)$.

3. Main Results

In this section, we prove the following theorems mentioned the relations between the summability methods $|C,0,0|_k$, $\left|\mathscr{A}_f\right|_k$, for several case of k.

Theorem 3.1. Let $1 < k < \infty$ and $\mathscr{A}$ be factorable matrix given by (2.1). Then, $|C,0,0|_k \Rightarrow \left|\mathscr{A}_f\right|$ if and only if

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij} \left(\left|a_i^{(1)} a_j^{(2)}\right| \sum_{n=i}^{\infty} \sum_{m=j}^{\infty} |\hat{a}_n^{(1)} \hat{a}_m^{(2)}| \right)^{k^*} < \infty \tag{3.1}$$

Proof.

Let $y_{nm} = (nm)^{1/k^*} t_{nm}^{00}$ where $\left(t_{nm}^{00}\right)$ is as in (2.3), i.e.,

$$t_{nm}^{00} = \frac{1}{A_n^0 A_m^0} \sum_{i=0}^{n} \sum_{j=0}^{m} A_{n-i}^{-1} A_{m-j}^{-1} s_{ij} = x_{nm}$$

and

$$T_{nm} = \hat{a}_n^{(1)} \hat{a}_m^{(2)} \sum_{i=1}^{n} a_i^{(1)} \sum_{j=1}^{m} a_j^{(2)} x_{ij}$$

Then, the series $\sum\sum x_{ij}$ is summable $\left|\mathscr{A}_f\right|$ and $|C,0,0|_k$ if and only if $(T_{nm}) \in \mathscr{L}$ and $(y_{nm}) \in \mathscr{L}_k$, respectively, and

$$T_{nm} = \hat{a}_n^{(1)} \hat{a}_m^{(2)} \sum_{i=1}^{n} a_i^{(1)} \sum_{j=1}^{m} a_j^{(2)} (ij)^{-1/k^*} y_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{nmij} y_{ij}$$

where $C = (c_{nmij})$ is defined by

$$c_{nmij} = \begin{cases} (ij)^{-1/k^*} a_i^{(1)} a_j^{(2)} \hat{a}_n^{(1)} \hat{a}_m^{(2)}, & 1 \le i \le n, 1 \le j \le m \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $|C,0,0|_k \Rightarrow \left|\mathscr{A}_f\right|$ if and only if $C = (c_{nmij}) \in (\mathscr{L}_k, \mathscr{L})$, or equivalently, by Lemma 2.6,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij} \left(\left| a_i^{(1)} a_j^{(2)} \right| \sum_{n=i}^{\infty}\sum_{m=j}^{\infty} \left| \hat{a}_n^{(1)} \hat{a}_m^{(2)} \right| \right)^{k^*} < \infty$$

which concludes the proof.

Theorem 3.2. Let $1 \le k < \infty$ and $\mathscr{A}$ be factorable matrix such that $\hat{a}_n^{(1)}, a_n^{(1)}, \hat{a}_n^{(2)}, a_n^{(2)} \neq 0$ for all n. Then, $\left|\mathscr{A}_f\right| \Rightarrow |C,0,0|_k$ if and only if

$$\frac{(nm)^{k-1}}{\left|\hat{a}_n^{(1)} \hat{a}_m^{(2)}\right|^k} \left(\frac{1}{\left|a_n^{(1)}\right|^k} + \frac{1}{\left|a_{n+1}^{(1)}\right|^k} \right) \left(\frac{1}{\left|a_m^{(2)}\right|^k} + \frac{1}{\left|a_{m+1}^{(2)}\right|^k} \right) = O(1) \tag{3.2}$$

as $n, m \to \infty$.

Proof.
Let $T_{nm} = \widehat{a}_n \widehat{b}_m \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j x_{ij}$. A few basic calculations give

$$x_{nm} = \frac{1}{a_n^{(1)} a_m^{(2)}} \left(\frac{T_{nm}}{\hat{a}_n^{(1)} \hat{a}_m^{(2)}} - \frac{T_{n-1,m}}{\hat{a}_{n-1}^{(1)} \hat{a}_m^{(2)}} - \frac{T_{n,m-1}}{\hat{a}_n^{(1)} \hat{a}_{m-1}^{(2)}} + \frac{T_{n-1,m-1}}{\hat{a}_{n-1}^{(1)} \hat{a}_{m-1}^{(2)}} \right) \tag{3.3}$$

Moreover,

$$\begin{aligned} y_{nm} = (nm)^{1/k^*} t_{nm}^{00} &= (nm)^{1/k^*} \frac{1}{a_n^{(1)} a_m^{(2)}} \left(\frac{T_{nm}}{\hat{a}_n^{(1)} \hat{a}_m^{(2)}} - \frac{T_{n-1,m}}{\hat{a}_{n-1}^{(1)} \hat{a}_m^{(2)}} - \frac{T_{n,m-1}}{\hat{a}_n^{(1)} \hat{a}_{m-1}^{(2)}} + \frac{T_{n-1,m-1}}{\hat{a}_{n-1}^{(1)} \hat{a}_{m-1}^{(2)}} \right) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} d_{nmij} T_{ij} \end{aligned}$$

where $D = (d_{nmij})$

$$d_{nmij} = \begin{cases} (nm)^{1/k^*} \frac{1}{a_n^{(1)} a_m^{(2)} \hat{a}_n^{(1)} \hat{a}_m^{(2)}}, & i = n, j = m \\ -(nm)^{1/k^*} \frac{1}{a_n^{(1)} a_m^{(2)} \hat{a}_{n-1}^{(1)} \hat{a}_m^{(2)}}, & i = n-1, j = m \\ -(nm)^{1/k^*} \frac{1}{a_n^{(1)} a_m^{(2)} \hat{a}_n^{(1)} \hat{a}_{m-1}^{(2)}}, & i = n, j = m-1 \\ (nm)^{1/k^*} \frac{1}{a_n^{(1)} a_m^{(2)} \hat{a}_{n-1}^{(1)} \hat{a}_{m-1}^{(2)}}, & i = n-1, j = m-1 \end{cases}$$

Thus, we have $\left|\mathscr{A}_f\right| \Rightarrow |C,0,0|_k$ if and only if $D \in (\mathscr{L}, \mathscr{L}_k)$. It follows from Lemma 2.5 that Equation (3.2) holds.

Corollary 3.3. Let $1 < k < \infty$ and $\mathscr{A}$ be factorable matrix such that $\hat{a}_n^{(1)}, a_n^{(1)}, \hat{a}_n^{(2)}, a_n^{(2)} \neq 0$ for all n. Then, $\left|\mathscr{A}_f\right| \Leftrightarrow |C,0,0|_k$ if and only if Equations (3.1) and (3.2) hold.

Theorem 3.4. Let $1 < k < \infty$ and $\mathscr{A}$ be factorable matrix such that $\hat{a}_n^{(1)}, a_n^{(1)}, \hat{a}_n^{(2)}, a_n^{(2)} \neq 0$ for all n. Then, $\left|\mathscr{A}_f\right|_k \Rightarrow |C,0,0|$ if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\left\{\frac{1}{\left|\hat{a}_i^{(1)}\hat{a}_j^{(2)}\right|}\left(\frac{1}{\left|a_i^{(1)}a_j^{(2)}\right|}+\frac{1}{\left|a_{i+1}^{(1)}a_j^{(2)}\right|}+\frac{1}{\left|a_i^{(1)}a_{j+1}^{(2)}\right|}+\frac{1}{\left|a_{i+1}^{(1)}a_{j+1}^{(2)}\right|}\right)\right\}^{k^*}<\infty \tag{3.4}$$

Proof.
Let $T'_{nm} = (nm)^{1/k^*}T_{nm}$. Using (3.3), we have

$$\begin{aligned} t_{nm}^{00} &= \frac{1}{a_n^{(1)}a_m^{(2)}}\left(\frac{(nm)^{-1/k^*}T'_{nm}}{\hat{a}_n^{(1)}\hat{a}_m^{(2)}}-\frac{((n-1)m)^{-1/k^*}T'_{n-1,m}}{\hat{a}_{n-1}^{(1)}\hat{a}_m^{(2)}}-\frac{(n(m-1))^{-1/k^*}T'_{n,m-1}}{\hat{a}_n^{(1)}\hat{a}_{m-1}^{(2)}}+\frac{((n-1)(m-1))^{-1/k^*}T'_{n-1,m-1}}{\hat{a}_{n-1}^{(1)}\hat{a}_{m-1}^{(2)}}\right) \\ &= \sum_i\sum_j e_{nmij}T'_{nm} \end{aligned}$$

where

$$e_{nmij}=\begin{cases} \frac{(nm)^{-1/k^*}}{a_n^{(1)}a_m^{(2)}\hat{a}_n^{(1)}\hat{a}_m^{(2)}}, & n=i, m=j \\ \frac{((n-1)m)^{1/k^*}}{a_n^{(1)}a_m^{(2)}\hat{a}_{n-1}^{(1)}\hat{a}_m^{(2)}}, & n-1=i, m=j \\ \frac{(n(m-1))^{1/k^*}}{a_n^{(1)}a_m^{(2)}\hat{a}_n^{(1)}\hat{a}_{m-1}^{(2)}}, & n=i, m-1=j \\ \frac{((n-1)(m-1))^{1/k^*}}{a_n^{(1)}a_m^{(2)}\hat{a}_{n-1}^{(1)}\hat{a}_{m-1}^{(2)}}, & n-1=i, m-1=j \end{cases}$$

Then, we get that $\sum\sum x_{ij}$ is summable $|C,0,0|$ whenever $\sum\sum x_{ij}$ is summable $\left|\mathscr{A}_f\right|_k$ if and only if $E = (e_{nmij}) \in (\mathscr{L}_k, \mathscr{L})$. So, we imply Equation (3.4) with Lemma 2.6. This concludes the proof.

Theorem 3.5. Let $1 \leq k < \infty$ and $\mathscr{A}$ be factorable matrix. Then, $|C,0,0| \Rightarrow \left|\mathscr{A}_f\right|_k$ if and only if

$$\sum_{n=i}^{\infty}\sum_{m=j}^{\infty}(nm)^{k-1}\left|\hat{a}_n^{(1)}\hat{a}_m^{(2)}a_i^{(1)}a_j^{(2)}\right|^k = O(1) \text{ as } i,j\to\infty \tag{3.5}$$

Since the theorem can be proved by the similar way with Theorem 3.2, it has been left to reader.

Corollary 3.6. Let $1 < k < \infty$ and $\mathscr{A}$ be factorable matrix such that $\hat{a}_n^{(1)}, a_n^{(1)}, \hat{a}_n^{(2)}, a_n^{(2)} \neq 0$ for all n. Then, $\left|\mathscr{A}_f\right|_k \Leftrightarrow |C,0,0|$ if and only if Equations (3.4) and (3.5) hold.

It may be noticed that if we take $\hat{a}_n^{(1)} = p_n \backslash P_nP_{n-1}$, $a_n^{(1)} = P_{n-1}$ and $\hat{a}_n^{(2)} = q_n \backslash Q_nQ_{n-1}$, $a_n^{(2)} = Q_{n-1}$, then the summability method $\left|\mathscr{A}_f\right|_k$ is reduced to the double absolute Riesz summability method $\left|R,p_n,q_n\right|_k$. Hence, we get the following results:

Corollary 3.7. Let $k \geq 1$. Then, $|C,0,0| \Rightarrow \left|R,p_n,q_n\right|_k$ if and only if

$$\sum_{n=i}^{\infty}\sum_{m=j}^{\infty}(nm)^{k-1}\left(\frac{p_nq_m}{P_nP_{n-1}Q_mQ_{m-1}}\right)^k = O\left(\frac{1}{\left(P_{i-1}Q_{j-1}\right)^k}\right)$$

Corollary 3.8. Let $k \geq 1$. Then, $\left|R,p_n,q_m\right| \Rightarrow |C,0,0|_k$ if and only if

$$(ij)^{1/s^*}P_iQ_j = O\left(p_iq_j\right) \text{ as } i,j\to\infty$$

Moreover, Equations (3.1) and (3.4) are equivalent to

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}<\infty$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\left(\frac{P_i}{p_i}\right)^{k^*}\left(\frac{Q_j}{q_j}\right)^{k^*}<\infty$$

which are impossible. Thus, we have the following result.

Corollary 3.9. If $k>1$, then $\left|R,p_n,q_m\right|_k \not\Rightarrow |C,0,0|$ and also $|C,0,0|_k \not\Rightarrow \left|R,p_n,q_m\right|$.

4. Conclusion

This paper aimed to adapt the summability method $\left|\mathscr{A}_f\right|_k$ to double series and extend some theorems given for single series to double series. The relations between other summability methods and $\left|\mathscr{A}_f\right|_k$ are worth studying.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Different computational approach for Fourier transforms by using variational iteration method

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Abstract — In this paper, we present another method for computing Fourier transforms of functions considering the Variational Iteration Method (VIM). Through our procedure, the Fourier transforms of functions can be calculated precisely and without reference to complex integration.

Subject Classification (2020): 42A38, 35A15.

1. Introduction

The variational iteration technique is one of the evaluation techniques that can be employed to many linear and nonlinear problems and reduces the computational effort. Numerous authors have worked for years to develop the method of variation iteration [1–3]. Starting from the Inokuti-Sekine-Mura technique as a starting point [4], it has developed into a full-fledged theory thanks to the work of several academics, including Wazwaz [5] and Moghimi [6]. The variational iteration technique [1–3, 5] is suitable for the treatment of a large class of linear or nonlinear differential problems. It is a suitable computational method for applications in the sciences [7–10]. Gubes [11] performed the variational iteration method (VIM) to obtain Laplace and Sumudu transforms. Xu and Lee [12] implemented the (VIM) for solving the boundary layer equations of magnetohydrodynamic flow over a nonlinear stretched sheet. Wazwaz [13] also applied this method to solve the linear and nonlinear ODE with variable coefficients. This method does not require much time when applied on the computer. In recent years, Düz et al. [14] have implemented the differential transformation technique to obtain Fourier transform of functions.

This paper is about a new calculation of Fourier transforms of functions using the variational iteration method (VIM) with first order linear IVP.

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2. The Basic Definitions and Theorems

2.1. Variational Iteration Method

To provide the core concept of the technique, consider the following general nonlinear differential equation:

$$L[\beta(u)] + N[\beta(u)] = \psi(u) \tag{2.1}$$

where N, L are the non-linear and linear operator respectively, and $\psi(u)$ is a given continuous function.

The main advantage of this technique is that it creates a correction function for Equation (2.1), which can be represented as follows.

$$\beta_{J+1}(u) = \beta_J(u) + \int_0^u \lambda(u,s)\left[L\beta_J(s) + N\hat{\beta}_J(s) - \psi(s)\right] ds \tag{2.2}$$

Here, β_J is the J^{th} approximation solution, λ is a General lagrange multiplier that may be ideally found using variational theory and $\hat{\beta}_J$ is restricted variation.

The choice of the first approximation function $\beta_0(u)$ can affect the approximation positively or negatively.

As a result, the exact solution to Equation (2.1) can be obtained by:

$$\beta = \lim_{J\to\infty} \beta_J$$

2.2. Fourier Transform and Dirac Delta Distribution

To begin our discussion of the new computation of Fourier transforms of functions, we represent the Fourier transform of $h(u)$ as [14, 15]

$$\begin{aligned} \mathscr{F}[h(u)] &= \hat{h}(w) \\ &= \textstyle\int_{-\infty}^{\infty} h(u)e^{-iwu}du \end{aligned} \tag{2.3}$$

and inverse Fourier transform is:

$$\begin{aligned} h(u) &= \mathscr{F}^{-1}[\hat{h}(w)] \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty} \hat{h}(w)e^{iwu}dw \end{aligned}$$

The integral in Equation (2.3) is convergent to the function of the w if the following condition is satisfied

$$\int_{-\infty}^{\infty} |h(u)|du < \infty$$

The Fourier transform of exponential, polynomial and trigonometric functions are defined with the Dirac delta function, we will now introduce you to some properties of the Dirac delta function, which play a big role in Fourier transform.

Some properties of the dirac delta functions are as following: [14, 16, 17]

i. $\delta(w) = \begin{cases} 0, & w \neq 0 \\ \infty, & w = 0 \end{cases}$

ii. $\int_{-\infty}^{\infty} \delta(w)dw = 1$

iii. $\delta(bw) = \frac{1}{|b|}\delta(w), b \neq 0$

iv. $\delta(w^2 - b^2) = \frac{1}{2|b|}\left(\delta(w-b) + \delta(w+b)\right), b \neq 0$

v. $h(0) = \int_{-\infty}^{\infty} h(u)\delta(w)du$

There are a number of suitable functions that can estimate the delta function in the limiting process. One of them is the family of Gaussian curves, which are

$$\phi(w,b) = \frac{1}{\sqrt{2\pi b}} e^{\frac{-w^2}{2b}}, b > 0 \tag{2.4}$$

As $b \to 0^+$ family of $\phi(w,b)$ functions satisfies exact the same properties of delta function. Thus,

$$\delta(w) = \lim_{b\to 0^+} \phi(w,b) \tag{2.5}$$

The Fourier Transform of delta function [1] is

$$\mathscr{F}[\delta(w)] = \int_{-\infty}^{\infty} \delta(w)e^{-iwu}dw = 1$$

and it's inverse is

$$\mathscr{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1.e^{iwu}du = \delta(w)$$

The delta function is an even function: $\delta(-w) = \delta(w)$ and instead of the above integral expression of the delta function it is more usual to express :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwu}du = \delta(w) \tag{2.6}$$

From the Fourier Integral Theorem :

$$\begin{aligned} h(u) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(w)e^{iwu}dw \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(v)e^{-iwv}dv\right)e^{iwu}dw \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} h(v)\left(\int_{-\infty}^{\infty} e^{-i(v-u)w}dw\right)dv \end{aligned}$$

From the previous equation and Equation (2.6), $h(u)$ can be written as follows

$$h(u) = \int_{-\infty}^{\infty} h(v)\delta(v-u)dv$$

3. Results Using VIM

In this study, we tried to find some elementary functions Fourier transform using VIM, this kind of work was done using VIM and DTM for Laplace, Sumudu transforms and we were inspired by this work.

Theorem 3.1. Let $w \in \mathbb{C}$, $h(u)$ is an analytic function, and consider the linear IVP given as:

$$\beta' - iw\beta = ih(u), \beta(0) = 0 \tag{3.1}$$

then the Fourier Transforms of $h(u)$ is

$$\mathscr{F}[h(u)] = \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\frac{e^{-iwu}}{i}\lim_{J\to\infty}\beta_J\right]_{u=B}^{u=A}$$

Proof.
$\beta' - iw\beta = ih(u)$ this ODE can be represented as follows

$$(\beta e^{-iwu})' = ih(u)e^{-iwu}$$

by integrate both sides concerning u from $-\infty$ to ∞, we get the relationship between the Fourier Transform and previous equation as follows

$$\lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\beta e^{-iwu}\right]_{u=B}^{u=A} = i\int_{-\infty}^{\infty} h(u)e^{-iwu}du$$

that means :

$$\mathscr{F}[h(u)] = \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\frac{e^{-iwu}}{i}\beta\right]_{u=B}^{u=A} \tag{3.2}$$

Besides, by creating the VIM formula of Equation (3.1) at $\lambda = -1$, we will find the Fourier transform of $h(u)$ as:

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta'_J(s) - iw\beta_J(s) - ih(s)\right] ds \tag{3.3}$$

$$\beta = \lim_{J\to\infty}\beta_J \tag{3.4}$$

We substitute β into Equation (3.2) to get

$$\mathscr{F}[h(u)] = \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\frac{e^{-iwu}}{i}\lim_{J\to\infty}\beta_J\right]_{u=B}^{u=A} \tag{3.5}$$

Now, we provide some examples to find Fourier Transforms of some functions using the VIM

Example 3.2. Suppose that $h(u) = 1$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta'_J(s) - iw\beta_J(s) - i\right] ds$$

$\beta_0(u) = 0.$
We will now find some of $\beta_J(u)$

$$\begin{aligned} \beta_1(u) &= iu \\ \beta_2(u) &= iu + \tfrac{i^2wu^2}{2!}, \\ \beta_3(u) &= iu + \tfrac{i^2wu^2}{2!} + \tfrac{i^3w^2u^3}{3!}, \cdots \end{aligned} \tag{3.6}$$

from Equations (3.4) and (3.6), we get

$$\beta = \lim_{J\to\infty}\beta_J = \frac{e^{iwu}-1}{w} \tag{3.7}$$

We substitute β into Equation (3.2) to get the Fourier Transform of 1

$$\begin{aligned} \mathscr{F}[1] &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{e^{-iwu}}{i}\left(\tfrac{e^{iwu}-1}{w}\right)\right]_{u=B}^{u=A} \\ &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{1-e^{-iwu}}{iw}\right]_{u=B}^{u=A} \\ &= \int_{-\infty}^{\infty} e^{-iwu}du \end{aligned}$$

by Equation (2.6), $\mathscr{F}[1]=2\pi\delta(w)$.

Example 3.3. Suppose that $h(u)=e^{bu}$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u)=\beta_J(u)-\int_0^u\left[\beta_J'(s)-iw\beta_J(s)-ie^{bs}\right]ds$$

$\beta_0(u)=0$.

We will now find some of $\beta_J(u)$

$$\begin{aligned} \beta_1(u) &= \tfrac{i}{b}\left(e^{bu}-1\right) \\ \beta_2(u) &= \tfrac{i}{b}\left(e^{bu}-1\right)+\tfrac{i^2w}{b}\left(\tfrac{e^{bu}-1}{b}-u\right) \\ \beta_3(u) &= \tfrac{i}{b}\left(e^{bu}-1\right)+\tfrac{i^2w}{b}\left(\tfrac{e^{bu}-1}{b}-u\right)+\tfrac{i^3w^2}{b}\left(\tfrac{e^{bu}-1}{b^2}-\tfrac{u}{b}-\tfrac{u^2}{2!}\right),\cdots \end{aligned} \tag{3.8}$$

from Equations (3.4) and (3.8), we get

$$\beta=\lim_{J\to\infty}\beta_J=\frac{i\left(e^{bu}-e^{iwu}\right)}{b-iw} \tag{3.9}$$

We substitute β into Equation (3.2) to get the Fourier Transform of e^{bu}

$$\begin{aligned} \mathscr{F}[e^{bu}] &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{e^{-iwu}}{i}\left(\tfrac{i\left(e^{bu}-e^{iwu}\right)}{b-iw}\right)\right]_{u=B}^{u=A} \\ &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{-\left(e^{-i(ib+w)u}-1\right)}{i(ib+w)}\right]_{u=B}^{u=A} \\ &= \int_{-\infty}^{\infty} e^{-i(ib+w)u}du \end{aligned} \tag{3.10}$$

by Equation (2.6) $\mathscr{F}[e^{bu}]=2\pi\delta(ib+w)$.

Example 3.4. Suppose that $h(u)=u^J$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u)=\beta_J(u)-\int_0^u\left[\beta_J'(s)-iw\beta_J(s)-is^J\right]ds$$

$\beta_0(u)=0$.

We will now find some of $\beta_J(u)$

$$\begin{aligned} \beta_1(u) &= \tfrac{i}{(J+1)} u^{J+1} \\ \beta_2(u) &= \tfrac{i}{(J+1)} u^{J+1} + \tfrac{i^2 w}{(J+1)(J+2)} u^{J+2} \\ \beta_3(u) &= \tfrac{i}{(J+1)} u^{J+1} + \tfrac{i^2 w}{(J+1)(J+2)} u^{J+2} + \tfrac{i^3 w^2}{(J+1)(J+2)(J+3)} u^{J+3} \\ &= \tfrac{iJ!}{(J+1)!} u^{J+1} + \tfrac{i^2 wJ!}{(J+2)!} u^{J+2} + \tfrac{i^3 w^2 J!}{(J+3)!} u^{J+3}, \cdots \end{aligned} \tag{3.11}$$

from Equations (3.4) and (3.11), we get

$$\begin{aligned} \beta &= \lim_{J\to\infty} \beta_J \\ &= \tfrac{iJ!}{(iw)^{J+1}} \left(e^{iwx} - 1 - \tfrac{iwu}{1} - \tfrac{(iwu)^2}{2!} - \cdots - \tfrac{(iwu)^J}{J!} \right) \end{aligned} \tag{3.12}$$

We substitute β into Equation (3.2) to get the Fourier Transform of u^J

$$\begin{aligned} \mathscr{F}[u^J] &= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\tfrac{e^{-iwu}}{i} \left(\tfrac{iJ!}{(iw)^{J+1}} \left(e^{iwu} - 1 - \tfrac{iwu}{1} - \tfrac{(iwu)^2}{2!} - \cdots - \tfrac{(iwu)^J}{J!} \right) \right) \right]_{u=B}^{u=A} \\ &= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\tfrac{2\pi}{(-i)^J} \left(\tfrac{(-i)^J}{2\pi} (-1) e^{-iwu} \left(\tfrac{u^J}{iw} + \tfrac{Jx^{J-1}}{(iw)^2} + \cdots + \tfrac{J!u}{(iw)^J} + \tfrac{J!}{(iw)^{J+1}} \right) \right) \right]_{u=B}^{u=A} \\ &= 2\pi i^J \delta^{(J)}(w) \end{aligned}$$

Example 3.5. Suppose that $h(u) = rect(u) = \begin{cases} \frac{1}{a}, & -\frac{a}{2} \le u \le \frac{a}{2} \\ 0, & otherwise \end{cases}$ and by Theorem 3.1, where $rect(u)$ is Rectangular Function, then we get

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta_J'(s) - iw\beta_J(s) - irect(s) \right] ds$$

$\beta_0(u) = 0$. We will now find some of $\beta_J(u)$

$$\begin{aligned} \beta_1(u) &= \tfrac{iu}{a} \\ \beta_2(u) &= \tfrac{iu}{a} + \tfrac{i^2 wu^2}{a2!}, \beta_3(u) = \tfrac{iu}{a} + \tfrac{i^2 wu^2}{a2!} + \tfrac{i^3 w^2 u^3}{a3!} \cdots \end{aligned} \tag{3.13}$$

from Equations (3.4) and (3.13), we get

$$\beta = \lim_{J\to\infty} \beta_J = \frac{e^{iwu} - 1}{aw} \tag{3.14}$$

We substitute β into Equation (3.2) to get the Fourier Transform of $rect(u)$

$$\begin{aligned} \mathscr{F}[rect(u)] &= \left[\tfrac{e^{-iwu}}{i} \left(\tfrac{e^{iwu}-1}{aw} \right) \right]_{u=-\frac{a}{2}}^{u=\frac{a}{2}} \\ &= \left[\tfrac{1-e^{-iwu}}{iaw} \right]_{u=-\frac{a}{2}}^{u=\frac{a}{2}} = \tfrac{\sin(\frac{aw}{2})}{\frac{aw}{2}} \\ &= \operatorname{sinc}(\tfrac{aw}{2\pi}) \end{aligned}$$

The definition of sinc function in [18].

