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An Alternative Approach to the Axiomatic Characterization of the Interval Shapley Value

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Abstract — This study presents a new approach to the axiomatic characterization of the interval Shapley value. This approach aims to improve our comprehension of the particular characteristics of the interval Shapley value in a provided context. Furthermore, the research contributes to the related literature by expanding and applying the fundamental axiomatic principles used to define the interval Shapley value. The proposed axioms encompass symmetry, gain-loss, and differential marginality, offering a distinctive framework for understanding and characterizing the interval Shapley value. Through these axioms, the paper examines and interprets the intrinsic properties of the value objectively, presenting a new perspective on the interval Shapley value. The characterization highlights the importance and distinctiveness of the interval Shapley value.

Keywords *Cooperative interval games, uncertainty, interval Shapley value, axiomatic characterization*

Mathematics Subject Classification (2020) 91A12, 90B70

1. Introduction

The Shapley value, first introduced by Shapley [1] in 1953, is a significant solution concept in cooperative game theory. Over the years, the Shapley value has developed substantially and become a captivating concept. Originally devised for cooperative games with transferable utility (TU games), which involve a finite set of players and real-numbered coalition values, the Shapley value has not only sustained its significance but has also undergone substantial development over the years. This concept is a foundational framework for equitable reward or cost allocation. It transcends its initial mathematical underpinnings and finds applications across diverse disciplines, such as operations research (OR), economics, sociology, and computer science [2]. It delves into the intricacies of complex problems related to reward and cost-sharing, offering a nuanced approach to evaluating contributions within coalitions.

In many real-world situations, the intricacies that arise from interactions between individuals and organizations require modeling. Game theory is valuable for comprehending and analyzing complex situations within a rigorous mathematical framework. Consider the dynamics between two companies operating in a competitive market. Each company aims to increase its market share and maximize profitability. However, each company's decisions are inevitably influenced by the strategies employed

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by the others. If a company chooses to reduce its prices, the other may follow suit, potentially affecting the profitability of both businesses. Game theory can be utilized to model and analyze competitive market dynamics, an instance of resource sharing within a group. When a group must distribute a limited resource, everyone naturally tries to protect their interests. For example, when a team decides on project roles, each member aims to maximize their skills and contributions. However, game theory can facilitate finding an optimal solution and achieving equilibrium among competing interests. These examples highlight the broad range of situations where game theory can be applied. Nevertheless, individuals frequently encounter interval uncertainty, providing a new perspective on cooperative interval games. Particularly, it addresses scenarios where individuals or companies consider collaboration and need to formalize a contract. In such cases, it is difficult to determine exact coalition payoffs, and only the minimum and maximum values can be clearly defined with certainty.

Each cooperative interval game represents an interval payoff, the interval Shapley value. This value holds significant influence as an interval solution concept in cooperative interval game theory, particularly in real-world applications and OR situations. It is characterized by the special subclass of cooperative interval games. This paper aims to present a novel axiomatic approach to characterizing the interval Shapley value, which does not rely on additivity or marginality but instead incorporates interval data. This paper explores the interval Shapley value and its axiomatic characterizations within cooperative interval games, drawing inspiration from [3]. Several characterizations of the interval Shapley value and grey Shapley value can be found in the literature, as documented in [4–7]. Numerous studies have been conducted on the Shapley value. For example, [1] uses the axioms of additivity (ADD), efficiency (EFF), symmetry (SYM), and the null player property (NULL). [8] characterizes the Shapley value by using EFF, SYM, and strong monotonicity property (SMON). As characterized by [3], the Shapley value uses a new axiom called coalitional strategic equivalence (CSE). Moreover, numerous characterizations of the Shapley value can be found in the literature [9–12].

The manuscript aims to present an innovative axiomatic characterization of the interval Shapley value. Departing from the conventional reliance on additivity and marginality, this characterization introduces a novel approach using (a specific concept) to establish a new perspective. The research deals with Shapley value and its axiomatic characterizations, inspired by the scientific contributions of [3]. The motivation behind characterization is to redefine a value using different axioms. By using a specific set of principles based on Gain-Loss, differential marginality, and symmetry axioms, we can redefine the Shapley value in a way that is different from existing characterizations. These selected principles enhance our approach's originality and provide a unique perspective for understanding cooperative game theory. The selection of these axioms strengthens the innovative nature of our work, deviating from traditional frameworks and presenting a novel conceptualization of the Shapley value. The conscious choice of Gain-Loss, differential marginality, and symmetry as guiding principles set our characterization apart from conventional approaches, contributing a new and distinctive viewpoint to the ongoing discourse surrounding the Shapley value. In essence, axiomatic characterization aims to provide an interval value by introducing a different point of view through specific axioms. These axioms serve as tools to analyze the characterization. As a result, we derive new interval properties and define this value with particular characteristics. In this study, we propose a new alternative characterization. The rest of the paper is organized as follows. Section 2 provides basic information and materials on cooperative and interval game theory. In Section 3, the interval Shapley value is characterized axiomatically with the axioms of gain-loss, differential marginality, and symmetry axioms with interval data. We conclude our paper by offering a comprehensive evaluation with potential perspectives for future studies.

2. Preliminaries

A coalition game in coalition form is represented by an ordered pair $\langle N, v\rangle$ where N is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function. The set 2^N denotes the set of all the subsets of N, each element of which is referred to as a coalition. A coalition game in coalition form is often employed as a TU game. G^N denotes the cooperative players in coalition form. The characteristic function of a game, denoted as $v \in G^N$, assigns the payoff $v(S)$ to each coalition $S \in 2^N$. Throughout this study, the notation 's' represents the cardinality of the coalition S instead of the notation $|S|$ for the number of elements in S.

Example 2.1. Let $N = \{1, 2, 3\}$ denote the set of players. Players 1 and 2 want to produce left gloves, while Player 3 wants to produce right gloves. The game's contribution is zero when producing only left or only right gloves, and it is 30 units when producing gloves together. This situation can be represented by the game $\langle N, v\rangle$. Here, the characteristic functions can be formulated as follows:

$$v(\emptyset) = 0$$

$$v(1) = v(2) = v(3) = v(12) = 0$$

$$v(13) = v(23) = v(N) = 30$$

Shapley value, one of the key concepts in cooperative game theory, will be discussed. Single-point solutions are represented through the transformation $f : G^N \to \mathbb{R}$ [13, 14].

Definition 2.2. The Shapley value of a cooperative game, denoted as $v \in G^N$, is articulated through the mapping $f : G^N \to \mathbb{R}$. Specifically, the Shapley value for player i is expressed as:

$$f_i(v) = \sum_{i \in S} \frac{\Delta_v(s)}{s}$$

In this context, the term $\Delta_v(s) = \sum_{T \subseteq S}(-1)^{s-t}v(T)$ embodies the concept of marginal contribution, a measure originally delineated by [15]. The Shapley value provides a fair allocation of the total payoff among players by considering all possible permutations of players and their contributions within coalitions. The set of all the games are $(2^{|N|} - 1)$- dimensional linear space where unanimity games form a basis. The unanimity game with the coalition of S, $u_S : 2^N \to \mathbb{R}$ is defined by

$$u_S(T) = \begin{cases} 1, \ S \subseteq T \\ 0, \ \text{otherwise} \end{cases}$$

for $S \in 2^N \setminus \{\emptyset\}$. For more details, see [16].

This section provides an overview of the historical background for cooperative interval games [17–19]. An interval game is defined by $< N, w >$. Here, $N = \{1, 2, \cdots, n\}$ is the set of players, and the characteristic function is $w : 2^N \to I(\mathbb{R})$ where $I(\mathbb{R})$ is the set of all the closed intervals in $\mathbb{R}$. The interval set $w(S)$ has form $[\underline{w}(S), \overline{w}(S)]$ for each coalition $S \in 2^N$ where $\underline{w}(S)$ is the lower value and $\overline{w}(S)$ is the upper value. We denote the set of all the interval games with the player set N by IG^N.

We use another subtraction operator different from Moore's subtraction operator for this study [20]. We define $I - J$, only when $|I| \geq |J|$, as $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$ where $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. It is noted that I is weakly superior to J, denoted by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. For $w_1, w_2 \in IG^N$, we state that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for all $S \in 2^N$, and we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$, for all $S \in 2^N$, such that $\lambda \in \mathbb{R}^+$. Additionally, for $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$, for all $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by

$(w_1 - w_2)(S) = w_1(S) - w_2(S)$. Interval solutions are interval payoff vectors in $I(\mathbb{R})$. We denote the set of all the interval payoff vectors by $I(\mathbb{R})^N$. We designate a game $< N, w >$ as size monotonic if $< N, |w| >$ is monotonic such that $|w|(S) \leq |w|(T)$, for all $S, T \in 2^N$ with $S \subseteq T$. For further use, the class of size monotonic interval games with the player set N is denoted by $SMIG^N$ (for more details, see [16]).

Definition 2.3. The interval Shapley value, denoted by $\Phi : SMIG^N \to I(\mathbb{R})^N$, is as follows:

$$\Phi(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w)$$

An interval game $\langle N, Iu_S \rangle$ is defined by

$$(Iu_S)(T) = u_S(T)I$$

in which $I \in I(\mathbb{R})$ and u_S is the unanimity game in there $S \in 2^N \setminus \{\emptyset\}$, for all $T \in 2^N \setminus \{\emptyset\}$. The interval Shapley value of the interval game Iu_S is defined by

$$\Phi_i(Iu_S) = \begin{cases} I/|S|, & i \in S \\ [0,0], & i \notin S \end{cases}$$

The set of the additive cone generated by the set

$$K = \left\{ I_S u_S \mid S \in 2^N \setminus \{\emptyset\}, I_S \in I(\mathbb{R}) \right\}$$

is denoted by KIG^N. Therefore, each element in the cone is a finite sum of elements in K. We note that KIG^N is a subset of $SMIG^N$, and in the specific subclass of cooperative interval games, the interval Shapley value is axiomatically characterized within KIG^N.

3. Axiomatization

This section presents a novel characterization of the Shapley value defined for cooperative interval games. Furthermore, we recommend using precise axioms and the main theorem in this characterization. The interval solution, denoted by a function $f : IG^N \to I(\mathbb{R})^N$ is characterized by assigning a $|N|$-dimensional real vector to each interval game within the set N. The vector represents the distribution of interval payoffs that can be achieved through collaborative efforts among individual players in the game. Initially, we articulate the established axioms governing solutions $f : IG^N \to I(\mathbb{R})^N$.

The efficiency axiom is a fundamental principle of game theory, emphasizing the imperative of efficient resource utilization and maximization of total welfare. This axiom states that game outcomes must be economically efficient. Consequently, resources must be distributed optimally, and the condition of any player should not be improved through a more effective allocation of the existing resources. The efficiency axiom is an important feature in game theory, frequently used to balance payoffs and strategies. The efficiency axiom is extended by defining it in the context of the interval concept.

Axiom 3.1 (I-EFF): For all $w \in IG^N$, it holds that

$$\sum_{i \in N} f_i(w) = w(N)$$

Player $i \in N$ is a null player in $v \in G^N$ if $v(S) = v(S \setminus \{i\})$, for all $S \subseteq N$. The concept of a "null player" refers to a player who has no impact on the strategies of other players and exerts no influence on the outcomes of the game. This condition is employed in analyses to denote situations where the participation or influence of a specific player can be considered negligible. The null player axiom is recognized as a significant tool in modeling equilibrium and outcome analyses within game theory. We expand the null axiom by defining it in the context of the interval concept.

Axiom 3.2 (I-NULL): If $i \in N$ is a null player in-game $w \in IG^N$, then $f_i(w) = [0, 0]$.

If $v(S \cup \{i\}) = v(S \cup \{j\})$, for all $S \subseteq N \backslash \{i, j\}$, then two players $i, j \in N$ are called symmetric in $v \in G^N$.

The symmetry axiom states that a player or situation is symmetrical with others. This axiom contributes to the understanding of equality and equilibrium in analyses. In game theory, symmetry typically describes situations where players share similar roles or strategies. The symmetry axiom is widely used to determine game equilibrium points and optimal game strategies. We extend the symmetry axiom by defining within the framework of the interval concept.

Axiom 3.3 (I-SYM): If $i,\, j \in N$ are symmetric in $w \in IG^N$, then $f_i(w) = f_j\,(w)$.

The additivity axiom asserts that an individual player's payoffs contribute to the total payoff of the game. This axiom emphasizes the idea that the combined contributions of individuals are reflected in the aggregate of game outcomes. As players aim to maximize their individual payoffs by making strategic decisions, the additivity axiom is used to determine the game's overall success. In game theory, the additivity axiom is a foundational principle essential for analyzing games' aggregate outcomes and reaching optimal strategies. We define the additivity axiom within the framework of the interval concept, extending its scope.

Axiom 3.4 (I-ADD): For all $w, w' \in IG^N$,

$$f\,(w + w') = f(w) + f(w')$$

where $(w + w') \in IG^N$ is provided by

$$(w + w')(S) = w(S) + w'(S)$$

for all $S \subseteq N$.

In game theory, the gain-loss axiom is pivotal in explaining how players assess their situations regarding acquired gains and incurred losses. Players begin their strategic journeys from a designated starting point, analyzing this point against the gains and losses accumulated as the game progresses. However, the evaluation process depends on the magnitude of gains and losses and their ability to improve the current circumstances. In essence, players carefully observe whether there is an equal amount of gain or loss, highlighting the importance of this dynamic. The gain-loss axiom is extended by defining within the framework of the interval concept.

Axiom 3.5 (I-GL): For all $w, w' \in IG^N$ and $i \in N$ such that

$$w(N) = w'(N) \quad \text{and} \quad f_i(w) \succnsim f_i(w')$$

there is some $j \in N$ such that $f_j(w) \precnsim f_j(w')$.

The marginality axiom is a fundamental principle in game theory that directs attention to the impact of a player's marginal contribution on the total payoff. This axiom is foundational in scrutinizing players' strategic choices by emphasizing the decisive effect of marginal changes in evaluating a player's decisions and contributions. Pursuing increased returns through marginal contributions is a prominent tenet of this axiom. We define the marginality axiom within the framework of the interval concept.

Axiom 3.6 (I-M): For all $w, w' \in IG^N$ and $i \in N$ such that

$$w(S \cup \{i\}) - w(S) = w'(S \cup \{i\}) - w'(S)$$

for all $S \subseteq N \backslash \{i\}$, $f_i(w) = f_i(w')$.

According to this axiom, the variation in the game's payoffs is determined by the marginal change in a player's contribution. Following this principle, if a new game is introduced to symmetric players, any equal change in their marginal contributions should have an equal impact on their payoffs. The principle states that any equal adjustment in players' marginal contributions should be fairly reflected in the corresponding payoffs. The definition of the differential marginality axiom is extended by providing it within the framework of the interval concept.

Axiom 3.7 (**I-DM**): For all $w, w' \in IG^N$ and i, $j \in N$ such that

$$w(S \cup \{i\}) - w(S \cup \{j\}) = w'(S \cup \{i\}) - w'(S \cup \{j\})$$

for all $S \subseteq N \backslash \{i, j\}$, $f_i(w) - f_j(w) = f_i(w') - f_j(w')$.

Axiomatic characterization involves redefining a concept by specifying its properties through axioms. Therefore, we have presented the axioms required to define the interval Shapley value. Three of these axioms serve as the foundational elements for the main theorem. We axiomatically characterize the interval Shapley value by utilizing these three properties. The relevant theorem will be presented, and its proof will be provided.

Theorem 3.1. The value that satisfies the Axioms I-GL, I-DM, and I-SYM is referred to as the interval Shapley value on KIG^N.

Proof. Interval Shapley value obeys I-GL and I-SYM axioms by the definition of this value. Moreover, [21] demonstrates that the Shapley value satisfies DM. Therefore, the interval Shapley value satisfies I-DM. We aim to establish the converse. Interval Shapley value obeys I-GL and I-SYM. Let w and w' belong to KIG^N. Consider the symmetric game $w' \in KIG^N$ where w' is uniformly zero across all coalitions, i.e., $w'_i(S) = [0, 0]$, for all $i \in S$. According to I-SYM, $f_i(w') = f_j(w')$, for all $i \neq j$, and

$$\sum_{i=1}^{n} f_i(w') = [0, 0]$$

follows from I-GL. Consequently, $f_i(w') = [0, 0]$, for all $i \in N$. By I-DM, it can be deduced that for any interval game $w \in KIG^N$ and any player $i \in N$,

$$w_i(S) = [0, 0], \text{ for all } S \subseteq N, \text{ implies } f_i(w) = [0, 0] \tag{3.1}$$

That is, null players get nothing. In other words, players with null contributions receive no payoff. We use Shapley's insight that any game v can be explained as the sum of primitive games, allowing for a detailed analysis of its fundamental components and strategic foundations. This concept can be redefined by extending it into the domain of interval games, as explained below:

$$w = \sum_{S \subseteq N : S \neq \emptyset} \lambda_S u_S \tag{3.2}$$

where

$$\lambda_S u_S(T) = \begin{cases} \lambda_S, & \text{if } S \subseteq T \\ [0, 0], & \text{otherwise} \end{cases}$$

The interval Shapley value finds its formulation in the following manner:

$$f_i(w) = \sum_{S \subseteq N : S \neq \emptyset} f_i(\lambda_S u_S) = \sum_{S : i \in S} \frac{\lambda_S}{|S|}$$

Define the index I of w as the minimal quantity of non-zero terms requisite in an expression delineating

w in the form specified by (3.2). The theorem is established through induction on the set I. If $I = 0$, then every is a null and hence $f_i(w) = [0,0]$ by (3.1). For all $i, j \in S$, I-SYM implies that $f_i(w) = f_j(w)$; supplemented by the prerequisite that $\sum_{i=1}^{n} f_i(w) = w(N)$. Consequently, we deduce that $f_i(w) = \frac{\lambda_S}{|S|}$, for all $i \in S$. Thus, $f(w)$ constitutes the interval Shapley value for the case of $I \in \{0, 1\}$. We assume that $f(w)$ represents the interval Shapley value for any index up to I. Consider w with an index of $I + 1$, expressed as

$$w = \sum_{l=1}^{I+1} \lambda_{S_l} u_{S_l}$$

where $\lambda_{S_l} \neq [0,0]$, for all I. Let $S = \bigcap_{l=1}^{I+1} S_l$ and suppose that $i \notin S$. Define the game

$$w' = \sum_{l:i \in S_l} \lambda_{S_l} u_{S_l}$$

The index of w is at most I and $w'_i(T) = w_i(T)$, for all S. Consequently, by induction and I-DM, it follows that

$$f_i(w) = f_i(w') = \sum_{l=i \in S_l} \frac{\lambda_S}{|S|}$$

which represents the interval Shapley value of i. We still need to demonstrate that $f_i(w)$ is the interval Shapley value when $i \in S = \bigcap_{l=1}^{I+1} S_l$. According to I-SYM, $f_i(w)$ is a constant c, for all members of S; similarly, the interval Shapley value is some constant c', for all members of S. By I-GL, it follows that $c = c'$. $\square$

The following example illustrates how to construct a model and compute the interval Shapley value in a real-life operational research scenario, as presented by [14]. Consider an inventory situation characterized by interval data and formulate an associated interval game. Player 3 owns a storage facility with a capacity for a single container, while Players 1 and 2 each possess one container. If Player 1 is permitted to store their container, they will receive a benefit between 20 and 40. If Player 2 is allowed to store their container, the corresponding benefit falls within the range of [60, 80].

Example 3.2. The situation described above corresponds to the interval game $\langle N, w \rangle$ with $N = \{1, 2, 3\}$ and $w(S) = [0,0]$ if $3 \notin S, w(\emptyset) = [0,0], w(1,3) = [20,40], w(2,3) = [60,80]$, and $w(N) = [80,100]$, i.e., a big boss interval game with Player 3 as a big boss. Then, the interval marginal vectors are provided in the following table where $\sigma : N \to N$ is identified with $(\sigma(1), \sigma(2), \sigma(3))$. Firstly, for $\sigma_1 = (1, 3, 2)$, we calculate the interval marginal vectors. Then,

$$m_1^{\sigma_1}(w) = w(1) = [0,0]$$

$$m_2^{\sigma_1}(w) = w(N) - w(13) = [80,100] - [20,40] = [60,60]$$

and

$$m_3^{\sigma_1}(w) = w(13) - w(1) = [20,40] - [0,0] = [20,40]$$

The others can be calculated similarly, which are shown in Table 1.

Table 1. Interval marginal vectors

σ	$m_1^{\sigma}(w)$	$m_2^{\sigma}(w)$	$m_3^{\sigma}(w)$
$\sigma_1 = (1,2,3)$	$m_1^{\sigma_1}(w) = [0,0]$	$m_2^{\sigma_1}(w) = [0,0]$	$m_3^{\sigma_1}(w) = [80,100]$
$\sigma_2 = (1,3,2)$	$m_1^{\sigma_2}(w) = [0,0]$	$m_2^{\sigma_2}(w) = [60,60]$	$m_3^{\sigma_2}(w) = [20,40]$
$\sigma_3 = (2,1,3)$	$m_1^{\sigma_3}(w) = [0,0]$	$m_2^{\sigma_3}(w) = [0,0]$	$m_3^{\sigma_3}(w) = [80,100]$
$\sigma_4 = (2,3,1)$	$m_1^{\sigma_4}(w) = [20,20]$	$m_2^{\sigma_4}(w) = [0,0]$	$m_3^{\sigma_4}(w) = [60,80]$
$\sigma_5 = (3,1,2)$	$m_1^{\sigma_5}(w) = [20,40]$	$m_2^{\sigma_5}(w) = [60,60]$	$m_3^{\sigma_5}(w) = [0,0]$
$\sigma_6 = (3,2,1)$	$m_1^{\sigma_6}(w) = [20,20]$	$m_2^{\sigma_6}(w) = [60,80]$	$m_3^{\sigma_6}(w) = [0,0]$

Table 1 illustrates the interval marginal vectors of the cooperative interval game in Example 3.2. The average of the six interval marginal vectors is the interval Shapley value of this game, which can be observed as:

$$\Phi(w) = \left(\left[10, \frac{40}{3}\right], \left[30, \frac{100}{3}\right], \left[40, \frac{160}{3}\right]\right)$$

4. Conclusion

This study aims to provide an axiomatic characterization of the Shapley value using the axioms above. It is argued that these axioms uniquely define the Shapley value. The paper surveys cooperative game theory in the literature, focusing on two specific subtraction operators: Moore's subtraction operator and the special subtraction operator. In the last decade, several axiomatic characterizations of the Shapley value have been using the special subtraction operator. Shortly, we plan to introduce new axiomatic characterizations for the Shapley value using Moore's subtraction operator. The classical game's dividends, initially introduced by Hars [15], play a pivotal role in characterizing the classical Shapley value. The utilization of dividends enables the characterization of the interval Shapley value in cooperative interval games with compact real-valued coalitional interval values.

In conclusion, further exploration of this idea shows promise as a potential avenue for future research. This approach offers a valuable perspective for comprehending and assessing the Shapley value. Regarding future research, further exploration of this concept presents an exciting and fruitful area for characterizing the Shapley value. Furthermore, a more comprehensive examination of Grey Game Theory, which considers uncertainty and incomplete information, enhances the accuracy of modeling cooperative games. Within this framework, characterization methods and the Shapley value enable a better understanding of collaboration dynamics among players, especially in situations involving uncertainty. This characterization has the potential to provide new insights for further characterizations and is amenable to extension within the domain of grey games.

Furthermore, it acts as a guiding framework to facilitate characterizations of other relevant values, such as the Banzhaf and T-value. In this respect, it offers a nuanced understanding of Shapley value and guides researchers seeking to characterize different values. As a result, this characterization emerges as a comprehensive framework that directs research toward the Shapley value and guides research into related values within the cooperative game theory. Therefore, future investigations could deepen understanding and knowledge in this field by conducting a comprehensive analysis incorporating both cooperative games and Grey Game Theory.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Fractional Curvatures of Equiaffine Curves in Three-Dimensional Affine Space

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Abstract — This paper presents a method for computing the curvatures of equiaffine curves in three-dimensional affine space by utilizing local fractional derivatives. First, the concepts of α-equiaffine arc length and α-equiaffine curvatures are introduced by considering a general local involving conformable derivative, V-derivative, etc. In fractional calculus, equiaffine Frenet formulas and curvatures are reestablished. Then, it presents the relationships between the equiaffine curvatures and α-equiaffine curvatures. Furthermore, graphical representations of equiaffine and α-equiaffine curvatures illustrate their behavior under various conditions.

Keywords *Fractional derivative, equiaffine curvatures, affine space*

Mathematics Subject Classification (2020) 26A33, 53A04

1. Introduction

Differential geometry, investigating the properties of curves, surfaces, and manifolds, provides a profound understanding of mathematical objects' intrinsic characteristics. Recently, the intersection of differential geometry with fractional derivatives has opened up new avenues of research and exploration. Fractional derivatives, which extend the classical notion of derivatives to non-differentiable functions, find application in various fields. This paper deals with the fascinating realm where local fractional derivatives and differential geometry meet, particularly in analyzing equiaffine curves.

The concept of fractional calculus, encompassing fractional derivatives, can be traced back to the endeavors of mathematicians, such as L'Hospital and Leibniz, in the 18th century. During this period, they explored the feasibility of extending the notion of derivatives to non-integer orders. However, their pioneering ideas did not gain widespread acceptance at that time. It was not until the 19th century that significant advancements were made in this field. Notable contributions were made by Cauchy, Weierstrass, and Liouville, who played key roles in shaping the theory of fractional derivatives. Cauchy, for instance, introduced the fundamental concept of fractional derivatives and integrals, laying the groundwork for their mathematical treatment. The efforts of Liouville and Riemann in the mid-19th century were pivotal in advancing the theory of fractional calculus. The Riemann-Liouville fractional derivatives, developed during this era, are extensively employed today. Fractional calculus progressed further throughout the 20th century, with significant contributions from renowned mathematicians; for more details, see [1].

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Beyond its applications in pure mathematics, fractional calculus has gained substantial significance in recent years. In [2], plane curves in equiaffine geometry are examined by considering fractional derivatives. The authors [3] have studied the geometry of curves possessing fractional-order tangent vectors and Frenet-Serret formulas. In [4], the authors obtained some characterizations of curves of fractional order in three-dimensional Euclidean space. The authors [5] provide the Frenet frame compatible with the conformable derivative. In [6], the authors obtained special fractional curve pairs and some characterizations. However, the fractional differential geometry of curves and surfaces is explicitly provided in [7]. In calculus, the chain rule, which holds great importance, is discussed regarding fractional derivatives in [8]. Moreover, fractional derivatives find frequent use in numerical analysis, as evidenced by studies [9, 10]. The applications of fractional calculus extend to various scientific and engineering domains. In the 20th century, fractional derivatives were applied across diverse disciplines, including physics, engineering, signal processing, and control theory, particularly in modeling systems with memory and intricate dynamics. In recent times, fractional calculus has experienced a resurgence of interest, with applications extending to finance, materials science, and bioengineering.

Recent advancements in computational techniques have simplified the handling of fractional derivatives in practical applications. In the present day, fractional calculus is a well-established branch of mathematics, with its applications continuously expanding into diverse scientific and engineering domains. These applications offer valuable tools for modeling and analyzing intricate systems. Notably, these applications span across fields such as medicine [11], bioengineering [12], viscoelasticity [13], and dynamical systems [14, 15]. In contrast to the straightforward expressions of integer-order derivatives and integrals, various more intricate fractional derivatives and integrals exist. Riemann-Liouville, Caputo, and Riesz fractional derivatives are prominent examples of non-local fractional derivatives [16]. On the other hand, conformable [17], truncated M- and V-fractional derivatives [18, 19] represent distinct types of local derivatives.

When dealing with non-local fractional derivatives, the conventional characteristics observed in integer order derivatives, such as the standard Leibniz and chain rules, are not satisfied. The absence of these features presents a significant challenge when formulating a theory of differential geometry, given their crucial importance. To elaborate, opting for a non-local fractional derivative rather than an integer-order derivative constrains the application of techniques derived from Riemannian geometry due to the lack of the Leibniz and chain rules. Many foundational concepts within Riemannian geometry heavily depend on these rules. Consequently, utilizing local fractional derivatives that exhibit the mentioned crucial properties facilitates calculations in differential geometry. For this reason, the study is crafted to focus on incorporating local fractional derivatives.

Our motivation arises from defining a general local fractional derivative operator compatible with all fractional derivatives, as outlined in (2.3). This study investigates equiaffine curves by examining this generalized local fractional derivative. Additionally, we introduce the concepts of equiaffine arc length and curvature. The rationale for considering equiaffine invariants is as follows: If one employs local fractional derivatives to analyze Frenet invariants of a curve, the Frenet frame remains unaffected. However, the utilization of local fractional derivatives impacts the equiaffine Frenet frame. In addition, the α-equiaffine Frenet frame is a new and different frame from the classical Frenet frame. This Frenet frame is obtained by considering the local fractional derivative and is a new study area.

The construction of this paper proceeded as follows: Section 2 provides fundamental concepts related to fractional derivatives. Section 3 discusses equiaffine invariants of equiaffine curves in the 3-dimensional affine space $\mathbb{R}^3$ by considering a general local fractional derivative. Section 4 presents the main results and introduces the α-equiaffine Frenet frame and α-equiaffine curvatures of a curve with equiaffine arc length. Moreover, the section establishes a relationship between the curve's equiaffine curvatures and α-equiaffine curvatures. Besides, it plots graphs of α-equiaffine curvature functions for some values of equiaffine curvatures. Finally, it discusses the need for further research.

2. Preliminaries

This section presents the notion of the V-fractional derivative, initially introduced in [19], due to its wider applicability when contrasted with the local fractional derivatives detailed in Section 1. For a visual depiction, please consult the informative diagram on page 23 of the referenced paper. Consequently, any statement valid for the $\mathcal{V}$-fractional derivative inherently generalizes to the other types of derivatives. Denoting by $\Gamma(\alpha)$ the Euler gamma function defined for the parameter α as described in [20]

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

Let $\mathbb{R}^+$ be the set of all positive real numbers. Suppose γ, β, ρ, and δ are complex numbers with positive real components, and p and q are elements of $\mathbb{R}^+$. In this context, we present a six-parameter Mittag–Leffler function denoted as

$${}_i\mathbb{E}^{\rho,\delta,q}_{\gamma,\beta,p}(z) = \sum_{k=0}^{i} \frac{(\rho)_{qk}}{(\delta)_{pk}} \frac{z^k}{\Gamma(\gamma k+\beta)}, z \in \mathbb{C}, Re(z) > 0$$

as detailed in [19]. The function incorporates $\Gamma(\rho)$, representing the gamma function involving ρ, and $(\rho)_{qk}$, which is an extension of the Pochhammer symbol defined by $(\rho)_{qk} = \frac{\Gamma(\rho+qk)}{\Gamma(\rho)}$.

Consider a real number α such that $0 < \alpha \leq 1$, $t \to f(t)$, and $t \in I \subset \mathbb{R}^+$. In this scenario, the truncated $\mathcal{V}$-fractional derivative of $f(t)$ is expressed as

$$ {}^{\rho}_{i}\mathcal{V}^{\rho,\delta,q}_{\gamma,\beta,p} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t_i H^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon t^{-\alpha}) - f(t)\right)}{\varepsilon} \tag{2.1}$$

where ${}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon t^{-\alpha})(z) = \Gamma(\beta){}_i\mathbb{E}^{\rho,\delta,q}_{\gamma,\beta,p}(z)$.

A function is α-differentiable if the limit in (2.1) exists. Furthermore, the truncated $\mathcal{V}$-fractional derivative operates as a linear operator, following the Leibniz and chain rules, by the principles expounded in [19]

$$ {}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(t) = \frac{t^{1-\alpha}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} \frac{df(t)}{dt} \tag{2.2}$$

By varying the specific the parameters γ, β, δ, p, and q various other local derivatives such as truncated M-fractional derivatives, alternative and conformable, etc., can be obtained. As indicated in (2.2), the $\mathcal{V}$-fractional derivative maintains a linear relationship with the standard integer order derivative. Consequently, a comprehensive definition of a general local fractional derivative operator is established as follows: Consider a function $C(\alpha,t)$ of class C^4 defined as

$$t \to C(\alpha,t) \in \mathbb{R}^+, \ t \in I \subset \mathbb{R}^+$$

where $C(\alpha,t)$ equals 1 specifically when $\alpha = 1$. Since the coefficient of the term $\frac{df(t)}{dt}$ in (2.2) is function of t, next we generally introduce

$$\frac{d^\alpha}{dt^\alpha} = C(\alpha,t)\frac{d}{dt} \tag{2.3}$$

where as $\frac{d}{dt}$ is the standard derivative operator. For example, (2.3) suggests conformable derivative if $C(\alpha,t) = t^{1-\alpha}$ and a truncated M-fractional derivative if $C(\alpha,t) = \frac{t^{1-\alpha}}{\Gamma(1+\gamma)}$, $(\gamma \in \mathbb{R}^+)$ and etc.

3. Equiaffine Invariants of Space Curves

This section provides brief information from [21–24]. Let $\mathbb{R}^3$ be 3-dimensional affine space and $Mat(3,\mathbb{R})$ be the set of all the square matrices of order 3. We write

$$SL(\mathbb{R}^3) = \{A \in Mat(3,\mathbb{R}) : det(A) = 1\}$$

Then, by an *equiaffine invariant*, we mean an unchanged feature under the actions of $SL(\mathbb{R}^3)$ and the translations of $\mathbb{R}^3$. For example, the volume is an equiaffine invariant. Let $[u_1\ u_2\ u_3]$ is the determinant of vectors $u_1, u_2, u_3 \in \mathbb{R}^3$. Then, the value of $[u_1\ u_2\ u_3]$ is an equiaffine invariant because it measures the volume of parallelopipedon determined by u_1, u_2, and u_3.

If $n = 3$, then the following expressions [24] are written as follows:

Let $t \to y(t)$, $t \in I \subset \mathbb{R}$, is smooth parametrized curve in $\mathbb{R}^3$. $y(t)$ is the *non-degenerate* if, for every $t \in I$,

$$\left[\frac{dy}{dt}(t)\ \ \frac{d^2y}{dt^2}(t)\ \ \frac{d^3y}{dt^3}(t)\right] \neq 0$$

For the sake of simplicity, when we refer to a curve in this paper, we mean a non-degenerate smooth parameterized curve. Subsequently, the *equiaffine arc length function* is defined as

$$\mu(t) = \int^t \left[\frac{dy}{du}(u)\ \ \frac{d^2y}{du^2}(u)\ \ \frac{d^3y}{du^3}(u)\right]^{\frac{1}{6}} du$$

We refer to the curve as being parameterized by *equiaffine arc length* if, for every $\mu \in J \subset \mathbb{R}$,

$$\left[\frac{dy}{d\mu}(\mu)\ \ \frac{d^2y}{d\mu^2}(\mu)\ \ \frac{d^3y}{d\mu^3}(\mu)\right] = 1 \tag{3.1}$$

The set $\left\{\frac{dy}{d\mu}(\mu), \frac{d^2y}{d\mu^2}(\mu), \frac{d^3y}{d\mu^3}(\mu)\right\}$ is referred to as the *equiaffine Frenet frame* of $y(\mu)$. When we take the derivative (3.1) according to the parameter μ, it is apparent that

$$\left[\frac{dy}{d\mu}(\mu)\ \ \frac{d^2y}{d\mu^2}(\mu)\ \ \frac{d^4y}{d\mu^4}(\mu)\right] = 0$$

where the set are linearly dependent for every $\mu \in J$, given by

$$\left\{\frac{dy}{d\mu}(\mu), \frac{d^2y}{d\mu^2}(\mu), \frac{d^4y}{d\mu^4}(\mu)\right\}$$

Therefore, this implies the existence of smooth functions κ and τ on J such that

$$\frac{d^4y}{d\mu^4}(\mu) + \kappa(\mu)\frac{dy}{d\mu}(\mu) + \tau(\mu)\frac{d^2y}{d\mu^2}(\mu) = 0$$

where

$$\kappa(\mu) = -\left[\frac{d^2y}{d\mu^2}(\mu)\ \ \frac{d^3y}{d\mu^3}(\mu)\ \ \frac{d^4y}{d\mu^4}(\mu)\right] \tag{3.2}$$

and

$$\tau(\mu) = \left[\frac{dy}{d\mu}(\mu)\ \ \frac{d^3y}{d\mu^3}(\mu)\ \ \frac{d^4y}{d\mu^4}(\mu)\right] \tag{3.3}$$

The function $\kappa(\mu)$ and $\tau(\mu)$ are called *equiaffine curvature* and *equiaffine torsion* of the curve $y(\mu)$, respectively. The equiaffine curvature and equiaffine torsion are the equiaffine invariants in $\mathbb{R}^3$.

Let y be a non-degenerate smooth curve in $\mathbb{R}^3$ parameterized by equiaffine arc length μ. As a result, the equiaffine equations of Frenet type are presented in matrix form as

$$\begin{bmatrix} \frac{d^2y}{d\mu^2}(\mu) \\ \frac{d^3y}{d\mu^3}(\mu) \\ \frac{d^4y}{d\mu^4}(\mu) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa(\mu) & -\tau(\mu) & 0 \end{bmatrix} \begin{bmatrix} \frac{dy}{d\mu}(\mu) \\ \frac{d^2y}{d\mu^2}(\mu) \\ \frac{d^3y}{d\mu^3}(\mu) \end{bmatrix}$$

4. Fractional Equiaffine Curvatures

This section establish a relationship between the equiaffine curvatures and a curve's α-equiaffine curvatures. It then presents graphical representations of the α-equiaffine curvature functions corresponding to certain equiaffine curvature values.

Proposition 4.1. Let y be a non-degenerate smooth curve in $\mathbb{R}^3$ parameterized by the equiaffine arc length μ. Moreover, let

$$\mu(s) = \int^s (C(\alpha,t))^{-\frac{1}{2}}\, dt \tag{4.1}$$

Then, the parameter s is the equiaffine arc length parameter concerning (2.3). Here, $\mu(s)$ is called α-equiaffine arc length.

Proof. Using (2.3), we have

$$\frac{d^\alpha y}{ds^\alpha} = C(\alpha,s)\frac{dy}{d\mu}\frac{d\mu}{ds}$$

or

$$\frac{d^\alpha y}{ds^\alpha} = (C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} \tag{4.2}$$

If we take the standard derivative of (4.2) with s, then

$$\frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + (C(\alpha,s))^{\frac{1}{2}}\frac{d^2y}{d\mu^2}\frac{d\mu}{ds}$$

or

$$\frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + \frac{d^2y}{d\mu^2} \tag{4.3}$$

If we take the standard derivative of (4.3) with s, then

$$\frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d^2}{ds^2}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}\frac{d^2y}{d\mu^2}\frac{d\mu}{ds} + \frac{d^3y}{d\mu^3}\frac{d\mu}{ds}$$

or

$$\frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d^2}{ds^2}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}(C(\alpha,s))^{-\frac{1}{2}}\frac{d^2y}{d\mu^2} + (C(\alpha,s))^{-\frac{1}{2}}\frac{d^3y}{d\mu^3} \tag{4.4}$$

Thus,

$$\left[\frac{d^\alpha y}{ds^\alpha}\ \ \frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right)\ \ \frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right)\right] = (C(\alpha,s))^{\frac{1}{2}}(C(\alpha,s))^{-\frac{1}{2}}\left[\frac{dy}{d\mu}\ \ \frac{d^2y}{d\mu^2}\ \ \frac{d^3y}{d\mu^3}\right] = 1$$

□

Let y(s), $s \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. Then, the set $\left\{e_1^{\{\alpha\}}, e_2^{\{\alpha\}}, e_3^{\{\alpha\}}\right\}$ is called α-equiaffine Frenet frame of $y(s)$, where $e_1^{\{\alpha\}} = \frac{d^\alpha y}{ds^\alpha}$, $e_2^{\{\alpha\}} = \frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right)$, and $e_3^{\{\alpha\}} = \frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right)$. Note that when $\alpha = 1$ the set $\left\{e_1^{\{\alpha\}}, e_2^{\{\alpha\}}, e_3^{\{\alpha\}}\right\}$ is the standard equiaffine Frenet frame of $y(s)$. Therefore,

$$\left[e_1^{\{\alpha\}}\ e_2^{\{\alpha\}}\ e_3^{\{\alpha\}}\right] = 1$$

and if we take the standard derivative of the last equation according to s, then

$$\left[e_1^{\{\alpha\}} \;\; e_2^{\{\alpha\}} \;\; \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right] = 0$$

where it can be seen that the set $\left\{e_1^{\{\alpha\}}, e_2^{\{\alpha\}}, \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right\}$ is linearly dependent for every $s \in (c,d)$. Then, there are some smooth functions on (c,d) denoted by $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ such that

$$\kappa^{\{\alpha\}}(s)\, e_1^{\{\alpha\}} + \tau^{\{\alpha\}}(s)\, e_2^{\{\alpha\}} + \frac{d}{ds}\left(e_3^{\{\alpha\}}\right) = 0$$

Hence, the functions $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are defined *α-equiaffine curvatures* of $y(s)$, where

$$\kappa^{\{\alpha\}}(s) = -\left[e_2^{\{\alpha\}} \;\; e_3^{\{\alpha\}} \;\; \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right] \tag{4.5}$$

and

$$\tau^{\{\alpha\}}(s) = \left[e_1^{\{\alpha\}} \;\; e_3^{\{\alpha\}} \;\; \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right] \tag{4.6}$$

According to the above discussion, we have the counterpart of equiaffine Frenet formulas as follows:

$$\begin{bmatrix} \frac{d}{ds}\left(e_1^{\{\alpha\}}\right) \\ \frac{d}{ds}\left(e_2^{\{\alpha\}}\right) \\ \frac{d}{ds}\left(e_3^{\{\alpha\}}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa^{\{\alpha\}} & -\tau^{\{\alpha\}} & 0 \end{bmatrix} \begin{bmatrix} e_1^{\{\alpha\}} \\ e_2^{\{\alpha\}} \\ e_3^{\{\alpha\}} \end{bmatrix}$$

We may present the relationship between the equiaffine frames of $y(s)$. For this, let $\mathcal{B}^{\{\alpha\}} = \left[e_1^{\{\alpha\}} \; e_2^{\{\alpha\}} \; e_3^{\{\alpha\}}\right]$ and $\mathcal{B} = [e_1 \; e_2 \; e_3]$, for $\mathcal{B}^{\{\alpha\}}, \mathcal{B} \in SL(3,\mathbb{R})$. Thus, using (4.2), (4.3), and (4.4),

$$\mathcal{B}^{\{\alpha\}} = \begin{bmatrix} (C(\alpha,s))^{\frac{1}{2}} & 0 & 0 \\ \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}} & 1 & 0 \\ \frac{d^2}{ds^2}(C(\alpha,s))^{\frac{1}{2}} & (C(\alpha,s))^{-\frac{1}{2}} \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}} & (C(\alpha,s))^{-\frac{1}{2}} \end{bmatrix} \mathcal{B}$$

Theorem 4.2. Let $y(s)$ be a non-degenerate smooth curve in $\mathbb{R}^3$ parameterized by equiaffine arc length parameter concerning (2.3). Then,

$$\begin{aligned} \kappa^{\{\alpha\}}(s) = &-\frac{3}{4}(C(\alpha,s))^{-3}\left(\frac{d}{ds}(C(\alpha,s))\right)^3 \\ &+\frac{5}{4}(C(\alpha,s))^{-2}\frac{d}{ds}(C(\alpha,s))\frac{d^2}{ds^2}(C(\alpha,s)) - \frac{1}{2}(C(\alpha,s))^{-1}\frac{d^3}{ds^3}(C(\alpha,s)) \\ &+(C(\alpha,s))^{-\frac{3}{2}}\kappa(\mu(s)) - \frac{1}{2}(C(\alpha,s))^{-2}\frac{d}{ds}(C(\alpha,s))\,\tau(\mu(s)) \end{aligned} \tag{4.7}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{3}{4}(C(\alpha,s))^{-2}\left(\frac{d}{ds}(C(\alpha,s))\right)^2 - (C(\alpha,s))^{-1}\frac{d^2}{ds^2}(C(\alpha,s)) + (C(\alpha,s))^{-1})\tau(\mu(s)) \tag{4.8}$$

Here, κ, τ and $\kappa^{\{\alpha\}}$, $\tau^{\{\alpha\}}$ denote the equiaffine and α-equiaffine curvatures, respectively.

