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Rings Whose Pure-Projective Modules Have Maximal or Minimal Projectivity Domain

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Abstract — In this study, we investigate the projectivity domain of pure-projective modules. A pure-projective module is called special-pure-projective (s-pure-projective) module if its projectivity domain contains only regular modules. First, we describe all rings whose pure-projective modules are s-pure-projective, and we show that every ring with an s-pure-projective module. Afterward, we research rings whose pure-projective modules are projective or s-pure-projective. Such rings are said to have $*$-property. We determine the right Noetherian rings have $*$-property.

Keywords *Projectivity domain, pure-projective module, s-pure-projective module, von Neumann regular rings, right Goldie torsion rings*

Mathematics Subject Classification (2020) 16D10, 16D40

1. Introduction

Let R be an associative ring with identity throughout the article, and unless otherwise indicated, any module be a right R-module. Projectivity has been investigated from various angles in the recent studies [1–15]. The class $\{Y \in \mathcal{M}od\text{-}R : X \text{ is } Y\text{-projective}\}$ for a module X is referred to as the projectivity domain of X and is represented by $\mathfrak{Pr}^{-1}(X)$ [16]. It is clear that X is projective if and only if $\mathfrak{Pr}^{-1}(X) = \mathcal{M}od\text{-}R$. Projectively poor (p-poor) modules whose projectivity domains contain only semisimple modules in $\mathcal{M}od\text{-}R$ and rings with no right p-middle class whose modules are projective or p-poor were explored in [1].

We study pure-projective modules in their projectivity domain. Initially, we address the presence of special pure-projective (s-pure-projective) modules, and we prove that an s-pure-projective module exists for every ring. Subsequently, we examine rings, each of whose pure-projective modules is s-pure-projective; these rings are specifically von Neumann regular rings or vNr rings for short. We study rings that have $*$-property, that is, their pure-projective modules are projective or s-pure-projective. For example, semisimple Artinian rings and vNr rings have $*$-property. Additionally, a quasi-Frobenius ring R with a homogeneous right socle and $J(R)^2 = 0$ is also such a ring, for more details, see [5]. We provide the structure of rings that have $*$-property over right Noetherian rings (Theorem 4.10): if R is a ring with $*$-property, then $R \cong \Lambda' \times \Lambda$ where Λ' is semisimple Artinian, and Λ is either zero or an indecomposable ring, which satisfies one of the following cases:

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i. Λ is a right Artinian, and right SI-ring with $J(\Lambda) \neq 0$

ii. Λ is a right Artinian and right Goldie torsion ring with $J(\Lambda) \neq 0$

iii. Λ is a right Artinian ring, and $\mathrm{Soc}(\Lambda_\Lambda) = Z_r(\Lambda) = J(\Lambda) \neq 0$

iv. Λ is a prime ring with $J(\Lambda) = \mathrm{Soc}(\Lambda_\Lambda) = 0$

Finally, we include examples for the cases of Theorem 4.10 as well as a partial answer for the converse of Theorem 4.10.

2. Preliminaries

This section provides some basic notions to be required the following sections. Let X be an R-module. If Y is a submodule, essential submodule, or direct summand of X, we denote $Y \leq X$, $Y \leq_e X$, or $Y \leq_d X$, respectively. For a module X and a ring R, $\mathrm{Rad}(X)$, $J(R)$, $\mathrm{Soc}(X)$, $\mathrm{Soc}(R_R)$, $Z(X)$, $Z_r(R)$, and $Z_2(X)$ stand for the Jacobson radical of X, the Jacobson radical of the ring R, the socle of X, the right socle of the ring R, the singular part of X, the right singular part of the ring R, and the second singular part of X, respectively.

Definition 2.1. A submodule Y of a module X is called pure if there is a monomorphism $i \otimes_R 1_A : Y \otimes_R A \to X \otimes_R A$, for all left R-module A.

Definition 2.2. A module T is called regular if every submodule of T is pure.

The set of all the regular modules is represented by $\mathcal{R}egular$.

Definition 2.3. A short exact sequence

$$\mathbb{E} : 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow Z \longrightarrow 0$$

is called a pure short exact sequence if $\mathrm{Im}(f)$ is a pure submodule of Y.

Definition 2.4. A module T is called pure-projective if it is projective relative to any pure short exact sequence.

$\mathcal{P}$ is our abbreviation for the collection of all pure-projective modules. A module T is pure-projective if and only if T is a summand of a direct sum of finitely presented modules.

Remark 2.5. [1] For a ring R, the following are equivalent.

i. R is semisimple Artinian

ii. Every right R-module is p-poor

iii. There exists a projective p-poor module

iv. $\{0\}$ is p-poor

v. R is p-poor

Proposition 2.6. [1] If R is a (non-semisimple Artinian) quasi-Frobenius ring with homogeneous right socle and $J(R)^2 = 0$, then R has no right p-middle class and $\mathfrak{In}^{-1}(M) = \mathfrak{Pr}^{-1}(M)$ for all right R-module M.

A ring R is called right pure-semisimple if any pure submodule of a module is a direct summand, right SI-ring if every singular right module is injective. A ring R is called semi-primary if $R/J(R)$ is semisimple Artinian and $J(R)$ is a nilpotent ideal, prime if for any two ideals A and B of R, $AB = 0$ implies $A = 0$ or $B = 0$, semiprime if there is no a nonzero nilpotent ideal in R, right Goldie torsion if R is equal to its second singular submodule. For more details, see [16–19].

3. Existence of s-pure-projective Modules

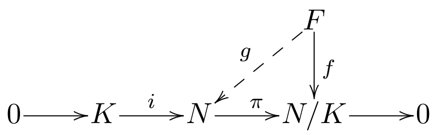
Let X represent an R-module. Then, $\mathfrak{Pr}^{-1}(X)$ is closed under submodules, finite direct sums, and epimorphic images [16]. It is clear from definitions that if T is a regular R-module and X is a pure-projective R-module, then $T \in \mathfrak{Pr}^{-1}(X)$.

Proposition 3.1. $\bigcap_{X \in \mathcal{P}} \mathfrak{Pr}^{-1}(X) = \mathcal{R}egular$.

PROOF. Let

$$N \in \bigcap_{X \in \mathcal{P}} \mathfrak{Pr}^{-1}(X)$$

It suffices to show that N is a regular module. Let $K \leq N$. Let F be a finitely presented R-module and $f : F \to N/K$ be any R-module homomorphism. Then, there exists $g : F \to N$, since F is a pure-projective module. The following diagram can be constructed.



where i is a canonical monomorphism and π is a canonical epimorphism. Hence, K is a pure submodule of N. □

Definition 3.2. A pure-projective module X is called s-pure-projective if $\mathfrak{Pr}^{-1}(X) = \mathcal{R}egular$.

The existence problem of s-pure-projective modules, or whether they exist in each ring, is the first question. The following proposition provides a favorable response to this query.

Proposition 3.3. S-pure-projective module is present for any ring R.

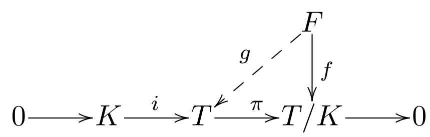
PROOF. A full set of representations of the isomorphism class of finitely presented R-modules is denoted by $\{X_\gamma \mid \gamma \in \Gamma\}$. Let

$$X = \bigoplus_{\gamma \in \Gamma} X_\gamma$$

It is obvious that X is a pure-projective module. Let $T \in \mathfrak{Pr}^{-1}(X)$ and $K \leq T$. Consider the short exact sequence

$$\mathbb{E} : 0 \to K \to T \to T/K \to 0$$

It suffices to show that $\mathbb{E}$ is a pure short exact sequence. Take any R-module homomorphism $f : F \to T/K$, where F is a finitely presented module. Then, F is a T-projective module since X is a T-projective module. Hence, there exists $g : F \to T$ such that $\pi \circ g = f$, that is, we can construct the following commutative diagram



where i is a canonical monomorphism and π is a canonical epimorphism. Hence, $\mathbb{E}$ is a pure, short, exact sequence, as desired. □

We can observe that

$$\mathfrak{Pr}^{-1}(X) = \mathfrak{Pr}^{-1}\left(\bigoplus X\right) = \bigcap \mathfrak{Pr}^{-1}(X)$$

In addition, we collect some useful properties of s-pure-projective modules.

Remark 3.4. Let X be a pure-projective R-module and $\{X_i\}_{i\in I}$ be a family of pure-projective R-modules. Then,

i. X is an s-pure-projective module if and only if $\bigoplus X$ is an s-pure-projective module

ii. $\bigoplus_{i\in I}(X_i)^{(J_i)}$ is an s-pure-projective module if and only if $\bigoplus_{i\in I} X_i$ is an s-pure-projective module

Proposition 3.5. Let X be an s-pure-projective R-module. If X is a direct summand of a pure-projective R-module Y, then Y is s-pure-projective.

Proof. Let $Y = X \oplus N$ where N is a module. Let Y be a T-projective. Then, N is pure-projective and X is T-projective, thus T is a regular module by our assumption. Hence,

$$\mathfrak{Pr}^{-1}(X \oplus N) = \mathcal{R}egular$$

that is, Y is s-pure-projective. □

Corollary 3.6. The arbitrary direct sum of s-pure-projective modules is s-pure-projective.

Lemma 3.7. Let R represent a ring. The expressions below are equivalent.

i. Any pure-projective R-module is s-pure-projective

ii. Any finitely presented R-module is s-pure-projective

iii. A projective s-pure-projective R-module exists

iv. $\{0\}$ is an s-pure-projective R-module

v. R is an s-pure-projective R-module

vi. R is a vNr ring

Corollary 3.8. A ring R is an s-pure-projective R-module if and only if $M_n(R)$ is an s-pure-projective $M_n(R)$-module.

Proposition 3.5 is incorrect in its opposite sense, meaning that a direct summand of an s-pure-projective module is not always s-pure-projective.

Example 3.9. A full set of representations of the isomorphism class of finitely presented R-modules is denoted by $\{X_\gamma \mid \gamma \in \Gamma\}$. Let

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma$$

X is an s-pure-projective module by Proposition 3.3. But if R is not vNr, then R is not s-pure-projective see Lemma 3.7. Hence, a copy of R in X as a summand is not s-pure-projective.

In closing this section, we investigate the relationship between p-poor and s-pure-projective modules for pure-projective R-modules. It is clear that if a pure-projective R-module X is p-poor, then it is s-pure-projective. The converse is not true in general; for example, a vNr ring R which is not semisimple Artinian is s-pure-projective but not p-poor, see Remark 2.2 in [1] and Lemma 3.7. As for the converse for a pure semisimple ring R, a pure-projective module X is s-pure-projective if and only if X is p-poor.

4. Rings Whose Pure-Projective Modules are Either s-pure-projective or Projective

This section addresses rings claimed to have $*$-property, meaning that their pure-projective modules are either projective or s-pure-projective. If a ring is not one of these rings, it is considered to has no $*$-property.

Note that a ring R is called right hereditary if every submodule of a projective right module is projective.

Proposition 4.1. Let R be a right hereditary ring. If R has $*$-property, then any pure-projective module that contains an s-pure-projective submodule is s-pure-projective.

Proof. If R is a vNr ring, then any pure-projective R-module is s-pure-projective by Lemma 3.7. Therefore, we may suppose R is not vNr without losing generality. Let X be an s-pure-projective R-module and $X \leq X'$, where X' is a pure-projective R-module. X' is projective or s-pure-projective by our assumption. If X' is projective, then X is projective by our right hereditary assumption. This is impossible by Lemma 3.7 since R is not vNr. Hence, X' must be an s-pure-projective module. □

Lemma 4.2. Let R be a ring with $*$-property and $0 \neq A$ be a finitely generated two-sided ideal of R. Then, $A \leq_d R$ or R/A is a vNr ring.

Proof. R/A is a finitely presented R-module; therefore, it is pure-projective. Then, R/A is projective or s-pure-projective by our assumption, that is, $A \leq_d R$ or $(R/A)_R$ is an s-pure-projective module. If $(R/A)_R$ is an s-pure-projective module, then $(R/A)_R$ is regular since A is fully invariant, and thus R/A is a (R/A)-projective (or quasi-projective) R-module. Hence, $(R/A)_{R/A}$ is regular for any two-sided ideal of R. □

Remark 4.3. Let A be a two-sided ideal of R, a ring with $*$-property. Then,

i. If $X_{R/A}$ is $N_{R/A}$-(R/A)-projective, then X_R is N_R-projective

ii. If $X_{R/A}$ is non-regular, then X_R is non-regular

Factor rings inherit the $*$-property as indicated by the following assertion.

Lemma 4.4. Let A be a two-sided ideal of R, a ring with $*$-property. Therefore, R/A has $*$-property.

Proof. Let X be a pure-projective (R/A)-module, which is not s-pure-projective. It is clear that X is a pure-projective R-module. There exists a non-regular (R/A)-module N such that $X_{R/A}$ is $N_{R/A}$-(R/A)-projective since X is a (R/A)-module, which is not s-pure-projective. This implies that X is N-projective, which means that X_R is not s-pure-projective since N is also a non-regular R-module. Then, X must be a projective R-module by assumption. Hence, X is a projective (R/A)-module. □

Note that a semilocal ring R is a ring, for which $R/J(R)$ is a semisimple Artinian ring. We can easily see that if R is a vNr and right Noetherian ring, then R is a semisimple Artinian ring. Note that a right Noetherian right semiartian ring is right Artinian. Therefore, the next result can be easily obtained with the help of Lemma 4.2.

Corollary 4.5. Let R be a right Noetherian ring with $*$-property, and A be a nonzero two-sided ideal of R. Then, the following are hold:

i. $A \leq_d R$ or R/A is a semisimple Artinian ring

ii. $\mathrm{Soc}(R_R) \leq_d R$ or R is a right Artinian ring

iii. If $J(R)$ is nonzero, then R is a semilocal ring

Lemma 4.6. Let R be a right Noetherian ring with $J(R) \neq J(R)^2$ and has $*$-property. Then, $J(R)^2 = 0$.

Proof. Suppose the contrary that $J(R)^2 \neq 0$. Then, $R/J(R)^2$ is a semisimple Artinian ring by Corollary 4.5. This implies that $J(R) = J(R)^2$, which provides a contradiction. Hence, $J(R) = 0$. □

Lemma 4.7. Let R be a right Noetherian ring with $*$-property. Then, R is either a right Artinian ring with $J(R)^2 = 0$ or a semiprime ring.

PROOF. Suppose that R is not semiprime. Let A be a nilpotent two-sided ideal of R. Then, R/A s-pure-projective by assumption, and nilpotent ideals are small in R. It is clear that R/A is a semisimple Artinian ring by Corollary 4.5. Then, it is clear that $J(R) = A$. Thus, R is a semilocal ring by Corollary 4.5; therefore, a semiprimary ring since $J(R)$ is a nilpotent ideal. Hence, R is a right Artinian ring by Hopkins-Levitzki theorem. Furthermore, $J(R)^2 = 0$ by Lemma 4.6. □

Lemma 4.8. Let R be an indecomposable right Noetherian ring with $*$-property. After that, $\mathrm{Soc}(R_R) \leq_e R$ with either $J(R)^2 = 0$ or $\mathrm{Soc}(R_R) = 0$.

PROOF. $R/\mathrm{Soc}(R_R)$ is a projective or an s-pure-projective R-module by assumption. If $R/\mathrm{Soc}(R_R)$ is a projective R-module, then

$$R = \mathrm{Soc}(R_R) \oplus \Lambda$$

for some right ideal Λ of R. It is obvious that

$$\mathrm{Hom}_R(\mathrm{Soc}(R_R), \Lambda) = 0$$

and

$$\mathrm{Hom}_R(\Lambda, \mathrm{Soc}(R_R)) = 0$$

since Λ is a socle-free R-module and $\mathrm{Soc}(R_R)$ is projective. Hence, Λ is a two-sided ideal of R. Then,

$$\mathrm{Soc}(R_R) = 0 \quad \text{or} \quad \mathrm{Soc}(R_R) = R$$

since R is an indecomposable ring, that is, $\mathrm{Soc}(R_R) \leq_e R$ with $J(R)^2 = 0$ or $\mathrm{Soc}(R_R) = 0$. If $R/\mathrm{Soc}(R_R)$ is an s-pure-projective R-module, then $R/\mathrm{Soc}(R_R)$ is semisimple Artinian, and thus R is a right Artinian by Corollary 4.5. Moreover, this implies that $\mathrm{Soc}(R_R) \leq_e R$ with $J(R)^2 = 0$ or $\mathrm{Soc}(R_R) = 0$ since if $J(R) = 0$, then R is semisimple Artinian, and if $J(R) \neq 0$, then $J(R)$ is nilpotent and $J(R) \neq J(R)^2$, then $J(R)^2 = 0$ by Lemma 4.6. □

Lemma 4.9. Let R be a semiprime right Noetherian ring that is indecomposable and has $*$-property. Then, R is a semisimple Artinian ring if it is not prime.

PROOF. Let A be a nonzero two-sided ideal of R. R/A is a projective R-module or R/A is an s-pure-projective R-module by our assumption. If R/A is a projective R-module, then a right ideal B exists, such as $R = A \oplus B$. $BA = 0$ because of the direct sum property and $AB = 0$ since $(AB)^2 = 0$ and R is semiprime. Then, $A = R$ since R is an indecomposable ring. If R/A is an s-pure-projective R-module, then R/A is semisimple Artinian by Corollary 4.5. We observe that R/A is a semisimple Artinian ring for any nonzero two-sided ideal A of R. Suppose that R is not prime. Let I_1 and I_2 be nonzero two-sided ideals of R such that $I_1 I_2 = 0$ and $I_1 \cap I_2 = 0$ since R is a semiprime ring. Then, there is an R-monomorphism $R_R \to R/I_1 \oplus R/I_2$. Hence, R is a semisimple Artinian ring since R/I_1 and R/I_2 are semisimple Artinian rings by our first observation. □

Theorem 4.10. Let R be a right Noetherian ring. If R has $*$-property, then $R \cong \Lambda' \times \Lambda$ where Λ' is semisimple Artinian ring and Λ is either zero or an indecomposable ring, which satisfies one of the following cases.

i. Λ is a right Artinian, and right SI-ring with $J(\Lambda) \neq 0$

ii. Λ is a right Artinian and right Goldie torsion ring with $J(\Lambda) \neq 0$

iii. Λ is a right Artinian ring, and $\mathrm{Soc}(\Lambda_\Lambda) = Z_r(\Lambda) = J(\Lambda) \neq 0$

iv. Λ is a prime ring with $J(\Lambda) = \mathrm{Soc}(\Lambda_\Lambda) = 0$

PROOF. We can write

$$R = R_1 \oplus R_2 \oplus ... \oplus R_n$$

where $n \in \mathbb{Z}^+$ and R_i is an indecomposable ring, for all $i \in \{1, 2, ..., n\}$. Either R is a semisimple Artinian, or there exists an $i \in \{1, 2, ..., n\}$ such that R_i is not a semisimple Artinian ring. Let A be a right ideal of R_j, for $i \neq j$. By our assumption, R_j/A is either s-pure-projective or projective since R_j/A is a pure-projective R-module. It is clear that

$$\mathrm{Hom}_R(R_j/A, R_i/B) = 0$$

for any submodule B of R_i. Then,

$$R_i \in \mathfrak{Pr}^{-1}(R_j/A) \neq \mathcal{R}egular$$

since R_i is not regular, that is, R_j/A is not an s-pure-projective R-module. Then, R_j/A must be projective; therefore, R_j is a semisimple Artinian ring. Thus, we have

$$R \cong \Lambda' \times \Lambda$$

where Λ' is semisimple Artinian, and Λ is either zero or an indecomposable ring. Λ is a ring with $*$-property, and Λ is either right Artinian with $J(R)^2 = 0$ or semiprime by Lemma 4.4 and Lemma 4.7.

We divide the proof into three cases.

First case: Suppose that Λ is a right Artinian with $J(R)^2 = 0$ and $Z_2(R) = 0$. Then, Λ is a right SI-ring by Proposition 3.5 [19]. This case provides us (i) of Theorem 4.10.

Second case: Suppose that Λ is a right Artinian with $J(R)^2 = 0$ and $Z_2(R) \neq 0$. Then, $R/Z_2(R)$ is projective or s-pure-projective. If $R/Z_2(R)$ is projective, then $Z_2(R) = R$. Hence, R is a right Goldie torsion ring. This case provides (ii) of Theorem 4.10.

If $R/Z_2(R)$ is not projective, then $R/Z_2(R)$ is s-pure-projective and thus semisimple Artininan by Corollary 4.5. Hence, $J(\Lambda) \leq Z_r(\Lambda)$, since $J(\Lambda)$ is a semisimple R-module and $Z_r(\Lambda) \leq_e Z_2(R)$. Λ has a decomposition

$$\Lambda = \bigoplus_{i=1}^{n} \Lambda_i$$

where each Λ_i is a local module since Λ is Artinian. It is clear that

$$J(\Lambda_i) \subseteq Z_r(\Lambda_i) \neq \Lambda_i$$

where $i \in \{1, 2, ..., n\}$. This implies $J(\Lambda) \subseteq Z_r(\Lambda)$, that is, $J(\Lambda) = Z_r(\Lambda)$. Suppose the contrary that

$$J(\Lambda) \neq \mathrm{Soc}(\Lambda_\Lambda)$$

Then, for some Λ_i is simple. We can suppose that the simple components of the decomposition are $\Lambda_1, \Lambda_2, \cdots, \Lambda_k$ where $1 \leq k \leq n$. Let

$$I_1 = \{i \in \{1, 2, ..., n\} \mid \Lambda_i \cong \Lambda_t\}$$

and

$$I_2 = \{1, 2, ..., n\} - I_1$$

Let

$$D = \bigoplus_{i \in I_1} \Lambda_i$$

Let $i \in I_1$ and $j \in I_2$. Clearly, $\mathrm{Hom}_R(\Lambda_i, \Lambda_j) = 0$. Moreover, $\mathrm{Hom}_R(\Lambda_j, \Lambda_i) = 0$ by the singularity of $J(\Lambda)$. This is a contradiction since Λ indecomposable; therefore, $J(\Lambda) = \mathrm{Soc}(\Lambda_\Lambda)$, that is, we get (*iii*) of Theorem 4.10.

Third case: Let Λ be a semiprime ring. It is clear that Λ is a prime ring by Lemma 4.9. Assume that $\mathrm{Soc}(\Lambda_\Lambda) = 0$. If $J(\Lambda) \neq 0$, then $J(\Lambda)^2 = J(\Lambda)$ by Lemma 4.6. By Nakayama's Lemma, $J(\Lambda)_\Lambda$ is infinitely generated, which is impossible by the right Noetherianity; therefore, $J(\Lambda) = 0$. This provides the last case of Theorem 4.10. But what about the case; where Λ is a semiprime ring and $\mathrm{Soc}(\Lambda_\Lambda) \neq 0$. $\mathrm{Soc}(\Lambda_\Lambda) \leq_e \Lambda$ and $J(\Lambda)^2 = 0$ by Lemma 4.8, and $J(\Lambda) = 0$ by assumption. We must have $Z_r(\Lambda) = 0$ since Λ is prime, and $\mathrm{Soc}(\Lambda_\Lambda) \neq 0$. Besides, we have that

$$Z_r(\Lambda)\,\mathrm{Soc}(\Lambda_\Lambda) = 0$$

Then, Λ is a right SI-ring by Corollary 3.7 [19]. According to Proposition 10.15 [16], if Λ has a finitely generated socle, it becomes a semisimple Artinian ring. This is impossible since Λ is not a semisimple Artinian ring. Further, $\mathrm{Soc}(\Lambda_\Lambda)$ can not be infinitely generated since Λ is a right Noetherian ring. Hence, we have no extra cases for Theorem 4.10. This completes the proof of Theorem 4.10. □

In the next example, we illustrate for each case in Theorem 4.10.