Example 3.6. Suppose that $h(u) = \cos(au)$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta'_m(s) - iw\beta_J(s) - i\cos(bs)\right] ds$$

The previous formula can be represented as:

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta'_J(s) - iw\beta_J(s) - \frac{i}{2}\left(e^{ibs} + e^{-ibs}\right)\right] ds$$

$\beta_0(u) = 0$. We will now find some of $\beta_J(u)$

$$\begin{aligned}
\beta_1(u) &= \frac{1}{2b}\left(e^{ibu} - e^{-ibu}\right) \\
\beta_2(u) &= \frac{1}{2b}\left(e^{ibu} - e^{-ibu}\right) + \frac{w}{2b^2}\left(e^{ibu} + e^{-ibu}\right) \\
\beta_3(u) &= \left(\frac{1}{2b} + \frac{w^2}{2b^3}\right)\left(e^{ibu} - e^{-ibu}\right) + \frac{w}{2b^2}\left(e^{ibu} + e^{-ibu}\right) \\
\beta_4(u) &= \left(\frac{1}{2b} + \frac{w^2}{2b^3}\right)\left(e^{ibu} - e^{-ibu}\right) + \left(\frac{w}{2b^2} + \frac{w^3}{2b^4}\right)\left(e^{ibu} + e^{-ibu}\right), \ldots
\end{aligned} \tag{3.15}$$

from Equations (3.4) and (3.15) we get

$$\begin{aligned}
\beta &= \lim_{J\to\infty} \beta_J \\
&= \frac{e^{ibu}}{2(b-w)} - \frac{e^{-ibu}}{2(b+w)}
\end{aligned} \tag{3.16}$$

We substitute β into Equation (3.2) to get the Fourier Transform of $cos(bu)$

$$\begin{aligned}
\mathscr{F}[\cos(bu)] &= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\frac{e^{-iwu}}{i}\left(\frac{e^{ibu}}{2(b-w)} - \frac{e^{-ibu}}{2(b+w)}\right)\right]_{u=B}^{u=A} \\
&= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\frac{e^{iu(b-w)}}{2i(b-w)} - \frac{e^{-iu(b+w)}}{2i(b+w)}\right]_{u=B}^{u=A} \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left[e^{-i(w-b)u} + e^{-i(w+b)u}\right] dx
\end{aligned} \tag{3.17}$$

by Equation (2.6), $\mathscr{F}[\cos(bu)] = \pi[\delta(w-b) + \delta(w+b)]$.

The main advantage of VIM when calculating Fourier transforms of functions is that it can expand the convergence region for iterative solutions and reduce the size of the calculations. Besides, the solutions obtained using VIM do not need much effort and time compared to DTM [14], as explained in the above examples .

4. Conclusion

We have implemented the variational technique (VIM) to compute the Fourier transforms of functions differently. Furthermore, the results have shown that the method we have presented is a comprehensive and accurate scientific method that does not require much effort in determining Fourier transforms.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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Determination of the functional properties of some Nigerian and imported rice varieties

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Keywords:
Rice flour,
Varieties,
Functional properties,
Nigerian rice.

Abstract — This study determined the comparative study of the functional properties of some Nigerian and imported rice varieties. The result showed a significant difference in functional properties at (p>0.05). Illa (Nigerian) rice had the highest Water Absorption Capacity (WAC) (353.76%) and Oil Absorption Capacity (OAC) (218.32%) than the imported rice variety. The bulk density of both samples had less significantly different values. The percentage values of dispersibility (75.00%) were higher in Ofada (Nigerian) rice. Nigerian rice had a higher swelling index (10.91%). The amylopectin contents of tested rice varieties were 76.26, 76.55, 75.58, 74.37, and 77.55% for Illa, Abakaliki, Jemila, Ofada, and Imported rice, respectively. The result revealed that Imported rice had the highest value (77.55%) of amylopectin, followed by Abakaliki rice which recorded 76.55%, and Illa rice variety from the Southern region had 76.28%. In comparison, the Ofada rice sample had the lowest value (74.37%) of amylopectin content, followed by Jemila rice which recorded 75.58% of amylopectin content. The amylose content for Illa, Abakaliki, Jemila, Ofada, and Imported rice varieties was 23.73, 23.45, 24.42, 25.64, and 22.45%, respectively. Rice flour, with low amylose content, gives moistness, chewiness, and softness to the textures of the product. It showed that Ofada rice had the highest value (25.64%) of amylose content, followed by Jemila rice which recorded 24.42% of amylose content. The lowest value (22.45%) was found in imported rice, followed by Abakaliki rice with 23.45% and Illa rice sample with 23.73% of amylose content. It can be concluded that Nigerian rice is compared handsomely with imported rice in terms of functional properties, which determines the end-use and general acceptability of flour samples.

Subject Classification (2020):

1. Introduction

According to [1], the quantities that determine the applications and final uses of numerous food products' materials are known as functional properties. Their use in industries and food production depends on their various functional properties [2,3]. Foods' essential physicochemical properties are their functional properties, which reflect the intricate interactions between food components' structures, molecular conformations, compositions, and physicochemical properties with the nature of

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the environment and conditions in which these are measured and associated [4,7]. Functional properties are highly needed to predict likely and precisely determine how new carbohydrates, such as starch and sugars, fibre, and fat, may react in food systems, as well as to establish whether such can be used to stimulate or replace common protein, carbohydrate, fat, and fibre [5,7].

The functionality of rice starch is the arrangement of properties which influence rice characteristics and procedural effectiveness. These properties vary significantly for various agricultural products and processes, and this may be calculated by chemical process or evaluation testing [1]. Functional properties may be categorized corresponding to the method of design on three main groups. According to [1], these categories includes; firstly, properties that are concerned with the protein composition and rheological characteristics, properties that are concerned with hydration, and lastly, properties that are concerned with the protein surface.

Functional characteristics also determine the behaviour of ingredients in the preparation and cooking of food, and how the finished products are affected by finished food materials, as regards how it looks, feel, and tastes. According to [1,4], functional characteristics' parameters include, but are not limited to, water absorption capacity, oil absorption capacity, swelling power, foam capacity, emulsion stability, foam stability, emulsion activity, and others.

The most important cereal for humans is rice (*Oryza sativa L.*), which feeds almost half of the world's population as a staple food and has dominated its position as one of the world's leading food crops [8,10]. While it remained an important economic crop for household food security, ceremonies, nutritional diversification, income generation, and employment, it supplied 23% of the world's calorie supply [8]. Rice is widely cultivated worldwide and is the second most important cereal crop in terms of cultivation after wheat, mainly in Nigeria [11]. It belongs to the family of Poaceae, and its domestication started from ancient civilisation native to Southeast Asia [9]. After cooking, rice is mostly consumed as a whole grain, accounting for between 40% and 80% of calories [12]. Since the beginning of time, rice landraces have been grown and consumed, and they play a crucial role in ensuring food security and livelihood [9,13]. The rice landraces have varied agro-morphological characteristics, and some of the landraces are prosperous in terms of yield [14]. According to [9], the traditional landraces have been used for their higher nutritional value than the hybrid and have been used as alternative medicine in treatments of some different diseases like diarrhoea, fever, vomiting, haemorrhage, burns, improve eyesight, vocal clarity, fertility [9].

Rice consumption and demand are anticipated to rise in tandem with Nigeria's expanding population. According to estimates, Nigeria consumes 40 kilograms of rice per person each year [15,16]. National demand for rice was estimated to be 6. Three million metric tons in 2016, while the domestic supply was 2.3 million metric tons [16,17]. Imports were anticipated to fill the deficit of 4 million metric tons. The importation of rice is bad for Nigeria's economy because it means that the country will run out of foreign exchange (forex) earnings and run out of money. [16]. Even though a wide range of rice varieties are grown in Nigeria, the country is increasingly influenced by imported rice varieties from other countries. The "Aroso" variety of parboiled rice is one popular foreign variety produced in Thailand and is widely consumed in Nigeria [8]. The need to mechanise the process and make local rice more competitive with almighty imported rice is recognised, primarily considering understanding some of its pasting and functional properties. Other research literature has revealed that Nigeria's local domestic rice varieties exhibit numerous benefits that are superior to imported rice [18]. Although Nigerians are aware of locally produced rice, most consumers are more familiar with rice imported from other countries [19,20].

2. Materials and Methods

2.1. Source of Rice Samples

The rice samples used for this research work were four Nigerian rice selected from four regions (North, South, East, and West) and imported (Indonesian) rice. The selection is determined based on patronage preference. The four local Nigerian rice were Ofada from West obtained from Du farm in Federal University of Agricultural Abeokute, Ogun State, Illah (FARRO 44) from South obtained from Nature's farm, Illah Ugbolu Local Government Area, Delta State, Jemila (FARRO 44) from North obtained from Da-Elgreen farm in Chikun Local Government Area, Kaduna State and Abakaliki (NERICA) rice from South-East obtained from Modery Community Farm in Ministry of Agriculture, Ezzamgbo branch, Ebonyi State while the imported rice was obtained from DU farm, Federal University of Agricultural Abeokute, Ogun State.

2.2. Preparation of Samples

The rice grains that were sourced were cleaned, sorted, and washed. The dried rice grains were milled using an attrition mill, and the milled grains were sieved using a 300 μm mesh-size sieve to obtain fine flour. They were steeped in water for 12 hours, drained, and dried in a hot air oven at 700C [3]. The processed flour was packaged with the proper label in an airtight plastic container and moved to various laboratories, where the properties were determined.

2.3 Determination of Functional Properties of Nigerian and Imported Rice Varieties

2.3.1 Oil Absorption Capacity

A centrifuge tube containing 1 gram of flour and 10 ml of refined corn oil was set to room temperature for 60 minutes. It was centrifuged for 20 minutes at 1600 g. The free oil's volume was measured and poured. The amount of fat absorbed by 100 grams of dried flour was measured as ml of oil bound to it. [21].

2.3.2 Water Absorption Capacity

The mixture of 1 g of flour with 10 ml refined corn oil in a centrifuge tube was allowed to stand at room temperature of 30 ± 2 °C for 60 mins. It was shaken on a platform tube rocker for 1 minute at room temperature. The sample was allowed to stand for 30 minutes and centrifuged at 1200 g for 30 minutes. The volume of free water was read directly from the centrifuge tube [21,22].

2.3.3 Swelling Power

The flour sample was divided into three grams, according to [23], and each portion of the dried flour was transferred into clean, dry, and calibrated 50 ml cylinders. The volumes of the flour samples were recorded after they had been slightly levelled. Each sample was diluted with 30 ml of distilled water, and the cylinder was left to swirl and stand for 60 minutes before the swelling power (volume change) was measured at 15 minutes. At each volume raised, the swelling power of each flour sample was calculated after 15 minutes.

2.3.4 Solubility

To determine solubility, 1 g (db) of flour was mixed with up to 10 ml of distilled water in a 5 ml measuring cylinder using the cold-water extraction method. The sample was allowed for 60 minutes with 10 minutes of stirring. After 2 ml of the supernatant was measured in a dry Petri dish, evaporated to dryness, and reweighed, the sample was allowed to settle for 15 minutes. The total soluble solids, which was determined using the equation presented in [24], was the change in mass.

$$Solubility = TSS(\%)\frac{VsMe - Md}{2MS1}100 \tag{4.1}$$

TSS is the total soluble solute; Vs is the total filtrate or supernatant; Md is the mass of a dry, empty Petri dish; while Me is the mass of the Petri dish in addition to the solid that remained after evaporative drying.

2.3.4 Determination of Dispersibility

This was determined by the method described by [25]. To reach the 100 ml mark, distilled water was added to a 200 ml measuring cylinder in which ten grams of each sample were suspended. The setup was subtracted from 100 after being vigorously stirred. The percentage of dispersibility was used to describe the difference.

$$\text{Dispersibility} = 100 - \text{Volumes of Settled Particle} \tag{4.2}$$

2.4 Determination of Amylose and Amylopectin

The ISO 6647 method was used to estimate amylose and amylopectin [26]. In a 50 ml conical flask, 100 mg (db) of the sample was treated with 1 mL of ethyl alcohol, slowly stirred, treated with 9 mL of 1 N sodium hydroxide, and heated for 10 minutes with occasional stirring in boiling water. After being chilled to room temperature, the sample was transferred to a volumetric flask of 100 millilitres, washed, transferred, and finally filled to volume with water. After taking 5 ml of the dispersion, 50 ml of water and 1 ml of 1 N acetic acid were added, and the entire mixture was shaken; 2 mL of a solution of 2% potassium iodide and 0.2% iodine were added, diluted with water to volume, and kept at 27 °C for 20 minutes. A blank (without a sample) UV-2010 spectrophotometer from Hitachi, Japan, read the colour at 620 nm. To estimate the amount of amylose in the unidentified sample, a regression equation was created.

3. Results and Discussion

3.1 Functional Properties of Nigerian and Imported Rice Flour

The functionality of foods is the characteristics of food ingredients other than nutritional quality, which significantly influences its utilisation [27,28]. It is presented in Table 1 and Figure 1.

3.2 Water Absorption Capacity

Table 1 and Figure 1 present the functional properties of Nigerian and imported rice flour varieties. It was found that the values of WAC for Illa, Abakaliki, Jemila, Ofada, and Imported rice flour varieties were 353.76, 185.79, 155.60, 158.62, and 216.80%, respectively.

Table 1. Comparison of functional properties of milled Nigerian and imported rice flour

Properties	Illa Rice	Abakaliki Rice	Jemila Rice	Ofada Rice	Imported Rice
WAC (%)	353.76 ± 12.64	185.79 ± 3.28	155.60 ± 7.68	158.62 ± 1.68	216.80 ± 15.42
OAC (%)	218.32 ± 0.75	175.65 ± 0.43	170.56 ± 1.15	115.54 ± 0.64	187.64 ± 2.05
Bulk density	0.10 ± 0.00	0.25 ± 0.07	0.25 ± 0.07	0.20 ± 0.00	0.10 ± 0.00
Dispersibility (%)	53.00 ± 0.00	72.00 ± 1.41	73.50 ± 2.12	75.00 ± 0.00	70.50 ± 0.70
Swelling power	8.36 ± 0.15	10.40 ± 0.30	10.39 ± 0.42	10.91 ± 0.01	8.69 ± 0.37
Solubility index	10.69 ± 0.22	12.10 ± 0.42	9.54 ± 0.52	10.01 ± 0.26	12.34 ± 0.27
Amylose (%)	23.73 ± 0.39	23.45 ± 0.01	24.42 ± 0.03	25.64 ± 0.01	22.45 ± 0.01
Amylopectin (%)	76.28 ± 0.39	76.55 ± 0.01	75.58 ± 0.03	74.37 ± 0.01	77.55 ± 0.01

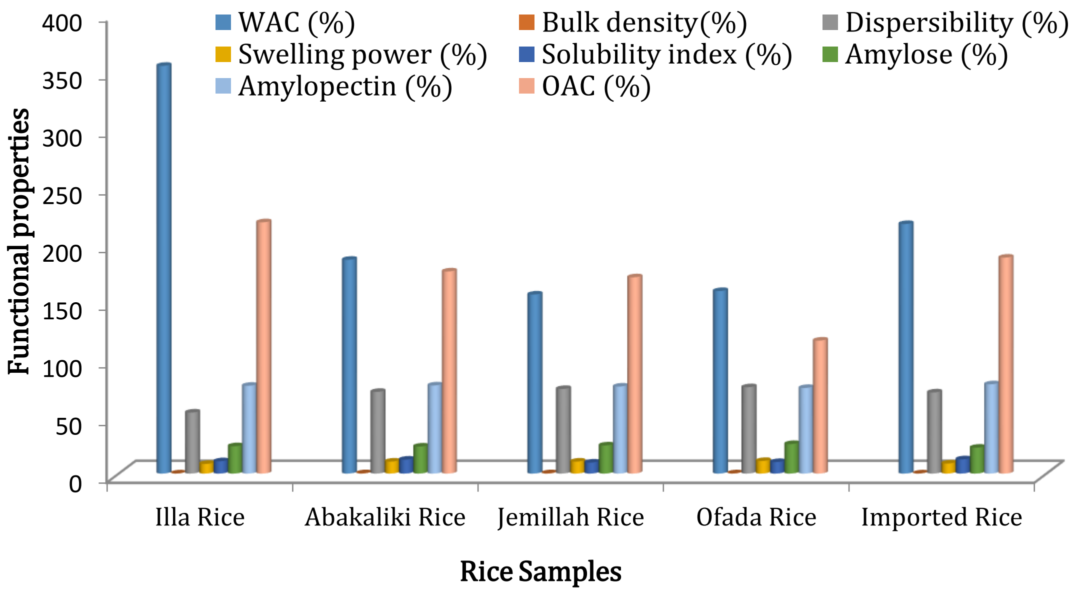

Figure 1. Comparison of functional properties of milled Nigerian and imported rice flour

During the gelatinisation of starch, water has a significant impact on a variety of significant changes as well as other functional properties. It is important to determine WAC because it helps make bread and soups, among other foods [29,31]. The findings showed that Illa rice had the highest value (353.76%) of water absorption capacity, followed by imported rice with 216.80% WAC, and Abakaliki rice which had 185.79% WAC. The lowest value (155.60%) of WAC was found in the varieties of Jemila rice, followed by Ofada rice, which recorded 158.62% WAC. These findings were comparable to the report [29] on the functional and pasting properties of Malaysia's imported and locally grown exotic rice varieties. Rice flour flours may differ in their capacity to absorb water due to the presence of hydrophobic amino acids, which hinder the rice starch's ability to absorb water [27,31]. The loose association of amylose and amylopectin in the native starch granules and their weak binding forces may be to blame for this effect [32,33].

3.3 Oil Absorption Capacity

The results of the OAC of investigated rice flour varieties are presented in Table 1. The values of Illa, Abakaliki, Jemila, Ofada, and Imported rice flour varieties were 218.32, 175.65, 170.56, 115.54, and 187.64 %. The findings showed that Illa rice had the highest value (218.32%), followed by imported rice which had 187.64%, and Abakaliki rice which recorded 175.65% of OAC. The lowest value (115.54%) of OAC was noticed in Ofada rice, followed by Jemila rice flour with 170.56 OAC. Oil absorption capacity measures proteins' capability to bind fat physically by capillary gravitation. It can influence the oil retention in food products and determine the level of flavour retention and mouth feel [29,34]. The difference in OAC of Illa, Abakaliki, Jemila, Ofada, and Imported rice varieties could be attributed to the amylose and amylopectin ratio variation and their chain length distribution [29,35].

3.4 Bulk Density

The bulk densities of studied rice varieties are presented in Table 1. The Illa rice had 0.10g/ml, Abakaliki rice had 0.25g/ml, Jemila rice recorded 0.25g/ml, and Ofada rice had 0.20g/ml, and Imported rice recorded at 0.10g/ml. It was observed that Abakaliki and Jemila rice flour had the highest value (0.25g/ml), followed by Ofada rice which had 0.20g/ml of bulk density. Illa and Imported rice had the lowest value (0.10g/ml) of bulk density. The bulk densities obtained were within the range discussed in [27,36] for composite flour and yam-soy blend. Particle size and starch polymer structure influence bulk density. The starch polymers' fluid structure may cause a low bulk density. In rice flour, the samples (Illa and imported rice) with a low bulk density (0.10g/ml) are preferred because they are easier to package and transport [37].

3.5 Dispersibility

The results of the investigated rice flour are presented in Table 1. The values obtained for Illa, Abakaliki, Jemila, Ofada, and imported rice varieties were 53.00, 72.00, 73.50, 75.00, and 70.50%, respectively. The dispersibility of Ofada rice flour (75.00%) was the highest, followed by Jemila rice sample flour with 73.50%, and Abakaliki had 72.00% of dispersibility. The lowest value (53.00%) was observed in the Illa rice sample, followed by imported rice with 70.50% of dispersibility. The values obtained in the research were higher to compare the report of [35] on rice flour and very similar to what [39] reported on wheat composite flour. Varietal differences may be to blame for the differences in studied rice flour's dispersibility. The tendency of flour to detach from water molecules and reveal its hydrophobic action is determined by the property of dispersibility. The rice samples (Ofada and Jemila) with the highest values (75.00% and 73.50%) of dispersibility were easily reconstituting to give fine consistent dough during mixing [40].

3.6 Swelling Power

Table 1 displays the swelling power result, which indicates the degree to which a flour sample expands relative to its initial volume when soaked in water. Illa rice had 8.36% swelling power, and Abakaliki rice recorded 10.40%, Jemila rice was observed to be 10.39% swelling power and Ofada rice from western Nigeria had 10.91% of swelling power, and imported rice had 8.69% of swelling power. This result obtained in swelling power of varieties of rice flour tested was like [41] reported on wheat composite flour and higher than the report of [38] on rice flour. Swelling power is a weight measurement of swollen starch granules and their occluded water; therefore, it measures hydration capacity. The swollen starch granules' water retention is frequently linked to food quality. The Nigerian rice flour recorded better values of swelling power than almighty imported rice flour. Consequently, Nigerian rice varieties are preferred to imported rice.

3.7 Solubility Index

From the result of functional properties presented in Table 1, the values for the solubility index of tested rice samples were 10.69, 12.10, 9.54, 10.01, and 12.34% for Illa, Abakaliki, Jemila, Ofada, and Imported rice flour varieties. It was observed that imported rice had the highest solubility value (12.34%), followed by Abakaliki rice, which had 12.20%, and Illa rice, which had 10.69%. The lowest value (9.54%) was found in Jemila rice, followed by Ofada rice which recorded 10.01%. These solubility index values were higher than the report of [41] on wheat flour and in range with the report of [38] on rice flour. The shorter chain lengths of the starch molecules in the sample (Imported rice) would weaken the hydrogen bonds that hold the granules together, resulting in the sample's higher solubility [38].

3.8 Amylose

The result of the amylose content of tested rice flour varieties which determine the gelatinisation temperature, pasting behaviour, and viscoelastic properties of flour samples, were presented in Table 1. The amount of amylose in the raw materials is a significant factor in determining the product's intended use [38]. The amylose contents for Illa, Abakaliki, Jemila, Ofada, and Imported rice varieties were 23.73, 23.45, 24.42, 25.64, and 22.45%, respectively. According to [42], low-amylose rice flour provides moistness, softness, and chewiness to product textures. The findings showed that Ofada rice had the highest value (25.64%) of amylose content, followed by Jemila rice, which recorded 24.42% of amylose content. The lowest value (22.45%) was found in imported rice, followed by Abakaliki rice with 23.45% and Illa rice sample with 23.73% of amylose content. These values of amylose content of the studied rice varieties agreed with the report of [41] on wheat flour. The rice will become stickier the less amylose it contains. When cooked, the extremely low amylose content of sticky or glutinous rice causes it to be sticky [43]. When properly cooked, rice samples with an amylose content of 25% do not stick to one another [43]. These outcomes demonstrated that the rice samples analysed did not contain gluten. There are four types of non-sticky rice: rice with a high amylose content (25 to 33 per cent), rice with a medium amylose content (20 to 25 per cent), and rice with a low amylose content (9 to 20 per cent). Moreover, rice with very little amylose (between 2 and 9 per cent) [44]. Ofada rice falls into rice with a high amylose content, while other varieties fall into rice with a medium amylose content. The application of fertiliser (nitrogen content), the time of year, the location of the growing areas, and the amylose and starch content of various rice varieties could be to blame [45].

3.9 Amylopectin

The amylopectin content of tested rice varieties was 76.26, 76.55, 75.58, 74.37, and 77.55% for Illa, Abakaliki, Jemila, Ofada, and Imported rice, respectively. These findings agreed with what [41] reported on wheat flour. The result revealed that Imported rice had the highest value (77.55%) of amylopectin, followed by Abakaliki rice which recorded 76.55% and the Illa rice sample from the Southern region had 76.28%. In comparison, the Ofada rice sample had the lowest value (74.37%) of amylopectin content, followed by Jemila rice which recorded 75.58% of amylopectin content. The higher the content of amylopectin, the stickier the rice will be. Their distinct origins could be the cause of the observed difference. The present investigation has revealed that amylopectin is influenced by flour's starch and amylose content. This means that one is influenced by the other, and both properties are necessary for food preparation and development. Starch's high viscosity and waxiness are caused by amylopectin [46].

4. Conclusion

The investigation conducted on four different Nigerian rice and one imported rice variety showed a significant difference in functional properties at ($p>0.05$). Illa (Nigerian) rice had the highest WAC (353.76%) and OAC (218.32%) than the imported rice variety. The bulk density of both samples had less significantly different vales. The percentage values of dispersibility (75.00%) were higher in Ofada (Nigerian) rice. Nigerian (Ofada) rice had a higher swelling index (10.91%), the amylose value (rice) had the highest value (25.64%), while the amylopectin value (77.55%) was found to be more prominent in imported rice. It can be concluded that Nigerian rice is compared handsomely with imported rice in terms of functional properties, determining the end-use of flour samples.

Author Contributions

All the authors contributed equally to the paper. They all read and approved the last version of the paper.

Conflict of Interest

The authors declare no conflict of interest.

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Finite coproducts in the category of quadratic modules of Lie algebras

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Keywords
Quadratic modules, Quasi-quadratic modules, Coproduct objects

Abstract — In this study, we will construct finite coproduct objects in the category of quadratic modules of Lie algebras with a new approach using the idea of quasi-quadratic modules.

Subject Classification (2020): 18A40, 18G45.

1. Introduction

The concept of the crossed module is an algebraic model described by Whitehead for classifying homotopy 2-types [1]. It has attracted the attention of many researchers. This notion initially introduced in groups has also naturally appeared in various algebraic cases as commutative and associative algebras, Lie and Lie-Rinehart algebras, etc, [2–9]. Kassel and Loday studied the classification of central extensions of Lie algebras and crossed modules of Lie algebras in [10]. In [11], Casas and Ladra studied some properties of the category of crossed modules of Lie algebras. Ellis constructed the coproduct of crossed modules of Lie algebras with the same base of Lie algebras [12]. D. Conduché introduced one of the models beyond the algebraic 2-type and called 2-crossed modules [13] (For studies of homotopy, see [14–16]). In [17], Carrasco and Porter developed the coproduct of 2-crossed modules. Some of the related works for algebraic models associated with homotopy 3-type can be found in [18–21].