Proof. Differentiating (4.4) with respect to s, we have

$$\frac{d}{ds}e_3^{\{\alpha\}} = \phi(s)e_1^{\{\alpha\}} + \zeta(s)e_2^{\{\alpha\}} \tag{4.9}$$

where

$$\phi(s) = \frac{d^3}{ds^3}(C(\alpha,s))^{\frac{1}{2}} - (C(\alpha,s))^{-1}\kappa(\mu(s))$$

and

$$\zeta(s) = 2\frac{d^2}{ds^2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\left(C(\alpha,s)\right)^{-\frac{1}{2}} + \frac{d}{ds}\left(C(\alpha,s)\right)^{\frac{1}{2}}\frac{d}{ds}\left(C(\alpha,s)\right)^{-\frac{1}{2}} - \left(C(\alpha,s)\right)^{-1}\tau(\mu(s))$$

If we consider (4.3), (4.4), and (4.9) in (4.5), after some calculations, we get (4.7). Analogously, if we consider (4.2), (4.4), and (4.9) in (4.6), we get (4.8). □

Remark 4.3. In particular if $C(\alpha,s) = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}$ as in [24], then Theorem 4.2 reduces to [24, Theorem 4.1].

Remark 4.4. If the conformable derivative is taken into consideration as our derivative, since $C(\alpha,s) = s^{1-\alpha}$, Theorem 4.2 can be given as follows:

$$\kappa^{\{\alpha\}}(s) = \frac{(\alpha+3)(\alpha-1)}{4}s^{-3} + s^{\frac{3\alpha-3}{2}}\kappa(\mu(s)) - \frac{(1-\alpha)}{2}s^{\alpha-2}\tau(\mu(s)) \tag{4.10}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{(\alpha+3)(1-\alpha)}{4}s^{-2} + s^{\alpha-1}\tau(\mu(s)) \tag{4.11}$$

Suppose that $C(\alpha,s) = s^{1-\alpha}$, namely our derivative is conformable type. Let $\kappa(\mu(s)) = -2, \tau(\mu(s)) = -3$ (respectively, $\kappa(\mu(s)) = 2, \tau(\mu(s)) = 3$ and $\kappa(\mu(s)) = 0, \tau(\mu(s)) = 0$). Then, the graphs of (4.10) are as in Figure 1. Similarly, the graphs of (4.11) are as in Figure 2.

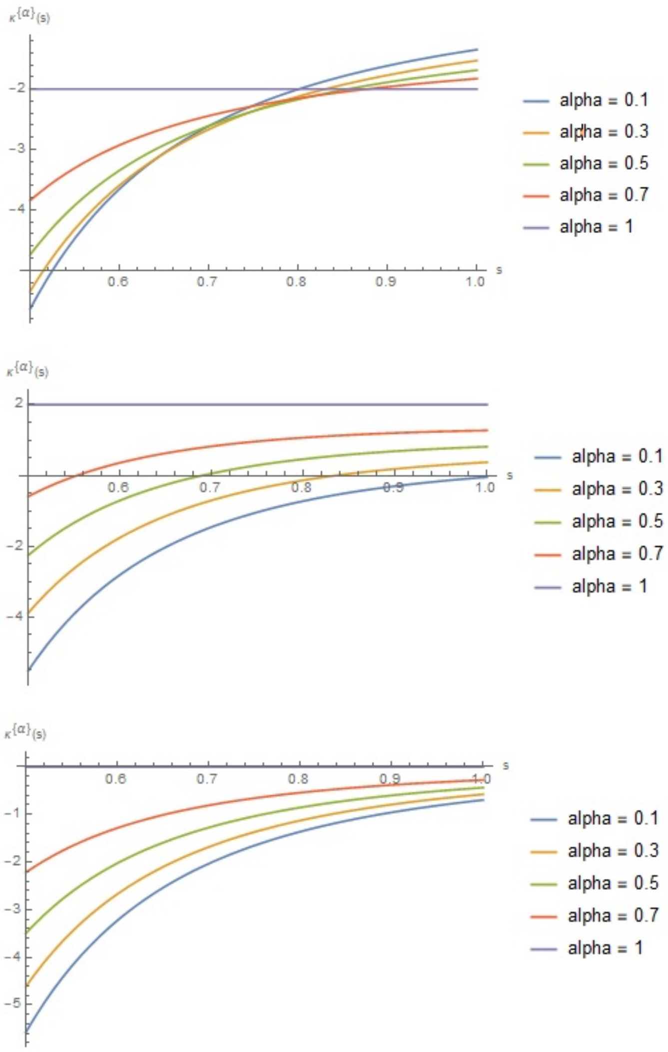

Figure 1. Graphs of $\kappa^{\{\alpha\}}(s)$ for $\kappa(\mu(s)) = -2, \tau(\mu(s)) = -3$ (respectively, $\kappa(\mu(s)) = 2, \tau(\mu(s)) = 3$ and $\kappa(\mu(s)) = 0, \tau(\mu(s)) = 0$) and $s \in [0.5, 1]$

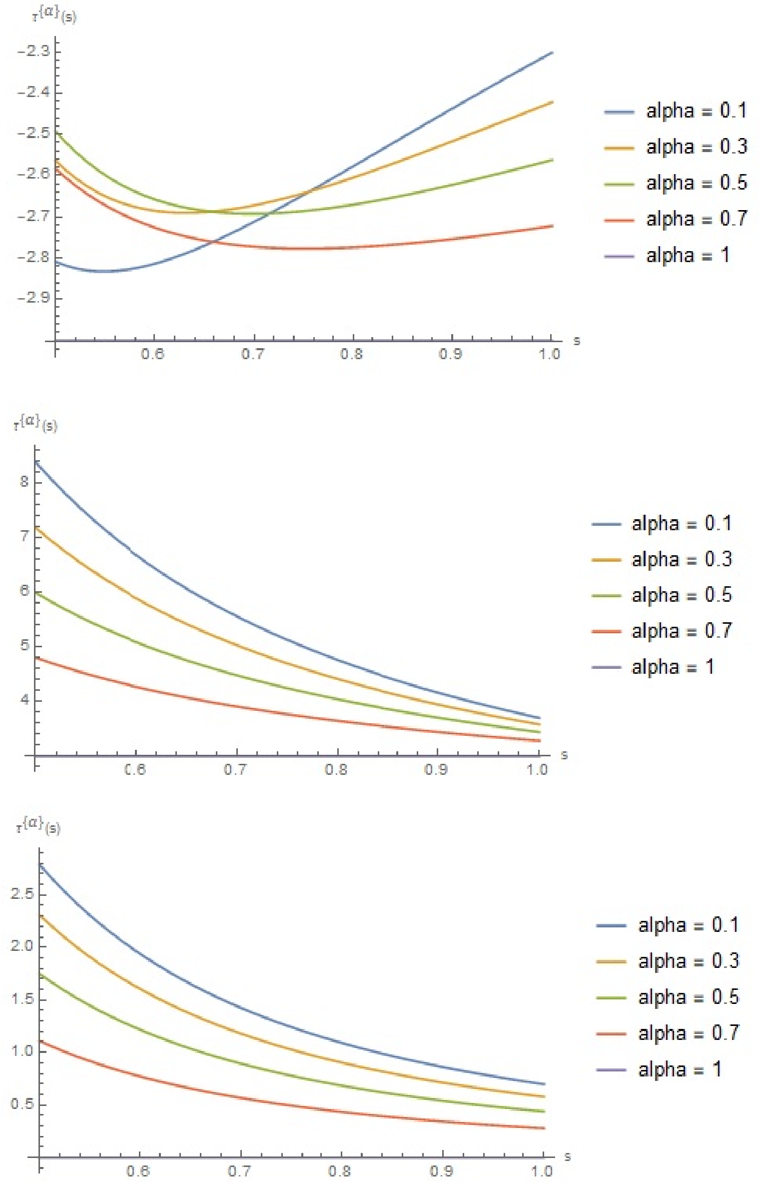

Figure 2. Graphs of $\tau^{\{\alpha\}}(s)$ for $\tau(\mu(s)) = -3$ (respectively, $\tau(\mu(s)) = 3, \tau(\mu(s)) = 0$) and $s \in [0.5, 1]$

Example 4.5. Consider conformable derivative as the derivative and the following curve $y(\mu)$ in $\mathbb{R}^3$ (see Figure 3)

$$y(\mu) = (\cos \mu, \sin \mu, \mu), \mu \in (a, b), 0 < a < b$$

where μ is the equiaffine arc length parameter of $y(\mu)$. Considering (3.2) and (3.3), $\kappa(\mu) = 0$ and $\tau(\mu) = 1$, for $y(\mu)$. Suppose that $C(\alpha, s) = s^{1-\alpha}$, namely our derivative is conformable type. Then, $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are obtained as follows:

$$\kappa^{\{\alpha\}}(s) = \frac{(\alpha+3)(\alpha-1)}{4} s^{-3} - \frac{1-\alpha}{2} s^{\alpha-2}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{(\alpha+3)(1-\alpha)}{4} s^{-2} + s^{\alpha-1}$$

The graphs of $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ for $y(\mu) = (\cos \mu, \sin \mu, \mu)$ are as in Figure 4.

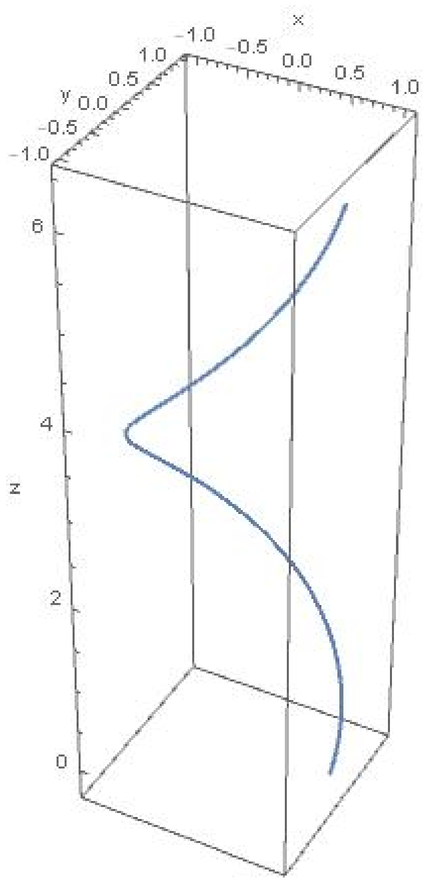

Figure 3. Graphs of the $y(\mu) = (\cos\mu, \sin\mu, \mu)$ function) and $s \in [0.5, 1]$

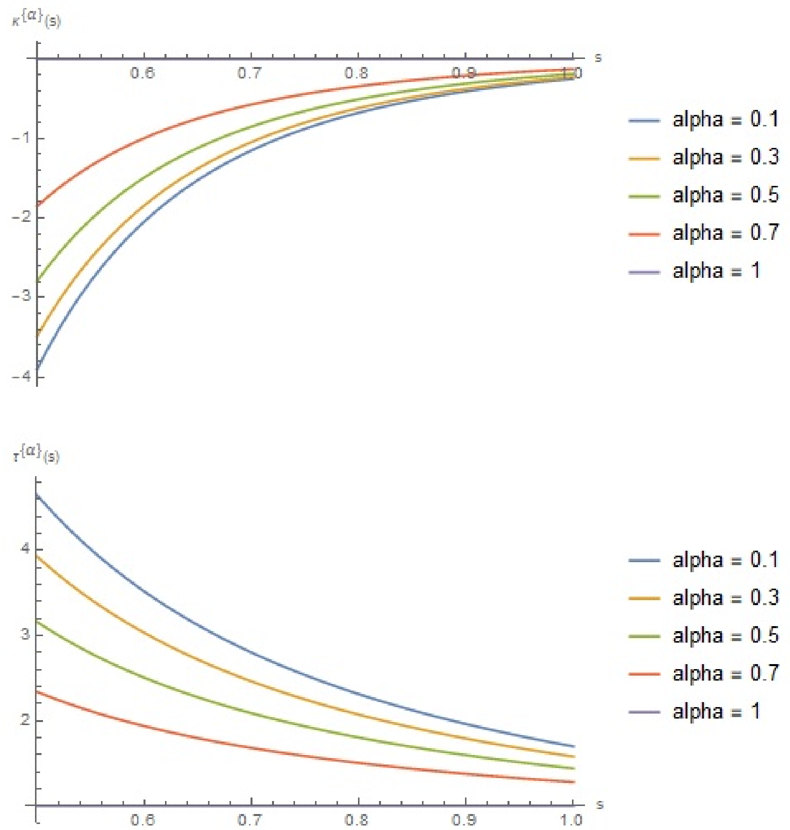

Figure 4. Graphs of α-equiaffine curvatures of $y(\mu) = (\cos\mu, \sin\mu, \mu)$ for $s \in [0.5, 1]$

Corollary 4.6. Let $y(s)$, $s \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. If the equiaffine curvatures of $y(s)$ vanish identically, then

$$\begin{aligned}\kappa^{\{\alpha\}}(s) &= -\tfrac{3}{4}\left(C(\alpha,s)\right)^{-3}\left(\tfrac{d}{ds}\left(C(\alpha,s)\right)\right)^{3}\tfrac{5}{4}\left(C(\alpha,s)\right)^{-2}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\\ &\quad -\tfrac{1}{2}\left(C(\alpha,s)\right)^{-1}\tfrac{d^3}{ds^3}\left(C(\alpha,s)\right)\end{aligned}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{3}{4}\left(C(\alpha,s)\right)^{-2}\left(\frac{d}{ds}\left(C(\alpha,s)\right)\right)^{2} - \left(C(\alpha,s)\right)^{-1}\frac{d^2}{ds^2}\left(C(\alpha,s)\right)$$

PROOF. It follows by (4.7) and (4.8). □

Corollary 4.7. Let $y(\mu)$, $\mu \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. Then,

$$\begin{aligned}\kappa(\mu) &= \tfrac{1}{4}\left(C(\alpha,s)\right)^{-\frac{3}{2}}\left(\tfrac{d}{ds}\left(C(\alpha,s)\right)\right)^{3} - \tfrac{3}{4}\left(C(\alpha,s)\right)^{-\frac{1}{2}}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\\ &\quad +\tfrac{1}{2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right) + \tfrac{1}{2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tau^{\{\alpha\}}(s) + \left(C(\alpha,s)\right)^{\frac{3}{2}}\kappa^{\{\alpha\}}(s)\end{aligned} \tag{4.12}$$

and

$$\tau(\mu) = \frac{d^2}{ds^2}\left(C(\alpha,s)\right) - \frac{3}{4}\left(C(\alpha,s)\right)^{-1}\left(\frac{d}{ds}\left(C(\alpha,s)\right)\right)^{2} + \left(C(\alpha,s)\right)\tau^{\{\alpha\}}(s) \tag{4.13}$$

PROOF. It follows by (4.7) and (4.8). □

Corollary 4.8. Let $y(\mu)$, $\mu \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. If $\kappa^{\{\alpha\}}(s) = 0$ and $\tau^{\{\alpha\}}(s) = 0$, then

$$\begin{aligned}\kappa(\mu) &= \tfrac{1}{4}\left(C(\alpha,s)\right)^{-\frac{3}{2}}\left(\tfrac{d}{ds}\left(C(\alpha,s)\right)\right)^{3} - \tfrac{3}{4}\left(C(\alpha,s)\right)^{-\frac{1}{2}}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\\ &\quad +\tfrac{1}{2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\end{aligned}$$

and

$$\tau(\mu) = \frac{d^2}{ds^2}\left(C(\alpha,s)\right) - \frac{3}{4}\left(C(\alpha,s)\right)^{-1}\left(\frac{d}{ds}\left(C(\alpha,s)\right)\right)^{2}$$

PROOF. It is obvious from (4.12) and (4.13). □

5. Conclusion

By incorporating a general local fractional derivative, this paper enhances the theory of equiaffine curves in the 3-dimensional affine space $\mathbb{R}^3$. We introduce novel invariants for these curves and establish connections between these new invariants and the conventional ones. Furthermore, we derive new results for equiaffine plane curves belonging to $\kappa^{\{\alpha\}}$ and $\tau^{\{\alpha\}}$. These bring forth a different perspective in affine geometry, leveraging the characteristics of fractional calculus. Additionally, Figures 1 and 2 show graphs indicating the relationship between specially selected equiaffine curvatures and α-equiafine curvatures of a curve. Moreover, Figures 3 and 4 show a provided curve in Example 4.5 and the graph of α-equiafine curvatures of this curve. These graphs show the behavior of α-equiafine curvatures for different α-values. It is clear that if the approach and calculations discussed in this study are considered in studies to be carried out in fractional derivative and affine space, a more general version of the characterizations obtained will be obtained. Therefore, this situation is an open problem, especially for researchers who will study invariants of curves. Hence, we can pose the following problem: To find the relations in higher dimensions between the fractional and standard equiaffine curvatures, akin to (4.7) and (4.8). Specifically, the primary objective of this problem is to formulate one equation expressing the relations between the fractional and standard equiaffine curvatures.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Whiskered Groupoids and Crossed Modules with Diagrams

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Abstract — In this study, we investigate the relationships between the category of crossed modules of groups and the category of whiskered groupoids. Our first aim is to construct a crossed module structure over groups from a whiskered groupoid with the objects set – a group (regular groupoid) – using the usual functor between the categories of crossed modules and cat groups. Conversely, the second aim is to construct a whiskered groupoid structure with the objects set, which is a group, from a crossed module of groups. While establishing this relationship, we frequently used arrow diagrams representing morphisms to make the axioms more comprehensible. We provide the conditions for the bimorphisms in a whiskered groupoid and give the relations between this structure and internal groupoids in the category of whiskered groupoids with the objects set as a group.

Keywords *Groupoid, crossed module, whiskered categories, bimorphism*

Mathematics Subject Classification (2020) 16E45, 18G45

1. Introduction

The notion of whiskering on a groupoid originally comes from the concept of tensor product in the category of crossed complexes over groupoids defined by Brown and Higgins [1]. If C is a crossed complex of groupoids together with the tensor product over itself, $w : C \otimes C \to C$, then a 1-truncation of C with the biactions of the objects set on the morphisms set gives a whiskered groupoid. In this case, we have the operations $w_{01} : C_0 \times C_1 \to C_1$, $\quad w_{10} : C_1 \times C_0 \to C_1$, and $w_{00} : C_0 \times C_0 \to C_0$ called whiskerings where C_0 is the set of objects and C_1 is the set of morphisms between objects. The operations w_{01} and w_{10} give the left and right actions of C_0 on C_1, respectively. Furthermore, the operation w_{00} gives a monoid structure over C_0. A crossed complex C over groupoids together with the tensor product $\otimes$ over C can be regarded as a crossed differential graded algebra defined by Baues in [2] and further studied by Baues-Tonks in [3]. Thus, we can say that the first component of a crossed differential graded algebra also gives a whiskered groupoid.

The purpose of defining whiskering operations is to explore the conditions under which the composition of morphisms has the commutativity for any given category. For a group G, if each commutator is identity in G, then G is an Abelian group. To define the notion of commutativity for any category $\mathcal{C}$, considering the whiskering operations in $\mathcal{C}$, the left and right multiplications have been introduced by Brown in [4]. In the case $\mathcal{C} := (C_1, C_0)$ is a groupoid together with the whiskering $w_{10} : C_1 \times C_0 \to C_1$

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and $w_{01} : C_0 \times C_1 \to C_1$, the commutator of $a : x \to y$ and $b : u \to v$ in $\mathcal{C}$ can be defined by

$$[a, b] = w_{10}(a, u)^{-1} \circ w_{10}(y, b)^{-1} \circ w_{10}(a, v) \circ w_{01}(x, b)$$

In this equality, the left and right multiplications are given by $l(a, b) = w_{01}(y, b) \circ w_{10}(a, u)$ and $r(a, b) = w_{10}(a, v) \circ w_{01}(x, b)$. Thus, the commutator of the morphisms a, b in $\mathcal{C}$ is $[a, b] = l(a, b)^{-1} r(a, b)$. In the case $l(a, b) = r(a, b)$, the groupoid $\mathcal{C}$ is called a commutative groupoid [4], and then $\mathcal{C}$ is a strict monoidal category.

On the other hand, if C is a groupoid, then the automorphism structure $Aut(C)$ is equivalent to a crossed module introduced by Whitehead in [5]; $\partial : Sc(C) \to Aut(C)$ where $Sc(C)$ is the set of sections of the source map s and the target map is a bijection on C_0. Then, the set $Sc(C)$ has a group structure with the Ehresmannian composition. Using this composition, we give the relationship between crossed modules and whiskered groupoids with the objects set is a group. The crossed module category is equivalent to the category of $\mathcal{G}$-groupoids [6]. The notion of $\mathcal{G}$-groupoid is also defined to be a group-groupoid [7]. Since the set of morphisms is not a group in a whiskered or regular groupoid, this structure is not equivalent to the group-groupoids or cat^1-groups.

As a 2-dimensional analog, we can say that if C is crossed module, then $Aut(C)$ has a braided regular crossed module structure defined by Brown and Gilbert [8], see also [9] for this structure. For the reduced cases of this structure in other contexts, see [10, 11]. Then, this structure can be considered as a whiskered 2-groupoid with the objects set as a group. Brown in [4] has also defined the notion of whiskering for any R-category. Since an R-algebroid can be considered as a small R-category, using the result of [12], it can be studied the R-algebroid version of the results herein.

2. Preliminaries

In this section, we recall the basic definitions of the whiskered categories and crossed modules of groups. For further details, see to [4, 8, 13, 14]. The following sources [15–18] also cover various aspects of this area, including the simplicial objects within categories of some algebraic structures, and could be valuable for the reader's reference.

2.1. Whiskered Groupoids

Suppose that $\mathfrak{C}$ is a (small) category with the set of morphisms (or 1-cells) written by C_1 and the set of objects (or 0-cells) written by C_0. In C_1, particularly, the set of morphisms $a : x \to y$ from x to y is denoted by $C_1(x, y)$, and x and y are called the source and target of the morphism a, respectively. The source and target maps are written $s, t : C_1 \to C_0$. Then, for $a \in C_1(x, y)$, we have $s(a) = x$ and $t(a) = y$.

The category composition in $\mathfrak{C}$ of morphisms $a : x \longrightarrow y$ and $b : y \longrightarrow z$ can be defined by $b \circ a : x \longrightarrow z$. In this case, clearly, $s(b \circ a) = s(a)$ and $t(b \circ a) = t(b)$. We write $C_1(x, x)$ as $C_1(x)$. Brown [4] introduced the notion of 'whiskering' for any category $\mathfrak{C}$ and gave the notions of left and right multiplications for a whiskered category $\mathfrak{C}$ as follows:

Definition 2.1. A whiskering on a category $\mathfrak{C} := (C_1, C_0)$ consists of operations

$$w_{i,j} : C_i \times C_j \longrightarrow C_{i+j}, \ i, j = 0, 1, \ i + j \leqslant 1$$

satisfying the following axioms:

Whisk 1) $w_{0,0}$ gives a monoid structure on C_0;

Whisk 2) $w_{0,1} : C_0 \times C_1 \longrightarrow C_1$ is a left action of the monoid C_0 on the category $\mathfrak{C}$ in the sense that, if $x \in C_0$ and $a : u \longrightarrow v$ in C_1, then

$$w_{0,1}(x, a) : w_{0,0}(x, u) \longrightarrow w_{0,0}(x, v)$$

in $\mathfrak{C}$, so that:

$$w_{0,1}(1, a) = a,\ \ w_{0,1}(w_{0,0}(x, y), a) = w_{0,1}(x, w_{0,1}(y, a))$$

$$w_{0,1}(x, a \circ b) = w_{0,1}(x, a) \circ w_{0,1}(x, b),\ \ w_{0,1}(x, 1_y) = 1_{xy}$$

Whisk 3) $w_{1,0} : C_1 \times C_0 \longrightarrow C_1$ is a right action of the monoid C_0 on C_1 with analogous rules.

Whisk 4)

$$w_{0,1}(x, w_{1,0}(a, y)) = w_{1,0}(w_{0,1}(x, a), y)$$

for all $x, y, u, v \in C_0$, $a, b \in C_1$.

Here, a category $\mathfrak{C}$ together with a whiskering is called a whiskered category.

In a whiskered category, for $a : x \to y$, $b : u \to v$, there are two multiplications given by

$$l(a, b) := m_{01}(y, b) \circ m_{10}(a, u) \quad \text{and} \quad r(a, b) := m_{10}(a, v) \circ m_{01}(x, b)$$

These multiplications can be denoted pictorially by

$$\begin{tikzcd}
xu \arrow[r, "m_{01}(x{,}b)"] \arrow[d, "m_{10}(a{,}u)"'] \arrow[dr, "r(a{,}b)", "l(a{,}b)"'] & xv \arrow[d, "m_{10}(a{,}v)"] \\
yu \arrow[r, "m_{01}(y{,}b)"'] & yv
\end{tikzcd}$$

It is well-known that a groupoid is a small category in which every arrow (or morphisms or 1-cells) is an isomorphism. That is, for any morphism a, there is a (necessarily unique) morphism a^{-1} such that $a \circ a^{-1} = e_{s(a)}$ and $a^{-1} \circ a = e_{t(a)}$ where $e : C_0 \to C_1$ gives the identity morphism at any object. We denote a groupoid as $\mathfrak{C} := (C_1, C_0)$, where C_0 is the set of objects and C_1 is the set of morphisms. For any groupoid $\mathfrak{C}$, if $C_1(x, y)$ is empty whenever x, y are distinct (that is, if $s = t$) then $\mathfrak{C}$ is called totally disconnected groupoid. A groupoid $\mathfrak{C} := (C_1, C_0)$ together with the whiskering operations $w_{i,j} : C_i \times C_j \to C_{i+j}$ for $i + j \leqslant 1$ satisfying the above conditions is called a whiskered groupoid. We denote a whiskered groupoid by $(\mathfrak{C}, w)$. In a whiskered groupoid, if the object set C_0 is a group with the multiplication given by w_{00}, we say that (C_1, C_0) is a regular groupoid as defined by Gilbert in [19]. We use the notation $\mathcal{RG}$ to denote the category of whiskered groupoids whose set of objects is a group with the operation w_{00}, or shortly of regular groupoids.

Example 2.2. Let $C_3 = \{1, x, x^2\} =< x >$ and $C_2 = \{1, y\} =< y >$ be cyclic groups. The action of C_2 on C_3 is given by

$${}^{1}1 = 1, {}^{1}x = x, {}^{1}(x^2) = x^2 \quad \text{and} \quad {}^{y}1 = 1, {}^{y}x = x^2, {}^{y}(x^2) = x$$

Using this action, we can create the semidirect product

$$C_3 \rtimes C_2 = \{(1, 1), (x^2, y), (x, y), (1, y), (x, 1), (x^2, 1)\}$$

with the multiplication of elements given by

$$\begin{aligned}
(x,y)(x,y) &= (x^y x, y^2) = (x^3, y^2) = (1,1)\\
(x^2,y)(x^2,y) &= (x^2({}^y(x^2)), y^2) = (1,1)\\
(1,y)(1,y) &= (1,1)\\
(x^2,1)(x^2,1) &= (x^2({}^1(x^2)), 1) = (x,1)\\
(x,1)(x,1) &= (x^1(x), 1) = (x^2,1)
\end{aligned}$$

and

$$\begin{aligned}
(x,y)(x^2,1) &= (x^y(x^2), y) = (x^2, y)\\
(x^2,1)(x,y) &= (1,y)\\
(x,y)(x,1) &= (x^y(x), y) = (xx^2, y) = (1,y)\\
(x,1)(x,y) &= (x^2,y)
\end{aligned}$$

It can be observed that this is a non-Abelian group and isomorphic to S_3. In this case, we can consider $C_3 \rtimes C_2$ as the set of morphisms G_1 and C_2 as the set of objects G_0. The elements of $C_3 \rtimes C_2$ can be regarded as morphisms.

$$(x^2,y),(1,y),(x,y) : y \to y \text{ and } (1,1),(x,1),(x^2,1) : 1 \to 1$$

The compositions of these morphisms are defined by

$$\begin{aligned}
(x^2,y) \circ (1,y) &= (x^2,y)\\
(x^2,y) \circ (x,y) &= (1,y)\\
(1,y) \circ (x,y) &= (x,y)\\
(x,1) \circ (x^2,1) &= (1,1)
\end{aligned}$$

The idendity map $e : G_0 \to G_1$ is defined on elements by $e(1) = (1,1)$ and $e(y) = (1,y)$. The whiskering operation $w_{01} : G_0 \times G_1 \to G_1$ is defined on elements by

$$w_{01}(y,(x,y)) = ({}^y x, y^2) = (x^2,1), \quad w_{01}(y,(x^2,y)) = ({}^y(x^2), y^2) = (x,1)$$
$$w_{01}(y,(1,y)) = ({}^y 1, y^2) = (1,1)$$

and

$$w_{01}(y,(1,1)) = ({}^y 1, y) = (1,y), \quad w_{01}(y,(x,1)) = ({}^y x, y) = (x^2,y)$$

and

$$w_{01}(y,(x^2,1)) = ({}^y(x^2), y) = (x,y)$$

and the whiskering operation $w_{10} : G_1 \times G_0 \to G_1$ is defined on elements by

$$w_{10}((x,y),y) = (x,y^2) = (x,1), \quad w_{10}((x^2,y),y) = (x^2,y^2) = (x^2,1)$$
$$w_{10}((1,y),y) = (1,y^2) = (1,1)$$

and

$$w_{10}((1,1),y) = (1,y), \quad w_{10}((x,1),y) = (x,y), \quad w_{10}((x^2,1),y) = (x^2,y)$$

It can be easily seen that these maps satisfy the whiskering axioms. Therefore, we obtain a regular groupoid (G_1, G_0, w_{ij}) which is also isomorphic to (S_3, H, w_{ij}) where $C_2 \cong H = \{I, (12)\}$ is the subgroup of S_3.

Example 2.3. With the same groups as previous example, define the action of C_2 on C_3 by ${}^y x = x$. Then, we have

$$^11 = 1, ^1x = x, ^1(x^2) = x^2 \text{ and } ^y1 = 1, ^yx = x, ^y(x^2) = x^2$$

Using this action, in the semidirect product $C_3 \rtimes C_2$, we have, for $g = (x, y) \in C_3 \rtimes C_2$

$$\begin{aligned} g^2 &= (x, y)(x, y) = (x^2, 1) \\ g^3 &= (x^2, 1)(x, y) = (1, y) \\ g^4 &= (1, y)(x, y) = (x, 1) \\ g^5 &= (x, 1)(x, y) = (x^2, y) \\ g^6 &= (x^2, y)(x, y) = (x^2({}^1x), y^2) = (1, 1) \end{aligned}$$

and then $C_3 \rtimes C_2 =< (x, y) >\cong C_6 = \{1, g, g^2, g^3, g^4, g^5\}$ is a cyclic group. The elements of $C_3 \rtimes C_2$ can be regarded as morphisms

$$g = (x, y), g^3 = (1, y), g^5 = (x^2, y) : y \to y \text{ and } g^6 = (1, 1), g^2 = (x^2, 1), g^4 = (x, 1) : 1 \to 1$$

The compositions of these morphisms are defined by

$$\begin{aligned} g^5 \circ g^3 &= (x^2, y) \circ (1, y) = (x^2, y) = g^5 \\ g^5 \circ g &= (x^2, y) \circ (x, y) = (1, y) = g^3 \\ g^3 \circ g &= (1, y) \circ (x, y) = (x, y) = g \\ g^4 \circ g^2 &= (x, 1) \circ (x^2, 1) = (1, 1) = g^6 = g^2 \circ g^4 \end{aligned}$$

The idendity map $e : G_0 \to G_1$ is defined on elements by $e(1) = g^6 = 1$ and $e(y) = g^3$. The whiskering operation $w_{01} : G_0 \times G_1 \to G_1$ is defined on elements by

$$w_{01}(y, g)) = ({}^yx, y^2) = (x, 1) = g^4$$

$$w_{01}(y, g^5) = ({}^y(x^2), y^2) = (x^2, 1) = g^2$$

$$w_{01}(y, g^3)) = ({}^y1, y^2) = (1, 1) = g^6$$

and

$$w_{01}(y, g^6)) = ({}^y1, y) = (1, y) = g^3$$

$$w_{01}(y, g^4) = ({}^yx, y) = (x, y) = g$$

$$w_{01}(y, g^2) = ({}^y(x^2), y) = (x^2, y) = g^5$$

and the whiskering operation $w_{10} : G_1 \times G_0 \to G_1$ is defined on elements by

$$w_{10}(g, y) = (x, y^2) = (x, 1) = g^4$$

$$w_{10}(g^5, y) = (x^2, y^2) = (x^2, 1) = g^2$$

$$w_{10}(g^3, y) = (1, y^2) = (1, 1) = g^6$$

and

$$w_{10}(g^6, y) = (1, y) = g^3$$

$$w_{10}(g^4, y) = (x, y) = g$$

$$w_{10}(g^2, y) = (x^2, y) = g^5$$

It can be easily seen that these maps satisfy the whiskering axioms. Therefore, we obtain a regular groupoid (C_6, C_2, w_{ij}).

Example 2.4. Consider the Klein 4-group $K = \{1, a, b, c\}$ with $a^2 = b^2 = c^2 = 1$ and the subgroup $N = \{1, b\}$. Let $G_0 = K$. Using the action of K on N given on elements by

$$^c1 = 1, \quad ^b1 = 1, \quad ^a1 = 1 \quad \text{and} \quad ^ab = aba = ac = b, \quad ^bb = b, \quad ^cb = cbc = ca = b$$

We can create a semidirect product group

$$G_1 = N \rtimes K = \{(1,1), (1,a), (1,b), (1,c), (b,1), (b,a), (b,b), (b,c)\}$$

which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The elements of this group can be regarded as morphisms:

$$(1,a) : a \to a, \quad (1,b) : b \to b, \quad (1,c) : c \to c, \quad (b,1) : 1 \to b$$

and

$$(b,a) : a \to ba = c, \quad (b,b) : b \to 1, \quad (b,c) : c \to a, \quad (1,1) : 1 \to 1$$

The compositions are defined on morphisms by, for example,

$$(b,c) \circ (b,a) = (b^2, a) = (1,a) : a \to a \quad \text{and} \quad (b,b) \circ (b,1) = (1,1)$$

Then, we can define the whiskering operations. The operations w_{01} and w_{10} are defined for $a \in K$ by

$$w_{01}(a, (1,a))) = (1,1) = w_{10}((1,a), a), \quad w_{01}(a, (1,b)) = (1, ab) = (1,c) = w_{10}((1,b), a)$$

$$w_{01}(a, (1,c))) = (1, ac) = (1,b) = w_{10}((1,c), a)$$

and

$$w_{01}(a, (b,1)) = (b,a) = w_{10}((b,1), a), \quad w_{01}(a, (b,b)) = (b,c) = w_{10}((b,b), a)$$

$$w_{01}(a, (b,c)) = (b,b) = w_{10}((b,c), a)$$

The whiskering operations can be defined similarly for elements $b, c \in K$. Thus, we have a regular groupoid $(N \rtimes K, K, w_{ij})$ which is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, w_{ij})$.

3. Crossed Modules and Regular Groupoids

In this section, we provide the close relationship between the category of crossed modules of groups and the category of regular groupoids. Crossed modules were introduced by Whitehead in [5]. This structure is an algebraic model for homotopy connected 2-types of topological spaces. Recall that a crossed module is a group homomorphism $\partial : M \to P$ together with an action of P on M, written ^pm, for $p \in P$ and $m \in M$, satisfying the conditions $\partial(^pm) = p\partial(m)p^{-1}$ and $^{\partial m}m' = mm'm^{-1}$, for all $m, m' \in M, p \in P$. We denote the category of crossed modules of groups by $\mathcal{XM}$. For further work about some categorical and algebraic properties of crossed modules in various settings and their examples, see to [20–24].

Example 3.1. Some algebraic examples of crossed modules are as follows:

i. The automorphism map $\phi : G \to Aut(G)$ defined by $\phi(g) = I_g$, for $g \in G$ is a crossed module, where I_g is the inner automorphism of G.

ii. If M is a P-module, there is a well-defined P-action on M. This, together with the zero homomorphism $0 : M \to P$, yields a crossed module.

iii. Let N be normal subgroup of G. Then, G acts on N by conjugation. This action and the inclusion map $i : N \to G$ form a crossed module.

3.1. From Crossed Modules to Regular Groupoids

Let $\partial : M \to N$ be a crossed module. We obtain a whiskered groupoid $\mathcal{C} := (C_1, C_0)$ together with the operations w_{10} and w_{01}. Let $C_0 = N$. By using the action of N on M, we can consider the semidirect product group $M \rtimes N$ with the group operation given by $(m,n)(m',n') = (m({}^{n}m'), nn')$, for $m, m' \in M$ and $n, n' \in N$. Then, by taking $C_0 = N$ and $C_1 = M \rtimes N$, we can create a whiskered groupoid as follows: The source and target maps from C_1 to C_0 are given by $s(m,n) = n$ and $t(m,n) = \partial(m)n$ for all $(m,n) \in C_1$. The groupoid composition is given by $(m',n') \circ (m,n) = (m'm, n)$ if $n' = \partial(m)n$. Finally, the whiskering operations w_{01} and w_{10} are given by respectively $w_{01}(p,(m,n)) = ({}^{p}m, pn)$ and $w_{10}((m,n),p) = (m, np)$, for all $m \in M, n, p \in N$. For these operations, we have

$$s(w_{01}(p,(m,n))) = s({}^{p}m, pn) = pn = ps(m,n)$$

and

$$\begin{aligned} t(w_{01}(p,(m,n))) &= t({}^{p}m, pn) \\ &= \partial({}^{p}m)pn \\ &= p\partial(m)p^{-1}pn \quad \text{(Since } \partial \text{ cros. mod)} \\ &= p\partial(m)n = pt(m,n) \end{aligned}$$

Similarly, we obtain easily that $s(w_{10}((m,n),p)) = s(m,np) = np = s(m,n)p$ and $t(w_{10}((m,n),p)) = t(m,np) = \partial(m)np = t(m,n)p$ for all $(m,n) \in C_1$ and $n, p \in C_0$.

Consequently, we obtain a whiskered groupoid. In this structure, the operation w_{00} can be taken as the group operation of $C_0 = N$. Thus, we can define a functor from the category of crossed modules of groups to the category of regular groupoids. We denote it by $S : \mathcal{XM} \to \mathcal{RG}$.

3.2. From Regular Groupoids to Crossed Modules

Let $\mathcal{C} := (C_1, C_0, w_{i,j})$ be a whiskered groupoid with the set of objects C_0 is a group according to the multiplication given by the operation w_{00}. In this case, from [4,8] we can say, using the Ehresmannian composition, that the set $K = \{a \in C_1 : s(a) = 1_{C_0}\}$ is a group with the group operation given by $a \odot b = w_{10}(a, t(b)) \circ b$, for any $a : 1_{C_0} \to y$ and $b : 1_{C_0} \to v$ in K, $y, v \in C_0$, and the target map t from K to C_0 is a homomorphism of groups. We can show this multiplication pictorially by

$$a \odot b := \begin{tikzcd} 1_{C_0} \arrow[r, "b"] \arrow[rr, bend right, "w_{10}(a,tb)\circ b"'] & v \arrow[r, "w_{10}(a,tb)"] & yv \end{tikzcd}$$

For $1_{C_0} \in C_0$, we have $e(1_{C_0}) : 1_{C_0} \to 1_{C_0}$ is the identity element of K. Indeed for any $a : 1_{C_0} \to y \in K$, we obtain

$$a \odot e(1_{C_0}) = w_{10}(a, 1_{C_0}) \circ 1_{C_0} = a = e(1_{C_0}) \odot a$$

The inverse of $a : 1_{C_0} \to y$ is $a^{-1} : 1_{C_0} \to y^{-1}$ where y^{-1} is the inverse of y in the group C_0. Thus, we have $a \odot a^{-1} = w_{10}(a, y^{-1}) \circ a^{-1} = e(1_{C_0})$. This can be represented by the diagram:

$$a \odot a^{-1} := \begin{tikzcd} 1_{C_0} \arrow[r, "a^{-1}"] \arrow[rr, bend right, "e(1_{C_0})"'] & y^{-1} \arrow[r, "w_{10}(a,y^{-1})"] & y^{-1}y \end{tikzcd}$$

We show that the target map t is a homomorphism of groups from K to C_0. For $a : 1_{C_0} \to y$ and $b : 1_{C_0} \to v$ in K, and $y, v \in C_0$, we obtain $t(a \odot b) = t(w_{10}(a, tb)) \circ b = yv = t(a)t(b)$.

The group action of $p \in C_0$ on $a : 1_{C_0} \to y \in K$ is given by ${}^{p}a = w_{01}(p, w_{10}(a, p^{-1})) = w_{10}(w_{01}(p, a), p^{-1})$.