Example 4.11. *i.* Let F be a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. It is well-known that R is an Artinian SI-ring and $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$

ii. Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. R is an Artinian ring and we can easily see that $Z_r(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$ and $Z_2(R_R) = R$. Thus, R is a right Goldie torsion ring

iii. Let p be a prime number and $R = \mathbb{Z}/p^2\mathbb{Z}$. Then, R is an Artinian ring with

$$\mathrm{Soc}(R_R) = Z_r(R) = J(R) = p\mathbb{Z}/p^2\mathbb{Z}$$

iv. Let $R = \mathbb{Z}$. Then, it is well known that R is a prime ring with $J(R) = \mathrm{Soc}(R) = 0$

The following proposition provides a partial answer to the converse of Theorem 4.10. The answer follows from any pure-projective p-poor module is s-pure-projective and Proposition 3.14 [1].

Proposition 4.12. If R is a (non-semisimple Artinian) quasi-Frobenius ring with a homogeneous right socle and $J(R)^2 = 0$, then R has $*$-property.

5. Conclusion

We study pure-projective modules in their projectivity domains. After showing that each ring with an s-pure-projective module, we characterize rings all of whose modules are s-pure-projective as vNr rings. Afterwards, we investigate rings whose pure-projective modules are projective or s-pure-projective, called rings has $*$-property. Semisimple Artinian rings and vNr rings are examples of these rings. We provide the structure of right Noetherian rings that have $*$-property (Theorem 4.10). Furthermore, we present a partial answer for the converse of Proposition 4.12. Consequently, the results can be generalized to non-Noetherian rings, and the full characterization of Theorem 4.10 can be studied. Since it is common and important to study certain classes of modules in module theory; in addition, one can continue to look at the projectivity domains of some other special classes of modules.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Lattice of Subinjective Portfolios of Modules

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Abstract — Given a ring R, we study its right subinjective profile $\mathfrak{si}\mathfrak{P}(R)$ to be the collection of subinjectivity domains of its right R-modules. We deal with the lattice structure of the class $\mathfrak{si}\mathfrak{P}(R)$. We show that the poset $(\mathfrak{si}\mathfrak{P}(R), \subseteq)$ forms a complete lattice, and an indigent R-module exists if $\mathfrak{si}\mathfrak{P}(R)$ is a set. In particular, if R is a generalized uniserial ring with $J^2(R) = 0$, then the lattice $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is Boolean.

Keywords *Subinjectivity domain, subinjective profile, complete lattice of subinjectivity domains*

Mathematics Subject Classification (2020) 16D50, 06B10

1. Introduction

Throughout this paper, every ring R is associative with unity, and all modules are unitary. $Mod-R$ stands for the category of right R-modules. Flat modules, injective modules, and projective modules are among the most studied structures of module and ring theory, and they occur naturally in many algebra fields, such as homological algebra, category theory, representation theory, and algebraic geometry. Researchers conduct numerous studies on projective, injective, and flat modules. Many of these studies explore ideas based on relative projectivity, injectivity, and flatness. Recently, instead of simply categorizing modules as having a specific homological property, each module is allocated a relative domain that gauges the degree to which it possesses that particular property. In particular, several research papers have been devoted to the study of the injectivity, flatness, and projectivity level of modules [1–11].

Subinjectivity domain of a module (Definition 2.2) was originally introduced in [12] in order to study in a way the degree of injectivity of modules. In this article, we shift our focus from the subjective domain of modules to examining the collection of these domains using a fresh approach. This collection is called the (right) subinjective profile (si-profile, for short) of R, and is denoted by $\mathfrak{si}\mathfrak{P}(R)$. $\mathfrak{si}\mathfrak{P}(R) = \{Mod-R\}$ if and only if R is a semisimple Artinian ring if and only if there exists an injective indigent right (or left) R-module. Semisimple Artinian rings stand out as the most straightforward type of rings regarding their subinjective characteristics. Another straightforward case arises from rings that are not semisimple Artinian; these rings exhibit only two possible domains of subinjectivity: injective modules and all modules. Such rings have no subinjective middle class [12, 13].

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We organize the paper as subsequent. In Section 2, we provide brief definitions and properties. In Section 3, we study the class $\mathfrak{si}\mathfrak{P}(R)$ under the condition that it is a set. We show that if $\mathfrak{si}\mathfrak{P}(R)$ is a set, then the class $\mathcal{IN}$ of injective modules is an si-portfolio and $\mathfrak{si}\mathfrak{P}(R)$ is closed under intersections. R has no subinjective middle class if and only $|\mathfrak{si}\mathfrak{P}(R)| = 2$. We show that the poset $(\mathfrak{si}\mathfrak{P}(R), \subseteq)$ forms a complete lattice if $\mathfrak{si}\mathfrak{P}(R)$ is a set (Theorem 3.6). In particular, if R is a generalized uniserial ring with $J^2(R) = 0$, then the lattice $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is Boolean (Theorem 3.9).

2. Preliminaries

This section provides some basic notions to be required for the following section. The paper uses the books [14–16] for the basic definitions. The references [7, 8, 12, 17] cover various aspects of this topic.

Definition 2.1. A module E is injective if for any morphism $f : A \to E$ and any monomorphism $q : A \to B$, f factors through q by some morphism $B \to E$.

By fixing the module A, the notion of A-subinjective module is introduced in [12].

Definition 2.2. A module X is called B-subinjective if for every monomorphism $h : B \to K$ and every homomorphism $f : B \to X$, there exists a homomorphism $g : K \to X$ such that $gh = f$.

Definition 2.3. For an R-module X, the subinjectivity domain of X, denoted as $\underline{\mathfrak{In}}^{-1}(X)$, encompasses all modules with respect to which X exhibits subinjective properties, i.e.,

$$\underline{\mathfrak{In}}^{-1}(X) = \{N \in Mod - R : X \text{ is } N\text{-subinjective}\}$$

Every subinjectivity domain contains the class $\mathcal{IN}$ of injective modules. Therefore, the class $\mathcal{IN}$ serves as a minimum benchmark for the subinjectivity domains of R-modules. The following result follows from Lemma 2.2 in [12].

Proposition 2.4. An R-module X is injective if and only if $\underline{\mathfrak{In}}^{-1}(X) = Mod - R$ if and only if X is X-subinjective.

Proposition 2.4 naturally leads to considering the degree to which a specific module exhibits injectiveness, as injective modules epitomize the highest level of injectiveness. The notion of indigent modules was introduced by Aydoğdu and López-Permouth [12].

Definition 2.5. A module M is called indigent if $\underline{\mathfrak{In}}^{-1}(M) = \mathcal{IN}$.

The existence of indigent modules within any arbitrary ring remains uncertain, although an affirmative answer is established for certain rings, such as Noetherian rings (for more details, see Proposition 3.4 in [2]). When considering the degree of injectivity in modules, we encounter two extremes: at one end, we find injective modules, and at the other, we have what are known as indigent modules.

Example 2.6. This example exhibits an indigent module. Let R be a commutative hereditary Noetherian ring. Let U be the direct sum of a representative set of all (nonprojective) simple modules. U is indigent module by [18, Proposition 2.12].

In this article, we focus on the study of the class of subinjectivity domains.

Definition 2.7. [19] A class $\mathcal{A}$ of R-modules is called si-portfolio if there exists an R-module M such that $\mathcal{A} = \underline{\mathfrak{In}}^{-1}(M)$.

Definition 2.8. [19] The class $\{\mathcal{A} \subseteq Mod - R : \mathcal{A} \text{ is an sp-portfolio}\}$ is called the (right) subinjective profile (si-profile, for short) of R and is denoted by $\mathfrak{si}\mathfrak{P}(R)$.

The class $Mod - R$ is an obvious example of an si-portfolio. Note that it is still unknown whether $\mathcal{IN}$ is an si-portfolio on non-Noetherian rings.

For a module T, we denote its injective hull, singular submodule, and radical by $E(T)$, $Z(T)$, and $\text{Rad}(T)$, respectively. The Jacobson radical of a ring R is denoted by $J(R)$. We use the notations $\leq$ and $\subseteq$ in order to indicate submodules and set inclusion, respectively.

3. Lattice Structure

The poset of si-portfolios is denoted by $(\mathfrak{siP}(R), \subseteq)$ where the partial order is given by containment $\subseteq$. Note that $\mathfrak{siP}(R)$ need not actually form a set but we still use the term poset by abuse of language when $\mathfrak{siP}(R)$ is a class. The poset $(\mathfrak{siP}(R), \subseteq)$ always contains a unique maximal element, the class of all the modules $Mod-R$. It is unknown whether $\mathcal{IN}$ is an si-portfolio. Moreover, it is unknown whether $\mathfrak{siP}(R)$ is closed under intersections.

Theorem 3.1. If $\mathfrak{siP}(R)$ is a set, then $\mathcal{IN}$ is an si-portfolio and $\mathfrak{siP}(R)$ is closed under intersections.

PROOF. Assume that $\mathfrak{siP}(R)$ is a set. Consider the function $\underline{\mathfrak{In}}^{-1} : Mod{-}R \to \mathfrak{siP}(R)$, $A \to \underline{\mathfrak{In}}^{-1}(A)$. The function $\underline{\mathfrak{In}}^{-1}$ is onto. Since $\mathfrak{siP}(R)$ is a set, there is a set $\mathcal{I}$ of R-modules such that $\underline{\mathfrak{In}}^{-1}_{|\mathcal{I}}$ is one-to-one and onto. We show that the R-module

$$\text{O} := \bigoplus_{M \in \mathcal{I}} M$$

is indigent. Let C be an R-module from $\underline{\mathfrak{In}}^{-1}(\text{O})$. Then, by Proposition 2.4 [12],

$$C \in \bigcap_{M \in \mathcal{I}} \underline{\mathfrak{In}}^{-1}(M)$$

Moreover, since

$$\underline{\mathfrak{In}}^{-1}(C) \in \mathfrak{siP}(R)$$

and

$$\underline{\mathfrak{In}}^{-1}(C) = \underline{\mathfrak{In}}^{-1}(X)$$

for a module $X \in \mathcal{I}$. Then, $C \in \underline{\mathfrak{In}}^{-1}(C)$, and thus C is injective R-module. This implies that O is indigent.

Let $\mathcal{M}$ be a family of si-portfolios. Since $\mathfrak{siP}(R)$ is a set, $\mathcal{M}$ is a set. Let I be a complete set of non-isomorphic modules whose subprojectivity domains are in $\mathcal{M}$. Set

$$\mathcal{O} := \bigoplus_{M \in I} M$$

Then,

$$\bigcap_{\mathcal{A} \in \mathcal{M}} \mathcal{A} = \underline{\mathfrak{In}}^{-1}(\mathcal{O})$$

by Proposition 2.4 in [12]. □

In [13], the authors investigate rings for which the si-profile consists of $\mathcal{IN}$ and $Mod-R$. They called these rings as having no subinjective middle class. By Theorem 3.1, we have the following result.

Corollary 3.2. R has no subinjective middle class if and only $|\mathfrak{siP}(R)| = 2$.

The subprojectivity domains of two non-isomorphic modules may be the same. For example,

$$\underline{\mathfrak{In}}^{-1}(0) = \underline{\mathfrak{In}}^{-1}(R) = Mod-R$$

For the remaining discussions, let δ be a complete set of representatives of non-isomorphic non-injective simple modules.

Proposition 3.3. Let $\mathcal{I}$ and $\mathcal{J}$ be subsets of δ. Then,

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) \quad \text{if and only if} \quad \mathcal{I} = \mathcal{J}$$

PROOF. To show the necessity, assume that

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) \quad \text{and} \quad \mathcal{I} \neq \mathcal{J}$$

Without loss of generality, we may assume that a simple R-module A exists in $\mathcal{I} \setminus \mathcal{J}$. Then, since $A \notin \mathcal{J}$, $\mathrm{Hom}(A, S) = 0$, for all $S \in \mathcal{J}$. Thus,

$$A \in \bigcap_{S\in\mathcal{J}} \underline{\mathfrak{In}}^{-1}(S) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right)$$

Then, by assumption,

$$A \in \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \bigcap_{S_i\in\mathcal{I}} \underline{\mathfrak{In}}^{-1}(S)$$

Since $A \in \mathcal{I}$, $A \in \underline{\mathfrak{In}}^{-1}(A)$, and thus A is injective, a contradiction. The sufficiency is clear. □

A ring R is called a semilocal ring if $R/J(R)$ is semisimple Artinian. Note that any semilocal ring has only finitely many simple R-modules up to isomorphism [14]. Define

$$\mathfrak{siP}(\delta) := \left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right) : \mathcal{I} \subseteq \delta\right\}$$

Corollary 3.4. If R is a semilocal ring, then $|\mathfrak{siP}(\delta)| = 2^{|\delta|}$.

PROOF. Since R is a semilocal ring, R has only finitely many simple R-modules up to isomorphism. Put $\delta := \{S_1, S_2, \ldots, S_n\}$ and let

$$\mathfrak{siP}(\delta) := \left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right) : \mathcal{I} \subseteq \delta\right\}$$

Since $|\delta|= n$, $|\mathfrak{siP}(\delta)| = 2^n$ by Proposition 3.3. □

Note that R is a generalized uniserial ring with $J^2(R) = 0$ if and only if every right (or left) R-module is a direct sum of a semisimple module and an injective module [20].

Lemma 3.5. Let R be a generalized uniserial ring with $J^2(R) = 0$. Then,

$$\mathfrak{siP}(R) = \left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) : \mathcal{I} \subseteq \delta\right\} \quad \text{and} \quad |\mathfrak{siP}(R)| = 2^{|\delta|}$$

PROOF. Since R is a semilocal ring, R has only finitely many simple R-modules up to isomorphism. Put $\delta := \{S_1, S_2, \ldots, S_n\}$. By Corollary 3.4,

$$\mathfrak{siP}(\delta) = \left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right) : \mathcal{I} \subseteq \delta\right\}$$

and $|\mathfrak{siP}(\delta)| = 2^n$. We claim that $\mathfrak{siP}(\delta) = \mathfrak{siP}(R)$. Clearly, $\mathfrak{siP}(\delta) \subseteq \mathfrak{siP}(R)$. Note that, for $\mathcal{I} = \emptyset$,

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right) = \underline{\mathfrak{In}}^{-1}(\{0\}) = Mod - R \in \mathfrak{siP}(\delta)$$

Let M be any R-module. If M is injective, then $\underline{\mathfrak{In}}^{-1}(M) = Mod - R$, and thus $\underline{\mathfrak{In}}^{-1}(M)$ is in $\mathfrak{siP}(\delta)$. If M is not injective, then, by [20], $M = A \oplus E$, where A is semisimple and E is injective. Without loss of generality, we may assume that A has no injective direct summands. Further, we have that

$$\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(A) \cap \underline{\mathfrak{In}}^{-1}(E) = \underline{\mathfrak{In}}^{-1}(A) \cap Mod - R = \underline{\mathfrak{In}}^{-1}(A)$$

Let $\mathcal{C}$ be a complete set of non-isomorphic simple submodules of A. Since

$$\underline{\mathfrak{In}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{C_i \in \mathcal{C}} C_i\right)$$

and each $C_i \in \mathcal{C}$ is isomorphic to one of S_γ in δ, $\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(A)$ must be in $\mathfrak{siP}(\delta)$. Thus, $\mathfrak{siP}(\delta) = \mathfrak{siP}(R)$, as claimed. □

A partially ordered set P is called a lattice if every pair of elements a and b in P has both a supremum $a \vee b$ (called join) and an infimum $a \wedge b$ (called meet). A partially ordered set P is called a complete lattice if its subsets have a join and a meet. A lattice P is said to be bounded if it has the greatest element and the least element [21].

Theorem 3.6. Assume that $\mathfrak{siP}(R)$ is a set. The poset $(\mathfrak{siP}(R), \subseteq)$ forms a complete lattice under the following meet and join operations:

i. The meet $\wedge$ is defined by $P_1 \wedge P_2 = P_1 \cap P_2$.

ii. The join $\vee$ is defined by $P_1 \vee P_2 = \bigcap \{P \in \mathfrak{siP}(R) : P_1 \subseteq P \text{ and } P_2 \subseteq P\}$.

Proof. $(\mathfrak{siP}(R), \subseteq, \wedge)$ is a meet-semilattice by Proposition 2.4 in [12]. Using the same technique in Theorem 3.1, it can be easily seen that $(\mathfrak{siP}(R), \subseteq, \vee)$ is a join-semilattice. By Theorem 3.1, every subset of $\mathfrak{siP}(R)$ has a meet. Again, by the same idea used in Theorem 3.1, it can be seen that every subset of $\mathfrak{siP}(R)$ has a join. Hence, $(\mathfrak{siP}(R), \subseteq, \vee)$ is a complete lattice. The class $\mathcal{IN}$ is the minimal element of $\mathfrak{siP}(R)$ by Theorem 3.1. On the other hand, for any injective module E, $\underline{\mathfrak{In}}^{-1}(E) = Mod - R$, and thus $Mod - R$ is the maximal element of $\mathfrak{siP}(R)$. □

Let P be a lattice with 0, 1, and $t \in P$. An element $t' \in P$ is called a complement of t if $t \wedge t' = 0$ and $t \vee t' = 1$. P is called complemented if each element in P has at least one complement. A complemented lattice is called Boolean if it is distributive. We claim that $(\mathfrak{siP}(R), \subseteq)$ is Boolean if R is a generalized uniserial ring with $J^2(R) = 0$.

Lemma 3.7. Let R be a generalized uniserial ring with $J^2(R) = 0$. Let $\mathcal{I}$ and $\mathcal{J}$ be any two subsets of δ. Then,

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \vee \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}} S\right) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I} \cap \mathcal{J}} S\right)$$

Proof. If

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

for a non-injective simple T, then $T \cong S$, for some $S \in \mathcal{I}$, since otherwise, $\mathrm{Hom}(T, S) = 0$, for every $S \in \mathcal{I}$, and hence

$$T \in \bigcap_{S \in \mathcal{I}} \underline{\mathfrak{In}}^{-1}(S) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

which implies T is injective, a contradiction. Therefore,

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \vee \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}} S_\gamma\right) \subseteq \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I} \cap \mathcal{J}} S\right)$$

Suppose that

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \vee \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}} S_\gamma\right) = \underline{\mathfrak{In}}^{-1}(X)$$

for a module X. As noted in the proof of Lemma 3.5,

$$\underline{\mathfrak{In}}^{-1}(X) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{T}} S\right)$$

where $\mathcal{T} \subseteq \delta$. The containment

$$\underline{\mathfrak{In}}^{-1}(X) \subset \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}\cap\mathcal{J}}{\oplus} S\right)$$

follows by the definition of $\vee$. Repeating the first paragraph, it can be shown that $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{T} \subseteq \mathcal{I}$ and $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{T} \subseteq \mathcal{J}$. Then, $\mathcal{I} \cap \mathcal{J} = \mathcal{T}$, which proves the assertion. □

For the sake of completeness, we provide the following result.

Corollary 3.8. [12, Theorem 4.2] If R is right-left hereditary Artinan serial ring, then

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\delta}{\oplus} S\right) = \mathcal{IN}$$

Theorem 3.9. If R is a generalized uniserial ring with $J^2(R) = 0$, then the lattice $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is Boolean.

Proof. Recall that $\mathfrak{si}\mathfrak{P}(R)$ is a set by Lemma 3.5. We first show that $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is complemented. Let $\mathcal{P} \in \mathfrak{si}\mathfrak{P}(R)$. If either $\mathcal{P} = \mathcal{IN}$ or $\mathcal{P} = Mod - R$, then the proof is completed. Assume that neither $\mathcal{P} = \mathcal{IN}$ nor $\mathcal{P} = Mod - R$. Then, a non-injective module H exists, such as $\underline{\mathfrak{In}}^{-1}(H) = \mathcal{P}$. Since R is a generalized uniserial ring with $J^2(R) = 0$, we get $H = A \oplus B$ where A is a direct sum of non-injective simple modules, and B is an injective module by [20]. Then,

$$\underline{\mathfrak{In}}^{-1}(H) = \underline{\mathfrak{In}}^{-1}(A) \cap \underline{\mathfrak{In}}^{-1}(P) = \underline{\mathfrak{In}}^{-1}(A) \cap Mod - R = \underline{\mathfrak{In}}^{-1}(A)$$

Let $\mathcal{C}$ be an exact set of non-isomorphic simple direct summands of A. Define a subset

$$\mathcal{I} := \{S \in \delta : S \cong C, C \in \mathcal{C}\} \subseteq \delta$$

Then,

$$\underline{\mathfrak{In}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}\left(\underset{C\in\mathcal{C}}{\oplus} C\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right)$$

We note that $\mathcal{J} := \delta - \mathcal{I} \neq \emptyset$, since otherwise, we would have

$$\underline{\mathfrak{In}}^{-1}(H) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \mathcal{IN}$$

by Corollary 3.8, a contradiction. By Proposition 2.4 in [12] and Corollary 3.8,

$$\underline{\mathfrak{In}}^{-1}(H) \wedge \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\delta}{\oplus} S\right) = \mathcal{IN}$$

To show that

$$\underline{\mathfrak{In}}^{-1}(H) \vee \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) = Mod - R$$

we assume that

$$\underline{\mathfrak{In}}^{-1}(H) \vee \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}(X)$$

for some module X. If X is injective, then we are done. Assume that X is not injective. Since R is a generalized uniserial ring with $J^2(R) = 0$, by [20], X has a non-injective simple direct summand, say T. Since

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}(H) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

and

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

$T \in \mathcal{J} \cap \mathcal{I}$. But $\mathcal{J} \cap \mathcal{I} = \emptyset$, and this means that X has no non-injective simple direct summand, and so it is injective by [20].

For the distributive property, we only need to show that

$$\left(\underline{\mathfrak{In}}^{-1}(A) \vee \underline{\mathfrak{In}}^{-1}(V)\right) \wedge \underline{\mathfrak{In}}^{-1}(Z) = \left(\underline{\mathfrak{In}}^{-1}(A) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right) \vee \left(\underline{\mathfrak{In}}^{-1}(V) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right)$$

for any modules A, V, and Z. Without lost of generality we may assume that A, V, and Z have no projective simple direct summands. Since R is a generalized uniserial ring with $J^2(R) = 0$,

$$\underline{\mathfrak{In}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}_A} S\right)$$

$$\underline{\mathfrak{In}}^{-1}(V) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}_V} S\right)$$

and

$$\underline{\mathfrak{In}}^{-1}(Z) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}_Z} S\right)$$

by Lemma 3.5 where $\mathcal{I}_A$, $\mathcal{I}_V$, and $\mathcal{I}_Z$ are subsets of δ. By Proposition 2.4 in [12] and Lemma 3.7,

$$\left(\underline{\mathfrak{In}}^{-1}(A) \vee \underline{\mathfrak{In}}^{-1}(V)\right) \wedge \underline{\mathfrak{In}}^{-1}(Z) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}_1} S\right)$$

where

$$\mathcal{J}_1 = (\mathcal{I}_A \cap \mathcal{I}_V) \cup \mathcal{I}_Z$$

Similarly,

$$\left(\underline{\mathfrak{In}}^{-1}(A) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right) \vee \left(\underline{\mathfrak{In}}^{-1}(V) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}_2} S\right)$$

where

$$\mathcal{J}_2 = (\mathcal{I}_A \cup \mathcal{I}_Z) \cap (\mathcal{I}_V \cup \mathcal{I}_Z)$$

Obviously, $\mathcal{J}_1 = \mathcal{J}_2$, which proves the assertion.□

Example 3.10. Let K be any field. Let $R = T_3(K)$ denote the ring of all upper triangular 3×3 matrices with entries in K and let S denote the left socle of R. R/S is a generalized uniserial ring with $J^2(R/S) = 0$ by Example 13.6 in [20]. Then, $(\mathfrak{si}\mathfrak{P}(R/S), \subseteq)$ is a Boolean lattice by Theorem 3.9.