In this study, we focus on quadratic modules of Lie algebras, one of the algebraic 3-type model, developed by Baues for group case, and whose homotopy structure is defined [22]. The Lie algebra version of this model was introduced in the [23] studied by Ulualan and Uslu, while the studies in [24, 25] rely on quadratic module of commutative algebras. A different homotopy relation for quadratic modules of Lie algebras is constructed in [26, 27]. We will construct the finite coproduct objects in the category of quadratic modules suggested in Remark 3 given in [17]. For the construction of the coproduct of quadratic modules of Lie algebras with the same base of $nil(2)$-module, we will follow a construction technique similar to that used

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to built the coproduct structure of crossed modules of Lie algebras. The coproduct of two crossed modules with the same base object is the associated crossed module: see [28, Chapter 4]. The tool that we will use while upgrading the dimension of this construction will be the concept of quasi-quadratic modules given in [29]. Any quadratic module is a quasi-quadratic module. In more detail, the category of quadratic modules is a reflexive subcategory of the category of quasi-quadratic modules, and an associated quadratic module functor is defined as follows:

$$(-)^{cr} : \mathscr{Q}\mathscr{Q}\mathscr{M}_L \longrightarrow \mathbf{QM_L}$$

In our study, using the above functor, which is left adjoint to the inclusion functor, we will define the coproduct of two quadratic modules with the same base of $nil(2)$-module as the associated quadratic module to their coproduct in the category of quasi-quadratic modules.

2. Preliminaries

Let k be a commutative ring with unit and we will refer to a Lie algebra over k as a Lie algebra, and the Lie bracket multiplication will be denoted as $[-,-]$.

2.1. Lie Algebra Actions

Let Z and Y be Lie algebras over k, a k-bilinear map $Z \times Y \to Y, (z,y) \mapsto z * y$, is called a Lie algebra action of Z on Y, if the below equations are verified:

L1) $z * [y,y'] = [z * y, y'] + [y, z * y']$

L2) $[z,z'] * y = z * (z' * y) - z' * (z * y)$

for each $z, z' \in Z$ and $y, y' \in Y$.

2.2. Crossed Modules of Lie Algebras

A crossed module of Lie algebras, $\left(Y \xrightarrow{\partial} Z\right)$, consists of Lie algebras Y and Z with a left Lie algebra action "$*_1$" of Z on Y, and a Lie algebra homomorphism $\partial : Y \to Z$ satisfying the following conditions:

XMod$_L$1: $\partial(z *_1 y) = [z, \partial(y)]$, for all $z \in Z$ and $y \in Y$

XMod$_L$2: $\partial(y) *_1 y' = [y, y']$, for all $y, y' \in Y$

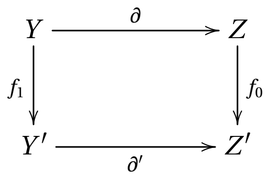
Note that "**XMod$_L$2**" is called the Peiffer identity, [10].

Example 2.1. Let I be a Lie ideal of a Lie algebra Z with $i : I \to Z$ the inclusion, in this case Z acts on the left I by conjugation and the inclusion Lie homomorphism i makes $\left(I \xrightarrow{i} Z\right)$, into a crossed module of Lie algebra.

Let $\left(Y \xrightarrow{\partial} Z\right)$ and $\left(Y' \xrightarrow{\partial'} Z'\right)$ are crossed modules of Lie algebras, a morphism, $f = (f_1, f_0) : \left(Y \xrightarrow{\partial} Z\right) \longrightarrow \left(Y \xrightarrow{\partial} Z\right)$ of crossed modules consists of Lie algebra homomorphisms $f_1 : Y \to Y'$ and $f_0 : Z \to Z'$ such that

- $\partial' f_1 = f_0 \partial$
- $f_1(z *_1 y) = f_0(z) *'_1 f_1(y)$

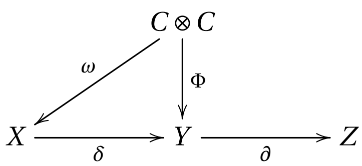
for all $Z \in Z$ and $y \in Y$. Thus, this means that the "f" morphism $*_1$ preserves the Lie algebra action, and the diagram below makes it commutative:



Together with these definitions, we can define the category of crossed modules over Lie algebras by denoting it as $\mathbf{XMod_L}$. If we fix the base of the crossed module, the Z Lie algebra, then $\mathbf{XMod_L/Z}$ will be the category of crossed Z-modules, which is a subcategory of $\mathbf{XMod_L}$.

2.3. Quadratic Modules of Lie Algebras

A quadratic module of Lie algebras $\mathscr{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ is a diagram:



of Lie algebra homomorphisms between Lie algebras such that $\mathbf{QM_L1}$, $\mathbf{QM_L2}$, $\mathbf{QM_L3}$, and $\mathbf{QM_L4}$ hold:

$\mathbf{QM_L1}$: The homomorphism $\partial : Y \to Z$ is a $nil(2)$-module and $Y \twoheadrightarrow C = Y^{cr}/[Y^{cr}, Y^{cr}]$ is defined by $y \mapsto [y]$ and Φ is defined by

$$\Phi([y_1] \otimes [y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

for $y_1, y_2 \in Y$,

$\mathbf{QM_L2}$: The boundary Lie homomorphisms composition of ∂ and δ satisfy $\partial\delta = 0$ and the quadratic map ω is a lift of the Peiffer commutator map Φ, that is $\delta\omega = \Phi$ or equivalently

$$\delta\omega = \Phi([y_1] \otimes [y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

for $y_1, y_2 \in Y$,

$\mathbf{QM_L3}$: X is a Lie Z-algebra, all of the homomorphisms in the diagram are Z-equivariant, and the action of Z on X also holds the following equality

$$\partial(y) *_3 x = \omega([\delta(x)] \otimes [y] + [y] \otimes [\delta(x)])$$

for $x \in X$ and $y \in Y$,

$\mathbf{QM_L4}$: For all $x_1, x_2 \in X$ commutators in X satisfy the formula

$$\omega([\delta(x_1)] \otimes [\delta(x_2)]) = [x_2, x_1]$$

Remark 2.2. It should be noted that $\left(X \xrightarrow{\delta} Y\right)$ is a crossed module, with

$$y *_2 x = \omega([\delta(x)] \otimes [y])$$

for each $y \in Y$ and $x \in X$. On the other hand, generally, $\left(Y \xrightarrow{\partial} Z\right)$ is only a $nil(2)$-module.

Remark 2.3. By $\mathbf{QM}_L\mathbf{3}$, we have:

$$\partial(y) *_3 x - y *_2 x = \omega([y] \otimes [\delta(x)])$$

where $*_2$ is a Lie action of Y on X.

Lemma 2.4. Let $\mathscr{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ be a quadratic module of Lie algebras and consider the "$*_2$" and "$*_3$" Lie algebra actions. Then for all $z \in Z$ and $y_1, y_2, y_3 \in Y$, we have:

$$z *_3 \omega([y_1] \otimes [y_2]) = \omega([z *_1 y_1] \otimes [y_2]) + \omega([y_1] \otimes [z *_1 y_2]) \tag{2.1}$$

$$\begin{aligned} \omega([[y_1, y_2]] \otimes [y_3]) &= \partial(y_1) *_3 \omega([y_2] \otimes [y_3]) + \omega([y_1] \otimes [[y_2, y_3]]) \\ &\quad -\partial(y_2) *_3 \omega([y_1] \otimes [y_3]) - \omega([y_2] \otimes [[y_1, y_3]]) \end{aligned} \tag{2.2}$$

$$\omega([y_1] \otimes [[y_2, y_3]]) = y_2 *_2 \omega([y_1] \otimes [y_3]) - y_3 *_2 \omega([y_1] \otimes [y_2]) \tag{2.3}$$

Example 2.5. Thanks to any $nil(2)$-module $\left(Y \xrightarrow{\partial} Z\right)$, we define a quadratic module as follows:

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{1} & \downarrow{\Phi} & & \\ X & \xrightarrow[\Phi]{} & Y & \xrightarrow[\partial]{} & Z \end{array}$$

Example 2.6. If $\mathscr{L} = (X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-]))$ is a quadratic module, then $Im\delta$ is a Lie ideal of Y and we have there is an induced crossed module structure on

$$Y / Im\delta \xrightarrow{\partial} Z.$$

Let $\mathscr{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ and $\mathscr{L}' = \left(X' \xrightarrow{\delta'} Y' \xrightarrow{\partial'} Z', \omega'([-] \otimes [-])\right)$ be two quadratic modules, a morphism of quadratic modules of Lie algebras given by a diagram

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{\omega} & X & \xrightarrow{\delta} & Y & \xrightarrow{\partial} & Z \\ {\scriptstyle \varphi_* \otimes \varphi_*}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C' \otimes C' & \xrightarrow[\omega']{} & X' & \xrightarrow[\delta']{} & Y' & \xrightarrow[\partial']{} & Z' \end{array}$$

where (f_1, f_0) is a morphism between $nil(2)$-modules which induced $\varphi_* : C \to C'$ and where

$$\begin{aligned} f_1(z *_1 y) &= f_0(z) *'_1 f_1(y) \\ f_2(z *_3 x) &= f_0(z) *'_3 f_2(x) \\ f_2(\omega([y_1] \otimes [y_2])) &= \omega'([f_1(y_1)] \otimes [f_1(y_2)]) \end{aligned}$$

for all $z \in Z$, $y, y_1, y_2 \in Y$ and $x \in X$. We will denote by $\mathbf{QM_L}$ the category of quadratic modules of Lie algebras and by $\mathbf{QM_L}/(\mathbf{Y} \xrightarrow{\partial} \mathbf{Z})$ the subcategory of quadratic modules of fixed $nil(2)$-module $\left(Y \xrightarrow{\partial} Z\right)$.

2.4. Quasi-Quadratic Modules of Lie Algebras

A quasi-quadratic module of Lie algebras is a semiexact sequence

$$X \xrightarrow{\ \delta\ } Y \xrightarrow{\ \partial\ } Z$$

of Z-Lie algebras together with an ω quadratic map

$$\omega([-]\otimes[-]) : C \otimes C \longrightarrow X$$

such that $\mathcal{Q}\mathcal{Q}\mathcal{M}_L 1$, $\mathcal{Q}\mathcal{Q}\mathcal{M}_L 2$, $\mathcal{Q}\mathcal{Q}\mathcal{M}_L 3$, and $\mathcal{Q}\mathcal{Q}\mathcal{M}_L 4$ hold:

$\mathcal{Q}\mathcal{Q}\mathcal{M}_L 1$: For all $y_1, y_2, y_3 \in Y$,

$$\delta\omega([y_1]\otimes[y_2]) = \Phi([y_1]\otimes[y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

$\mathcal{Q}\mathcal{Q}\mathcal{M}_L 2$:

$$\begin{aligned} \omega([[y_1,y_2]]\otimes[y_3]) \quad &= \quad \partial(y_1) *_3 \omega([y_2]\otimes[y_3]) + \omega([y_1]\otimes[[y_2,y_3]])| \\ &\quad -\partial(y_2) *_3 \omega([y_1]\otimes[y_3]) - \omega([y_2]\otimes[[y_1,y_3]]) \end{aligned}$$

$\mathcal{Q}\mathcal{Q}\mathcal{M}_L 3$:

$$\omega([y_1]\otimes[[y_2,y_3]]) = y_2 *_2 \omega([y_1]\otimes[y_3]) - y_3 *_2 \omega([y_1]\otimes[y_2])$$

$\mathcal{Q}\mathcal{Q}\mathcal{M}_L 4$:

$$[\omega([y_1]\otimes[y_2]), \partial(y_1) *_3 (y_2 *_2 x)] = \omega([\partial(y_1) *_1 [y_2, \delta x]] \otimes [\delta\omega([y_1]\otimes[y_2])])$$

Quasi-quadratic module morphisms are defined in the same way as quadratic module morphisms. We will denote the category of quasi-quadratic module of Lie algebras by $\mathcal{Q}\mathcal{Q}\mathcal{M}_L$.

Furthermore, any quadratic module of Lie algebras is a quasi-quadratic module of Lie algebras, and we can construct a quadratic module of Lie algebras associated with a quasi-quadratic module of Lie algebras (see in [29, Lemma 3.2 and Lemma 3.4]). There exists an adjunction as below:

$$\mathcal{Q}\mathcal{Q}\mathcal{M}_L \underset{\hookleftarrow}{\overset{(-)^{cr}}{\rightleftarrows}} \mathbf{QM_L}$$

3. Finite Coproducts in the Category of Quadratic Modules of Lie Algebras

3.1. The Coproduct of Crossed Modules of Lie Algebras

In this section, we recall the definition of a coproduct object of crossed modules of Lie algebras given by [12].

Let $\left(Y \xrightarrow{\partial_Y} Z, *_Y\right)$ and $\left(X \xrightarrow{\partial_X} Z, *_X\right)$ be two crossed Z-modules of Lie algebras. As follows, Y has a left Lie algebra action on X:

$$\begin{aligned} Y \times X &\longrightarrow X \\ (y,x) &\longmapsto y \star x = \partial_Y(y) *_X x \end{aligned} \tag{3.1}$$

Thus, we can define $X \rtimes Y$ a semidirect product Lie algebra. Considering the left Lie algebra action of Z on $X \rtimes Y$, we define the following Lie algebra homomorphism:

$$\begin{aligned} \partial : X \rtimes Y &\longrightarrow Z \\ (x,y) &\longmapsto \partial((x,y)) = \partial_x(x) + \partial_y(y) \end{aligned}$$

for $(x,y) \in X \rtimes Y$.

Naturally, $\left(X \rtimes Y \xrightarrow{\partial} Z, *\right)$ is a pre-crossed Z-module:

$$\begin{aligned} \partial(z*(x,y)) &= \partial((z *_X x, z *_Y y)) \\ &= \partial(z *_X x, z *_Y y) \\ &= [z, \partial_X(x)] + [z, \partial_Y(y)] \\ &= [z, \partial_X + \partial_Y(y)] \\ &= [z, \partial((x,y))] \end{aligned}$$

for all $z \in Z$ and $(x,y) \in X \rtimes Y$. Let I be an ideal of $X \rtimes Y$ generated by the elements below

$$[(x,y),(x',y')] - \partial((x,y)) * (x',y')$$

for all $(x,y),(x',y') \in X \rtimes Y$.

Moreover, we have:

$$\begin{aligned} [(x,y),(x',y')] - \partial((x,y)) * (x',y') &= ([x,x'] + y \star x' - y' \star x, [y,y']) - ((\partial_X(x) + \partial_Y(y)) * (x',y')) \\ &= ([x,x'] + \partial_Y(y) *_X x' - \partial_Y(y') *_X x, [y,y']) \\ &\quad -(\partial_X(x) *_X x' + \partial_Y(y) *_X x', \partial_X(x) *_X y' + \partial_Y(y) *_Y y') \\ &= ([x,x'] + \partial_Y(y) *_X x' - \partial_Y(y') *_X x - \partial_X(x) *_X x' - \partial_Y(y) *_X x', [y,y'] \\ &\quad -\partial_X(x) *_Y y' - \partial_Y(y) *_Y y') \\ &= (-\partial_Y(y') *_X x, -\partial_X(x) *_Y y') \end{aligned}$$

This means I is generated by the elements:

$$(-\partial_Y(y') *_X x, -\partial_X(x) *_Y y')$$

Consider $\partial(I) = 0$. Then, for each $(x, y) \in X \rtimes Y$, we get the following induced morphism:

$$\begin{aligned} \bar{\partial} : (X \rtimes Y)/I &\longrightarrow Z \\ ((x, y) + I) &\longmapsto \bar{\partial}((x, y) + I) = \partial_x(x) + \partial_y(y) \end{aligned}$$

For all $(x, y), (x', y') \in X \rtimes Y$, this structure gives us the crossed module definition:

$$\begin{aligned} \bar{\partial}((x, y) + I) * ((x', y') + I) &= (\partial_X(x) + \partial_Y(y)) * ((x', y') + I) \\ &= (\partial_X(x) *_X x' + \partial_Y(y) *_X x', \partial_X(x) *_Y y' \\ &\quad +\partial_Y(y) *_Y y') + I \\ &= ([x, x'] + \partial_Y(y) *_Y x' - \partial_Y(y') *_X x, [y, y']) + I \\ &= [(x, y) + I, (x', y') + I] \quad (\because \mathbf{XMod_L 2}) \end{aligned}$$

With these structures, we have the crossed module of Lie algebra $\left((X \rtimes Y)/I \xrightarrow{\bar{\partial}} Z\right)$, which is the coproduct object in $\mathbf{XMod_L}/\mathbf{Z}$.

3.2. The Coproduct of Quadratic Modules of Lie Algebras

Let

$$\mathscr{L}_1 = \left(\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega_1} & \downarrow \Phi & & \\ X_1 & \xrightarrow[\delta_1]{} & Y & \xrightarrow[\partial]{} & Z \end{array} \right) \text{ and } \mathscr{L}_2 = \left(\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega_2} & \downarrow \Phi & & \\ X_2 & \xrightarrow[\delta_2]{} & Y & \xrightarrow[\partial]{} & Z \end{array} \right)$$

be two quasi-quadratic module over $(Y \xrightarrow{\partial} Z)$. Thus we have

$$\delta_1\omega_1([y_1] \otimes [y_2]) = \delta_2\omega_2([y_1] \otimes [y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

for all $y_1, y_2 \in Y$. Then let I be the Lie ideal of $X_1 \rtimes X_2$ generated by the elements:

$$\left(q_1\omega_1([y_1] \otimes [y_2]), q_2\omega_2([y_1] \otimes [y_2])\right)$$

where $q_i = \pm 1$ and $q_1 \neq q_2$.

Now let us define quadratic map.

$$\begin{aligned} \bar{\omega}([-] \otimes [-]) : C \otimes C &\longrightarrow (X_1 \rtimes X_2)/I \\ (y_1, y_2) &\longmapsto \bar{\omega}([y_1] \otimes [y_2]) = (\omega_1([y_1] \otimes [y_2]), 0) + I \\ &\qquad\qquad = (0, \omega_2([y_1] \otimes [y_2])) + I \end{aligned}$$

by considering $(\omega_1([y_1] \otimes [y_2]), -\omega_2([y_1] \otimes [y_2])) \in I$.

Moreover, Y acts on $X_1 \rtimes X_2/I$ as follows:

$$\diamond : Y \times (X_1 \rtimes X_2/I) \longrightarrow X_1 \rtimes X_2/I$$

$$(y,((x,x')+I)) \longmapsto y\diamond((x,x')+I) = (y *_2 x, y *_2 x') + I$$

for all $y \in Y$ and $(x,x') \in X_1 \rtimes X_2$. Additionally, using $\bar{\delta}(I) = 0$, we have induced morphism as follows:

$$\bar{\delta} : (X_1 \rtimes X_2)/I \longrightarrow Y$$

$$((x_1,x_2)+I) \longrightarrow \bar{\delta}((x_1,x_2)+I) = \delta_1(x_1) + \delta_2(x_2)$$

Theorem 3.1. The pair of $\bar{\mathscr{L}} = \big((X_1 \rtimes X_2)/I \xrightarrow{\bar{\delta}} Y \xrightarrow{\partial} Z, \bar{\omega}([-]\otimes[-]), \partial, \diamond\big)$ is a quasi-quadratic module which is the coproduct object of $\mathscr{L}_1$ and $\mathscr{L}_2$ in the category of $\mathscr{QQM}_L/(Y \xrightarrow{\partial} Z)$.

Proof.
First we need to show that $\bar{\mathscr{L}}$ is a quasi-quadratic module. For this we need to provide $\mathscr{QQM}_L1$, $\mathscr{QQM}_L2$, $\mathscr{QQM}_L3$, and $\mathscr{QQM}_L4$ axioms:

$\mathscr{QQM}_L$**1:**

$$\begin{aligned}
\bar{\delta}\bar{\omega}([y_1]\otimes[y_2]) &= \bar{\delta}((\omega_1([y_1]\otimes[y_2]),0)+I)\\
&= \bar{\delta}(\omega_1([y_1]\otimes[y_2]),0)+\bar{\delta}(I)\\
&= \delta_1\omega_1([y_1]\otimes[y_2])\\
&= \partial(y_1) *_1 y_2 - [y_1,y_2]
\end{aligned}$$

$\mathscr{QQM}_L$**2:**

$$\begin{aligned}
\bar{\omega}([[y_1,y_2]]\otimes[y_3]) &= (\omega_1([[y_1,y_2]]\otimes[y_3]),0)+I\\
&= (\partial(y_1) *_3 \omega_1([y_2]\otimes[y_3] + \omega_1([y_1]\otimes[[y_2,y_3]])\\
&\quad -\partial(y_2) *_3 \omega_1([y_1]\otimes[y_3]) - \omega([y_2]\otimes[[y_1,y_3]]),0)+I\\
&= (\partial(y_1) *_3 \omega_1([y_2]\otimes[y_3]),0)+I+(\omega_1([y_1])\otimes[[y_2,y_3]]),0)+I\\
&\quad +(-\partial(y_2) *_3 \omega_1([y_1]\otimes[y_3]),0)+I+(-\omega_1([y_2])\otimes[[y_1,y_3]]),0)+I\\
&= \partial(y_1) *_3 \bar{\omega}([y_2]\otimes[y_3]) + \bar{\omega}([y_1]\otimes[[y_2,y_3]])\\
&\quad -\partial(y_2) *_3 \bar{\omega}([y_1]\otimes[y_3]) - \bar{\omega}([y_2]\otimes[[y_1,y_3]])
\end{aligned}$$

$\mathscr{QQM}_L$**3:**

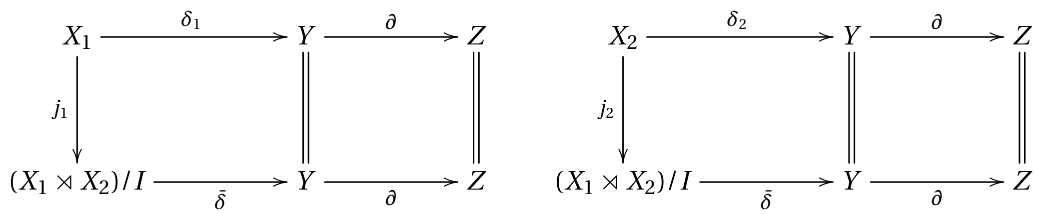
$$\begin{aligned}
\bar{\omega}([y_1]\otimes[[y_2,y_3]]) &= (\omega_1([y_1]\otimes[[y_2,y_3]]),0)+I\\
&= (y_2 *_2 \omega_1([y_1]\otimes[y_3]) - y_3 *_2 \omega_1([y_1]\otimes[y_2]),0)+I\\
&= (y_2 *_2 \omega_1([y_1]\otimes[y_3],0)+I-(y_3 *_2 \omega_1([y_1]\otimes[y_2]),0)+I\\
&= y_2 *_2 (\omega_1([y_1]\otimes[y_3],0)+I - y_3 *_2 (\omega_1([y_1]\otimes[y_2]),0)+I\\
&= y_2\diamond\bar{\omega}([y_1]\otimes[y_3]) - y_3\diamond\bar{\omega}([y_1]\otimes[y_2])
\end{aligned}$$

$\mathscr{Q}\mathscr{Q}\mathscr{M}_L 4$**:**

$$
\begin{aligned}
[\bar{\omega}([y_1]\otimes[y_2]), \partial(y_1) *_3 (y_2 \diamond (x_1,x_2)+I)] &= [(\omega_1([y_1]\otimes[y_2]),0)+I, (\partial(y_1) *_3 (y_2 *_2 x_1), \partial(y_1) *_3 (y_2 *_2 x_2))+I] \\
&= ([\omega_1([y_1]\otimes[y_2]), \partial(y_1) *_3 (y_2 *_2 x_1)] + 0 \star (\partial(y_1) *_3 (y_2 *_2 x_1)) \\
&\quad -(\partial(y_1) *_3 (y_2 *_2 x_2)) \star \omega_1([y_1]\otimes[y_2]), [0, \partial(y_1) *_3 (y_2 *_2 x_2)]) + I \\
&= (\omega_1([\partial(y_1) *_1 [y_2, \delta_1(x_1)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]) \\
&\quad -\delta_2(\partial(y_1) *_3 (y_2 *_3 x_2)) *_2 \omega_1([y_1]\otimes[y_2]),0) + I \\
&= (\omega_1([\partial(y_1) *_1 [y_2, \delta_1(x_1)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]) \\
&\quad +\omega_1([\partial(y_1) *_1 [y_2, \delta_2(x_2)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]),0) + I \\
&= (\omega_1([\partial(y_1) *_1 [y_2, \delta_1(x_1)+\delta_2(x_2)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]),0) + I \\
&= \bar{\omega}([\partial(y_1) *_1 [y_2, \bar{\delta}((x_1,x_2)+I)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]])
\end{aligned}
$$

for all $y_1, y_2, y_3 \in Y$ and $(x_1,x_2)+I \in (X_1 \rtimes X_2)/I$.

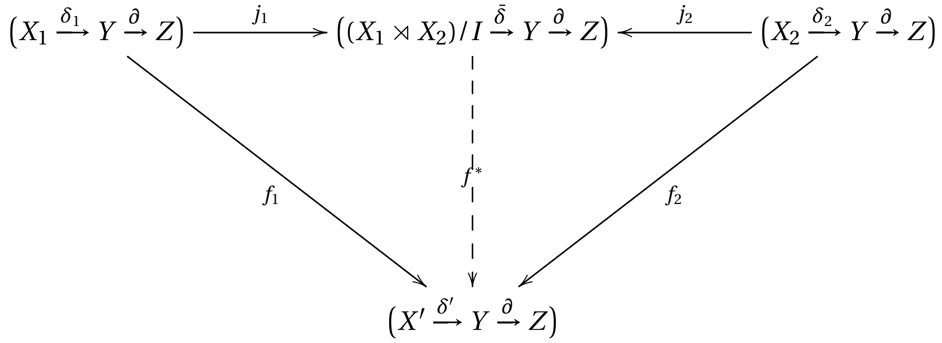
Furthermore, the canonical morphisms are given by

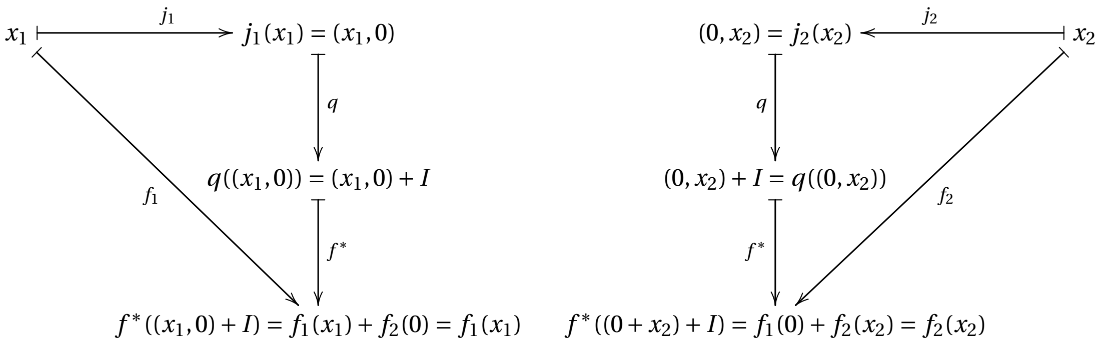


where $j_t, t=1,2$, composition $X_t \xrightarrow{i_t} (X_1 \rtimes X_2) \to (X_1 \rtimes X_2)/I$, i_t is canonical inclusion in the semidirect product and second morphism is quotient homomorhism:

$$
\begin{aligned}
f^* : (X_1 \rtimes X_2)/I &\longrightarrow X' \\
(x_1,x_2)+I &\longmapsto f^*((x_1,x_2)+I) = f_1(x_1) + f_2(x_2)
\end{aligned}
$$

which satisfies the universal property of coproduct object with the following commutative diagram completes the proof.