The group C_0 is acting on itself by conjugation. This action can be represented pictorially by

$$\begin{tikzcd} * \arrow[r, "p"] & * \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "v"'] \arrow[r, phantom, "\Downarrow a"] & * \arrow[r, "p^{-1}"] & * \end{tikzcd} := \begin{tikzcd} * \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "p\cdot v\cdot p^{-1}"'] \arrow[r, phantom, "\Downarrow a"] & * \end{tikzcd}$$

Thus, we obtain that the homomorphism t is C_0-equivariant relative to the action of C_0 on K given above. Indeed, we have

$$t({}^{p}a) = pvp^{-1} = pt(a)p^{-1}$$

for $p \in C_0$ and $a \in K$, and so t is a pre-crossed module of groups.

Furthermore, for any $a : 1_{C_0} \to y, b : 1_{C_0} \to v \in K$, we have

$$a \odot b \odot a^{-1} = w_{10}(a, vy^{-1}) \circ w_{10}(b, y^{-1}) \circ a^{-1} = w_{01}(t(a), w_{10}(b, (ta)^{-1}))$$

This can be represented pictorially by

$$\begin{tikzcd} 1_{C_0} \arrow[r, "w_{10}(b,y^{-1})\circ a"] \arrow[rr, bend right, "a\odot b\odot a^{-1}"'] & vy^{-1} \arrow[r, "w_{10}(a,vy^{-1})"] & yvy^{-1} \end{tikzcd}$$

Therefore, we obtain ${}^{t(a)}b = a \odot b \odot a^{-1}$ and this is second crossed module axiom. So, we can say that t is a crossed module of groups. Thus, we have a crossed module $t : K \to C_0$ from the regular groupoid $(\mathcal{C}, w) := (C_1, C_0, w_{i,j})$. We can define a functor from the category of regular groupoids to the category of crossed modules as $F : \mathcal{RG} \to \mathcal{XM}$.

Remark 3.2. We see that there are functors between the categories $\mathcal{RG}$ and $\mathcal{XM}$. However, these functors do not give an equivalence between these categories. Consider a regular groupoid (C_1, C_0, w_{ij}). In this structure, we know that C_0 is a group with the multiplication given by w_{00}, and the set of morphisms C_1 is not a group. If we apply the functor $F : \mathcal{RG} \to \mathcal{XM}$ to this regular groupoid, we obtain $F((C_1, C_0, w_{ij})) := K \to C_0$ and we see that this is a crossed module of groups. If we apply the functor $S : \mathcal{XM} \to \mathcal{RG}$ to this crossed module, we have $S(K \to C_0) := (K \rtimes C_0, C_0, w_{ij})$. Since in the regular groupoid (C_1, C_0, w_{ij}), the set of morphisms C_1 is not a group, there is no isomorphism between $K \rtimes C_0$ and C_1. That is, $K \rtimes C_0 \ncong C_1$; therefore, we can say that these categories are not equivalent.

4. Bimorphisms within Whiskered (Regular) Groupoids

In this section, using the axioms of the crossed module, we give the bimorphism conditions in the regular groupoid obtained from a crossed module. We know from [4] that for the ordered set $I = \{-, +\}$ with $- < +$, a square or a 2-cube, in any category C is a functor $f : I^2 \to C$ and this is written as a diagram

$$\begin{tikzcd} x \arrow[r, "\partial_1^- f"] \arrow[d, "\partial_2^- f"'] & \cdot \arrow[d, "\partial_2^+ f"] \\ \cdot \arrow[r, "\partial_1^+ f"'] & y \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

where $Sf = x$, $tf = y$. The squares in C form a double category $\square C$ with compositions $\circ_1, \circ_2$ as given in [4].

Definition 4.1. [4] Let C be a category. A bimorphism $m : (C, C) \to \square C$ assigns to each pair of morphisms $a, b \in C$ a square $m(a, b) \in \square C$ such that if ad, bc are defined in C then

$$m(ad, c) = m(a, c) \circ_1 m(d, c)$$

$$m(a, bc) = m(a, b) \circ_2 m(a, c)$$

Remark 4.2. If we assume that C and D are crossed complexes of groupoids as provided in [1], the tensor product $C \otimes D$ of crossed complexes C, D constructed by Brown and Higgins in [1], is given by the universal bimorphism $(C, D) \to C \otimes D$.

Proposition 4.3. [4] If C is a whiskered category then a bimorphism

$$* : (C, C) \to \square C$$

is defined for $a : x \to y, b : u \to v$ by

$$a * b = \begin{pmatrix} & w_{01}(x, b) & \\ w_{10}(a, u) & & w_{10}(a, v) \\ & w_{01}(y, b) & \end{pmatrix}$$

According to the results obtained above, we can give the following proposition.

Proposition 4.4. For the regular groupoid

$$(C_1, C_0) := \; C_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightleftarrows}} C_0 = N \;, \circ, w_{ij})$$

which is obtained from the crossed module $\partial : M \to N$, the multiplication $a * b$ given by

$$m(a, b) = a * b = \begin{pmatrix} & w_{01}(n, (m', n')) & \\ w_{10}((m, n), n') & & w_{10}((m, n), \partial(m')n') \\ & w_{01}(\partial(m)n, (m', n')) & \end{pmatrix}$$

is a bimorphism for $a = (m, n), b = (m', n') \in M \rtimes N$.

Proof. We must show that

$$m(a \circ d, c) = m(a, c) \circ_1 m(d, c), \quad \text{and } m(a, b \circ c) = m(a, b) \circ_2 m(a, c)$$

For $a = (m, n) : n \to \partial(m)n$ and $b = (m', n') : n' \to \partial(m')n'$, we have already obtained the following diagram:

$$\begin{tikzcd} nn' \arrow[r, "w_{01}(n,b)"] \arrow[d, "w_{10}(a,n')"'] \arrow[dr, phantom, "a * b"] & n\partial(m')n' \arrow[d, "w_{10}(a,\partial(m')n')"] \\ \partial(m)nn' \arrow[r, "w_{01}(\partial(m)n,b)"'] & \partial(m)n\partial(m')n' \end{tikzcd}$$

and then we have $l(a, b) = r(a, b)$. To prove the above equality suppose that $a = (m, n) : n \to \partial(m)n$, $\quad d = (m', \partial(m, n)) : \partial(m)n \to \partial(m')\partial(m)n$. In this case, we have $a \circ d = (m'm, n) : n \to \partial(m')\partial(m)n$. For $c = (m'', n'') : n'' \to \partial(m'')n''$, we can draw the multiplication $m(a \circ d, c)$ by the following picture

$$\begin{tikzcd}
nn'' \arrow[r, "x\cdot c"] \arrow[d, "(a\circ d)\cdot u"'] & n\partial(m'')n'' \arrow[d, "(a\circ d)\cdot v"] \\
\partial(m'm)nn'' \arrow[r, "y\cdot c"'] & \partial(m'm)n\partial(m'')n''
\end{tikzcd}$$

where

$$(a \circ d) \cdot u = (m'm, n) \cdot n'' = (m'm, nn'')$$

$$(a \circ d) \cdot v = (m'm, n) \cdot \partial(m'')n'' = (m'm, n\partial(m'')n'')$$

$$x \cdot c = n \cdot (m'', n'') = ({}^{n}(m''), nn'')$$

$$\begin{aligned} y \cdot c = \partial(m'm)n \cdot (m'', n'') &= ({}^{\partial(m'm)}({}^{n}m''), \partial(m'm)nn'') \\ &= (m'm({}^{n}m'')(m'm)^{-1}), \partial(m'm)nn'') \end{aligned}$$

Thus,

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}m'',nn'')"] \arrow[d, "(m'm,nn'')"'] \arrow[dr, phantom, "(a\circ d)*c"] & n\partial(m'')n'' \arrow[d, "(m'm,n\partial(m'')n'')"] \\
\partial(m'm)nn'' \arrow[r, "(m'm({}^{n}m'')(m'm)^{-1}),\partial(m'm)nn'')"'] & \partial(m'm)n\partial(m'')n''
\end{tikzcd}$$

On the other hand, we investigate the notion of $m(a, c) \circ_1 m(d, c)$. In the following diagram, we show the multiplication $m(a, c)$.

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}m'',nn'')"] \arrow[d, "(m,nn'')"'] \arrow[dr, phantom, "(a*c)"] & n\partial(m'')n'' \arrow[d, "(m,n\partial(m'')n'')"] \\
\partial(m)nn'' \arrow[r, "(m{}^{n}m''m^{-1},\partial(m)nn'')"'] & \partial(m)n\partial(m'')n''
\end{tikzcd}$$

Similarly, we have

$$\begin{tikzcd}
\partial(m)nn'' \arrow[r, "({}^{\partial(m)}({}^{n}m''),\partial(m)nn'')"] \arrow[d, "(m',\partial(m)nn'')"'] \arrow[dr, phantom, "(d*c)"] & \partial(m)n\partial(m'')n'' \arrow[d, "(m',\partial(m)n\partial(m'')n'')"] \\
\partial(m'm)nn'' \arrow[r, "(m'{}^{\partial(m)n}(m'')(m')^{-1},\partial(m'm)nn'')"'] & \partial(m'm)n\partial(m'')n''
\end{tikzcd}$$

Since ∂ is a crossed module, in $d * c$, we have

$$s(d*c) = ({}^{\partial(m)}({}^{n}m''), \partial(m)nn'') = t(a*c) = (m{}^{n}m''m^{-1}, \partial(m)nn'')$$

and

$$t(d*c) = (m'{}^{\partial(m)n}(m'')(m')^{-1}, \partial(m')\partial(m)nn'') = (m'm({}^{n}m'')m^{-1}(m')^{-1}, \partial(m')\partial(m)nn'')$$

Then, we can compose them as

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}m'',nn'')"] \arrow[d, "(m,nn'')"'] \arrow[dr, phantom, "(a*c)"] & n\partial(m'')n'' \arrow[d, "(m,n\partial(m'')n'')"] \\
\partial(m)nn'' \arrow[r, "(m{}^{n}m''m^{-1},\partial(m)nn'')"'] \arrow[d, "(m',\partial(m)nn'')"'] \arrow[dr, phantom, "(d*c)"] & \partial(m)n\partial(m'')n'' \arrow[d, "(m',\partial(m)n\partial(m'')n'')"] \\
\partial(m'm)nn'' \arrow[r, "(m'm({}^{n}m'')m^{-1}(m')^{-1},\partial(m'm)nn'')"'] & \partial(m'm)n\partial(m'')n''
\end{tikzcd}$$

Therefore, from these diagrams, the composition $(a * c) \circ_1 (d * c)$ is given pictorially by

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}m'',nn'')"] \arrow[d, "(m'm,nn'')"'] \arrow[dr, phantom, "(a*c)\circ_1(d*c)"] & n\partial(m'')n'' \arrow[d, "(m'm,n\partial(m'')n'')"] \\
\partial(m'm)nn'' \arrow[r, "(m'm({}^{n}m'')(m'm)^{-1},\partial(m'm)n\partial(m'')n'')"'] & \partial(m'm)n\partial(m'')n''
\end{tikzcd}$$

Thus,

$$(a \circ d) * c = (a * c) \circ_1 (d * c)$$

We must show that

$$a * (b \circ c) = (a * b) \circ_2 (a * c)$$

For $b = (m', n') : n' \to \partial(m')n'$ and $c = (m'', n'') : n'' \to \partial(m'')n''$ with $n'' = \partial(m')n'$, we have

$$b \circ c = (m', n') \circ (m'', \partial(m')n') = (m''m', n')$$

We can draw the following diagram for the multiplication $a * (b \circ c)$:

$$\begin{tikzcd}
nn' \arrow[r, "({}^{n}(m''m'){,}nn')"] \arrow[d, "(m{,}nn')"'] \arrow[dr, phantom, "a*(b\circ c)"] & n\partial(m''m')n' \arrow[d, "(m{,}n\partial(m''m')n')"] \\
\partial(m)nn' \arrow[r, "(m{}^{n}(m''m')m^{-1}{,}\partial(m)nn')"'] & \partial(m)n\partial(m''m')n'
\end{tikzcd}$$

On the other hand, we have

$$\begin{tikzcd}
nn' \arrow[r, "({}^{n}(m'){,}nn')"] \arrow[d, "(m{,}nn')"'] \arrow[dr, phantom, "a*b"] & n\partial(m')n' \arrow[d, "(m{,}n\partial(m')n')"] \\
\partial(m)nn' \arrow[r, "({}^{\partial m}({}^{n}m'){,}\partial(m)nn')"'] & \partial(m)n\partial(m')n'
\end{tikzcd}$$

where since ∂ is a crossed module, we obtain

$$t(a*b) = ({}^{\partial m}({}^{n}m'), \partial(m)nn') = (m({}^{n}m')m^{-1}, \partial(m)nn')$$

Similarly,

$$\begin{tikzcd}
n\partial(m')n' \arrow[r, "({}^{n}(m''){,}n\partial(m')n')"] \arrow[d, "(m{,}n\partial(m')n')"'] \arrow[dr, phantom, "a*c"] & n\partial(m''m')n' \arrow[d, "(m{,}n\partial(m''m')n')"] \\
\partial(m)n\partial(m')n' \arrow[r, "({}^{\partial m}({}^{n}m''){,}\partial(m)n\partial(m')n')"'] & \partial(m)n\partial(m''m')n'
\end{tikzcd}$$

where

$$s(a*c) = (m, n\partial(m')n') = t(a*b)$$

For the horizontal composition $\circ_2$, we obtain the following diagram:

$$\begin{tikzcd}
nn' \arrow[r, "({}^{n}(m'){,}nn')"] \arrow[d, "(m{,}nn')"'] \arrow[dr, phantom, "a*b"] & n\partial(m')n' \arrow[r, "({}^{n}m''{,}n\partial(m')n')"] \arrow[d, "(m{,}n\partial(m')n')"] \arrow[dr, phantom, "a*c"] & n\partial(m''m')n' \arrow[d, "(m{,}n\partial(m''m')n')"] \\
\partial(m)nn' \arrow[r, "({}^{\partial m}({}^{n}m'){,}\partial(m)nn')"'] & \partial(m)n\partial(m')n' \arrow[r, "(m({}^{n}m'')m^{-1}{,}\partial(m)n\partial(m')n')"'] & \partial(m)n\partial(m''m')n'
\end{tikzcd}$$

From this diagram, we have

$$({}^{n}m', nn') \circ_2 ({}^{n}m'', n\partial(m')n') = ({}^{n}(m''m'), nn')$$

and

$$\begin{aligned}(m^n m' m^{-1}, \partial(m)nn') \circ_2 (m^n m'' m^{-1}, \partial(m)n\partial(m')n') &= (m^n m'' m^{-1} m^n m' m^{-1}, \partial(m)nn') \\ &= (m^n (m'' m') m^{-1}, \partial(m)nn')\end{aligned}$$

Thus, we obtain $(a * b) \circ_2 (a * c) = a * (b \circ c)$. Therefore, in the regular groupoid

$$(C_1, C_0) := (\ C_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} C_0 = N\ , \circ, w_{ij})$$

associated to the crossed module $\partial : M \to N$, the multiplication given by the following diagram

$$\begin{tikzcd} nn' \arrow[r, "({}^{n}(m'),nn')"] \arrow[d, "(m,nn')"'] \arrow[dr, phantom, "a*b"] & n\partial(m')n' \arrow[d, "(m,n\partial(m')n')"] \\ \partial(m)nn' \arrow[r, "({}^{\partial m}({}^{n}m'),\partial(m)nn')"'] & \partial(m)n\partial(m')n' \end{tikzcd}$$

is a bimorphism for $a = (m, n)$ and $b = (m', n')$. □

Proposition 4.5. In the regular groupoid

$$(C_1, C_0) := (\ C_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} C_0 = N\ , \circ, w_{ij})$$

associated to the crossed module $\partial : M \to N$, we have $l(a, b) = r(a, b)$ so this category is a strict monoidal category.

PROOF. For $a = (m, n)$ and $b = (m', n') \in M \rtimes N$,

$$\begin{aligned} l(a,b) =& m_{0,1}(\partial(m)n, b) \circ m_{1,0}(a, n') \\ =& ({}^{\partial(m)n}m', \partial(m)nn') \circ (m, nn') \\ =& ({}^{\partial(m)n}m'm, nn') \\ =& (m({}^{n}m')m^{-1}m, nn') \\ =& (m, n\partial(m')n') \circ ({}^{n}m', nn') \\ =& m_{1,0}(a, \partial(m')n') \circ m_{0,1}(n, b) \\ =& r(a, b) \end{aligned}$$

Therefore, for $a, b \in C_1$, $ab = l(a, b) = r(a, b)$ and thus the regular groupoid (C_1, C_0) is a strict monoidal category. We can illustrate this equality in the following diagram:

$$\begin{tikzcd} nn' \arrow[r, "w_{01}(n,b)"] \arrow[d, "m_{10}(a,n')"'] \arrow[dr, "r(a,b)", "l(a,b)"'] & n\partial(m')n' \arrow[d, "w_{10}(a,\partial(m')n')"] \\ \partial(m)nn' \arrow[r, "w_{01}(\partial(m)n,b)"'] & \partial(m)n\partial(m')n' \end{tikzcd}$$

□

Proposition 4.6. In the regular groupoid

$$(C_1, C_0) := (\ C_1 = M \rtimes N \overset{s,t}{\underset{e}{\rightleftarrows}} C_0 = N\ , \circ, m_{ij})$$

associated to the crossed module $\partial : M \to N$, the interchange law is hold,

$$(a \circ c) * (b \circ d) = (a * b) \circ (c * d)$$

Thus, (C_1, C_0) is an internal category in the category of regular groupoids.

Proof. For $a = (m, n)$, $c = (m', \partial(m)n)$, $b = (m'', n'')$, and $d = (m''', \partial(m'')n'')$, we have $a \circ c = (m'm, n)$ and $b \circ d = (m'''m'', n'')$ and then

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}(m'''m''),nn'')"] \arrow[d, "(m'm,nn'')"'] \arrow[dr, phantom, "(a\circ c)*(b\circ d)"] & n\partial(m'''m'')n'' \arrow[d, "(m'm,n\partial(m'''m'')n'')"] \\
\partial(m'm)nn'' \arrow[r, "({}^{\partial(m'm)n}(m'''m''),\partial(m'm)nn'')"'] & \partial(m'm)n\partial(m'''m'')n''
\end{tikzcd}$$

where

$$({}^{\partial(m'm)}({}^{n}(m'''m''), \partial(m')\partial(m)nn'') = (m'm^{n}(m'''m'')(m'm)^{-1}, \partial(m'm)nn'')$$

and thus,

$$t(m'm^{n}(m'''m'')(m'm)^{-1}, \partial(m'm)nn'') = \partial(m')\partial(m)n\partial(m''')\partial(m'')n''$$

On the other hand, for $a = (m, n)$ and $b = (m'', n'')$,

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}(m''),nn'')"] \arrow[d, "(m,nn'')"'] \arrow[dr, phantom, "(a*b)"] & n\partial(m'')n'' \arrow[d, "(m,n\partial(m'')n'')"] \\
\partial(m)nn'' \arrow[r, "({}^{\partial(mn)}(m''),\partial(m)nn'')"'] & \partial(m)n\partial(m'')n''
\end{tikzcd}$$

and for $c = (m', \partial(m)n)$ and $d = (m''', \partial(m'')n'')$,

$$\begin{tikzcd}
\partial(m)n\partial(m'')n'' \arrow[r, "({}^{\partial(m)n}(m'''),\partial(m)n\partial(m'')n'')"] \arrow[d, "(m',\partial(m)n\partial(m'')n'')"'] \arrow[dr, phantom, "(c*d)"] & \partial(m)n\partial(m'''m'')n'' \arrow[d, "(m',\partial(m)n\partial(m'''m'')n'')"] \\
\partial(m'm)n\partial(m'')n'' \arrow[r, "({}^{\partial(m'mn)}(m'''),\partial(m'm)n\partial(m'')n'')"'] & \partial(m'm)n\partial(m'''m'')n''
\end{tikzcd}$$

and thus,

$$(a \circ c) * (b \circ d) = (a * b) \circ (c * d)$$

We can illustrate this result in the following diagram,

$$\begin{tikzcd}
nn'' \arrow[r, "({}^{n}(m''){,}nn'')"] \arrow[d, "(m{,}nn'')"'] \arrow[dr, phantom, "(a*b)"] & n\partial(m'')n'' \arrow[r, dotted, "({}^{n}m'''{,}n\partial(m'')n'')"] \arrow[d, "(m{,}n\partial(m'')n'')"] \arrow[dr, phantom, "(a*d)"] & n\partial(m'''m'')n'' \arrow[d, dotted] \\
\partial(m)nn'' \arrow[r, "({}^{\partial(mn)}(m''){,}\partial(m)nn''))"'] \arrow[d, dotted] \arrow[dr, phantom, "(c*b)"] & \partial(m)n\partial(m'')n'' \arrow[r, "(m^{n}m'''m^{-1}{,}\partial(m)n\partial(m'')n'')"'] \arrow[d, "(m'{,}\partial(m)n\partial(m'')n'')"'] \arrow[dr, phantom, "(c*d)"] & \partial(m)n\partial(m'''m'')n'' \arrow[d, "(m'{,}\partial(m)n\partial(m'''m'')n'')"'] \\
\partial(m'm)n'' \arrow[r, dotted, "({}^{\partial(m'm)n}m''{,}\partial(m'm)nn'')"'] & \partial(m'm)n\partial(m'')n'' \arrow[r, "(m'm^{n}m''(m'm)^{-1}{,}\partial(m'm)n\partial(m'')n'')"'] & \partial(m'm)n\partial(m'''m'')n''
\end{tikzcd}$$

In this diagram, we can see that

$$((a*b)\circ_2(c*b))\circ_1((a*d)\circ_2(c*d))=((a*b)\circ_1(a*d))\circ_2((c*b)\circ_1(c*d))$$

□

5. Conclusion

In this study, we give the close relationship between crossed modules and whiskered groupoids with the objects set in a group. Thus, we see that a whiskered groupoid can be regarded as a crossed module of groups. If C is a crossed module, then the automorphism structure $Aut(C)$ defined by Brown and Gilbert in [8] has a braided regular crossed module structure. Then, this structure can be considered as a whiskered 2-groupoid with the objects set in a group. For a future study, as a two-dimensional analog of the result herein, the notion of a whiskered 2-groupoid can be defined using the properties of the braiding map on a crossed module. An R-algebroid can be considered as a small R-category, using the results of [12] and [23]; it can also be studied as the R-algebroid version of the results.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis, supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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On Some New Normed Narayana Sequence Spaces

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Abstract — In this paper, we first establish the regular matrix N using Narayana numbers. Then, we create new normed sequence spaces $Z(N)$ using the matrix N and demonstrate that these spaces are linearly isomorphic to Z where $Z \in \{c_0, c, \ell_p, \ell_\infty\}$. Additionally, we provide inclusion relations for the spaces $c_0(N)$, $c(N)$, $\ell_p(N)$, and $\ell_\infty(N)$. Furthermore, we construct the Schauder bases of the $c_0(N)$, $c(N)$, and $\ell_p(N)$. Finally, we compute the α-, β-, and γ-duals of these spaces and characterize the classes $(Z(N), X)$ for the certain choice of the sequence space X.

Keywords *Narayana numbers, Narayana matrices, sequence spaces, matrix transformations*

Mathematics Subject Classification (2020) 40C05, 46A45

1. Introduction

Narayana studied the problem of a herd of cows and calves in the fourteenth century [1, 2]. This is analogous to the Fibonacci rabbit problem, which can be expressed as follows: Every year, a cow gives birth to one calf, and starting in the fourth year, each calf has its annual production. The question is, after 20 years, how many calves are there in total? This problem can be solved in the same way as the Fibonacci rabbit problem [3]. The Narayana issue can be modeled if s is the year by recurrence

$$n_{s+3} = n_{s+2} + n_s$$

with $s \geq 0$, $n_0 = 0$, $n_1 = 1$, and $n_2 = 1$ [1]. Thus,

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \cdots$$

are the first few terms in this sequence [4]. This sequence is known as the Narayana sequence, commonly called the Fibonacci-Narayana sequence or the Narayana's cows sequence. There has been a lot of interest in the Narayana sequence and its generalizations recently. For more details, see [1,5–11].

We can provide some fundamental details about sequence spaces and summability theory. Each Γ subset of ω is referred to as a sequence space, and ω denotes the space of all real or complex sequences. We use the symbols ℓ_∞, c, and c_0 to symbolize the spaces of all bounded, convergent, and null sequences. Moreover, we designate the spaces of all convergent, bounded, absolutely, and p-absolutely convergent series by cs, bs, ℓ_1, and ℓ_p, $1 < p < \infty$, respectively.

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A K-space is a sequence space with a linear topology where each of the mappings $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$, for all $i \in \mathbb{N}$, is continuous. An FK-space is a K-space and a complete linear metric space. An FK-space with a normable topology is a BK-space.

If there are real entries in an infinite matrix $A = (a_{rs})$, then A_r represents the r_{th} row, for all $r \in \mathbb{N}$. If the series is convergent, for all $r \in \mathbb{N}$, then the A-transform of $u = (u_s) \in \omega$ is provided as follows:

$$(Au)_r = \sum_{s=0}^{r} a_{rs} u_s$$

If $Au \in \Psi$, then it is stated that A is a matrix transformation from Υ to Ψ, for all $u \in \Upsilon$. The notation (Υ, Ψ) denotes the class of all the matrices that transform from Υ to Ψ. The matrix domain of A in Υ is the set of all the vectors $u = (u_s)$ in ω such that $Au \in \Upsilon$. The approach of constructing a sequence space using the domain of an infinite matrix was first made by Ng and Lee [12]. In the following years, many studies were published in the literature. The papers [13–17] and the books [18, 19] can be investigated to get detailed information about them.

If Υ and Ψ are two sequence spaces, then the multiplier set $\mathfrak{D}(\Upsilon : \Psi)$ is described as

$$\mathfrak{D}(\Upsilon : \Psi) = \{x = (x_s) \in \omega : xu = (x_s u_s) \in \Psi, \text{ for all } (u_s) \in \Upsilon\}$$

Thus, α-, β-, and γ-duals of Υ are described as

$$\Upsilon^\alpha = \mathfrak{D}(\Upsilon : \ell_1), \quad \Upsilon^\beta = \mathfrak{D}(\Upsilon : cs), \quad \text{and} \quad \Upsilon^\gamma = \mathfrak{D}(\Upsilon : bs)$$

Recently, special integer sequences have been intensively studied in the Fibonacci, Lucas, Padovan, Catalan, Bell, and Schröder sequence spaces theory. For example, in 2022, Dağlı defined a new regular matrix using Schröder numbers and constructed new sequence spaces with the help of this matrix. For related works, the reader can refer to [20–28]. In line with the works mentioned above, our article aims to use Narayana numbers in the theory of sequence spaces.

In this paper, we define Narayana sequence spaces and investigate some of their basic topological properties. Moreover, we derive Schauder bases and compute the alpha, beta, and gamma duals of Narayana sequence spaces. In addition, we characterize some matrix transformations.

2. Narayana Sequence Spaces

This section presents the Narayana sequence spaces' definitions and characteristics. The Narayana matrix $N = (n_{rs})$ is defined by the following equation:

$$n_{rs} = \begin{cases} \dfrac{n_s}{n_{r+3} - 1}, & 1 \leqslant s \leqslant r \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}$. Equivalently,

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The inverse of $N = (n_{rs})$ is given by:

$$n_{rs}^{-1} = \begin{cases} (-1)^{r-s}\dfrac{n_{s+3}-1}{n_r}, & r \leqslant s \leqslant r+1 \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}$. Lately, Soykan [11] has researched the basic properties of Narayana numbers and discovered several interesting identities, such as

$$\sum_{s=0}^{r} n_s = n_{r+3} - 1$$

$$\sum_{s=0}^{r} n_{2s} = \frac{1}{3}\left(n_{2r+2} + n_{2r+1} + 2n_{2r} - 2\right)$$

and

$$\sum_{s=0}^{r} n_{2s+1} = \frac{1}{3}\left(2n_{2r+2} + 2n_{2r+1} + n_{2r} - 1\right)$$

where $r \in \mathbb{N}_0$. In this study, we define Narayana sequence spaces using Narayana numbers. The definitions of the Narayana sequence spaces $c_0(N)$, $c(N)$, $\ell_p(N)$, and $\ell_\infty(N)$ are as follows, respectively:

$$c_0(N) = \left\{ x = (x_s) \in \omega \ : \ \lim_{r\to\infty} \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s = 0 \right\}$$

$$c(N) = \left\{ x = (x_s) \in \omega \ : \ \lim_{r\to\infty} \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \text{ exists} \right\}$$

$$\ell_p(N) = \left\{ x = (x_s) \in \omega \ : \ \sum_{r} \left| \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \right|^p < \infty \right\}$$

and

$$\ell_\infty(N) = \left\{ x = (x_s) \in \omega \ : \ \sup_{r\in\mathbb{N}} \left| \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \right| < \infty \right\}$$

The previously mentioned sequence spaces can be redefined by

$$c_0(N) = (c_0)_N, \quad c(N) = (c)_N, \quad \ell_p(N) = (\ell_p)_N, \quad \text{and} \quad \ell_\infty(N) = (\ell_\infty)_N \tag{2.1}$$

using the notation of the matrix domain. The N-transform of a sequence $x = (x_s)$ is defined as $y = (y_r)$ by

$$y_r = (Nx)_r = \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \tag{2.2}$$

for each $r \in \mathbb{N}$. Throughout the remainder of the article, the sequences x and y are related to (2.2). Therefore, for all $s \in \mathbb{N}$,

$$x_s = (Ny)_s = \sum_{i=s-1}^{s} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} y_i \tag{2.3}$$

Theorem 2.1. The space $\ell_p(N)$ is a BK-space with the norm

$$\|(Nx)_r\|_p = \|x\|_{\ell_p(N)} = \left(\sum_{r} |(Nx)_r|^p \right)^{1/p}, \quad 1 \leqslant p < \infty$$

and the spaces $\ell_\infty(N)$, $c_0(N)$, and $c(N)$ are BK-spaces with the norm

$$\|(Nx)_r\|_{\ell_\infty} = \|x\|_{\ell_\infty(N)} = \|x\|_{c_0(N)} = \|x\|_{c(N)} = \sup_{r\in\mathbb{N}} |(Nx)_r|$$

Proof. The matrices N are triangular since (2.1) is true. Then, from Wilansky's Theorem 4.3.12 of [29], the spaces $\ell_p(N)$ are BK-spaces with the given norms where $1 \leqslant p \leqslant \infty$. Moreover, the spaces $c_0(N)$ and $c(N)$ are BK-spaces with the given norms from Wilansky's Theorem 4.3.2 of [29]. □

We may state the theorems relating to the inclusion relations concerning the spaces $c_0(N)$, $c(N)$, $\ell_p(N)$, and $\ell_\infty(N)$.

Theorem 2.2. Let Z represent any one of the four standard sequence spaces (c_0, c, ℓ_p, or ℓ_∞). The inclusion $Z \subset Z(N)$ strictly holds.

Proof. It is obvious that the inclusion $Z \subset Z(N)$ holds. Assume that $Z = c$. Besides, consider the sequence $g = (g_s)$ defined by $g = (1, 0, 1, 0, 1, \cdots)$. Then,

$$(Ng)_r = \sum_{s=1}^{r} \frac{n_s}{n_{r+3} - 1} g_s = \frac{n_1 + n_3 + \cdots + n_r}{n_{r+3} - 1} \tag{2.4}$$

is converges. Thus, we immediately observe that g is in $c(N)$ but not in c. Because at least one sequence in $c(N) \setminus c$, the inclusion $c(N) \subset c$ is strict. The proofs of the other inclusions are similar. □

Theorem 2.3. The inclusions $\ell_p(N) \subset c_0(N) \subset c(N) \subset \ell_\infty(N)$ strictly hold.

Proof. Since the matrix N is regular and the inclusion $\ell_p \subset c_0 \subset c \subset \ell_\infty$ holds, the inclusion part holds. Consider the sequence $h = (h_s)$ defined by $h = (1, 1, 1, 1, \cdots)$. Then,

$$(Nh)_r = \sum_{s=1}^{r} \frac{n_s}{n_{r+3} - 1} h_s = 1$$

for all $r \in \mathbb{N}$. Then, $Nh \in c \setminus c_0$. That is, $h \in c(N) \setminus c_0(N)$. This verifies the strictness of the inclusion $c_0(N) \subset c(N)$. Similarly, the strictness of other inclusions can be established. □

Theorem 2.4. If $1 \leq p < q$, then $\ell_p(N) \subset \ell_q(N)$.

Proof. Suppose $1 \leq p < q$. The matrix N is known to be regular, and the inclusion $\ell_p \subset \ell_q$ holds. These imply that the inclusion part holds. Then, consider a sequence $k = (k_s) \in \ell_q \setminus \ell_p$. Define a sequence $l = (l_s)$ by

$$l_s = \left(\frac{k_s(n_{s+3} - 1) - k_{s-1}(n_{s+2} - 1)}{n_s} \right)$$

such that $s \in \mathbb{N}$. Then,

$$\begin{aligned} (Nl)_r &= \frac{1}{n_{r+3} - 1} \sum_{s=1}^{r} n_s l_s \\ &= \frac{1}{n_{r+3} - 1} \sum_{s=1}^{r} [k_s(n_{s+3} - 1) - k_{s-1}(n_{s+2} - 1)] \\ &= \frac{1}{n_{r+3} - 1} k_r(n_{r+3} - 1) \\ &= k_r \end{aligned} \tag{2.5}$$

for each $r \in \mathbb{N}$ where the negative subscripted terms are considered to be zero and $k_0 = 0$. Therefore, we conclude that $Nl = k \in \ell_q \setminus \ell_p$, which means that $l \in \ell_q(N) \setminus \ell_p(N)$. Consequently, there is at least one sequence in $\ell_q(N)$ that is not in $\ell_p(N)$. □

Theorem 2.5. Let $Z \in \{c_0, c, \ell_p, \ell_\infty\}$. Then, $Z(N) \cong Z$.

Proof. Define the mapping $\tau : \ell_p(N) \to \ell_p$ by $\tau x = y = Nx$, for all x in $\ell_p(N)$. It is obvious that τ is linear and one-to-one. Assume that $x = (x_s)$ is defined as (2.3) such that $y = (y_r)$ is any sequence in ℓ_p. Moreover, let $1 \leqslant p < \infty$. Since $y \in \ell_p$,

$$\begin{aligned}
\|x\|_{\ell_p(N)} &= \left(\sum_{r=1}^{\infty} |(Nx)_r|^p\right)^{1/p} \\
&= \left(\sum_{r=1}^{\infty} \left|\sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s\right|^p\right)^{1/p} \\
&= \left(\sum_{r=1}^{\infty} \left|\sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} \sum_{i=s-1}^{s} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} y_i\right|^p\right)^{1/p} \\
&= \left(\sum_{r} |y_r|^p\right)^{1/p} \\
&= \|y\|_p < \infty
\end{aligned}$$

and

$$\|x\|_{\ell_\infty(N)} = \sup_{r\in\mathbb{N}} |(Nx)_r| = \|y\|_\infty < \infty$$

Consequently, x is a sequence in $\ell_p(N)$, and the mapping τ is onto and norm preserving. The proofs for the other spaces are similar. □

The result that ℓ_p space is not a Hilbert space for $p \neq 2$ is also valid for $\ell_p(N)$ space for $p \neq 2$ (see Theorem 2.6).

Theorem 2.6. The sequence space $\ell_p(N)$ is not a Hilbert space for $p \neq 2$.

Proof. We first prove that the space $\ell_2(N)$ is the only Hilbert space among the $\ell_p(N)$ spaces for $1 < p < \infty$. Consider the sequences $u = (u_s)$ and $v = (v_s)$ provided by

$$(u_s) = (1, 1, 0, 0, \cdots) \quad \text{and} \quad (v_s) = (1, -3, 0, 0, \cdots)$$

Thereby, $Nu = (1, 1, 0, 0, 0, \cdots)$ and $Nv = (1, -1, 0, 0, 0, \cdots)$. Since N is linear, so $N(u + v) = (2, 0, 0, 0, \cdots)$ and $N(u - v) = (0, 2, 0, 0, \cdots)$. Hence,

$$\|u+v\|^2_{\ell_p(N)} + \|u-v\|^2_{\ell_p(N)} = 8 \neq 2^{2(1+(1/p))} = 2\left(\|u\|^2_{\ell_p(N)} + \|v\|^2_{\ell_p(N)}\right) \tag{2.6}$$

Thus, it is evident that the norm $\|\cdot\|_{\ell_p(N)}$ violates the parallelogram identity when $p \neq 2$. □

It is understood that a matrix domain Z_A where A is triangular has a basis only if Z has a basis [30]. Thus, from Theorem 2.5, the following outcome can be deduced:

Theorem 2.7. Consider the sequence $b^{(s)} = \left(b^{(s)}\right)_{s\in\mathbb{N}}$ of the elements of the space $\ell_p(N)$ by

$$b_r^{(s)} = \begin{cases} (-1)^{r-s}\dfrac{n_{s+3}-1}{n_r}, & r \leqslant s \leqslant r+1 \\ 0, & s > r \end{cases}$$

for every fixed $s \in \mathbb{N}$ and $1 \leqslant p < \infty$. The following claims are accurate:

i. The sequence $\left(b^{(s)}\right)_{s\in\mathbb{N}_0}$ is a basis for the spaces $c_0(N)$ and $\ell_p(N)$, and any $x \in c_0(N)$ and $x \in \ell_p(N)$ has a unique representation of the form

$$x = \sum_s y_s b^{(s)}$$

ii. The sequence $\left(e, b^{(s)}\right)_{s\in\mathbb{N}}$ is a basis for the space $c(N)$, and any $x \in c(N)$ has a unique representation of the form

$$x = le + \sum_s [y_s - l] b^{(s)}$$

where $y_s = (N(x))_s \to l$ as $s \to \infty$.

iii. The space $\ell_\infty(N)$ does not have a basis.

3. Dual Spaces

In this section, the alpha, beta, and gamma duals of our novel sequence spaces are determined. In the following, the set of all the bounded subsets of $\mathbb{N}$ is denoted by N and assume that p^* denotes the conjugate of p, i.e., $p^{-1} + p^{*-1} = 1$. In this section, the lemmas used to prove the theorems are presented.

Lemma 3.1. [31] The following claims are accurate:

i. $A = (a_{rs}) \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ if and only if

$$\sup_{K \in N} \sum_{r=0}^{\infty} \left| \sum_{s \in K} a_{rs} \right| < \infty \tag{3.1}$$

ii. $A = (a_{rs}) \in (c_0 : c) = (c : c)$ if and only if

$$\exists \alpha_s \in \mathbb{C} \ni \lim_{r \to \infty} a_{rs} = \alpha_s, \quad \text{for all } s \in \mathbb{N}_0 \tag{3.2}$$

and

$$\sup_{r \in \mathbb{N}} \sum_{s=1}^{\infty} |a_{rs}| < \infty \tag{3.3}$$

iii. $A = (a_{rs}) \in (\ell_\infty : c)$ if and only if (3.2) holds and

$$\lim_{r \to \infty} \sum_{s=1}^{\infty} |a_{rs}| = \sum_{s=1}^{\infty} |\lim_{r \to \infty} a_{rs}| \tag{3.4}$$

iv. $A = (a_{rs}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ if and only if (3.3) holds.

Lemma 3.2. [31] The followings are valid:

i. Let $1 < p < \infty$. Then, $A = (a_{rs}) \in (\ell_p : \ell_\infty)$ if and only if

$$\sup_{r \in \mathbb{N}} \sum_{s=1}^{\infty} |a_{rs}|^{p^*} < \infty \tag{3.5}$$

ii. Let $1 < p < \infty$. Then, $A = (a_{rs}) \in (\ell_p : c)$ if and only if (3.2) and (3.5) hold.

iii. $A = (a_{rs}) \in (\ell_p : \ell_1)$ if and only if

$$\sup_{\mathsf{N} \in N} \sup_{s \in \mathbb{N}} \left| \sum_{r \in \mathsf{N}} a_{rs} \right|^p < \infty, \quad 0 < p \leq 1 \tag{3.6}$$

and

$$\sup_{\mathsf{N} \in N} \sum_{s=1}^{\infty} \left| \sum_{r \in \mathsf{N}} a_{rs} \right|^{p^*} < \infty, \quad 1 < p < \infty \tag{3.7}$$

iv. Let $0 < p \leq 1$. Then, $A = (a_{rs}) \in (\ell_p : \ell_\infty)$ if and only if

$$\sup_{r,s \in \mathbb{N}} \left| a_{rs} \right|^p < \infty \tag{3.8}$$

v. Let $0 < p \leq 1$. Then, $A = (a_{rs}) \in (\ell_p : c)$ if and only if (3.2) and (3.8) hold.