4. Conclusion

The objective of this paper is to commence exploration into an alternative viewpoint regarding the subinjective profile of rings. Differing from recent examinations focusing on the subinjective profile of rings, our approach delves into the lattice theoretical perspective of this concept. In future studies, researchers can consider how profile properties may determine the rings' structure. Specifically, they can investigate the necessary and sufficient conditions for rings to exhibit distributive and modular properties within this profile.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Mestré's Finite Field Method for Searching Elliptic Curves with High Ranks

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Research Article

Abstract — The theory of elliptic curves is one of the popular topics of recent times with its unsolved problems and interesting conjectures. In 1922, Mordell proved that the group of $\mathbb{Q}$-rational points on an elliptic curve is finitely generated. However, the rank of this group, signifying the number of independent generators, can be arbitrarily high for certain curves, a fact yet to be definitively proven. This study leverages the computer algebra system Magma to investigate curves with potentially high ranks using a technique developed by Mestré.

Keywords *Mestré method, elliptic curve, rank, height matrix, magma*

Mathematics Subject Classification (2020) 14H52, 20K15

1. Introduction

Elliptic curves possess a fascinating and yet unsolved property: their rank. The set of rational points on an elliptic curve forms a finitely generated abelian group. Mordell's Theorem guarantees this group is isomorphic to a finite direct sum of cyclic groups. The rank of the elliptic curve signifies the number of these cyclic groups with infinite order, which directly corresponds to the number of independent points of infinite order within the group. Determining the rank of an elliptic curve presents a significant challenge. There currently exists no known algorithm for calculating it efficiently. Additionally, a fundamental open question in number theory revolves around the existence of an upper bound for the rank of elliptic curves. While it is widely believed that elliptic curves can possess a rank of any non-negative integer, a definitive proof remains elusive. This lack of a proven upper bound fuels the ongoing search for elliptic curves with the highest possible rank.

Several key advancements have been made in determining the maximum possible rank of elliptic curves. Penney and Pomerance [1,2] established lower bounds, proving that the rank can be greater than 6 and 7, respectively. Subsequently, Grunewald and Zimmert [3] pushed this bound further by demonstrating the existence of curves with a rank exceeding 8. Brumer and Kramer [4] achieved a rank greater than 9. Mestré [5–8] introduced a breakthrough with two novel methods for estimating elliptic curve rank. He significantly raised the known lower bounds by applying these methods and showcasing specific examples. His works [5–9] demonstrably showed that the rank of certain elliptic curves can be greater than 11, 12, 14 and 15. Building upon Mestré's groundwork, Nagao and Fermigier [10–13] achieved

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further progress. They specialized in a family of curves introduced by Mestré over the field of rational functions in one variable, $\mathbb{Q}(t)$. This specialization allowed them to prove that the rank over the rational field $\mathbb{Q}$ is greater than 17 and $19 - 22$. Notably, the current record for the highest discovered rank, at 28, was achieved by Noam Elkies. It's worth mentioning that Martin and McMillen [14] also played a crucial role by independently discovering specific elliptic curves with ranks of 23 and 24, respectively.

Recent research in rank studies has tackled three key areas: calculating ranks for specific families of curves [15–19], exploring how rank behaves for curves constructed from special number sequences [20–28] and analyzing rank distributions within families and across field extensions [29–31]. Dujella [32] provided an enumeration of the strategies for generating high-rank Diaphontine elliptic curves. The contributions of Elkies and Klagsburn [33] are also noteworthy in that they established new rank records for specific torsion groups. Another noteworthy work is Kazlicki's attempt to develop a rank classification system using deep neural networks [34]. Although various studies continue to be conducted in different ways, there is not yet a complete method that yields high-rank curves. The majority of existing studies employ the Mestré method and Mestré's sum [7]. Therefore, the finite field method of Mestré was selected as the subject of this study.

This paper is divided into three distinct sections. The initial section establishes the fundamental groundwork by presenting the essential definitions and theorems relevant to elliptic curve ranks. Following the foundational elements, we introduce the method developed by Mestré for finding elliptic curves with high ranks. Finally, the third section analyzes the data obtained through the custom codes we developed using the MAGMA software program.

2. Preliminaries

We begin our discussion with the definition of elliptic curves.

Definition 2.1. [35] Let $\mathbb{K}$ be a field. An elliptic curve over $\mathbb{K}$ can be defined as

i. A genus one curve with one $\mathbb{K}$-rational point,

ii. A plane cubic with a $\mathbb{K}$-rational point or

iii. A Weierstrass cubic, $y^2 = x^3 + px + q$.

Example 2.2. The curve

$$YZ^2 = X^3 - XZ^2$$

over $\mathbb{Q}$ is an elliptic curve with the point at infinity denoted $\mathcal{O} = [0, 1, 0]$ in homogeneous coordinates. If we write $x = X/Z$ and $y = Y/Z$ in the equation, then we obtain the Weierstrass form of the equation as

$$y^2 = x^3 - x$$

The set of $\mathbb{K}$ rational points on the curve is given by

$$E(\mathbb{K}) = \{\mathcal{O}\} \cup \{(x, y) \in \mathbb{K}\}$$

Theorem 2.3. [36] The group $E(\mathbb{K})$ is finitely generated.

Mordell proved this theorem for the field $\mathbb{Q}$ in 1922 and Weil generalized it to any field $\mathbb{K}$ in 1928. It can be stated by the Mordell Theorem along with the general structure theory of finitely generated abelian groups that

$$E(\mathbb{K}) \cong E(\mathbb{K})_{tors} \times \mathbb{Z}^r$$

The group $E(\mathbb{K})$ is called the Mordell-Weil group. The subgroup $E(\mathbb{K})_{tors}$ consists of points with finite order and is referred to as the torsion subgroup. Formally, this subgroup is defined as follows:

$$E(\mathbb{K})_{tors} = \{P \in E(\mathbb{K}) : \exists n \in \mathbb{N} \text{ that } nP = \mathcal{O}\}$$

The free part of the Mordell-Weil group is generated by r points of $E(\mathbb{K})$ with infinite order. Here r is called the rank of $E(\mathbb{K})$.

Theorem 2.4 (Mazur Theorem, Conjecture of Ogg)**.** [37, 38] Let $E(\mathbb{Q})_{tors}$ be the torsion subgroup of the Mordell-Weil group of an elliptic curve over $\mathbb{Q}$. Then, $E(\mathbb{Q})_{tors}$ is isomorphic to one of the following fifteen groups:

i. $\mathbb{Z}/m\mathbb{Z}$, $1 \leq m \leq 10$ or $m = 12$

ii. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2v\mathbb{Z}$, $1 \leq v \leq 4$

The Conjecture of Ogg was proven by Barry Mazur [37] in 1977. However, many unsolved questions still exist with on ranks of elliptic curves. Determining the rank of an elliptic curve remains a significant challenge. No known algorithm efficiently calculates the rank for any curve. Additionally, it is unproven whether an upper bound exists on the rank. While the possibility of arbitrarily high ranks is widely accepted, complete proof remains elusive [33]. This study utilizes one of Mestré's influential methods, which were the first to construct elliptic curves with demonstrably high ranks. Notably, in 1982, Mestré [6] introduced a groundbreaking method to construct elliptic curves over the rational numbers ($\mathbb{Q}$) with demonstrably high ranks. This method allowed him to find curves with 8 and $10-12$ ranks in [5, 6]. These achievements marked a significant step forward in the field. Mestré presented two methods for searching elliptic curves. The method, which is the subject of this article, is known as the "Finite Field Method" as proposed by Campbell [39].

3. Mestré's Finite Field Method

We introduce a key conjecture underpinning Mestré's method. Intriguing and far-reaching, the Birch and Swinnerton-Dyer conjecture (BSD) is a central pillar in studying elliptic curves, offering a profound connection between their arithmetic and analytic properties. Indeed, the BSD conjecture plays a fundamental role in understanding the underlying principles of our approach.

Definition 3.1. [40] Let E be an elliptic curve defined over the field of rational numbers, $\mathbb{Q}$. Denote its conductor by N. We define its associated $L-$function by $L(E, s)$, where s is a complex number:

$$\begin{aligned} L(E,s) &= \sum_n a_n n^{-s} \\ &= \prod_{p|N} \left(1 - a_p p^{-s}\right)^{-1} \prod_{p\nmid N} \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1} \end{aligned}$$

Conjecture 3.2 (Birch-Swinnerton-Dyer Conjecture)**.** [41] The function L of an elliptic curve is extendable into a holomorphic function in the neighborhood of 1 and its order in 1 is equal to the rank of the Mordell-Weil group of E over $\mathbb{Q}$.

The Birch and Swinnerton-Dyer Conjecture (BSD Conjecture) is relevant to the context of elliptic curves and their $L-$functions, but it doesn't directly provide a bound for the rank of an elliptic curve. However, it establishes a connection between the rank and the behavior of the $L-$function at a specific point, which can be used to infer information about the rank under certain conditions.

Mestré's algorithm is the following [5, 6]:

Let

$$E : y^2 + y = x^3 + a_4 x + a_6$$

be an elliptic curve over $\mathbb{Q}$, p be a good reduction prime for E and N_p be several points of E modulo p. An analysis of Weil's exponential formulas [6] applied to the $L-$function of the elliptic curve E reveals a potential discrepancy between the rank, r, implied by the large size of N_p for numerous prime numbers p, and a potentially higher rank suggested by the analytic behavior of the $L-$ function under this analysis. As a result, to obtain elliptic curves with high ranks, one builds curves such that N_p is maximal for all p inferior or equal to P_0, which is an integer depending on the computing capabilities available. Then,

$$\Delta = -(4a_4)^3 - 27(1 + 4a_6)^2$$

We provide four integers P_0, P_1, k_0, and k'_0 to our search.

i. Let M be an integer and

$$M_0 = \prod_{p \leq M} p$$

The congruences modulo p that coefficients a_4 and a_6 of an elliptic curve attain when N_p is maximized for each $p \leq P_0$ are calculated.

ii. The congruences (a_4, a_6) modulo M_0, which ensures the maximal value of N_P for all integers $p \leq P_0$, are derived by a simple application of the Chinese remainder theorem.

iii. For each pair of congruences (a_4, a_6), the negative value a_4 of minimum absolute value congruent to a_4 and for each of the values $a'_4 = a_4 - kM_0$, $0 \leq k \leq k_0$, are searched, a'_6 congruent to a_6 and minimizing $|\Delta|$ are calculated. Then, each curve with coefficients a'_4 and $a''_6 = a'_6 + kM_0$, $| \ k \ | \leq k'_0$, are considered.

iv. For each of these curve N_P are calculated for $P_0 \leq p \leq P_1$, then

$$S = \sum_{P_0 \leq p \leq P_1} \left(\frac{p-1}{N_p} - 1 \right) \log p$$

Curves such that S is greater than a constant S_0 dependent only on P_0 and P_1 are rejected.

v. If E is such that $S \leq S_0$, integer points of this curve are searched for example in the interval

$$[e_1, e_1 + 5000]$$

where e_1 being the abscissa of an order 2 point of E.

vi. If we do not find an integer point, the curve is rejected; otherwise, the matrix of the heights of the points obtained and the rank of the height matrix are calculated.

Mestré [5] obtained many curves with rank 6, 7, 8, and 9 for $P_0 = 17$, $P_1 = 50$, $k_0 = 20$, and $k'_0 = 50$. He also obtained a curve with rank 12 for $P_0 = 37$, $P_1 = 101$, $k_0 = 1$, and $k'_0 = 8$.

4. Results on our Search for Finding Curves with High Ranks

We implemented Mestré's finite field method [5] in Magma software to estimate the rank of elliptic curves. Running our code with parameters $P_0 = 17$, $P_1 = 50$, $k_0 = 20$, and $k'_0 = 50$, we successfully reproduced Mestré's results of sieves with ranks $7 - 9$. Our code identified numerous elliptic curves with rank 7. Due to space limitations, we omit specific examples of these curves here, but they are available upon request. The results of our investigation are presented in Table 1.

Table 1. Searching results

Rank	TorsionSubgruop	Curve
7	trivial	$y^2 + y = x^3 - 1201837x - 28298094$
7	trivial	$y^2 + y = x^3 - 3243877x - 44634414$
7	trivial	$y^2 + y = x^3 - 1832467x - 37997784$
7	trivial	$y^2 + y = x^3 - 4385017x - 37997784$
7	trivial	$y^2 + y = x^3 - 4234867x - 26491674$
7	trivial	$y^2 + y = x^3 - 2733367x - 46401564$
7	trivial	$y^2 + y = x^3 - 1321957x - 37212384$
7	trivial	$y^2 + y = x^3 - 3363997x - 33638814$
7	trivial	$y^2 + y = x^3 - 1199107x - 42174684$
7	trivial	$y^2 + y = x^3 - 8256157x - 33496014$
7	trivial	$y^2 + y = x^3 - 9907807x - 32985504$
7	trivial	$y^2 + y = x^3 - 1078987x - 47515404$
7	trivial	$y^2 + y = x^3 - 5163067x - 35773674$
7	trivial	$y^2 + y = x^3 - 598507x - 34242144$
7	trivial	$y^2 + y = x^3 - 6214117x - 45473364$
7	trivial	$y^2 + y = x^3 - 7235137x - 29137044$
7	trivial	$y^2 + y = x^3 - 1739647x - 38326224$
8	trivial	$y^2 + y = x^3 - 2667847x - 25888344$
8	trivial	$y^2 + y = x^3 - 2842567x - 50714124$
8	trivial	$y^2 + y = x^3 - 3688867x - 49646694$
8	trivial	$y^2 + y = x^3 - 3224767x - 37444434$
9	trivial	$y^2 + y = x^3 - 3151057x - 34517034$

The curve

$$E : y^2 + y = x^3 - 3151057x - 34517034$$

has 9 independent points on $\mathbb{Q}$. The Torsion Subgroup is trivial. Points of E are

$$P_1 = (90641/4, 27205059/8)$$

$$P_2 = (20221, -2864331)$$

$$P_3 = (8945/4, 512371/8)$$

$$P_4 = (1325657/484, -1160700207/10648)$$

$$P_5 = (741102/361, -317934960/6859)$$

$$P_6 = (1636928/289, -1988611043/4913)$$

$$P_7 = (3097593/361, -5333671306/6859)$$

$$P_8 = (5325049/64, -12285322125/512)$$

and

$$P_9 = (3872137/676, -7243647215/17576)$$

Applying our implementation of Mestré's method to primes $p = 19$ and $p = 23$, we were not able to identify any elliptic curves with a rank greater than 6. These findings are consistent with the observations reported in Mestré's work [5]. The curve discovered during the scan does not correspond to any entries on Dujella's website [14, 42], which serves as a repository for elliptic curve ranks. Our Magma implementation encountered errors and did not complete the computations for primes 29, 31 and 37. This suggests that analyzing these cases might require more extensive computational resources than those available on personal computers.

5. Conclusion

In this study, we implemented Mestrés method for searching elliptic curves with high ranks using a Magma code. This approach yielded a comprehensive list of elliptic curves with ranks $7-9$. Notably, these curves were not previously documented in the referenced literature [5,6] or on Dujella's website, a leading resource for rank records. While we have not identified curves exceeding rank 9, our exploration has exposed potential limitations in our current Magma code regarding time and memory efficiency. Future efforts can be focused on optimizing the code to handle computations for even higher ranks.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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Relationship Between a Homoderivation and a Semi-Derivation

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Abstract — Let $\wp$ be a ring. It is shown that if an additive mapping ϑ is a zero-power valued on $\wp$, then $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a bijective mapping of $\wp$. The main aim of this study is to prove that ϑ is a homoderivation of $\wp$ if and only if $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ is a semi-derivation associated with α, where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$. Moreover, if ϑ is a zero-power valued homoderivation on $\wp$, then ϑ is a semi-derivation associated with α, where $\alpha : \wp \to \wp$ is an automorphism of $\wp$ such that $\alpha = \vartheta + 1$.

Keywords *Ring, semi-derivation, homoderivation*

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1. Introduction

The definition of homoderivation is given by El Sofy in [1]. Many problems have been solved using homoderivation since the definition of homoderivation is given, but in solving the problems addressed, the necessity of the function being zero-power valued has emerged most of the time. For this reason, the most general results in the literature are generally found when homoderivation is zero-power valued. This situation brings to mind whether there is a relationship between being a zero-power valued mapping and being a bijective mapping. In this study, it is first shown that if ϑ is both an additive and a zero-power valued mapping on $\wp$, then $\alpha = \vartheta + 1$ is one-to-one and onto mapping on $\wp$. However, it is shown that a homoderivation ϑ is a semi-derivation of the ring $\wp$ associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ where 1 is the identity mapping of $\wp$. In [2], it is proved by Chang that if there exists a nonzero semi-derivation f of a prime ring $\wp$ associated with a not necessarily surjective function g, then g is a homomorphism of $\wp$. It is shown in this paper that every homoderivation ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ and α is a homomorphism without the condition of primeness of $\wp$. In addition, if ϑ is a zero-power valued homoderivation on $\wp$, then ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an automorphism without the condition of primeness of $\wp$. This means that the results have been provided for semi-derivation in the literature [2–6], but associated functions have to be surjective can be used for a zero-power valued homoderivation [7–11].

Almost every result for semi-derivation in the literature becomes applicable to homoderivation, but it is necessary to ensure that each condition is actually needed. To see this better, the results in [2],

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and in [5] for semi-derivation can be compared with El Sofy's results in [1] for homoderivation. For example, Herstein is proved in [12] that if d is a nonzero derivation of a prime ring $\wp$ with characteristic, not 2 and $a \in \wp$ is such that $[a, d(x)] = 0$, for all $x \in \wp$, then a must be in the center of $\wp$. This result of Herstein is generalized by Theorem 4 in [2] by using a nonzero semi-derivation f associated with a surjective function g of $\wp$ and by Theorem 3.3.1 in [1] by using a nonzero homoderivation ϑ of $\wp$. Every homoderivation ϑ of $\wp$ is a semi-derivation associated with $\alpha = \vartheta + 1$, but α doesn't have to be a surjective mapping of $\wp$. If ϑ is a nonzero homoderivation and $a \in \wp$ is such that $[a, \vartheta(x)] = 0$, for all $x \in \wp$, then it holds that, for all $x \in \wp$,

$$[a, \alpha(x)] = [a, \vartheta(x) + x] = [a, \vartheta(x)] + [a, x] = [a, x]$$

The last equation is in equipoise that α must be a surjective function in the proof of Theorem 4 in [2]. Hence, it is obtained from Theorem 4 in [2] that if ϑ is a nonzero homoderivation of a prime ring $\wp$ with characteristic not 2 and $a \in \wp$ is such that $[a, \vartheta(x)] = 0$, for all $x \in \wp$, then a must be in the center of $\wp$. This means that the result of Chang is more general than the result of El Sofy. A similar situation is observed between Theorem 6 [2] and Theorem 3.3.3 [1]. Moreover, the results of in [3] for semi-derivation on an ideal of the ring can be compared with El Sofy's results in [1] for homoderivation on an ideal of the ring. Other examples can also be found.

In [4] and [5], it is proved that if f is a semi-derivation associated with function g of a prime ring $\wp$, then f is an ordinary derivation of $\wp$ or f satisfies that $f(x) = \lambda(1 - g)(x)$, for all $x \in \wp$, where λ is an element in the extended centroid of $\wp$ and g is an endomorphism of $\wp$. Since every homoderivation ϑ is a nonzero semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ and α is a homomorphism of $\wp$ without the condition of primeness of $\wp$, it is clear that homoderivation ϑ satisfies the result of Bresar and Chuang in case of $\lambda = -1$ without the condition of primeness of $\wp$.

It is shown in [6] that every semi-derivation $f : R \to R$ associated with a function g is both a $(1, g)$-derivation and a $(g, 1)$-derivation. Since every homoderivation ϑ is a semi-derivation associated with $\alpha = \vartheta + 1$, then ϑ is both a $(1, \alpha)$-derivation and $(\alpha, 1)$-derivation. To illustrate, let $\wp$ be a noncommutative ring, β be a $\wp$-module homomorphism, and α be a non-additive mapping of $\wp$. Assume that $f, g : \wp \times \wp \to \wp \times \wp$ are defined as $f((x, y)) = (0, \beta(y))$ and $g((x, y)) = (\alpha(x), 0)$, respectively. Then, f is both a $(1, g)$-derivation and $(g, 1)$-derivation where 1 is the identity mapping of $\wp$ but f is neither a semi-derivation associated with g of $\wp$ nor a homoderivation of $\wp$.

In the main section, it is proved that every homoderivation ϑ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a homomorphism of $\wp$ but the converse is generally not true (see Examples 3.7-3.9). Thus, it is shown that a homoderivation ϑ is equivalent to the function given, for example, of semi-derivation by Bergen in [13]. It is clear that homoderivation ϑ satisfies the result of Bresar and Chuang in the case of $\lambda = -1$ without the condition of primeness of $\wp$.

2. Preliminaries

Let $\wp$ be a ring. Then, $\wp$ is called prime if a, b in $\wp$ such that $a\wp b = (0)$ implies that either a or b is zero. An additive mapping $\alpha : \wp \to \wp$ is called a homomorphism if $\alpha(uv) = \alpha(u)\alpha(v)$, for all $u, v \in \wp$. A mapping ϑ such that $\vartheta(A) \subseteq A$ is called a zero-power valued on A if there is a positive integer $n(a) > 1$ such that $\vartheta^{n(a)}(a) = 0$, for all $a \in A$.

In [14], derivation is defined on $\wp$ as follows: An additive mapping $d : \wp \to \wp$ is called a derivation if $d(uv) = d(u)v + ud(v)$, for all $u, v \in \wp$. Semi-derivation [13] is defined on $\wp$ as follows: An additive mapping ϑ is called a semi-derivation if there is a function $\alpha : \wp \to \wp$ such that

i. $\vartheta(uv) = \vartheta(u)\alpha(v) + u\vartheta(v) = \vartheta(u)v + \alpha(u)\vartheta(v)$, for all $u, v \in \wp$

ii. $\vartheta(\alpha(u)) = \alpha(\vartheta(u))$, for all $u \in \wp$

It is clear that any derivation is a semi-derivation associated with 1 which is the identitiy mapping of $\wp$. Conversely, in the the same article, it is shown by Bergen that $\alpha : \wp \to \wp$ such that $\alpha \neq 1$ is a homomorphism, $\vartheta = \alpha - 1$ is a semi-derivation which is not a derivation of $\wp$. $(1, g)$-derivation and $(g, 1)$-derivation are defined on $\wp$ in [6] as follows, respectively: An additive mapping ϑ is called a $(1, g)$-derivation if there is a function $g : \wp \to \wp$ such that $\vartheta(uv) = \vartheta(u)v + g(u)\vartheta(v)$, for all $u, v \in \wp$, and an additive mapping ϑ is called a $(g, 1)$-derivation if there is a function $g : \wp \to \wp$ such that $\vartheta(uv) = \vartheta(u)g(v) + u\vartheta(v)$, for all $u, v \in \wp$. The definition of homoderivation is introduced in [1] as follows: An additive mapping $\vartheta : \wp \to \wp$ is a homoderivation if $\vartheta(uv) = \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)$, for all $u, v \in \wp$.