Corollary 3.2. Consider the "$(-)^{cr}$" functor and adjunction given in [29]. If $\mathscr{L}_1 = (X_1 \xrightarrow{\delta_1} Y \xrightarrow{\partial} Z, \omega_1([-]\otimes[-]))$ and $\mathscr{L}_2 = (X_2 \xrightarrow{\delta_2} Y \xrightarrow{\partial} Z, \omega_2([-]\otimes[-]))$ are $\mathbf{QM_L}/(\mathbf{Y} \xrightarrow{\partial} \mathbf{Z})$, then applying functor $(-)^{cr}$ to $((X_1 \rtimes X_2)/I \xrightarrow{\bar{\delta}} Y \xrightarrow{\partial} Z, \bar{\omega}([-]\otimes[-]))$ with the morphism $u_t : X_t \xrightarrow{j_t} (X_1 \rtimes X_2)/I \to ((X_1 \rtimes X_2)/I)^{cr}$, gives the coproduct object of $\mathscr{L}_1$ and $\mathscr{L}_2$ in the category of $\mathbf{QM_L}/(\mathbf{Y} \xrightarrow{\partial} \mathbf{Z})$.

We denote the coproduct object of $\mathscr{L}_1$ and $\mathscr{L}_2$ by

$$\left(\mathscr{L}_1 \rtimes_* \mathscr{L}_2\right) = X_1 \rtimes_* X_2 \xrightarrow{(\delta_1 \rtimes_* \delta_2)} Y \xrightarrow{\quad\partial\quad} Z$$

4. Conclusion

In this paper, we constructed finite coproduct objects in the category of quadratic modules of Lie algebras with the same base as the $nil(2)$-module. This structure can be generalised by changing the base. This construction can be defined for other algebraic cases and thus may reveal important structures for nonabelian algebraic topology or categorification.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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A web scraping-based approach for fundamental analysis platform in financial assets

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Keywords:
Django,
Financial forecasting,
Fundamental analysis,
Selenium,
Web scraping

Abstract — There are two main benefits of using fundamental analysis for investors and portfolio managers. First, investing in a company with good ratios has lower risks. The second reason is that it is possible to evaluate share prices with internal valuation methods based on ratios. These price valuations can be more meaningful when combined with technical analysis data. Many data terminals provide processes such as fundamental analysis data and price valuation on a paid and licensed basis. However, the balance sheet data of publicly traded markets are publicly available and can be obtained and interpreted by web scraping methods. This study presents an approach in which basic analysis and price evaluation are made with balance sheets and ratios using open-source tools and web scraping.

Subject Classification (2020): 91G15, 91G80.

1. Introduction

Two main approaches are used in financial markets: technical analysis and fundamental analysis. Technical analysis mainly prefers by short-term investors called traders [1] and uses statistical methods-based approaches on time series consisting of opening, closing, lowest, highest price, and volume information of financial assets. These approaches are based on moving averages and indicators obtained with similar approaches to these averages [2]. On the other hand, fundamental analysis is an approach based on the ratios announced quarterly. These ratios are the company's quarterly profit or loss, indebtedness, stock status, cash flow, Price/Earnings ratio (P/E), and market Price-to-Book value (P/B) ratio [3]. Unlike technical analysis data, fundamental analysis data makes sense for long-term investment. For example, while a company's profitability increases, the decrease in the stock holding period and the decrease in the receivable collection period can indicate a significant acceleration in the company's growth figures. Using these ratios, the company's intrinsic value, that is, the required price can be determined. Different approaches are used for this valuation process. [4]. It is common to choose the statistically most probable of these approaches or to average them. In this way, if there are significant differences between the instant value of the company in the market in which it is traded and its real value, investment opportunities arise in the markets that are traded in two directions.

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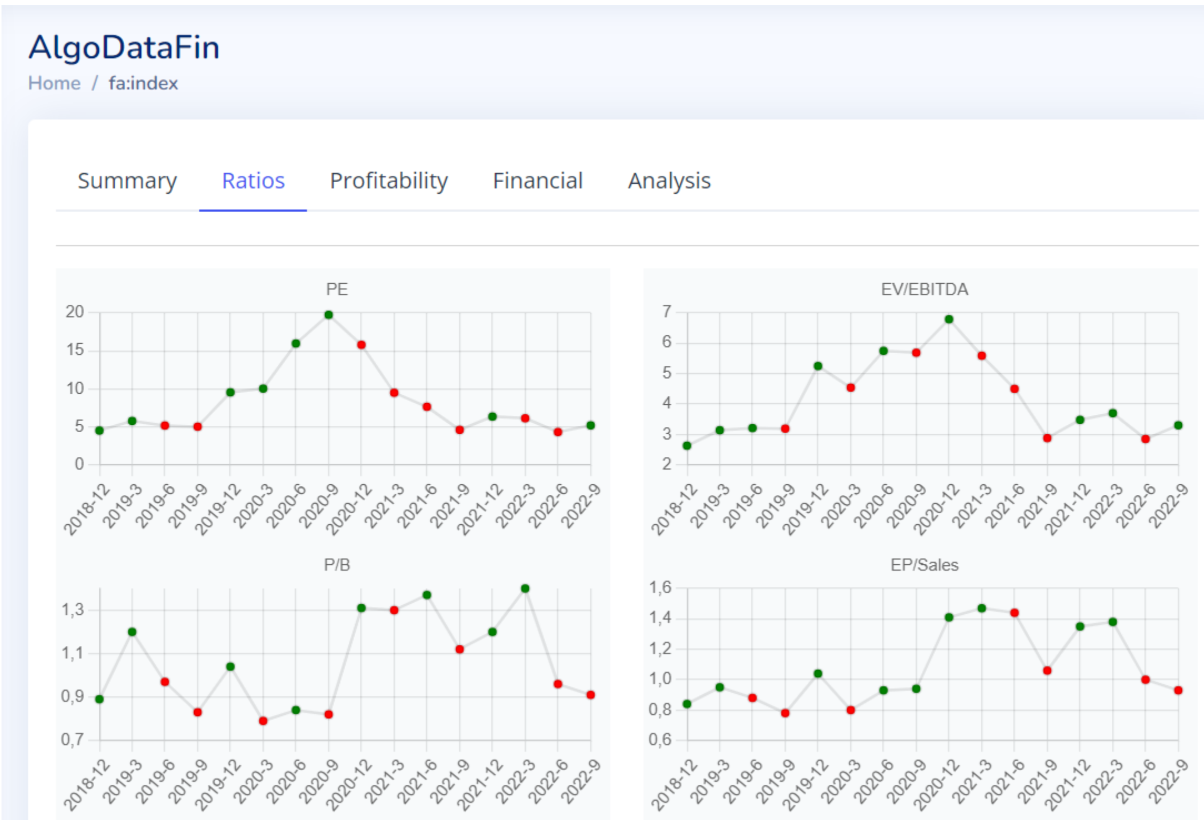
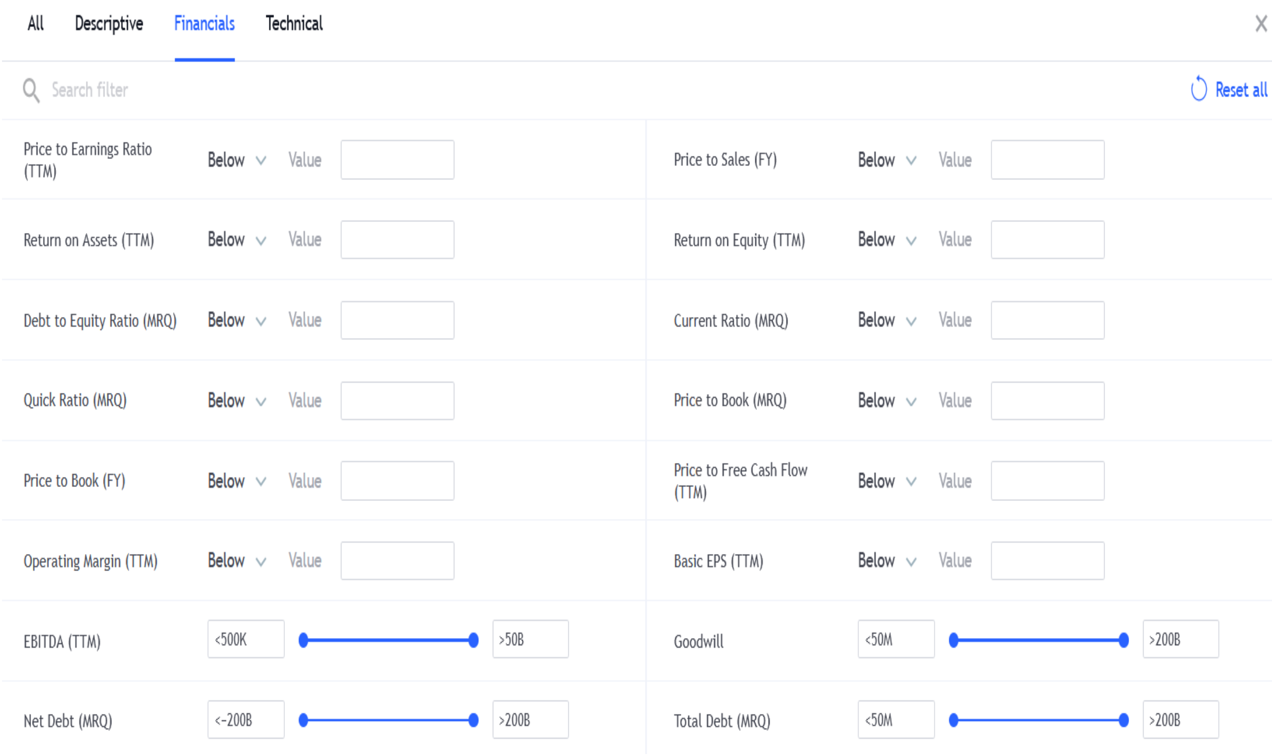
The instant price of the company, which is traded on the stock market, is meaninglessly above these internal values, which provides an opportunity for short selling or short trading in the futures markets. As an opposite example, if the company's current price is below the stock market, there is an opportunity for a long transaction in the futures market [5]. Subfigures obtained from data terminals for technical and fundamental analysis are given in Figure.1. While technical analysis is used to determine the trend direction and momentum with price and price averages, fundamental analysis filters out hundreds of financial assets traded and identifies companies to invest in.

(a)



(b)



(c)

Figure 1. Fundamental analysis screener (a) Tradingview technical analysis [6] (b) Fundamental analysis example (c) Tradingview platform fundamental analysis filtering [6]

The main contribution of this study to the literature is the analysis of financial assets based on basic ratios and the establishment of a backend structure for integrating artificial intelligence in the future. The study is organized as Background, Material and Method, Results and Discussion and Conclusion sections.

2. Background

Fundamental analysis data of a company is obtained from the quarterly balance sheets announced by the company. Since it is generally prepared as an excel table, the ratios can be created automatically with a programming language. However, it can sometimes occur in extraordinary situations on data such as profitability, debt, stock transfer, and balance sheet. For example, the sale of an immovable property the company already owns can be seen as a profit in the balance sheet period. Still, the company may even have sold the immovable property for debt payment. Even if the balance sheet figures appear positive, such exceptional cases, called footnotes, should also be interpreted.

Similarly, in a situation where the amount of borrowing of the company increases, the company may have grown through acquisitions. Therefore, just like in technical analysis for financial assets, it cannot be expected that the forecasts to be made with the results to be obtained from the figures and formulas for the future in the fundamental analysis will show 100% consistency. For this purpose, text processing and interpretation approaches can be used. In this case, the most general approach that can be used is to make sense of the related footnotes by reading them automatically and classifying the emotions [7].

The ratios used in the basic analysis are given in groups in Figure.2 below. There are dozens of ratio data. All ratios can be obtained from excel tables, easily analysed and visualised with data science approaches, and converted into .csv format. As a general opinion, it is accepted that the P/E and P/B values are small, and the Earnings Before Interest, Tax, Depreciation and Amortisation (EBITDA) ratio grows positively. However, no single ratio is meaningful on its own. For example, investing in a company whose P/E value has decreased only according to this ratio may involve high risk. The main point here is to extract the association rules between the ratios, interpret them by experts, and make a price valuation [8].

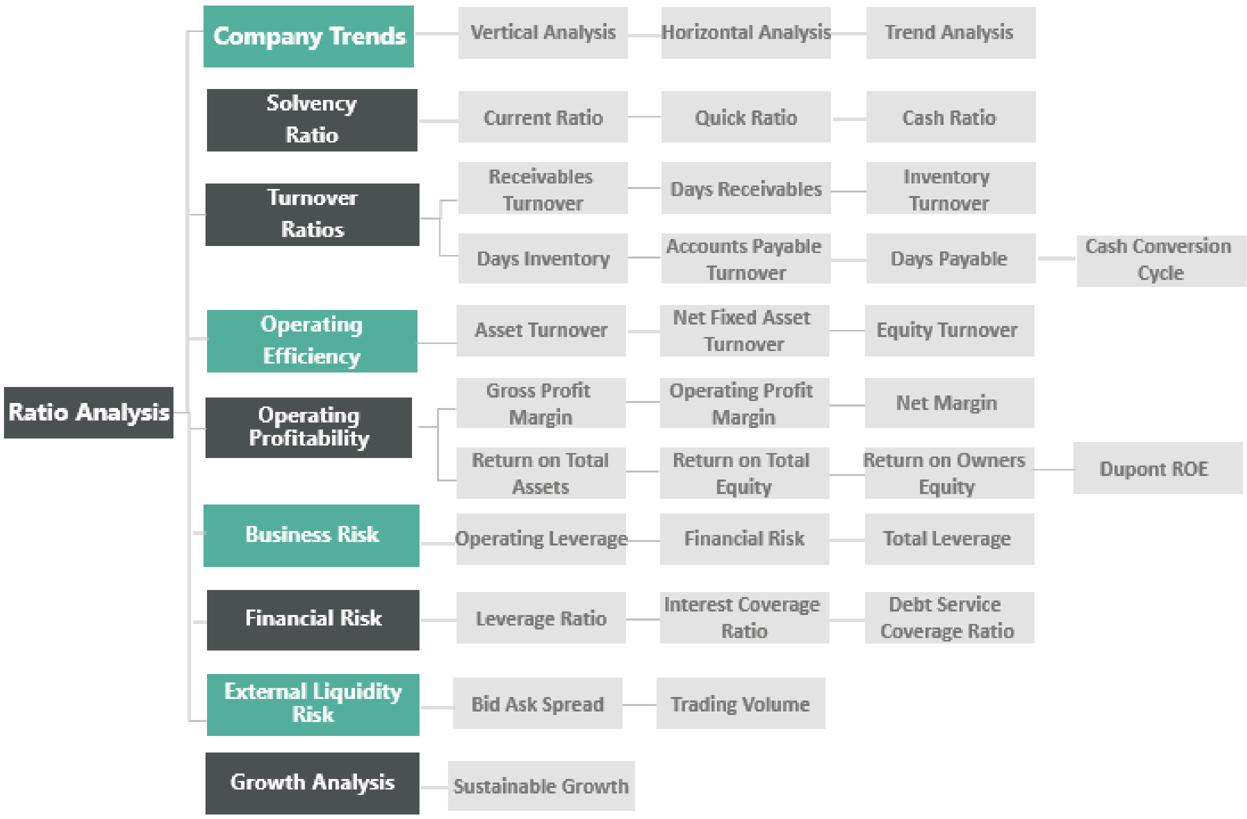


Figure 2. Grouped fundamental analysis ratios [9]

3. Material and Method

In the development of this approach, web-based software has been developed using the Python programming language and Selenium library to visualise and price valuation by pulling automated data from the Public Disclosure Platform (KAP), Yahoo Finance and similar sources that contain financial data and balance sheets [10-12]. The general diagram of the developed approach is given in Figure 3.

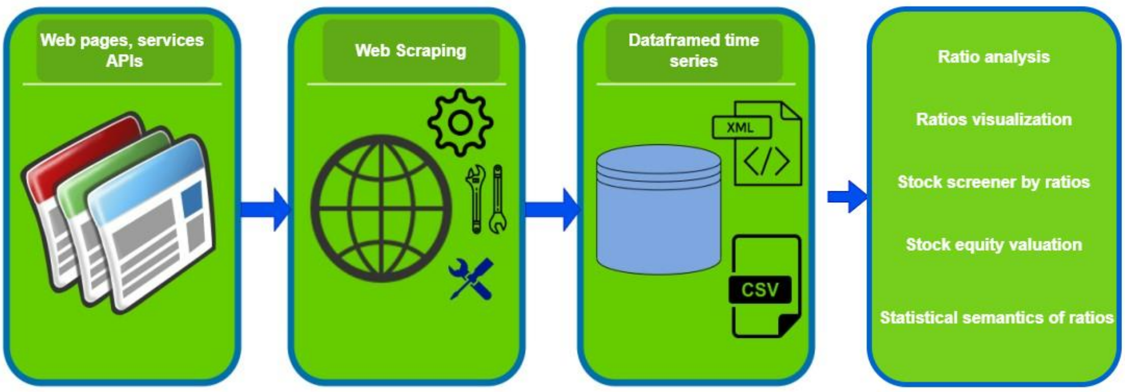


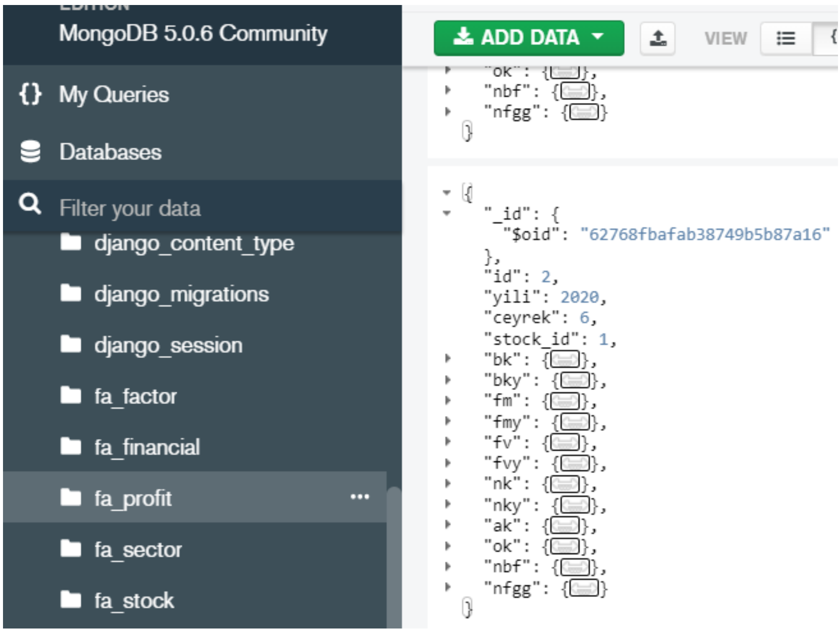
Figure 3. The architecture of the proposed model

Collecting data from websites and sorting through the collected data is called web scraping. Selenium is a web browser testing and automation software. It performs the operations according to the desired conditions. It is generally used for automated tests, but thanks to these features, it can be used for downloading and saving data from web resources within certain rules [13]. In Figure 4, raw data and the balance sheet, a typical excel document, are transformed into a structured data structure. For this purpose, MongoDB was used in the No-SQL structure. MongoDB keeps data in a .json/.bson structure [14, 15].

Balance Sheet Forecast

Assets	Beginning DD/MM/YY	Projected DD/MM/YY
Cash in bank	$ 2.00	$ 3.00
Accounts receivable	$ 2.00	$ 3.00
Inventory	$ 2.00	$ 3.00
Prepaid expenses	$ 2.00	$ 3.00
Other current assets	$ 2.00	$ 4.00
Total Current Assets	$ 10.00	$ 16.00
Fixed Assets		
Machinery & equipment	$ 1.00	$ 1.00
Furniture & fixtures	$ 1.00	$ 1.00
Leasehold improvements	$ 1.00	$ 1.00
Land & buildings	$ 1.00	$ 1.00
Other fixed assets	$ 1.00	$ 1.00
(Less accumulated depreciation on all fixed assets)	$ 1.00	$ 1.00
Total Fixed Assets	$ 6.00	$ 6.00
Other Assets		
Intangibles	$ 1.00	$ 1.00
Deposits	$ 1.00	$ 1.00
Goodwill	$ 1.00	$ 1.00
Other	$ 1.00	$ 1.00
Total Other Assets	$ 4.00	$ 4.00

(a)



(b)

Figure 4. Stock raw data and structured database image (a) Balance sheet raw data (b) No-Sgl MongoDB

In this study, the ratios were obtained with Selenium, and their structural storage was performed with MongoDB. Matplotlib was used in the Python environment for the visualisation of the ratios. Python and MongoDB map-reduce paradigm is used for stock filtering [16]. For the filtering process, the 1-Many structure has been designed to establish relations such as stocks, the sector it belongs to and indices. In Figure 5, the Django web framework and the model file of the application are given. Below is an example of obtaining a semantic data summary based on statistical data using the Map-Reduce paradigm and Aggregation framework. Pandas library data frame structure is used for filtering. This structure, which is kept as a data frame in the server environment, is sent to the client in .json format. Data sending is done only once. Filtering according to the selected criteria is performed on the client using Javascript. In this context, dynamic filtering and sorting can be done by combining date, string, integer, and float fields with and/or conditions with numerous criteria such as small, large, between, containing or not.

```
class Stock(models.Model):
    son = models.FloatField(null=True, blank=True)
    name = models.CharField(max_length=6, null=False)
    title = models.CharField(max_length=50, null=False)
    web = models.CharField(max_length=50, null=True, blank=True)
    f_k = models.FloatField(null=True,blank=True)
    pd_dd = models.FloatField(null=True,blank=True)
    fd_favok = models.FloatField(null=True,blank=True)
    hao = models.FloatField(null=True,blank=True)
    ypay = models.FloatField(null=True,blank=True)
    pd = models.FloatField(null=True,blank=True)

    sector=models.ForeignKey(Sector, on_delete=models.CASCADE)
    slug = models.SlugField(null=True, unique=True, db_index=True,default=0, allow_unicode=True)

    def save(self, *args, **kwargs):
        self.slug=slugify(self.name)
        super().save(*args, **kwargs)

    def __str__(self) -> str:
        return self.name
```

Figure 5. Example of using Django model and Aggregation framework Map-Reduce

The last application carried out within the scope of the study is the share valuation scenarios, which are very important for investors. In this context, three basic approaches are used in the example given. It can be assumed that the ratio presented here is P/E or P/B. In the first approach given in Equality 3.1, the share valuation is done by taking the proportional value according to the ratio. The approach used here is the simple ratio-proportion approach. In Equality 3.2, the sector average of the relevant ratio value is found instead of taking the constant of 1 as in Equality 3.1. For example, there is a relative value of the stock according to the average of all stocks in the "Food" sector. This approach can be extended recursively to the European and world averages. There is multiple valuation management according to the selected ratio feature. Equality valuation can be obtained by taking the average or weighted average of all valuations, as in Equality 3.3, with the simplest approach [4]. The approaches presented are based on statistical calculations. The motivation of this study is expanded with artificial intelligence-based prediction models in the future. For this purpose, recent literature studies will be used in price valuation of financial assets based on basic ratios [17-19].

$$Value\ of\ Equity\ (VoE) = Stock_{price}\ \frac{1}{Ratio_{stock}} \tag{3.1}$$

$$VoE = Stock_{price} \frac{\mu(Ratio_{sector})}{Ratio_{stock}} \tag{3.2}$$

$$VoE = \mu(VoE_1, VoE_2, \ldots, VoE_n) \tag{3.3}$$

4. Results

In this study, a software library has been introduced to enable investors to perform the basic analysis transactions they need for financial assets using the Python programming language and the Django web library. With the implemented application, a backend was created with MongoDB and Django. Then, stock information and balance sheets were automated with Selenium. The application has the features of obtaining and visualising the basic analysis ratios, making price valuations according to the basic ratios and making sense of them according to statistical values. Figure 6 displays basic information such as the last price, most frequently used ratios, free float, foreign share and price valuation based on more than one ratio value.

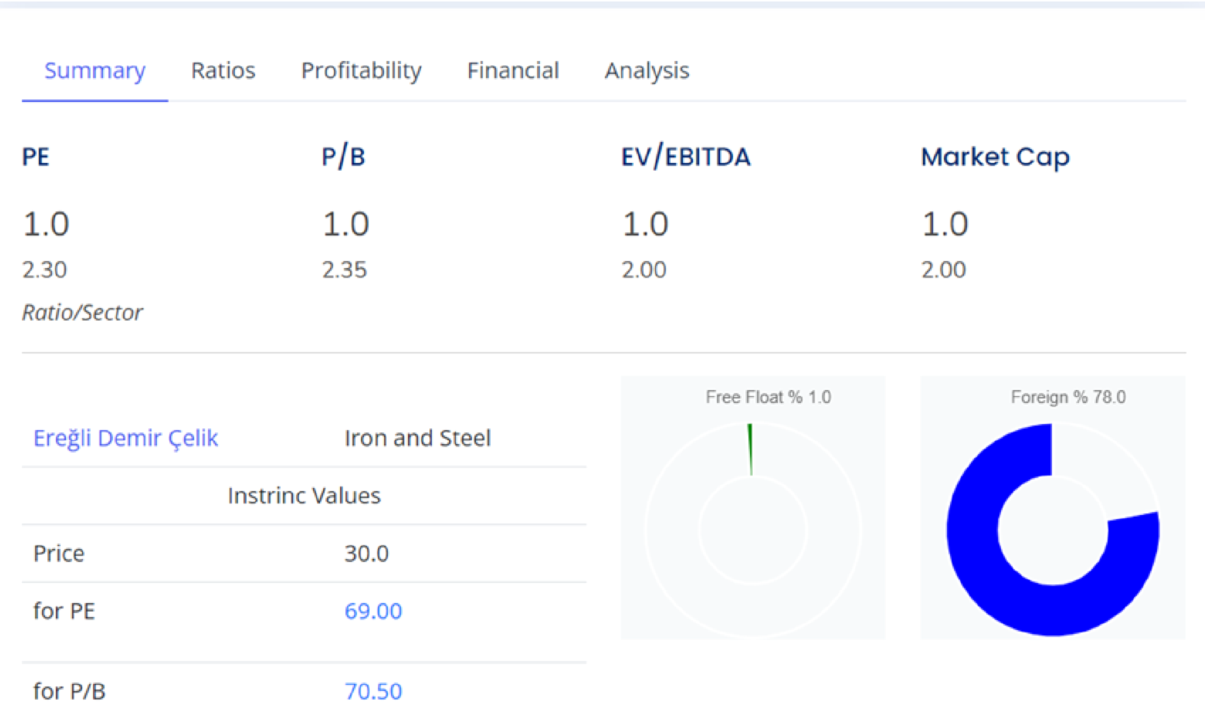


Figure 6. Display of stock basic ratios and price valuation according to more than one ratio

In Figure 6, the basic ratios and the groups they belong to are given in tabs. The ratios obtained from the quarterly balance sheets can be visually displayed in each tab. In Figure 7, ratios such as market value, earnings per share, dividend per share, stock holding period, receivable collection period, debt and collection period are given in this group of multipliers.