Theorem 3.3. Let $t=(t_r)\in\omega$. Define the matrix $T=(t_{rs})$ by

$$t_{rs}=\begin{cases}(-1)^{r-s}\dfrac{n_{s+3}-1}{n_r}t_r, & 0\leqslant s\leqslant r\\ 0, & s>r\end{cases}$$

for all $s,r\in\mathbb{N}$. Then, $\{c_0(N)\}^\alpha=\{c(N)\}^\alpha=\{\ell_\infty(N)\}^\alpha=\mathfrak{c}_1$ where $\mathfrak{c}_1$ defined by

$$\mathfrak{c}_1 \quad = \quad \left\{t=(t_s)\in w\ :\ \sup_{K\in N}\sum_{r=0}^{\infty}\left|\sum_{s\in K}t_{rs}\right|<\infty\right\}$$

Proof. We give the proof only for the sequence space $c_0(N)$. Thus,

$$t_rx_r=\sum_{s=r-1}^{r}(-1)^{r-s}\frac{n_{s+3}-1}{n_r}t_ry_s=(Tx)_r, r\in\mathbb{N} \tag{3.9}$$

It follows from (3.9), $tx=(t_rx_r)\in\ell_1$, for $x\in c_0(N)$, if and only if $Ty\in\{c_0(N)\}$, for $y\in c_0$. Hence, by Lemma 3.1 from (3.1), it is concluded that $\{c_0(N)\}^\alpha=\mathfrak{c}_1$. □

Theorem 3.4. Let the sets $\mathfrak{c}_2$ and $\mathfrak{c}_3$ be as follows:

$$\mathfrak{c}_2=\left\{t=(t_s)\in\omega\ :\ \sup_{\mathsf{N}\in N}\sup_{s\in\mathbb{N}}\left|\sum_{r\in\mathsf{N}}t_{rs}\right|^p<\infty\right\}$$

and

$$\mathfrak{c}_3=\left\{t=(t_s)\in\omega\ :\ \sup_{\mathsf{N}\in N}\sum_{s=1}^{\infty}\left|\sum_{r\in\mathsf{N}}t_{rs}\right|^{p^*}<\infty\right\}$$

respectively. Then,

$$\{\ell_p(N)\}^\alpha=\begin{cases}\mathfrak{c}_2, & 0<p\le 1\\ \mathfrak{c}_3, & 1<p<\infty\end{cases}$$

Proof. This is accomplished by using the same procedure as in the proof of Theorem 3.3 but substituting (3.6) and (3.7) of Lemma 3.2 (*iii*) for (3.1) of Lemma 3.1 (*i*) with t_{rs} rather than a_{rs}. □

Theorem 3.5. Consider the definition of $D=(d_{rj})$ using the sequence $a=(a_j)$ by

$$d_{rs}=\begin{cases}\displaystyle\sum_{i=s}^{r}(-1)^{s-i}\frac{n_{i+3}-1}{n_s}a_i, & 0\leqslant s\leqslant r\\ 0, & s>r\end{cases} \tag{3.10}$$

and define the following sets

$$\mathfrak{d}_1=\left\{a=(a_s)\in\omega\ :\ \sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|d_{rs}|<\infty\right\}$$

$$\mathfrak{d}_2=\left\{a=(a_s)\in\omega\ :\ \lim_{r\to\infty}d_{rs}=\alpha_s\right\}$$

$$\mathfrak{d}_3=\left\{a=(a_s)\in\omega\ :\ \lim_{r\to\infty}\sum_{s=1}^{\infty}|d_{rs}|=\sum_{s=1}^{\infty}\left|\lim_{r\to\infty}d_{rs}\right|\right\}$$

$$\mathfrak{d}_4=\left\{a=(a_s)\in\omega\ :\ \lim_{r\to\infty}\sup_{s\in\mathbb{N}}\sum_{s=1}^{\infty}|d_{rs}|<\infty\right\}$$

and

$$\mathfrak{d}_5=\left\{a=(a_s)\in\omega\ :\ \sup_{r,s\in\mathbb{N}}|d_{rs}|^p<\infty\right\}$$

Then,

i. $\{c_0(N)\}^\beta = \mathfrak{b}_1 \cap \mathfrak{b}_2$ and $\{c_0(N)\}^\gamma = \mathfrak{b}_1$

ii. $\{c(N)\}^\beta = \mathfrak{b}_1 \cap \mathfrak{b}_2$ and $\{c(N)\}^\gamma = \mathfrak{b}_1$

iii. $\{\ell_\infty(N)\}^\beta = \mathfrak{b}_2 \cap \mathfrak{b}_3$ and $\{\ell_\infty(N)\}^\gamma = \mathfrak{b}_1$

iv. $\{\ell_p(N)\}^\beta = \begin{cases} \mathfrak{b}_2 \cap \mathfrak{b}_4, & 0 \leqslant p < 1 \\ \mathfrak{b}_2 \cap \mathfrak{b}_5, & 1 \leqslant p < \infty \end{cases}$ and $\{\ell_p(N)\}^\gamma = \begin{cases} \mathfrak{b}_4, & 0 \leqslant p < 1 \\ \mathfrak{b}_5, & 1 \leqslant p < \infty \end{cases}$

Proof. We give the proof only for the $\beta-$dual of the sequence space $\ell_p(N)$. Consider the equation

$$\begin{aligned}\sum_{s=1}^{r} d_s x_s &= \sum_{s=1}^{r} \left[\sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} y_i\right] d_s \\ &= \sum_{s=1}^{r} \left[\sum_{i=s}^{r} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} d_i\right] y_s \\ &= (Dy)_r\end{aligned}$$

for any $r \in \mathbb{N}_0$. This equation states that if x is an element of $\ell_p(N)$, then dx is an element of cs if and only if Dy is an element of c, for x in ℓ_p. This means that D is an element of $(\ell_p : c)$. As a consequence, by Lemma 3.2 from (3.2) and (3.5), it is deduced that

$$\{\ell_p(N)\}^\beta = \begin{cases} \mathfrak{b}_2 \cap \mathfrak{b}_4, & 0 \leqslant p < 1 \\ \mathfrak{b}_2 \cap \mathfrak{b}_5, & 1 \leqslant p < \infty \end{cases}$$

□

4. Matrix Transformations

In this section, let $\lambda \in \{c_0(N), c(N), \ell_p(N), \ell_\infty(N)\}$ and $\mu \in \{c_0, c, \ell_\infty, \ell_1\}$. We provide necessary and sufficient conditions for matrix mappings from the spaces λ to any one of the spaces μ and from the spaces μ to the spaces λ.

Theorem 4.1. Define, for all $s, r \in \mathbb{N}_0$, $\mathcal{Z}^{(r)} = (z_{ms}^{(r)})$ and $\mathcal{Z} = (z_{rs})$ by

$$z_{ms}^{(r)} = \begin{cases} \sum_{i=s}^{m} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} a_{ri}, & 0 \leqslant s \leqslant m \\ 0, & s > m \end{cases}$$

and

$$z_{rs} = \sum_{i=s}^{m} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} a_{ri}$$

Then, $\mathcal{A} = (a_{rs}) \in (\ell_p(N) : \mu)$ if and only if $\mathcal{Z}^{(r)} \in (\ell_p : c)$, for all $r \in \mathbb{N}_0$, and $\mathcal{Z} \in (\ell_p : \mu)$.

Proof. Let $\mathcal{A} \in (\ell_p(N) : \mu)$ and $y = (y_s) \in \ell_p(N)$. Thus,

$$\sum_{s=1}^{\infty} a_{rs} x_s = \sum_{s=1}^{\infty} \left[\sum_{i=s}^{r} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} a_{ri}\right] y_s \tag{4.1}$$

for all $m, r \in \mathbb{N}$. $\mathcal{A}y$ exists, therefore $Z^{(r)} \in (\ell_p : c)$. Moreover, $\mathcal{A}y = \mathcal{Z}x$ by using $m \to \infty$ as in (4.1). Given that $\mathcal{A}y \in \mu$, $\mathcal{Z}x \in \mu$ follows, with the result that $\mathcal{Z} \in (\ell_p : \mu)$.

Conversely, suppose that $\mathcal{Z}^{(r)} \in (\ell_p : c)$, for all $r \in \mathbb{N}$, and that $\mathcal{Z} \in (\ell_p : \mu)$. Let $y = (y_s) \in \ell_p(N)$. Consequently, for all $r \in \mathbb{N}$, $\{a_{rs}\}_{s=1}^{\infty} \in \ell_p^\beta$, which means that $\{a_{rs}\}_{s=1}^{\infty} \in (\ell_p(N)^\beta$, for all $r \in \mathbb{N}$. From (4.1), $\mathcal{A}y = \mathcal{Z}x$ by as $m \to \infty$. Hence, $\mathcal{A} \in (\ell_p(N) : \mu)$. □

Theorem 4.2. Let $A = (a_{rs})$ be an infinite matrix and define the matrix $B = (b_{rs})$ by

$$b_{rs} = \sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} a_{ri} \tag{4.2}$$

for all $s, r \in \mathbb{N}$, and μ be a sequence space. Then, $A \in (\mu : \ell_p(N))$ if and only if $B \in (\mu : \ell_p)$.

PROOF. Let $z = (z_s) \in \mu$. Then,

$$\begin{aligned} \sum_{s=1}^{r} b_{rs} z_s &= \sum_{s=1}^{r} \left(\sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} a_{ri} \right) z_s \\ &= \sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} \left(\sum_{s=1}^{r} a_{is} z_s \right) \end{aligned}$$

for all $m, r \in \mathbb{N}$. Since $r \to \infty$, $(Bz)_m = (\Delta(Az))_m$, for all $m \in \mathbb{N}$. Thus, $z \in \mu$. Hence, $Az \in \ell_p(N)$ if and only if $Bz \in \ell_p$. $\square$

Combining Theorem 4.1 and the matrix mapping characterization findings presented in [31], we arrive at the following conclusions:

Corollary 4.3. The following claims are accurate:

i. $\mathcal{A} \in (\ell_p(N) : c_0)$ if and only if

$$\sup_{m \in \mathbb{N}_0} \sum_{s=1}^{\infty} \left| z_{ms}^{(r)} \right|^{p^*} < \infty \tag{4.3}$$

$$\lim_{m \to \infty} z_{ms}^{(r)} \text{ exists, for all } s \in \mathbb{N}_0 \tag{4.4}$$

and

$$\lim_{r \to \infty} z_{rs} = 0, \text{ for all } s \in \mathbb{N}_0$$

ii. $\mathcal{A} \in (\ell_p(N) : c)$ if and only if (4.3) and (4.4) hold,

$$\sup_{r \in \mathbb{N}_0} \sum_{s=1}^{\infty} |z_{rs}|^{p^*} < \infty \tag{4.5}$$

and

$$\lim_{r \to \infty} z_{rs} \text{ exists, for all } s \in \mathbb{N}_0$$

iii. $\mathcal{A} \in (\ell_p(N) : \ell_\infty)$ if and only if (4.3)-(4.5) hold.

iv. $\mathcal{A} \in (\ell_p(N) : \ell_1)$ if and only if (4.3) and (4.4) hold and

$$\sup_{N} \sum_{s=1}^{\infty} \left| \sum_{r \in N} z_{rs} \right|^{p^*} < \infty$$

Then, combining Theorem 4.2 and the matrix mapping characterization findings presented in [31], we arrive at the following conclusions:

Corollary 4.4. The following claims are accurate:

i. $\mathcal{A} \in (c_0 : \ell_p(N)) = (c : \ell_p(N)) = (\ell_\infty : \ell_p(N))$ if and only if $\sup_K \sum_{r=0}^{\infty} \left| \sum_{s \in K} b_{rs} \right|^p < \infty$.

ii. $\mathcal{A} \in (\ell_1 : \ell_p(N))$ if and only if $\sup_s \sum_{r=0}^{\infty} |b_{rs}|^p < \infty$.

5. Conclusion

Fibonacci, Lucas, Padovan, Catalan, Bell, and Schröder sequence spaces have been studied by many authors in the past and today. Hence, an introduction to sequence spaces was made within this study with the help of Narayana numbers. For this reason, this study is expected to serve as a guide for studies on the additivity theory. By benefiting from this study, researchers can consider similar results regarding Narayana difference sequence spaces, paranormed Narayana sequence spaces, the compactness measure of $\ell_p(N)$ space, and the convergence field of the Narayana matrix.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Characterization of Two Specific Cases with New Operators in Ideal Topological Spaces

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Abstract — This research deals with new operators $\wedge_\Gamma$, $\underline{\vee}_\Gamma$, and $\bar{\wedge}_\Gamma$, defined using Γ-local closure function and Ψ_Γ-operator in ideal topological spaces. It investigates the main features of these operators and their relationships with each other. The paper also analyzes their behaviors in some special ideals. Besides, it explores whether these operators preserve some set operations. Then, the study researches the properties of some special sets using these operators and proposes their characterizations. Additionally, it interprets some characterizations of the case $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ and the closure compatibility by means of these new operators.

Keywords *Ideal, Γ-local closure function, Ψ_Γ-operator*

Mathematics Subject Classification (2020) 54A05, 54A99

1. Introduction

After the emergence of the concept of ideal in [1, 2], this topic has been discussed by many authors in the literature. The local function [1] obtained using ideals and the *-topology [2] obtained with the help of this function are the most tackled topics by researchers. One of the most notable studies about these topics is the work of Janković and Hamlett in [3]. Furthermore, Ψ-operator [4], extensions of ideal [5], I-open sets [6], $PC^\star$-closed sets [7], and weakly I_{rg}-open sets [8] are the other examples of these topics. Apart from these topics, Selim et al. [9, 10] and Modak and Selim [11] also studied various set operators acquired by local function and Ψ-operator.

Afterward, Al-Omari and Noiri [12] introduced the Γ-local closure function and presented various properties of this operator. Furthermore, they defined the Ψ_Γ-operator with the Γ-local closure function and formed two topologies called σ and σ_0 owing to the operator Ψ_Γ. Many new studies based on Al-Omari and Noiri's works have been produced. For instance, Pavlović [13] investigated the similarities and differences between local functions with Γ-local closure functions and researched the cases under which they coincide. Tunç and Özen Yıldırım [14] made additions to Pavlović's conditions, and hence they [15, 16] defined some special sets using local closure functions and Ψ_Γ-operators. Furthermore, they [17] obtained a Γ-boundary operator through local closure functions and researched its properties.

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In this study, we build new set operators termed $\wedge_\Gamma$, $\veebar_\Gamma$, and $\bar{\wedge}_\Gamma$ via local closure function and Ψ_Γ-operator using different methods in ideal topological spaces. We research their main properties and the relationships of these operators with each other. We characterize the case $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ and interpret the closure compatibility by means of these operators.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following sections. Throughout this study, (Y, τ) represents a topological space. In (Y, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and the interior of a subset A of Y, respectively. $P(Y)$ represents the family of all the subsets of Y. An ideal $\Im$ [1] on a topological space (Y, τ) is a nonempty collection of subsets of Y satisfying the following conditions:

i. if $A \in \Im$ and $B \subseteq A$, then $B \in \Im$ (heredity).

ii. if $A \in \Im$ and $B \in \Im$, then $A \cup B \in \Im$ (finite additivity).

An ideal topological space $(Y, \tau, \Im)$ is a topological space (Y, τ) with an ideal $\Im$ on Y.

Let $(Y, \tau, \Im)$ be an ideal topological space. For a subset A of Y,

$$\Gamma(A)(\Im, \tau) = \{x \in Y \mid A \cap \mathrm{cl}(U) \notin \Im, \text{ for all } U \in \tau(x)\}$$

is called the local closure function of A with respect to $\Im$ and τ where $\tau(x) = \{U \in \tau \mid x \in U\}$ [12]. It is shortly denoted by $\Gamma(A)$ instead of $\Gamma(A)(\Im, \tau)$. An operator $\Psi_\Gamma : P(Y) \mapsto \tau$ is defined by $\Psi_\Gamma(A) = Y \setminus \Gamma(Y \setminus A)$, for all $A \in P(Y)$ [12]. A subset A of Y is called $\Im_\Gamma$-perfect (respectively, Γ-dense-in-itself, L_Γ-perfect, R_Γ-perfect, and $\Im_\Gamma$-dense) if $A = \Gamma(A)$ (respectively, $A \subseteq \Gamma(A)$, $A \setminus \Gamma(A) \in \Im$, $\Gamma(A) \setminus A \in \Im$, and $\Gamma(A) = Y$) [15]. A subset A of Y is called C_Γ-perfect if A is both L_Γ-perfect and R_Γ-perfect [15]. A subset A of Y is called $\theta^{\Im}$-closed if $\Gamma(A) \subseteq A$ [18].

For a topological space (Y, τ) and a subset A of Y, $\mathrm{cl}_\theta(A) = \{x \in Y : \mathrm{cl}(U) \cap A \neq \emptyset \text{ for each } U \in \tau(x)\}$ is called the θ-closure of A [19]. The θ-interior of A [20], denoted $\mathrm{int}_\theta(A)$, consists of those points x of A such that $U \subseteq \mathrm{cl}(U) \subseteq A$ for some open set U containing x. Furthermore, $Y \setminus \mathrm{int}_\theta(A) = \mathrm{cl}_\theta(Y \setminus A)$ [21]. A subset A is called θ-closed [19] if $A = \mathrm{cl}_\theta(A)$. The complement of a θ-closed set is called θ-open [19]. The family of all θ-open sets in (Y, τ) is denoted by τ_θ. Moreover, τ_θ is a topology on Y and it is coarser than τ.

As mentioned above, Al-Omari and Noiri [12] have defined the two topologies on Y as follows: $\sigma = \{A \subseteq Y : A \subseteq \Psi_\Gamma(A)\}$ and $\sigma_0 = \{A \subseteq Y : A \subseteq \mathrm{int}(\mathrm{cl}(\Psi_\Gamma(A)))\}$. They have shown that $\tau_\theta \subseteq \sigma \subseteq \sigma_0$ in $(Y, \tau, \Im)$. A subset A of Y is called σ-open (σ_0-open) set if $A \in \sigma$ ($A \in \sigma_0$). The topology τ is said to be closure compatible with the ideal $\Im$, denoted by $\tau \sim_\Gamma \Im$, if the following condition is held, for all subset A of Y: if, for all $x \in A$, there exists a $U \in \tau(x)$ such that $\mathrm{cl}(U) \cap A \in \Im$, then $A \in \Im$ [12].

Theorem 2.1. [12] Let $(Y, \tau, \Im)$ be an ideal topological space and $A, B \subseteq Y$. Then,

i. $\Gamma(\emptyset) = \emptyset$

ii. If $A \in \Im$, then $\Gamma(A) = \emptyset$.

iii. $\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B)$

iv. $\Psi_\Gamma(A \cap B) = \Psi_\Gamma(A) \cap \Psi_\Gamma(B)$

v. $\Gamma(A) = \mathrm{cl}(\Gamma(A)) \subseteq \mathrm{cl}_\theta(A)$

vi. If $A \subseteq B$, then $\Psi_\Gamma(A) \subseteq \Psi_\Gamma(B)$.

vii. If $A \subseteq B$, then $\Gamma(A) \subseteq \Gamma(B)$.

Corollary 2.2. [12] Let $(Y, \tau, \Im)$ be an ideal topological space. If $B \in \Im$, then $\Gamma(A \cup B) = \Gamma(A) = \Gamma(A \setminus B)$ in $(Y, \tau, \Im)$, for all $A, B \subseteq Y$.

Definition 2.3. [22] Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, the θ-closure of A with respect to an ideal $\Im$ is defined by $\mathrm{cl}_{\Im_\theta}(A) = A \cup \Gamma(A)(\Im, \tau)$. If $A = \mathrm{cl}_{\Im_\theta}(A)$, then A is called to be $\Im_\theta$-closed. Moreover, $\mathrm{Int}_{\Im_\theta}(A)$ is defined by $\mathrm{Int}_{\Im_\theta}(A) = Y \setminus \mathrm{cl}_{\Im_\theta}(Y \setminus A)$. If $A = \mathrm{Int}_{\Im_\theta}(A)$, then A is called to be $\Im_\theta$-open.

Remark 2.4. [17] Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then,

$$A \text{ is } \Im_\theta\text{-closed } \Leftrightarrow A = \mathrm{cl}_{\Im_\theta}(A) = A \cup \Gamma(A) \Leftrightarrow \Gamma(A) \subseteq A \Leftrightarrow A \text{ is } \theta^{\Im}\text{-closed}$$

Thus, the concept of $\Im_\theta$-closed set in [22] and the concept of $\theta^{\Im}$-closed set in [18] are identical.

Proposition 2.5. [17] Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then,

i. A is $\Im_\theta$-open $\Leftrightarrow Y \setminus A$ is $\Im_\theta$-closed

ii. A is $\Im_\theta$-open $\Leftrightarrow A \subseteq \Psi_\Gamma(A)$

iii. A is σ-open $\Leftrightarrow A$ is $\Im_\theta$-open

Theorem 2.6. [12] Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ such that $\mathrm{cl}(\tau) = \{\mathrm{cl}(G) : G \in \tau\}$ if and only if $Y = \Gamma(Y)$.

Theorem 2.7. [12] Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\tau \sim_\Gamma \Im$

ii. For all subset A of Y, $A \setminus \Gamma(A) \in \Im$

Theorem 2.8. [14] Let $(Y, \tau, \Im)$ be an ideal topological space such that $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, $\Psi_\Gamma(A) \subseteq \Gamma(A)$, for all $A \subseteq Y$.

Theorem 2.9. [15] Let $(Y, \tau, \Im)$ be an ideal topological space. Every $\Im_\Gamma$-dense set is Γ-dense-in-itself.

3. The Operator $\wedge_\Gamma$

This section defines the operator $\wedge_\Gamma$ and investigates its basic properties.

Definition 3.1. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the operator $\wedge_\Gamma : P(Y) \to P(Y)$ is defined by $\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus A$, for all $A \subseteq Y$.

Proposition 3.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

i. If $\Im = \{\emptyset\}$, then $\Gamma(K) = \mathrm{cl}_\theta(K)$. Therefore,

$$\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = (Y \setminus \mathrm{cl}_\theta(Y \setminus K)) \setminus K = \mathrm{int}_\theta(K) \setminus K = \emptyset$$

ii. If $\Im = P(Y)$, then $\Gamma(Y \setminus K) = \emptyset$. Thus,

$$\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = (Y \setminus \Gamma(Y \setminus K)) \setminus K = Y \setminus K$$

Remark 3.3. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $M \subseteq N$ implies that neither $\wedge_\Gamma(M) \subseteq \wedge_\Gamma(N)$ nor $\wedge_\Gamma(N) \subseteq \wedge_\Gamma(M)$, for all $M, N \subseteq Y$.

Example 3.4. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{q\}, \{r\}, \{q, r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. Suppose that $M = \{p\}$, $N = \{p, r\}$, and $K = \{p, s\}$. Although, $M \subseteq N$ in the ideal topological space $(Y, \tau, \Im)$, $\wedge_\Gamma(M) \nsubseteq \wedge_\Gamma(N)$. Similarly, although $M \subseteq K$, $\wedge_\Gamma(K) \nsubseteq \wedge_\Gamma(M)$.

Theorem 3.5. Let $(Y, \tau, \Im)$ be an ideal topological space and $M, N \subseteq Y$. Then, the following are held.

i. $\wedge_\Gamma(\emptyset) = Y \setminus \Gamma(Y)$

ii. $\wedge_\Gamma(Y) = \emptyset$

iii. If $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ and $M \in \Im$, then $\wedge_\Gamma(M) = \emptyset$.

iv. $\wedge_\Gamma(M) = (Y \setminus M) \setminus \Gamma(Y \setminus M)$

v. $\wedge_\Gamma(M) \cap \wedge_\Gamma(N) \subseteq \wedge_\Gamma(M \cup N)$

vi. $\wedge_\Gamma(M \cap N) = (\wedge_\Gamma(M) \cap \Psi_\Gamma(N)) \cup (\wedge_\Gamma(N) \cap \Psi_\Gamma(M)) \subseteq \wedge_\Gamma(M) \cup \wedge_\Gamma(N)$

vii. $\wedge_\Gamma(\wedge_\Gamma(M)) \subseteq \wedge_\Gamma(\Psi_\Gamma(M)) \cup \Psi_\Gamma(\Psi_\Gamma(M))$

viii. $\Gamma(\wedge_\Gamma(M)) \subseteq \Gamma(\Psi_\Gamma(M))$

ix. $\wedge_\Gamma(M) \cap M = \emptyset$ and thus $\wedge_\Gamma(M) \subseteq Y \setminus M$

x. $\wedge_\Gamma(M) \cup \wedge_\Gamma(N) \subseteq (M \cap \wedge_\Gamma(N)) \cup \wedge_\Gamma(M \cup N) \cup (\wedge_\Gamma(M) \cap N)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $M, N \subseteq Y$.

i. $\wedge_\Gamma(\emptyset) = \Psi_\Gamma(\emptyset) \setminus \emptyset = Y \setminus \Gamma(Y)$

ii. $\wedge_\Gamma(Y) = \Psi_\Gamma(Y) \setminus Y = \emptyset$

iii. Let $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ and $M \in \Im$. Then, by Corollary 2.2,

$$\wedge_\Gamma(M) = \Psi_\Gamma(M) \setminus M = (Y \setminus \Gamma(Y \setminus M) \setminus M = (Y \setminus \Gamma(Y)) \setminus M$$

Moreover, $\Gamma(Y) = Y$ from Theorem 2.6. Thus,

$$\wedge_\Gamma(M) = (Y \setminus Y) \setminus M = \emptyset$$

iv. $\wedge_\Gamma(M) = \Psi_\Gamma(M) \setminus M = (Y \setminus M) \cap (Y \setminus \Gamma(Y \setminus M)) = (Y \setminus M) \setminus \Gamma(Y \setminus M)$.

v.

$$\begin{aligned}
\wedge_\Gamma(M) \cap \wedge_\Gamma(N) &= (\Psi_\Gamma(M) \setminus M) \cap (\Psi_\Gamma(N) \setminus N) \\
&= (\Psi_\Gamma(M) \cap \Psi_\Gamma(N)) \cap [(Y \setminus M) \cap (Y \setminus N)] \\
&= (\Psi_\Gamma(M) \cap \Psi_\Gamma(N)) \cap [Y \setminus (M \cup N)]
\end{aligned}$$

From Theorem 2.1 (*iv.*),

$$\wedge_\Gamma(M) \cap \wedge_\Gamma(N) = \Psi_\Gamma(M \cap N) \cap [Y \setminus (M \cup N)] = \Psi_\Gamma(M \cap N) \setminus (M \cup N)$$

From Theorem 2.1 (*vi.*),

$$\Psi_\Gamma(M \cap N) \setminus (M \cup N) \subseteq \Psi_\Gamma(M \cup N) \setminus (M \cup N) = \wedge_\Gamma(M \cup N)$$

Therefore,

$$\wedge_\Gamma(M) \cap \wedge_\Gamma(N) \subseteq \wedge_\Gamma(M \cup N)$$

vi. By Theorem 2.1 (*iv.*),

$$\begin{aligned}
\wedge_\Gamma(M \cap N) &= \Psi_\Gamma(M \cap N) \setminus (M \cap N) \\
&= (\Psi_\Gamma(M) \cap \Psi_\Gamma(N)) \setminus (M \cap N) \\
&= [\Psi_\Gamma(M) \cap \Psi_\Gamma(N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(M) \cap \Psi_\Gamma(N) \cap (Y \setminus N)]
\end{aligned}$$

$$
\begin{aligned}
&= ([\Psi_\Gamma(M) \cap (Y \setminus M)] \cap \Psi_\Gamma(N)) \cup (\Psi_\Gamma(M) \cap [\Psi_\Gamma(N) \cap (Y \setminus N)]) \\
&= (\wedge_\Gamma(M) \cap \Psi_\Gamma(N)) \cup (\wedge_\Gamma(N) \cap \Psi_\Gamma(M)) \\
&\subseteq \wedge_\Gamma(M) \cup \wedge_\Gamma(N)
\end{aligned}
$$

vii. By Theorem 2.1 (*iv.*),

$$
\begin{aligned}
\wedge_\Gamma(\wedge_\Gamma(M)) &= \Psi_\Gamma(\wedge_\Gamma(M)) \setminus \wedge_\Gamma(M) \\
&= \Psi_\Gamma(\Psi_\Gamma(M) \setminus M) \setminus (\Psi_\Gamma(M) \setminus M) \\
&= (\Psi_\Gamma(\Psi_\Gamma(M)) \cap \Psi_\Gamma(Y \setminus M)) \cap [(Y \setminus \Psi_\Gamma(M)) \cup M] \\
&\subseteq \Psi_\Gamma(\Psi_\Gamma(M)) \cap [(Y \setminus \Psi_\Gamma(M)) \cup M] \\
&= [\Psi_\Gamma(\Psi_\Gamma(M)) \cap (Y \setminus \Psi_\Gamma(M))] \cup (\Psi_\Gamma(\Psi_\Gamma(M)) \cap M) \\
&\subseteq (\Psi_\Gamma(\Psi_\Gamma(M)) \setminus \Psi_\Gamma(M)) \cup \Psi_\Gamma(\Psi_\Gamma(M)) \\
&= \wedge_\Gamma(\Psi_\Gamma(M)) \cup \Psi_\Gamma(\Psi_\Gamma(M))
\end{aligned}
$$

viii. By Theorem 2.1 (*vii.*), $\Gamma(\wedge_\Gamma(M)) = \Gamma(\Psi_\Gamma(M) \setminus M) \subseteq \Gamma(\Psi_\Gamma(M))$.

ix. $\wedge_\Gamma(M) \cap M = (\Psi_\Gamma(M) \setminus M) \cap M = \emptyset$ and thus $\wedge_\Gamma(M) \subseteq Y \setminus M$.

x.

$$
\begin{aligned}
\wedge_\Gamma(M) &= \Psi_\Gamma(M) \setminus M \\
&= \Psi_\Gamma(M) \cap (Y \setminus M) \cap [(Y \setminus N) \cup N] \\
&= [\Psi_\Gamma(M) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N]
\end{aligned}
$$

By Theorem 2.1 (*vi.*),

$$
\wedge_\Gamma(M) \subseteq [\Psi_\Gamma(M \cup N) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N] \tag{3.1}
$$

Moreover,

$$
\begin{aligned}
\wedge_\Gamma(N) &= \Psi_\Gamma(N) \setminus N \\
&= \Psi_\Gamma(N) \cap (Y \setminus N) \cap [(Y \setminus M) \cup M] \\
&= [\Psi_\Gamma(N) \cap (Y \setminus N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M]
\end{aligned}
$$

By Theorem 2.1 (*vi.*),

$$
\wedge_\Gamma(N) \subseteq [\Psi_\Gamma(M \cup N) \cap (Y \setminus N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M] \tag{3.2}
$$

From (3.1) and (3.2),

$$
\begin{aligned}
\wedge_\Gamma(M) \cup \wedge_\Gamma(N) \subseteq\ & ([\Psi_\Gamma(M \cup N) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N]) \cup \\
& ([\Psi_\Gamma(M \cup N) \cap (Y \setminus N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M]) \\
=\ & [\Psi_\Gamma(M \cup N) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M] \\
=\ & [\Psi_\Gamma(M \cup N) \setminus (M \cup N)] \cup [(\Psi_\Gamma(M) \setminus M) \cap N] \cup [(\Psi_\Gamma(N) \setminus N) \cap M] \\
=\ & \wedge_\Gamma(M \cup N) \cup (\wedge_\Gamma(M) \cap N) \cup (\wedge_\Gamma(N) \cap M)
\end{aligned}
$$

□

Theorem 3.6. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ if and only if $\wedge_\Gamma(\emptyset) = \emptyset$.

The proof of Theorem 3.6 is obvious by Theorem 2.6 and Theorem 3.5 (*i.*).

Theorem 3.7. Let $(Y, \tau, \Im)$ be an ideal topological space. If A is a θ-closed (or an $\Im_\theta$-closed) subset of Y, then $\wedge_\Gamma(A) \subseteq Y \setminus \Gamma(Y)$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and A be a θ-closed (or an $\Im_\theta$-closed) subset of Y in $(Y, \tau, \Im)$. Then, $\Gamma(A) \subseteq A = \mathrm{cl}_\theta(A)$ by Theorem 2.1 (*v.*) (or $\Gamma(A) \subseteq A$). It follows

$$\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus A \subseteq \Psi_\Gamma(A) \setminus \Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A)$$

By Theorem 2.1 (*iv.*),

$$\wedge_\Gamma(A) \subseteq \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) = \Psi_\Gamma(A \cap (Y \setminus A)) = \Psi_\Gamma(\emptyset) = Y \setminus \Gamma(Y)$$

□

Remark 3.8. The inverses of the above requirements may not be true in general.

Example 3.9. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{p\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, suppose that $A = \{p\}$ and $B = \{q\}$. Then, $\wedge_\Gamma(A) = \emptyset \subseteq Y \setminus \Gamma(Y)$ but A is not θ-closed. Similarly, $\wedge_\Gamma(B) = \emptyset \subseteq Y \setminus \Gamma(Y)$ but B is not $\Im_\theta$-closed.

Theorem 3.10. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\wedge_\Gamma(K) \in \tau$, for all closed (or θ-closed) $K \subseteq Y$.

The proof of Theorem 3.10 is obvious by Theorem 2.1 (*v.*).

Corollary 3.11. Let $(Y, \tau, \Im)$ be an ideal topological space, $A \subseteq Y$, and $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. If A is θ-closed (or $\Im_\theta$-closed), then $\wedge_\Gamma(A) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space, A be a θ-closed (or an $\Im_\theta$-closed) subset of Y, and $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, by Theorem 3.7, $\wedge_\Gamma(A) \subseteq Y \setminus \Gamma(Y)$. It implies that $\wedge_\Gamma(A) = \emptyset$ from Theorem 2.6. □

Theorem 3.12. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $Y \setminus K$ is Γ-dense-in-itself if and only if $\wedge_\Gamma(K) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then,

$$\begin{aligned} Y \setminus K \text{ is } \Gamma\text{-dense-in-itself} &\Leftrightarrow Y \setminus K \subseteq \Gamma(Y \setminus K) \\ &\Leftrightarrow Y \setminus \Gamma(Y \setminus K) \subseteq K \\ &\Leftrightarrow \Psi_\Gamma(K) \subseteq K \\ &\Leftrightarrow \Psi_\Gamma(K) \setminus K = \wedge_\Gamma(K) = \emptyset \end{aligned}$$

□

Corollary 3.13. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If $Y \setminus A$ is $\Im_\Gamma$-dense, then $\wedge_\Gamma(A) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space, $A \subseteq Y$, and $Y \setminus A$ be $\Im_\Gamma$-dense in $(Y, \tau, \Im)$. Then, by Theorem 2.9, $Y \setminus A$ is Γ-dense-in-itself. Thus, $\wedge_\Gamma(A) = \emptyset$ by Theorem 3.12. □

Remark 3.14. The reverse of the above requirement may not be true in general.

Example 3.15. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{q, s\}$, then $\wedge_\Gamma(A) = \emptyset$ but $Y \setminus A$ is not $\Im_\Gamma$-dense.

Theorem 3.16. Let $(Y, \tau, \Im)$ be an ideal topological space.

i. If K is $\Im_\Gamma$-perfect, then $\wedge_\Gamma(K) = Y \setminus \Gamma(Y)$, for all $K \subseteq Y$.

ii. If $Y \setminus K$ is $\Im_\Gamma$-perfect, then $\wedge_\Gamma(K) = \emptyset$, for all $K \subseteq Y$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space.

i. Let K be an $\Im_\Gamma$-perfect set. Then,

$$\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = (Y \setminus \Gamma(Y \setminus K)) \setminus K = Y \setminus (K \cup \Gamma(Y \setminus K)) = Y \setminus (\Gamma(K) \cup \Gamma(Y \setminus K))$$

From Theorem 2.1 (*iii.*), $\Gamma(K) \cup \Gamma(Y \setminus K) = \Gamma(Y)$. As a result, $\wedge_\Gamma(K) = Y \setminus \Gamma(Y)$.

ii. Let $Y \setminus K$ be an $\Im_\Gamma$-perfect set. Then,

$$\Psi_\Gamma(K) = Y \setminus \Gamma(Y \setminus K) = Y \setminus (Y \setminus K) = K$$

As a result, $\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = \emptyset$.

□

Remark 3.17. The reverse of the above requirements may not be true in general.

Example 3.18. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{p\}$, then $\wedge_\Gamma(A) = \emptyset = Y \setminus \Gamma(Y)$ but A and $Y \setminus A$ are not $\Im_\Gamma$-perfect.

Theorem 3.19. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, A is L_Γ-perfect $\Leftrightarrow$ $\wedge_\Gamma(Y \setminus A) \in \Im$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. From Theorem 3.5 (*iv.*), $\wedge_\Gamma(Y \setminus A) = A \setminus \Gamma(A)$. Thus, the proof is obvious. □

Corollary 3.20. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If A is C_Γ-perfect, then $\wedge_\Gamma(Y \setminus A) \in \Im$.

The proof is obvious by Theorem 3.19.

Remark 3.21. In an ideal topological space, the reverse of the above requirement may not be true in general.

Example 3.22. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{p\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{r\}$, then $\wedge_\Gamma(Y \setminus A) = \emptyset \in \Im$. However, A is not R_Γ-perfect and thus it is not C_Γ-perfect.

Theorem 3.23. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $\wedge_\Gamma(Y \setminus K) = K$ if and only if $K \cap \Gamma(K) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then,

$$\begin{aligned} \wedge_\Gamma(Y \setminus K) = K &\Leftrightarrow \Psi_\Gamma(Y \setminus K) \cap K = K \\ &\Leftrightarrow K \subseteq \Psi_\Gamma(Y \setminus K) = Y \setminus \Gamma(K) \\ &\Leftrightarrow K \cap \Gamma(K) = \emptyset \end{aligned}$$

□

Theorem 3.24. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\tau \sim_\Gamma \Im$

ii. For all subset A of Y, $\wedge_\Gamma(A) \in \Im$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space. Then,

$$\begin{aligned} \wedge_\Gamma(A) \in \Im, \text{ for all subset } A \text{ of } Y &\Leftrightarrow \wedge_\Gamma(Y \setminus A) \in \Im, \text{ for all subset A of } Y \\ &\Leftrightarrow \Psi_\Gamma(Y \setminus A) \setminus (Y \setminus A) = A \setminus \Gamma(A) \in \Im, \text{ for all subset A of } Y \\ &\Leftrightarrow \tau \sim_\Gamma \Im \text{ from Theorem 2.7} \end{aligned}$$

□

4. The Operator $\veebar_\Gamma$

This section propounds the operator $\veebar_\Gamma$ and analyzes its basic properties.

Definition 4.1. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the operator $\veebar_\Gamma : P(Y) \to P(Y)$ is defined by $\veebar_\Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A)$, for all $A \subseteq Y$.

Theorem 4.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$. Then, the following are held.

i. $\veebar_\Gamma(C) = Y \setminus \Gamma(Y)$

ii. $\veebar_\Gamma(C) \in \tau$

iii. $\veebar_\Gamma(C) = \Psi_\Gamma(C) \setminus \Gamma(C)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$.

i. By Theorem 2.1 (*iv.*), $\veebar_\Gamma(C) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \Psi_\Gamma(C \cap (Y \setminus C)) = \Psi_\Gamma(\emptyset) = Y \setminus \Gamma(Y)$.

ii. By Theorem 2.1 (*v.*), $Y \setminus \Gamma(Y)$ is in τ. As a result, from (*i.*), $\veebar_\Gamma(C) \in \tau$.

iii. $\veebar_\Gamma(C) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \Psi_\Gamma(C) \cap (Y \setminus \Gamma(C)) = \Psi_\Gamma(C) \setminus \Gamma(C)$

□

Proposition 4.3. Let $(Y, \tau, \Im)$ be an ideal topological space and $D \subseteq Y$.

i. If $\Im = \{\emptyset\}$, then $\veebar_\Gamma(D) = Y \setminus \Gamma(Y) = Y \setminus \mathrm{cl}_\theta(Y) = \emptyset$.

ii. If $\Im = P(Y)$, then $\veebar_\Gamma(D) = Y \setminus \Gamma(Y) = Y \setminus \emptyset = Y$.

The proof is obvious by Theorem 4.2 (*i.*).

Corollary 4.4. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\Psi_\Gamma(A) \subseteq \Gamma(A)$ if and only if $\veebar_\Gamma(A) = \emptyset$.

The proof is obvious by Theorem 4.2 (*iii.*).

Corollary 4.5. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$

ii. $\veebar_\Gamma(A) = \emptyset$, for all $A \subseteq Y$

iii. $\veebar_\Gamma(A) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$, for all $A \subseteq Y$

iv. $\Psi_\Gamma(A) \subseteq \Gamma(A)$, for all $A \subseteq Y$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space.

(*i.*) $\Rightarrow$ (*ii.*) Let $A \subseteq Y$ and $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. By Theorem 2.8, $\Psi_\Gamma(A) \subseteq \Gamma(A)$. From Corollary 4.4, $\veebar_\Gamma(A) = \emptyset$.

(*ii.*) $\Rightarrow$ (*i.*) Let $\veebar_\Gamma(A) = \emptyset$, for all $A \subseteq Y$. Then, by Theorem 4.2 (*i.*), $\veebar_\Gamma(Y) = Y \setminus \Gamma(Y) = \emptyset$. It implies that $Y = \Gamma(Y)$ and thus $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ from Theorem 2.6.

(*i.*) $\Rightarrow$ (*iii.*) Let $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ and $A \subseteq Y$. Then, $\veebar_\Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$ from Theorem 2.8.

(*iii.*) $\Rightarrow$ (*i.*) Let $\veebar_\Gamma(A) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$, for all subset A of Y. We know that $\veebar_\Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) = (Y \setminus \Gamma(Y \setminus A)) \cap (Y \setminus \Gamma(A))$ and by the hypothesis $(Y \setminus \Gamma(Y \setminus A)) \cap (Y \setminus \Gamma(A)) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$. Therefore, $(Y \setminus \Gamma(Y \setminus A)) \cap (Y \setminus \Gamma(A))$ must be an empty set, namely $\veebar_\Gamma(A) = \emptyset$. From the equivalence (*i.*) $\Leftrightarrow$ (*ii.*), $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$

(*i.*) $\Rightarrow$ (*iv.*) It is obvious by Theorem 2.8.

(*iv.*) $\Rightarrow$ (*ii.*) Let $\Psi_\Gamma(A) \subseteq \Gamma(A)$, for all $A \subseteq Y$. Then, $\veebar_\Gamma(A) = \emptyset$, for all $A \subseteq Y$, from Corollary 4.4. □

Theorem 4.6. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. $\veebar_\Gamma(A) = Y \Leftrightarrow$ there exists a $U \in \tau(x)$ such that $\mathrm{cl}(U) \in \Im$, for all $x \in Y$.

ii. If there exists a nonempty set A such that $\veebar_\Gamma(A) = A$, then $\mathrm{cl}(\tau) \cap \Im \neq \{\emptyset\}$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i.

$$\begin{aligned}
\veebar_\Gamma(A) = Y &\Leftrightarrow Y \setminus \Gamma(Y) = Y \text{ from Theorem 4.2 } (i.) \\
&\Leftrightarrow \Gamma(Y) = \emptyset \\
&\Leftrightarrow x \notin \Gamma(Y), \text{ for all } x \in Y \\
&\Leftrightarrow \text{there exists a } U \in \tau(x) \text{ such that } \mathrm{cl}(U) \cap Y = \mathrm{cl}(U) \in \Im, \text{ for all } x \in Y
\end{aligned}$$

ii. Let A be a nonempty set such that $\veebar_\Gamma(A) = A$. Then, from Theorem 4.2 (*i.*), $A = Y \setminus \Gamma(Y)$ and thus $Y \setminus \Gamma(Y) \neq \emptyset$. It implies that $\Gamma(Y) \neq Y$. By Theorem 2.6, $\mathrm{cl}(\tau) \cap \Im \neq \{\emptyset\}$.