Lemma 2.1. [2] Let $\wp$ be a prime ring. If f is a nonzero semi-derivation of a prime ring $\wp$ associated with a not necessarily surjective function g, then g is a homomorphism of $\wp$.

Lemma 2.2. [1] Let ϑ be a homoderivation of $\wp$. Then, $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an endomorphism of $\wp$.

3. Main Results

Unless otherwise stated throughout this paper, $\wp$ is a noncommutative ring, and 1 is the identity mapping of $\wp$.

Lemma 3.1. Let ϑ be an additive mapping of $\wp$. Then, $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an additive mapping and $\alpha\vartheta = \vartheta\alpha$.

Proof. Assume that $\alpha : \wp \to \wp$ is defined as $\alpha = \vartheta + 1$ where ϑ is an additive mapping of $\wp$. Therefore, it holds that, for all $u, v \in \wp$

$$\begin{aligned} \alpha(u+v) &= (\vartheta + 1)(u+v) \\ &= \vartheta(u+v) + u + v \\ &= \vartheta(u) + \vartheta(v) + u + v \\ &= \alpha(u) + \alpha(v) \end{aligned}$$

which implies that α is an additive mapping of $\wp$. Moreover,

$$\begin{aligned} (\alpha\vartheta)(u) &= \alpha(\vartheta(u)) \\ &= \vartheta(\vartheta(u)) + \vartheta(u) \\ &= \vartheta^2(u) + \vartheta(u) \end{aligned}$$

and using that ϑ is an additive mapping

$$(\vartheta\alpha)(u) = \vartheta(\alpha(u)) = \vartheta(\vartheta(u) + u) = \vartheta(\vartheta(u)) + \vartheta(u) = \vartheta^2(u) + \vartheta(u)$$

for all $u \in \wp$. Thus, it implies that

$$(\alpha\vartheta)(u) = (\vartheta\alpha)(u)$$

for all $u \in \wp$, which means that

$$\alpha\vartheta = \vartheta\alpha$$

□

Theorem 3.2. Let ϑ be an additive mapping of $\wp$ and $\alpha : \wp \to \wp$ be defined as $\alpha = \vartheta + 1$ where 1 is the identity mapping of $\wp$. Assume that $(A, +)$ is a nonzero subgroup of $\wp$.

i. If ϑ is a zero-power valued on A, then α is a surjective mapping on A

ii. If ϑ is a zero-power valued on A and $\ker\alpha \subset A$, then α is an injective mapping of $\wp$

Proof. *i.* Assume that ϑ is an additive and a zero-power valued mapping on A. Then, $\vartheta(A) \subset A$ and there exists a positive integer $n(u) > 1$ such that $\vartheta^{n(u)}(u) = 0$, for all $u \in A$. Since ϑ satisfies that $\vartheta(A) \subset A$, it holds that, for all $u \in A$,

$$\alpha(u) = \vartheta(u) + u \subset A$$

that is,

$$\alpha(A) \subset A$$

For all $u \in A$,

$$\begin{aligned}\alpha(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) =& \vartheta(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) \\ & + u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u) \\ =& \vartheta(u) - \vartheta^2(u) + \vartheta^3(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)}(u) \\ & + u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u) \\ =& u + (-1)^{n(u)-1}\,\vartheta^{n(u)}(u)\end{aligned}$$

Thus,

$$\alpha(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) = u + (-1)^{n(u)-1}\,\vartheta^{n(u)}(u)$$

for all $u \in A$. Using that ϑ is a zero-power valued mapping of A, it implies that, for all $u \in A$,

$$\alpha(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) = u$$

Since there exists at least one $a = u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u) \in A$ such that $\alpha(a) = u$, for all $u \in A$. Then, it is obtained that α is a surjective mapping on A.

ii. Let ϑ is a zero-power valued on A and $\ker\alpha \subset A$. Assume that $a \in \ker\alpha \subset A$. Then, it holds that

$$\begin{aligned}\alpha(a) = 0 &\Rightarrow \vartheta(a) + a = 0 \\ &\Rightarrow \vartheta(a) = -a\end{aligned}$$

Thus,

$$\vartheta(a) = -a$$

which means that

$$\vartheta(\ker\alpha) \subset \ker\alpha \tag{3.1}$$

Moreover, using that ϑ is an additive mapping of $\wp$

$$\vartheta^2(a) = \vartheta\left(\vartheta(a)\right) = \vartheta(-a) = -\vartheta(a) = -(-a) = a$$
$$\vartheta^3(a) = \vartheta\left(\vartheta^2(a)\right) = \vartheta(a) = -a$$
$$\vartheta^4(a) = \vartheta\left(\vartheta^3(a)\right) = \vartheta(-a) = -(\vartheta(a)) = -(-a) = a$$
$$\vdots$$
$$\vartheta^{n(a)}(a) = (-1)^{n(a)}a$$

for all $a \in \ker\alpha$. Thus, it follows that, for all $a \in \ker\alpha$,

$$\vartheta^{n(a)}(a) = (-1)^{n(a)}a \tag{3.2}$$

Since ϑ is a zero-power valued mapping on A and $\ker\alpha \subset A$, there is a positive integer $n(u) > 1$ such that $\vartheta^{n(u)}(u) = 0$, for all $u \in \ker\alpha$. In addition, it holds that $\vartheta(\ker\alpha) \subset \ker\alpha$ from (3.1) which means that ϑ is a zero-power valued mapping on $\ker\alpha$. Therefore, it follows from (3.2), that for all $u \in \ker\alpha$, a positive integer $n(u) > 1$

$$\vartheta^{n(u)}(u) = (-1)^{n(u)}u = 0$$

Then, it is obtained that $u = 0$, for all $u \in \ker\alpha$, which implies that $\ker\alpha = (0)$. Thus, α is an injective mapping of $\wp$. □

Corollary 3.3. Let ϑ be an additive mapping of $\wp$. If ϑ is a zero-power valued on $\wp$, then $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a bijective mapping of $\wp$.

The proof is clear from Theorem 3.2.

Theorem 3.4. If $\vartheta : \wp \to \wp$ is a homoderivation, then ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$.

Proof. Let $\vartheta : \wp \to \wp$ be a homoderivation. If the definition of homoderivation is rearranged by using that $\alpha = \vartheta + 1$, it holds that, for all $u, v \in \wp$,

$$\begin{aligned}\vartheta(uv) &= \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)\\ &= \vartheta(u)(\vartheta(v) + v) + u\vartheta(v)\\ &= \vartheta(u)\alpha(v) + u\vartheta(v)\end{aligned}$$

and

$$\begin{aligned}\vartheta(uv) &= \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)\\ &= \vartheta(u)v + (\vartheta(u) + u)\vartheta(v)\\ &= \vartheta(u)v + \alpha(u)\vartheta(v)\end{aligned}$$

Thus, ϑ is written as, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\alpha(v) + u\vartheta(v) = \vartheta(u)v + \alpha(u)\vartheta(v) \tag{3.3}$$

In addition, it implies that from Lemma 3.1,

$$(\alpha\vartheta)(u) = (\vartheta\alpha)(u), \text{ for all } u \in \wp \tag{3.4}$$

Thus, (3.3) and (3.4) mean that ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$. □

It is shown in the next corollary that Lemma 2.1 is satisfied without the condition of primeness of ring $\wp$.

Corollary 3.5. If $\vartheta : \wp \to \wp$ is a homoderivation, then ϑ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a homomorphism of $\wp$.

The proof is clear from Lemma 2.2 and Theorem 3.4.

Corollary 3.6. Let ϑ be a zero-power valued homoderivation on $\wp$. Then, ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an automorphism of $\wp$.

Proof. Assume that ϑ is a zero-power valued homoderivation on $\wp$. ϑ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a homomorphism of $\wp$ from Corollary 3.5. Thus, α is a bijective mapping of $\wp$ from Theorem 3.2 which means that α is an automorphism of $\wp$. Hence, ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an automorphism of $\wp$. □

Every homoderivation is a semi-derivation, but it is possible to find examples that are semi-derivation but not homoderivation:

Example 3.7. Let $\wp_1$ and $\wp_2$ be two rings. Let $\wp = \wp_1 \times \wp_2$ be a ring with the operations such that

$$(p_1, p_2) + (a_1, a_2) = (p_1 + a_1, p_2 + a_2)$$

and

$$(p_1, p_2)(a_1, a_2) = (p_1 a_1, p_2 a_2)$$

for all $(p_1, p_2), (a_1, a_2) \in \wp$. If $f : \wp \to \wp$ is defined as $f((u_1, u_2)) = (0, u_2)$ and $g : \wp \to \wp$ is defined as $g((u_1, u_2)) = (u_1, 0)$, for all $(u_1, u_2) \in \wp$, then f is a semi-derivation of $\wp$ associated with function g but f is not a homoderivation of $\wp$.

Let $p = (p_1, p_2)$ and $a = (a_1, a_2)$ be elements of $\wp$. It holds that

$$f(pa) = f((p_1, p_2)(a_1, a_2)) = f((p_1 a_1, p_2 a_2)) = (0, p_2 a_2) \tag{3.5}$$

Moreover,

$$\begin{aligned} f(p)g(a) + pf(a) &= f((p_1, p_2))g((a_1, a_2)) + (p_1, p_2)f((a_1, a_2)) \\ &= (0, p_2)(a_1, 0) + (p_1, p_2)(0, a_2) \\ &= (0, 0) + (0, p_2 a_2) \\ &= (0, p_2 a_2) \end{aligned}$$

and

$$\begin{aligned} f(p)a + g(p)f(a) &= f((p_1, p_2))(a_1, a_2) + g((p_1, p_2))f((a_1, a_2)) \\ &= (0, p_2)(a_1, a_2) + (p_1, 0)(0, a_2) \\ &= (0, p_2 a_2) + (0, 0) \\ &= (0, p_2 a_2) \end{aligned}$$

for all $p = (p_1, p_2), a = (a_1, a_2) \in \wp$. Thus, it implies that, for $p, a \in \wp$,

$$f(pa) = f(p)g(a) + pf(a) = f(p)a + g(p)f(a) \tag{3.6}$$

Besides, it holds that, for all $p = (p_1, p_2) \in \wp$,

$$(fg)(p) = (fg)((p_1, p_2)) = f(g((p_1, p_2))) = f((p_1, 0)) = (0, 0)$$

and

$$(gf)(p) = (gf)((p_1, p_2)) = g(f((p_1, p_2))) = g((0, p_2)) = (0, 0)$$

Therefore, it means that, for all $p \in \wp$,

$$(fg)(p) = (gf)(p)$$

Thus,

$$fg = gf \tag{3.7}$$

(3.6) and (3.7) imply that f is a semi-derivation associated with function g of $\wp$. But

$$\begin{aligned} f(p)f(a)+f(p)a+pf(a) &= f((p_1,p_2))f\left((a_1,a_2)\right)+f((p_1,p_2))(a_1,a_2)+(p_1,p_2)f((a_1,a_2)) \\ &= (0,p_2)(0,a_2)+(0,p_2)(a_1,a_2)+(p_1,p_2)(0,a_2) \\ &= (0,p_2a_2)+(0,p_2a_2)+(0,p_2a_2) \\ &= (0,3p_2a_2) \end{aligned}$$

for all $p=(p_1,p_2), a=(a_1,a_2)\in\wp$. Hence, it holds that, for $p=(p_1,p_2), a=(a_1,a_2)\in\wp$,

$$f(p)f(a)+f(p)a+pf(a)=(0,3p_2a_2) \tag{3.8}$$

(3.5) and (3.8) imply that, for all $p,a\in\wp$,

$$f(pa)\neq f(p)f(a)+f(p)a+pf(a)$$

which means that f is not a homoderivation of $\wp$.

Example 3.8. Let $\wp$ be a ring and $\vartheta:\wp\times\wp\to\wp\times\wp$ be a mapping such that $\vartheta\left((u,v)\right)=(u,0)$ and $\alpha:\wp\times\wp\to\wp\times\wp$ be a mapping such that $\alpha\left((u,v)\right)=(0,v)$, for all $(u,v)\in\wp\times\wp$. Then, ϑ is a semi-derivation associated with the function α but ϑ is not a homoderivation of $\wp\times\wp$.

Let $p=(p_1,p_2)$ and $a=(a_1,a_2)$ be elements of $\wp\times\wp$. It holds that

$$\vartheta(pa)=\vartheta\left((p_1,p_2)(a_1,a_2)\right)=\vartheta\left((p_1a_1,p_2a_2)\right)=(p_1a_1,0) \tag{3.9}$$

Besides,

$$\begin{aligned} \vartheta(p)\alpha(a)+p\vartheta(a) &= \vartheta((p_1,p_2))\alpha\left((a_1,a_2)\right)+(p_1,p_2)\vartheta((a_1,a_2)) \\ &= (p_1,0)(0,a_2)+(p_1,p_2)(a_1,0) \\ &= (0,0)+(p_1a_1,0) \\ &= (p_1a_1,0) \end{aligned}$$

and

$$\begin{aligned} \vartheta(p)a+\alpha(p)\vartheta(a) &= \vartheta((p_1,p_2))(a_1,a_2)+\alpha((p_1,p_2))\vartheta((a_1,a_2)) \\ &= (p_1,0)(a_1,a_2)+(0,p_2)(a_1,0) \\ &= (p_1a_1,0)+(0,0) \\ &= (p_1a_1,0) \end{aligned}$$

for all $p=(p_1,p_2)$ and $a=(a_1,a_2)\in\wp\times\wp$. Thus, it implies that, for $p,a\in\wp\times\wp$,

$$\vartheta(pa)=\vartheta(p)\alpha(a)+p\vartheta(a)=\vartheta(p)a+\alpha(p)\vartheta(a) \tag{3.10}$$

Moreover, it holds that, for all $p=(p_1,p_2)\in\wp\times\wp$,

$$(\vartheta\alpha)(p)=(\vartheta\alpha)\left((p_1,p_2)\right)=\vartheta\left(\alpha((p_1,p_2))\right)=\vartheta\left((0,p_2)\right)=(0,0)$$

and

$$(\alpha\vartheta)(p)=(\alpha\vartheta)((p_1,p_2))=\alpha\left(\vartheta((p_1,p_2))\right)=\alpha\left((p_1,0)\right)=(0,0)$$

Therefore, it means that, for all $p\in\wp\times\wp$,

$$(\vartheta\alpha)(p)=(\alpha\vartheta)(p)$$

Thus,

$$\vartheta\alpha = \alpha\vartheta \tag{3.11}$$

(3.10) and (3.11) imply that ϑ is a semi-derivation associated with function α of $\wp \times \wp$. But

$$\begin{aligned}\vartheta(p)\vartheta(a) + \vartheta(p)a + p\vartheta(a) &= \vartheta((p_1,p_2))\vartheta\left((a_1,a_2)\right) + \vartheta((p_1,p_2))(a_1,a_2) + (p_1,p_2)\vartheta((a_1,a_2))\\ &= (p_1,0)(a_1,0) + (p_1,0)(a_1,a_2) + (p_1,p_2)(a_1,0)\\ &= (p_1a_1,0) + (p_1a_1,0) + (p_1a_1,0)\\ &= (3p_1a_1,0)\end{aligned}$$

for all $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$. Hence, it holds that, for $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$,

$$\vartheta(p)\vartheta(a) + \vartheta(p)a + p\vartheta(a) = (3p_1a_1,0) \tag{3.12}$$

(3.9) and (3.12) imply that, for $p, a \in \wp \times \wp$,

$$\vartheta(pa) \neq \vartheta(p)\vartheta(a) + \vartheta(p)a + p\vartheta(a)$$

which means that ϑ is not a homoderivation of $\wp \times \wp$.

Example 3.9. Let $\wp$ be a ring and $d : \wp \to \wp$ be a derivation. Let $f : \wp \times \wp \to \wp \times \wp$ be a mapping such that $f\left((u,v)\right) = (d(u),0)$ and $\sigma : \wp \times \wp \to \wp \times \wp$ be a mapping such that $\sigma\left((u,v)\right) = (u,v)$, for all $(u,v) \in \wp \times \wp$. Then, f is a semi-derivation associated with function σ but f is not a homoderivation of $\wp \times \wp$.

Assume that d is a derivation of $\wp$. Let $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$. It holds that

$$f(pa) = f\left((p_1,p_2)(a_1,a_2)\right) = f((p_1a_1,p_2a_2)) = (d(p_1a_1),0) \tag{3.13}$$

Further, using that d is a derivation

$$\begin{aligned}f(p)\sigma(a) + pf(a) &= f((p_1,p_2))\sigma\left((a_1,a_2)\right) + (p_1,p_2)f((a_1,a_2))\\ &= (d(p_1),0)(a_1,a_2) + (p_1,p_2)(d(a_1),0)\\ &= (d(p_1)a_1,0) + (p_1d(a_1),0)\\ &= (d(p_1)a_1 + p_1d(a_1),0)\\ &= (d(p_1a_1),0)\end{aligned}$$

and

$$\begin{aligned}f(p)a + \sigma(p)f(a) &= f((p_1,p_2))(a_1,a_2) + \sigma((p_1,p_2))f((a_1,a_2))\\ &= (d(p_1),0)(a_1,a_2) + (p_1,p_2)(d(a_1),0)\\ &= (d(p_1)a_1,0) + (p_1d(a_1),0)\\ &= (d(p_1)a_1 + p_1d(a_1),0)\\ &= (d(p_1a_1),0)\end{aligned}$$

for all $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$. Thus, it implies that, for $p, a \in \wp \times \wp$,

$$f(pa) = f(p)\sigma(a) + pf(a) = f(p)a + \sigma(p)f(a) \tag{3.14}$$

Moreover, it holds that, for all $p = (p_1, p_2) \in \wp \times \wp$,

$$(f\sigma)(p) = (f\sigma)\left((p_1, p_2)\right) = f\left(\sigma((p_1, p_2))\right) = f\left((p_1, p_2)\right) = (d(p_1), 0)$$

and

$$(\sigma f)(p) = (\sigma f)((p_1, p_2)) = \sigma(f((p_1, p_2))) = \sigma\left((d(p_1), 0)\right) = (d(p_1), 0)$$

Therefore, it means that, for all $p \in \wp \times \wp$,

$$(f\sigma)(p) = (\sigma f)(p)$$

Thus,

$$f\sigma = \sigma f \tag{3.15}$$

(3.14) and (3.15) imply that f is a semi-derivation associated with function σ of $\wp \times \wp$. But

$$\begin{aligned} f(p)f(a) + f(p)a + pf(a) &= f((p_1, p_2))f\left((a_1, a_2)\right) + f((p_1, p_2))(a_1, a_2) + (p_1, p_2)f((a_1, a_2)) \\ &= (d(p_1), 0)(d(a_1), 0) + (d(p_1), 0)(a_1, a_2) + (p_1, p_2)(d(a_1), 0) \\ &= (d(p_1)d(a_1), 0) + (d(p_1)a_1, 0) + (p_1 d(a_1), 0) \\ &= (d(p_1)d(a_1) + d(p_1)a_1 + p_1 d(a_1), 0) \end{aligned}$$

for all $p = (p_1, p_2), a = (a_1, a_2) \in \wp \times \wp$. Thus, it holds that, for all $p = (p_1, p_2), a = (a_1, a_2) \in \wp \times \wp$,

$$f(p)f(a) + f(p)a + pf(a) = (d(p_1)d(a_1) + d(p_1)a_1 + p_1 d(a_1), 0) \tag{3.16}$$

(3.13) and (3.16) imply that, for all $p, a \in \wp \times \wp$,

$$f(pa) \neq f(p)f(a) + f(p)a + pf(a)$$

which means that f is not a homoderivation of $\wp \times \wp$.

These examples explain that the definition of semi-derivation is more general than homo-derivation on a ring.

Theorem 3.10. If $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$, then ϑ is a homoderivation of $\wp$.

Proof. Let $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ be a semi-derivation associated with α where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$. Then, it holds that, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\alpha(v) + u\vartheta(v) = \vartheta(u)v + \alpha(u)\vartheta(v)$$

Since α satisfies that $\alpha = \vartheta + 1$, it is obtained that, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\left(\vartheta + 1\right)(v) + u\vartheta(v) = \vartheta(u)v + \left(\vartheta + 1\right)(u)\vartheta(v)$$

which implies that, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)$$

That is, ϑ is a homoderivation of $\wp$. □

Corollary 3.11. ϑ is a homoderivation of $\wp$ if and only if $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$.

The proof is clear from Theorems 3.4 and 3.10.

4. Conclusion

The fact that every homoderivation ϑ of $\wp$ is a semi-derivation associated with $\alpha = \vartheta + 1$ and α is a homomorphism of an arbitrary ring $\wp$ shows that some results can be achieved without needing the primeness of the ring or being surjective of the associated function. Examples of this situation are provided in the introduction. Based on the examples, the reader may be advised to think about the relationship between homoderivation and semi-derivation ϑ such that $\vartheta = \alpha - 1$. Semi-derivation evidences typically require to be surjective of the ring's primeness or the related function. Although a function that is the form of $\vartheta = \alpha - 1$ is a semi-derivation associated with homomorphism α, there may be no need for being surjective of the associated function or the primeness of the ring while doing the proof. It is proved in this paper that the definition of homoderivation is equivalent to the definition of semi-derivation, the form of $\vartheta = \alpha - 1$ where $\alpha : \wp \to \wp$ such that $\alpha \neq 1$ is a homomorphism. Therefore, while generalizing the problems in the literature for homoderivation, using homoderivation is the same as using the semi-derivation, the form of $\vartheta = \alpha - 1$ associated with homomorphism α.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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A Global Krylov Subspace Method for the Sylvester Quaternion Matrix Equation

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Abstract — This study concerns the Sylvester matrix equation in the quaternion setting when the coefficient matrices as well as the unknown matrix have quaternion entries. We propose a global Generalized Minimal Residual (GMRES) method for the solution of such a matrix equation. The proposed approach works directly with the Sylvester operator to generate orthonormal bases for Krylov subspaces formed of matrices. Then, the best approximate matrix solution to the Sylvester equation at hand in such a Krylov subspace is constructed from a matrix minimizing the Frobenius norm of the residual. We describe how this minimization of the residual norm can be carried out efficiently and report numerical results on real examples related to image restoration.

Keywords *Sylvester quaternion matrix equation, quaternion Krylov subspace, global GMRES, quaternion Arnoldi process*

Mathematics Subject Classification (2020) 15A24, 15B33

1. Introduction

The Sylvester matrix equation is of the form

$$AX + XB = C \tag{1.1}$$

where X is the $n \times m$ unknown matrix, and A, B, and C are given matrices with appropriate sizes [1]. Such a matrix equation and its special case, a Lyapunov matrix equation [2], arise in fields, such as control theory, eigenstructure assignment, model reduction, image restoration problems, numerical solutions of ordinary differential equations [3–9]. On the other side, quaternions have applications in various fields, including those from computer science, quantum mechanics, signal and color image processing [10, 11]. Due to these wide ranges of applications for Sylvester equations, as well as quaternions, the problem of obtaining solutions to (1.1), specifically over the skew-field of quaternions, has attracted considerable attention [12–15]. Matrix equations other than Sylvester equations over quaternions have also been studied in the literature [16–21].