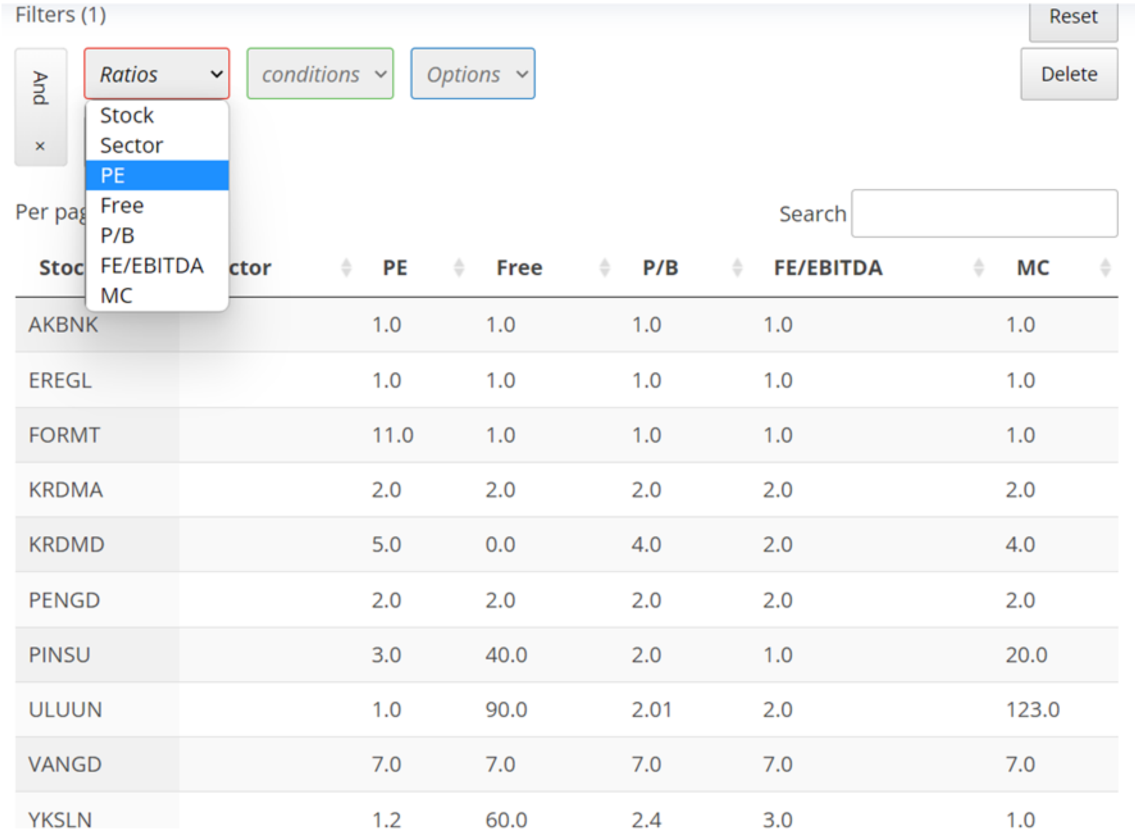
Figure 7. Visualisation of stock base ratios

Figure 8 shows the sectoral analysis screen. The selected sector or index shares in this tab can be sorted practically according to the selected ratio value.

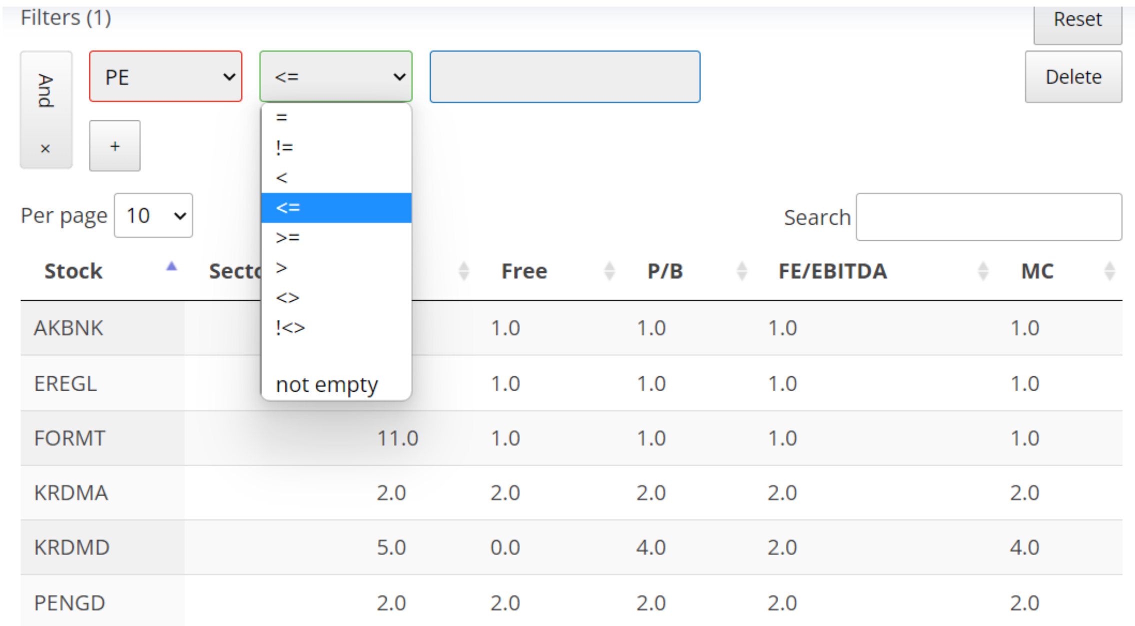
Stock	Last	P/E ▾	P/B	FE/EBITDA	Free Float	Foreign	Market Cap
#EREGL	30.0	1.0	1.0	1.0	1.0	78.0	1.0
#YKSLN	24.0	1.2	2.4	3.0	60.0	20.0	1.0
#KRDMA	23.0	2.0	2.0	2.0	2.0	2.0	2.0
#KRDMD	16.2	5.0	4.0	2.0	0.0	40.0	4.0

Figure 8. Analysis of basic share ratios by sectors

In Figure 9, on the other hand, the shares can be filtered according to the criteria of more than one ratio by combining the and/or conditions according to the sector, index and date. Among these criteria, there are many options, such as being small, large, inclusion, not containing, being in between and not being empty.



(a)



(b)

Figure 9. Advanced stock filtering (a) and (b) based on fundamental ratios

In the last tab of the application, statistical data (such as the smallest, largest and above average) and simple semantic data are given in Figure 10.

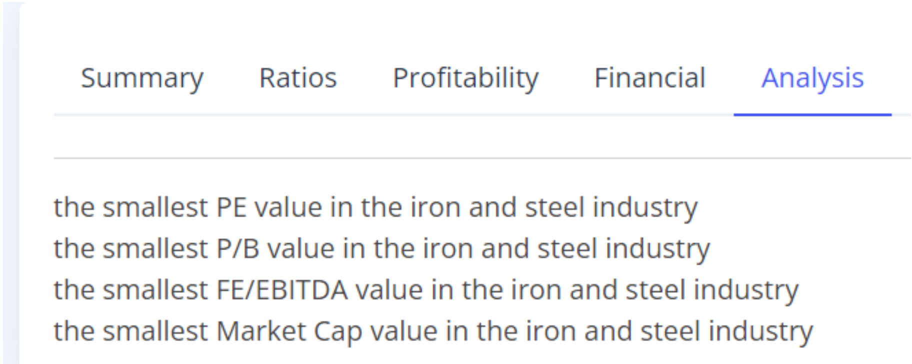


Figure 10. Obtaining simple statistical semantics

5. Discussion and Conclusion

The main contribution of this study to the literature is the analysis of financial assets in basic ratios and the establishment of a backend structure for integrating artificial intelligence in the future. The biggest disadvantage of working is that it currently works on localhost. It is not put into service on the cloud or any platform. experimental developments continue to be developed in the local environment. This study proposed an approach to obtain and visualise publicly traded companies' ratios, statistically interpret companies' relative and within their sectors, find intrinsic value, and make real-time and semantic queries among dozens of ratios.

The proposed approach is coded in Python and developed as a Django platform. This study will be designed to make semantic queries and artificial intelligence-supported stock suggestions based on fundamental analysis in the future. At this stage of the study, the MongoDB database and scientific python environment are designed as a backend structure. Artificial intelligence models developed within the scope of the project will be integrated into this structure in the future and will work as a smart recommendation platform at near time.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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Rough statistical convergence of double sequences in intuitionistic fuzzy normed spaces

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Keywords
Rough convergence, Statistical convergence, Rough statistical convergence, Intuitionistic fuzzy normed spaces

Abstract — This paper proposes rough convergence and rough statistical convergence of a double sequence in intuitionistic fuzzy normed spaces. It then defines the rough statistical limit points and rough statistical cluster points of a double sequence in these spaces. Afterwards, this paper examines some of their basic properties. Finally, it discusses the need for further research.

Subject Classification (2020): 40A35, 40G15.

1. Introduction

Based on the concept of density of positive natural numbers, statistical convergence was independently defined by Fast [1] and Steinhaus [2] in 1951. Moreover, Zygmund [3] studied the concept of statistical convergence under the name of almost all convergence in 1939. Afterward, in 2003, Mursaleen and Edely [4] investigated the concept of statistical convergence in double sequence space.

Phu [5] has defined the concept of rough convergence in finite dimensional normed spaces as a natural generalization of ordinary convergence. He has also shown that the set LIM_x^r, the set of all the rough limit points, is bounded, closed, and convex. Using the concept of natural density, Aytar [6] has defined the concept of rough statistical convergence. Furthermore, Malik and Maity [7, 8] have studied the concepts of rough convergence and rough statistical convergence of double sequences in normed linear spaces, respectively. Besides, many studies on these concepts have been conducted [9–11].

The theory of fuzzy sets was introduced by Zadeh [12] in 1965. Then, the concept of fuzzy norms on a linear space was proposed by Cheng and Mordeson [13], and some properties of the fuzzy norm have been studied [14]. Atanassov [15, 16] has proposed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Park [17] has suggested the concept of intuitionistic fuzzy metric space. After, Saadati and Park

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[18] defined the concept of intuitionistic fuzzy normed space (IFNS). Moreover, they have also introduced the concepts of convergence and the Cauchy sequences in an IFNS. Afterward, Karakuş et al. [19] proposed and investigated the concept of statistical convergence in an IFNS. Savaş and Gürdal [20] have suggested a generalization of statistical convergence in an IFNS. Mursaleen [21] has defined the concept of statistical convergence of double sequences in an IFNS. Moreover, Antal et al. [22] have proposed the concept of rough statistical convergence in an IFNS.

The present paper can be summarized as follows: In the second part of the present study, some basic definitions and properties to be required for the next section are provided. Section 3 proposes the concepts of rough convergence and rough statistical convergence of double sequences in an IFNS and studies some of their basic properties. Moreover, it defines the concepts of rough statistical (r-st) limit points and r-st cluster points of a double sequence in an IFNS and investigates some of their basic properties. The final section discusses the need for further research.

2. Preliminary

This section presents some of the basic definitions to be required in the next sections.

Definition 2.1. [7] Let (x_{jk}) be a double sequence in a normed space $(\mathbb{X}, \|\cdot\|)$ and $r \geq 0$. Then, the double sequence (x_{jk}) is said to be rough convergent (r-convergent) to $x_0 \in \mathbb{X}$, if for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $j, k \geq N_\varepsilon$,

$$\left\| x_{jk} - x_0 \right\| < r + \varepsilon$$

It is denoted by $x_{jk} \xrightarrow{r} x_0$. The element x_0 is called an r-limit point of the double sequence (x_{jk}).

Definition 2.2. [4] The double natural density of the set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined by

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|}{mn}$$

where $|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|$ denotes the number of elements of A not exceeding m and n, respectively. It can be observed that if the set A is finite, then $\delta_2(A) = 0$.

Definition 2.3. [8] Let (x_{jk}) be a double sequence in a normed space $(\mathbb{X}, \|\cdot\|)$ and $r \geq 0$. Then, (x_{jk}) is referred to as r-statistically convergent to $x_0 \in \mathbb{X}$, if for all $\varepsilon > 0$,

$$\delta_2\left(\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left\| x_{jk} - x_0 \right\| \geq r + \varepsilon\right\}\right) = 0$$

In this case we write $x_{jk} \xrightarrow{r-st_2} x_0$. The element x_0 is called an r-statistical limit point of the double sequence (x_{jk}).

Definition 2.4. [22] An intuitionistic fuzzy normed space (IFNS) is the triplet $(\mathbb{X}, \mu, \nu)$ with linear space $\mathbb{X}$ and fuzzy sets μ, ν on $\mathbb{X} \times \mathbb{R}$, if the following conditions for all $x, y \in \mathbb{X}$ and $s, u \in \mathbb{R}$ are valid:

i. $\mu(x;u) = 0$ and $\nu(x;u) = 1$, for $u \notin \mathbb{R}^+$

ii. $\mu(x;u) = 1$ and $\nu(x;u) = 0$, for $u \in \mathbb{R}^+ \Leftrightarrow x = 0$

iii. $\mu(\alpha x;u) = \mu\left(x; \frac{u}{|\alpha|}\right)$ and $\nu(\alpha x;u) = \nu\left(x; \frac{u}{|\alpha|}\right)$, for $\alpha \neq 0$

iv. $\min\{\mu(x;s), \mu(y;u)\} \leq \mu(x+y; s+u)$ and $\max\{\nu(x;s), \nu(y;u)\} \geq \nu(x+y; s+u)$

v. $\lim_{u\to\infty}\mu(x;u)=1$, $\lim_{u\to 0}\mu(x;u)=0$, $\lim_{u\to\infty}\nu(x;u)=0$, and $\lim_{u\to 0}\nu(x;u)=1$

Moreover, the ordered pair (μ,ν) is called an intuitionistic fuzzy norm.

Example 2.5. [18] Let $(\mathbb{X},\|\cdot\|)$ be a normed space and for all $u>0$ and $x\in\mathbb{X}$,

$$\mu(x;u)=\frac{u}{u+\|x\|} \text{ and } \nu(x;u)=\frac{\|x\|}{u+\|x\|}$$

Since the conditions in Definition 2.4 are valid, the triplet $(\mathbb{X},\mu,\nu)$ is an IFNS.

Definition 2.6. [18] Let $(\mathbb{X},\mu,\nu)$ be an IFNS, $x\in\mathbb{X}$, $\varepsilon\in(0,1)$, and $u>0$. Then, the open ball with center x and radius ε is the set $B(x,\varepsilon,u)=\{y\in\mathbb{X}:\mu(x-y;u)>1-\varepsilon \text{ and } \nu(x-y;u)<\varepsilon\}$.

Definition 2.7. [21] Let $(\mathbb{X},\mu,\nu)$ be an IFNS. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is called convergent to $x_0\in\mathbb{X}$ with respect to the intuitionistic fuzzy norm (μ,ν), if there exists $N_\varepsilon\in\mathbb{N}$ for all $u>0$ and $\varepsilon\in(0,1)$,

$$\mu\left(x_{jk}-x_0;u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-x_0;u\right)<\varepsilon, \text{ for all } j,k\geq N_\varepsilon$$

and denoted by $(\mu,\nu)-\lim_{j,k\to\infty}x_{jk}=x_0$ or $x_{jk}\xrightarrow{(\mu,\nu)}x_0$.

Definition 2.8. [21] Let $(\mathbb{X},\mu,\nu)$ be an IFNS. Then, a double sequence $\left(x_{jk}\right)$ is said to be statistically convergent to $x_0\in\mathbb{X}$ with respect to the intuitionistic fuzzy norm (μ,ν), for all $u>0$ and $\varepsilon\in(0,1)$, if

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-x_0;u\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{jk}-x_0;u\right)\geq\varepsilon\right\}\right)=0$$

and denoted by $st_2^{(\mu,\nu)}-\lim_{j,k\to\infty}x_{jk}=x_0$.

3. Rough Statistical Convergence

This section defines the concepts of rough convergence and rough statistical convergence of double sequences in an IFNS and examines some of their basic properties.

Definition 3.1. Let $(\mathbb{X},\mu,\nu)$ be an IFNS and $r\geq 0$. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is said to be r-convergent to $x_0\in\mathbb{X}$, with respect to the norm (μ,ν), if there exists $N_\varepsilon\in\mathbb{N}$ for all $u>0$ and $\varepsilon\in(0,1)$ such that

$$\mu(x_{jk}-x_0;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-x_0;r+u)<\varepsilon, \text{ for all } j,k\geq N_\varepsilon$$

In this case, we write $r_{(\mu,\nu)}-\lim_{j,k\to\infty}x_{jk}=x_0$ or $x_{jk}\xrightarrow{r_{(\mu,\nu)}}x_0$, where x_0 is called an $r_{(\mu,\nu)}$−limit point of the double sequence (x_{jk}).

Note 3.2. For $r=0$, the concept of rough convergence in IFNSs becomes the concept of ordinary convergence in IFNSs.

The $r_{(\mu,\nu)}$−limit point of a double sequence may not be unique. Therefore, the set of all the $r_{(\mu,\nu)}$-limit points for a double sequence (x_{jk}) is as follows:

$${}_{\nu}^{\mu}LIM_{x_{jk}}^{r}:=\left\{x_0\in\mathbb{X}:x_{jk}\xrightarrow{r_{(\mu,\nu)}}x_0\right\}$$

If ${}_{\nu}^{\mu}LIM_{x_{jk}}^{r}\neq\emptyset$, the double sequence (x_{jk}) is rough convergent.

Definition 3.3. Let $(\mathbb{X},\mu,\nu)$ be an IFNS and $r\geq 0$. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is referred to as rough statistically convergent to $x_0\in\mathbb{X}$ with respect to the norm (μ,ν), for all $u>0$ and $\varepsilon\in(0,1)$, if

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\})=0$$

and denoted by $r-_{\nu}^{\mu}st_2-\lim\limits_{j,k\to\infty}x_{jk}=x_0$ or $x_{jk}\xrightarrow{r-_{\nu}^{\mu}st_2}x_0$.

Note 3.4. If $r=0$, then rough statistical convergence coincide with statistical convergence in IFNSs

The rough statistical limit of a double sequence may not be unique. Hence, the set of rough statistical limit points is denoted as follows:

$$st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}:=\left\{x_0\in\mathbb{X}:x_{jk}\xrightarrow{r-_{\nu}^{\mu}st_2}x_0\right\}$$

Let (x_{jk}) be a unbounded sequence. Then, ${}_{\nu}^{\mu}LIM_{x_{jk}}^{r}$ is empty set. However, this is not achived in the case of rough statistical convergence. Hence, $st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$ may not be empty set.

Example 3.5. Let us consider a real normed space $(\mathbb{X},\|\cdot\|)$ and, for all $u>0$ and $x\in\mathbb{X}$,

$$\mu(x,u)=\frac{u}{u+\|x\|} \text{ and } \nu(x,u)=\frac{\|x\|}{u+\|x\|}$$

Then, the triplet $(\mathbb{X},\mu,\nu)$ is an IFNS. For all $j,k\in\mathbb{N}$, define

$$x_{jk}=\begin{cases}(-1)^{j+k}, & j \text{ and } k \text{ are non-squares}\\ jk, & \text{otherwise}\end{cases}$$

Then,

$$st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}=\begin{cases}\emptyset, & r<1\\ [1-r,r-1], & r\geq 1\end{cases}$$

and $LIM_{x_{jk}}^{r}=\emptyset$, for all $r\geq 0$.

Afterward, the concept of rough statistically bounded sequence in an IFNS is as follows:

Definition 3.6. Let $(\mathbb{X},\mu,\nu)$ be an IFNS and $r\geq 0$. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is said to be statistical bounded with respect to the norm (μ,ν), if there exists a real number $M>0$ for all $u>0$ and $\varepsilon\in(0,1)$ such that

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk};M)\leq 1-\varepsilon \text{ or } \nu(x_{jk};M)\geq\varepsilon\})=0$$

Definition 3.7. [8] A double subsequence $x'=\left(x_{j_pk_q}\right)$ of a double sequence $x=\left(x_{jk}\right)$ is called a dense subsequence, if $\delta_2\left(\left\{\left(j_p,k_q\right)\in\mathbb{N}\times\mathbb{N}:p,q\in\mathbb{N}\right\}\right)=1$.

Example 3.8. Let us consider the IFNS in Example 3.5 and, for all $j,k\in\mathbb{N}$, define

$$x_{jk}=\begin{cases}jk, & j \text{ and } k \text{ are squares}\\ 0, & \text{otherwise}\end{cases}$$

Thus, $st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}=[-r,r]$. Moreover, for the subsequence $x'=(x_{j_tk_t})$ of (x_{jk}) such that j_t and k_t are squares, $st_2-_{\nu}^{\mu}LIM_{x'}^{r}=\emptyset$. It can be seen that $st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}\nsubseteq st_2-_{\nu}^{\mu}LIM_{x'}^{r}$. However, this inclusion for the rough statistical convergent sequences and their dense subsequences in an IFNS is valid. The following theorem explains this state.

Theorem 3.9. Let $(\mathbb{X},\mu,\nu)$ be an IFNS. If $x' = \left(x_{j_p k_q}\right)$ is a dense subsequence of $x = \left(x_{jk}\right)$, then

$$st_2 -_\nu^\mu LIM_x^r \subseteq st_2 -_\nu^\mu LIM_{x'}^r$$

Proof.
The proof is obvious.

Theorem 3.10. Let $(\mathbb{X},\mu,\nu)$ be an IFNS. A double sequence (x_{jk}) in $\mathbb{X}$ is statistically bounded if and only if $st_2 -_\nu^\mu LIM_{x_{jk}}^r \neq \emptyset$, for all $r>0$.

Proof.
Let $(\mathbb{X},\mu,\nu)$ be an IFNS.

$(\Rightarrow)$: Let a double sequence (x_{jk}) be statistically bounded in the IFNS. Then, for all $u>0$, $\varepsilon \in (0,1)$, and $r>0$, there exists a real number $M>0$ such that

$$\delta_2(\{(j,k)\in \mathbb{N}\times\mathbb{N} : \mu(x_{jk};M) \le 1-\varepsilon \text{ or } \nu(x_{jk};M)\ge \varepsilon\}) = 0$$

Let $K = \{(j,k)\in \mathbb{N}\times\mathbb{N} : \mu(x_{jk};M) \le 1-\varepsilon \text{ or } \nu(x_{jk};M)\ge \varepsilon\}$. For $k\in K^c$, $\mu(x_{jk};M) > 1-\varepsilon$ and $\nu(x_{jk};M)<\varepsilon$. Moreover,

$$\mu(x_{jk}; r+M) \ge \min\{\mu(0;r),\mu(x_{jk};M)\} = \min\{1,\mu(x_{jk};M)\} > 1-\varepsilon$$

and

$$\nu(x_{jk}; r+M) \le \max\{\nu(0;r),\nu(x_{jk};M)\} = \max\{0,\nu(x_{jk};M)\} < \varepsilon$$

Hence, $0\in st_2 -_\nu^\mu LIM_{x_{jk}}^r$. Consequently, $st_2 -_\nu^\mu LIM_{x_{jk}}^r \neq \emptyset$.

$(\Leftarrow)$: Let $st_2 -_\nu^\mu LIM_{x_{jk}}^r \neq \emptyset$, for all $r>0$. Then, there exists $x_0\in\mathbb{X}$ such that $x_0 \in st_2 -_\nu^\mu LIM_{x_{jk}}^r$. For all $u>0$ and $\varepsilon\in(0,1)$,

$$\delta_2(\{(j,k)\in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-x_0; r+u) \le 1-\varepsilon \text{ or } \nu(x_{jk}-x_0; r+u)\ge \varepsilon\}) = 0$$

Therefore, almost all x_{jk} are contained in some ball with center x_0 which implies that double sequence (x_{jk}) is statistically bounded in an IFNS. Theorem 3.11 shows that rough statistical convergence of a double sequences in an IFNS has many arithmetic properties similar to those of ordinary convergence.

Theorem 3.11. Let (x_{jk}) and (y_{jk}) be two double sequences in an IFNS. Then, for all $r\ge 0$, the following holds:

i. If $x_{jk} \xrightarrow{r-_\nu^\mu st_2} x_0$ and $\alpha\in\mathbb{F}$, then $\alpha x_{jk} \xrightarrow{r-_\nu^\mu st_2} \alpha x_0$.

ii. If $x_{jk} \xrightarrow{r-_\nu^\mu st_2} x_0$ and $y_{jk} \xrightarrow{r-_\nu^\mu st_2} y_0$, then $x_{jk}+y_{jk} \xrightarrow{r-_\nu^\mu st_2} x_0+y_0$.