□

Theorem 4.7. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$. Then, the following are held.

i. If C is $\Im_\theta$-open, then $C \setminus \Gamma(C) \subseteq \veebar_\Gamma(C)$.

ii. If C is θ-open, then $C \setminus \Gamma(C) \subseteq \veebar_\Gamma(C)$.

iii. $\veebar_\Gamma(C) \subseteq \Psi_\Gamma(\veebar_\Gamma(C))$

iv. $\veebar_\Gamma(C) \cap C = \mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) = \mathrm{Int}_{\Im_\theta}(C) \setminus \Gamma(C)$

v. $\veebar_\Gamma(C) \setminus C = \mathrm{Int}_{\Im_\theta}(Y \setminus C) \cap \Psi_\Gamma(C) = \Psi_\Gamma(C) \setminus \mathrm{cl}_{\Im_\theta}(C)$

vi. If $C \in \Im$, then $\veebar_\Gamma(C) = \Psi_\Gamma(C)$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$.

i. Let C be $\Im_\theta$-open. Then, $C \subseteq \Psi_\Gamma(C)$ from Proposition 2.5 (*ii.*). Thus,

$$C \cap (Y \setminus \Gamma(C)) \subseteq \Psi_\Gamma(C) \cap (Y \setminus \Gamma(C))$$

namely

$$C \setminus \Gamma(C) \subseteq \Psi_\Gamma(C) \cap (Y \setminus \Gamma(Y \setminus (Y \setminus C))) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \veebar_\Gamma(C)$$

ii. Let C be θ-open. Then, it is σ-open as $\tau_\theta \subseteq \sigma$. By Proposition 2.5 (*iii.*) and from (*i.*) in this theorem, the proof is obvious.

iii. By Theorem 2.1 (*vi.*), as $\emptyset \subseteq \veebar_\Gamma(C)$, $\Psi_\Gamma(\emptyset) \subseteq \Psi_\Gamma(\veebar_\Gamma(C))$. Then, by Theorem 2.1 (*iv.*),

$$\veebar_\Gamma(C) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \Psi_\Gamma(C \cap (Y \setminus C)) = \Psi_\Gamma(\emptyset) \subseteq \Psi_\Gamma(\veebar_\Gamma(C))$$

iv.

$$\mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) = \mathrm{Int}_{\Im_\theta}(C) \cap (Y \setminus \Gamma(C)) = \mathrm{Int}_{\Im_\theta}(C) \setminus \Gamma(C)$$

Moreover,

$$\begin{aligned}
\mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) &= (Y \setminus \mathrm{cl}_{\Im_\theta}(Y \setminus C)) \cap \Psi_\Gamma(Y \setminus C) \\
&= (Y \setminus [\Gamma(Y \setminus C) \cup (Y \setminus C)]) \cap \Psi_\Gamma(Y \setminus C) \\
&= [(Y \setminus \Gamma(Y \setminus C)) \cap C] \cap \Psi_\Gamma(Y \setminus C) \\
&= (\Psi_\Gamma(C) \cap C) \cap \Psi_\Gamma(Y \setminus C) \\
&= (\Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C)) \cap C \\
&= \veebar_\Gamma(C) \cap C
\end{aligned}$$

Consequently, $\mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) = \mathrm{Int}_{\Im_\theta}(C) \setminus \Gamma(C) = \veebar_\Gamma(C) \cap C$.

v. From (*iv.*),

$$\veebar_\Gamma(Y \setminus C) \cap (Y \setminus C) = \mathrm{Int}_{\Im_\theta}(Y \setminus C) \cap \Psi_\Gamma(C)$$

From Theorem 4.2 (*i.*),

$$\veebar_\Gamma(Y \setminus C) \cap (Y \setminus C) = \veebar_\Gamma(C) \cap (Y \setminus C) = \veebar_\Gamma(C) \setminus C$$

and thus

$$\veebar_\Gamma(C) \setminus C = \mathrm{Int}_{\Im_\theta}(Y \setminus C) \cap \Psi_\Gamma(C) = (Y \setminus \mathrm{cl}_{\Im_\theta}(C)) \cap \Psi_\Gamma(C) = \Psi_\Gamma(C) \setminus \mathrm{cl}_{\Im_\theta}(C)$$

vi. If $C \in \Im$, then $\Gamma(C) = \emptyset$ by Theorem 2.1 (*ii.*). From Theorem 4.2 (*iii.*),

$$\veebar_\Gamma(C) = \Psi_\Gamma(C) \setminus \Gamma(C) = \Psi_\Gamma(C) \setminus \emptyset = \Psi_\Gamma(C)$$

□

Theorem 4.8. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $\veebar_\Gamma(K) = \Psi_\Gamma(Y \setminus K)$ if and only if $Y \setminus \Gamma(K) \subseteq \Psi_\Gamma(K)$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then,

$$\begin{aligned}
\veebar_\Gamma(K) = \Psi_\Gamma(Y \setminus K) &\Leftrightarrow \Psi_\Gamma(K) \cap \Psi_\Gamma(Y \setminus K) = \Psi_\Gamma(Y \setminus K) \\
&\Leftrightarrow Y \setminus \Gamma(K) = \Psi_\Gamma(Y \setminus K) \subseteq \Psi_\Gamma(K)
\end{aligned}$$

□

Theorem 4.9. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If A is $\Im_\Gamma$-perfect, then $\veebar_\Gamma(A) = \wedge_\Gamma(A)$.

Proof. Let $A \subseteq Y$ in $(Y, \tau, \Im)$. If A is $\Im_\Gamma$-perfect, then $\Gamma(A) = A$. Hence, $\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus A = \Psi_\Gamma(A) \setminus \Gamma(A)$. Consequently, from Theorem 4.2 (*iii.*), $\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus \Gamma(A) = \veebar_\Gamma(A)$. □

Remark 4.10. The reverse of Theorem 4.9 may not be true in general.

Example 4.11. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{p\}$, then $\wedge_\Gamma(A) = \emptyset = \veebar_\Gamma(A)$ but A is not $\Im_\Gamma$-perfect.

Theorem 4.12. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

i. If K is Γ-dense-in-itself, then $\veebar_\Gamma(K) \subseteq \Psi_\Gamma(K) \setminus K$.

ii. If K is $\Im_\Gamma$-dense, then $\veebar_\Gamma(K) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

i. If K is Γ-dense-in-itself, then $K \subseteq \Gamma(K)$. Therefore, by Theorem 4.2 (*iii.*), $\veebar_\Gamma(K) = \Psi_\Gamma(K) \setminus \Gamma(K) \subseteq \Psi_\Gamma(K) \setminus K$.

ii. If K is $\Im_\Gamma$-dense, then $\Gamma(K) = Y$. Thus, by Theorem 4.2 (*iii.*), $\veebar_\Gamma(K) = \Psi_\Gamma(K) \setminus \Gamma(K) = \Psi_\Gamma(K) \setminus Y = \emptyset$.

□

Remark 4.13. The reverse of the above requirements may not be true in general.

Example 4.14. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $K = \{r\}$, then $\veebar_\Gamma(K) = \emptyset = \Gamma(K)$. Although $\emptyset = \veebar_\Gamma(K) \subseteq \Psi_\Gamma(K) \setminus K$, K is neither Γ-dense-in-itself nor $\Im_\Gamma$-dense.

Corollary 4.15. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ if and only if there is an $\Im_\Gamma$-dense set $A \subseteq Y$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space.

$(\Rightarrow)$: Let $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. From Theorem 2.6, $\Gamma(Y) = Y$. Consequently, Y is $\Im_\Gamma$-dense.

$(\Leftarrow)$: Let there be an $\Im_\Gamma$-dense set $A \subseteq Y$. Hence, $\veebar_\Gamma(A) = \emptyset$ by Theorem 4.12 (*ii.*). It is known that $\veebar_\Gamma(A) = Y \setminus \Gamma(Y)$ by Theorem 4.2 (*i.*). Thereby, $Y \setminus \Gamma(Y) = \emptyset$ and thus $Y = \Gamma(Y)$. Consequently, by Theorem 2.6, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. □

5. The Operator $\barwedge_\Gamma$

This section proposes the operator $\barwedge_\Gamma$ and researches its basic properties.

Definition 5.1. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the operator $\barwedge_\Gamma : P(Y) \to P(Y)$ is defined by $\barwedge_\Gamma(A) = A \setminus \Gamma(A)$, for all $A \subseteq Y$.

Theorem 5.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$. Then, the following are held.

i. $\wedge_\Gamma(Y \setminus F) = \barwedge_\Gamma(F)$

ii. $\wedge_\Gamma(F) \cap \barwedge_\Gamma(F) = \emptyset$

iii. $\wedge_\Gamma(F) \cap \wedge_\Gamma(Y \setminus F) = \emptyset$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$.

i. $\wedge_\Gamma(Y \setminus F) = \Psi_\Gamma(Y \setminus F) \setminus (Y \setminus F) = \Psi_\Gamma(Y \setminus F) \cap F = (Y \setminus \Gamma(F)) \cap F = F \setminus \Gamma(F) = \barwedge_\Gamma(F)$

ii. $\wedge_\Gamma(F) \cap \barwedge_\Gamma(F) = (\Psi_\Gamma(F) \cap (Y \setminus F)) \cap (F \cap (Y \setminus \Gamma(F)) = (F \cap (Y \setminus F)) \cap (\Psi_\Gamma(F) \cap (Y \setminus \Gamma(F))) = \emptyset$

iii. It is obvious from Theorem 5.2 (*i.*) and (*ii.*).

□

Proposition 5.3. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. If $\Im = \{\emptyset\}$, then $\barwedge_\Gamma(A) = \emptyset$.

ii. If $\Im = P(Y)$, then $\barwedge_\Gamma(A) = A$.

Theorem 5.4. Let $(Y,\tau,\Im)$ be an ideal topological space and $K \subseteq Y$. If an element x of Y is in $\overline{\wedge}_{\Gamma}(K)$, then $\{x\} \in \Im$.

Proof. Let $(Y,\tau,\Im)$ be an ideal topological space and $K \subseteq Y$. Suppose that an element x of Y is in $\overline{\wedge}_{\Gamma}(K)$, i.e., $x \in K \setminus \Gamma(K)$. Then, $x \in K$ but $x \notin \Gamma(K)$. Therefore, there exists a $G \in \tau(x)$ such that $\mathrm{cl}(G) \cap K \in \Im$. It implies that $x \in \mathrm{cl}(G) \cap K \in \Im$. Hence, $\{x\} \in \Im$ by the heredity of the ideal. □

Theorem 5.5. Let $(Y,\tau,\Im)$ be an ideal topological space and $x \in Y$. Then, $x \in \overline{\wedge}_{\Gamma}(\{x\})$ if and only if $\{x\} \in \Im$.

Proof. Let $(Y,\tau,\Im)$ be an ideal topological space and $x \in Y$.

($\Rightarrow$): It is obvious by Theorem 5.4.

($\Leftarrow$): It is known that $Y \in \tau(x)$. Let $\{x\} \in \Im$. Then, $\mathrm{cl}(Y) \cap \{x\} = \{x\} \in \Im$, for $Y \in \tau(x)$, and thus $x \notin \Gamma(\{x\})$. Consequently, $x \in \{x\} \setminus \Gamma(\{x\}) = \overline{\wedge}_{\Gamma}(\{x\})$. □

Remark 5.6. In an ideal topological space $(Y,\tau,\Im)$, it is obvious that from Example 3.4 and Theorem 5.2 (*i.*), if $M \subseteq N \subseteq Y$, then neither $\overline{\wedge}_{\Gamma}(M) \subseteq \overline{\wedge}_{\Gamma}(N)$ nor $\overline{\wedge}_{\Gamma}(N) \subseteq \overline{\wedge}_{\Gamma}(M)$.

Theorem 5.7. Let $(Y,\tau,\Im)$ be an ideal topological space.

i. $\overline{\wedge}_{\Gamma}(G) \in \tau$, for all $G \in \tau$ (or $G \in \tau_{\theta}$)

ii. $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\} \Leftrightarrow \overline{\wedge}_{\Gamma}(Y) = \emptyset$

The proofs are obvious by Theorem 2.1 (*v.*) and Theorem 2.6, respectively.

Theorem 5.8. Let $(Y,\tau,\Im)$ be an ideal topological space and $K, L \subseteq Y$. Then, the following are held.

i. $\overline{\wedge}_{\Gamma}(\emptyset) = \emptyset$

ii. If K is in $\Im$, then $\overline{\wedge}_{\Gamma}(K) = K$.

iii. $\overline{\wedge}_{\Gamma}(\overline{\wedge}_{\Gamma}(K)) \subseteq \overline{\wedge}_{\Gamma}(K)$

iv. $\overline{\wedge}_{\Gamma}(K) \cap \Gamma(K) = \emptyset$

v. $\overline{\wedge}_{\Gamma}(K \cup L) = (\overline{\wedge}_{\Gamma}(K) \setminus \Gamma(L)) \cup (\overline{\wedge}_{\Gamma}(L) \setminus \Gamma(K))$

vi. $\overline{\wedge}_{\Gamma}(\overline{\wedge}_{\Gamma}(K)) \subseteq K$

vii. $\overline{\wedge}_{\Gamma}(K) \cap \overline{\wedge}_{\Gamma}(L) = (K \cap L) \setminus \Gamma(K \cup L)$

Proof. Let $(Y,\tau,\Im)$ be an ideal topological space and $K, L \subseteq Y$.

i. $\overline{\wedge}_{\Gamma}(\emptyset) = \emptyset \setminus \Gamma(\emptyset) = \emptyset$

ii. Let $K \in \Im$. Then, from Theorem 2.1 (*ii.*), $\overline{\wedge}_{\Gamma}(K) = K \setminus \Gamma(K) = K \setminus \emptyset = K$.

iii. $\overline{\wedge}_{\Gamma}(\overline{\wedge}_{\Gamma}(K)) = \overline{\wedge}_{\Gamma}(K) \setminus \Gamma(\overline{\wedge}_{\Gamma}(K)) \subseteq \overline{\wedge}_{\Gamma}(K)$

iv. $\overline{\wedge}_{\Gamma}(K) \cap \Gamma(K) = (K \setminus \Gamma(K)) \cap \Gamma(K) = \emptyset$

v. From Theorem 2.1 (*iii.*),

$$\begin{aligned}
\overline{\wedge}_{\Gamma}(K \cup L) &= (K \cup L) \setminus \Gamma(K \cup L) \\
&= (K \cup L) \setminus (\Gamma(K) \cup \Gamma(L)) \\
&= (K \cup L) \cap (Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L)) \\
&= [K \cap (Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L))] \cup [L \cap (Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L))] \\
&= [\overline{\wedge}_{\Gamma}(K) \cap (Y \setminus \Gamma(L))] \cup [\overline{\wedge}_{\Gamma}(L) \cap (Y \setminus \Gamma(K))] \\
&= (\overline{\wedge}_{\Gamma}(K) \setminus \Gamma(L)) \cup (\overline{\wedge}_{\Gamma}(L) \setminus \Gamma(K))
\end{aligned}$$

vi. The proof is obvious by (*iii.*) in this theorem.

vii. From Theorem 2.1 (*iii.*),

$$\begin{aligned}\overline{\curlywedge}_\Gamma(K) \cap \overline{\curlywedge}_\Gamma(L) &= (K \setminus \Gamma(K)) \cap (L \setminus \Gamma(L)) \\ &= (K \cap L) \cap [(Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L))] \\ &= (K \cap L) \cap [Y \setminus (\Gamma(K) \cup \Gamma(L))] \\ &= (K \cap L) \cap (Y \setminus \Gamma(K \cup L)) \\ &= (K \cap L) \setminus \Gamma(K \cup L)\end{aligned}$$

□

Theorem 5.9. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, the following are equivalent.

i. $\overline{\curlywedge}_\Gamma(K) = \emptyset$

ii. $\mathrm{cl}_{\Im_\theta}(K) = \Gamma(K)$

iii. K is Γ-dense-in-itself.

PROOF. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

(*i.*) $\Leftrightarrow$ (*ii.*) $\overline{\curlywedge}_\Gamma(K) = \emptyset \Leftrightarrow K \setminus \Gamma(K) = \emptyset \Leftrightarrow K \subseteq \Gamma(K) \Leftrightarrow \mathrm{cl}_{\Im_\theta}(K) = K \cup \Gamma(K) = \Gamma(K)$.

(*i.*) $\Leftrightarrow$ (*iii.*): It is obvious from Theorem 3.12 and Theorem 5.2 (*i.*). □

Theorem 5.10. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, the following are held.

i. If A is $\Im_\Gamma$-perfect, then $\overline{\curlywedge}_\Gamma(A) = \emptyset$.

ii. If A is $\Im_\Gamma$-dense, then $\overline{\curlywedge}_\Gamma(A) = \emptyset$.

PROOF. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. It is obvious from Theorem 3.16 (*ii.*) and Theorem 5.2 (*i.*).

ii. It is obvious from Corollary 3.13 and Theorem 5.2 (*i.*).

□

Theorem 5.11. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, A is L_Γ-perfect $\Leftrightarrow$ $\overline{\curlywedge}_\Gamma(A) \in \Im$.

The proof is obvious from Theorem 5.2 (*i.*) and Theorem 3.19.

Corollary 5.12. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If A is C_Γ-perfect, then $\overline{\curlywedge}_\Gamma(A) \in \Im$.

The proof is obvious by Theorem 5.11.

Remark 5.13. It is obvious that the reverse of the above requirement may not be true from Example 3.22 and Theorem 5.2 (*i.*).

Theorem 5.14. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $\overline{\curlywedge}_\Gamma(K) = K$ if and only if $K \cap \Gamma(K) = \emptyset$.

The proof is obvious from Theorem 3.23 and Theorem 5.2 (*i.*).

Theorem 5.15. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\tau \sim_\Gamma \Im$

ii. For all subset A of Y, $\overline{\wedge}_\Gamma(A) \in \Im$

The proof is obvious from Theorem 2.7.

6. Various Relations

This section investigates various relations between the operators defined herein.

Theorem 6.1. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$. Then, the following are held.

i. $\underline{\vee}_\Gamma(F) \cap \wedge_\Gamma(F) = \underline{\vee}_\Gamma(F) \setminus F$

ii. $\underline{\vee}_\Gamma(F) \cap \overline{\wedge}_\Gamma(F) = \Psi_\Gamma(F) \cap \overline{\wedge}_\Gamma(F)$

iii. $\underline{\vee}_\Gamma(F) \setminus \wedge_\Gamma(F) = \underline{\vee}_\Gamma(F) \cap F$

iv. $\underline{\vee}_\Gamma(F) \setminus \overline{\wedge}_\Gamma(F) = \underline{\vee}_\Gamma(F) \setminus F$

v. $\overline{\wedge}_\Gamma(F) \setminus \underline{\vee}_\Gamma(F) = \overline{\wedge}_\Gamma(F) \setminus \Psi_\Gamma(F)$

vi. $\wedge_\Gamma(F) \cup \underline{\vee}_\Gamma(F) = \Psi_\Gamma(F) \setminus (F \cap \Gamma(F))$

vii. $\underline{\vee}_\Gamma(F) \cup \overline{\wedge}_\Gamma(F) = (\Psi_\Gamma(F) \cup F) \setminus \Gamma(F)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$.

i.

$$\begin{aligned}
\underline{\vee}_\Gamma(F) \cap \wedge_\Gamma(F) &= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap (\Psi_\Gamma(F) \cap (Y \setminus F)) \\
&= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap (Y \setminus F) \\
&= \underline{\vee}_\Gamma(F) \setminus F
\end{aligned}$$

ii. By Theorem 4.2 (*iii.*),

$$\begin{aligned}
\underline{\vee}_\Gamma(F) \cap \overline{\wedge}_\Gamma(F) &= (\Psi_\Gamma(F) \cap (Y \setminus \Gamma(F))) \cap (F \cap (Y \setminus \Gamma(F))) \\
&= (Y \setminus \Gamma(F)) \cap (F \cap \Psi_\Gamma(F)) \\
&= \Psi_\Gamma(F) \cap (F \cap (Y \setminus \Gamma(F))) \\
&= \Psi_\Gamma(F) \cap \overline{\wedge}_\Gamma(F)
\end{aligned}$$

iii.

$$\begin{aligned}
\underline{\vee}_\Gamma(F) \setminus \wedge_\Gamma(F) &= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap [Y \setminus (\Psi_\Gamma(F) \cap (Y \setminus F))] \\
&= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap [(Y \setminus \Psi_\Gamma(F)) \cup F] \\
&= [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap (Y \setminus \Psi_\Gamma(F))] \cup [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap F] \\
&= (\Psi_\Gamma(Y \setminus F) \cap [\Psi_\Gamma(F) \cap (Y \setminus \Psi_\Gamma(F))]) \cup [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap F] \\
&= (\Psi_\Gamma(Y \setminus F) \cap \emptyset) \cup (\underline{\vee}_\Gamma(F) \cap F) \\
&= \underline{\vee}_\Gamma(F) \cap F
\end{aligned}$$

iv.

$$\begin{aligned}
\veebar_\Gamma(F) \setminus \barwedge_\Gamma(F) &= \veebar_\Gamma(F) \cap (Y \setminus \barwedge_\Gamma(F)) \\
&= \veebar_\Gamma(F) \cap [Y \setminus (F \setminus \Gamma(F))] \\
&= \veebar_\Gamma(F) \cap (Y \setminus [F \cap (Y \setminus \Gamma(F))]) \\
&= \veebar_\Gamma(F) \cap [(Y \setminus F) \cup \Gamma(F)] \\
&= [\veebar_\Gamma(F) \cap (Y \setminus F)] \cup (\veebar_\Gamma(F) \cap \Gamma(F)) \\
&= [\veebar_\Gamma(F) \cap (Y \setminus F)] \cup [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap \Gamma(F)] \\
&= [\veebar_\Gamma(F) \cap (Y \setminus F)] \cup [\Psi_\Gamma(F) \cap (\Psi_\Gamma(Y \setminus F) \cap \Gamma(F))] \\
&= [\veebar_\Gamma(F) \cap (Y \setminus F)] \cup (\Psi_\Gamma(F) \cap [(Y \setminus \Gamma(F)) \cap \Gamma(F)]) \\
&= [\veebar_\Gamma(F) \cap (Y \setminus F)] \cup (\Psi_\Gamma(F) \cap \emptyset) \\
&= \veebar_\Gamma(F) \cap (Y \setminus F) \\
&= \veebar_\Gamma(F) \setminus F
\end{aligned}$$

v.

$$\begin{aligned}
\barwedge_\Gamma(F) \setminus \veebar_\Gamma(F) &= \barwedge_\Gamma(F) \cap (Y \setminus \veebar_\Gamma(F)) \\
&= \barwedge_\Gamma(F) \cap [Y \setminus (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F))] \\
&= \barwedge_\Gamma(F) \cap [(Y \setminus \Psi_\Gamma(F)) \cup (Y \setminus \Psi_\Gamma(Y \setminus F))] \\
&= [\barwedge_\Gamma(F) \cap (Y \setminus \Psi_\Gamma(F))] \cup [\barwedge_\Gamma(F) \cap (Y \setminus \Psi_\Gamma(Y \setminus F))] \\
&= [(\barwedge_\Gamma(F) \setminus \Psi_\Gamma(F)) \cup ([F \cap (Y \setminus \Gamma(F))] \cap (Y \setminus \Psi_\Gamma(Y \setminus F))) \\
&= (\barwedge_\Gamma(F) \setminus \Psi_\Gamma(F)) \cup [(F \cap \Psi_\Gamma(Y \setminus F)) \cap (Y \setminus \Psi_\Gamma(Y \setminus F))] \\
&= (\barwedge_\Gamma(F) \setminus \Psi_\Gamma(F)) \cup (F \cap [\Psi_\Gamma(Y \setminus F) \cap (Y \setminus \Psi_\Gamma(Y \setminus F))]) \\
&= (\barwedge_\Gamma(F) \setminus \Psi_\Gamma(F)) \cup (F \cap \emptyset) \\
&= \barwedge_\Gamma(F) \setminus \Psi_\Gamma(F)
\end{aligned}$$

vi.

$$\begin{aligned}
\wedge_\Gamma(F) \cup \veebar_\Gamma(F) &= (\Psi_\Gamma(F) \setminus F) \cup (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \\
&= [\Psi_\Gamma(F) \cap (Y \setminus F)] \cup (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \\
&= \Psi_\Gamma(F) \cap [(Y \setminus F) \cup \Psi_\Gamma(Y \setminus F)] \\
&= \Psi_\Gamma(F) \cap [(Y \setminus F) \cup (Y \setminus \Gamma(F))] \\
&= \Psi_\Gamma(F) \cap [Y \setminus (F \cap \Gamma(F))] \\
&= \Psi_\Gamma(F) \setminus (F \cap \Gamma(F))
\end{aligned}$$

vii. By Theorem 4.2 (*iii.*),

$$\begin{aligned}
\veebar_\Gamma(F) \cup \barwedge_\Gamma(F) &= (\Psi_\Gamma(F) \setminus \Gamma(F)) \cup (F \setminus \Gamma(F)) \\
&= [\Psi_\Gamma(F) \cap (Y \setminus \Gamma(F))] \cup [F \cap (Y \setminus \Gamma(F))] \\
&= (\Psi_\Gamma(F) \cup F) \cap (Y \setminus \Gamma(F)) \\
&= (\Psi_\Gamma(F) \cup F) \setminus \Gamma(F)
\end{aligned}$$

□

Theorem 6.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $H \subseteq Y$. Then, the following are held.

i. $\overline{\wedge}_\Gamma(\underline{\vee}_\Gamma(H)) = Y \setminus \Gamma(Y)$

ii. $\wedge_\Gamma(\underline{\vee}_\Gamma(H)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$

iii. $\wedge_\Gamma(\underline{\vee}_\Gamma(H)) = \emptyset$ if $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$

iv. $\overline{\wedge}_\Gamma(\wedge_\Gamma(H)) = \wedge_\Gamma(H)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $H \subseteq Y$.

i. By Theorem 4.2 (*i.*),

$$\begin{aligned}
\overline{\wedge}_\Gamma(\underline{\vee}_\Gamma(H)) &= \overline{\wedge}_\Gamma(Y \setminus \Gamma(Y)) \\
&= (Y \setminus \Gamma(Y)) \setminus \Gamma(Y \setminus \Gamma(Y)) \\
&= (Y \setminus \Gamma(Y)) \cap (Y \setminus \Gamma(Y \setminus \Gamma(Y))) \\
&= Y \setminus (\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y)))
\end{aligned}$$

From Theorem 2.1 (*iii.*),

$$Y \setminus (\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y))) = Y \setminus \Gamma(Y \cup (Y \setminus \Gamma(Y)) = Y \setminus \Gamma(Y)$$

As a result, $\overline{\wedge}_\Gamma(\underline{\vee}_\Gamma(H)) = Y \setminus \Gamma(Y)$.

ii. By Theorem 4.2 (*i.*), $\wedge_\Gamma(\underline{\vee}_\Gamma(H)) = \wedge_\Gamma(Y \setminus \Gamma(Y))$. Then, by Theorem 3.5 (*iv.*),

$$\wedge_\Gamma(\underline{\vee}_\Gamma(H)) = \wedge_\Gamma(Y \setminus \Gamma(Y)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$$

iii. Let $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, from Theorem 2.6 and Theorem 6.2 (*ii.*),

$$\wedge_\Gamma(\underline{\vee}_\Gamma(H)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) = \Gamma(Y) \setminus \Gamma(Y) = \emptyset$$

iv. By Theorem 2.1 (*iii.*),

$$\begin{aligned}
\overline{\wedge}_\Gamma(\wedge_\Gamma(H)) &= \wedge_\Gamma(H) \setminus \Gamma(\wedge_\Gamma(H)) \\
&= (\Psi_\Gamma(H) \setminus H) \setminus \Gamma(\Psi_\Gamma(H) \setminus H) \\
&= [(Y \setminus \Gamma(Y \setminus H)) \cap (Y \setminus H)] \cap (Y \setminus \Gamma(\Psi_\Gamma(H) \setminus H)) \\
&= (Y \setminus H) \cap [(Y \setminus \Gamma(Y \setminus H)) \cap (Y \setminus \Gamma(\Psi_\Gamma(H) \setminus H))] \\
&= (Y \setminus H) \cap [Y \setminus (\Gamma(Y \setminus H) \cup \Gamma(\Psi_\Gamma(H) \setminus H))] \\
&= (Y \setminus H) \cap (Y \setminus \Gamma((Y \setminus H) \cup (\Psi_\Gamma(H) \setminus H)))
\end{aligned}$$

Then,

$$\begin{aligned}
\overline{\wedge}_\Gamma(\wedge_\Gamma(H)) &= (Y \setminus H) \cap (Y \setminus \Gamma((Y \setminus H) \cup (\Psi_\Gamma(H) \cap (Y \setminus H)))) \\
&= (Y \setminus H) \cap (Y \setminus \Gamma(Y \setminus H)) \\
&= (Y \setminus H) \cap \Psi_\Gamma(H) \\
&= \Psi_\Gamma(H) \setminus H \\
&= \wedge_\Gamma(H)
\end{aligned}$$

□

Remark 6.3. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Although $\wedge_\Gamma(\underline{\vee}_\Gamma(A)) = \emptyset$, $\mathrm{cl}(\tau) \cap \Im$ may not be equal to $\{\emptyset\}$.

Example 6.4. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{q\}, \{s\}, \{q, s\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = Y$, then $\wedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$ but $\mathrm{cl}(\tau) \cap \Im \neq \{\emptyset\}$.

Theorem 6.5. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\barwedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$ if and only if $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

$(\Rightarrow)$: Let $\barwedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$. Then,

$$\veebar_\Gamma(A) \setminus \Gamma(\veebar_\Gamma(A)) = (Y \setminus \Gamma(Y)) \setminus \Gamma(Y \setminus \Gamma(Y)) = \emptyset$$

from Theorem 4.2 (*i.*). Therefore,

$$\veebar_\Gamma(A) \setminus \Gamma(\veebar_\Gamma(A)) = Y \setminus (\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y))) = \emptyset$$

and thus $\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y)) = Y$. From Theorem 2.1 (*iii.*), $\Gamma(Y \cup (Y \setminus \Gamma(Y))) = \Gamma(Y) = Y$. By Theorem 2.6, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$.

$(\Leftarrow)$: The proof is obvious by Theorem 6.2 (*i.*) and Theorem 2.6. $\square$

Theorem 6.6. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\veebar_\Gamma(A) \subseteq \Psi_\Gamma(\wedge_\Gamma(A))$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. It is obvious that $\Psi_\Gamma(\emptyset) \subseteq \Psi_\Gamma(\wedge_\Gamma(A))$ by Theorem 2.1 (*vi.*). Since from Theorem 2.1 (*iv.*),

$$\Psi_\Gamma(\emptyset) = \Psi_\Gamma(A \cap (Y \setminus A)) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) = \veebar_\Gamma(A)$$

and thus $\veebar_\Gamma(A) \subseteq \Psi_\Gamma(\wedge_\Gamma(A))$. $\square$

Theorem 6.7. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, the following are held.

i. $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = \barwedge_\Gamma(Y \setminus \Gamma(Y)) = \veebar_\Gamma(A) = Y \setminus \Gamma(Y)$

ii. $\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A))) = \wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = \barwedge_\Gamma(Y \setminus \Gamma(Y))$ by Theorem 4.2 (*i.*). Moreover, $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = Y \setminus \Gamma(Y)$ by Theorem 6.2 (*i.*). As a result, from Theorem 4.2 (*i.*),

$$\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = \barwedge_\Gamma(Y \setminus \Gamma(Y)) = \veebar_\Gamma(A) = Y \setminus \Gamma(Y)$$

ii. By Theorem 6.2 (*iv.*), $\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \wedge_\Gamma(\veebar_\Gamma(A))$. Moreover, from Theorem 6.2 (*ii.*), $\wedge_\Gamma(\veebar_\Gamma(A)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$. Thus,

$$\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) \tag{6.1}$$

and

$$\wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) \tag{6.2}$$

from Theorem 6.2 (*ii.*). From Theorem 6.2 (*i.*),

$$\wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A)) = \wedge_\Gamma(Y \setminus \Gamma(Y)) = \Psi_\Gamma(Y \setminus \Gamma(Y)) \setminus (Y \setminus \Gamma(Y)) = (Y \setminus \Gamma(\Gamma(Y))) \cap \Gamma(Y) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) \tag{6.3}$$

Consequently, from (6.1)-(6.3),

$$\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A))) = \wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$$

$\square$

Corollary 6.8. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ if and only if each of the following is empty.

i. $\underline{\vee}_\Gamma(\wedge_\Gamma(\underline{\vee}_\Gamma(A)))$

ii. $\underline{\vee}_\Gamma(\underline{\vee}_\Gamma(\wedge_\Gamma(A)))$

iii. $\underline{\vee}_\Gamma(\overline{\wedge}_\Gamma(\wedge_\Gamma(A)))$

iv. $\underline{\vee}_\Gamma(\wedge_\Gamma(\overline{\wedge}_\Gamma(A)))$

v. $\overline{\wedge}_\Gamma(\underline{\vee}_\Gamma(\wedge_\Gamma(A)))$

vi. $\overline{\wedge}_\Gamma(Y \setminus \Gamma(Y))$

vii. $\underline{\vee}_\Gamma(A)$

The proof is obvious by Theorem 6.7 and Corollary 4.5.

Corollary 6.9. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$, then each of the following is empty.

i. $\overline{\wedge}_\Gamma(\wedge_\Gamma(\underline{\vee}_\Gamma(A)))$

ii. $\wedge_\Gamma(\underline{\vee}_\Gamma(\overline{\wedge}_\Gamma(A)))$

iii. $\wedge_\Gamma(\overline{\wedge}_\Gamma(\underline{\vee}_\Gamma(A)))$

iv. $\Gamma(Y) \setminus \Gamma(\Gamma(Y))$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space, $A \subseteq Y$, and $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, from Theorem 2.6, $Y = \Gamma(Y)$ and thus

$$\Gamma(Y) \setminus \Gamma(\Gamma(Y)) = Y \setminus \Gamma(Y) = \emptyset$$

Hence, the proof is obvious by Theorem 6.7 (*ii.*). □

7. Conclusion

In this study, new set operators were presented via Ψ_Γ-operator and Γ-local closure function in ideal topological spaces, and their behavioral properties were analyzed. It was investigated whether these set operators preserve some set operations. In future studies, different set operators can be presented, and their relations with these newly studied operators can be researched.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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New Conformable P-Type $(3+1)$-Dimensional Evolution Equation and its Analytical and Numerical Solutions

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Abstract — The paper examines the conformable nonlinear evolution equation in $(3+1)$-dimensions. First, basic definitions and characteristics for the conformable derivative are given. Then, the modified extended tanh-function and $\exp(-\phi(\xi))$-expansion techniques are utilized to determine the exact solutions to this problem. The consequences of some of the acquired data's physical 3D and 2D contour surfaces are used to demonstrate the findings, providing insight into how geometric patterns are physically interpreted. These solutions help illustrate how the studied model and other nonlinear representations in physical sciences might be used in real-world scenarios. It is clear that these methods have the capacity to solve a large number of fractional differential equations with beneficial outcomes.

Keywords $(3+1)$*-dimensional evolution equation, modified extended tanh-function method,* $\exp(-\phi(\xi))$*-expansion method, residual power series method, conformable derivative*

Mathematics Subject Classification (2020) 35R11, 65J15

1. Introduction

In several fields of the social and fundamental sciences, as well as engineering, fractional differential equations are encountered. Their significance in several disciplines requiring complex physical processes, from electrical circuits and control theory to wave propagation, has earned more attention in recent years. Many engineering issues are modeled and designed using them. Solutions to these equations have been helpful since they highlight nonlinear physical properties more clearly and provide a path for further study. In mathematical physics, nonlinear wave equations play a role in several fields, notably chemical kinetics, solid-state physics, optical fibers, fluid mechanics, and plasma physics.

A particular kind of partial differential equation that depicts how a system changes over time is called an evolution partial differential equation(PDE). PDEs are equations involving functions and their partial derivatives concerning several independent variables in mathematics and science. Time is one of these factors that evolution PDEs particularly include, so they represent how a system changes or evolves. Dynamic processes are frequently modeled using evolution PDEs in physics, engineering, biology, and economics, among other disciplines. The wave equation, the heat equation, and the Schrödinger equation in quantum mechanics are a few examples of the evolution of PDEs. These equations, essential to comprehend physical systems' behavior, explain how variables like temperature, displacement, or wave function change over time and space.

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Much focus has been placed on the nonlinear evolution equations recently. It is becoming more and more attractive to look for PDE solutions directly. In applied research and sciences, the mathematical modeling of physical occurrences is an essential tool for analysis. Various mathematical methods have been used to search for solutions and an advanced knowledge of these equations. Many analytical and numerical methods have been used to solve these equations and gain an excellent grasp of them. A few of these analytical methods are: Unified Ansätze Method [1] for the optical solitons and traveling wave solutions to Kudryashov's equation, Sub-equation Method [2] for the generalized Benjamin, modified generalized multidimensional Kadomtsev–Petviashvili, modified generalized multidimensional Kadomtsev–Petviashvili–Benjamin–Bona–Mahony, and the variant Boussinesq system of equations, Sardar Sub-equation Method [3] for the Korteweg–de Vries–Zakharov–Kuznetsov equation, Jacobi Elliptic Function Expansion Method [4] for the Korteweg–de Vries, Boussinesq, Klein–Gordon, and variant Boussinesq equations, Exp-function Method [5] for the generalized shallow water-like equation, Boiti–Leon–Manna–Pempinelli, generalized variable-coefficient B-type Kadomtsev–Petviashvili, and Caudrey-Dodd-Gibbon-Kotera-Sawada equations, Extended sinh-Gordon Equation Expansion Method [6] for the Kundu–Eckhaus equation, Modified Kudryashov Method [7] for the Kuramoto–Sivashinsky and seventh-order Sawada–Kotera equations, Modified Exponential Function Method [8] for the modified Benjamin-Bona-Mahony and Sharma-Tasso-Olver equations, (G'/G)-expansion Method [9] for the the higher order Broer–Kaup, breaking soliton, and asymmetric Nizhnik–Novikov–Vesselov equations, Modified Simple Equation Method [10] for the Kaup-Newell equation, the Extended Trial Equation Method [11] for the $B(n+1,1,n)$ equation, and the Variational Direct Method [12] for the complex Ginzburg–Landau equation.

Scientists became deeply interested in inventing fractional models and discovering approximations to the generated problems. Scientists also place extensive attention on the creation and use of different methods to get these solutions. Multiple methods, frequently discovered in literature, are used to find numerical solutions for FDEs. These include the Residual Power Series Method(RPSM) [13], Homotopy Analysis Method [14], Homotopy-Perturbation, and Variational Iteration Methods [15]. It has become clear that no single method can strictly and universally solve every nonlinear problem. Many techniques were created as the result of this insight, such as Modified Extended tanh-function Method [16] and $\exp(-\phi(\xi))$-expansion Method [17].

For fractional differential equations, multiple definitions of derivative have been put upward, such as the Riemann-Liouville [18], Caputo [19] and conformable derivatives [20]. The Riemann-Liouville and Caputo fractional derivatives are notable for often used in modern mathematical discourse. Similarly, the conformable fractional derivative technique is prominent due to its dependability and ease of use.

Recently, Mohan et al. [21] has presented a new $(3+1)$-dimensional P-type evolution equation as

$$u_{xxxy} + \alpha_1 u_{yt} + \alpha_2 (uux)_y + \alpha_3 u_{xx} + \alpha_4 u_{zz} = 0 \tag{1.1}$$

The authors in this work present the Painlevé integrability analysis of the model. Using Cole-Hopf transformation and symbolic computation, they obtain the rogue waves up to the third order. Finally, they introduce dispersive-soliton solutions to this equation.

In this paper, we address some new analytical and numerical solutions of the model that do not exist in the literature. The structure of the paper is as follows: Section 2 provides some basic definitions to be needed for the following sections. Section 3 details the modified, extended tanh-function method. Section 4 describes $\exp(-\phi(\xi))$-expansion method in detail. Section 5 presents the approximation approach known as the residual power series method(RPSM). Section 6 contains analytical and numerical solutions to the underlying equation. Section 7 discusses the need for further research.

2. Preliminaries

This section provides some basic notions to be needed for the following sections.

Definition 2.1. [24] The following defines the conformable derivative of a function of order ω, $j:[0,\infty)\to\mathbb{R}$, $t>0$, $\omega\in(0,1)$,

$$\mathscr{D}_t^\omega(j)(t)=\lim_{\delta\to 0}\frac{j\left(t+\delta t^{1-\omega}\right)-j(t)}{\delta}$$

In addition, if $\lim_{t\to 0^+}\mathscr{D}_t^\omega(j)(t)$ exists and j is ω-differentiable in the range $(0,k)$ for $k>0$, the definition becomes

$$\mathscr{D}_t^\omega(j)(0)=\lim_{t\to 0^+}\mathscr{D}_t^\omega(j)(t)$$

Lemma 2.2. [22–24] For $0<\omega\leq 1$, let j_1 and j_2 be ω-differentiable at $t>0$. Then,

i. $\mathscr{D}_t^\omega(t^{p_1})=p_1t^{p_1-\omega}$, $p_1\in\mathbb{R}$

ii. $\mathscr{D}_t^\omega(p_1j_1+p_2j_2)=p_1\mathscr{D}_t^\omega(j_1)+p_2\mathscr{D}_t^\omega(j_2)$, $p_1,p_2\in\mathbb{R}$

iii. $\mathscr{D}_t^\omega(\frac{j_1}{j_2})=\frac{j_2.\mathscr{D}_t^\omega(j_1)-j_1\mathscr{T}_t^\omega(j_2)}{j_2^2}$

iv. $\mathscr{D}_t^\omega(j_1.j_2)=j_1\mathscr{D}_t^\omega(j_2)+j_2\mathscr{D}_t^\omega(j_1)$

v. $\mathscr{D}_t^\omega(j_1)(t)=t^{1-\omega}\frac{dj_1(t)}{dt}$

vi. $\mathscr{D}_t^\omega(S)=0$, if S is a constant

Definition 2.3. [25] Let $j(y_1,y_2,\cdots,y_n)$ be the function with n variables. Following is the partial derivatives of j in y_i of order $\omega\in(0,1]$.

$$\frac{d^\omega}{dy_i^\omega}j(y_1,y_2,\cdots,y_n)=\lim_{\delta\to 0}\frac{j(y_1,y_2,\cdots,y_{i-1},y_i+\delta y_i^{1-\omega},y_n)-j(y_1,y_2,\cdots,y_n)}{\delta}$$

The following sections will introduce modified extended tanh-function, $\exp(-\phi(\xi))$-expansion, and RPS methods.