In general, direct or iterative numerical methods are employed to find the solutions of (1.1) depending on the size of A and B. When the coefficient matrices have sizes of a few hundred at most, the problem is referred to as a small- or medium-scale problem. For these problems, the most efficient method is a direct method proposed by Bartels and Stewart [22]. This method is based on the Schur decompositions of the coefficient matrices A and B, resulting in a Sylvester matrix equation in a

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simplified form that is easily solved by back substitution. When both coefficient matrices are small, but one is significantly smaller than the other, Golub et al. have presented a variant of Bartels and Stewart algorithm by means of the Hessenberg decomposition of the larger coefficient matrix [23]. A classical alternative approach is to turn the Sylvester matrix equation into a linear system by using the Kronecker product and vec operator. Then, the LU factorization with partial pivoting can be applied to this linear system for finding the solution. Apart from these methods, if only one of n and m is large, several approaches are available based on a decomposition of the smaller coefficient matrix. However, these approaches are not useful when both n and m are large (i.e., typically larger than 200). In the case of such large problems, commonly employed techniques to solve Sylvester equations are alternate direction implicit (ADI) iteration and projection methods. For instance, Krylov subspace methods are commonly employed projection methods for such large Sylvester equations when coefficients matrices are sparse. For large Sylvester matrix equations over real or complex fields, block and global Krylov subspace methods have attracted substantial interest in the literature [24–34]. While a block Arnoldi process constructs orthonormal bases for several subspaces of $\mathbb{C}^n$ or $\mathbb{R}^n$, simultaneously, the global Arnoldi process constructs an orthonormal basis for a subspace of a space of matrices.

Linear matrix equations over quaternions rather than over real or complex numbers come up with additional challenges, especially as the multiplication of two quaternion scalars is not commutative. It is possible to convert a quaternion matrix equation into a real or a complex matrix equation by employing real or complex representations of quaternion matrices. However, this conversion is usually not desirable since it results in matrices in the converted matrix equation that are twice or four times as large as the matrices in the original problem. On the other hand, structure-preserving Krylov subspace methods have become popular recently to overcome the increase in the size of the matrices when such a conversion is applied [19–21]. Some other approaches for solving linear quaternion matrix equations work on a right or a left Hilbert space over quaternions equipped with a proper inner product [17,18].

In this study, we consider (1.1) with non-Hermitian and nonsingular coefficient matrices A and B of size $n \times n$ and $m \times m$, respectively, when n and m are large. We aim to find the solution by means of a global Generalized Minimal Residual (GMRES) algorithm operating on the Krylov subspaces of the space of $n \times m$ quaternion matrices. We directly work with the original quaternion matrices without using real or complex representations of quaternion matrices and exploit a real inner product defined in the space of $n \times m$ quaternion matrices. Our approach applies the Sylvester operator $X \mapsto AX + XB$ at every iteration when adding a new direction to the Krylov subspace.

We present our study in the following order. In Section 2, the preliminaries for quaternion matrices, useful identities, and problem reformulation are presented. In Section 3, the global Arnoldi process to construct an orthonormal basis for a matrix Krylov subspace is described, and then the global GMRES method to retrieve the best approximation in this matrix Krylov subspace is presented. Finally, in Section 4, the efficiency and accuracy of the proposed approach are illustrated with examples related to image restoration.

2. Preliminaries

In this section, we summarize quaternions and some of their properties. The division ring $\mathbb{H}$ of quaternions is given by

$$\mathbb{H} = \{q_0 + q_1 i + q_2 j + q_3 k \mid i^2 = j^2 = k^2 = -1, \;\; ij = -ji = k, \;\; q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

For $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, the conjugate and the modulus of q are

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$$

and

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

respectively.

Since the multiplication of the quaternion units i, j, and k are non-commutative, the multiplication of a quaternion scalar $p \in \mathbb{H}$ with another quaternion scalar $q \in \mathbb{H}$ is usually not commutative. Thus, the n-tuples of $\mathbb{H}$ denoted by $\mathbb{H}^n$ can be regarded either as a right vector space or a left vector space over the division ring $\mathbb{H}$, depending on whether the multiplication in $\mathbb{H}^n$ with quaternion scalars is defined from the right or from the left, respectively. In this study, we consider $\mathbb{H}^n$ together with the multiplication with scalars from the right, that is, $\mathbb{H}^n$ as a right vector space. A possible real inner product on $\mathbb{H}^n$ is

$$\langle u, v\rangle \ = \sum_{i=1}^{n} \operatorname{Re}\left(\overline{v_i}\, u_i\right) \tag{2.1}$$

for $u, v \in \mathbb{H}^n$. The right vector space $\mathbb{H}^n$ with (2.1) is commonly referred to as a right quaternionic Hilbert space. We define the norm of a vector $u \in \mathbb{H}^n$ in this right quaternionic Hilbert space as

$$\|u\| = \sqrt{\sum_{i=1}^{n} \operatorname{Re}\left(\overline{u_i}\, u_i\right)} = \sqrt{\sum_{i=1}^{n} |\ u_i\ |^2} \tag{2.2}$$

We denote the set of $n \times m$ matrices with quaternion entries with $\mathbb{H}^{n\times m}$, which can also be regarded as a right vector space over $\mathbb{H}$ together with the multiplication with quaternion scalars from the right. The basic linear algebra terminology, definitions, and standard notations for a vector space over complex numbers also apply to a right vector space over quaternions. In particular, the notations X^* and v^* are reserved for the conjugate transposes of $X \in \mathbb{H}^{n\times m}$ and $v \in \mathbb{H}^n$, respectively. Multiplication of two quaternion matrices of suitable sizes is defined analogously to the multiplication of two complex matrices. If a matrix $X \in \mathbb{H}^{n\times n}$ satisfies $X^*X = I$, then X is called a unitary matrix. On the other hand, if $X \in \mathbb{H}^{n\times n}$ satisfies $X^* = X$, then X is called Hermitian. Moreover, $X \in \mathbb{H}^{n\times n}$ is invertible if there exits $X^{-1} \in \mathbb{H}^{n\times n}$ such that $XX^{-1} = X^{-1}X = I$. In this work, we make use of the real inner product on $\mathbb{H}^{n\times m}$ defined as

$$\langle X, Y\rangle_F = \operatorname{Re}\left(\operatorname{tr}\left(Y^*X\right)\right) \tag{2.3}$$

for $X, Y \in \mathbb{H}^{n\times m}$. The norm of $X \in \mathbb{H}^{n\times m}$ induced by (2.3) is

$$\|X\|_F = \sqrt{\operatorname{Re}\left(\operatorname{tr}\left(X^*X\right)\right)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |\ x_{ij}\ |^2} \tag{2.4}$$

Note that (2.4) is an extension of the Frobenius norm defined for complex matrices to quaternion matrices.

We provide a generalization of the definition of a block-partitioned matrix product introduced originally by Bouyouli et al. [35] to the setting of quaternion matrices equipped with (2.3).

Definition 2.1. Let $A = [A_1 A_2 \cdots A_p] \in \mathbb{H}^{n\times mp}$ and $B = [B_1 B_2 \cdots B_l] \in \mathbb{H}^{n\times ml}$ with $A_i, B_j \in \mathbb{H}^{n\times m}$ for $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$. Then, the $p \times l$ matrix $A^*\diamond B$ is defined by $(A^*\diamond B)_{ij} = \langle A_i, B_j\rangle_F$, for $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$.

If the product of $A^*\diamond A$ is equal to the $p \times p$ identity matrix, that is if

$$\langle A_i, A_j\rangle_F = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $i, j \in \{1, 2, \ldots, p\}$, then the matrix A is called *F-orthonormal.*

In the next subsection, we formally introduce our problem, as well as basic notions concerning the important ingredients of the problem, including the Sylvester operator.

2.1. Problem Reformulation

Any $A \in \mathbb{H}^{n\times m}$ can be uniquely expressed as

$$A = A_1 + A_2 j \quad \text{or} \quad A = \mathrm{Re}(A_1) + \mathrm{Im}(A_1)i + \mathrm{Re}(A_2)j + \mathrm{Im}(A_2)k$$

for some $A_1, A_2 \in \mathbb{C}^{n\times m}$. For a matrix $A \in \mathbb{H}^{n\times n}$, a scalar $\lambda \in \mathbb{H}$ is called a right eigenvalue of A if

$$Ax = x\lambda$$

holds for some nonzero $x \in \mathbb{H}^n$. We remark that if $\lambda \in \mathbb{H}$ is a non-real right eigenvalue, then $Axs = xs(s^{-1}\lambda s)$, for all nonzero $s \in \mathbb{H}$, therefore $s^{-1}\lambda s$ is also an eigenvalue of A. Hence, we refer to the set

$$\mathcal{E}_A(\lambda) \ := \ \left\{ s^{-1}\lambda s \ : \ s \in \mathbb{H}, \ s \neq 0 \right\}$$

as the equivalence class of $\lambda \in \mathbb{H}$. Consequently, if a quaternion matrix has a non-real eigenvalue, then it has infinitely many non-real eigenvalues. The equivalence class of a non-real eigenvalue has only one pair of complex conjugate scalars, i.e., $\mathcal{E}_A(\lambda) \cap \mathbb{C} = \left\{ \lambda, \overline{\lambda} \right\}$. If the imaginary part of a right complex eigenvalue is nonnegative, it is called a standard eigenvalue. Any $n \times n$ quaternion matrix has exactly n standard eigenvalues counting the multiplicities [18, 36].

Consider the Sylvester matrix equation

$$AX + XB = C \tag{2.5}$$

such that $A \in \mathbb{H}^{n\times n}$, $B \in \mathbb{H}^{m\times m}$, and $C \in \mathbb{H}^{n\times m}$. It follows from Theorem 2.2.4.1 in [37] that (2.5) has a unique solution if and only if

$$\Lambda(A) \cap \Lambda(-B) = \emptyset$$

where $\Lambda(\cdot)$ denotes the set of standard eigenvalues of its quaternion matrix argument. In other words, (2.5) has a unique solution if and only if the quaternion matrices A and $-B$ do not have any common standard eigenvalue.

Associated with every Sylvester equation, there is a Sylvester operator. Formally, the Sylvester operator $S : \mathbb{H}^{n\times m} \to \mathbb{H}^{n\times m}$ for given $A \in \mathbb{H}^{n\times n}$ and $B \in \mathbb{H}^{m\times m}$ is defined as

$$S(X) = AX + XB \tag{2.6}$$

From (2.5) and (2.6),

$$S(X) = C \tag{2.7}$$

We define the norm of the operator S by

$$\|S\| = \max_{\|X\|_F = 1} \|S(X)\|_F$$

where $\|\cdot\|_F$ is defined by (2.4). The adjoint of S is denoted by S^* and is given by

$$S^*(Y) = A^*Y + YB^*$$

for $Y \in \mathbb{H}^{n\times m}$. Given $X \in \mathbb{H}^{n\times m}$ and $Y \in \mathbb{H}^{n\times m}$, the equality $\langle S(X), Y\rangle_F = \langle X, S^*(Y)\rangle_F$ holds where the inner product $\langle \cdot, \cdot \rangle_F$ is defined as in (2.3).

In the rest of this paper, we focus on the solution of (2.5) by means of a global Krylov subspace method assuming that (2.5) has a unique solution. Our approach makes use of the associated Sylvester operator frequently. In the next section, we describe a global Arnoldi process to construct an orthonormal basis for a Krylov subspace, as well as a global GMRES method to find the best solution of (2.5) in a least-squares sense in an affine space associated with this Krylov subspace.

3. Solution of the Sylvester Quaternion Matrix Equation by Global GMRES

3.1. The Global Arnoldi Process

Suppose that $X_0 \in \mathbb{H}^{n\times m}$ is an approximate solution of (2.7), and $R_0 = C - S(X_0)$ is the corresponding residual. The quaternion matrix Krylov subspace $\mathcal{K}_k\left(S, R_0\right) \subset \mathbb{H}^{n\times m}$ associated with (2.6) and the residual R_0 that we will be dealing with is given by

$$\begin{aligned} \mathcal{K}_k\left(S, R_0\right) &:= \operatorname{span}\{R_0, S(R_0), \dots, S^{k-1}(R_0)\} \\ &= \{\alpha_0 R_0 + \alpha_1 S(R_0) + \cdots + \alpha_{k-1} S^{k-1}(R_0) \mid \alpha_0, \alpha_1, \dots, \alpha_{k-1} \in \mathbb{R}\} \end{aligned} \tag{3.1}$$

for a prescribed integer k. Note that in (3.1) the operator $S^i(R_0)$ is defined recursively by $S(S^{i-1}(R_0))$, for $i \in \{1, 2, \dots, k-1\}$, and $S^0(R_0) = R_0$. We remark that the set $\mathbb{H}^{n\times m}$ over the field of real numbers is indeed a real vector space. Moreover, $\mathcal{K}_k\left(S, R_0\right)$, a subset of $\mathbb{H}^{n\times m}$, equipped with real scalars, is a real vector space as well, hence a subspace of $\mathbb{H}^{n\times m}$.

The global Arnoldi process, described formally in Algorithm 1, is a procedure that constructs an F-orthonormal basis for the Krylov subspace $\mathcal{K}_k\left(S, R_0\right)$. At termination, the process generates the set of matrices $\{Q_1, Q_2, \dots, Q_k\}$ that forms an orthonormal basis for $\mathcal{K}_k\left(S, R_0\right)$ with respect to the inner product $\langle X, Y\rangle_F = \operatorname{Re}(\operatorname{tr}(Y^*X))$, for $X, Y \in \mathbb{H}^{n\times m}$.

Algorithm 1 Global Arnoldi Process

$R_0 \leftarrow C - S(X_0)$

Set $Q_1 = \frac{R_0}{\|R_0\|_F}$

for $j = 1$ to k **do**

  $V \leftarrow S(Q_j)$

  for $i = 1$ to j **do**

    $h_{ij} \leftarrow \langle Q_i, V\rangle_F$

    $V \leftarrow V - Q_i h_{ij}$

  end for

  $h_{(j+1)j} \leftarrow \|V\|_F$. If $h_{(j+1)j} = 0$, then stop.

  $Q_{j+1} \leftarrow \frac{V}{h_{(j+1)j}}$

end for

$\widetilde{Q}_k \leftarrow [Q_1 Q_2 \dots Q_k]$, $\widetilde{Q}_{k+1} \leftarrow [\widetilde{Q}_k\; Q_{k+1}]$ and $\widetilde{H}_k$ is as in (3.4).

From the global Arnoldi process above, the recurrence for $j \in \{1, 2, \ldots, k\}$,

$$S(Q_j) = \sum_{i=1}^{j+1} Q_i h_{ij} \tag{3.2}$$

is immediate. Moreover, it can be verified in a straightforward manner that (3.2) above yields the relation

$$S(\widetilde{Q}_k) = \widetilde{Q}_{k+1}(\widetilde{H}_k \otimes I) \tag{3.3}$$

where I is the $m \times m$ identity matrix and $\widetilde{H}_k \in \mathbb{R}^{(k+1)\times k}$ is the Hessenberg matrix whose the entry (i, j) is h_{ij} produced by the global Arnoldi process,

$$\widetilde{H}_k = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1(k-1)} & h_{1k} \\ h_{21} & h_{22} & \cdots & h_{2(k-1)} & h_{2k} \\ 0 & h_{32} & \cdots & h_{3(k-1)} & h_{3k} \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & h_{k(k-1)} & h_{kk} \\ 0 & 0 & \cdots & 0 & h_{(k+1)k} \end{bmatrix} \tag{3.4}$$

Here and throughout the rest of this paper, $F \otimes G$ represents the Kronecker product of the real matrices F and G. In the next subsection, we present the global GMRES method for retrieving the solution of (2.7) by making use of $\widetilde{Q}_k$ and $\widetilde{H}_k$ generated by the global Arnoldi process.

3.2. The Global GMRES Method

For a given initial estimate $X_0 \in \mathbb{H}^{n\times m}$ for the solution of (2.7), our global GMRES method at the kth iteration finds X_k minimizing $\|C - S(X)\|_F$ over all $X \in X_0 + \mathcal{K}_k(S, R_0)$. For all $X_k \in X_0 + \mathcal{K}_k(S, R_0)$ can be expressed as

$$X_k = X_0 + \sum_{i=1}^{k} y_i^{(k)} Q_i$$

for some real scalars $y_i^{(k)}$ for $i \in \{1, \ldots, k\}$, or equivalently

$$X_k = X_0 + \widetilde{Q}_k Y_k$$

for $Y_k = y^{(k)} \otimes I \in \mathbb{R}^{km\times m}$ where $y^{(k)} := \left[y_1^{(k)} y_2^{(k)} \ldots y_k^{(k)}\right]^T$. Thus, recalling $R_0 = C - S(X_0)$, the residual $R_k = C - S(X_k)$ can be written as

$$R_k = R_0 - S(\widetilde{Q}_k Y_k) \tag{3.5}$$

The minimization of $\|C - S(X)\|_F$ overall $X \in X_0 + \mathcal{K}_k(S, R_0)$ is equivalent to the minimization of $\|R_k\|_F$ with R_k of (3.5) and $Y_k = y^{(k)} \otimes I$, for some $y^{(k)} \in \mathbb{R}^k$. In other words, we would like to solve the following minimization problem over $y^{(k)} \in \mathbb{R}^k$:

$$\left\| R_0 - S\left(\widetilde{Q}_k(y^{(k)} \otimes I)\right)\right\|_F = minimum$$

Using the linearity of the operator S and (3.3), the last minimization can be rewritten as

$$\left\| R_0 - \widetilde{Q}_{k+1}(\widetilde{H}_k \otimes I)(y^{(k)} \otimes I)\right\|_F = minimum \tag{3.6}$$

It follows from the description in Algorithm 1 that $R_0 = \beta Q_1$ for $\beta := \|R_0\|_F$, or equivalently $R_0 = \widetilde{Q}_{k+1}(\beta e_1 \otimes I)$ where e_1 is the first column of the $(k+1) \times (k+1)$ identity matrix. Hence, (3.6) can further be simplified as

$$\left\|\widetilde{Q}_{k+1}\left((\beta e_1 - \widetilde{H}_k y^{(k)}) \otimes I\right)\right\|_F = \left\|(\beta e_1 - \widetilde{H}_k y^{(k)}) \otimes I\right\|_F = minimum \tag{3.7}$$

The first equality in (3.7) follows from the fact that $\widetilde{Q}_{k+1} = [Q_1 Q_2 \cdots\ Q_{k+1}]$ where $\{Q_1, Q_2, \cdots, Q_{k+1}\}$ is an orthonormal set with respect to the inner product $\langle \cdot, \cdot \rangle_F$, inducing the Frobenius norm $\|\cdot\|_F$. As a result, our least squares problem reduces to finding $y^{(k)}$ such that

$$\left\|\beta e_1 - \widetilde{H}_k y^{(k)}\right\| = minimum \tag{3.8}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{k+1}$. At iteration k of the global GMRES method, we solve this real least-squares problem over the variable $y^{(k)} \in \mathbb{R}^k$.

A typical approach to solve (3.8) efficiently is triangularizing $\widetilde{H}_k$ unitarily. Specifically, we transform the Hessenberg matrix $\widetilde{H}_k$ into an upper triangular matrix $\widetilde{U}$ by applying k square unitary matrices $W_1, W_2, \ldots, W_k$ from left, that is

$$W_k W_{k-1} \ldots W_1 \widetilde{H}_k = \widetilde{U} = \begin{bmatrix} \widetilde{u}_{11} & \widetilde{u}_{12} & \cdots & \widetilde{u}_{1k} \\ 0 & \widetilde{u}_{22} & \ldots & \widetilde{u}_{2k} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & \widetilde{u}_{kk} \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

where $W_j \in \mathbb{R}^{(k+1)\times(k+1)}$, for $j \in \{1, 2, \ldots, k\}$, is given by

$$W_j = \begin{bmatrix} I_{j-1} & 0 & 0 \\ 0 & P_j & 0 \\ 0 & 0 & I_{k-j} \end{bmatrix} \tag{3.9}$$

for a Givens rotator $P_j \in \mathbb{R}^{2\times 2}$ and I_ℓ denoting the identity matrix of size $\ell \times \ell$. For completeness, an efficient realization of these ideas to turn the Hessenberg matrix $\widetilde{H}_k$ into an upper triangular form $\widetilde{U}$ is given in Algorithm 2.

Algorithm 2 Triangularization of the Hessenberg Matrix $\widetilde{H}_k$

for $j = 1$ to k **do**

$\quad y^{(j)} \leftarrow \widetilde{H}_k(j, j+1 : j)$

$\quad \widetilde{y} \leftarrow \frac{y^{(j)}}{\|y^{(j)}\|}$

$\quad u \leftarrow \widetilde{H}_k(j+1, j : k)$

$\quad \widetilde{H}_k(j+1, j+1 : k) \leftarrow -\widetilde{y}_2 \widetilde{H}_k(j, j+1 : k) + \widetilde{y}_1 \widetilde{H}_k(j+1, j+1 : k))$

$\quad \widetilde{H}_k(j+1, j) \leftarrow 0$

$\quad \widetilde{H}_k(j, j : k) \leftarrow \widetilde{y}_1 \widetilde{H}_k(j, j : k) + \widetilde{y}_2 u$

end for

$\widetilde{U} \leftarrow \widetilde{H}_k$

In the description in Algorithm 2, the notation $\widetilde{H}_k(\ell, \ell_1 : \ell_2)$ is reserved for the row vector formed of the entries of $\widetilde{H}_k$ on the ℓth row with column indices from ℓ_1 to ℓ_2. Moreover, the unitary matrices $W_1, W_2, \ldots, W_k$ triangularizing $\widetilde{H}_k$ can be formed using the vectors $y^{(1)}, y^{(2)}, \ldots, y^{(k)}$ generated by Algorithm 2. Specifically, W_j is as in (3.9) with the Givens rotator P_j defined as

$$P_j = \frac{1}{\|y^{(j)}\|} \begin{bmatrix} y_1^{(j)} & y_2^{(j)} \\ -y_2^{(j)} & y_1^{(j)} \end{bmatrix}$$

for $j \in \{1, 2, \ldots, k\}$.

Once $\widetilde{H}_k$ is triangularized into $\widetilde{U}$, (3.8) can be solved efficiently. In particular, as (2.2) is invariant under unitary transformations, (3.8) can equivalently be expressed as

$$\left\| \begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \vdots \\ \widehat{u}_k \\ \widehat{u}_{k+1} \end{bmatrix} - \begin{bmatrix} \widetilde{u}_{11} & \widetilde{u}_{12} & \cdots & \widetilde{u}_{1k} \\ 0 & \widetilde{u}_{22} & \ldots & \widetilde{u}_{2k} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ldots & \widetilde{u}_{kk} \\ 0 & 0 & \ldots & 0 \end{bmatrix} y^{(k)} \right\| = minimum$$

where

$$[\widehat{u}_1 \; \widehat{u}_2 \; \ldots \; \widehat{u}_{k+1}]^T := W_k W_{k-1} \ldots W_1(\beta e_1)$$

can be obtained by applying the rotators $P_1, P_2, \ldots, P_k$ in this order to βe_1. It follows that the solution $y_*^{(k)}$ of (3.8) is the solution of the upper triangular system

$$\begin{bmatrix} \widetilde{u}_{11} & \widetilde{u}_{12} & \ldots & \widetilde{u}_{1k} \\ 0 & \widetilde{u}_{22} & \ldots & \widetilde{u}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{u}_{kk} \end{bmatrix} y = \begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \vdots \\ \widehat{u}_k \end{bmatrix} \tag{3.10}$$

and can be retrieved by back substitution.

Once we have $y_*^{(k)}$ at hand, the best approximate solution X_k in $X_0 + \mathcal{K}_k(S, R_0)$ for $AX + XB = C$ that is X_k minimizing $\|C - (AX + XB)\|_F$ over all $X \in X_0 + \mathcal{K}_k(S, R_0))$, is given by $X_k = X_0 + \widetilde{Q}_k(y_*^{(k)} \otimes I)$. An outline of the overall global GMRES method is provided in Algorithm 3.