Proof.
Let (x_{jk}) and (y_{jk}) be two double sequences in an IFNS and $r\ge 0$.

i. Let $x_{jk} \xrightarrow{r-_\nu^\mu st_2} x_0$ and $\alpha\in\mathbb{F}$. Therefore, if for all $u>0$ and $\varepsilon\in(0,1)$,

$$K = \left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \mu\left(x_{jk}-x_0; \frac{r+u}{|\alpha|}\right) \le 1-\varepsilon \text{ or } \nu\left(x_{jk}-x_0; \frac{r+u}{|\alpha|}\right) \ge \varepsilon\right\}$$

then $\delta_2(K) = 0$. Let $(t,s) \in K^c$. Then, $\mu\left(x_{ts} - x_0; \frac{r+u}{|\alpha|}\right) > 1-\varepsilon$ and $\nu\left(x_{ts} - x_0; \frac{r+u}{|\alpha|}\right) < \varepsilon$. Hence,

$$\mu(\alpha x_{ts} - \alpha x_0; r+u) = \mu\left(x_{ts} - x_0; \frac{r+u}{|\alpha|}\right) > 1-\varepsilon \tag{3.1}$$

and

$$\nu(\alpha x_{ts} - \alpha x_0; r+u) = \nu\left(x_{ts} - x_0; \frac{r+u}{|\alpha|}\right) < \varepsilon \tag{3.2}$$

Let $H = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(\alpha x_{jk} - \alpha x_0; r+u) > 1-\varepsilon \text{ and } \nu(\alpha x_{jk} - \alpha x_0; r+u) < \varepsilon\}$. From Equations (3.1) and (3.2), $(t,s) \in H$. Therefore, $K^c \subseteq H$. Consequently, $\alpha x_{jk} \xrightarrow{r-_{\nu}^{\mu} st_2} \alpha x_0$.

ii. Let $x_{jk} \xrightarrow{r-_{\nu}^{\mu} st_2} x_0$ and $y_{jk} \xrightarrow{r-_{\nu}^{\mu} st_2} y_0$. Therefore, if for all $u > 0$ and $\varepsilon \in (0,1)$,

$$A = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - x_0; \frac{r+u}{2}\right) \le 1-\varepsilon \text{ or } \nu\left(x_{jk} - x_0; \frac{r+u}{2}\right) \ge \varepsilon\right\}$$

and

$$B = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(y_{jk} - y_0; r+u) \le 1-\varepsilon \text{ or } \nu(y_{jk} - y_0; r+u) \ge \varepsilon\}$$

Then, $\delta_2(A) = 0$ and $\delta_2(B) = 0$. Let $(t,s) \in A^c \cap B^c$. Then,

$$\mu\left(x_{ts} - x_0; \frac{r+u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(x_{ts} - x_0; \frac{r+u}{2}\right) < \varepsilon$$

and

$$\mu\left(y_{ts} - y_0; \frac{r+u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(y_{ts} - y_0; \frac{r+u}{2}\right) < \varepsilon$$

Hence,

$$\mu(x_{ts} + y_{ts} - (y_0 + x_0); r+u) \ge \min\left\{\mu\left(x_{ts} - x_0; \frac{r+u}{2}\right), \mu\left(y_{ts} - y_0; r + \frac{r+u}{2}\right)\right\} > 1-\varepsilon \tag{3.3}$$

and

$$\nu(x_{ts} + y_{ts} - (y_0 + x_0); r+u) \ge \max\left\{\nu\left(x_{ts} - x_0; \frac{r+u}{2}\right), \nu\left(y_{ts} - y_0; \frac{r+u}{2}\right)\right\} < \varepsilon \tag{3.4}$$

Let $C = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} + y_{jk} - (x_0 + y_0); r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} + y_{jk} - (x_0 + y_0); r+u\right) < \varepsilon\right\}$. From the Equations (3.3) and (3.4), $(t,s) \in C$. Therefore, $C \subseteq A^c \cap B^c$. Consequently, $x_{jk} + y_{jk} \xrightarrow{r-_{\nu}^{\mu} st_2} x_0 + y_0$.

Theorem 3.12 and Theorem 3.13 prove some topological properties of the r-statistical limit set of a double sequence in IFNSs.

Theorem 3.12. Let (x_{jk}) be a double sequence in an IFNS $(\mathbb{X}, \mu, \nu)$ and $r \ge 0$. Then, the set $st_2 -_{\nu}^{\mu} LIM_{x_{jk}}^r$ is closed.

Proof.
Let (x_{jk}) be a double sequence in an IFNS and $r \ge 0$. If $st_2 -_{\nu}^{\mu} LIM_{x_{jk}}^r = \emptyset$, then the theorem is valid. Therefore, let $st_2 -_{\nu}^{\mu} LIM_{x_{jk}}^r \neq \emptyset$, for all $r \ge 0$ and $y_0 \in \overline{st_2 -_{\nu}^{\mu} LIM_{x_{jk}}^r}$. Then, $y_{jk} \in st_2 -_{\nu}^{\mu} LIM_{x_{jk}}^r$ such that $y_{jk} \xrightarrow{(\mu,\nu)} y_0$. Then, for all $u > 0$ and $\varepsilon \in (0,1)$, there exists a $k_1 \in \mathbb{N}$ such that

$$\mu\left(y_{jk} - y_0; \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(y_{jk} - y_0; \frac{u}{2}\right) < \varepsilon, \text{ for all } j,k \ge k_1$$

Let $y_{mn} \in st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$ for $m,n > k_1$ such that

$$\delta_2\left((j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_{mn}; r + \frac{u}{2}\right) \leq 1-\varepsilon \text{ or } \nu\left(x_{jk} - y_{mn}; r + \frac{u}{2}\right) \geq \varepsilon\right) = 0$$

For $(t,s) \in A = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right) > 1-\varepsilon$ and $\nu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right) < \varepsilon\}$

$$\mu\left(x_{ts} - y_{mn}, r + \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(x_{ts} - y_{mn}, r + \frac{u}{2}\right) < \varepsilon$$

Then,

$$\mu(x_{ts} - y_0, r+u) \geq \min\left\{\mu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right), \mu\left(y_{mn} - y_0, \frac{u}{2}\right)\right\} > 1-\varepsilon \tag{3.5}$$

and

$$\nu(x_{ts} - y_0, r+u) \leq \max\left\{\mu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right), \mu\left(y_{mn} - y_0, \frac{u}{2}\right)\right\} < \varepsilon \tag{3.6}$$

Let $B = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_0, r+u\right) > 1-\varepsilon$ and $\nu\left(x_{jk} - y_0, r+u\right) < \varepsilon\}$. From the Equations (3.5) and (3.6), $(t,s) \in B$. Since $A \subseteq B$, then $\delta_2(A) \leq \delta_2(B)$. Consequently,

$$y_0 \in st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$$

Theorem 3.13. Let (x_{jk}) be a double sequence in an IFNS $(\mathbb{X},\mu,\nu)$ and $r \geq 0$. Then, the set $st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$ is convex.

Proof.
Let (x_{jk}) be a double sequence in an IFNS, $r \geq 0$, and $x_1, x_2 \in st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$. For the convexity of the set $st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$, we should show that $[(1-\lambda)x_1 + \lambda x_2] \in st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$, for all $\lambda \in (0,1)$. For all $u > 0$ and $\varepsilon \in (0,1)$, let

$$M_1 = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - x_1; \frac{r+u}{2(1-\lambda)}\right) \leq 1-\varepsilon \text{ or } \nu\left(x_{jk} - x_1; \frac{r+u}{2(1-\lambda)}\right) \geq \varepsilon\right\}$$

and

$$M_2 = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - x_2; \frac{r+u}{2\lambda}\right) \leq 1-\varepsilon \text{ or } \nu\left(x_{jk} - x_2; \frac{r+u}{2\lambda}\right) \geq \varepsilon\right\}$$

By assumption we have $\delta_2(M_1) = 0$ and $\delta_2(M_2) = 0$. For $k \in M_1^c \cap M_2^c$,

$$\begin{aligned}
\mu(x_{jk} - [(1-\lambda)x_1 + \lambda x_2]; r+u) &= \mu((1-\lambda)(x_{jk} - x_1) + \lambda(x_{jk} - x_2); r+u) \\
&\geq \min\left\{\mu\left((1-\lambda)(x_{jk} - x_1); \frac{r+u}{2}\right)\mu\left(\lambda(x_{jk} - x_2); \frac{r+u}{2}\right)\right\} \\
&= \min\left\{\mu\left(x_{jk} - x_1; \frac{r+u}{2(1-\lambda)}\right), \mu\left(x_{jk} - x_2; \frac{r+u}{2\lambda}\right)\right\} \\
&> 1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}\nu(x_{jk}-[(1-\lambda)x_1+\lambda x_2];r+u)&=\nu((1-\lambda)(x_{jk}-x_1)+\lambda(x_{jk}-x_2);r+u)\\&\geq\max\left\{\nu\left((1-\lambda)(x_{jk}-x_1);\frac{r+u}{2}\right),\nu\left(\lambda(x_{jk}-x_2);\frac{r+u}{2}\right)\right\}\\&=\max\left\{\nu\left(x_{jk}-x_1;\frac{r+u}{2(1-\lambda)}\right),\nu\left(x_{jk}-x_2;\frac{r+u}{2\lambda}\right)\right\}\\&>\varepsilon\end{aligned}$$

Thus,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-[(1-\lambda)x_1+\lambda x_2];r+u)\leq 1-\varepsilon \text{ or } \nu(x_k-[(1-\lambda)x_1+\lambda x_2];r+u)\geq 1-\varepsilon\})=0$$

Consequently, $[(1-\lambda)x_1+\lambda x_2]\in st_2-^{\mu}_{\nu} LIM^r_{x_{jk}}$ and so $st_2-^{\mu}_{\nu} LIM^r_{x_{jk}}$ is a convex set.

Theorem 3.14. Let (x_{jk}) be a double sequence in an IFNS $(\mathbb{X},\mu,\nu)$ and $r\geq 0$. If there exists a double sequence (y_{jk}) in $\mathbb{X}$, statistically convergent to $x_0\in\mathbb{X}$ with respect to the norm (μ,ν) and, for all $\varepsilon\in(0,1)$ and $j,k\in\mathbb{N}$, $\mu(x_{jk}-y_{jk};r)>1-\varepsilon$ and $\nu(x_{jk}-y_{jk};r)<\varepsilon$, then (x_{jk}) is rough statistically convergent to $x_0\in\mathbb{X}$ with respect to the norm (μ,ν).

Proof.

Let (x_{jk}) be a double sequence in an IFNS, $r\geq 0$, $u>0$ and there exists a double sequence (y_{jk}) in $\mathbb{X}$ such that $y_{jk}\xrightarrow{st_2^{(\mu,\nu)}} x_0$ and $\mu(x_{jk}-y_{jk};r)>1-\varepsilon$ and $\nu(x_{jk}-y_{jk};r)<\varepsilon$, for all $j,k\in\mathbb{N}$. For given $\varepsilon\in(0,1)$, let

$$A=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(y_{jk}-x_0;u)\leq 1-\varepsilon \text{ or } \nu(y_{jk}-x_0;u)\geq\varepsilon\}$$

and

$$B=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-y_{jk};r)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-y_{jk};r)\geq\varepsilon\}$$

Clearly, $\delta_2(A)=0$ and $\delta_2(B)=0$. For $(j,k)\in A^c\cap B^c$,

$$\mu(x_{jk}-x_0;r+u)\geq\min\{\mu(x_{jk}-y_{jk};r),\mu(y_{jk}-x_0;u)\}>1-\varepsilon$$

and

$$\nu(x_{jk}-x_0;r+u)\leq\max\{\nu(x_{jk}-y_{jk};r),\nu(y_{jk}-x_0;u)\}<\varepsilon$$

Then, for all $(j,k)\in A^c\cap B^c$,

$$\mu(x_{jk}-x_0;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-x_0;r+u)<\varepsilon$$

This implies that

$$\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\}\subseteq A\cup B$$

Then,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\})\leq\delta_2(A)+\delta_2(B)$$

Hence, $\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\})=0$. Consequently, $x_{jk}\xrightarrow{r-_{\nu}^{\mu}st_2}x_0$.

Theorem 3.15. Let $x=(x_{jk})$ be a sequence in an IFNS and $r>0$. Then, there does not exist $y,z\in st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$ such that $\mu(y-z;mr)\leq 1-\varepsilon$ or $\nu(y-z;mr)\geq\varepsilon$, for $\varepsilon\in(0,1)$ and $m>2$.

Proof.

Let (x_{jk}) be a sequence in an IFNS and $r>0$. Assume that there exists $y,z\in st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$ such that for $m>2$,

$$\mu(y-z;mr)\leq 1-\varepsilon \text{ or } \nu(y-z;mr)\geq\varepsilon$$

For given $\varepsilon\in(0,1)$ and $u>0$. K_1 and K_2 are denoted as follows:

$$K_1=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-y;r+\frac{u}{2}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{jk}-y;r+\frac{u}{2}\right)\geq\varepsilon\right\}$$

and

$$K_2=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-z;r+\frac{u}{2}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{jk}-z;r+\frac{u}{2}\right)\geq\varepsilon\right\}$$

Hence, $\delta(K_1)=0$ and $\delta(K_2)=0$. For $(j,k)\in K_1^c\cap K_2^c$,

$$\mu(y-z;2r+u)\geq\min\left\{\mu\left(x_{jk}-z;r+\frac{u}{2}\right),\mu\left(x_{jk}-y;r+\frac{u}{2}\right)\right\}>1-\varepsilon$$

and

$$\nu(y-z;2r+u)\leq\max\left\{\nu\left(x_{jk}-z;r+\frac{u}{2}\right),\nu\left(x_{jk}-y;r+\frac{u}{2}\right)\right\}<\varepsilon$$

Hence, $\mu(y-z;2r+u)>1-\varepsilon$ and $\nu(y-z;2r+u)<\varepsilon$. Then,

$$\mu(y-z;mr)>1-\varepsilon \text{ or } \nu(y-z;mr)<\varepsilon, \text{ for } m>2$$

which is contradiction to the hypothesis. Therefore, there does not exists $y,z\in st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$ such that $\mu(y-z;mr)\leq 1-\varepsilon$ or $\nu(y-z;mr)\geq\varepsilon$, for $\varepsilon\in(0,1)$ and $m>2$.

Next, the concept of rough statistical cluster points of a double sequence in an IFNS is defined, and some related results are proposed.

Definition 3.16. Let $(\mathbb{X},\mu,\nu)$ be an IFNS, $\gamma\in\mathbb{X}$, and $r\geq 0$. Then, γ is called rough statistical cluster point of the double sequence (x_{jk}) in $\mathbb{X}$ with respect to the norm (μ,ν) (briefly, $r-_{\nu}^{\mu}st_2$-cluster point of(x_{jk})) if for all $u>0$ and $\varepsilon\in(0,1)$

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-\gamma;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-\gamma;r+u)<\varepsilon\})>0$$

The set of all the $r-_{\nu}^{\mu}st_2$-cluster points of $x=(x_{jk})$ in an IFNS is denoted by $\Gamma_{(\mu,\nu)_2}^{r}(x)$. If $r=0$, then $\Gamma_{(\mu,\nu)_2}^{r}(x)=\Gamma_{(\mu,\nu)_2}(x)$.

Theorem 3.17. Let $x=(x_{jk})$ be a double sequence in an IFNS and $r\geq 0$. Then, $\Gamma_{(\mu,\nu)_2}^{r}(x)$ is a closed set.

Proof.

Let (x_{jk}) be a double sequence in an IFNS and $r\geq 0$.

i. If $\Gamma_{(\mu,\nu)_2}^{r}(x)=\emptyset$, then the theorem is valid.

ii. Let $\Gamma^r_{(\mu,\nu)_2}(x) \neq \emptyset$ and $y_0 \in \overline{\Gamma^r_{(\mu,\nu)_2}(x)}$. Then, there is a double sequence (y_{jk}) in $\Gamma^r_{(\mu,\nu)_2}(x)$ such that $y_{jk} \xrightarrow{(\mu,\nu)} y_0$, for all $j,k \in \mathbb{N}$. It is sufficient to show that $y_0 \in \Gamma^r_{(\mu,\nu)_2}(x)$. As $y_{jk} \xrightarrow{(\mu,\nu)} y_0$, for all $u > 0$ and $\varepsilon \in (0,1)$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\mu\left(y_{jk} - y_0; \frac{u}{2}\right) > 1-\varepsilon$ and $\nu\left(y_{jk} - y_0; \frac{u}{2}\right) < \varepsilon$, for all $j,k \geq k_\varepsilon$. Let $j_0, k_0 \in \mathbb{N}$ such that $j_0, k_0 \geq k_\varepsilon$. Then,

$$\mu\left(y_{j_0k_0} - y_0; \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(y_{j_0k_0} - y_0; \frac{u}{2}\right) < \varepsilon$$

Since $y_{jk} \in \Gamma^r_{(\mu,\nu)_2}(x)$, $y_{j_0k_0} \in \Gamma^r_{(\mu,\nu)_2}(x)$. Thus,

$$\delta_2\left(\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(y_{jk} - y_{j_0k_0}; r + \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(y_{jk} - y_{j_0k_0}; r + \frac{u}{2}\right) < \varepsilon\right\}\right) > 0$$

Let

$$A = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_{j_0k_0}; r + \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - y_{j_0k_0}; r + \frac{u}{2}\right) < \varepsilon\right\}$$

Choose $(t,s) \in A$. Then, $\mu\left(x_{ts} - y_{j_0k_0}; r + \frac{u}{2}\right) > 1-\varepsilon$ and $\nu\left(x_{jk} - y_{j_0k_0}; r + \frac{u}{2}\right) < \varepsilon$. Therefore,

$$\mu\left(x_{ts} - y_0; r+u\right) \geq \min\left\{\mu\left(x_{ts} - y_{j_0k_0}; r + \frac{u}{2}\right), \mu\left(y_{j_0k_0} - y_0; \frac{u}{2}\right)\right\} > 1-\varepsilon \tag{3.7}$$

and

$$\nu\left(x_{ts} - y_0; r+u\right) \leq \max\left\{\nu\left(x_{ts} - y_{j_0k_0}; r + \frac{u}{2}\right), \nu\left(y_{j_0k_0} - y_0; \frac{u}{2}\right)\right\} < \varepsilon \tag{3.8}$$

Let $B = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_0; r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - y_0; r + \frac{u}{2}\right) < \varepsilon\right\}$. From the Equations (3.7) and (3.8), $(t,s) \in B$. Thereby, $A \subseteq B$ and so $\delta_2(A) \leq \delta_2(B)$. Therefore,

$$\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_0; r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - y_0; r+u\right) < \varepsilon\}) > 0$$

Consequently, $y_0 \in \Gamma^r_{(\mu,\nu)_2}(x)$.

Theorem 3.18. Let $x = (x_{jk})$ be a double sequence in an IFNS. Then, for an arbitrary $\gamma \in \Gamma_{(\mu,\nu)_2}(x)$ and $\varepsilon \in (0,1)$, $\mu(\xi - \gamma; r) > 1-\varepsilon$ and $\nu(\xi - \gamma; r) < \varepsilon$, for all $\xi \in \Gamma^r_{(\mu,\nu)_2}(x)$.

Proof.

Let $x = (x_{jk})$ be a double sequence in an IFNS and $\gamma \in \Gamma_{(\mu,\nu)_2}(x)$. Then, for all $u > 0$ and $\varepsilon \in (0,1)$,

$$\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - \gamma; u) > 1-\varepsilon \text{ and } \nu(x_{jk} - \gamma; u) < \varepsilon\}) > 0$$

Let $A = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - \gamma; u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - \gamma; u\right) < \varepsilon\right\}$. Choose $(t,s) \in A$.

Then, $\mu\left(x_{ts} - \gamma; u\right) > 1-\varepsilon$ and $\nu\left(x_{ts} - \gamma; u\right) < \varepsilon$. Thus,

$$\mu\left(x_{ts} - \xi; r+u\right) \geq \min\left\{\mu\left(x_{ts} - \gamma; u\right), \mu(\xi - \gamma; r)\right\} > 1-\varepsilon \tag{3.9}$$

and

$$\nu\left(x_{ts} - \xi; r+u\right) \leq \max\left\{\nu\left(x_{ts} - \gamma; u\right), \nu(\xi - \gamma; r)\right\} < \varepsilon \tag{3.10}$$

Let $B = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - \xi; r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - \xi; r+u\right) < \varepsilon\right\}$. From the Equations (3.9) and (3.10),

$(t,s)\in B$. Thereby $A\subseteq B$, $\delta_2(A)\leq\delta_2(B)$. Therefore,

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-\xi;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-\xi;r+u)<\varepsilon\right\}\right)>0$$

Consequently, $\xi\in\Gamma^r_{(\mu,\nu)_2}(x)$.

Theorem 3.19. Let $x=(x_{jk})$ be a double sequence in an IFNS, $r>0$ and $c\in\mathbb{X}$. Then,

$$\Gamma^r_{(\mu,\nu)_2}(x)=\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$$

Proof.

Let $x=(x_{jk})$ be a double sequence in an IFNS and $r>0$. Let $\gamma\in\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$, then there exists $c\in\Gamma_{(\mu,\nu)_2}(x)$ such that for all $r>0$ and given $\varepsilon\in(0,1)$, $\mu(c-\gamma;r)>1-\varepsilon$ and $\nu(c-\gamma;r)<\varepsilon$. Fix $u>0$. Since $c\in\Gamma_{(\mu,\nu)_2}(x)$, there exists a set

$$K=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-c;u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-c;u\right)<\varepsilon\right\}$$

such that $\delta_2(K)>0$. For $(j,k)\in K$,

$$\mu\left(x_{jk}-\gamma;r+u\right)\geq\min\left\{\mu\left(x_{jk}-c;u\right),\mu(c-\gamma;r)\right\}>1-\varepsilon$$

and

$$\nu\left(x_{jk}-\gamma;r+u\right)\leq\max\left\{\nu\left(x_{jk}-c;u\right),\nu(c-\gamma;r)\right\}<\varepsilon$$

This implies that $\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\gamma;r+u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-\gamma;r+u\right)<\varepsilon\right\}\right)>0$. Hence, $\gamma\in\Gamma^r_{(\mu,\nu)_2}(x)$. Therefore, $\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\lambda,r)}\subseteq\Gamma^r_{(\mu,\nu)_2}(x)$.

Conversely, let $\gamma\in\Gamma^r_{(\mu,\nu)_2}(x)$ and $\gamma\notin\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$ and so $\gamma\notin\overline{B(c,\varepsilon,r)}$, for all $c\in\Gamma_{(\mu,\nu)_2}(x)$. Then,

$$\mu(\gamma-c;r)\leq1-\varepsilon \text{ or } \nu(\gamma-c;r)\geq\varepsilon, \text{ for all } c\in\Gamma_{(\mu,\nu)_2}(x)$$

By Theorem 3.18, for $\gamma\in\Gamma_{(\mu,\nu)_2}(x)$, $\mu(\gamma-c;r)>1-\varepsilon$ and $\nu(\gamma-c;r)<\varepsilon$, for all $c\in\Gamma^r_{(\mu,\nu)_2}(x)$ which is a contradiction to the assumption. Therefore, $\gamma\in\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$. Hence, $\Gamma^r_{(\mu,\nu)_2}(x)\subseteq\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$.

Theorem 3.20. Let $x=(x_{jk})$ be a double sequence in an IFNS and $r>0$. Then, for all $\varepsilon\in(0,1)$,

i. If $c\in\Gamma_{(\mu,\nu)_2}(x)$, then $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\subseteq\overline{B(c,\varepsilon,r)}$.

ii. $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}=\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}=\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$

Proof.

Let $x=(x_{jk})$ be a double sequence in an IFNS.

i. Consider $\xi\in st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$ and $c\in\Gamma_{(\mu,\nu)_2}(x)$. For all $u>0$ and $\varepsilon\in(0,1)$, let

$$A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\xi:r+u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-\xi;r+u\right)<\varepsilon\right\}$$

and

$$B=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-c;u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-c;u\right)<\varepsilon\right\}$$

Thus, $\delta_2\left(A^c\right)=0$ and $\delta_2(B)\neq 0$. For $(j,k)\in A\cap B$,

$$\mu(\xi-c;r)\geq\min\left\{\mu\left(x_{jk}-c;u\right),\mu\left(x_{jk}-\xi;r+u\right)\right\}>1-\varepsilon$$

and

$$\nu(\xi-c;r)\leq\max\left\{\nu\left(x_{jk}-c;u\right),\nu\left(x_{jk}-\xi;r+u\right)\right\}<\varepsilon$$

Therefore, $\xi\in\overline{B(c,\varepsilon,r)}$. Hence, $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\subseteq\overline{B(c,\varepsilon,r)}$.

ii. From the statement *i.*, $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\subseteq\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$. Let $y\in\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$. Then, $\mu(y-c;r)\geq 1-\varepsilon$ and $\nu(y-c;r)\leq\varepsilon$, for all $c\in\Gamma_{(\mu,\nu)_2}(x)$. This implies that

$$\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(y,\varepsilon,r)}$$

and so

$$\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}\subseteq\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$$

Further, let $y\notin st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$. Then, for $u>0$,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-y;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-y;r+u)\geq\varepsilon\})\neq 0$$

which implies that a statistical cluster point c exists for the sequence x such that

$$\mu(y-c;r+u)\leq 1-\varepsilon \text{ or } \nu(y-c;r+u)\geq\varepsilon$$

Thus, $\Gamma_{(\mu,\nu)_2}(x)\nsubseteq\overline{B(y,\varepsilon,r)}$ and $y\notin\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$. Therefore,

$$\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}\subseteq st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$$

and so $\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}\subseteq st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$. Consequently,

$$st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}=\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}=\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$$

Theorem 3.21. Let (x_{jk}) be a double sequence in an IFNS. If (x_{jk}) is statistically convergent to $\xi\in\mathbb{X}$ with respect to the norm (μ,ν), then for all $\varepsilon\in(0,1)$ and $r>0$ $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}=\overline{B(\xi,\varepsilon,r)}$ is hold.

Proof.

Let (x_{jk}) be a double sequence in an IFNS and (x_{jk}) be statistically convergent to $\xi\in\mathbb{X}$ with respect to the norm (μ,ν) and $u>0$. Since $x_{jk}\xrightarrow{st_2^{(\mu,\nu)}}\xi$, then there exists a set

$$A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\xi;u\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{jk}-\xi;u\right)\geq\varepsilon\right\}$$

such that $\delta_2(A) = 0$. Let $y \in \overline{B(\xi,\varepsilon,r)} = \{y \in \mathbb{X} : \mu(y-\xi;r) \geq 1-\varepsilon, \nu(y-\xi;r) \leq \varepsilon\}$. For $(j,k) \in A^c$,

$$\mu\left(x_{jk} - y; r+u\right) \geq \min\left\{\mu\left(x_{jk}-\xi;u\right), \mu(y-\xi;r)\right\} > 1-\varepsilon$$

and

$$\nu\left(x_{jk} - y; r+u\right) \leq \max\left\{\nu\left(x_{jk}-\xi;u\right), \nu(y-\xi;r)\right\} < \varepsilon$$

This implies that $y \in st_2-_{\nu}^{\mu}LIM^r_{x_{jk}}$, i.e., $\overline{B(\xi,\varepsilon,r)} \subseteq st_2-_{\nu}^{\mu}LIM^r_{x_{jk}}$. On the other hand, $st_2-_{\nu}^{\mu}LIM^r_{x_{jk}} \subseteq \overline{B(\xi,\varepsilon,r)}$. Hence, $st_2-_{\nu}^{\mu}LIM^r_{x_{jk}} = \overline{B(\xi,\varepsilon,r)}$.