3. Modified Extented tanh-function Method

The primary stages of the modified extended tanh-function method [25–27] are explained in this section as follows. Suppose we have a nonlinear evolution equation of the type

$$\beta\left(u,u_t,u_x,u_y,u_{xx},u_{yy},\cdots\right)=0 \tag{3.1}$$

where β is a polynomial in $u(x,y,\cdots,t)$ and nonlinear components are found in its partial derivatives. Utilizing the transformation,

$$u(x,y,\cdots,t)=u(\xi),\ \xi=kx+wy+\cdots+\frac{ct^\omega}{\omega}$$

will turn (3.1) to an ODE as

$$\beta\left(u(\xi),u'(\xi),u''(\xi),\cdots\right)=0 \tag{3.2}$$

Suppose that the form of the solution of (3.1),

$$u(\xi)=A_0+\sum_{m=1}^{N}\left(A_m\phi^m(\xi)+B_m\phi^{-m}(\xi)\right),\ m\in\{0,1,2,\cdots,N\} \tag{3.3}$$

where $A_N\neq 0, B_N\neq 0$, and A_m and B_m are constants that have to be found and $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi)=\sigma+\phi(\xi)^2 \tag{3.4}$$

In this case, σ is an unknown parameter. Numerous solutions can be found for (3.4), as illustrated below

i. If $\sigma < 0$, then

$$\phi(\xi) = -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right) \text{ or } \phi(\xi) = -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right)$$

ii. If $\sigma > 0$, then

$$\phi(\xi) = \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right) \text{ or } \phi(\xi) = -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right)$$

iii. If $\sigma = 0$, then

$$\phi(\xi) = -\frac{1}{\xi}$$

In (3.3), the positive integer N is obtained by balancing the biggest nonlinear variable and the highest-order derivatives.

By replacing (3.3), its derivative, and (3.4) into (3.2), as well as collecting all the terms of the same power ϕ^m, $(m \in \{0, 1, 2, \cdots, N\})$ and equating them to zero, one can use a symbolic computation tool to determine the values of A_m and B_m. By entering these values and the solutions to (3.4) into (3.3), we can obtain the exact solutions to (3.1).

4. exp$(-\phi(\xi))$-expansion Method

Examine the nonlinear evolution equation presented in the following manner

$$\mathscr{D}(u, \mathscr{D}_t^\omega, \mathscr{D}_x u, \mathscr{D}_y u, \mathscr{D}_x^2 u, \mathscr{D}_y^2 u, \cdots) = 0 \tag{4.1}$$

The arbitrary order conformable derivative operator is represented by $\mathscr{D}_t^\omega$ in this case. $u = u(x, y, \cdots, t)$ is an unknown function, and the subscripts stand for partial derivatives. When using $\exp(-\phi(\xi))$-expansion method [28–30] in order to obtain wave solutions for (4.1), the following steps must be carried out.

i. The real variables $x, y, z, \cdots, t$ are combined using a compound variable named ξ as

$$\xi = kx + wy + \cdots + \frac{ct^\omega}{\omega}, \quad u(x, y, z, \cdots, t) = u(\xi)$$

ii. The next ordinary differential equation may be obtained by reducing (4.1)

$$\mathscr{G}(u(\xi), u'(\xi), u''(\xi), \cdots) = 0 \tag{4.2}$$

iii. As the following finite series, the exact solutions may be constructed:

$$u(\xi) = \sum_{r=0}^{N} B_r(\exp(\xi(-\phi)))^r, \quad B_N \neq 0, 0 \leq r \leq N \tag{4.3}$$

iv. The following ordinary differential equation can be solved for $\phi = \phi(\xi)$

$$\phi'(\xi) = \exp(-\phi(\xi)) + \eta\exp(\phi(\xi)) + \lambda \tag{4.4}$$

v. The following are the possible solutions to (4.4) for $\lambda^2 - 4\eta > 0$ and $\eta \neq 0$, depending on the pertinent parameters.

$$u_1(\xi) = \frac{\ln\left(-\sqrt{(\lambda^2 - 4\eta)}\tanh\left(\frac{\sqrt{(\lambda^2-4\eta)}}{2}(\xi + h)\right) - \lambda\right)}{2\eta}$$

when $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$ are present,

$$u_2(\xi) = \frac{\ln\left(\sqrt{(4\eta - \lambda^2)}\tanh\left(\frac{\sqrt{(4\eta-\lambda^2)}}{2}(\xi + h)\right) - \lambda\right)}{2\eta}$$

when $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$, and $\eta = 0$ are present,

$$u_3(\xi) = -\ln\left(\frac{\lambda}{\sinh(\lambda(\xi + h)) + \cosh(\lambda(\xi + h)) - 1}\right)$$

when $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$, and $\eta \neq 0$ are present,

$$u_4(\xi) = \ln\left(-\frac{2(\lambda(\xi + h) + 2)}{\lambda^2(h + \xi)}\right)$$

when $\lambda^2 - 4\eta = 0$, $\lambda = 0$, and $\eta = 0$ are present,

$$u_5(\xi) = \ln(\xi + h)$$

in which h serves as the integration constant.

vi. The positive integer N is determined by considering the homogeneous balance between the highest order derivatives of $u(\xi)$ as given in (4.2) and the biggest nonlinear term. When (4.2) is replaced by (4.3) along with (4.4), and terms with the same powers of $\exp(-\phi)$ are combined, the left side of (4.2) becomes a polynomial. A series of algebraic equations in terms of $B_r (r \in \{0, 1, 2, 3, \cdots, N\})$, c, λ, and η are produced. We get solutions for (4.2) by equating all of this polynomial's coefficients to zero, solving the ensuing system of algebraic equations, and then substituting the solutions back into (4.3).

5. Residual Power Series Method(RPSM)

Examine the following nonlinear fractional differential equation to illustrate the basis of the RPS method [31, 32].

$$h(x, y, z, t) = \mathscr{D}_\omega u(x, y, z, t) + R[x, y, z]u(x, y, z, t) + N[x, y, z]u(x, y, z, t) \tag{5.1}$$

The initial condition is

$$u(x, y, z, 0) = f_0(x, y, z) = f(x, y, z) \tag{5.2}$$

$R[x, y, z]$ is a linear operators and $N[x, y, z]$ is a nonlinear operators. The RPS method requires expanding the unknown function to a fractional series at $t = 0$ to find the approximate solutions to (5.1), subject to (5.2).

Thus, the solution may be represented as follows using a series expansion

$$u(x, y, z, t) = \sum_{n=0}^{\infty} f_n(x, y, z)\frac{t^{n\omega}}{\omega^n n!}$$

Consequently, for $0 \leq t < \mathbb{R}^{\frac{1}{v}}$ and $0 < \omega \leq 1$, the k-th series of $u(x, y, z, t)$, or $u_k(x, y, z, t)$, is determined to be as follows

$$u_k(x, y, z, t) = f(x, y, z) + \sum_{n=1}^{k} f_n(x, y, z)\frac{t^{n\omega}}{\omega^n n!}, \quad k \in \{1, 2, 3, \cdots\} \tag{5.3}$$

Then, we express the residual function and the coefficient k-th residual function as

$$\text{Res}u_k(x, y, z, t) = \mathscr{D}_\omega u_k(x, y, z, t) + R[x, y, z]u_k(x, y, z, t) + N[x, y, z]u_k(x, y, z, t) - h(x, y, z, t) \tag{5.4}$$

where $k \in \{1, 2, 3, \cdots\}$. For $\text{Res}u(x, y, z, t) = 0$ and $\lim_{k\to\infty} \text{Res}u_k(x, y, z, t) = \text{Res}u(x, y, z, t)$, it is obvious that $t \geq 0$.

Calculating out $\text{Res}u_1(x,y,z,0) = 0$, yields the first unknown function, $f_1(x,y,z)$. The fractional derivative of a constant is 0 in the conformable sense, hence $\mathscr{D}_t^{(n-1)\omega}\text{Res}u_k(x,y,z,t) = 0$ relative to $n \in \{1,2,3,\cdots,k\}$. The desired $f_n(x,y,z)$ coefficients are obtained by solving this equation for $t = 0$. Thus, $u_n(x,y,z,t)$ solutions may be determined, respectively.

6. Solutions for the Equation

Examine the conformable version of (1.1) in the specific situation provided as follows for the next two analytical methods

$$u_{xxxy} + \mathscr{D}_t^{\omega}\alpha_1 u_y + \alpha_2(uux)_y + \alpha_3 u_{xx} + \alpha_4 u_{zz} = 0 \tag{6.1}$$

A conformable fractional derivative is a mathematical concept that extends the notion of classical derivatives to non-integer orders in a more flexible and generalized manner. It is a relatively recent development in the field of fractional calculus [24]. Compared to classical fractional derivatives, conformable fractional derivative has two essential advantages. First, most of the properties of the classical derivative, including linearity, quotient rule, product rule, power rule, and chain rule, are satisfied by the conformable fractional derivative. Second, differential equations with a conformable fractional derivative are more straightforward to solve numerically than those involving the Riemann-Liouville or Caputo fractional derivatives, making it very convenient to model many physical problems.

After doing the transformation as $u(x,y,z,t) = u(\xi)$ with $\xi = kx + wy + sz + \frac{ct^{\omega}}{\omega}$, the following ODE is obtained

$$u(\xi) = \left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + k^3 w u''(\xi) + \frac{1}{2}\alpha_2 kwu(\xi)^2$$

By balancing, $u^2 = 2N$, $u'' = N + 2$, and $N = 2$ is calculated. The exact solutions are obtained by substituting them into (3.3) and (4.3).

6.1. Modified Extended tanh-function Method Solutions

For $N = 2$, (3.3) takes the following form,

$$u = A_0 + A_1\phi(\xi) + B_1\phi(\xi)^{-1} + A_2\phi(\xi)^2 + B_2\phi(\xi)^{-2}$$

When combined with (3.4), the following algebraic equation system is created.

$$\alpha_2 A_1 B_1 kw + \alpha_2 A_2 B_2 kw + A_0\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 2A_2 k^3\sigma^2 w + \frac{1}{2}\alpha_2 A_0^2 kw + 2B_2 k^3 w = 0$$

$$A_2\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 8A_2 k^3\sigma w + \frac{1}{2}\alpha_2 A_1^2 kw + \alpha_2 A_0 A_2 kw = 0$$

$$2A_1 k^3 w + \alpha_2 A_1 A_2 kw = 0$$

$$6A_2 k^3 w + \frac{1}{2}\alpha_2 A_2^2 kw = 0$$

$$\alpha_2 A_2 B_1 kw + A_1\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 2A_1 k^3\sigma w + \alpha_2 A_0 A_1 kw = 0$$

$$\alpha_2 A_0 B_2 kw + B_2\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 8B_2 k^3\sigma w + \frac{1}{2}\alpha_2 B_1^2 kw = 0$$

$$\alpha_2 A_0 B_1 kw + \alpha_2 A_1 B_2 kw + B_1\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 2B_1 k^3\sigma w = 0$$

$$2B_1 k^3\sigma^2 w + \alpha_2 B_1 B_2 kw = 0$$

and

$$6B_2 k^3\sigma^2 w + \frac{1}{2}\alpha_2 B_2^2 kw = 0$$

For A_0, A_1, A_2, B_1, B_2, and c, we have six cases and six sets of solutions

Case 1.

$$A_0{=}-\frac{24k^2\sigma}{\alpha_2},\quad A_1{=}0,\quad A_2{=}-\frac{12k^2}{\alpha_2},\quad B_1{=}0,\quad B_2{=}-\frac{12k^2\sigma^2}{\alpha_2},\quad \text{and}\quad c{=}-\frac{k^2\left(\alpha_3-16k\sigma w\right)+\alpha_4 s^2}{\alpha_1 w}$$

Set 1.

For $\sigma<0$,

$$\begin{aligned}u_1(x,y,z,t)\;=&\;-\frac{24k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\&+\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\end{aligned}\tag{6.2}$$

or

$$\begin{aligned}u_2(x,y,z,t)\;=&\;-\frac{24k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\&+\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\end{aligned}$$

For $\sigma>0$,

$$\begin{aligned}u_3(x,y,z,t)\;=&\;-\frac{24k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\&-\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\end{aligned}$$

or

$$\begin{aligned}u_4(x,y,z,t)\;=&\;-\frac{24k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\&-\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\end{aligned}$$

For $\sigma=0$,

$$u_5(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(-\frac{t^\omega(\alpha_3k^2+\alpha_4s^2)}{\alpha_1w\omega}+kx+sz+wy\right)^2}$$

Case 2.

$$A_0{=}-\frac{12k^2\sigma}{\alpha_2},\quad A_1{=}0,\quad A_2{=}0,\quad B_1{=}0,\quad B_2{=}-\frac{12k^2\sigma^2}{\alpha_2},\quad \text{and}\quad c{=}-\frac{k^2\left(\alpha_3-4k\sigma w\right)+\alpha_4 s^2}{\alpha_1 w}$$

Set 2.

For $\sigma<0$,

$$u_6(x,y,z,t)=\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-4k\sigma w)+\alpha_4s^2\right)}{\alpha_1w\omega}+kx+sz+wy\right)\right)}{\alpha_2}-\frac{12k^2\sigma}{\alpha_2}$$

or

$$u_7(x,y,z,t) = \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{12k^2\sigma}{\alpha_2} \tag{6.3}$$

For $\sigma > 0$,

$$u_8(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \cot^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

or

$$u_9(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \tan^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{10}(x,y,z,t) = 0$$

which is a trivial solution.

Case 3.

$$A_0 = -\frac{12k^2\sigma}{\alpha_2}, \quad A_1 = 0, \quad A_2 = -\frac{12k^2}{\alpha_2}, \quad B_1 = 0, \quad B_2 = 0, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 - 4k\sigma w\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 3.

For $\sigma < 0$,

$$u_{11}(x,y,z,t) = \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{12k^2\sigma}{\alpha_2}$$

or

$$u_{12}(x,y,z,t) = \frac{12k^2\sigma \coth^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{12k^2\sigma}{\alpha_2}$$

For $\sigma > 0$,

$$u_{13}(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \tan^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

or

$$u_{14}(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \cot^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{15}(x,y,z,t) = -\frac{12k^2}{\alpha_2\left(-\frac{t^{\omega}(\alpha_3 k^2+\alpha_4 s^2)}{\alpha_1 w\omega} + kx + sz + wy\right)^2}$$

Case 4.

$$A_0 = -\frac{4k^2\sigma}{\alpha_2}, \quad A_1 = 0, \quad A_2 = 0, \quad B_1 = 0, \quad B_2 = -\frac{12k^2\sigma^2}{\alpha_2}, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 + 4k\sigma w\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 4.

For $\sigma < 0$,

$$u_{16}(x,y,z,t) = \frac{12k^2\sigma \coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

or

$$u_{17}(x,y,z,t) = \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

For $\sigma > 0$,

$$u_{18}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

or

$$u_{19}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{20}(x,y,z,t) = 0$$

which is a trivial solution.

Case 5.

$$A_0 = -\frac{4k^2\sigma}{\alpha_2}, \quad A_1 = 0, \quad A_2 = -\frac{12k^2}{\alpha_2}, \quad B_1 = 0, \quad B_2 = 0, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 + 4k\sigma w\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 5.

For $\sigma < 0$,

$$u_{21}(x,y,z,t) = \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

or

$$u_{22}(x,y,z,t) = \frac{12k^2\sigma \coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

For $\sigma > 0$,

$$u_{23}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

or

$$u_{24}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{25}(x,y,z,t) = -\frac{12k^2}{\alpha_2\left(-\frac{t^\omega(\alpha_3 k^2+\alpha_4 s^2)}{\alpha_1 w\omega} + kx + sz + wy\right)^2}$$

Case 6.

$$A_0=\frac{8k^2\sigma}{\alpha_2}, \quad A_1=0, \quad A_2=-\frac{12k^2}{\alpha_2}, \quad B_1=0, \quad B_2=-\frac{12k^2\sigma^2}{\alpha_2}, \quad \text{and} \quad c=-\frac{k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2}{\alpha_1 w}$$

Set 6.

For $\sigma<0$,

$$u_{26}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$+\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

or

$$u_{27}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$+\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

For $\sigma>0$,

$$u_{28}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$-\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

or

$$u_{29}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$-\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

For $\sigma=0$,

$$u_{30}(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(-\frac{t^\omega(\alpha_3 k^2+\alpha_4 s^2)}{\alpha_1 w\omega}+kx+sz+wy\right)^2}$$

6.2. exp $(-\phi(\xi))$-expansion Method Solutions

Considering that $N=2$, (4.3) is as follows:

$$u=B_0+B_1\exp(-\phi(\xi))+B_2\exp(-\phi(\xi))^2$$

The algebraic system of equations next develops when (4.4) is included

$$\alpha_1 B_0 cw+2B_2\eta^2k^3w+B_1\eta\lambda k^3w+\alpha_3 B_0k^2+\frac{1}{2}\alpha_2 B_0^2kw+\alpha_4 B_0 s^2=0$$

$$\alpha_1 B_1 cw + 6B_2\eta\lambda k^3 w + 2B_1\eta k^3 w + B_1\lambda^2 k^3 w + \alpha_3 B_1 k^2 + \alpha_2 B_0 B_1 kw + \alpha_4 B_1 s^2 = 0$$

$$\alpha_1 B_2 cw + 8B_2\eta k^3 w + 4B_2\lambda^2 k^3 w + 3B_1\lambda k^3 w + \alpha_3 B_2 k^2 + \frac{1}{2}\alpha_2 B_1^2 kw + \alpha_2 B_0 B_2 kw + \alpha_4 B_2 s^2 = 0$$

$$10B_2\lambda k^3 w + 2B_1 k^3 w + \alpha_2 B_1 B_2 kw = 0$$

and

$$6B_2 k^3 w + \frac{1}{2}\alpha_2 B_2^2 kw = 0$$

Case 7.

$$B_0 = -\frac{12\eta k^2}{\alpha_2}, \quad B_1 = -\frac{12k^2\lambda}{\alpha_2}, \quad B_2 = -\frac{12k^2}{\alpha_2}, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 + kw\left(\lambda^2 - 4\eta\right)\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 7.

For $\lambda^2 - 4\eta > 0$, $\eta \neq 0$,

$$\begin{aligned} v_1(x,y,z,t) = &-\frac{48\eta^2 k^2}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G - \frac{t^\omega\left(k^2(\alpha_3+kw(\lambda^2-4\eta))+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)^2} \\ &-\frac{24\eta k^2\lambda}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G - \frac{t^\omega\left(k^2(\alpha_3+kw(\lambda^2-4\eta))+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)} \\ &-\frac{12\eta k^2}{\alpha_2} \end{aligned} \tag{6.4}$$

For $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$,

$$\begin{aligned} v_2(x,y,z,t) = &-\frac{48\eta^2 k^2}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G - \frac{t^\omega\left(k^2(\alpha_3+kw(\lambda^2-4\eta))+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)^2} \\ &-\frac{24\eta k^2\lambda}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G - \frac{t^\omega\left(k^2(\alpha_3+kw(\lambda^2-4\eta))+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)} \\ &-\frac{12\eta k^2}{\alpha_2} \end{aligned}$$

For $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$\begin{aligned} v_3(x,y,t,z) = &-\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G - \frac{t^\omega\left(k^2(\alpha_3+k\lambda^2 w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)+\cosh\left(\lambda\left(G - \frac{t^\omega\left(k^2(\alpha_3+k\lambda^2 w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-1\right)} \\ &-\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G - \frac{t^\omega\left(k^2(\alpha_3+k\lambda^2 w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)+\cosh\left(\lambda\left(G - \frac{t^\omega\left(k^2(\alpha_3+k\lambda^2 w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-1\right)^2} \end{aligned}$$

For $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$ and $\eta \neq 0$,

$$v_4(x,y,z,t) = -\frac{3k^2\lambda^4\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)^2}{\alpha_2\left(\lambda\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)+2\right)^2} + \frac{6k^2\lambda^3\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)}{\alpha_2\left(\lambda\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)+2\right)} - \frac{12\eta k^2}{\alpha_2}$$

For $\lambda^2-4\eta=0,\ \lambda=0$ and $\eta=0$,

$$v_5(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(G-\frac{t^\omega(k^2(\alpha_3+4\eta kw)+\alpha_4s^2)}{\alpha_1w\omega}\right)^2}$$

Case 8.

$$B_0=-\frac{2k^2\left(2\eta+\lambda^2\right)}{\alpha_2},\quad B_1=-\frac{12k^2\lambda}{\alpha_2},\quad B_2=-\frac{12k^2}{\alpha_2},\quad \text{and}\quad c=\frac{k^3w\left(\lambda^2-4\eta\right)-\alpha_3k^2-\alpha_4s^2}{\alpha_1w}$$

Set 8.

For $\lambda^2-4\eta>0$, $\eta\neq0$,

$$\begin{aligned}v_6(x,y,z,t)=&-\frac{48\eta^2k^2}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)^2}\\&-\frac{24\eta k^2\lambda}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)}\\&-\frac{2k^2\left(2\eta+\lambda^2\right)}{\alpha_2}\end{aligned}$$

For $\lambda^2-4\eta<0$ and $\eta\neq0$,

$$\begin{aligned}v_7(x,y,z,t)=&-\frac{48\eta^2k^2}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)^2}\\&-\frac{24\eta\lambda k^2}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)}\\&-\frac{2k^2\left(2\eta+\lambda^2\right)}{\alpha_2}\end{aligned}$$

For $\lambda^2-4\eta>0,\ \lambda\neq0$ and $\eta=0$,

$$\begin{aligned}v_8(x,y,z,t)=&-\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)+\cosh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-1\right)}\\&-\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)+\cosh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-1\right)^2}\\&-\frac{2k^2\lambda^2}{\alpha_2}\end{aligned}$$

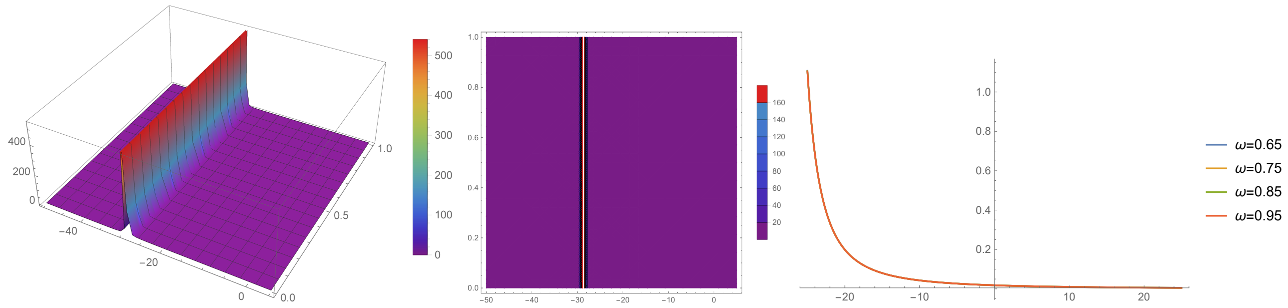
For $\lambda^2-4\eta=0,\ \lambda\neq0$ and $\eta\neq0$,

$$v_9(x,y,z,t)=-\frac{3k^2\lambda^4\left(G+\frac{t^\omega\left(\alpha_3\left(-k^2\right)-\alpha_4s^2\right)}{\alpha_1w\omega}\right)^2}{\alpha_2\left(\lambda\left(G+\frac{t^\omega(\alpha_3(-k^2)-\alpha_4s^2)}{\alpha_1w\omega}\right)+2\right)^2}+\frac{6k^2\lambda^3\left(G+\frac{t^\omega\left(\alpha_3\left(-k^2\right)-\alpha_4s^2\right)}{\alpha_1w\omega}\right)}{\alpha_2\left(\lambda\left(G+\frac{t^\omega(\alpha_3(-k^2)-\alpha_4s^2)}{\alpha_1w\omega}\right)+2\right)}-\frac{12\eta k^2}{\alpha_2}$$

For $\lambda^2-4\eta=0,\ \lambda=0$, and $\eta=0$,

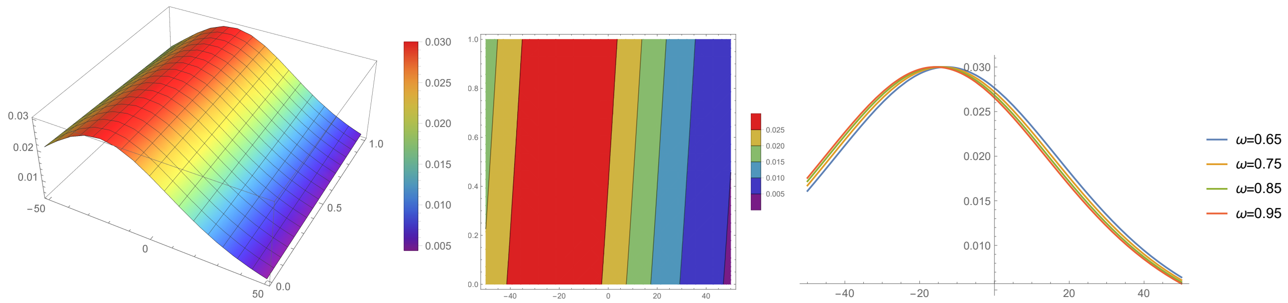
$$v_{10}(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(G+\frac{t^\omega(4\eta k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)^2}-\frac{8\eta k^2}{\alpha_2}$$

where, $G=h+kx+sz+wy$.



(a) **(b)** **(c)**

Figure 1. Tanh-function method solution $u_1(x,y,z,t)$ of (6.2) in three dimensions(a), contour(b), and two dimensions(c)



(a) **(b)** **(c)**

Figure 2. (a) 3D,(b) contour and (c) 2D plots of $\exp(-\phi(\xi))$-expansion method solution $v_1(x,y,z,t)$ of (6.4)

6.3. RPSM Solutions

First, consider an initial condition for $t=0$, using any of the previously obtained exact solutions. If (6.3) is taken as the exact solution, the initial condition is becomes

$$u_7(x,y,z,0)=\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}-\frac{12k^2\sigma}{\alpha_2}$$

The RPSM solution takes the form of (5.3) for the approximate solutions to the $(3+1)$-dimensional P-type evolution (6.1), where $u=u(x,y,z,t)$ and $t\geq 0,\;\; 0<\omega\leq 1$ the generic form of the $k-th$ residual function of the time-fractional equation may be shown using (5.4) as follows:

$$\mathrm{Res}u_k(x,y,z,t)=u_{xxxy}+\mathscr{D}_t^{\omega}\alpha_1 u_y+\alpha_2(uux)_y+\alpha_3 u_{xx}+\alpha_4 u_{zz}=0$$

It is required to determine $f_1(x,y,z)$ for a known $f(x,y,z)$ function in order to establish $\mathrm{Res}u_1(x,y,z,t)$. In considering it, $\mathrm{Res}u_1(x,y,z,t)$ is obtained as

$$\begin{aligned}\mathrm{Res}u_1(x,y,z,t) &= \alpha_1\left(f_1\right)_y+\alpha_2\left(\left(\frac{(f_1)_x t^{\omega}}{\omega}+(f)_x\right)\left(\frac{(f_1)_y t^{\omega}}{\omega}+(f)_y\right)+\left(\frac{(f_1)\, t^{\omega}}{\omega}+f\right)\left(\frac{(f_1)_{xy} t^{\omega}}{\omega}+(f)_{xy}\right)\right)\\ &\quad+\alpha_3\left(\frac{(f_1)_{xx} t^{\omega}}{\omega}+(f)_{xx}\right)+\left(\frac{(f_1)_{xxxy} t^{\omega}}{\omega}\right)+\alpha_4\left(\frac{(f_1)_{zz} t^{\omega}}{\omega}+(f)_{zz}\right)+(f)_{xxxy}\end{aligned}$$

when $f_1=f_1(x,y,z)$ and $f=f(x,y,z)$ occur. Thus, the first unknown coefficient is calculated when $t=0$.

$$f_1=-\frac{24k^2\sigma^2\left(4k^3\sigma w-\alpha_3k^2-\alpha_4 s^2\right)\tanh\left(\sqrt{-\sigma}(kx+sz+wy)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w}$$

Hence, the first approximate RPSM solution $u_1 = u_1(x, y, z, t)$ is subsequently obtained as

$$
\begin{aligned}
u_1 = & -\frac{12k^2\sigma}{\alpha_2} + \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_2} \\
& -\frac{24k^2\sigma^2 t^\omega \left(4k^3\sigma w - \alpha_3 k^2 - \alpha_4 s^2\right)\tanh\left(\sqrt{-\sigma}(kx+sz+wy)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega}
\end{aligned}
$$

Likewise, to get the second unknown parameter, the second residual function is established as

$$
\begin{aligned}
\mathrm{Res}u_2 = & \frac{(f_2)_{xxxy}\, t^{2\omega}}{2\omega^2} + \alpha_1 t^{1-\omega}\left((f_1)_y\, t^{\omega-1} + \frac{(f_2)_y\, t^{2\omega-1}}{\omega}\right) + \frac{(f_1)_{xxxy}\, t^\omega}{\omega} \\
& +\alpha_2\left(\left(\frac{(f_2)_x\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_x\, t^\omega}{\omega} + (f)_x\right)\left(\frac{(f_2)_y\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_y\, t^\omega}{\omega} + (f)_y\right)\right) \\
& +\alpha_3\left(\frac{(f_2)_{xx}\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_{xx}\, t^\omega}{\omega} + (f)_{xx}\right) \\
& +\alpha_2\left(\frac{(f_2)\, t^{2\omega}}{2\omega^2} + \frac{(f_1)\, t^\omega}{\omega} + f\right)\left(\frac{(f_2)_{xy}\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_{xy}\, t^\omega}{\omega} + (f)_{xy}\right) \\
& +\alpha_4\left(\frac{(f_2)_{zz}\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_{zz}\, t^\omega}{\omega} + (f)_{zz}\right) + (f)_{xxxy}
\end{aligned}
$$

where $f_2(x, y, z) = f_2$. Taking the first order derivative, we can get the second unknown parameter for $t = 0$ as follows:

$$
f_2 = \frac{24\sigma^2\left(-4k^4\sigma w + \alpha_3 k^3 + \alpha_4 k s^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+sz+wy)\right) - 2\right)\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1^2\alpha_2 w^2}
$$

As a result, the second approximation of $u_2 = u_2(x, y, z, t)$ becomes

$$
\begin{aligned}
u_2 = & -\frac{12k^2\sigma}{\alpha_2} + \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_2} \\
& +\frac{12\sigma^2 t^{2\omega}\left(-4k^4\sigma w + \alpha_3 k^3 + \alpha_4 k s^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+sz+wy)\right) - 2\right)\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1^2\alpha_2 w^2\omega^2} \\
& -\frac{24k^2\sigma^2 t^\omega \left(4k^3\sigma w - \alpha_3 k^2 - \alpha_4 s^2\right)\tanh\left(\sqrt{-\sigma}(kx+sz+wy)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega}
\end{aligned}
$$

Likewise, the following approximate solutions appear

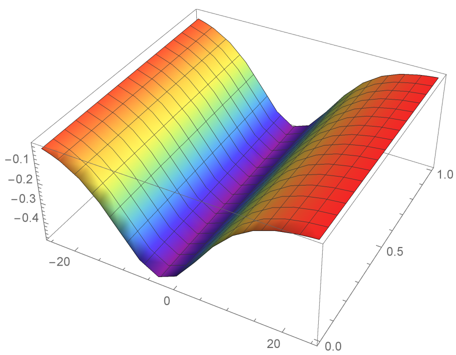
$$
\begin{aligned}
u_3 = & -\frac{12k^2\sigma}{\alpha_2} + \frac{12k^2\sigma \tanh^2((A))}{\alpha_2} \\
& +\frac{12\sigma^2 t^{2\omega}\left(-4k^4\sigma w + \alpha_3 k^3 + \alpha_4 k s^2\right)^2\left(\cosh(2(A)) - 2\right)\operatorname{sech}^4((A))}{\alpha_1^2\alpha_2 w^2\omega^2} \\
& -\frac{8k^2\sigma^3 t^{3\omega}\left(-4k^3\sigma w + \alpha_3 k^2 + \alpha_4 s^2\right)^3\left(\cosh(2(A)) - 5\right)\tanh((A))\operatorname{sech}^4((A))}{\alpha_1^3\alpha_2\sqrt{-\sigma}w^3\omega^3} \\
& -\frac{24k^2\sigma^2 t^\omega\left(4k^3\sigma w - \alpha_3 k^2 - \alpha_4 s^2\right)\tanh((A))\operatorname{sech}^2((A))}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega}
\end{aligned}
$$

$$
\begin{aligned}
u_4 = & \frac{12k^2\sigma \tanh^2(A)}{\alpha_2} + \frac{12\sigma^2(\cosh(2A)-2)\text{sech}^4(A)t^{2\omega}\left(-4k^4\sigma w+\alpha_3 k^3+\alpha_4 k s^2\right)^2}{\alpha_1^2\alpha_2 w^2\omega^2} \\
& + \frac{k^2\sigma^3(-26\cosh(2A)+\cosh(4A)+33)\text{sech}^6(A)t^{4\omega}\left(4k^3\sigma w-\alpha_3k^2-2\alpha_4 s^2\right)\left(-4k^3\sigma w+\alpha_3k^2+\alpha_4s^2\right)^3}{2\alpha_1^4\alpha_2 w^4\omega^4} \\
& - \frac{8k^2\sigma^3(\cosh(2A)-5)\tanh(A)\text{sech}^4(A)t^{3\omega}\left(-4k^3\sigma w+\alpha_3k^2+\alpha_4s^2\right)^3}{\alpha_1^3\alpha_2\sqrt{-\sigma}w^3\omega^3} \\
& - \frac{24k^2\sigma^2\tanh(A)\text{sech}^2(A)t^{\omega}\left(4k^3\sigma w-\alpha_3k^2-\alpha_4s^2\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega} - \frac{12k^2\sigma}{\alpha_2} \qquad (6.5)
\end{aligned}
$$

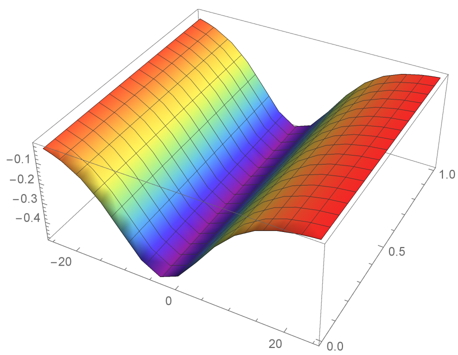
where, $A = \sqrt{-\sigma}(kx + sz + wy)$.

Table 1. Comparison of specific numerical values of RPSM approximate solution u_4 of (6.5) and Modified Extended tanh-function Method exact solution u_7 of (6.3)

	$\omega = 0.55$			$\omega = 0.85$			$\omega = 0.95$		
t	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*
0.0	−0.418977	−0.418977	0.00000	−0.418977	−0.418977	0.00000	−0.418977	−0.418977	0.00000
0.1	−0.421380	−0.421380	4.4640×10^{-11}	−0.419759	−0.419759	4.5158×10^{-13}	−0.419533	−0.419533	1.1368×10^{-13}
0.2	−0.422485	−0.422485	2.1718×10^{-10}	−0.420385	−0.420385	4.9379×10^{-12}	−0.420050	−0.420050	1.6314×10^{-12}
0.3	−0.423351	−0.423351	5.5319×10^{-10}	−0.420961	−0.420961	2.0229×10^{-11}	−0.420553	−0.420553	7.8329×10^{-12}
0.4	−0.424090	−0.424090	1.0792×10^{-9}	−0.421508	−0.421508	5.5392×10^{-11}	−0.421045	−0.421045	2.4008×10^{-11}
0.5	−0.424747	−0.424747	1.8182×10^{-9}	−0.422032	−0.422032	1.2156×10^{-10}	−0.421531	−0.421531	5.7522×10^{-11}
0.6	−0.425345	−0.425345	2.7903×10^{-9}	−0.422539	−0.422539	2.3187×10^{-10}	−0.42201	−0.42201	1.1791×10^{-10}
0.7	−0.425898	−0.425898	4.0145×10^{-9}	−0.423033	−0.423033	4.0137×10^{-10}	−0.422484	−0.422484	2.1700×10^{-10}
0.8	−0.426415	−0.426415	5.5082×10^{-9}	−0.423514	−0.423514	6.4705×10^{-10}	−0.422953	−0.422953	3.6900×10^{-10}
0.9	−0.426901	−0.426901	7.2881×10^{-9}	−0.423986	−0.423986	9.8777×10^{-10}	−0.423418	−0.423418	5.9063×10^{-10}
1.0	−0.427363	−0.427363	9.3700×10^{-9}	−0.424448	−0.424448	1.4443×10^{-9}	−0.423880	−0.423880	9.0121×10^{-10}



(a) RPSM solution



(b) Exact solution

Figure 3. Comparison of surface plots of RPSM approximate solution u_4 of (6.5) and Modified Extended tanh-function Method exact solution u_7 of (6.3)

The surface plots show some novel solutions to the present equation that might be useful for other types of differential equations of arbitrary order. Figures 1 and 2 display some of the physical characteristics of the acquired analytical solutions in 3D, 2D, and contour representations. Besides, Figure 3 compares the surface graphics of the approximate and exact solutions obtained in 3D. Concurrently, for the given Table 1 and the mentioned figures, the following values and ranges are used for the exact and approximate solutions.

i. Figure 1: $k = 0.01$, $s = 0.01$, $\alpha_1 = 0.9$, $\alpha_2 = -0.8$, $\alpha_3 = 0.7$, $\alpha_4 = 0.4$, $\sigma = -0.04$, $y = 0.3$, $z = 0.2$, $w = 0.95$, and $\omega = 0.95$, $-50 \le x \le 5$, for (a) and (b); $t = 0.99$, for (c).

ii. Figure 2: $k = 0.2$, $w = 1$, $s = 0.01$, $y = 0.1$, $z = 0.5$, $h = 0.1$, $\eta = 0.05$, $\lambda = 0.5$, $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.1$, $\alpha_4 = 0.1$, $\omega = 0.95$, $-50 \leq x \leq 50$, for (a) and (b); $t = 0.99$, for (c).

iii. Table 1: $\alpha_1 = 0.3$, $\alpha_2 = -0.2$, $\alpha_3 = 0.01$, $\alpha_4 = 0.4$, $k = 0.3$, $s = 0.1$, $\sigma = -0.09$, $w = 0.9$, $x = 1$, $\omega = 0.95$, $y = 1$, $z = 1$, and $0 \leq t \leq 1$.

iv. Figure 3: $\alpha_1 = 0.3$, $\alpha_2 = -0.2$, $\alpha_3 = 0.01$, $\alpha_4 = 0.4$, $k = 0.3$, $s = 0.1$, $\sigma = -0.09$, $w = 0.9$, $x = 1$, $\omega = 0.95$, $y = 1$, $z = 1$ and $\omega = 0.95$, $-25 \leq x \leq 25$, for (a) and (b); $0 \leq t \leq 1$.

7. Conclusion

In the main study [21], the authors presented the Painlevé integrability analysis of the model. Additionally, they can acquire the rogue waves up to the third order by using symbolic computation and the Cole-Hopf transformation. Dispersive-soliton solutions to this equation are finally introduced. Next, very recently, multi-wave, breather, and other localized wave solutions via the Hirota bilinear method have been presented in [33]. In this paper, using modified extended tanh-function and the $\exp(-\phi(\xi))$-expansion methods, solutions to the $(3+1)$-dimensional P-type evolution equation with conformable derivative were explored in this study. The residual power series method(RPSM) was also employed to get approximate solutions. Modified extended tanh-function and $\exp(-\phi(\xi))$-expansion methods produced several accurate exact solutions with low processing complexity. Furthermore, there is no requirement for discretization, translation, or perturbation when applying the RPSM to the governing equation. 3D, 2D, and contour plots were illustrated to visually present the solutions discovered. Besides, a comparison table is presented to compare the approximate solutions with the exact solutions. These solutions have important physical characteristics that have not been previously reported in the literature and are unique. According to some interpretations of the figures, the exact solutions' physical behavior appears for particular values. Comprehending these applications is essential for their possible practical uses. Thus, analytical and numerical solutions are essential to understanding real-world scenarios. As a result, further fractional order differential equations may be handled and solved using the suggested methods in later research.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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On Strong (i,j)-Semi*-Γ-Open Sets in Ideal Bitopological Space

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Research Article

Abstract — In this study, we introduce the concepts of (i,j)-semi*-Γ-open sets within the context of ideal bitopological spaces. This concept is demonstrated to be weaker than the established the notion of (i,j)-semi-Γ-open sets. Subsequently, we define strong (i,j)-semi*-Γ-open sets in ideal bitopological spaces, elucidating some of their essential characteristics. Furthermore, leveraging this newly introduced concept, we establish the notions of strong (i,j)-semi*-Γ-interior and strong (i,j)-semi*-Γ-closure.

Keywords *Ideal bitopological spaces, generalized open sets, (i,j)-Γ-open sets, (i,j)-semi*-Γ-open sets*

Mathematics Subject Classification (2020) 54E55, 54A05

1. Introduction

Recent studies have focused on bitopological spaces $(X,\mathcal{V}_1,\mathcal{V}_2)$, a nonempty set X endowed with two topologies $\mathcal{V}_1$ and $\mathcal{V}_2$ [1–5]. In 2006, Noiri and Rajesh studied the generalized closed sets concerning an ideal in bitopological spaces [6]. An ideal on a topological space $(X,\mathcal{V})$ is a collection of subsets of X with the hereditary properties:

i. if $U\in\Gamma$ and $V\subset U$, then $V\in\Gamma$

ii. if $U\in\Gamma$ and $V\in\Gamma$, then $U\cup V\in\Gamma$

Let Γ be an ideal on X. Then, $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is termed an ideal bitopological space.

For $P(X)$ being the entire set of subsets of X, and for $i\in\{1,2\}$, an operator $(.)^*_i : P(X)\to P(X)$, referred to as the local function of $\mathcal{U}$ with respect to $\mathcal{V}_i$ and Γ, is defined as follows: for $U\subset X$, $U^*_i(\mathcal{V}_i,\Gamma)=\{x\in X \mid V\cap U\notin\Gamma, \text{ for every } V\in\mathcal{V}_i(x)\}$, where $\mathcal{V}_i(x)=\{V\in\mathcal{V}_i \mid x\in V\}$ [7].