Algorithm 3 The Global GMRES Method to Solve the Sylvester Quaternion Matrix Equation

1: **Apply Algorithm 1.**
In particular, form the matrix $\widetilde{Q}_k = [Q_1 Q_2 \cdots Q_k]$ such that $\{Q_1, Q_2, \ldots Q_k\}$ forms an orthonormal basis for $\mathcal{K}_k(A, R_0)$, as well as the Hessenberg matrix $\widetilde{H}_k$ as in (3.4) satisfying (3.3).

2: **Use Algorithm 2 to Triangularize $\widetilde{H}_k$.**
Specifically, unitarily transform $\widetilde{H}_k$ into the upper triangular matrix $\widetilde{U}$, and keep also the rotation vectors $y^{(1)}, y^{(2)}, \ldots, y^{(k)}$ that define the unitary transformation.

3: **Apply Unitary Transformation from Step 2 to βe_1.**
Apply the unitary transformation from the previous step to βe_1 by making use of $y^{(1)}, y^{(2)}, \ldots, y^{(k)}$ to obtain $[\widehat{u}_1 \; \widehat{u}_2 \; \ldots \widehat{u}_{k+1}]^T$.

4: **Find the Solution $y_*^{(k)}$ of the Upper Triangular System in (3.10).**

5: **Form $X_k = X_0 + \widetilde{Q}_k(y_*^{(k)} \otimes I)$, Which is the Best Approximate Solution of $AX + XB = C$ in $X_0 + \mathcal{K}_k(S, R_0)$.**

4. Numerical Examples

In this section, we demonstrate the effectiveness of the proposed global GMRES approach for (2.5) by conducting numerical experiments on examples related to color image restoration.

A color image can be encoded as an $n \times m$ quaternion matrix of the form

$$Q = Ri + Gj + Bk$$

where R, G and B are $n \times m$ real matrices represent the color image's red, green, and blue components. Let A and B be the blurring quaternion matrices of the form

$$A = A_1 + A_2 i + A_3 j + A_4 k \quad \text{and} \quad B = B_1 + B_2 i + B_3 j + B_4 k$$

for some real matrices A_j and B_j such that $j \in \{1, 2, 3, 4\}$. The constant parts A_1 and B_1 of blurring matrices A and B are specified as

$$a_{ij}, d_{ij} = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(i-j)^2}{2\sigma^2}}, & |i-j| \leq r \\ 0, & \text{otherwise} \end{cases}$$

whereas the non-constant parts, i.e., the coefficient matrices for i, j, and k parts, of A and B are given by

$$a_{ij}, d_{ij} = \begin{cases} \frac{1}{10} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(i-j)^2}{2\sigma^2}}, & |i-j| \leq r \\ 0, & \text{otherwise} \end{cases}$$

for prescribed positive real number r and σ. By applying the Sylvester operator we obtain $AX + XB = C$ where C is the quaternion matrix corresponding to the blurred image. On the other hand, given A, B, and C, the solution X to the Sylvester equation $AX + XB = C$ is the quaternion matrix corresponding to the original color image that we would like to restore back.a as as a sas a sa

Example 4.1. We report results illustrating the effectiveness of our proposed approach on such Sylvester equations obtained from the two original color images depicted in Figures 1(a) and 2(a). The images are stored as $n \times m$ quaternion matrices, with sizes 583×500 and 500×752, respectively. We set $\sigma = 10$ and $r = 10$ in both of the examples, and the resulting blurred images C by the application of the Sylvester operator are shown in Figures 1(b) and 2(b). We apply Algorithm 3 to solve $AX + XB = C$ approximately by setting the number of iterations equal to $k = 2$, $k = 4$, $k = 10$, and $k = 50$. The restored images corresponding to the approximate solutions after so many iterations are shown in 1(c)-(f) and 2(c)-(f). Finally, the convergence of the algorithm is illustrated in Figure 3 by plotting the residual norms as a function of number of iterations k. To be precise, the plot on the top in Figure 3 depicts the residual norm $\|R_k\|_F$ for the approximate solution X_k by Algorithm 3 as a function of the number of iterations k for the parrot example. The plot at the bottom does the same for the tiger example.

(a) Original image **(b)** Degraded image **(c)** Restored image for k=2 **(d)** Restored image for k=4 **(e)** Restored image for k=10 **(f)** Restored image for k=50

Figure 1. This concerns the parrot example. Original and blurred images are illustrated in (a) and (b), respectively. The restored images obtained by applications of Algorithm 3 with $k = 2$, $k = 4$, $k = 10$, and $k = 50$ iterations are depicted in (c)-(f)

(a) Original image **(b)** Degraded image **(c)** Restored image for k=2 **(d)** Restored image for k=4 **(e)** Restored image for k=10 **(f)** Restored image for k=50

Figure 2. The images are similar to those in Figure 1 but now for the tiger example. In particular, (a) and (b) are the original and blurred images, whereas (c)-(f) are restored images retrieved by applying Algorithm 3 with k iterations

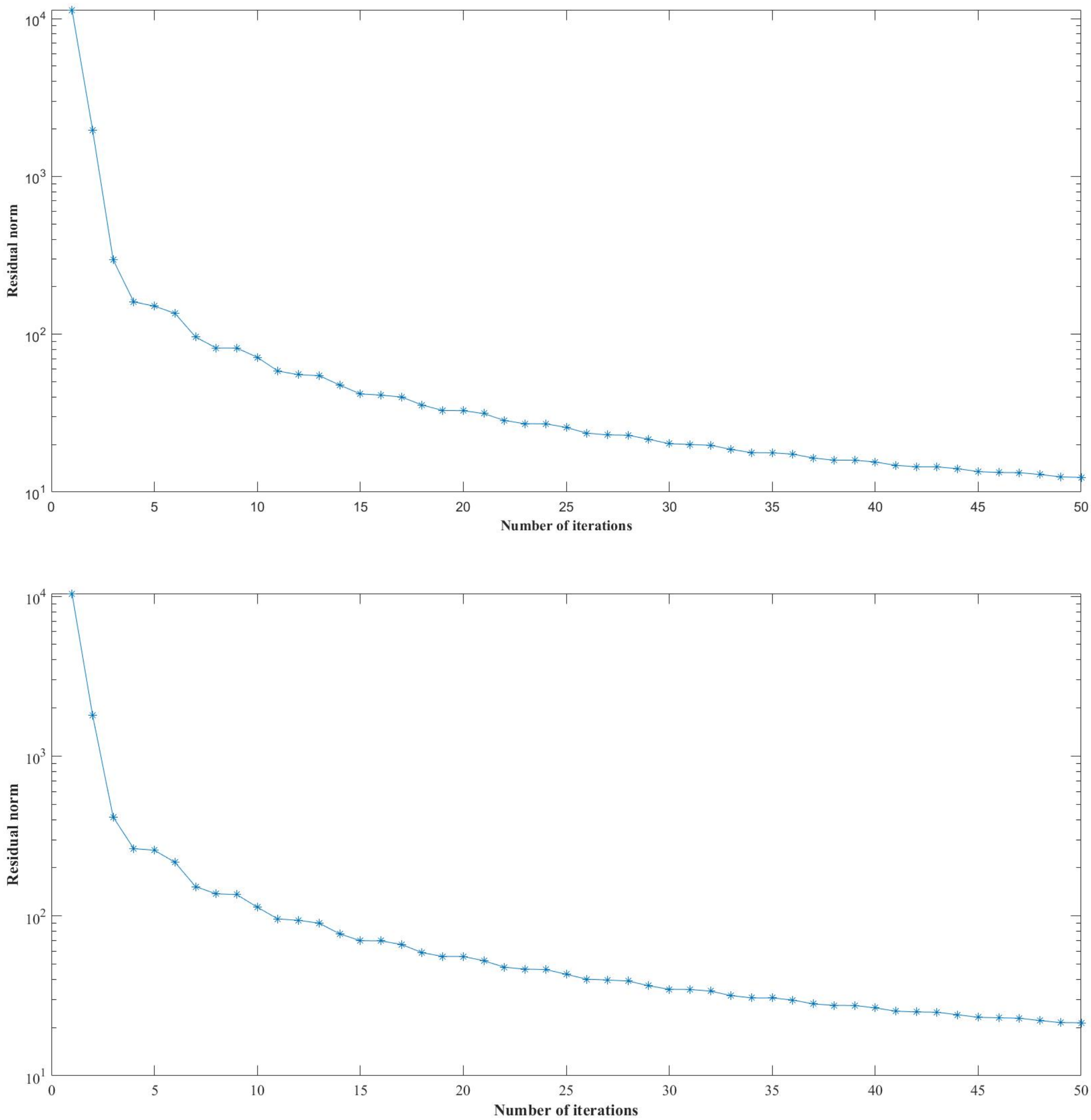

Figure 3. The residual norms for the approximate solutions by Algorithm 3 are plotted as a function of number of iterations k. The top and bottom plots concern the parrot and tiger examples, respectively

5. Conclusion

We have proposed an iterative algorithm for solving a Sylvester quaternion matrix equation, especially the large-scale setting when at least one of the coefficient matrices is large. The proposed algorithm is a global GMRES method operating on a Krylov subspace of a vector space of quaternion matrices. An Arnoldi process based on repeated applications of the Sylvester operator is presented to construct an orthonormal basis for this Krylov subspace. We have also discussed how the determination of the best solution to the Sylvester equation in this Krylov subspace minimizing the Frobenius norm of the residual can be converted into a standard least-squares problem in $\mathbb{R}^n$, which in turn paves the way for efficient computation of the best solution. Finally, we have illustrated on numerical examples that the proposed approach works effectively on Sylvester equations that need to be solved to retrieve the originals of degraded color images. A natural extension of the approach introduced here that could be considered as future work is a conjugate gradient method for solving Lyapunov quaternion matrix equations.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Some Properties of the Generalized Leonardo Numbers

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Research Article

Abstract — In this study, various properties of the generalized Leonardo numbers, which are one of the generalizations of Leonardo numbers, have been investigated. Additionally, some identities among the generalized Leonardo numbers have been obtained. Furthermore, some identities between Fibonacci numbers and generalized Leonardo numbers have been provided. In the last part of the study, binomial sums of generalized Leonardo numbers have been derived. The results obtained for generalized Leonardo numbers are reduced to Leonardo numbers.

Keywords *Binet's formula, Fibonacci numbers, Leonardo numbers*

Mathematics Subject Classification (2020) 11B37, 11B39

1. Introduction

Number sequences are one of the fundamental areas of study within mathematics. Amongst number sequences, the Fibonacci sequence holds a place of importance. This sequence has comprehensive applications in various fields, including mathematics, biology, art, and finance. Many authors have studied different mathematical properties of Fibonacci numbers in [1–6].

The Lucas sequence is another significant number sequence. The Lucas sequence has similar properties with the Fibonacci sequence in [5, 6]. The studies of these sequences involve investigating their properties, relationships, and applications. Mathematicians continue to investigate new properties of number sequences.

In recent years, researchers have been studying Leonardo numbers, which are similar to the recurrence relation of Fibonacci numbers. Catarino and Borges defined the Leonardo sequence in [7]. Moreover, some identities of Leonardo numbers were obtained in [8]. Recent studies on Leonardo numbers have investigated various generalizations of Leonardo numbers in [9–19].

This study investigates the k-Leonardo numbers as defined by Kuhapatanakul and Chobsorn in [13]. Some identities, including binomial sums for k-Leonardo numbers, are obtained. Additionally, some relationships between Fibonacci and k-Leonardo numbers are provided. All the results obtained in this study are reduced to Leonardo numbers for $k = 1$.

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2. Preliminaries

In this section, some definitions and identities of Fibonacci, Lucas and Leonardo numbers are provided.

Definition 2.1. [1] The Fibonacci numbers are characterized, for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0$ and $F_1 = 1$.

Fibonacci numbers correspond A000045 in OEIS [20].

Proposition 2.2. [1] The Binet's formula for Fibonacci sequence is provided as follows:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Definition 2.3. [1] The Lucas numbers are provided the following reccurence relation, for $n \geq 2$,

$$L_n = L_{n-1} + L_{n-2}$$

with $L_0 = 2, L_1 = 1$.

Lucas numbers correspond A000032 in OEIS, [20].

Proposition 2.4. [1] The Binet's formula for Lucas sequence is provided as follows:

$$L_n = \alpha^n + \beta^n \tag{2.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Some identities [5, 6] relating to Fibonacci and Lucas numbers are as follows:

$$F_{m-1} + F_{m+1} = L_m \tag{2.3}$$

$$L_{m-1} + L_{m+1} = 5F_m \tag{2.4}$$

$$F_{s+t} + (-1)^t F_{s-t} = L_t F_s \tag{2.5}$$

$$F_{s+t} - (-1)^t F_{s-t} = F_t L_s \tag{2.6}$$

$$L_{s+t} + (-1)^t L_{s-t} = L_t L_s \tag{2.7}$$

$$L_{s+t} - (-1)^t L_{s-t} = 5F_s F_t \tag{2.8}$$

$$F_m F_n - F_{m+k} F_{n-k} = (-1)^{n-k} F_{m+k-n} F_k \tag{2.9}$$

$$L_{2h} - 2(-1)^h = 5F_h^2 \tag{2.10}$$

$$F_{s+t} = F_{s+1} F_{t+1} - F_{s-1} F_{t-1} \tag{2.11}$$

$$L_{2m} L_{2n} = 5(F_{m+n}^2 + F_{m-n}^2) + 4(-1)^{m+n} \tag{2.12}$$

$$\sum_{i=0}^{2n} \binom{2n}{i} F_{2i} = 5^n F_{2n} \tag{2.13}$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{2i} = 5^n L_{2n+1} \tag{2.14}$$

Definition 2.5. [7] The Leonardo sequence has the following recurrence relation, for $n \geq 2$,

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

and the initial conditions of this recurrence relation are $Le_0 = Le_1 = 1$.

These numbers correspond A001595 in OEIS [20].

Proposition 2.6. [7] The Binet's formula of Leonardo sequence is

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Definition 2.7. [13] The generalized Leonardo numbers has the following recurrence:

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k$$

for $k \in \mathbb{N}$ and $n \geq 2$. In addition, the initial conditions are $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$.

Proposition 2.8. [13] The relation between Fibonacci numbers and generalized Leonardo numbers is provided as follows:

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k \tag{2.15}$$

Proposition 2.9. [14] The Binet's formula of the generalized Leonardo sequence is

$$\mathcal{L}_{k,n} = (k+1)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - k \tag{2.16}$$

Where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Table 1. Several terms of the Fibonacci, Leonardo, Lucas, and generalized Leonardo numbers

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13
Le_n	1	1	3	5	9	15	25	41
L_n	2	1	3	4	7	11	18	29
$\mathcal{L}_{k,n}$	1	1	$2+k$	$3+2k$	$5+4k$	$8+7k$	$13+12k$	$21+20k$

3. Main Results

This section provides new identities of the generalized Leonardo numbers.

Proposition 3.1. For any non-negative integers r, s and $r \geq s$, the following identity is valid

$$\mathcal{L}_{k,r+s}^2 - \mathcal{L}_{k,r-s}^2 = (k+1)^2 F_{2r+2}F_{2s} - 2k(\mathcal{L}_{k,r+s} - \mathcal{L}_{k,r-s})$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

PROOF. Using (2.16) to left hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{r+s+1} - \beta^{r+s+1}}{\alpha - \beta}\right) - k\right)^2 - \left((k+1)\left(\frac{\alpha^{r-s+1} - \beta^{r-s+1}}{\alpha - \beta}\right) - k\right)^2$$

From (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}(L_{2r+2s+2} - L_{2r-2s+2}) - 2k(k+1)(F_{r+s+1} - F_{r-s+1})$$

Considering (2.8),

$$LHS = (k+1)^2 F_{2r+2}F_{2s} - 2k(k+1)(F_{r+s+1} - F_{r-s+1})$$

Using (2.15), the result is obtained. □

Taking $k = 1$ in Proposition 3.1, the identity [8] for Leonardo numbers is as follows:

$$Le_{r+s}^2 - Le_{r-s}^2 = 2(2F_{2r+2}F_{2s} - Le_{r+s} + Le_{r-s})$$

Proposition 3.2. For any non-negative integers r and s such that $r \geq s+4$,

$$\begin{aligned}\mathcal{L}_{k,r+s}\mathcal{L}_{k,r+s-2} + \mathcal{L}_{k,r-s}\mathcal{L}_{k,r-s-2} =& \mathcal{L}_{k,r+s-1}^2 + \mathcal{L}_{k,r-s-1}^2 + 2(-1)^{r+s}(k+1)^2 \\ &- k(\mathcal{L}_{k,r+s-4} + \mathcal{L}_{k,r-s-4}) - 2k^2\end{aligned}$$

where $\mathcal{L}_{k,r}$ is rth generalized Leonardo number.

Proof. Using (2.16) to the left-hand side (LHS),

$$\begin{aligned}LHS =& \left((k+1)\left(\frac{\alpha^{r+s+1}-\beta^{r+s+1}}{\alpha-\beta}\right)-k\right)\left((k+1)\left(\frac{\alpha^{r+s-1}-\beta^{r+s-1}}{\alpha-\beta}\right)-k\right)+ \\ &\left((k+1)\left(\frac{\alpha^{r-s+1}-\beta^{r-s+1}}{\alpha-\beta}\right)-k\right)\left((k+1)\left(\frac{\alpha^{r-s-1}-\beta^{r-s-1}}{\alpha-\beta}\right)-k\right)\end{aligned}$$

From (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}\left(L_{2r+2s} + L_{2r-2s} + 6(-1)^{r+s}\right) - k(k+1)(F_{r+s+1} + F_{r+s-1} + F_{r-s+1} + F_{r-s-1})$$

Using (2.3) and (2.7),

$$LHS = \frac{(k+1)^2}{5}(L_{2r}L_{2s} + 6(-1)^{r+s}) - k(k+1)(L_{r+s} + L_{r-s})$$

Considering (2.12),

$$LHS = (k+1)^2(F_{r+s}^2 + F_{r-s}^2) - k(k+1)(L_{r+s} + L_{r-s}) + 2(-1)^{r+s}(k+1)^2$$

In the final step, from (2.15),

$$LHS = \mathcal{L}_{k,r+s-1}^2 + \mathcal{L}_{k,r-s-1}^2 + 2(-1)^{r+s}(k+1)^2 - k(\mathcal{L}_{k,r+s-4} + \mathcal{L}_{k,r-s-4}) - 2k^2$$

□

Taking $k = 1$ in Proposition 3.2, the following identity [8] of Leonardo numbers is obtained:

$$Le_{r+s}Le_{r+s-2} + Le_{r-s}Le_{r-s-2} = Le_{r+s-1}^2 + Le_{r-s-1}^2 - Le_{r+s-4} - Le_{r-s-4} + 8(-1)^{r-s} - 2$$

Proposition 3.3. For any non-negative integers r and s, the following identity holds true:

$$\mathcal{L}_{k,r}\mathcal{L}_{k,s} = (\mathcal{L}_{k,r} + k)(\mathcal{L}_{k,s} + k) - k(\mathcal{L}_{k,r} + \mathcal{L}_{k,s}) - k^2$$

where $\mathcal{L}_{k,r}$ is rth generalized Leonardo number.

Proof. Using (2.16) to LHS,

$$\mathcal{L}_{k,r}\mathcal{L}_{k,s} = \left((k+1)\left(\frac{\alpha^{r+1}-\beta^{r+1}}{\alpha-\beta}\right)-k\right)\left((k+1)\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)-k\right)$$

From (2.1) and (2.2),

$$\mathcal{L}_{k,r}\mathcal{L}_{k,s} = \frac{(k+1)^2}{5}(L_{r+s+2} - (-1)^{s+1}L_{r-s}) - k(k+1)(F_{r+1} + F_{s+1}) + k^2$$

Using (2.8) and (2.15), the result is obtained. □

If we take $2r$ and $2s$ instead of r and s, respectively, and take $k=1$, we obtain the following identity [8] of Leonardo numbers:

$$Le_{2r}Le_{2s} = (Le_{r+s}+1)^2 + (Le_{r-s-1}+1)^2 - Le_{2r} - Le_{2s} - 1$$

Proposition 3.4. For non-negative integers m, r, and s, the following holds:

$$\mathcal{L}_{k,m+r}\mathcal{L}_{k,m+s} - \mathcal{L}_{k,m}\mathcal{L}_{k,m+r+s} = (k+1)^2(-1)^{m+1}F_rF_s - k\mathcal{L}_{k,m+r} - k\mathcal{L}_{k,m+s} + k\mathcal{L}_{k,m} + k\mathcal{L}_{k,m+r+s}$$

where F_m and $\mathcal{L}_{k,m}$ are mth Fibonacci and generalized Leonardo numbers, respectively.

Proof. Using (2.16) to the left-hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{m+r+1}-\beta^{m+r+1}}{\alpha-\beta}\right)-k\right)\left((k+1)\left(\frac{\alpha^{m+s+1}-\beta^{m+s+1}}{\alpha-\beta}\right)-k\right) -$$

$$\left((k+1)\left(\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\right)-k\right)\left((k+1)\left(\frac{\alpha^{m+r+s+1}-\beta^{m+r+s+1}}{\alpha-\beta}\right)-k\right)$$

Considering (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}(-1)^{m+1}(L_{r+s}-(-1)^sL_{r-s}) + k(k+1)F_{m+1} +$$

$$k(k+1)(F_{m+r+s+1} - F_{m+r+1} - F_{m+s+1})$$

From (2.8),

$$LHS = (k+1)^2(-1)^{m+1}F_rF_s + k(k+1)(F_{m+r+s+1} + F_{m+1} - F_{m+r+1} - F_{m+s+1})$$

Considering (2.15), the result is clear. □

Taking $k=1$, the following identity [8] for Leonardo numbers holds true:

$$Le_{m+r}Le_{m+s} - Le_mLe_{m+r+s} = 4(-1)^{m+1}F_rF_s - Le_{m+r} - Le_{m+s} + Le_m + Le_{m+r+s}$$

Proposition 3.5. For any non-negative integers $r \geq 1$ and $s \geq r$, the following identities are valid:

$$\mathcal{L}_{k,s+r} + (-1)^r\mathcal{L}_{k,s-r} = L_r(\mathcal{L}_{k,s}+k) - k(1+(-1)^r)$$

and

$$\mathcal{L}_{k,s+r} - (-1)^r\mathcal{L}_{k,s-r} = L_{s+1}(\mathcal{L}_{k,r-1}+k) - k(1-(-1)^r)$$

where L_r and $\mathcal{L}_{k,r}$ are rth Lucas and generalized Leonardo numbers, respectively.

Proof. From (2.15),

$$\mathcal{L}_{k,s+r} + (-1)^r\mathcal{L}_{k,s-r} = (k+1)(F_{s+r+1} + (-1)^rF_{s-r+1}) - k(1+(-1)^r)$$

Using (2.5), the first identity is obtained. Similarly, the other identity is derived by using (2.15) and (2.6). □

For $k=1$, we obtain the following identities [8] of Leonardo numbers:

$$Le_{s+r} + (-1)^rLe_{s-r} = L_r(Le_s+1) - (1+(-1)^r)$$

and

$$Le_{s+r} - (-1)^mLe_{s-r} = L_{s+1}(Le_{r-1}+1) - (1-(-1)^r)$$

Proposition 3.6. For any non-negative integers $r \geq 1$ and $s \geq 1$,

$$\mathcal{L}_{k,r+1}\mathcal{L}_{k,s+1} - \mathcal{L}_{k,r-1}\mathcal{L}_{k,s-1} = (k+1)\mathcal{L}_{k,r+s+1} - k(\mathcal{L}_{k,r} + \mathcal{L}_{k,s}) - k^2 + k$$

where $\mathcal{L}_{k,r}$ is rth generalized Leonardo number.