Theorem 3.22. Let $x = (x_{jk})$ be a double sequence in an IFNS. If x is statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm (μ,ν), then $\Gamma^r_{(\mu,\nu)_2}(x) = st_2-_{\nu}^{\mu}LIM^r_{x_{jk}}$ for some $r > 0$.

Proof.
Let $x = (x_{jk})$ be a double sequence in an IFNS and x be statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm (μ,ν). Then, $\Gamma_{(\mu,\nu)_2}(x) = \{\xi\}$. By Theorem 3.19, for some $r > 0$ and $\varepsilon \in (0,1)$, $\Gamma^r_{(\mu,\nu)_2}(x) = \overline{B(\xi,\varepsilon,r)}$. Moreover, by Theorem 3.21, $\overline{B(\xi,\varepsilon,r)} = st_2-_{\nu}^{\mu}LIM^r_{x_{jk}}$. Hence, $\Gamma^r_{(\mu,\nu)_2}(x) = st_2-_{\nu}^{\mu}LIM^r_{x_{jk}}$.

4. Conclusion

This paper studies the concept of rough statistical convergence, a generalization of rough convergence, and statistical convergence in an IFNS. Then, it defines the concepts of r-st limit and r-st cluster points' sets and investigates some of their basic properties.

In the future studies, researchers can study the concepts proposed herein for triple sequences. Moreover, they can define the concept of rough ideal convergence of a double sequence in an IFNS and examines its basic properties.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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Determination of the physicomechanical characteristics of black chickpea grains

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Sphericity,
Colour characteristics,
Friction of coefficient,
Hardness

Abstract — In this study, some physicomechanical characteristics of black chickpea grains, which have rich content, were investigated. Dimensional and volumetric properties, colour characteristics, and frictional and mechanical properties were examined within the physicomechanical properties of the grains. Compression tests were carried out for the behaviour of black chickpea against mechanical force, both force and deformation characteristics of black chickpea grains were determined, and they were used at three different loading speeds (30 mm min^{-1}, 50 mm min^{-1}, 70 mm min^{-1}). The size dimension, such as width, length, and thickness values, were found as 6.68 mm, 8.85 mm, and 6.26 mm, and the mass was 0.25 g, respectively. The highest a^* and b^*, L^* values were determined as 4.31, −0.27, and 18.87, respectively. The values of the static friction coefficient of the black chickpea grains on different friction surfaces (PVC, galvanized sheet, laminate, plywood, and rubber) were observed on the rubber surface with the highest. In the mechanical test results, the highest force value was found on the 50 mm min^{-1} width (Y-) axis, and the highest hardness value was found on the thickness (Z-) axis at the 30 mm min^{-1} loading speed.

Subject Classification (2020):

1. Introduction

The black chickpea is a nutritious plant with a fibrous structure and rich content. Black chickpeas, which are both delicious, satisfying, beneficial and widely used, have recently become very popular. Black chickpeas have rich vitamins and minerals. However, the amount of magnesium and folic acid is also quite high. The amount of iron that needs to be met to prevent anaemia is high in black chickpeas. It also contains higher levels of antioxidants than other legumes. In addition to being richer in terms of nutritional value, black chickpea is more delicious than other legumes in terms of taste [1].

The particles' size, shape, mass, and aerodynamic properties are generally considered in the equipment used for cleaning and separation processes [2]. In addition, knowing these properties is essential in designing and developing systems, machines, equipment, and models for special purposes such as processing, transmission, drying, transportation, and storage of grains [3-7]. In this respect, parameters

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such as grain size, sphericity, mass, projection area, true density, bulk density, friction coefficient and terminal velocity should be determined.

The physicomechanical properties of the black chickpea grains should be studied to design and develop relevant machinery and facilities for harvesting, storage, transportation, and processing. The size dimension, colour, shape, frictional characteristics, and mechanical behaviour of chickpea grains are important in grinding, separation, sizing, transport structures and storage design. Physical properties are essential to optimize the various factors of black chickpea grains, threshing yield, pneumatic conveying, and storage.

The fact that black chickpeas have been very popular recently increases production and creates a desire to obtain quality results. When the articles about black chickpeas were examined, there were few studies related to the subject, and this study was carried out in which physicomechanical properties were combined. This study examines black chickpea grains' physicomechanical properties, linear dimensions, volumetric properties, static friction coefficient on different surfaces, mechanical behaviour, and colour characteristics.

2. Materials and Methods

The broken and immature grains, foreign matter, dust, and dirt in the grains used in the study were manually cleaned. 100 black chickpea grains were used to be used in the study. Length (Ln), width (Wd), and thickness (Th) as three basic dimensional properties among the physical and mechanical properties of the grains, measured with a digital caliper [8] with an accuracy of 0.01 mm, was shown in Figure 1.

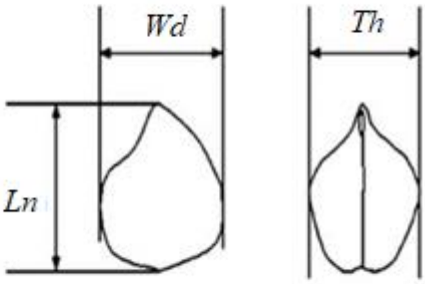

Figure 1. Basic dimensions of a sample black chickpea grain

A precision scale (accuracy of 0.01 g) was used to obtain the mass of each grain. To get a thousand-grain mass (Tm), 100 grains were weighed (accuracy of 0.001 g) by an electronic scale [9].

To determine the moisture content of the grains, they were dried in an oven at 105°C for 24 hours and calculated concerning the dry basis [10].

$$M_d = \frac{mw-md}{md}100$$

Here,
mw: wet mass of the grain
md: dry mass of the grain
M_d: moisture content (dry basis %)

The following equations were used to calculate the geometric mean diameter, sphericity, surface area, and volume of the grains [11].

$$Gm = (LnWdTh)^{\frac{1}{3}} \tag{2.1}$$

$$Sa = \pi(Gm)^2 \tag{2.2}$$

$$Sp = \frac{Gm}{Ln}100 \tag{2.3}$$

$$Vl = \frac{\pi}{6}LnWdTh \tag{2.4}$$

Here,

Gm: Geometric mean diameter (mm)

Ln: Length (mm)

Wd: Width (mm)

Th: Thickness (mm)

Sa: Surface area (mm^2)

Sp: Sphericity (%)

Vl: Volume (mm^3)

To determine the bulk density (Bd) of the grain, the hectoliter method was used, and to determine the true density (Td), the liquid displacement method was used [11]. Pure water was used as the fluid. The porosity value (Pr) was calculated according to Mohsenin [11], considering the bulk density and true density values.

The colour characteristics of the grains, a^* $[green(-\infty) - red\ (\infty)]$, $b^*[blue(-\infty) - yellow(\infty)]$, and $L^*[brightness\ (0 - 100)]$ were determined with a colourimeter [Minolta (CR- 3000)]. Chroma (Cr) defines the purity and saturation of the colour, and hue angle describes the angle of the colour [12]. Chroma and hue angle (ha) values [13] stated the following equations were obtained.

$$Cr = [(a^*)^2 + (b^*)^2]^{\frac{1}{2}} \tag{2.5}$$

$$ha = \tan^{-1}\frac{b^*}{a^*} \tag{2.6}$$

In addition, the ΔE value for the general colour value of black chickpeas was calculated using Equation (2.7) [14]. The lower the ΔE value, the darker the colour.

$$\Delta E = \sqrt{(L^{*2} + a^{*2} + b^{*2})} \tag{2.7}$$

PVC, galvanized, laminate, plywood and rubber materials were used as different friction surfaces for the mechanical properties of the grains. Static friction coefficients were found using a friction measurement scheme. The friction coefficient value (μ) was calculated by considering the angle of inclination ($\tan\alpha$) at the moment the grains started moving from the surface inclined with a lever [15]. Trials were applied with ten repetitions.

A biological material test measuring device is used for compression testing of mechanical properties and operated with a computer program. This device has a pressure and draws dynamometer (Sundoo draw dynamometer (Model SH-500, 0.1 N precision, China), a digital velocity unit, and a measuring ruler stand. It is a motorized and automatically controlled device. Experiments carried out by applying mechanical force to the grains at three different deformation rates (30 mm min^{-1}, 50 mm min^{-1} and 70 mm min^{-1}) and in three different axes (length, thickness, and width) are provided in Figure 2. The values are read by keeping the speeds determined by the speed adjustment and fixation panel in the test device constant and are manifested in Figure 3. The values of rupture energy (Re), hardness (Hr) and rupture force (Rp) are determined with the help of the following equations [16-17].

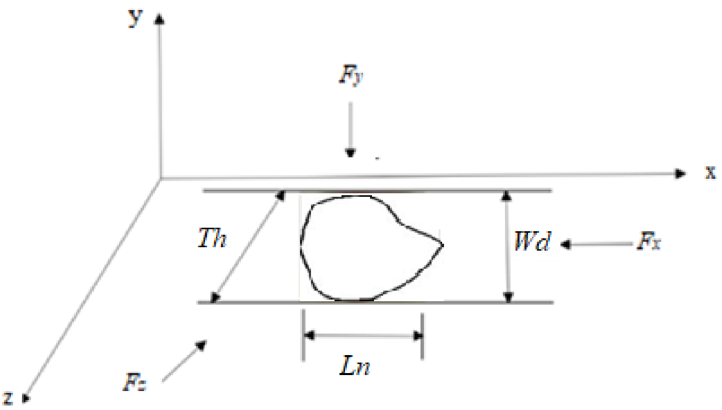

Figure 2. Representation of the axial dimensions (F_x, F_y, F_z) forces of a sample black chickpea grain

$$Re = \frac{RfDf}{2} \tag{2.8}$$

$$Hr = \frac{Rf}{Df} \tag{2.9}$$

$$Rp = \frac{ReLs}{60000\,Df} \tag{2.10}$$

Here,

Hr: Hardness

Re: Rupture energy (N mm)

Rf: Rupture force (N)

Df: Deformation (mm)

Rp: Rupture power (W)

Ls: Loading speed (mm min^{-1})

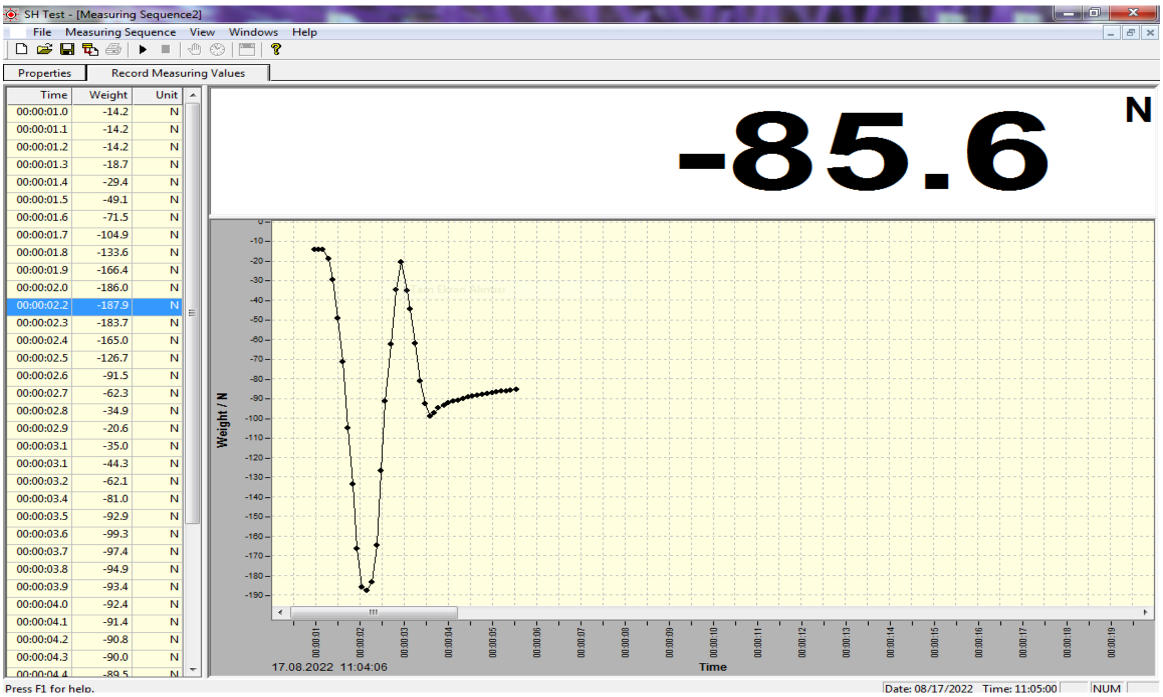

Figure 3. Test measurement of the rupture force of black chickpea grain

In the statistical evaluations of the research results, SPSS (Statistical Package for Social Sciences) program was used. Apart from general statistical calculations, an analysis of variance was performed since loading speed and axes were also used in force, deformation, hardness, energy, and rupture power, in which mechanical behaviour was determined. A multiple comparison (Duncan) test was also applied to determine the differences related to the parameters examined.

3. Results and Discussions

The initial moisture content of the black chickpea grains was determined as 12.01% (db). Data on the geometric properties of black chickpea grains are given in Table 1. When we look at the results table, the highest porosity and surface area values were found to be 93.31% and 221.29 mm^2.

Table 1. Some physical properties of black chickpea grains

Physical Properties	Mean (*)	Maximum	Minimum	Variation coefficient	Standard error
Ln (mm)	8.85 ± 0.61	10.36	6.83	6.83	0.19
Wd (mm)	6.68 ± 0.39	7.75	5.53	5.81	0.12
Th (mm)	6.26 ± 0.49	7.70	5.01	7.89	0.16
Gm (mm)	7.16 ± 0.38	8.39	6.23	5.30	0.12
Sp (%)	81.07 ± 3.83	93.31	72.66	4.72	1.21
Sa (mm^2)	161.44 ± 17.11	221.29	121.91	10.60	5.42
m (g)	0.25 ± 0.04	0.36	0.14	17.18	0.01
Tm (g)	229.08 ± 4.90	234.20	222.55	2.14	1.55
Vl (mm^3)	194.84 ± 31.12	311.51	127.27	15.97	9.85
Bd (kg m^{-3})	632.26 ± 9.61	647.02	615.08	1.52	3.04
Td (kg m^{-3})	1241.05 ± 113.98	1434.60	1049.43	9.18	36.07
Pr (%)	48.68 ± 4.66	56.53	40.05	9.57	1.47

Here, ± values indicate standard deviation. Moreover, *Ln*: Length (mm). *Wd*: Width (mm) *Th*: Thickness (mm). *Gm*: Geometric mean diameter (mm). *Sp*: Sphericity (%). *Sa*: Surface area (mm^2). *m*: grain mass (g). *Tm*: Thousand-grain mass. *Vl*: Volume (mm^3). *Bd*: Bulk density (kg m^{-3}). *Td*: True density (kg m^{-3}). *Pr*: Porosity (%).

Eissa et al. [18] determined some physical properties of moisture content for two types of chickpea grains (Giza 3 and Giza 195) and used four moisture content levels ranging from 11.6% db to 25.4% db. The width, length, thickness and geometric mean diameter between 6.10 ± 0.04 mm and 6.37 ± 0.04 mm, between 7.92 ± 0.04 mm and 8.14 ± 0.04 mm, between 6.43 ± 0.04 mm and 6.84 ± 0.04 mm, between 6.77 ± 0.07 mm and 7.08 ± 0.07 mm, the sphericity ranged from between 85.53 ± 0.19 and 87.00 ± 0.19, respectively. They determined the mean measured surface area as (144.73 ± 1.55) mm^2. Although there are differences in terms of variety and moisture content, close and similar values, have emerged when the grains are compared with the study.

In the same study, Eissa et al. [18] determined that the physical properties of chickpea grains at 11.6% db moisture content, namely bulk density, true density, and porosity, were 730.05 kg m^{-3}, 1.308.02 kg m^{-3} and, 44.13%, respectively.

The colour values (L^*, a^*, b^*, *Chroma, hue angle*) of the black chickpea grain measured over the shell surface are given in Table 2. The mean L^*, a^*, and b^* colour values of black chickpea grains were measured as 17.86, 3.64, and -0.95, respectively.

Table 2. Colour characteristics of black chickpea grains.

Colour characteristics	Mean (*)	Maximum	Minimum	Variation coefficient	Standard error
L^*	17.86 ± 0.76	18.87	16.70	4.26	0.24
a^*	3.64 ± 0.70	4.31	2.30	19.33	0.22
b^*	-0.95 ± 0.40	−0.27	−1.50	−42.32	0.13
Cr	3.77 ± 0.73	4.35	2.38	19.22	0.23
ha	-14.79 ± 5.22	−3.58	−20.99	−35.30	1.65
ΔE	62.65 ± 31.10	102.46	21.17	49.63	9.84

Here, ± values indicate standard deviation.

The results of the static friction coefficient values on PVC (Polyvinyl Chloride), laminate, galvanized metal, plywood, and rubber surfaces as different friction surfaces of black chickpea grains are given in Figure 4.

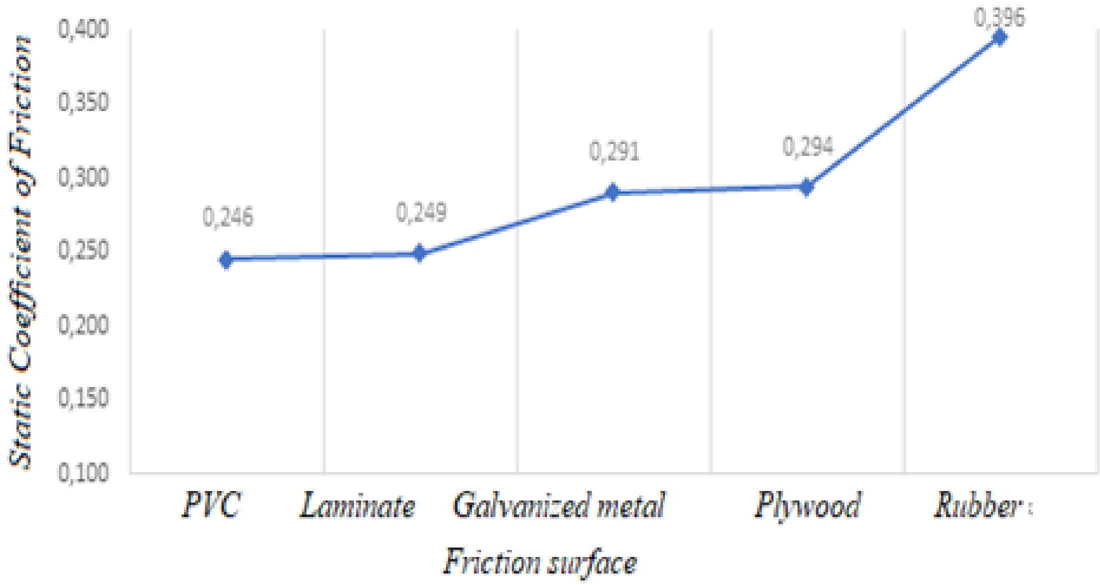

Figure 4. Static friction coefficient values of black chickpea grains on different surfaces.

Tabatabaeefar et al. [19] found the highest static friction value of chickpeas on the rubber surface as 0.33 in a study carried out by the chickpea cultivar grown in Iran to separate the foreign materials in the chickpea grain. Looking at the results, the black chickpea grain also shows the highest value on the rubber surface.

In Table 3, in the statistical results of black chickpea grains, the rupture force and the power required for rupture were obtained in the width axis at the highest speed of 50 mm min^{-1}. Statistically, significant differences in rupture force, hardness and rupture power values of black chickpea grains were obtained at different loading speeds and axes. In their study, Jaliliantabar and Lorestani [20] determined the rupture force of chickpea grains in F_y and F_z to be 318 N and 230.27 N, respectively, at a loading speed of 5 mm min^{-1}. When compared with the results of the study, close values were found in both studies due to their rupture forces.

Table 3. The mechanical characteristics of black chickpea grains vary according to loading speeds and axes

Loading speeds (Ls, mm min^{-1})	Loading axess	Rupture force (Rf, N)	Deformation (Df, mm)	Rupture energy (Re, N mm)	Hardness (Hr, N mm^{-1})	Rupture power (Rp, W)
30	*X-*	$230.01 \pm 40.03b^{**}$	4.31 ± 0.74^{ns}	503.84 ± 147.39^{ns}	$54.34 \pm 10.34b^{*}$	$0.058 \pm 0.010b^{**}$
	Y-	$329.38 \pm 52.60a^{**}$	4.01 ± 1.09^{ns}	658.97 ± 206.03^{ns}	$89.16 \pm 32.95a^{*}$	$0.082 \pm 0.013a^{**}$
	Z-	$317.64 \pm 81.33a^{**}$	3.77 ± 1.18^{ns}	603.88 ± 262.88^{ns}	$94.03 \pm 44.26a^{*}$	$0.079 \pm 0.020a^{**}$
	F value	**8.05**	**0.71**	**1.39**	**4.46**	**8.05**
50	*X-*	$243.67 \pm 57.16a^{**}$	$6.10 \pm 0.44a^{**}$	739.80 ± 173.85^{ns}	$40.34 \pm 10.44b^{**}$	$0.122 \pm 0.029b^{**}$
	Y-	$350.10 \pm 55.82b^{**}$	$5.39 \pm 0.54b^{**}$	947.74 ± 209.01^{ns}	$65.28 \pm 10.68a^{**}$	$0.175 \pm 0.028a^{**}$
	Z-	$348.66 \pm 79.58b^{**}$	$4.71 \pm 0.66c^{**}$	828.64 ± 237.10^{ns}	$74.56 \pm 17.08a^{**}$	$0.174 \pm 0.040a^{**}$
	F value	**8.79**	**15.91**	**2.51**	**18.25**	**8.79**
70	*X-*	$222.71 \pm 71.87b^{**}$	$5.71 \pm 0.68a^{*}$	$607.39 \pm 291.01b^{*}$	$39.08 \pm 11.93b^{*}$	$0.162 \pm 0.063b^{**}$
	Y-	$345.03 \pm 28.11a^{**}$	$4.95 \pm 0.53b^{*}$	$851.42 \pm 97.36a^{*}$	$70.59 \pm 9.99a^{*}$	$0.259 \pm 0.021a^{**}$
	Z-	$327.62 \pm 37.98a^{**}$	$1.08 \pm 0.34b^{*}$	$749.70 \pm 194.52ab^{*}$	$77.67 \pm 32.88a^{*}$	$0.246 \pm 0.028a^{**}$
	F value	**17.76**	**5.10**	**3.41**	**9.57**	**15.98**

***: $p < 0.01$. *: $p < 0.05$, ns: non significant, $\pm$ values indicate standard deviation.*

4. Conclusion

Within the scope of the research, the physicomechanical properties of black chickpea grains were examined.

- The moisture content of the black chickpea grains used in the study was 12.01% compared to the dry basis.

- The length, width, and thickness values of the size properties were determined as 8.85 mm, 6.68 mm, and 6.26 mm, and their mass was 0.25 g, respectively.

- The bulk density, true density and porosity values were determined as 632.26 kg m^{-3}, 1241.05 kg m^{-3}, and 48.68%, respectively.

- The highest L^*, $a^*$and b^*values were determined as 18.87, 4.31, and -0.27, respectively.

-The static friction coefficient values of black chickpea grains on different friction surfaces (PVC, galvanized sheet, laminate, plywood, and rubber) were shown on the rubber surface with the highest.

- When the compression test was applied, the highest force value in the grains was measured as 350.10 N in the 50 mm min^{-1} width (*Y*-) axis. The highest hardness value was found with 94.03 N mm^{-1} in the thickness (*Z*-) axis at a loading speed of 30 mm min^{-1}.

Considering the importance of the black chickpea, which is progressing towards becoming popular in our country, the results of the biotechnological features of the equipment of the systems and facilities to be used in the cleaning, classification according to the dimensions, packaging and packaging of the product in industrial applications, depending on the production areas, can be determined by the results of harvest and post-harvest product quality of black chickpea. It is thought that together, they can increase their commercial value.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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On the Richard and Raoul numbers

Orhan Dişkaya[1], **Hamza Menken**[2]

Keywords
Padovan numbers,
Perrin numbers,
Generating functions,
Binet-like formula,
Binomial sum

Abstract — In this study, we define and examine the Richard and Raoul sequences and we deal with, in detail, two special cases, namely, Richard and Raoul sequences. We indicate that there are close relations between Richard and Raoul numbers and Padovan and Perrin numbers. Moreover, we present the Binet-like formulas, generating functions, summation formulas, and some identities for these sequences.

Subject Classification (2020): 11B39, 05A15.

1. Introduction

There has been a considerable deal of interest in the existing literature on the study of integer sequences such as Fibonacci, Lucas, Pell, and Jacobsthal and their applications in various scientific disciplines. One of the most studied sequences is the Padovan sequence $\{P_n\}_{n\geq 0}$, which is defined by the recurrence relation

$$P_{n+3} = P_{n+1} + P_n \tag{1.1}$$

with $P_0 = 1$, $P_1 = 1$, and $P_2 = 1$. The first few Padovan numbers are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28. This sequence corresponds to the sequence A000931 in the online encyclopedia of integer sequences (OEIS) in [1].

This and similar sequences have been presented in various math articles, including numbers theory, analysis, calculus, applied mathematics, algebra, and statistics, as well as architectural, physics, and various scientific articles [2]. In [3], the Padovan p-numbers are defined and various properties are discussed. In [4], some families of Toeplitz-Hessenberg determinants the entries of which are the Padovan numbers are investigated. In [5], the Fermat numbers are determined in the Padovan and Perrin sequences. Matrices formula and sequences for the Padovan and Perrin sequences are given in [6, 7]. In [8], the Padovan and Pell-Padovan quaternions are introduced and their some properties are investigated. The split (s,t)-Padovan and (s,t)-Perrin quaternions are studied in [9]. In [10], some geometric interpretations of the plastic ratio

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related to the Padovan numbers are given. A historical analysis of the Padovan numbers is studied in [11]. Now let us give the definition of Perrin numbers which have the same recurrence as the Padovan numbers, but they have different initial conditions than the initial conditions of the Padovan numbers. Edouard Lucas (1876) studied Perrin numbers for the first time. However, the sequence was later named after Raoul Perrin, who worked on this sequence in 1899. The Perrin sequence $\{R_n\}_{n\geq 0}$ is defined by the recurrence

$$R_{n+3} = R_{n+1} + R_n \tag{1.2}$$

with $R_0 = 3$, $R_1 = 0$, and $R_2 = 2$. The first few Perrin numbers are 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39. This sequence corresponds to the sequence A001608 in the online encyclopedia of integer sequences (OEIS) in [12].