For each ideal topological space $(X,\mathcal{V},\Gamma)$, there exists a topology $\mathcal{V}^*(\Gamma)$ that is more refined than $\mathcal{V}$, generated by the base $\mathcal{B}(\Gamma,\mathcal{V})=\{V-I \mid V\in\mathcal{V} \text{ and } I\in\Gamma\}$. However, it is worth noting that $\mathcal{B}(\Gamma,\mathcal{V})$ is not always a topology [8]. Moreover, we can observe that $\mathrm{Cl}^*_i(U)=U\cup U^*_i(\mathcal{V}_i,\Gamma)$ defines a Kuratowski closure operator for $\mathcal{V}^*_i(\Gamma)$.

Ekici and Noiri [9] introduced the notion of semi-Γ-open sets in ideal topological spaces. Çaldaş et al. [10] introduced the notion of (i,j)-semi-Γ-open sets in ideal bitopological spaces. Finally, Aqeel and Bin-Kuddah [11] established the concept of strong semi*-Γ-open sets in ideal topological spaces.

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Throughout this paper, we use the notation U_i^* for $U_i^*(\mathcal{V}_i,\Gamma)$. Moreover, $\mathrm{Int}_i(U)$ ($\mathrm{Cl}_i(U)$) and $\mathrm{Int}_i^*(U)$ ($\mathrm{Cl}_i^*(U)$) denote the interior (closure) of U respect to $\mathcal{V}_i$ and $\mathcal{V}_i^*$, respectively.

2. Preliminaries

In this paper, we consistently refer to $(X,\mathcal{V}_1,\mathcal{V}_2)$ as a bitopological space without assuming any separation axioms. Additionally, $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is considered as an ideal bitopological space, denoted by the abbreviation IBS. We use OS and CS as abbreviations for open sets and closed sets, respectively.

Definition 2.1. [11] A subset U of an ideal topological space $(X,\mathcal{V},\Gamma)$ is called as:

i. Semi-$\Gamma-$OS if $U\subset \mathrm{Cl}^*(\mathrm{Int}(U))$

ii. Semi*-$\Gamma-$OS if $U\subset \mathrm{Cl}(\mathrm{Int}^*(U))$

iii. Strong semi*-$\Gamma-$OS if $U\subset \mathrm{Cl}^*(\mathrm{Int}^*(U))$

Definition 2.2. [12] For any IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$, Γ is called as codense if $\mathcal{V}_i\cap\Gamma=\{\emptyset\}$, for $i\in\{1,2\}$.

Lemma 2.3. [11] For any IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$, if Γ is a codense ideal, then the following hold:

i. $\mathrm{Cl}_j(U)=\mathrm{Cl}_j^*(U)$, for all j-open set $U\subset X$

ii. $\mathrm{Int}_i(F)=\mathrm{Int}_i^*(F)$, for all j-closed set $F\subset X$

Theorem 2.4. [12] Let U be a subset of IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. For $i,j\in\{1,2\}$ and $i\neq j$,

i. If $\Gamma=\emptyset$, then $U_j^*(\Gamma)=\mathrm{Cl}_j(U)$

ii. If $\Gamma=P(X)$, then $U_j^*(\Gamma)=\emptyset$

iii. $U_j^*\subset \mathrm{Cl}_j(U)$

Lemma 2.5. [12] Let U be an (i,j)-Γ-OS in $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, $U_j^*=\left(\mathrm{Int}_i(U_j^*)\right)_j^*$.

Lemma 2.6. [12] Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ and $V\in\mathcal{V}_j$. Then, $V\cap \mathrm{Cl}_j^*(U)\subset \mathrm{Cl}_j^*(V\cap U)$.

Definition 2.7. [12–14] A subset U of an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is called as:

i. (i,j)-Γ-OS if $U\subset \mathrm{Int}_i(U_j^*)$

ii. (i,j)-semi-Γ-OS if $U\subset \mathrm{Cl}_j^*\left(\mathrm{Int}_i(U)\right)$

iii. (i,j)-semi-Γ-CS if $\mathrm{Int}_j^*\left(\mathrm{Cl}_i(U)\right)\subset U$

iv. (i,j)-α-Γ-OS if $U\subset \mathrm{Int}_i(\mathrm{Cl}_j^*(\mathrm{Int}_i(U)))$

v. (i,j)-pre-Γ-OS if $U\subset \mathrm{Int}_i(\mathrm{Cl}_j^*(U))$

vi. (i,j)-pre-Γ-CS if $\mathrm{Cl}_j(\mathrm{Int}_i^*(U))\subset U$

3. On (i,j)-Semi*-Γ-Open Set

This section defines the concept of (i,j)-semi*-Γ-open sets in an ideal bitopological space and presents some associated properties.

Definition 3.1. A subset U in an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is called as an (i,j)-semi*-Γ-OS if $U\subset \mathrm{Cl}_j\left(\mathrm{Int}_i^*(U)\right)$, for $i,j\in\{1,2\}$ and $i\neq j$. The set of all the (i,j)-semi*-Γ-open sets in X is denoted by $S_{ij}^*\Gamma O(X)$.

Example 3.2. Consider an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ where $X=\{\alpha,\beta,\gamma\}$, $\Gamma=\{\emptyset,\{\beta\}\}$, $\mathcal{V}_1=\{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\},X\}$, and $\mathcal{V}_2=\{\emptyset,\{\beta\},X\}$. Then,

$$\mathcal{V}_1^*=\{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\},\{\alpha,\gamma\},X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\beta\}, \{\alpha,\gamma\}, X\}$$

Therefore, $\{\beta,\gamma\}$ is a $(1,2)$-semi*-Γ-OS.

Proposition 3.3. Following properties are valid for any IBS $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$:

i. Every (i,j)-semi-Γ-OS is an (i,j)-semi*-Γ-OS.

ii. Every (i,j)-α-Γ-OS is an (i,j)-semi*-Γ-OS.

Proof. *i.* Assume $U \subset X$ is an (i,j)-semi-Γ-OS. Since $\mathcal{V}_i \subset \mathcal{V}_i^*$, then $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i(U)) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$.

ii. The proof follows from (i).

□

Lemma 3.4. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$ be an (i,j)-semi*-Γ-OS. Then, $\mathrm{Cl}_j(U)$ is (i,j)-semi*-Γ-open set.

Proof. Since U is an (i,j)-semi*-Γ-OS, then $U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. Thus,

$$\mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Cl}_j(\mathrm{Int}_i^*(U))) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U)) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(\mathrm{Cl}_j(U)))$$

□

Theorem 3.5. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$. Then, U is an (i,j)-semi*-Γ-OS if and only if $\mathrm{Cl}_j(U) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$.

Proof. Let U be an (i,j)-semi*-Γ-OS. Then, according to Lemma 3.4, it follows that $\mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. Suppose $\mathrm{Cl}_j(U) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. This implies $U \subset \mathrm{Cl}_j(U)$. Using the hypothesis, we obtain $U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. □

Theorem 3.6. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $V \subset X$. Then, V is an (i,j)-semi*-Γ-OS if and only if, there exists an (i,j)-semi*-Γ-OS U such that $U \subset V \subset \mathrm{Cl}_j(U)$.

Proof. Let V is an (i,j)-semi*-Γ-OS. Then, $V \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(V))$. Let $U = \mathrm{Int}_i^*(V)$ be (i,j)-$\Gamma-$OS. In other words, U is an (i,j)-semi*-Γ-OS and we have $U = \mathrm{Int}_i^*(V) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(V)) = \mathrm{Cl}_j(U)$. In contrast, if U is an (i,j)-semi*-Γ-OS such that $U \subset V \subset \mathrm{Cl}_j(U)$, then since $U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$, it follows that $V \subset \mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Cl}_j(\mathrm{Int}_i^*(U))) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U)) = \mathrm{Cl}_j(\mathrm{Int}_i^*(V))$. Hence, V is an (i,j)-semi*-Γ-OS. □

Definition 3.7. A subset U in $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ is called as an (i,j)-semi*-Γ-CS if complement of U is an (i,j)-semi*-Γ-OS. The set of all the (i,j)-semi*-Γ-closed sets in X is denoted by $S_{ij}^*\Gamma C(X)$.

Theorem 3.8. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $\{U_\alpha \mid \alpha \in \Lambda\}$ be a family of subsets of X where Λ is an index set. Then, if $U_\alpha \in S_{ij}^*\Gamma O(X)$, for every $\alpha \in \Lambda$, then $\bigcup\limits_{\alpha\in\Lambda} U \in S_{ij}^*\Gamma O(X)$.

Proof. Let $U_\alpha \in S_{ij}^*\Gamma O(X)$, for every $\alpha \in \Lambda$. Then, $U_\alpha \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U_\alpha))$. Therefore,

$$\begin{aligned}\bigcup_{\alpha\in\Delta} U_\alpha \subset \bigcup_{\alpha\in\Lambda} \mathrm{Cl}_j(\mathrm{Int}_i^*(U_\alpha)) &\subset \mathrm{Cl}_j\left(\bigcup_{\alpha\in\Lambda} \mathrm{Int}_i^*(U_\alpha)\right)\\ &\subset \mathrm{Cl}_j\left(\mathrm{Int}_i^*\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right)\right)\end{aligned}$$

□

A finite intersection of (i,j)-semi*-Γ-open sets need not to be in $S_{ij}^*\Gamma O(X)$ in general as demonstrated by the following example.

Example 3.9. Let $X = \{\alpha, \beta, \gamma, \eta\}$, $\mathcal{V}_1 = \{\emptyset, X, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$, and $\mathcal{V}_2 = \{\emptyset, X\}$. Let $\Gamma = \{\emptyset, \{\gamma\}, \{\eta\}, \{\gamma, \eta\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \eta\}, X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \eta\}, X\}$$

Therefore, $\{\alpha, \eta\}$ and $\{\beta, \eta\}$ are $(1,2)$-semi*-Γ open sets. However, $\{\alpha, \eta\} \cap \{\beta, \eta\} = \{\eta\}$ is not $(1,2)$-semi*-Γ open.

Theorem 3.10. A subset U in $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ is an (i,j)-semi*-Γ-CS if and only if $\mathrm{Int}_j^*(\mathrm{Cl}_i(U)) \subset U$.

Proof. Let U is an (i,j)-semi*-Γ-CS. Then, $X - U$ is an (i,j)-semi*-Γ-OS. Therefore, $X - U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(X - U)) = X - \mathrm{Int}_j^*(\mathrm{Cl}_i(U))$. Consequently, $\mathrm{Int}_j^*(\mathrm{Cl}_i(U)) \subset U$.

In contrast, if $\mathrm{Int}_j^*(\mathrm{Cl}_i(U)) \subset U$, then $X - U \subset X - \mathrm{Int}_j^*(\mathrm{Cl}_i(U)) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(X - U))$. Therefore, $X - U$ is an (i,j)-semi*-Γ-OS. Thus, U is an (i,j)-semi*-Γ-CS. □

Theorem 3.11. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and Γ is codense. Then, any subset U is an (i,j)-semi*-Γ-CS if and only if $\mathrm{Int}_i(\mathrm{Cl}_j(U)) \subset U$.

Proof. Let U be an (i,j)-semi*-Γ-CS. Then, $\mathrm{Int}_i^*(\mathrm{Cl}_j(U)) \subset U$. Since $\mathrm{Int}(U) \subset \mathrm{Int}^*(U)$, then $\mathrm{Int}_i(\mathrm{Cl}_j(U)) \subset U$.

In contrast, let $U \subset X$ and $\mathrm{Int}_i(\mathrm{Cl}_j(U)) \subset U$. Since Γ is codense, this implies that $\mathrm{Int}_i^*(\mathrm{Cl}_j(U)) \subset U$. Therefore, U is an (i,j)-semi*-Γ-CS. □

4. On Strong (i,j)-semi*-Γ-Open Set

This section suggests strong (i,j)-semi*-Γ-open sets in ideal bitopological spaces.

Definition 4.1. A subset U of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ is named as a strong (i,j)-semi*-Γ-open set if $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. The collection comprised of all the strong (i,j)-semi*-Γ-open sets in X is denoted by $SS_{ij}^*\Gamma O(X)$.

Example 4.2. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS such that $X = \{\alpha, \beta, \gamma, d\}$, $\mathcal{V}_1 = \{\emptyset, \{\beta\}, \{\alpha, \gamma, \eta\}, X\}$, $\mathcal{V}_2 = \{\emptyset, \{\alpha, \beta\}, X\}$, and $\Gamma = \{\emptyset, \{\gamma\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset, \{\beta\}, \{\alpha, \eta\}, \{\alpha, \beta, \eta\}, \{\alpha, \gamma, \eta\}, X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\alpha, \beta\}, \{\alpha, \beta, \eta\}, X\}$$

Therefore, $\{\beta, \gamma\}$ is a strong $(1,2)$-semi*-Γ-OS but $\{\alpha, \gamma\}$ is not.

Proposition 4.3. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$.

i. Every (i,j)-semi-Γ-OS is a strong (i,j)-semi*-Γ-OS.

ii. Every (i,j)-α-Γ-OS is a strong (i,j)-semi*-Γ-OS.

iii. Every strong (i,j)-semi*-Γ-OS is an (i,j)-semi*-Γ-OS.

The evidences come from Proposition 3.3 and Definition 4.1.

Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$. Then, we get this diagram:

$$(i,j)\text{-}\alpha\text{-}\Gamma\text{-OS} \rightarrow (i,j)\text{-semi-}\Gamma\text{-OS} \rightarrow \text{strong } (i,j)\text{-semi}^*\text{-}\Gamma\text{-}OS \rightarrow (i,j)\text{-semi}^*\text{-}\Gamma\text{-}OS$$

Generally, the opposites of Proposition 4.3 are inaccurate, as demonstrated by the next example.

Example 4.4. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS such that $X = \{\alpha, \beta, \gamma, \eta\}$, $\mathcal{V}_1 = \{\emptyset, \{\beta\}, \{\alpha, \gamma, \}, \{\alpha, \beta, \gamma\}, X\}$, $\mathcal{V}_2 = \{\emptyset, \{\gamma\}, X\}$, and $\Gamma = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset, \{\beta\}, \{\gamma\}, \{\alpha, \gamma\}, \{\gamma, \eta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \gamma, \eta\}, \{\beta, \gamma, \eta\}, X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\gamma\}, \{\gamma, \eta\}, \{\alpha, \gamma, \eta\}, \{\beta, \gamma, \eta\}, X\}$$

Therefore, $\{\alpha, \beta, \eta\}$ is a $(1,2)$-semi*-Γ-OS but it is not a strong $(1,2)$-semi*-Γ-open.

Example 4.5. In Example 4.2, $\{\alpha, \eta\}$ is a strong $(1,2)$-semi*-Γ-OS but it is not a $(1,2)$-semi-Γ-open.

Proposition 4.6. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$. Then, U is a strong (i,j)-semi*-Γ-OS if and only if $\mathrm{Cl}_j^*(U) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$.

Proof. Assume U is a strong (i,j)-semi*-Γ-OS, then $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. This implies that $\mathrm{Cl}_j^*(U) \subset \mathrm{Cl}_j^*(\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. Thus, $\mathrm{Cl}_j^*(U) \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. In contrast, assume $\mathrm{Cl}_j^*(U) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. Since $U \subset \mathrm{Cl}_j^*(U)$, then $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. □

Proposition 4.7. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $V \subset X$. Then, V is a strong (i,j)-semi*-Γ-OS if and only if there exists a strong (i,j)-semi*-Γ-OS U such that $U \subset V \subset \mathrm{Cl}_j^*(U)$.

Proof. Assume V is a strong (i,j)-semi*-Γ-OS, then $V \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V))$. Let $U = \mathrm{Int}_i^*(V)$. Then, $U \subset V \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V)) = \mathrm{Cl}_j^*(U)$. Moreover,

$$U \subset V \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V)) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(\mathrm{Int}_i^*(V)) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$$

Therefore, U is a strong (i,j)-semi*-Γ-OS.

In contrast, if U is a strong (i,j)-semi*-Γ-OS such that $U \subset V \subset \mathrm{Cl}_j^*(U)$, then $\mathrm{Cl}_j^*(U) = \mathrm{Cl}_j(V)$ and $\mathrm{Int}_i^*(U) \subset \mathrm{Int}_i^*(V)$. Besides, $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$ and hence

$$V \subset \mathrm{Cl}_j^*(U) \subset \mathrm{Cl}_j^*(\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V))$$

which V is a strong (i,j)-semi*-Γ-OS. □

Theorem 4.8. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $\{U_\alpha \subset X : \alpha \in \Delta\}$ be a family of subsets of X where Δ is an arbitrary index set.

i. If $U_\alpha \in SS_{ij}^*\Gamma O(X)$, for every $\alpha \in \Delta$, then $\bigcup\limits_{\alpha\in\Delta} \{U_\alpha : \alpha \in \Delta\} \in SS_{ij}^*\Gamma O(X)$.

ii. If $U \in SS_{ij}^*\Gamma O(X)$ and $V \in \mathcal{V}_j$, then $U \cap V \in SS_{ij}^*\Gamma O(X)$.

Proof. *i.* Since $U_\alpha \in SS_{ij}^*\Gamma O(X)$, for every $\alpha \in \Delta$, it follows that $U_\alpha \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U_\alpha))$. Consequently,

$$\begin{aligned}\bigcup_{\alpha\in\Delta} U_\alpha \subset \bigcup_{\alpha\in\Delta} \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U_\alpha)) &\subset \mathrm{Cl}_j^*\left(\bigcup_{\alpha\in\Delta} \mathrm{Int}_i^*(U_\alpha)\right) \\ &\subset \mathrm{Cl}_j^*\left(\mathrm{Int}_i^*\left(\bigcup_{\alpha\in\Delta} U_\alpha\right)\right)\end{aligned}$$

ii. Let $U \in SS_{ij}^*\Gamma O(X)$ and $V \in \mathcal{V}_j$. Since $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$, applying Lemma 2.6 yields:

$$U \cap V \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \cap V \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U) \cap V)$$

□

Generally, the intersection of strong (i,j)-semi-Γ-open sets need not be in $SS_{ij}^*\Gamma O(X)$ as demonstrated by the next example.

Example 4.9. Let $X = \{\alpha, \beta, \eta, \gamma\}$, $\mathcal{V}_1 = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, X\}$, and $\mathcal{V}_2 = \{\emptyset, X\}$. If $\Gamma = \{\emptyset, \{\eta\}, \{\gamma\}, \{\eta, \gamma\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \eta\}\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \eta\}\}$$

Therefore, $\{\alpha, \eta, \gamma\}$ and $\{\beta, \eta, \gamma\}$ are strong $(1, 2)$-semi*-Γ-OS; however, $\{\alpha, \eta, \gamma\} \cap \{\beta, \eta, \gamma\} = \{\eta, \gamma\}$, which is not a strong $(1, 2)$-semi*-Γ-OS.

Theorem 4.10. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS. The union of (i, j)-semi*-Γ-OS and a strong (i, j)-semi*-Γ-OS is an (i, j)-semi*-Γ-open.

Proof. Let $U \in SS^*_{ij}\Gamma O(X)$ and V is an (i, j)-semi-Γ-OS, then

$$\begin{aligned} U \cup V &\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \cup (\mathrm{Cl}_j(\mathrm{Int}_i^*(V)) \\ &\subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U)) \cup \mathrm{Cl}_j(\mathrm{Int}_i^*(V)) \\ &= \mathrm{Cl}_j(\mathrm{Int}_i^*(U) \cup \mathrm{Int}_i^*(V)) \\ &\subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U \cup V)) \end{aligned}$$

Hence, $U \cup V$ is an (i, j)-semi*-Γ-OS. □

Definition 4.11. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$. Then, U is called as a strong (i, j)-semi*-Γ-CS if its complement is a strong (i, j)-semi*-Γ-open. The set of all the strong (i, j)-semi*-Γ-closed sets in X is denoted by $SS^*_{ij}C(X)$.

Theorem 4.12. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$. Then, U is a strong (i, j)-semi*-Γ-CS if and only if $\mathrm{Int}_j^*(\mathrm{Cl}_i^*(U)) \subset U$.

Proof. Assume U is a strong (i, j)-semi*-Γ-CS of X. Then,

$$X - U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(X - U)) = X - \mathrm{Int}_j^*(\mathrm{Cl}_i^*(U))$$

Thus, $\mathrm{Int}_j^*(\mathrm{Cl}_i^*(U)) \subset U$. In contrast, assume U is any subset of X such that $\mathrm{Int}_j^*(\mathrm{Cl}_i^*(U)) \subset U$. This gives that $X - U \subset X - \mathrm{Int}_j^*(\mathrm{Cl}_i(U)) \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(X - U))$. Therefore, $X - U$ is a strong (i, j)-semi*-Γ-OS. □

Theorem 4.13. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and Γ is codense. Then, U is a strong (i, j)-semi*-Γ-CS if and only if $\mathrm{Int}_i(\mathrm{Cl}_j^*(U)) \subset U$.

Proof. Assume U is a strong (i, j)-semi*-Γ-CS of X. Then, $\mathrm{Int}_i^*(\mathrm{Cl}_j^*(U)) \subset U$. Thus, $\mathrm{Int}_i(\mathrm{Cl}_j^*(U)) \subset U$. In contrast, assume U is any subset of X such that $\mathrm{Int}_i(\mathrm{Cl}_j^*(U)) \subset U$. This suggests that $\mathrm{Int}_i^*(\mathrm{Cl}_j^*(U)) \subset U$, which gives that U is a strong (i, j)-semi*-Γ-CS. □

Theorem 4.14. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U, V \in SS^*_{ij}\Gamma C(X)$. Then, $U \cap V$ is a strong (i, j)-semi*-Γ-CS.

Proof. Assume $U, V \in SS^*_{ij}\Gamma C(X)$. Then,

$$\begin{aligned} U \cap V &\supset \mathrm{Int}_i^*(\mathrm{Cl}_j^*(U)) \cap \mathrm{Int}_i^*(\mathrm{Cl}_j^*(V)) \\ &\supset \mathrm{Int}_i^*(\mathrm{Cl}_j^*(U) \cap \mathrm{Cl}_j^*(V)) \\ &\supset \mathrm{Int}_i^*(\mathrm{Cl}_j^*(U \cap V)) \end{aligned}$$

Therefore, $U \cap V \in SS^*_{ij}\Gamma C(X)$. □

5. The Strong (i,j)-Semi*-Γ-Interior and Strong (i,j)-Semi*-Γ-Closure

This section defines the concept of strong (i,j)-semi*-Γ-interior and strong (i,j)-semi*-Γ-closure in an ideal bitopological space and establishes their varied characteristics.

Definition 5.1. Let U be a subset of an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. The strong (i,j)-semi*-Γ-interior of U, denoted by $ss^*_{i,j}\Gamma\text{-Int}(U)$, is defined as the union of all the strong (i,j)-semi*-Γ-open sets of X that are contained within U. In other words,

$$ss^*_{i,j}\Gamma\text{- Int}(U)=\bigcup\{V\subset U\mid V\in SS^*_{ij}\Gamma O(X)\}$$

Theorem 5.2. Let U is a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then,

$$ss^*_{i,j}\Gamma\text{- Int}(U)=U\cap \text{Cl}^*_j(\text{Int}^*_i(U))$$

Proof. Let $U\subset X$. Then,

$$\begin{aligned}U\cap \text{Cl}^*_j(\text{Int}^*_i(U)) &\subset \text{Cl}^*_j(\text{Int}^*_i(U))\\ &\subset \text{Cl}^*_j(\text{Int}^*_i(\text{Int}^*_i(U)))\\ &= \text{Cl}^*_j(\text{Int}^*_i(U\cap \text{Int}^*_i(U))\\ &\subset \text{Cl}^*_j(\text{Int}^*_i(U\cap \text{Cl}^*_j(\text{Int}^*_i(U))))\end{aligned}$$

Thus, $U\cap \text{Cl}^*_j(\text{Int}^*_i(U))$ is a strong (i,j)-semi*-Γ-OS contained in U, which means $U\cap \text{Cl}^*_j(\text{Int}^*_i(U))\subset ss^*_{ij}\Gamma\text{-Int}(U)$. Furthermore, since $ss^*_{i,j}\Gamma\text{-Int}(U)$ is a strong (i,j)-semi*-Γ-open, then

$$\text{ss}^*_{i,j}\Gamma\text{- Int}(U)\subset \text{Cl}^*_j(\text{Int}^*_i(\text{ss}^*_{i,j}\Gamma\text{- Int}(U)))\subset \text{Cl}^*_j(\text{Int}^*_i(U))$$

Consequently, $ss^*_{i,j}\Gamma\text{-Int}(U)\subset U\cap \text{Cl}^*_j(\text{Int}^*_i(U))$. □

Lemma 5.3. Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, U is a strong (i,j)-semi*-Γ-OS if and only if $ss^*_{i,j}\Gamma\text{-Int}(U)=U$.

Proof. Assume U is a strong (i,j)-semi*-Γ-OS. Then, $U\subset \text{Cl}^*_j(\text{Int}^*_i(U))$. Hence, $ss^*_{i,j}\Gamma\text{-Int}(U)=U\cap\text{Cl}^*_j(\text{Int}^*_i(U))=U$. In contrast, let $ss^*_{i,j}\Gamma\text{-Int}(U)=U$. Since $ss^*_{i,j}\Gamma\text{-Int}(U)=U\cap\text{Cl}^*_j(\text{Int}^*_i(U))=U$, then $U\subset \text{Cl}^*_j(\text{Int}^*_i(U))$. Hence, U is a strong (i,j)-semi*-Γ-OS. □

Definition 5.4. Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, the strong (i,j)-semi*-Γ-closure of U, denoted by $ss^*_{i,j}\Gamma\text{-Cl}(U)$, defined by the intersection of all the strong (i,j)-semi*-Γ-closed sets of X containing U. In other words,

$$ss^*_{ij}\Gamma-\text{Cl}(U)=\bigcap\{V\subset X: U\subset V, V\in SS^*_{ij}\Gamma C(X)\}$$

Theorem 5.5. Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, $ss^*_{ij}\Gamma-\text{Cl}(U)=U\cup\text{Int}^*_i(\text{Cl}^*_j(U))$.

Proof. Let $U\subset X$. Then,

$$\begin{aligned}U\cup \text{Int}^*_i(\text{Cl}^*_j(U)) &= \text{Int}^*_i(\text{Cl}^*_j(\text{Cl}^*_j(U)))\\ &= \text{Int}^*_i(\text{Cl}^*_j(U\cup \text{Cl}^*_j(U)))\\ &\supset \text{Int}^*_i(\text{Cl}^*_j(U\cup \text{Int}^*_i(\text{Cl}^*_j(U))))\end{aligned}$$

Thus, $U\cup\text{Int}^*_i(\text{Cl}^*_j(U))$ is as strong (i,j)-semi*-Γ-CS containing U. Therefore, $ss^*_{ij}\Gamma\text{- Cl}(U)\subset U\cup \text{Int}^*_i(\text{Cl}^*_j(U))$.

In contrast, let $ss^*_{ij}\Gamma\text{- Cl}(U)=U\cup\text{Int}^*_i(\text{Cl}^*_j(U))$. Since $ss^*_{i,j}\Gamma\text{- Cl}(U)$ is a strong (i,j)-semi*-Γ-CS, then

$$ss^*_{i,j}\Gamma\text{- Cl}(U)\supset \text{Int}^*_i(\text{Cl}^*_j(ss^*_{ij}\Gamma\text{- Cl}(U)))\supset \text{Int}^*_i(\text{Cl}^*_j(U))$$

Therefore, $ss^*_{ij}\Gamma\text{-Cl}(U)) \supset U \cup \text{Int}^*_i(\text{Cl}^*_j(U))$. Consequently, $ss^*_{ij}\Gamma\text{-Cl}(U)) = U \cup \text{Int}^*_i(\text{Cl}^*_j(U))$. □

Lemma 5.6. Let U be a subset of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$. Then, U is a strong (i, j)-semi*-Γ-CS if and only if $ss^*_{ij}\Gamma\text{-Cl}(U) = U$.

Proof. Assume U is a strong (i, j)-semi*-Γ-CS. This implies that $U \supset \text{Int}^*_i(\text{Cl}^*_j(U))$. Therefore, $ss^*_{ij}\Gamma\text{-Cl}(U) = U \cup \text{Int}^*_i(\text{Cl}^*_j(U)) = U$. In contrast, let $ss^*_{ij}\Gamma\text{-Cl}(U) = U$. Given that $ss^*_{ij}\Gamma\text{-Cl}(U) = U \cup \text{Int}^*_*(\text{Cl}^*_j(U))$, it follows that $U \supset \text{Int}^*_i(\text{Cl}^*_j(U))$. Consequently, U is a strong (i, j)-semi*-Γ-CS. □

Theorem 5.7. Let U be a subset of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$. Then, the following properties are held:

i. If U is an (i, j)-preΓ-OS, then $ss^*_{ij}\Gamma\text{-Cl}(U) = \text{Int}^*_i(\text{Cl}^*_j(U))$.

ii. If U is an(i, j)-preΓ-CS, then $ss^*_{ij}\Gamma\text{-Int}(U) = \text{Cl}^*_j(\text{Int}^*_i(U))$.

Proof. *i.* Let U be an (i, j)-preΓ-OS. Then, $U \subset \text{Int}_i(\text{Cl}^*_j(U)) \subset \text{Int}^*_i(\text{Cl}^*_j(U))$. This gives that

$$ss^*_{ij}\Gamma\text{- Cl}(U) = U \cup \text{Int}^*_i(\text{Cl}^*_j(U)) = \text{Int}^*_i(\text{Cl}^*_j(U))$$

ii. Let U be an (i, j)-preΓ-CS. Then, $\text{Cl}^*_j(\text{Int}^*_i(U)) \subset \text{Cl}_j(\text{Int}^*_i(U)) \subset U$. This suggests that

$$ss^*_{ij}\Gamma\text{- Int}(U)) = U \cap \text{Cl}^*_j(\text{Int}^*_i(U)) = \text{Cl}^*_j(\text{Int}^*_i(U))$$

□

Theorem 5.8. Let U be a subset of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$. The following properties are held:

i. $\text{Int}^*_i(ss^*_{ij}\Gamma\text{-Cl}(U)) = \text{Int}^*_i(\text{Cl}^*_j(U))$

ii. $\text{Cl}^*_j(ss^*_{ij}\Gamma\text{-Int}(U)) = \text{Cl}^*_j(\text{Int}^*_i(U))$

Proof. *i.*

$$\begin{aligned}\text{Int}^*_i(ss^*_{ij}\Gamma\text{- Cl}(U)) &= \text{Int}^*_i(U \cup \text{Int}^*_i(\text{Cl}^*_j(U)))\\ &\supset \text{Int}^*_i(U) \cup \text{Int}^*_i(\text{Int}^*_i(\text{Cl}^*_j(U)))\\ &= \text{Int}^*_i(U) \cup \text{Int}^*_i(\text{Cl}^*_j(U))\\ &= \text{Int}^*_i(\text{Cl}^*_j(U))\end{aligned}$$

In contrast,

$$\begin{aligned}\text{Int}^*_i(ss^*_{ij}\Gamma\text{- Cl}(U)) &= \text{Int}^*_i(U \cup \text{Int}^*_i(\text{Cl}^*_j(U)))\\ &\subset \text{Int}^*_i(\text{Cl}^*_j(U)) \cup \text{Int}^*_i(\text{Cl}^*_j(U))\\ &= \text{Int}^*_i(\text{Cl}^*_j(U))\end{aligned}$$

This indicates that $\text{Int}^*_i(ss^*_{ij}\Gamma\text{-Cl}(U)) = \text{Int}^*_i(\text{Cl}^*_j(U))$.

ii.

$$\begin{aligned}\text{Cl}^*_j(ss^*_{ij}\Gamma\text{- Int}(U)) &= \text{Cl}^*_j(U \cap \text{Cl}^*_j(\text{Int}^*_i(U))\\ &\subset \text{Cl}^*_j(\text{Cl}^*_j(U) \cap \text{Cl}^*_j(\text{Int}^*_i(U)))\\ &\subset \text{Cl}^*_j(\text{Cl}^*_j(\text{Int}^*_i(U)))\\ &= \text{Cl}^*_j(\text{Int}^*_i(U))\end{aligned}$$

In contrast,

$$\begin{aligned}\mathrm{Cl}_j^*(ss_{ij}^*\Gamma\text{- Int}(U)) &= \mathrm{Cl}_j^*(U \cap \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \\ &\supset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \cap \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \\ &= \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))\end{aligned}$$

This suggests that $\mathrm{Cl}_j^*(ss_{ij}^*\Gamma\text{-Int}(U)) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$.

□

6. Conclusion

In this paper, we introduced the notions of (i, j)-semi*-Γ-open sets and strong (i, j)-semi*-Γ-open sets in ideal bitopological spaces. We demonstrated that the concept of (i, j)-semi*-Γ-open set is weaker than (i, j)-open sets in ideal bitopological spaces. We discussed and proved several properties and relationships of (i, j)-semi*-Γ-open sets and strong (i, j)-semi*-Γ-open sets. Additionally, we introduced the notions of strong (i, j)-semi*-Γ-interior and strong (i, j)-semi*-Γ-closure, providing proofs for their properties.

In future studies, researchers can investigate more applications of (i, j)-semi*-Γ-open sets and strong (i, j)-semi*-Γ-open sets in ideal bitopological spaces. Furthermore, the concept of continuity can be studied in the light of the newly defined generalized open sets.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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Geodetic Domination Integrity of Thorny Graphs

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Abstract — The concept of geodetic domination integrity is a crucial parameter when examining the potential damage to a network. It has been observed that the removal of a geodetic set from the network can increase its vulnerability. This study explores the geodetic domination integrity parameter and presents general results on the geodetic domination integrity values of thorn ring graphs, n-sunlet graphs, thorn path graphs, thorn rod graphs, thorn star graphs, helm graphs, E_p^t tree graphs, dendrimer graphs, spider graphs, and bispider graphs, which are the frequently used graph classes in the literature.

Keywords *Geodetic domination integrity, geodetic dominating set, geodetic set, dominating set*

Mathematics Subject Classification (2020) 05C69, 05C76

1. Introduction

Measuring the stability and reliability of communication networks is critical in today's rapidly growing and changing communication infrastructures. The centers of a network can be modeled as the vertices and the connecting lines as the edges of the graph. An important question is how long the network's communication will last if some vertices or edges of the graph modeling a network are damaged. The measurement of the resilience of a network after the damage of some centers or connection lines in a communication network, until the communication is lost in the remaining network, is called vulnerability [1]. Various parameters have been defined to measure the vulnerability of networks. Some of these measurements are connectivity, integrity, domination integrity, and toughness [2]. Different versions of these parameters have been defined according to the features needed in the networks [2–4]. The geodetic domination integrity is one of the newly defined parameters [5]. Finding a geodesic path in any network to optimize time and cost plays an important role.

A geodetic path is the shortest path between two vertices. The combination of the shortest paths between the elements of the geodetic set of the graph modeling, the network covers the entire network. Transportation networks are required to pass through critical centers and to minimize the cost of logistics expenses. The analysis of this set plays an important role in optimizing traffic flow, planning public transport networks, and improving road safety. Geodetic set analysis helps to find solutions to problems, such as identifying areas with traffic congestion or determining alternative transportation routes. Damage to the geodetic dominating set can disrupt all communication in the network. The geodetic domination integrity is an important parameter to investigate the network-wide damage

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because removing a geodetic set from the network increases its vulnerability [6]. Therefore, geodetic domination integrity has a wide research area in graph theory, which motivated us to study geodetic domination integrity of graphs.

In this study, the geodetic domination integrity parameter is studied, and general results are obtained and proved for thorn graphs, dendrimer graphs, helm graphs, E_p^t tree graphs, and spider graphs, frequently used graph classes in the literature.

2. Preliminaries

This section provides some basic notions to be required for the following sections. Throughout the paper, simple graphs are considered, and the books [1, 7–10] are used for the basic definitions. For any graph $G = (V, E)$, the order is $n = |V(G)|$, and the size is $m = |E(G)|$. The set $N(v) = \{u_i : d(v, u_i) = 1, u_i \in V(G)\}$ is the open neighborhood of $v \in V(G)$, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is defined by $d(v) = |N(v)|$. For $X \subseteq V(G)$, let $G[X]$ be the subgraph of graph G induced by X, $N(X) = \sum_{x\in X} N(x)$, and $N[X] = \sum_{x\in X} N[x]$. The length of a shortest path $x - y$ in a connected graph G is the distance between x and y, denoted by $d(x, y)$, and the diameter of a graph is defined by $\operatorname{diam}(G) = \max_{x,y\in V(G)} \{d(x, y)\}$. An $x - y$ geodesic is a path of length $d(x, y)$, and the closed interval $I[x, y]$ consists of x, y, and all the vertices contained on some geodesic $x - y$ where $I[S] = \bigcup_{x,y\in S} I[x, y]$, for $S \subseteq V(G)$. If $I[S] = V(G)$, then S is a geodetic set. The minimum cardinality of a geodetic set is the geodetic number of G, denoted by $g(G)$. A geodetic set is called a $g(G)$-set if its cardinality is $g(G)$ [6]. $S \subseteq V(G)$ is a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S; in other words, if every vertex of G is dominated by some vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma(G)$ [1]. If a dominating set is also a geodetic set, then the set is called a geodetic dominating set, and the minimum cardinality of a geodetic dominating set in G is called the geodetic domination number, denoted by $\gamma_g(G)$ [11].

A communication network consists of centers and connection lines that enable these centers to communicate with each other. The graph's resistance following the breakdown of certain centers or connections is called vulnerability in a communication network. There are some parameters to measure vulnerability [2]. One of which, the geodetic domination integrity DI_g, was defined in 2021 by Balaraman et al. [5].

Definition 2.1. [5] The geodetic domination integrity of a graph G is defined by

$$DI_g(G) = \min_{S\subseteq V(G)} \{|S| + m(G - S)\}$$

where the order of the greatest component in $G - S$ is indicated by $m(G - S)$, and S is a geodetic dominating set of G. If $DI_g(G) = |S| + m(G - S)$, then a subset S of $V(G)$ is a DI_g-set.

Lemma 2.2. [5] The general results for the geodetic domination integrity of some known graphs are as follows:

i. $DI_g(K_n) = n$

ii. $DI_g(K_{1,n-1}) = n$

iii. $DI_g(K_{r,s}) = \min\{r, s\} + 1$, for $r, s \geq 2$

iv. $DI_g(W_n) = \left\lceil \frac{n-1}{2} \right\rceil + 2$, for $n \geq 5$

v. $DI_g(C_n) = \lceil \frac{n}{3} \rceil + 2, n \geq 6$

vi. $DI_g(P_n) = \left\lceil \frac{n+2}{3} \right\rceil + 2$

vii. $DI_g(G) = 6$ where G is the Petersen graph.

3. Geodetic Domination Integrity

In this section, the geodetic domination integrity values of thorn ring graphs, n-sunlet graphs, thorn path graphs, thorn rod graphs, thorn star graphs, helm graphs, E_p^t tree graphs, dendrimer graphs, regular dendrimer graphs, spider graphs, and bispider graphs, were analyzed, and general formulas were obtained based on the order of the graphs. Across this study, let $I_n := \{1, 2, 3, \cdots, n\}$.

Definition 3.1. [12] Let k be a non-negative integer. A thorn ring graph, denoted by $C_{n,k}$, is constructed by adding k pendant edges to each vertex of the cycle graph C_n.

Figure 1 illustrates the thorn ring $C_{8,3}$.

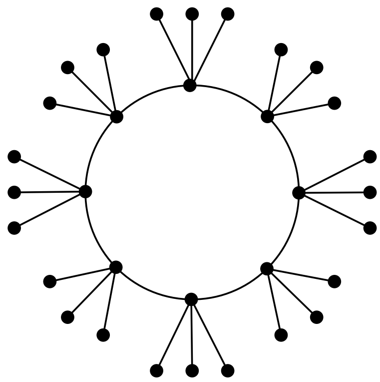

Figure 1. $C_{8,3}$ thorn ring

Theorem 3.2. Let $C_{n,k}$ be a thorn ring graph. Then,

$$DI_g(C_{n,k}) = nk + \lceil 2\sqrt{n} \rceil - 1$$

Proof. Let $C_{n,k}$ be a thorn ring graph where $\{x_i : \deg(x_i) = 1, i \in I_{nk}\}$ are pendant vertices, S be the geodetic dominating set, and $m(C_{n,k} - S)$ be the largest component in $C_{n,k} - S$. Let $X \subseteq V(C_{n,k})$ with $|X| = r$ such that $S = \{x_1, x_2, \cdots, x_{nk}\} \cup X$ is a geodetic dominating set. Then, $|S| = nk + r$ and the number of components in $C_{n,k} - S$, denoted by $\omega(C_{n,k} - S)$, is at most r. Hence, $m(C_{n,k} - S) \geq \dfrac{n + nk - (nk + r)}{r}$ which implies

$$DI_g(C_{n,k}) \geq \min_r \left\{ nk + r + \frac{n - r}{r} \right\}$$

For $r \geq 0$, the minimum integer value of the function $f(r) = nk + r + \frac{n-r}{r}$ is $nk + \lceil 2\sqrt{n} \rceil - 1$. Since geodetic domination integrity is an integer value,

$$DI_g(C_{n,k}) = nk + \lceil 2\sqrt{n} \rceil - 1$$

$\square$

Definition 3.3. [13] An n-sunlet graph is obtained from the cycle graph C_n by adding n pendant edges to each vertex of G and is denoted by S_n.

Figure 2 illustrates the 8-sunlet graph S_8.

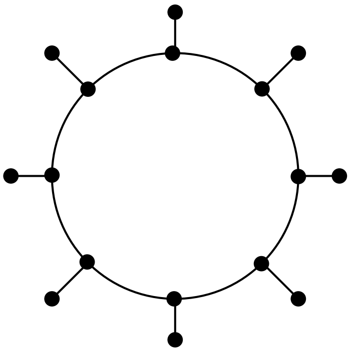

Figure 2. 8-sunlet graph

Theorem 3.4. Let S_n be an n-sunlet graph. Then,

$$DI_g(S_n) = n + \lceil 2\sqrt{n} \rceil - 1$$

Proof. Since $C_{n,1} \cong S_n$, it follows from Theorem 3.2 that $DI_g(S_n) = n + \lceil 2\sqrt{n} \rceil - 1$. □

Definition 3.5. [14] Let p and u be non-negative integers. A thorn path graph, denoted by $P_{n,p,u}$, is constructed by adding u pendant edges to both the initial and the terminal vertices of the path graph P_n, and p pendant edges to each internal vertex.

Figure 3 illustrates the thorn path $P_{6,2,3}$.