PROOF. Using (2.16) to the left-hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{r+2} - \beta^{r+2}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{s+2} - \beta^{s+2}}{\alpha - \beta}\right) - k\right) - \left((k+1)\left(\frac{\alpha^{r} - \beta^{r}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{s} - \beta^{s}}{\alpha - \beta}\right) - k\right)$$

Considering (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}(L_{r+s+4} - L_{r+s}) - k(k+1)(F_{r+1} + F_{s+1})$$

From (2.8) and (2.11), we obtain the result. □

Taking $k = 1$, we find the following identity [8] of Leonardo numbers:

$$Le_{r+1}Le_{s+1} - Le_{r-1}Le_{s-1} = 2Le_{r+s+1} - Le_r - Le_s$$

Proposition 3.7. Let r, t, and s be non-negative integers such that $r \geq t$ and $r \geq s$. Then, the following identity is valid:

$$\mathcal{L}_{k,r+t}\mathcal{L}_{k,r-t} - \mathcal{L}_{k,r+s}\mathcal{L}_{k,r-s} = (k+1)^2((-1)^{r-t}F_t^2 - (-1)^{r-s}F_s^2) - k(\mathcal{L}_{k,r+t} + \mathcal{L}_{k,r-t} - \mathcal{L}_{k,r+s} - \mathcal{L}_{k,r-s})$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

PROOF. By applying Binet's formula for the generalized Leonardo numbers to the left-hand side, we can derive the result. □

Taking $k = 1$, the following identity [8] can be found:

$$Le_{r+t}Le_{r-t} - Le_{r+s}Le_{r-s} = 4(-1)^r((-1)^tF_t^2 - (-1)^sF_s^2) + Le_{r+s} + Le_{r-s} - Le_{r+t} - Le_{r-t}$$

Proposition 3.8. For any non-negative integer r, the following holds:

$$\mathcal{L}_{k,r+1}F_{r+1} - \mathcal{L}_{k,r}F_r = \mathcal{L}_{k,r}F_{r+1} + kF_r$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

PROOF. Using (2.16) and (2.1) to the left-hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{r+2} - \beta^{r+2}}{\alpha - \beta}\right) - k\right)\left(\frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta}\right) - \left((k+1)\left(\frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta}\right) - k\right)\left(\frac{\alpha^{r} - \beta^{r}}{\alpha - \beta}\right)$$

From (2.1) and (2.2),

$$LHS = \frac{k+1}{5}(L_{2r+2} + 2(-1)^r) - kF_{r-1}$$

Considering (2.10), the following identity is obtained:

$$\mathcal{L}_{k,r+1}F_{r+1} - \mathcal{L}_{k,r}F_r = \mathcal{L}_{k,r}F_{r+1} + kF_r$$

□

For $k = 1$, we obtain the following identity [8] between Leonardo and Fibonacci number:

$$Le_{r+1}F_{r+1} - Le_rF_r = Le_rF_{r+1} + F_r$$

Proposition 3.9. For any non-negative integers s and r where $r \geq 1$ and $s \geq r+1$, the following identities are valid:

$$F_s\mathcal{L}_{k,r} - F_r\mathcal{L}_{k,s} = (-1)^r(\mathcal{L}_{k,s-r-1} + k) + k(F_r - F_s)$$

and

$$F_s\mathcal{L}_{k,r} + F_r\mathcal{L}_{k,s} = \mathcal{L}_{k,s+r-1} + F_s\mathcal{L}_{k,r-1} - kF_r + k$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

Proof. Using (2.15),

$$F_s\mathcal{L}_{k,r} - F_r\mathcal{L}_{k,s} = (k+1)(F_sF_{r+1} - F_{s+1}F_r) + k(F_r - F_s)$$

From (2.9), the first identity is obtained. Similarly, the second identity can be found. □

Taking $k = 1$, the following identities [8] between Leonardo and Fibonacci numbers can be obtained:

$$F_sLe_r - F_rLe_s = (-1)^r(Le_{s-r-1} + 1) + (F_r - F_s)$$

and

$$F_sLe_r + F_rLe_s = Le_{s+r-1} + F_sLe_{r-1} - F_r + 1$$

Proposition 3.10. For non-negative integer s,

$$\sum_{i=0}^{2s}\binom{2s}{i}\mathcal{L}_{k,2i-1} = 5^s(\mathcal{L}_{k,2s-1} + k) - 4^sk$$

and

$$\sum_{i=0}^{2s+1}\binom{2s+1}{i}\mathcal{L}_{k,2i-1} = 5^s(\mathcal{L}_{k,2s-1} + \mathcal{L}_{k,2s+1}) + 2k(5^s - 4^s)$$

where $\mathcal{L}_{k,s}$ is sth generalized Leonardo number.

Proof. Using (2.15),

$$\sum_{i=0}^{2s}\binom{2s}{i}\mathcal{L}_{k,2i-1} = \sum_{i=0}^{2s}\binom{2s}{i}((k+1)F_{2i} - k)$$

From (2.13), the first identity is obtained. Similarly, other identity can be found. □

Taking $k = 1$, the following binomial sums of Leonardo numbers are obtained:

$$\sum_{i=0}^{2s}\binom{2s}{i}Le_{2i-1} = 5^s(Le_{2s-1} + 1) - 4^s$$

and

$$\sum_{i=0}^{2s+1}\binom{2s+1}{i}Le_{2i-1} = 5^s(Le_{2s-1} + Le_{2s+1}) + 2(5^s - 4^s)$$

4. Conclusion

In this study, various identities for generalized Leonardo numbers have been obtained. Additionally, some identities between Fibonacci numbers and generalized Leonardo numbers have been provided. The results obtained in this study are reduced to identities among Leonardo numbers for $k = 1$. In future studies, a new generalization of Leonardo numbers can be defined, and some identities, similar to those provided in this study, can be established.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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Existence Results for ℵ-Caputo Fractional Boundary Value Problems with p-Laplacian Operator

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Abstract — This study delves into the investigation of positive solutions for a specific class of ℵ-Caputo fractional boundary value problems with the inclusion of the p-Laplacian operator. In this research, we use the theory of the fixed point theory within a cone to establish the existence results for solutions of nonlinear ℵ-Caputo fractional differential equations involving the p-Laplacian operator. These findings not only advance the theoretical understanding of fractional differential equations but also hold promise for applications in diverse scientific and engineering disciplines. Furthermore, we provide a clear and illustrative example that serves to reinforce the fundamental insights garnered from this investigation.

Keywords *Fractional differential equation, boundary value problem, p-Laplacian operator, fixed point theorem*

Mathematics Subject Classification (2020) 34K10, 34K37

1. Introduction

In the realm of fractional calculus, the ℵ-Caputo fractional derivatives [1–6] have recently emerged as a powerful tool for capturing complex dynamics with non-local memory effects. This, combined with the influence of the p-Laplacian operator [7–9], has opened up new avenues for investigating the behavior of systems characterized by fractional derivatives [10–12] and nonlinearity.

Fractional order models can be used to model anomalous diffusion processes. Such diffusion processes are more accurate than classical diffusion models, especially in heterogeneous environments. In the field of engineering, fractional-order models are widely used to study the behavior of viscoelastic materials. These materials can exhibit memory effects and dynamic load responses that cannot be explained by classical models. Moreover, biological systems, especially in areas, such as population dynamics and epidemiology, more accurately represent the more complex and memory-based nature of interactions between individuals. For examples, Metzler and Klafter [13] describes anomalous diffusion processes using fractional differential equations. This work has important applications in plasma physics and earth sciences. Bagley and Torvik [14] demonstrates the usability of fractional order differential equations in modeling viscoelastic materials. Magin [15] discusses how fractional differential equations can be used to model complex dynamics in biological tissues.

This paper delves into the examination of the existence of solutions for ℵ-Caputo fractional boundary value problems (CFBVP) with the inclusion of the p-Laplacian operator. The ℵ-Caputo fractional

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derivative introduces a parameter ℵ that allows for a more nuanced control over the memory effects, providing a flexible framework for modeling diverse phenomena. The interplay between ℵ-Caputo derivatives and the p-Laplacian operator adds a layer of complexity that is essential for understanding the intricacies of real-world systems. The central focus of this study is to establish rigorous results regarding the solvability and uniqueness of solutions for the formulated fractional boundary value problems. The investigation spans theoretical analyses, employing advanced mathematical tools, and computational methodologies to unveil the underlying dynamics.

As fractional calculus continues to gain prominence in various scientific disciplines, the findings of this research not only contribute to the theoretical foundations but also hold potential implications for applications in physics, engineering, and other fields. By exploring ℵ-CFBVP involving the p-Laplacian operator, in this paper, we aim to contribute to the ongoing dialogue in fractional calculus and its expanding role in understanding complex systems. Subsequent sections will delve into the mathematical formulations, methodologies, and results, providing a comprehensive exploration of the addressed problems.

Inspired by the previously explored investigations concerning p-Laplacian ℵ-CFBVP that incorporate both right and left-sided fractional derivatives, as well as left-sided integral operators with respect to a power function, we delve into the examination of uniqueness results. Employing the properties inherent in Green's functions, our focus is directed towards a mixed p-Laplacian boundary value problem. This particular problem involves ℵ-Caputo fractional derivatives and integrals, specifically in connection with a power function.

Bai et al. [8] considered the existence of solutions for the following boundary value problem of the fractional p-Laplacian equation

$$\begin{cases} (\varphi_p(D^{\alpha}_{0^+}u(t)))' + h(t,u(t)) = 0, \quad 0 < t < 1 \\ u(0) = D^{\alpha}_{0^+}u(0) = 0, \quad {}^{C}D^{\beta}_{0^+}u(0) = {}^{C}D^{\beta}_{0^+}u(1) = 0 \end{cases}$$

where $0 < \beta \leq 1$, $1 < \alpha \leq 2 + \beta$, $D^{\alpha}_{0^+}$ and $D^{\beta}_{0^+}$ are Riemann -Liouville fractional derivative and Caputo fractional derivative of orders α, β, $p > 1$, and $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

In [1], Mfadel et al. investigated the boundary value problem ψ-Caputo fractional differential equations involving the p-Laplacian operator provided by

$$\begin{cases} (\phi_p({}^{C}D^{\alpha,\psi}_{0^+}u(t)))' = f(t,u(t)), \quad t \in \Delta = [0,T] \\ u(0) = \sigma_1 u(T), \quad u'(0) = \sigma_2 u'(T) \end{cases}$$

where $T > 0$, $D^{\alpha,\psi}_{0^+}$ is the ψ-Caputo fractional derivative of order $\alpha \in (1,2)$, and ϕ_p is a p-Laplacian operator, i.e., $\phi_p(t) = t^{p-1}$ such that $p - 1 > 0$.

In this study, we concentrate on the existence results of the following p-Laplacian ℵ-CFBVP:

$$\begin{cases} {}^{C}D^{\beta,\aleph}_{b^-}(\varphi_p({}^{C}D^{\alpha,\aleph}_{b^-}y(t))) = f(t,y(t)), \quad t \in [a,b] \\ y(a) = D^{\alpha,\aleph}_{b^-}y(a) = 0, \quad y(b) = D^{\alpha,\aleph}_{b^-}y(b) = 0 \end{cases} \tag{1.1}$$

where φ_p is a p-Laplacian operator, i.e., $\varphi_p(t) = |t|^{p-2}t$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $1 < \alpha, \beta \leq 2$, $D^{\alpha,\aleph}_{b^-}$ and $D^{\beta,\aleph}_{b^-}$ denote the right-sided ℵ-Caputo fractional derivatives of orders α and β, respectively, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

In Section 2, we provide some important definitions, lemmas, and theorems that play a key role for the considered problems. In Section 3, we establish Green functions, and this part contains the main results for the provided problem. Using Krasnoselskii fixed point theorem, we can say existence results for the problems. Moreover, we provide an example in this section. In Section 4, we provide a conclusion part.

2. Preliminaries

In the preliminaries section, we provide some definitions, notations, theorems, and results for the generalized ℵ-fractional derivative and integral to be used throughout this paper.

Definition 2.1. [10] Let $\alpha > 0$, $f : [a, b] \to \mathbb{R}$ be an integrable function defined on $[a, b]$, and $\aleph \in C^1([a, b], \mathbb{R})$ an increasing differentiable function with $\aleph'(t) \neq 0$, for all $t \in [a, b]$. Then, the $\alpha-$th order right-sided ℵ-Riemann-Liouville fractional integral of a function f is provided by

$$I_{b^-}^{\alpha,\aleph} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \aleph'(s)(\aleph(s) - \aleph(t))^{\alpha-1} f(s) ds$$

where $\Gamma(.)$ is a gamma function.

Definition 2.2. [3] Let $\alpha > 0$ and $\aleph, f \in C^m([a, b], \mathbb{R})$ two functions such that ℵ is increasing and $\aleph'(t) \neq 0$, for all $t \in [a, b]$. Then, the left-sided ℵ-Caputo fractional derivative of f of order α is given by

$${}^C D_{b^-}^{\alpha,\aleph} f(t) = I_{b^-}^{m-\alpha,\aleph} \left[-\frac{1}{\aleph'(t)} \frac{d}{dt} \right]^m f(t)$$

where $m = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Theorem 2.3. [3] Let $f, g \in C^m([a, b], \mathbb{R})$ and $\alpha > 0$. Then,

$${}^C D_{b^-}^{\alpha,\aleph} f(t) = {}^C D_{b^-}^{\alpha,\aleph} g(t) \Leftrightarrow f(t) = g(t) + \sum_{k=0}^{m-1} d_k (\aleph(b) - \aleph(t))^k$$

where d_k are arbitrary constants.

Theorem 2.4. (Guo-Krasnosel'skii Fixed Point Theorem) Let Y be a Banach space and $P \subseteq Y$ be a cone. Assume that Θ_1 and Θ_2 are open subsets of P with $0 \in \Omega_1$ and $\overline{\Theta_1} \subset \Theta_2$. Suppose that $F : P \cap \left(\overline{\Theta_2} \setminus \Theta_1 \right) \to P$ is a completely continuous operator such that, either

i. $\|Fy\| \leq \|y\|$, for $y \in P \cap \partial\Theta_1$, $\|Fy\| \geq \|y\|$, for $y \in P \cap \partial\Theta_2$

ii. $\|Fy\| \geq \|y\|$, for $y \in P \cap \partial\Theta_1$, $\|Fy\| \leq \|y\|$, for $y \in P \cap \partial\Theta_2$

holds. Then, F has at least one fixed point in $P \cap (\overline{\Theta_2} \setminus \Theta_1)$.

3. Main Results

In this section, we establish Green function for the uniqueness of the solutions to (1.1). To that end, we first provide the following useful result which provides the solution of the linear form of (1.1).

We consider the following linear boundary value problem

$$\begin{cases} {}^C D_{b^-}^{\beta,\aleph} (\varphi_p({}^C D_{b^-}^{\alpha,\aleph} y(t))) = h(t), \quad t \in [a, b] \\ y(a) = D_{b^-}^{\alpha,\aleph} y(a) = 0, \quad y(b) = D_{b^-}^{\alpha,\aleph} y(b) = 0 \end{cases}$$

Lemma 3.1. Let $h \in C([a, b], \mathbb{R})$ and $1 < \alpha, \beta \leq 2$. Then, $y \in C[a, b]$ is a solution if and only if

$$y(t) = \frac{1}{\Gamma(\alpha)(\Gamma(\beta))^{q-1}} \int_a^b \aleph'(s) G_2(t, s) \varphi_q \Big(\int_a^b \aleph'(\tau) G_1(s, \tau) h(\tau) d\tau \Big) ds$$

where $G_1(t,s)$ and $G_2(t,s)$ provided by

$$G_1(t,s) = \begin{cases} \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1}, & s \leq t \\ \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(t))^{\beta-1}, & s \geq t \end{cases} \tag{3.1}$$

and

$$G_2(t,s) = \begin{cases} \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\alpha-1}, & s \leq t \\ \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\alpha-1} - (\aleph(s)-\aleph(t))^{\alpha-1}, & s \geq t \end{cases}$$

Proof. Let $-\varphi_p({}^C D_{b^-}^{\alpha,\aleph} y(t)) := v(t)$. Then, (1.1) changes to

$$\begin{cases} -{}^C D_{b^-}^{\beta,\aleph} v(t) = h(t) \\ v(a) = 0, \quad v(b) = 0 \end{cases}$$

and

$$\begin{cases} {}^C D_{b^-}^{\alpha,\aleph} y(t) = -\varphi_q v(t) := k(t) \\ y(a) = 0, \quad y(b) = 0 \end{cases} \tag{3.2}$$

Solving the equation ${}^C D_{b^-}^{\beta,\aleph} v(t) = -h(t)$,

$$v(t) = -\frac{1}{\Gamma(\beta)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\beta-1} h(s)ds + c_0 + c_1(\aleph(b)-\aleph(t))$$

where c_0 and c_1 are constants. Using the condition $v(b) = 0$ yields $c_0 = 0$. Since $v(a) = 0$, then

$$-\frac{1}{\Gamma(\beta)} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\beta-1} h(s)ds + c_1(\aleph(b)-\aleph(a)) = 0$$

where

$$c_1 = \frac{1}{\Gamma(\beta)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\beta-1} h(s)ds$$

Therefore,

$$\begin{aligned} v(t) &= \frac{\aleph(b)-\aleph(t)}{\Gamma(\beta)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\beta-1} h(s)ds - \frac{1}{\Gamma(\beta)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\beta-1} h(s)ds \\ &= \frac{1}{\Gamma(\beta)} \int_a^b \aleph'(s) G_1(t,s) h(s)ds \end{aligned}$$

where $G_1(t,s)$ is provided in (3.1). Applying the integral $I_{b^-}^{\alpha,\aleph}$ on both sides of the differential equation in (3.2),

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\alpha-1} k(s)ds + d_0 + d_1(\aleph(b)-\aleph(t))$$

where d_0 and d_1 are constants. Using the boundary conditions $y(a) = 0$ and $y(b) = 0$, we obtain $d_0 = 0$ and

$$d_1 = \frac{-1}{\Gamma(\alpha)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(b)-\aleph(s))^{\alpha-1} k(s)ds$$

Thus,

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \aleph'(s)(\aleph(s)-\aleph(t))^{\alpha-1} k(s)ds - \frac{\aleph(b)-\aleph(t)}{\Gamma(\alpha)(\aleph(b)-\aleph(a))} \int_a^b \aleph'(s)(\aleph(s)-\aleph(a))^{\alpha-1} k(s)ds \\ &= \frac{-1}{\Gamma(\alpha)} \int_a^b \aleph'(s) G_2(t,s) k(s)ds \end{aligned}$$

Hence,

$$\begin{aligned}
y(t) &= \frac{-1}{\Gamma(\alpha)}\int_a^b \aleph'(s)G_2(t,s)(-\varphi_q v(s))ds \\
&= \frac{1}{\Gamma(\alpha)}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_a^b \aleph'(\tau)G_1(s,\tau)h(\tau)d\tau\right)ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\left(\int_a^b \aleph'(\tau)G_1(s,\tau)h(\tau)d\tau\right)ds
\end{aligned}$$

□

Lemma 3.2. $G_1(t,s)$ and $G_2(t,s)$ have possess the following properties:

i. $G_1(t,s)$ and $G_2(t,s)$ are continuous functions on $[a,b]\times[a,b]$

ii. $G_1(t,s)$ and $G_2(t,s)$ are non-negative functions on $[a,b]\times[a,b]$

Proof. *i.* Since the function $\aleph$ is a continuous function on $[a,b]$, then the functions $G_1(t,s)$ and $G_2(t,s)$ are continuous on $[a,b]\times[a,b]$.

ii. For $s \geq t$,

$$\begin{aligned}
G_1(t,s) &= \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(t))^{\beta-1} \\
&= \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(b)-\aleph(t)+\aleph(a))^{\beta-1} \\
&= \frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}(\aleph(s)-\aleph(a))^{\beta-1} - (\aleph(s)-\aleph(a))^{\beta-1}\left[1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right]^{\beta-1} \\
&= (\aleph(s)-\aleph(a))^{\beta-1}\left[\frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)} - \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1}\right]
\end{aligned}$$

Since

$$\begin{aligned}
\aleph(s)\geq\aleph(t) &\Rightarrow \aleph(s)-\aleph(a)\geq\aleph(t)-\aleph(a) \\
&\Rightarrow 1\geq\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\geq\frac{\aleph(t)-\aleph(a)}{\aleph(b)-\aleph(a)} \\
&\Rightarrow 1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\leq 1-\frac{\aleph(t)-\aleph(a)}{\aleph(b)-\aleph(a)} \\
&\Rightarrow \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1}\leq\left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(b)-\aleph(a)}\right)^{\beta-1} \\
&\Rightarrow \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1}\leq\frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)}
\end{aligned}$$

then

$$\frac{\aleph(b)-\aleph(t)}{\aleph(b)-\aleph(a)} - \left(1-\frac{\aleph(t)-\aleph(a)}{\aleph(s)-\aleph(a)}\right)^{\beta-1}\geq 0$$

Thus, $G_1(t,s)\geq 0$. Similarly, $G_2(t,s)\geq 0$. □

We assume that the function $f(t,y)$ satisfies the following condition:

(H_1) $f:[a,b]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous.

We consider the Banach space $Y=\mathcal{C}[a,b]$ with maximum norm $\|y\|=\max_{t\in[a,b]}|y(t)|$, for $y\in Y$, and the cone $\mathcal{P}=\{y\in Y: y(t)\geq 0,\quad t\in[a,b]\}$. Define the operator $A:\mathcal{P}\longrightarrow Y$ with

$$Ay(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds,\quad t\in[a,b] \tag{3.3}$$

Lemma 3.3. $Ay\in\mathcal{P}$, for all $y\in\mathcal{P}$. Especially, the operator A leaves the cone P invaryant, i.e., $A(\mathcal{P})\subset\mathcal{P}$.

Proof. Using the condition (H_1) and the positivity of the functions $G_1(t,s)$ and $G_2(t,s)$, we have $Ay(t)\geq 0$, for all $y\in\mathcal{P}$ and $t\in[a,b]$. Therefore, $Ay\in\mathcal{P}$. □

Lemma 3.4. The operator A defined by (3.3) is a completely continuous operator in Y.