The recurrences (1.1) and (1.2) involve the characteristic equation

$$x^3 - x - 1 = 0 \tag{1.3}$$

If its roots are denoted by α, β, and γ then, the following equalities can be derived

$$\alpha + \beta + \gamma = 0$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -1$$

$$\alpha\beta\gamma = 1$$

Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \tag{1.4}$$

Here,

$$a = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)}, \quad \text{and} \quad c = \frac{(\alpha-1)(\beta-1)}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.5}$$

and the Binet-like formula for the Perrin sequence is

$$R_n = \alpha^n + \beta^n + \gamma^n \tag{1.6}$$

The negative indices of Padovan and Perrin numbers are obtained with the following recurrences, respectively:

$$P_n = P_{n+3} - P_{n+1} \text{ and } R_n = R_{n+3} - R_{n+1}$$

For $n \geq 0$, the Padovan and Perrin sequences the following identities are valid:

$$P_n = P_{n-1} + P_{n-5} \tag{1.7}$$

$$R_n = P_{n+1} + P_{n-10} \tag{1.8}$$

$$P_n^2 - P_{n+1}P_{n-1} = P_{-n-7} \tag{1.9}$$

$$\sum_{k=0}^{n} P_k = P_{n+5} - 2 \tag{1.10}$$

$$\sum_{k=0}^{n} P_{2k} = P_{2n+3} - 1 \tag{1.11}$$

$$\sum_{k=0}^{n} P_{2k+1} = P_{2n+4} - 1 \tag{1.12}$$

$$\sum_{k=0}^{n} P_k^2 = P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2 \tag{1.13}$$

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x + x^2}{1 - x^2 - x^3} \tag{1.14}$$

$$\sum_{n=0}^{\infty} R_n x^n = \frac{3 - x^2}{1 - x^2 - x^3} \tag{1.15}$$

$$\sum_{n=0}^{m} \binom{m}{n} P_n = P_{3m} \tag{1.16}$$

$$\sum_{k=0}^{m} \binom{m}{k} P_{n-k} = P_{n+2m} \tag{1.17}$$

The above identities and properties can be seen in [13–15].

2. The Richard, Raoul, and A023434 Numbers

In this part, we give definitions of the Richard (for honor Richard Padovan) and Raoul (for honor Raoul Perrin) sequences and investigate some identities of these sequences such as the Binet-like formula, generating functions, certain binomial sums and various identities. Moreover, we define a new sequence associated with Richard and Raoul sequences, and also examine some identities of this sequence. Studies similar to the numbers in this work, namely generalized Edouard, Ernst, Oresme, Pisano and John numbers, were studied by Soykan [14, 16–21]. Catarino and Borges [22] studied the Leonardo numbers.

Definition 2.1. The Richard sequence $\{\mathscr{P}_n\}_{n\geq 0}$ is defined by the following recurrence

$$\mathscr{P}_{n+3} = \mathscr{P}_{n+1} + \mathscr{P}_n + 1, \quad \mathscr{P}_0 = \mathscr{P}_1 = \mathscr{P}_2 = 1 \tag{2.1}$$

The first few terms of the Richard numbers are 1, 1, 1, **3**, **3**, **5**, **7**, 9, **13**, **17**, **23**, **31**, **41**, 55, **73**, **97**, 129, 171, **227**, 301, 399, 529, **701**, 929, **1231**, 1631, **2161**, 2863, **3793**, 5025, 6657, **8819**, 11683, 15477, 20503, 27161 where bold values are prime numbers and every Richard numbers are odd, which is easily verified. Furthermore, the unit digits of Richard numbers are periodic with period 24.

Lemma 2.2. For each $n \geq 0$, the Richard number $\mathscr{P}_n$ is an odd number.

Proof.

We prove this by using the induction method on n. For $n = 0,1,2$, the assertion is true. Assume that the assertion is valid for $2 < k \leq n$. Then, we may verify it for $n+1$. Since by recurrence (2.1) we have $\mathscr{P}_{n+4} = \mathscr{P}_{n+2} + \mathscr{P}_{n+1} + 1$ and as the sum of two odd numbers which are $\mathscr{P}_{n+2}$ and $\mathscr{P}_{n+1}$ by induction hypothesis is even and, in order, the sum of an even number with the number 1 is an odd number, The proof is finished.

The recurrence (2.1) can also be written as follows

$$\mathscr{P}_{n+4} = \mathscr{P}_{n+3} + \mathscr{P}_{n+2} - \mathscr{P}_n \tag{2.2}$$

In fact, by the equalities $\mathscr{P}_{n+4} = \mathscr{P}_{n+2} + \mathscr{P}_{n+1} + 1$ and $\mathscr{P}_{n+3} = \mathscr{P}_{n+1} + \mathscr{P}_n + 1$, we reach equality (2.2)

The recurrence (2.2) involves the characteristic equation

$$r^4 - r^3 - r^2 + 1 = 0 \tag{2.3}$$

The roots of the characteristic Equation (2.3) are 1, α, β, and γ where the other roots except 1 are the same as the roots of the characteristic Equation (1.3).

If we take the initial conditions of the above recurrence (2.2) as $\mathscr{P}_0 = \mathscr{P}_1 = \mathscr{P}_2 = 1$, and $\mathscr{P}_3 = 3$, we can easily reach the following Binet-like formula.

Theorem 2.3. The Binet-like formula for the Richard sequence is

$$\mathscr{P}_n = 2a\alpha^n + 2b\beta^n + 2c\gamma^n - 1 \tag{2.4}$$

where the values a, b, c are given in Equation (1.5).

Proof.

Assume that $\mathscr{P}_n = u\alpha^n + v\beta^n + w\gamma^n + k$. So, we have

$$\mathscr{P}_0 = u + v + w + k = 1$$

$$\mathscr{P}_1 = u\alpha + v\beta + w\gamma + k = 1$$

$$\mathscr{P}_2 = u\alpha^2 + v\beta^2 + w\gamma^2 + k = 1$$

$$\mathscr{P}_3 = u\alpha^3 + v\beta^3 + w\gamma^3 + k = 3$$

By performing the solution with the Gaussian elimination method, we find $u = 2a$, $v = 2b$, $w = 2c$, and $k = -1$.

Theorem 2.4. For $n \geq 0$, the following identity is valid:

$$\mathscr{P}_n = 2P_n - 1 \tag{2.5}$$

Proof.
Using te identities (2.4) and (1.4), we get

$$\begin{aligned}\mathscr{P}_n &= 2a\alpha^n + 2b\beta^n + 2c\gamma^n - 1\\ &= 2\left(a\alpha^n + b\beta^n + c\gamma^n\right) - 1\\ &= 2P_n - 1\end{aligned}$$

Definition 2.5. The Raoul sequence $\{\mathscr{R}_n\}_{n\geq 0}$ is defined by the following recurrence

$$\mathscr{R}_{n+3} = \mathscr{R}_{n+1} + \mathscr{R}_n + 1, \quad \mathscr{R}_0 = 3, \quad \mathscr{R}_1 = 0, \quad \text{and} \quad \mathscr{R}_2 = 2 \tag{2.6}$$

The first few terms of the Raoul numbers are 3, 0, 2, 4, 3, 7, 8, 11, 16, 20, 28, 37, 49, 66, 87, 116, 154, 204, 271. Similarly, the recurrence (2.6) can also be written as follows

$$\mathscr{R}_{n+4} = \mathscr{R}_{n+3} + \mathscr{R}_{n+2} - \mathscr{R}_n \tag{2.7}$$

If we take the initial conditions of the above recurrence (2.7) as $\mathscr{R}_0 = 3$, $\mathscr{R}_1 = 0$, $\mathscr{R}_2 = 2$, and $\mathscr{R}_3 = 4$, we can easily reach the following Binet-like formula.

Theorem 2.6. The Binet-like formula for the Raoul sequence is

$$\mathscr{R}_n = (a+1)\alpha^n + (b+1)\beta^n + (c+1)\gamma^n - 1 \tag{2.8}$$

where the values a, b, c are given in Equation (1.5).

Proof.
Assume that $\mathscr{R}_n = x\alpha^n + y\beta^n + z\gamma^n + l$. Therefore, we have

$$\mathscr{R}_0 = x + y + z + l = 3$$

$$\mathscr{R}_1 = x\alpha + y\beta + z\gamma + l = 0$$

$$\mathscr{R}_2 = x\alpha^2 + y\beta^2 + z\gamma^2 + l = 2$$

$$\mathscr{R}_3 = x\alpha^3 + y\beta^3 + z\gamma^3 + l = 4$$

By performing the solution with the Gaussian elimination method, we find $x = a+1$, $y = b+1$, $z = c+1$, and $l = -1$. Now, we give a new integers sequence (called the sequence A023434) which we will relate to some sequences such as Padovan, Perrin, Richard and Raoul sequences.

Definition 2.7. The sequence A023434 $\{\mathscr{T}_n\}_{n\geq 0}$ is defined by the following recurrence relation

$$\mathscr{T}_{n+4} = \mathscr{T}_{n+3} + \mathscr{T}_{n+2} - \mathscr{T}_n, \quad \mathscr{T}_0 = 0, \quad \mathscr{T}_1 = 0, \quad \mathscr{T}_2 = 0, \quad \text{and} \quad \mathscr{T}_3 = 1 \tag{2.9}$$

The first few numbers of the sequence A023434 are 0, 0, 0, 1, 1, 2, 3, 4, 6, 8, 11, 15, 20, 27, 36, 48, 64, 85. This sequence corresponds to the sequence A023434 in the online encyclopedia of integer sequences (OEIS) in [23].

Theorem 2.8. The Binet-like formula for the sequence A023434 is

$$\mathcal{T}_n = a\alpha^n + b\beta^n + c\gamma^n - 1 \tag{2.10}$$

where the values a, b, c are given in Equation (1.5).

Proof.
It is proved similarly to the proofs of Theorem 2.3 and Theorem 2.6.
The negative indices of Richard, Raoul and A023434 numbers are obtained with the following recurrences, respectively:

$$\mathcal{P}_n = \mathcal{P}_{n+3} - \mathcal{P}_{n+1} - 1$$
$$\mathcal{R}_n = \mathcal{R}_{n+3} - \mathcal{R}_{n+1} - 1$$
$$\mathcal{T}_n = \mathcal{T}_{n+3} + \mathcal{T}_{n+2} - \mathcal{T}_{n+4}$$

The relations of the above sequences with each other are given below.

Proposition 2.9. For $n \geq 0$, the following identities are valid:

i. $\mathcal{R}_n = P_n + R_n - 1$

ii. $\mathcal{P}_n = 2\mathcal{R}_n - 2R_n + 1$

iii. $\mathcal{T}_n = P_n - 1$

iv. $\mathcal{R}_n = R_n + \mathcal{T}_n$

v. $\mathcal{P}_n = 2\mathcal{T}_n + 1$

Proof.
By using the identities (1.4), (1.6), (2.4), (2.8), (2.9), and (2.10), the above identities are proved.

Proposition 2.10. For $n \geq 0$, the following identities are valid:

i. $\mathcal{P}_n = \mathcal{P}_{n-1} + \mathcal{P}_{n-5} + 1$

ii. $\mathcal{R}_n = \dfrac{\mathcal{P}_{n+3} + \mathcal{P}_{n-10}}{2}$

iii. $\mathcal{P}_n^2 - \mathcal{P}_{n+1}\mathcal{P}_{n-1} = 2\mathcal{P}_{-n-7} - 2\mathcal{P}_n + \mathcal{P}_{n+1} + \mathcal{P}_{n-1} + 2$

Proof.
By using the identities (1.7), (1.8), (1.9), and (2.5) the above identities are proved.

Proposition 2.11. For $n \geq 0$, the following sum formulas of terms of the Richard sequence are valid:

i. $\sum_{k=0}^{n} \mathcal{P}_k = \mathcal{P}_{n+5} - n - 3$

ii. $\sum_{k=0}^{n} \mathcal{P}_{2k} = \mathcal{P}_{2n+3} - n - 1$

iii. $\sum_{k=0}^{n} \mathcal{P}_{2k+1} = \mathcal{P}_{2n+4} - n - 1$

iv. $\sum_{k=0}^{n} \mathcal{P}_k^2 = \mathcal{P}_{n+2}^2 - \mathcal{P}_{n-1}^2 - \mathcal{P}_{n-3}^2 + 2\mathcal{P}_{n-2} - \mathcal{P}_{n+5} + n + 6$

Proof.

By using the identities (1.10), (1.11), (1.12), (1.13), and (2.5) the above identities are proved.

Theorem 2.12. The generating functions of the Richard, Raoul and A023434 sequences are as follows:

i. $\sum_{n=0}^{\infty} \mathscr{P}_n x^n = \frac{-x^3+x^2+2x-1}{(1-x)(1-x^2-x^3)}$

ii. $\sum_{n=0}^{\infty} \mathscr{R}_n x^n = \frac{x^3-2x+2}{(1-x)(1-x^2-x^3)}$

iii. $\sum_{n=0}^{\infty} \mathscr{T}_n x^n = \frac{2x^3+x^2}{(1-x)(1-x^2-x^3)}$

Proof.

By using the identities (1.14), (1.15), and (2.5) the above identities are proved.

The reader can look for similar results in the above for the Raoul sequence.

Theorem 2.13. The exponential generating functions of the Richard, Raoul, and A023434 sequences are as follows:

i. $\sum_{n=0}^{\infty} \mathscr{P}_n \frac{t^n}{n!} = 2ae^{\alpha t} + 2be^{\beta t} + 2ce^{\gamma t} - e^t$

ii. $\sum_{n=0}^{\infty} \mathscr{R}_n \frac{t^n}{n!} = (a+1)e^{\alpha t} + (b+1)e^{\beta t} + (c+1)e^{\gamma t} - e^t$

iii. $\sum_{n=0}^{\infty} \mathscr{T}_n \frac{t^n}{n!} = ae^{\alpha t} + be^{\beta t} + ce^{\gamma t} - e^t$

Proof.

By using the identities (2.4), (2.8), and (2.10) the above identities are proved.

Proposition 2.14. For $n \geq 0$, the following identities are valid:

i. $\sum_{n=0}^{m} \binom{m}{n} \mathscr{P}_n = 2\mathscr{P}_{3m} + 1 - 2^m$

ii. $\sum_{k=0}^{m} \binom{m}{k} \mathscr{P}_{n-k} = 2\mathscr{P}_{n+2m} + 1 - 2^m$

Proof.

By using the identities (1.16), (1.17), and (2.5) the above identities are proved.

The reader can look for similar results in the above for the Raoul and sequence A023434.

3. Conclusion

In this present work, the sequences of the Richard and Raoul numbers are introduced. In addition, the sequence A023434 associated with these sequences is identified. The Binet-like formulas, generating functions, and a few identities, among other characteristics affecting these sequences, are presented. There are also created a number of expressions including sums and products with the terms of these sequences. In the future, the various new properties of these sequences can be examined.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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Comparison of different helichrysum species in terms of agro-morphological characteristics and essential oil content

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Keywords:
Helichrysum stoechas, Helichrysum italicum, Immortelle, Goldengrass

Abstract — *Helichrysum* species belonging to the Asteraceae family are popularly known as goldengrass, mantuvar, guddeme flower and immortelle in our country. *Helichrysum* species, which are commonly found in nature in Anatolia, are consumed as herbal tea. *Helichrysum* essential oils are used in folk medicine to protect against colds, kidney stones, stomach ailments, coronary heart disease, stroke and some cancer derivatives, thanks to their antioxidant, antiviral, antifungal, antimicrobial, anti-inflammatory properties and phenolic compounds. This research was carried out to compare agro-morphological characteristics and essential oil ratio between cultured *Helichrysum italicum* (HI) and naturally distributed *Helichrysum stoechas* (HS) species. The experiment was established in Ege University Ödemiş Vocational School's land in April 2022 with HS and HI species according to the Random Blocks Trial Design with 3 replications. In the study, plant height, green herb yield, dry herb yield, fresh flower yield, dry flower yield and essential oil ratio parameters were investigated. The results of the analysis of variance with the data of the investigated characteristics, it was determined that the culture form HI may be a more suitable medicinal plant for the ecological conditions of İzmir-Ödemiş compared to HS, which spreads in the flora.

Subject Classification (2020): 62K10, 92F05.

1. Introduction

Helichrysum, a perennial herbaceous plant belonging to the Asteraceae family, is colloquially known as mantuvar, guddeme flower [1]. *Helichrysum* genus consists of more than 1000 species that are widely found in the world. There are 21 *Helichrysum* species and 27 taxa in the flora of Türkiye, and 14 of them are endemic species [2]. The flowers of the plant have been used in traditional medicine in Asia and Europe from past to present [3]. The biologically active and dominant compounds in the flowering parts of *Helichrysum* are flavonoids, flavanones, salipurposide, purine, and naringenin. Other highly present compounds are essential oils, carotenoids, and yellow pigments [4]. *Helichrysum* species are generally used to protect against colds, kidney stones, stomach ailments, stroke, and coronary heart disease thanks to their antioxidant, antiviral, antifungal, antimicrobial properties, and phenolic compounds [5]. It has been reported that Helichrysum genus generally shows anticarcinogen, carcinogen, and genotoxic effects and provides protection against cancer diseases [2]. *Helichrysum italicum* (HI) and *Helichrysum*

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stoechas (HS) are included as folk remedies due to their antiallergic and anti-inflammatory properties [6]. It is also known that HI has an anti-mutagenic effect [7]. Due to its anti-inflammatory properties, HI is preferred as a medicinal tea in traditional medicine for gallbladder disorders due to its bile regulating and diuretic properties. In many studies, antioxidant and inflammatory activity of aboveground of HI has been determined in various in vivo and in vitro experimental models [8,9]. In addition, the flowers of HI are traditionally used in the treatment of sunburns as well as their anti-inflammatory and anti-allergic effects [10].

HI generally grows at an altitude of 0-1800 m, on limestone maquis and limestone cliffs. It is grown in Cyprus, Southern Europe, Northwest Africa and the Mediterranean regions, and the plant's flowering time is usually April-May. *HS* is found at an altitude of 0-700 m, in lime-stony maquis, forest clearings and limestone rocks. It spreads in Italy, the Balkans, Cyprus, Lebanon, Syria, and Northwest Africa, and the plant usually blooms in March-June. HI can be distinguished from HS by its sparse hair cover, straw-coloured, regular imbricate arrangement, reverse pyramid-shaped involucrum bracts in the capitulum, and loose but very smooth corimbus. On the other hand, HS has dense hair cover, bright yellow coloured, loose imbricate arrangement in the capitulum [11].

In addition to all these, the essential oils of *Helichrysum* species contribute to the production of aromatherapy, perfumery and cosmetic products thanks to their pleasant smell [2]. According to the results of the essential oil analysis, the main components of HI were determined as α-pinene (10.2%), α-cedrene (9.6%), aromadendrene (4.4%), β-caryophyllene (4.2%) and limonene (3.8%) [12]. The main components of *HS* were determined as P-caryophyllene (27.9%), α-humulene (13.4%) and α-pinene (12.0%) [13].

Recently, interest in aromatherapy, which is applied using essential oils and fixed oils, has been increasing in our country and other countries. With the widespread use of aromatherapy, many fake and unidentified oils have also been introduced to the market. Among these oils, the essential oil of HI has an important place. Since the essential oil yield of HI is low, it is known that this oil is used as adulteration and put on the market as in other oils.

The purpose of the production of medicinal and aromatic plants is to obtain high quality and high yield besides the production of standard active substances. For this purpose, first of all, it is necessary to develop varieties suitable for different ecological regions and to determine modern cultivation techniques. There is no literature on the agro-morphological parameters of HI and HS species.

2. Material and Method

The research was carried out to compare some agro-morphological properties and essential oil content of HI and HS plants. It was carried out in 2022 at Ege University Ödemiş Vocational School Field Area (38°12'N latitude, 27°52'E longitude, 111 m altitude). HI and HS were used as plant material in the experiment, which was established in three replications according to the Random Blocks Trial Design. Seedlings of HI were obtained from Uludağ-Agro company and seedlings of HS were obtained from Bionorm Natural Products. The seedlings of the species used were planted on 12.04.2022 in 5 rows per plot, 3 m in length, with 20 plants in each row, 40 cm between rows and 20 cm above rows. The climatic data of the experimental area are given in Table 1.

Soil samples taken from different depths from the experimental area before planting were analysed in the laboratory of Ege University Faculty of Agriculture, Department of Soil and Plant Nutrition. The soil has a sandy loam structure with 1% organic matter, 0.08% total salt, 2.7% lime and pH 7.5 and 6 kg da^{-1} N was applied according to the analysis result. The cutting was started during the flowering period of the plants.

Table 1. Climatic data of the experimental area

Climatic Parameters	April	May	June	July	August
Average Temperature (°C)	16.2	21.5	26.9	29.3	28.6
Maximum Temperature (°C)	28.5	36.6	43.5	44.6	41.8
Minimum Temperature (°C)	7.8	8.3	14.0	14.3	16.2
Relative Humidity (%)	62.5	60.8	49.3	42.1	48.3
Total Rainfall (mm)	48.2	31.7	7.0	-	-

Before cutting, the height of 10 randomly selected plants from each plot from the soil surface to the tip of the plant was measured in cm and the averages were taken. Fresh herb yield and fresh flower yield were determined by cutting and weighing 10 plants at a height of 5 cm above the soil level. Dry herb and dried flower yields per plant were calculated by utilizing the % moisture losses calculated by drying these plants at 35 ºC [14]. The essential oil ratios were determined volumetrically with the Clevenger device according to the water distillation method in dried flowers at 35 ºC. The essential oil rate in the flower was calculated as ml/100 g (%) on dry matter.

2.1. Statistical Analysis

After the homogeneity test of all the data obtained in the research was done using SPSS 20.0 statistical package program, variance analysis was performed. Based on the significance of the analysis of variance, the groupings between the means were made using the Duncan multiple comparison test ($\alpha = 0.01$).

3. Results and Discussion

In terms of the parameters examined in the research, the F value, standard error, mean values and significance control of HS and HI are given in Table 2. As a result of the statistical analysis, it is seen that there is a 1% difference in plant height, fresh flower and dry flower values. In addition, it was determined that the examined fresh herb and dry herb yields did not reveal statistically significant results. It was determined that the average plant height was 25.00 cm in HS species and 29.93 cm in HI species. It was showed that the average wet herb yield was 120.90 g and the average dry herb yield was 43.41 g. While the average yield of fresh flowers was 19.39 g in HS species and 16.60 g in HI species, dry flower yield was stated as 7.06 g in HS species and 5.35 g in HI species.

The F value, standard error, mean values and significance control of the herba essential oil ratio and flower essential oil ratios of the species examined in the research are given in Table 2. Herba essential oil ratio was found to be 0.41% in HS species and 0.45% in HI species. In addition, it was stated that the flower essential oil ratio was 0.05% in HS species and 0.48% in HI species.

Helichrysum has very high thermal requirements, preferring warm regions and micro-habitats, as well as dry oligotrophic soils poor in organic matter [14]. The average plant height of Helichrysum was determined as 40.8 cm in the study carried out in Serbian conditions. In addition, the average plant height of plants growing near the sea was 39.4 cm and the height of plants growing inland was 42.2 cm, but these variations were not significant [15]. The existence of two different topographic conditions can be mentioned as the reason for the difference between the reported results and the plant height data obtained in this research.

Table 2. The mean, standard deviation, F value and significance levels of the examined parameters of the species used in the research

Species	Plant Height (cm)		Fresh Herb (g)		Dry Herb(g)		Fresh Flower(g)		Dry Flower(g)		Essential Oil (Herb)(%)		Essential Oil (Flower)(%)	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
H.stoechas	25.00	2.72	123.96	20.50	40.26	9.16	19.39	2.94	7.06	1.57	0.41	0.28	0.05	0.01
H.italicum	29.93	2.52	115.83	47.51	43.05	17.10	16.60	4.19	5.35	2.62	0.45	0.01	0.48	0.28
Total	27.46	3.61	120.90	36.09	43.41	13.78	17.99	3.83	6.20	2.30	0.43	0.25	0.26	2.38
F Value	24.365**		0.183 NS		1.080 NS		4.070*		4.422**		4.000*		676.000**	
CV (%)	9.96		12.43		11.02		13.99		15.98		4.73		7.70	
Comparison Test	H.*stoechas*-b H.*italicum*-a		-		-		H.*stoechas*-a H.*italicum*-b		H.*stoechas*-a H.*italicum*-b		H.*stoechas*-b H.*italicum*-a		H.*stoechas*-b H.*italicum*-a	

P< 0.05(*), P< 0.01(**), NS: Non-significant, Std. Dev.: Standart Deviation

Miloradovic et al. [15] was reported that *Helichrysum* species formed vegetative biomass and a large number of flower stalks and thus a high plant yield was obtained. It has been reported that this situation is caused by the optimum climatic conditions in the period between planting and cutting. It has been reported that this situation is caused by the optimum climatic conditions in the period between planting and cutting. And the same time, they reported that the average fresh herb yield per plant for the researchers varied between 210.3-232.9 grams. The reported results are higher than the results obtained in this study. This situation may have been caused by the cultivation under optimum conditions and the different plant species. Approximately 65% of the mass of the wet herb subjected to drying is lost during drying and the drug herb (dry herb) yield is determined. In the reported study and this research, it was determined that the dry herb yield did not create a significant variation.

When the essential oil ratios in dry herb were evaluated on the basis of species, it was determined that HI had 9% more essential oil than HS. When the essential oil ratios in dry flowers were evaluated on the basis of species, it was stated that HI had 9.6 times more essential oil than HI. The analyses showed that the essential oil content of HI flowers was higher than that of herbaceous, and the opposite was the case in the HS species. One of the most important components in the essential oil ratio is stability. It is thought that HI, which is cultivated, has a higher herb and flower essential oil ratio due to its more stable form.

4. Conclusion

The main purpose in the production of medicinal and aromatic plants is standard active ingredient, quality and high efficiency production. For this purpose, it is necessary to develop varieties suitable for different ecological regions and to determine modern cultivation techniques. Some agro-morphological characteristics and essential oil amounts of HI and HS species grown in İzmir-Ödemiş conditions were compared. As a result of the statistical analysis, it was stated that there were no significant results for both fresh herb and dry herb yields for both Helichrysum species. In addition, it was determined that HI showed superior results for İzmir, Ödemiş conditions in terms of plant height, herba essential oil amount and flower essential oil amount compared to HS. It was specified that HS showed superior results for İzmir, Ödemiş conditions in terms of fresh flower and dried flower yield compared to HI.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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