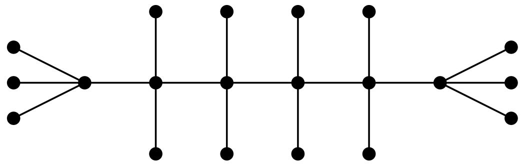

Figure 3. $P_{6,2,3}$ thorn path

Theorem 3.6. Let $P_{n,p,u}$ be a thorn path graph. Then,

$$DI_g(P_{n,p,u}) = (n-2)p + 2u + \lceil 2\sqrt{n+1} \rceil - 2$$

Proof. Let $P_{n,p,u}$ be a thorn path graph with pendant vertices $\{x_k : \deg(x_k) = 1, k \in I_{(n-2)p}\}$ and $\{y_j : \deg(y_j) = 1, j \in I_{2u}\}$, S be a geodetic dominating set, and $m(P_{n,p,u} - S)$ be the largest component in $P_{n,p,u} - S$. Let $X \subseteq V(P_{n,p,u})$ and $|X| = r$ such that $S = \{x_1, x_2, \cdots, x_{(n-2)p}\} \cup \{y_1, y_2, \cdots, y_{2u}\} \cup X$ be a geodetic dominating set removed. Hence, $|S| = (n-2)p + 2u + r$ and the number of components in $P_{n,p,u} - S$ is $\omega(P_{n,p,u} - S) \leq r + 1$, implying

$$m(P_{n,p,u} - S) \geq \frac{n + (n-2)p + 2u - ((n-2)p + 2u + r)}{r+1}$$

Therefore,

$$DI_g(P_{n,p,u}) \geq \min_r \left\{ (n-2)p + 2ur + \frac{n-r}{r+1} \right\}$$

For $r \geq 0$, $f(r) = (n-2)p + 2u + r + \frac{n-r}{r+1}$ is the function, and its lowest value is $(n-2)p + 2u + \lceil 2\sqrt{n+1} \rceil - 2$. Since the geodetic domination integrity is an integer value,

$$DI_g(P_{n,p,u}) = (n-2)p + 2u + \lceil 2\sqrt{n+1} \rceil - 2$$

□

Definition 3.7. [12] Let n and k be non-negative integers. A thorn rod graph, denoted by $P_{n,k}$, is constructed by adding k pendant edges to each of the initial and the terminal vertices of the path graph P_n.

Figure 4 illustrates the thorn rod $P_{6,3}$.

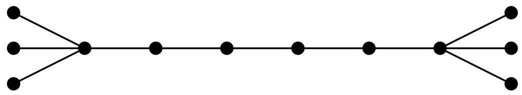

Figure 4. $P_{6,3}$ thorn rod

Theorem 3.8. Let $P_{n,k}$ be a thorn rod graph. Then,

$$DI_g\left(P_{n,k}\right) = 2k + \lceil 2\sqrt{n+1}\rceil - 2$$

PROOF. Since $P_{n,k} \cong P_{n,0,k}$, as a result of Theorem 3.6 that $DI_g\left(P_{n,k}\right) = 2k + \lceil 2\sqrt{n+1}\rceil - 2$. □

Definition 3.9. [14] Let p and k be non-negative integers. A thorn star graph, denoted by $S_{n,p,k}$, is constructed by attaching k pendant edges to each pendant vertex and p pendant edges to the central vertex of the complete bipartite graph $K_{1,n}$.

Figure 5 illustrates the thorn star $S_{4,2,3}$.

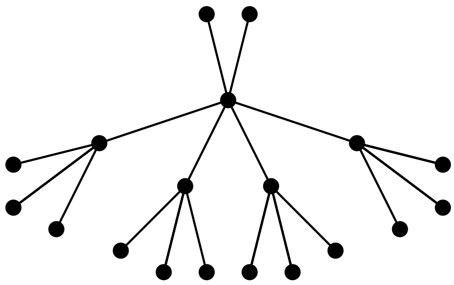

Figure 5. $S_{4,2,3}$ thorn star

Theorem 3.10. Let $S_{n,p,k}$ be a thorn star graph. Then,

$$DI_g\left(S_{n,p,k}\right) = 2 + p + nk$$

PROOF. Assume that $S_{n,p,k}$ is a thorn star graph, v is its central vertex, $\{x_i : \deg\left(x_i\right) = 1, i \in I_{nk}\}$, and $\{y_j : \deg\left(y_j\right) = 1, j \in I_p\}$ are pendant vertices. There is an equal number of pendant vertices and elements in the geodetic dominating set S, i.e., $|S| = p + nk$. Thus, $\gamma_g\left(S_{n,p,k}\right) = p + nk$. Removing the set $\{x_1, x_2, \cdots, x_{nk}\} \cup \{y_1, y_2, \cdots, y_p\}$ from the graph leaves a $K_{1,n}$ star graph with $n+1$ vertices. Removing the central vertex v from $K_{1,n}$ leaves n isolated vertices. Therefore, if the dominating set $S = \{x_1, x_2, \cdots, x_{nk}\} \cup \{y_1, y_2, \cdots, y_p\} \cup \{v\}$ is removed from $S_{n,p,k}$, then the largest component is $m\left(S_{n,p,k} - S\right) = 1$. Thus,

$$\begin{aligned} DI_g\left(S_{n,p,k}\right) &= |S| + m\left(S_{n,p,k} - S\right) \\ &= 1 + nk + p + 1 \\ &= 2 + p + nk \end{aligned}$$

The geodetic domination integrity value of the thorn star graph is

$$DI_g\left(S_{n,p,k}\right) = 2 + p + nk$$

□

Definition 3.11. [15] A Helm graph is constructed by adding a pendant edge to every vertex of the wheel graph W_n with the exception of the central vertex and denoted by H_n. H_n has $2n+1$ vertices and $3n$ edges.

Figure 6 illustrates the Helm graph H_6.

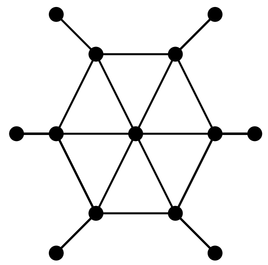

Figure 6. H_6 graph

Theorem 3.12. Let H_n be a Helm graph. Then,

$$DI_g(H_n) = n + \lceil 2\sqrt{n} \rceil$$

PROOF. Let H_n be a Helm graph with pendant vertices $\{x_i : \deg(x_i) = 1, i \in I_n\}$, v be the central vertex, S be a geodetic dominating set, and $m(H_n - S)$ be the largest component in $H_n - S$. The Helm graph H_n contains a cycle graph C_n with n vertices, and each vertex of C_n is adjacent to the central vertex v. Let $X \subseteq V(H_n)$ with $|X| = r$, and consider $S = \{v\} \cup \{x_1, x_2, \cdots, x_n\} \cup X$ as the dominating set. Then, $|S| = n + 1 + r$ and $\omega(H_n - S) \leq r$, implying

$$m(H_n - S) \geq \frac{2n + 1 - (n + 1 + r)}{r}$$

Hence,

$$DI_g(H_n) \geq \min_r \left\{ n + 1 + r + \frac{n - r}{r} \right\}$$

For $r \geq 0$, the function $f(r) = n + 1 + r + \frac{n-r}{r}$ has the minimum value $n + 2\sqrt{n}$. Since the geodetic domination integrity is an integer value,

$$DI_g(H_n) = n + \lceil 2\sqrt{n} \rceil$$

□

Definition 3.13. [10] The graph E_p^t is a graph with t legs, each containing p vertices.

Figure 7 illustrates the graph E_3^5.

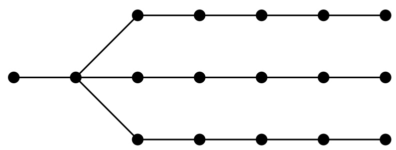

Figure 7. E_3^5 graph

Theorem 3.14. Let E_p^t be a tree graph. Then, the geodetic domination integrity of E_p^t is given by

$$DI_g\left(E_p^t\right) = \begin{cases} t\lfloor \frac{p+2}{3} \rfloor + 3, & n \equiv 1 \pmod 3 \\ t\lfloor \frac{p+2}{3} \rfloor + 4, & \text{otherwise} \end{cases}$$

PROOF. Let ${E_p}^t$ be a tree graph where x represents the vertex with degree 1, y represents the vertex with the maximum degree, and $u_1, u_2, \cdots, u_p$ represent the vertices on the paths (for every path t). Let S be a geodetic dominating set, $m({E_p}^t - S)$ be the largest component in ${E_p}^t - S$, and $I_G[S]$ denote the union of all the geodetic sets $I_G[a, b]$, for all $a, b \in S$.

i. For $n \equiv 0 \pmod 3$, let $S = \{u_{3k} : 1 \leq k \leq \frac{p}{3}\} \cup \{x\} \cup \{y\}$. Then, $|S| = t\left\lfloor \frac{p+2}{3} \right\rfloor + 2$. Since $u_{3k-2}, u_{3k-1} \in N(u_{3k})$ and $I_G[S] = V\left(E_p^t\right)$, S is a geodetic dominating set for E_p^t. Removing S from the graph yields $m\left(E_p^t - S\right) = 2$. Hence, $DI_g\left(E_p^t\right) = t\left\lfloor \frac{p+2}{3} \right\rfloor + 4$.

ii. For $n \equiv 1 \pmod 3$, let $S = \left\{u_{3k+1} : 0 \leq k \leq \frac{p-1}{3}\right\} \cup \{x\}$. Then, $|S| = t\left\lfloor\frac{p+2}{3}\right\rfloor + 1$. Since $u_{3k}, u_{3k+2} \in N\left(u_{3k+1}\right)$ and $I_G[S] = V\left(E_p^t\right)$, S is a geodetic dominating set for E_p^t. Removing S from the graph yields $m\left(E_p^t - S\right) = 2$. Hence, $DI_g\left(E_p^t\right) = t\left\lfloor\frac{p+2}{3}\right\rfloor + 3$.

iii. For $n \equiv 2 \pmod 3$, let $S = \left\{u_{3k+2} : 0 \leq k \leq \frac{p-2}{3}\right\} \cup \{x\} \cup \{y\}$. Then, $|S| = t\left\lfloor\frac{p+2}{3}\right\rfloor + 2$. Since $u_{3k}, u_{3k+1} \in N\left(u_{3k+2}\right)$ and $I_G[S] = V\left(E_p^t\right)$, S is a geodetic dominating set for E_p^t. Removing S from the graph yields $m\left(E_p^t - S\right) = 2$. Hence, $DI_g\left(E_p^t\right) = t\left\lfloor\frac{p+2}{3}\right\rfloor + 4$.

From *i*-*iii*,

$$DI_g\left(E_p^t\right) = \begin{cases} t\lfloor\frac{p+2}{3}\rfloor + 3, & n \equiv 1 \pmod 3 \\ t\lfloor\frac{p+2}{3}\rfloor + 4, & \text{otherwise} \end{cases}$$

□

Definition 3.15. [16] Let k and n be positive integers. Dendrimer graphs $D_{k,n}$ are constructed by adding k degree-one vertices to the vertices with degree one in the graph $D_{k,0}$, initially defined as D_0, for a total of n repetitions.

Figure 8 illustrates the graph D_0, while Figure 9 shows the graphs $D_{2,1}$ and $D_{2,2}$.

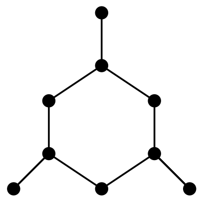

Figure 8. D_0 dendrimer

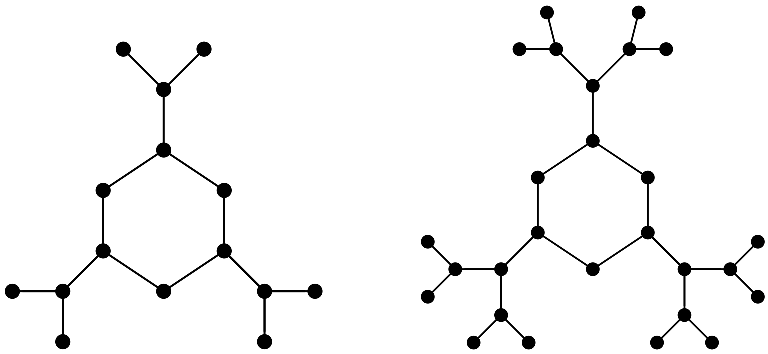

Figure 9. $D_{2,1}$ and $D_{2,2}$ dendrimer

Theorem 3.16. [4] Let H_n^k be a complete k-ary tree of height $n-1$. Then, the domination number is given by

$$\gamma\left(H_n^k\right) = \begin{cases} \frac{k\left(k^{(n/3)}-1\right)}{7}, & \text{if } n \equiv 0 \pmod 3 \\ 1 + \frac{k^2\left(k^{\left(\frac{n-1}{3}\right)}-1\right)}{7} & \text{if } n \equiv 1 \pmod 3 \\ 1 + \frac{k^3\left(k^{\left(\frac{n-2}{3}\right)}-1\right)}{7} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Theorem 3.17. Let $D_{k,n}$ be a dendrimer graph. For $k, n > 2$, the geodetic domination integrity of $D_{k,n}$ is

$$DI_g\left(D_{k,n}\right) = 3\left(\gamma\left(H_{n-2}^k\right) + k^n\right) + k + 4$$

PROOF. Let $D_{k,n}$ be a dendrimer graph with pendant vertices $\{x_i : \deg(x_i) = k^n, i \in I_{3(k^n)}\}$, S be a geodetic dominating set, and $m(D_{k,n} - S)$ be the largest component in $D_{k,n} - S$. The number of pendant vertices is equal to the number of vertices composing the smallest geodetic set of $D_{k,n}$. Therefore, $g(D_{k,n}) = 3k^n$. These vertices are the elements of the smallest geodetic dominating set S. They have the property of dominating the vertices that are at distance of n edges from the center, forming a C_6 graph. Thus, to find the dominating set of $D_{k,n}$, it suffices to find the minimum dominating set of $D_{k,n-2}$. The graph $D_{k,n-2}$ is formed by attaching three different H^k_{n-2} k-ary trees to the three vertices of a C_6 graph such that it is regular. Therefore, $|S| = 3\left(\gamma\left(H^k_{n-2}\right) + k^n\right) + 3$. In this case, $m(D_{k,n} - S) = k + 1$. Hence,

$$|S| + m(D_{k,n} - S) \geq 3\left(\gamma\left(H^k_{n-2}\right) + k^n\right) + 3 + (k + 1)$$

which leads to

$$DI_g(D_{k,n}) = 3\left(\gamma\left(H^k_{n-2}\right) + k^n\right) + k + 4$$

□

Definition 3.18. [16] A regular dendrimer graph is a tree consisting of a central vertex v and each non-pendant vertex has a degree d two or more. In regular dendrimers, the distance from the central vertex to each pendant vertex is called the radius and is denoted by k. Regular dendrimer graphs are denoted by $T_{k,d}$.

Figure 10 illustrates the regular dendrimers $T_{2,4}$ and $T_{3,4}$.

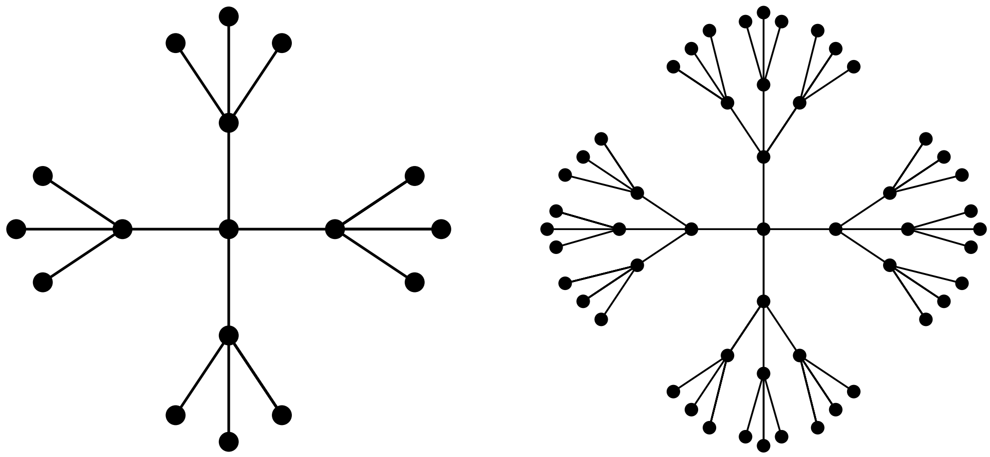

Figure 10. Regular dendrimers $T_{2,4}$ and $T_{3,4}$

Theorem 3.19. [17] Let $T_{k,d}$ be a regular dendrimer graph. Then, the domination number is given by

$$\gamma(T_{k,d}) = \begin{cases} 1 + \frac{(d-1)^k - d + 1}{d-2}, & k \text{ is odd} \\ \frac{(d-1)^k - 1}{d-2}, & k \text{ is even} \end{cases}$$

Theorem 3.20. For $k, d > 2$, the geodetic domination integrity of a regular dendrimer graph is

$$DI_g(T_{k,d}) = d + \gamma(T_{k-2,d}) + d(d-1)^{k-1}$$

PROOF. Suppose that $T_{k,d}$ is a regular dendrimer graph where $k, d > 2$, v is the central vertex, $\{x_i : d(v, x_i) = k, i \in I_{d(d-1)^{k-1}}\}$ are the pendant vertices, S is a geodetic dominating set, and $m(T_{k,d} - S)$ is the largest component in $T_{k,d} - S$. The number of vertices in $T_{k,d}$ forming the minimum geodetic set is equal to the number of pendant vertices. Hence, $g(T_{k,d}) = d(d-1)^{k-1}$. These vertices form the minimum geodetic dominating set S, and they dominate the vertices at distance $k-1$ from the central vertex. Therefore, to find the geodetic dominating set of $T_{k,d}$, it is sufficient to find the minimum dominating set of $T_{k-2,d}$. Hence, $|S| = \gamma(T_{k-2,d}) + d(d-1)^{k-1}$. In this case, $m(T_{k,d} - S) = d$. Therefore,

$$|S| + m(T_{k,d} - S) \geq \gamma(T_{k-2,d}) + d(d-1)^{k-1} + d$$

which leads to

$$DI_g(T_{k,d}) = d + \gamma(T_{k-2,d}) + d(d-1)^{k-1}$$

□

Definition 3.21. [18] A spider graph is constructed by adding a pendant edge to each pendant vertex of the $K_{1,k}$ graph and is denoted by S_k^*.

Figure 11 shows the spider graph S_3^*.

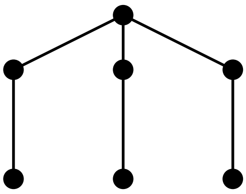

Figure 11. S_3^* graph

Theorem 3.22. Let S_k^* be a spider graph. Then, the geodetic domination integrity of S_k^* is

$$DI_g(S_k^*) = k + 2$$

Proof. Assume that S_k^* is a spider graph with a central vertex v, $\{x_i : \deg(x_i) = 1, i \in I_k\}$ are pendant vertices, and S is a geodetic dominating set. The number of vertices forming the minimum dominating set of S_k^* is equal to the number of pendant vertices. Hence, $g(S_k^*) = k$. Removing the set $\{x_1, x_2, \cdots, x_k\}$ from the graph leaves a star graph $K_{1,k}$ with $k+1$ vertices. Removing the central vertex v from $K_{1,k}$ leaves k isolated vertices. Therefore, if $S = \{x_1, x_2, \cdots, x_k\} \cup \{v\}$ is removed from S_k^*, then the largest component is $m(S_k^* - S) = 1$. Thus,

$$\begin{aligned} DI_g(S_k^*) &= |S| + m(S_k^* - S) \\ &= k + 1 + 1 \\ &= k + 2 \end{aligned}$$

Therefore, the geodetic domination integrity of the spider graph is

$$DI_g(S_k^*) = k + 2$$

□

Definition 3.23. [18] A bispider graph is constructed by adding one edge between the central vertices of two S_k^* graphs and is denoted by $S_{r,s}^*$.

Figure 12 shows the bispider graph $S_{3,3}^*$.

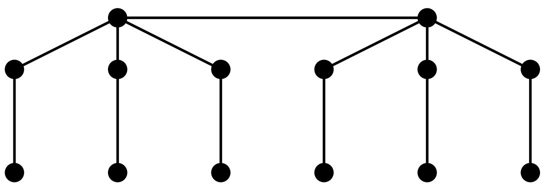

Figure 12. $S_{3,3}^*$ graph

Theorem 3.24. For a bispider graph $S^*_{r,s}$, the geodetic domination integrity is

$$DI_g\left(S^*_{r,s}\right) = r + s + 3$$

PROOF. Suppose that $S^*_{r,s}$ is a bispider graph with central vertices u and v, pendant vertices are $\{x_i : \deg(x_i) = 1, i \in I_r\}$ and $\{y_i : \deg(y_i) = 1, i \in I_s\}$, and S is a geodetic dominating set. The number of vertices forming the minimum geodetic set of $S^*_{r,s}$ is equal to the number of pendant vertices. Hence, $g\left(S^*_{r,s}\right) = r + s$. Removing the set $\{x_1, x_2, \cdots, x_r\} \cup \{y_1, y_2, \cdots, y_s\}$ from the graph leaves two star graphs connected to each other, each with $s+r+2$ vertices. Removing the central vertices u and v from this graph leaves $r+s$ isolated vertices. Therefore, if $S = \{x_1, x_2, \cdots, x_r\} \cup \{y_1, y_2, \cdots, y_s\} \cup \{u\} \cup \{v\}$ is removed from $S^*_{r,s}$, then the largest component is $m\left(S^*_{r,s} - S\right) = 1$. Thus,

$$\begin{aligned} DI_g\left(S^*_{r,s}\right) &= |S| + m\left(S^*_{r,s} - S\right) \\ &= r + s + 2 + 1 \\ &= r + s + 3 \end{aligned}$$

Then, the geodetic domination integrity of the bispider graph is

$$DI_g\left(S^*_{r,s}\right) = r + s + 3$$

□

4. Conclusion

In this study, a newly defined parameter geodetic domination integrity of some classes of graphs, thorn graphs, dendrimer graphs, helm graphs, E^t_p trees, spider graphs, and bispider graphs are investigated, and general formulas are obtained based on the order of the graphs. In future studies, it is recommended to apply this parameter on different types of graphs, especially on transformation graphs of thorny graphs. When a graph is transformed, it loses its old form, and a new graph structure is formed. If it is possible to decode the given graph from the encoded graph in polynomial time, such an operation can be used to analyze various structural properties of the original graph in terms of transformation graphs. As the geodetic domination integrity has a broad research area in graph theory and there is limited work on this newly defined parameter, it is expected that obtaining general results by applying this new parameter to transformation graphs will make a significant contribution to the literature.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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Some Results on Related Fixed Point Theorems in Two S-Metric Spaces

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Abstract — In this study, by considering the technique in the theorem of Bollenbacher and Hicks, we obtain some related fixed point theorems involving two mappings on two related complete S-metric spaces with certain conditions. As the applications of the main theorem, we then derive some new related fixed point theorems involving a pair of mappings in such spaces. We finally discuss the need for further research.

Keywords *S-metric space, fixed point, related fixed point, related complete S-metric space*

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1. Introduction

Recently, Sedghi et al. [1] have defined a new type of generalized metric space, S-metric space, and presented certain important properties of them. Further, they have proved a version of the famous Banach's theorem on complete S-metric spaces. This novel framework enriches the theoretical landscape and opens avenues for practical applications in diverse fields, such as optimization, computer science, and engineering. Afterward, many authors [2–8] have explored and established various fixed point theorems in S-metric spaces, further solidifying the significance of this emerging area of study.

Moreover, Fisher [9] has provided a theorem involving compositions of two mappings on two complete metric spaces and investigated the relation between the fixed points for these mappings. This theorem is the initial related fixed point result involving two mappings on two complete metric spaces. Several authors [10–17] have established various types of this theorem in many different directions.

In the present study, we introduced the concept related to completeness in two S-metric spaces. In section 3, by considering the technique in the theorem of Bollenbacher and Hicks [18], we establish certain related fixed point theorems involving two mappings in two S-metric spaces under certain conditions. In section 4, as applications of the main theorem, we derive some new related fixed point theorems involving a pair of mappings in such spaces. In addition, we provide an illustrative example of the main result. In the last section, we address whether additional research concerning the aforesaid notions is needed.

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2. Preliminaries

This section presents S-metric spaces and some of their basic properties.

Definition 2.1. [1] An S-metric space (W,S) is a nonempty set W with a non negative real-valued function S on W^3 such that, for all $u,v,y,z \in W$, the following conditions hold,

(SM1) $S(u,v,y)=0 \Leftrightarrow u=v=y$

(SM2) $S(u,v,y) \leq S(u,u,z)+S(v,v,z)+S(y,y,z)$

Here, the function S is said to be an S-metric on W.

Lemma 2.2. [1] If S is an S-metric on W, then $S(v,v,u)=S(u,u,v)$, for all $u,v \in W$.

Definition 2.3. [1] Let $\{v_k\}$ be a sequence in an S-metric space. Then,

i. $\{v_k\}$ is a convergent to $v \in W$ if, for all $\varepsilon>0$, there exists a $k_0 \in \mathbb{N}$ such that $S(v_k,v_k,v)<\varepsilon$ whenever $k \geq k_0$

ii. $\{v_k\}$ is a Cauchy sequence if, for all $\varepsilon>0$, there exists a $k_0 \in \mathbb{N}$ such that $S(v_k,v_k,v_l)<\varepsilon$ whenever $k,l \geq k_0$

Moreover, an S-metric space (W,S) is said to be complete if every Cauchy sequence in W is convergent in W.

Lemma 2.4. [1] Let (W,S) be an S-metric space.

i. If $\{v_k\}$ is a sequence in W such that $\lim\limits_{k\to\infty} v_k=v$, then v is unique

ii. If a sequence $\{v_k\}$ is convergent in W, then $\{v_k\}$ is a Cauchy sequence

iii. If $\{v_k\}$ and $\{w_k\}$ are two sequences in W such that $\lim\limits_{k\to\infty} v_k=v$ and $\lim\limits_{k\to\infty} w_k=w$, then

$$\lim_{k\to\infty} S(v_k,v_k,w_k)=S(v,v,w)$$

Definition 2.5. [17, 19] Let (V,S_1) and (W,S_2) be two S-metric spaces. A real-valued function $\delta: V\times W \to [0,\infty)$ is a weak lower semi-continuous (briefly WLSC) at $(v,w) \in V\times W$ if and only if $\{v_k\}$ and $\{w_k\}$ are sequences in V and W, respectively, and

$$\lim_{k\to\infty} v_k=v \wedge \lim_{k\to\infty} w_k=w \quad \Rightarrow \quad \delta(v,w) \leq \lim_{k\to\infty}\sup \delta(v_k,w_k)$$

Definition 2.6. [17, 19] Let (V,S_1) and (W,S_2) be two S-metric space and $A:V\to W$ and $B:W\to V$ be two mappings. The sequences $\{v_k\}$ in V and $\{w_k\}$ in W defined by

$$v_k=(BA)^k v_0 \quad \text{and} \quad w_k=A(BA)^{k-1}v_0, \quad k\in\mathbb{Z}^+$$

We use these sequences in the main results. Consider the following sets

$$\Re_V(v_0)=\left\{(BA)^k v_0 : k\in\mathbb{Z}^+\right\} \quad \text{and} \quad \Re_W(v_0)=\left\{A(BA)^{k-1}v_0 : k\in\mathbb{Z}^+\right\}$$

Then, the S-metric spaces (V,S_1) and (W,S_2) are said to be related complete if each Cauchy sequence in $\Re_V(v_0)$ and $\Re_W(v_0)$ converges to a point in V and converges to a point in W, respectively.

Note that two complete S-metric spaces (V,S_1) and (W,S_2) are related complete. However, the reverse is not generally true, as shown in the following example.

Example 2.7. [17] Let $V=(-1,1]$, $W=[0,1]$, and $|\,.\,|$ be absolute value function on V and W. Then, $S(u,v,y)=|u-y|+|v-y|$ is an S-metric on V and W. Let the mappings $A:V\to W$ and

$B : W \to V$ be defined by

$$A(v) = \begin{cases} 0, & -1 < v < 0 \\ 1, & 0 \leq v \leq 1 \end{cases} \quad \text{and} \quad B(w) = \begin{cases} -\frac{1}{2}, & 0 \leq w < \frac{1}{2} \\ 1, & \frac{1}{2} \leq w \leq 1 \end{cases}$$

Then, for any $v_0 \in (-1, 0)$, $\Re_V(v_0) = \left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \cdots\right\}$ and $\Re_W(v_0) = \{0, 0, 0, \cdots\}$. Thus, (V, S) and (W, S) are related complete. However, (V, S) is not complete.

3. Main Results

This section proposes the following related fixed point theorem in two related complete S-metric spaces.

Theorem 3.1. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$\max\{S_1(v, v, BAv), S_2(w, w, ABw)\} \leq \alpha(v) - \alpha(BAv) + \beta(w) - \beta(ABw) \tag{3.1}$$

for all $v \in \Re_V(v_0)$ and for all $w \in \Re_W(v_0)$ where $\alpha : V \to [0, \infty)$ and $\beta : W \to [0, \infty)$ are two mappings, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$.

Proof. *i.* From (3.1),

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \max\{S_1(v_n, v_n, v_{n+1}), S_2(w_n, w_n, w_{n+1})\} \\ &= \sum_{n=1}^{k} \max\{S_1(v_n, v_n, BAv_n), S_2(w_n, w_n, ABw_n)\} \\ &\leq \sum_{n=1}^{k} [\alpha(v_n) - \alpha(BAv_n) + \beta(w_n) - \beta(ABw_n)] \\ &= \sum_{n=1}^{k} [\alpha(v_n) - \alpha(v_{n+1}) + \beta(w_n) - \beta(w_{n+1})] \\ &= \alpha(v_1) - \alpha(v_{k+1}) + \beta(w_1) - \beta(w_{k+1}) \leq \alpha(v_1) + \beta(w_1) \end{aligned}$$

Thus, $\{s_k\}$ is bounded above. Moreover, $\{s_k\}$ is non-decreasing. Thus, it is convergent.

Let k and l be any two positive integers with $k < l$. Using (SM2) of S-metric and Lemma 2.2,

$$\begin{aligned} \max\{S_1(v_l, v_l, v_k), S_2(w_l, w_l, w_k)\} &\leq \max\left\{\sum_{n=k}^{l-1} 2S_1(v_n, v_n, v_{n+1}), \sum_{n=k}^{l-1} 2S_2(w_n, w_n, w_{n+1})\right\} \\ &\leq 2\sum_{n=k}^{l-1} \max\{S_1(v_n, v_n, v_{n+1}), S_2(w_n, w_n, w_{n+1})\} \end{aligned} \tag{3.2}$$

Since $\{s_k\}$ convergent, for any $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\sum_{n=k}^{\infty} \max\{S_1(v_n, v_n, v_{n+1}), S_2(w_n, w_n, w_{n+1})\} < \varepsilon/2$$

for all $k \geq n_0$. Hence, from (3.1),

$$\max\{S_1(v_l, v_l, v_k), S_2(w_l, w_l, w_k)\} < \varepsilon$$

for all $k, l \geq n_0$. Thus, $\{v_k\}$ and $\{w_k\}$ are two Cauchy sequence in $\Re_V(v_0)$ and $\Re_W(v_0)$, respectively. Since (V, S_1) and (W, S_2) are related complete, the sequence $\{v_k\}$ converges to a point $z \in V$ and the sequence $\{w_k\}$ converges to a point $u \in W$.

ii. Suppose that $Bu = z$ and $Az = u$ and $\{v_k\}$ and $\{w_k\}$ are sequences in V and W, respectively, such that $v_k \to z$ and $w_k \to u$. Then,

$$P(z, u) = S_1(z, z, Bu) = 0 \leq \limsup S_1(v_k, v_k, Bw_k) = \limsup P(v_k, w_k)$$

and

$$Q(z, u) = S_2(u, u, Az) = 0 \leq \limsup S_2(w_k, w_k, Av_k) = \limsup Q(v_k, w_k)$$

Thus, P and Q are WLSC at (z, u).

Conversely, let $v_k = (BA)^k v_0$, $w_k = A(BA)^{k-1} v_0$, and P and Q are WLSC at (z, u). It follows from *i*,

$$\begin{aligned} 0 \leq S_1(z, z, Bu) = P(z, u) &\leq \limsup P(v_k, w_k) \\ &= \limsup S_1(v_k, v_k, Bw_k) \\ &= \limsup S_1(v_k, v_k, v_k) = 0 \end{aligned}$$

and using Lemma 2.4,

$$\begin{aligned} 0 \leq S_2(u, u, Az) = Q(z, u) &\leq \limsup Q(v_k, w_k) \\ &= \limsup S_2(w_k, w_k, Av_k) \\ &= \limsup S_2(w_k, w_k, w_{k+1}) = 0 \end{aligned}$$

Thus, $Bu = z$ and $Az = u$.

Moreover, assume that *ii* holds. Then, $BAz = z$ and $ABu = u$. □

Remark 3.2. Since the inequality

$$\max\{S_1(v, v, BAv), S_2(w, w, ABw)\} \leq S_1(v, v, BAv) + S_2(w, w, ABw)$$

holds, for all $v \in V$ and for all $w \in W$, then Theorem 3.1 holds for the following inequality

$$S_1(v, v, BAv) + S_2(w, w, ABw) \leq \alpha(v) - \alpha(BAv) + \beta(w) - \beta(ABw) \tag{3.3}$$

If $(V, S_1) = (W, S_2) = (U, S)$, then from Theorem 3.1, we have the following result: The (U, S) is BA-orbitally complete, defined as each Cauchy sequence in the set

$$O_{BA}(v_0, \infty) = \{v_0, Av_0, BAv_0, \cdots, A(BA)^{k-1}v_0, (BA)^k v_0, \cdots\}$$

converges in U, where $v_0 \in U$.

Corollary 3.3. Let (U, S) be an S-metric space and A and B be two mappings of U into itself. If there exists a $v_0 \in U$ such that (U, S) is BA-orbitally complete and

$$\max\{S(v, v, BAv), S(w, w, ABw)\} \leq \alpha(v) - \alpha(BAv) + \beta(w) - \beta(ABw)$$

for all $v, w \in O_{BA}(v_0, \infty)$ where $\alpha, \beta : U \to [0, \infty)$ are two mappings, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : U \times U \to [0, \infty)$, defined by $P(v, w) = S(v, v, Bw)$, and $Q : U \times U \to [0, \infty)$, defined by $Q(v, w) = S(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$ and thus if $u = z$, then $Az = Bz = z$.

Note that a similar result can be proved for (3.3).

If $(V, S_1) = (W, S_2) = (U, S)$ and $A = B$, then the set $\Re_U(v_0) = \{A^k v_0 : k \in \mathbb{Z}^+\}$ is equal to $O_A(v_0, \infty)$, which is called orbit of $v_0 \in U$. Moreover, (U, S) is A-orbitally complete if each Cauchy sequence in $O_A(v_0, \infty)$ converges to U.

If $(V, S_1) = (W, S_2) = (U, S)$, $B = I$, and $\alpha = \beta$ where I is a identity mapping of U, then using (3.3), the following Bollenbacher and Hicks's result [18] in S-metric version is obtained.

Corollary 3.4. Let (U, S) be an S-metric space and $A : U \to U$ be a mapping. If there exists a $v_0 \in U$ such that (U, S) is A-orbitally complete and

$$S(v, v, Av) \leq \alpha(v) - \alpha(Av)$$

for all $v \in O_{BA}(v_0, \infty)$ where $\alpha : U \to [0, \infty)$ is a mapping, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} A^k v_0 = z$

ii. $Az = z$ if and only if $P : U \to [0, \infty)$, defined by $P(v) = S(v, v, Av)$, is WLSC at z.

4. Additional Results

This section drives some related fixed point theorems in two S-metric spaces using the Theorem 3.1.

Theorem 4.1. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$S_1(BAv, BAv, (BA)^2 v) + S_2(ABw, ABw, (AB)^2 w) \leq r[S_1(v, v, BAv) + S_2(w, w, ABw)] \tag{4.1}$$

for all $v \in \Re_V(v_0)$ and for all $w \in \Re_W(v_0)$ where $0 \leq r < 1$, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$.

Proof. Define the mappings $\Upsilon(v) = \frac{1}{1-r} S_1(v, v, BAv)$, for all $v \in V$, and $\Gamma(w) = \frac{1}{1-r} S_2(w, w, ABw)$, for all $w \in W$. Then, from (4.1),

$$S_1(v, v, BAv) + S_2(w, w, ABw) - r[S_1(v, v, BAv) + S_2(w, w, ABw)] \leq S_1(v, v, BAv) + S_2(w, w, ABw) \\ -[S_1(BAv, BAv, (BA)^2 v) + S_2(ABw, ABw, (AB)^2 w)]$$

and thus

$$\begin{aligned} S_1(v, v, BAv) + S_2(w, w, ABw) \leq & \frac{1}{1-r}[S_1(v, v, BAv) + S_2(w, w, ABw)] \\ & - \frac{1}{1-r}[S_1(BAv, BAv, (BA)^2 v) + S_2(ABw, ABw, (AB)^2 w)] \end{aligned}$$

Thus,

$$S_1(v, v, BAv) + S_2(w, w, ABw) \leq \Upsilon(v) - \Upsilon(BAv) + \Gamma(w) - \Gamma(ABw)$$

The results follow from (3.3) and Theorem 3.1. $\square$

Corollary 4.2. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$\max\{S_1(BAv, BAv, (BA)^2v), S_2(ABw, ABw, (AB)^2w)\} \leq \frac{r}{2}[S_1(v, v, BAv) + S_2(w, w, ABw)] \quad (4.2)$$

for all $v \in \Re_V(v_0)$ and for all $w \in \Re_W(v_0)$ where $0 \leq r < 1$, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$.

Proof. Using (4.2),

$$\begin{aligned} S_1(BAv, BAv, (BA)^2v) + S_2(ABw, ABw, (AB)^2w) &\leq 2\max\{S_1(BAv, BAv, (BA)^2v), S_2(ABw, ABw, (AB)^2w)\} \\ &\leq 2\frac{r}{2}[S_1(v, v, BAv) + S_2(w, w, ABw)] \\ &= r[S_1(v, v, BAv) + S_2(w, w, ABw)] \end{aligned}$$

Then, the results follow immediately from Theorem 4.1. □

Theorem 4.3. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$S_1(BAv, BAv, BAv') + S_2(ABw, ABw, ABw') \leq r[S_1(v, v, v') + S_2(w, w, w')] \quad (4.3)$$

for all $v, v' \in \Re_V(v_0)$ and for all $w, w' \in \Re_W(v_0)$, where $0 \leq r < 1$, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then z is a unique fixed point of BA and u is a unique fixed point of AB.

Proof. Put $v' = BAv$ and $w' = ABw$ in (4.3). Then, (4.1) holds. The results follow immediately from Theorem 4.1.

To prove uniqueness, if z' is a second fixed point of BA, then from (4.3), for $w = w'$,

$$\begin{aligned} S_1(z, z, z') &= S_1(BAz, BAz, BAz') + S_2(ABw, ABw, ABw) \\ &\leq r[S_1(z, z, z') + S_2(w, w, w)] \\ &= rS_1(z, z, z') \end{aligned}$$

proving that $z = z'$, since $r < 1$. Similarly, u is a unique fixed point of AB. □

Corollary 4.4. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$\max\{S_1(BAv, BAv, BAv'), S_2(ABw, ABw, ABw')\} \leq \frac{r}{2}[S_1(v, v, v') + S_2(w, w, w')] \quad (4.4)$$

for all $v, v' \in \Re_V(v_0)$ and for all $w, w' \in \Re_W(v_0)$ where $0 \leq r < 1$, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then z is a unique fixed point of BA and u is a unique fixed point of AB.

PROOF. Using (4.4),

$$\begin{aligned} S_1(BAv, BAv, BAv') + S_2(ABw, ABw, ABw') &\leq 2\max\{S_1(BAv, BAv, BAv'), S_2(ABw, ABw, ABw')\} \\ &\leq 2\frac{r}{2}[S_1(v, v, v') + S_2(w, w, w')] \\ &= r[S_1(v, v, v') + S_2(w, w, w')] \end{aligned}$$

The results follow immediately from Theorem 4.3. □

We provide an example relevant main result.

Example 4.5. Let $V = (-1, 1]$, $W = [0, 1]$, and $|\,.\,|$ is absolute value function on V and W. Then, $S(u, v, y) = |u - y| + |v - y|$ is an S-metric space on V and W. Define the mappings $A : V \to W$ and $B : W \to V$ by

$$A(v) = \begin{cases} 0, & -1 < v < 0 \\ \frac{1}{2}, & 0 \leq v \leq 1 \end{cases} \quad \text{and} \quad B(w) = \begin{cases} -\frac{1}{3}, & 0 \leq w < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leq w \leq 1 \end{cases}$$

We have

$$AB(w) = \begin{cases} 0, & 0 \leq w < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leq w \leq 1 \end{cases} \quad \text{and} \quad BA(v) = \begin{cases} -\frac{1}{3}, & -1 < v < 0 \\ \frac{1}{2}, & 0 \leq v \leq 1 \end{cases}$$

If $-1 < v_0 < 0$, then

$$\Re_V(v_0) = \left\{-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \cdots\right\} \quad \text{and} \quad \Re_W(v_0) = \{0, 0, 0, \cdots\}$$

Thus, (V, S) and (W, S) are related complete and (3.1) holds for $v \in \Re_V(v_0)$ and $w \in \Re_W(v_0)$ where $\alpha : V \to [0, \infty)$ and $\beta : W \to [0, \infty)$ are two mappings. Further,

$$\lim_{k\to\infty} (BA)^k v_0 = -\frac{1}{3} \quad \text{and} \quad \lim_{k\to\infty} A(BA)^{k-1} v_0 = 0$$

and $P(v, w) = 2|v - Bw|$ and $Q(v, w) = 2|w - Av|$ are WLSC at $\left(-\frac{1}{3}, 0\right) \in V \times W$. It can be observed that

$$B(0) = -\frac{1}{3}, \quad A\left(-\frac{1}{3}\right) = 0, \quad BA\left(-\frac{1}{3}\right) = -\frac{1}{3}, \quad \text{and} \quad AB(0) = 0$$

5. Conclusion

In this study, by considering the paper of Bollenbacher and Hicks [18], we proved some new related fixed point theorems for pair of mappings in two related complete and S-metric spaces under suitable conditions. Furthermore, we derived some related fixed-point theorems from the main results herein. Further, we introduced the concept related to completeness in two S-metric spaces. Note that two complete S-metric spaces are related complete. However, the reverse is not generally true. In future studies, considering the concept of related completeness, some related fixed point theorems can be proved in various types of metric spaces under suitable conditions, such as b-metric spaces, fuzzy metric spaces, and G^*-metric spaces.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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