Proof. Firstly, we show $A:\mathcal{P}\longrightarrow Y$ is well-defined, for all $y\in Y$. Let $y\in Y$. Then, we know that $y(t)\geq 0$. Let $\Omega\subset\mathcal{P}$ be bounded. Then, there exists a positive constant M such that $|f(\tau,y(\tau))|\leq M$, for all $\tau\in[a,b]$ and $y\in\Omega$. Moreover, by the continuity of $G_1(t,s)$ and $G_2(t,s)$ on $[a,b]\times[a,b]$, for fixed $s\in[a,b]$ and for any $\epsilon>0$, there exists a constant $\delta>0$, such that $t_1,t_2\in[a,b]$ and $|t_1-t_2|<\delta$ imply that the function $G_2(t,s)$ satisfies

$$\Big|G_2(t_1,s)-G_2(t_2,s)\Big|\leq\epsilon\Big[\frac{M^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b\aleph'(s)\varphi_q\Big(\int_a^b\aleph'(\tau)G_1(s,\tau)d\tau\Big)ds\Big]^{-1}$$

Thus, for all $y\in\Omega$,

$$\begin{aligned}\Big|Ay(t_1)-Ay(t_2)\Big| &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b\aleph'(s)|G_2(t_1,s)-G_2(t_2,s)|\Big|\varphi_q(\int_a^b\aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau)\Big|ds\\ &\leq \frac{M^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b\aleph'(s)|G_2(t_1,s)-G_2(t_2,s)|\varphi_q\Big(\int_a^b\aleph'(\tau)G_1(s,\tau)d\tau\Big)ds\\ &\leq\epsilon\end{aligned}$$

This means that $A(\Omega)$ is equicontinuous and by the Arzela-Ascoli Theorem, we obtain $A:\mathcal{C}[a,b]\longrightarrow\mathcal{C}[a,b]$ is completely continuous. □

Set

$$M_1:=\int_a^b\aleph'(s)\max_{t\in[a,b]}G_1(t,s)ds,\qquad m_1:=\int_\xi^\nu\aleph'(s)\min_{t\in[\xi,\nu]}G_1(t,s)ds$$

and

$$M_2:=\int_a^b\aleph'(s)\max_{t\in[a,b]}G_2(t,s)ds,\qquad m_2:=\int_\xi^\nu\aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)ds$$

such that $\xi,\nu\in(a,b)$ and $\xi<\nu$. We assume that the function $f(t,y)$ satisfies the following condition:

(H_2) There exist numbers $0<r_1<R_1<\infty$ such that for all $t\in[a,b]$,

$$f(t,y)\geq r_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1m_2^{p-1}},\quad 0\leq y\leq r_1$$

and

$$f(t,y)\leq R_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1M_2^{p-1}},\quad 0\leq y\leq R_1$$

The main results herein heavily rely on the fundamental and crucial Guo-Krasnosel'skii's fixed point theorem (Theorem 2.4).

Theorem 3.5. Assume that conditions (H_1) and (H_2) are hold. Then, (1.1) has at least one positive solution $y(t)$ such that

$$r_1 \leq \|y\| \leq R_1, \quad t \in [a,b]$$

Proof. For $y \in P$ with $\|y\| = r_1$, for $s \in [a,b]$, we have, for all $t \in [a,b]$,

$$\begin{aligned}
|Ay(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\right| \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)G_2(t,s)\varphi_q\Big(\int_\xi^\nu \aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)\min_{t\in[\xi,\nu]} G_2(t,s)\varphi_q\Big(\int_\xi^\nu \aleph'(\tau)\min_{s\in[\xi,\nu]} G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)\min_{t\in[\xi,\nu]} G_2(t,s)\varphi_q\Big(r_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}\int_\xi^\nu \aleph'(\tau)\min_{s\in[\xi,\nu]} G_1(s,\tau)d\tau\Big)ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} m_1^{q-1} m_2 \frac{r_1\Gamma(\alpha)\Gamma(\beta)^{q-1}}{m_1^{q-1}m_2} \\
&= r_1 \\
&= \|y\|
\end{aligned}$$

Let $\Omega_1 = \{y \in Y : \|y\| < r_1\}$. Then, this inequalities shows that

$$\|Ay\| \geq \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_1$$

Further, let $\Omega_2 = \{y \in Y : \|y\| \leq R_1\}$. Then, for all $t \in [a,b]$,

$$\begin{aligned}
|Ay(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)\max_{t\in[a,b]} G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)\max_{s\in[a,b]} G_1(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)\max_{t\in[a,b]} G_2(t,s)\varphi_q\Big(R_1^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}\int_a^b \aleph'(\tau)\max_{s\in[a,b]} G_1(s,\tau)d\tau\Big)ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}} M_1^{q-1} M_2 \frac{R_1\Gamma(\alpha)\Gamma(\beta)^{q-1}}{M_1^{q-1}M_2} \\
&= R_1 \\
&= \|y\|
\end{aligned}$$

Hence,

$$\|Ay\| \leq \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_2$$

Consequently, by Guo-Krasnosel'skii fixed point theorem, it follow that A has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ such that $r_1 \leq \|y\| \leq R_1$. □

Theorem 3.6. Let (H_1) and (H_2) are hold. Moreover, assume

(H_3) There exist numbers $r_k, R_k \in \mathbb{R}^+$, $k \in \{1,2,...,n\}$, and

$$0 < r_1 < R_1 < r_2 < R_2 < ... < r_n < R_n < \infty$$

such that for all $t \in [a,b]$,

$$f(t,y_k) \geq r_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}, \qquad 0 \leq y_k \leq r_k$$

and

$$f(t,y_k) \leq R_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}, \qquad 0 \leq y_k \leq R_k$$

Then, (1.1) has at least n positive solution such that $r_k \leq \|y_k\| \leq R_k$, $k \in \{1,2,...,n\}$.

Proof. Let $\Omega_{r_k} = \{y_k \in Y : \|y_k\| < r_k\}$ and for all $y_k \in \mathcal{P}$

$$\begin{aligned}
|Ay_k(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y_k(\tau))d\tau\Big)ds\right| \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)G_2(t,s)\varphi_q\Big(\int_\xi^\nu \aleph'(\tau)G_1(s,\tau)f(\tau,y_k(\tau))d\tau\Big)ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_\xi^\nu \aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)\varphi_q\Big(r_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}\int_\xi^\nu \aleph'(\tau)\min_{s\in[\xi,\nu]}G_1(s,\tau)d\tau\Big)ds \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\frac{[r_k^{p-1}]^{q-1}[\Gamma(\alpha)^{p-1}]^{q-1}\Gamma(\beta)^{q-1}}{m_1^{q-1}[m_2^{p-1}]^{q-1}}\int_\xi^\nu \aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)\varphi_q\Big(m_1\Big)ds \\
&= \frac{r_k m_1^{q-1}}{m_1^{q-1}m_2}\int_a^b \aleph'(s)\min_{t\in[\xi,\nu]}G_2(t,s)ds \\
&= r_k
\end{aligned}$$

which implies that $\|Ay_k\| \geq \|y_k\|$, for all $y_k \in \mathcal{P}\cap\Omega_{r_k}$. Further, for $\Omega_{R_k} = \{y_k \in Y : \|y_k\| < R_k\}$ and for all $y_k \in \mathcal{P}$

$$\begin{aligned}
|Ay_k(t)| &= \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)G_2(t,s)\varphi_q\Big(\int_a^b \aleph'(\tau)G_1(s,\tau)f(\tau,y_k(\tau))d\tau\Big)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\int_a^b \aleph'(s)\max_{t\in[a,b]}G_2(t,s)\varphi_q\Big(R_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}\int_a^b \aleph'(\tau)\max_{s\in[a,b]}G_1(s,\tau)d\tau\Big)ds \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)^{q-1}}\frac{[R_k^{p-1}]^{q-1}[\Gamma(\alpha)^{p-1}]^{q-1}\Gamma(\beta)^{q-1}}{M_1^{q-1}[M_2^{p-1}]^{q-1}}\int_a^b \aleph'(s)\max_{t\in[a,b]}G_2(t,s)\varphi_q\Big(M_1\Big)ds \\
&= \frac{R_k M_1^{q-1}}{M_1^{q-1}M_2}\int_a^b \aleph'(s)\max_{t\in[a,b]}G_2(t,s)ds \\
&= R_k
\end{aligned}$$

which implies that $\|Ay_k\| \leq \|y_k\|$, for all $y_k \in \mathcal{P}\cap\Omega_{R_k}$. By part (ii) of Theorem 2.4, it follows that A has a fixed point in $\mathcal{P}\cap(\overline{\Omega_{R_k}}\setminus\Omega_{r_k})$ that y_k, $k \in \{1,2,...,n\}$, are positive solutions such that $r_k \leq \|y_k\| \leq R_k$, $t \in [a,b]$. □

Theorem 3.7. Let (H_1) and (H_2) are hold. Assume

(H_4) There exist numbers $r_k, R_k \in \mathbb{R}^+$, $k \in \{1,2,...,n\}$ and

$$0 < r_1 < R_1 < r_2 < R_2 < ... < r_n < R_n < \infty$$

such that for all $t \in [a,b]$,

$$f(t,y_k) \geq R_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{m_1 m_2^{p-1}}, \qquad 0 \leq y_k \leq R_k \tag{3.4}$$

$$f(t,y_k) \leq r_k^{p-1}\frac{\Gamma(\alpha)^{p-1}\Gamma(\beta)}{M_1 M_2^{p-1}}, \qquad 0 \leq y_k \leq r_k \tag{3.5}$$

Then, (1.1) has at least n positive solutions such that $r_k \leq \|y_k\| \leq R_k$, $k \in \{1,2,...,n\}$.

The proof of Theorem 3.7 is similar to the proof of Theorem 3.6 by the part (i) of Theorem 2.4.

Example 3.8. Consider the following ℵ-CFBVP:

$$\begin{cases} {}^cD_{1^-}^{\frac{4}{3},t}(\varphi_p({}^cD_{1^-}^{\frac{3}{2},t}y(t))) = \sqrt{1+t}(\sin\frac{1}{1+y}+10^2), \quad t \in [0,1] \\ y(0) = D_{1^-}^{\frac{3}{2},t}y(0) = 0, \quad y(1) = D_{1^-}^{\frac{3}{2},t}y(1) = 0 \end{cases} \tag{3.6}$$

Note that (3.6) is a particular case of (1.1) with $\aleph(t) = t$, $f(t,y) = \sqrt{1+t}\left(\sin\frac{1}{1+y}+10^2\right)$ and $a = 0$, $b = 1$, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $\xi = \frac{1}{3}$, $\nu = \frac{1}{2}$, and $p = 2$. Thus,

$$G_1(t,s) = \begin{cases} (1-t)s^{\frac{1}{3}}, & s \leq t \\ (1-t)s^{\frac{1}{3}} - (s-t)^{\frac{1}{3}}, & s \geq t \end{cases}$$

and

$$G_2(t,s) = \begin{cases} (1-t)s^{\frac{1}{2}}, & s \leq t \\ (1-t)s^{\frac{1}{2}} - (s-t)^{\frac{1}{2}}, & s \geq t \end{cases}$$

Hence,

(H_1) $f(t,y) = \sqrt{1+t}\left(\sin\frac{1}{1+y}+10^2\right)$, for all $(t,y) \in [0,1]\times[0,\infty)$, is continuous

(H_2)

$$f(t,y) \leq R\frac{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}{M_1M_2} = R\frac{(0,8862)(0,8991)}{M_1M_2} \cong R\frac{0,7967}{M_1M_2}, \quad 0 \leq y \leq R$$

and

$$f(t,y) \geq r\frac{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}{m_1m_2} = r\frac{(0,8862)(0,8991)}{m_1m_2} \cong r\frac{0,7967}{m_1m_2}, \quad 0 \leq y \leq r$$

Therefore,

$$M_1 = \int_0^1 \max_{t\in[0,1]} G_1(t,s)ds = \frac{3}{4}$$

$$M_2 = \int_0^1 \max_{t\in[0,1]} G_2(t,s)ds = \frac{2}{3}$$

$$m_1 = \int_{\frac{1}{3}}^{\frac{1}{2}} \min_{t\in[\frac{1}{3},\frac{1}{2}]} G_1(t,s)ds \cong 0,0142$$

and

$$m_2 = \int_{\frac{1}{3}}^{\frac{1}{2}} \min_{t\in[\frac{1}{3},\frac{1}{2}]} G_2(t,s)ds \cong 0,0262$$

Thereby,

$$f(t,y) \leq \frac{2(0,7967)}{0,5} \cong 3,1868$$

and

$$f(t,y) \geq \frac{\frac{1}{2.10^3}(0,7967)}{0,00037} \cong 1,0765$$

with $r = \frac{1}{2.10^3}$ and $R = 2$. Thus, all the conditions of Theorem 3.5 are satisfied. Hence, (3.6) has a unique solution on $[0,1]$ such that $\frac{1}{2.10^3} \leq \|y\| \leq 2$.

4. Conclusion

In conclusion, this study delves into the investigation of ℵ-CFBVP involving the p-Laplacian operator. Through a meticulous examination of the problem setup and employing well-established mathematical techniques, we have derived significant existence results.

Firstly, by exploiting the properties of the p-Laplacian operator and leveraging the theory of fractional calculus, we formulated the ℵ-CFBVP. This problem encapsulates phenomena where the behavior of the system exhibits fractional-order dynamics, and the p-Laplacian operator accounts for nonlinear effects. Subsequently, by applying suitable fixed-point theorems and employing appropriate function spaces, we established the existence of solutions to the formulated boundary value problem. Our results not only confirm the existence of solutions but also provide conditions under which uniqueness can be guaranteed. These findings are crucial for understanding the behavior of systems governed by fractional differential equations with nonlinear operators. Moreover, our analysis sheds light on the intricate interplay between the fractional order, nonlinearity, and boundary conditions. By delineating the conditions under which solutions exist, we contribute to the theoretical framework underlying fractional boundary value problems with the p-Laplacian operator. As we navigate the diverse landscapes of physics, engineering, and applied mathematics, the outcomes of this research open avenues for further exploration. The ℵ-CFBVP with the p-Laplacian operator provide a rich framework for understanding the dynamics of systems with fractional derivatives and nonlinearities.

Furthermore, the results presented herein have potential implications in various fields, including mathematical physics, engineering, and biology. Systems exhibiting fractional-order dynamics with nonlinearities are prevalent in nature and engineering applications. The insights gained from this study can aid in modeling, analysis, and control of such systems, thereby facilitating advancements in diverse areas of science and technology. The application of fractional order p-Laplacian operators in more irregular and complex geometries can enable innovative designs in materials science and engineering. Fractional dynamical models can more accurately represent population dynamics and disease spread in biology.

In future research, extending this work to more complex scenarios and exploring applications in specific scientific domains could deepen our understanding and broaden the impact of the presented results. This study, thus, stands as a valuable contribution to the evolving field of fractional calculus, emphasizing the continued relevance and potential applications of these mathematical tools in addressing real-world challenges.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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On the Diophantine Equation $(9d^2+1)^x+(16d^2-1)^y=(5d)^z$ Regarding Terai's Conjecture

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Abstract — This study proves that the Diophantine equation $\left(9d^2+1\right)^x+\left(16d^2-1\right)^y=(5d)^z$ has a unique positive integer solution $(x,y,z)=(1,1,2)$, for all $d>1$. The proof employs elementary number theory techniques, including linear forms in two logarithms and Zsigmondy's Primitive Divisor Theorem, specifically when d is not divisible by 5. In cases where d is divisible by 5, an alternative method utilizing linear forms in p-adic logarithms is applied.

Keywords *Terai's conjecture, Diophantine equations, primitive divisor theorem*

Mathematics Subject Classification (2020) 11D61, 11D75

1. Introduction

The exponential Diophantine equation $e^x+f^y=g^z$ involves coprime positive integers e, f, and g greater than 1. Solutions (x,y,z) satisfying this equation are referred to as valid solutions to the provided equation [1]. In 1956, Sierpinski [2] demonstrated that by substituting exponential expressions for the sides of the Pythagorean theorem into variables, the exponential Diophantine equation $3^x+4^y=5^z$ has a unique solution, specifically $(2,2,2)$. Furthermore, Jeśmanowicz [3] extended this equation to various Pythagorean triples, affirming that for positive integers e, f, and g that satisfy the exponential Diophantine equation, the unique solution remains $(2,2,2)$.

In 1994, Terai [4] proposed that if the equation $e^k+f^l=g^m$ holds for positive constant integers k, l, and m with $m\geq 2$, multiple known solutions (k,l,m) exist for the equation, except for certain specific sets of triples (e,f,g). This conjecture is proved for many special cases. One of them is as follows:

$$\left(pd^2+1\right)^x+\left(ud^2-1\right)^y=(wd)^z \tag{1.1}$$

This study explores the solutions of the following exponential Diophantine equation

$$\left(9d^2+1\right)^x+\left(16d^2-1\right)^y=(5d)^z \tag{1.2}$$

(1.2) is a specific case derived from (1.1), particularly when the condition $p+u=w^2$ is satisfied. Several specific instances of (1.1) have been explored, confirming the validity of Terai's conjecture. Some of these are as follows:

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i. $(4d^2+1)^a+(5d^2-1)^b=(3d)^c$ [5]

ii. $(d^2+1)^a+(yd^2-1)^b=(zd)^c,\ 1+y=z^2$ [6]

iii. $(12d^2+1)^a+(13d^2-1)^b=(5d)^c$ [7]

iv. $(xd^2+1)^a+(yd^2-1)^b=(zd)^c,\ z|d$ [8]

v. $(xd^2+1)^a+(yd^2-1)^b=(zd)^c,\ d=\mp 1 \pmod 5$ [9]

vi. $(18d^2+1)^a+(7d^2-1)^b=(5d)^c$ [10]

vii. $((x+1)d^2+1)^a+(xd^2-1)^b=(zd)^c,\ 2x+1=z^2$ [11]

viii. $(3xd^2-1)^a+(x(x-3)d^2+1)^b=(xd)^c$ [12]

ix. $(4d^2+1)^a+(21d^2-1)^b=(5d)^c$ [13]

x. $(5pd^2-1)^a+(p(p-5)d^2+1)^b=(pd)^c$ [14]

xi. $(3d^2+1)^a+(bd^2-1)^b=(cd)^c$ [15]

xii. $(4d^2+1)^a+(45d^2-1)^b=(7d)^c$ [16]

xiii. $(6d^2+1)^a+(3d^2-1)^b=(3d)^c$ [17]

xiv. $(c(c-l)d^2+1)^a+(cld^2-1)^b=(cd)^c$ [18]

xv. $(44d^2+1)^a+(5d^2-1)^b=(7d)^c$ [19]

This research is dedicated to exploring and analyzing Terai's conjecture, focusing specifically on investigating the exponential Diophantine equation.

2. Preliminaries

This section presents some basic properties to be required in the following section.

Theorem 2.1. For any positive integer d, (1.2) possesses a sole and distinct positive integer solution, namely, $(x,y,z)=(1,1,2)$.

The proof of this theorem involves several important steps. Firstly, elementary methods, such as congruences and properties of the Jacobi symbol are employed to simplify the solution. Particular attention is given to the case where $x=1$, especially when $d\equiv \pm 2 \pmod 5$. Subsequently, a lower bound for linear forms in two logarithms, as established by Laurent [20], is utilized.

In cases where $d\equiv 0 \pmod 5$, a result concerning linear forms in p-adic logarithms, as detailed in Bugeaud's study [21], is applied. Conversely, for the case $d\equiv \pm 1 \pmod 5$, an earlier version of the Primitive Divisor Theorem, is attributed to Zsigmondy [22].

Definition 2.2. The expression of the absolute logarithmic height for any non-zero algebraic number α with degree m over $\mathbb{Q}$ is provided by the following

$$h(\alpha)=\frac{1}{m}\left(\log\left(|a_0|+\sum_{i=0}^{m}\log\left(\max\{1,|\alpha^{(i)}|\}\right)\right)\right)$$

Here, the symbol a_0 denotes the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(i)}$ represents the conjugates of α.

The linear form defined by $L=k_1\alpha_1+k_2\alpha_2$ is an expression involving two real algebraic numbers, α_1 and α_2, where the absolute values of both α_1 and α_2 are greater than or equal to 1. The coefficients

k_1 and k_2 are positive integers. The linear form is as follows:

$$\Lambda = k_2 \log \alpha_2 - k_1 \log \alpha_1$$

Let $D = [Q(\alpha_1, \alpha_2) : Q]$. Set

$$k' = \frac{k_1}{D \log K_2} + \frac{k_2}{D \log K_1}$$

where K_1 and K_2 are real numbers greater than 1, satisfying

$$\log K_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}, \qquad i \in \{1,2\}$$

The following proposition is a specific instance derived from Corollary 2 in [20], with the values $m = 10$ and $C_2 = 25.2$ chosen as indicated in Table 1 [20].

Proposition 2.3. [20] Given the previously defined variables Λ, α_i, D, K_i, and k' where $\alpha_i > 1$, for $i \in \{1,2\}$, and assuming that α_1 and α_2 are not multiplicatively related, the following inequality is valid:

$$\log|\Lambda| \geq -25.2\ D^4 \left(\max\left\{\log k' + 0.38, \frac{10}{D}, 1\right\}\right) \log K_1 \log K_2$$

In this context, a specific case is considered where $y_1 = y_2 = 1$ from Theorem 2 [21], referencing a result from [21]. Prior to investigating this result, it is pertinent to reintroduce some notations. Take an odd prime p and define v_p as the p-adic valuation normalized such that $v_p(p) = 1$. Consider two nonzero integers a_1 and a_2. The smallest positive integer g satisfying the following conditions is identified:

$$v_p({a_1}^g - 1) > 0 \quad \text{and} \quad v_p({a_2}^g - 1) > 0$$

Suppose that there exists a real number E such that

$$v_p({a_1}^g - 1) \geq E > \frac{1}{p-1}$$

The following theorem provides a specific upper bound for the p-adic valuation of

$$\Lambda = {a_1}^{k_1} - {a_2}^{k_2}$$

where k_1 and k_2 are positive integers.

Proposition 2.4. [21] Let $K_1, K_2 > 1$ be real numbers such that

$$\log K_i \geq \max\{\log|a_i|, E \log p\}, \quad i \in \{1,2\}$$

and put

$$t' = \frac{k_1}{\log K_2} + \frac{k_2}{\log K_1}$$

If a_1 and a_2 are multiplicatively independent then, the upper estimates can be expressed as follows

$$v_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4} \left(\max\{\log t' + \log(E \log p) + 0.4, 6E \log p, 5\}\right)^2 \log K_1 \log K_2$$

Proposition 2.5. [22] Consider relatively prime integers E and F with $E > F \geq 1$. Define the sequence $\{a_n\}_{n \geq 1}$ as

$$a_n = E^n + F^n$$

For $n > 1$, the sequence a_n has a prime factor not dividing $a_1 a_2 a_3 \cdots a_{n-1}$, except when $(E, F, n) \neq (2, 3, 1)$.

3. Main Results

This section presents the proof of Theorem 2.1, based on a series of lemmas.

Lemma 3.1. If (x, y, z) represents a positive integer solution of (1.2), then it follows that y must be an odd integer.

PROOF. If $z \leq 2$, the solution $(x, y, z) = (1, 1, 2)$ is clearly the only solution to (1.2). However, when assuming $z \geq 3$, taking (1.2) modulo d^2 results in $1 + (-1)^y \equiv 0 \pmod{d^2}$. This implies that y must be odd since $d^2 > 2$. □

Lemma 3.2. In (1.2), if d is even, then x is also even. Conversely, if d is odd, then x is odd as well.

PROOF. Applying modulo d^3 to (1.2), it follows that

$$1 + 9d^2x + (-1) + 16d^2y \equiv 0 \pmod{d^3}$$

and thus

$$9x + 16y \equiv 0 \pmod{d}$$

It can be seen from here that if d is even, then x is also even. Similarly, if d is odd, then x is also odd. □

Lemma 3.3. [23] Consider positive integers p, u, and w and $d > 1$ such that $p + u = w^2$. Suppose a positive integer solution (x, y, z) to the exponential Diophantine equation

$$\left(pd^2+1\right)^x + \left(ud^2-1\right)^y = (wd)^z$$

where $x \geq y$. The following inequalities hold true:

$$\left(2 - \frac{\log\left(\frac{w^2}{p}\right)}{\log(wd)}\right) x < z \leq 2x$$

Moreover, if y is the larger value, then

$$\left(2 - \frac{\log\left(\frac{w^2d^2}{ud^2-1}\right)}{\log(wd)}\right) y < z \leq 2y$$

In particular, when $M = \max\{x, y\} > 1$, it follows that

$$\left(2 - \frac{\log\left(\frac{w^2}{\min\left\{p, u - \frac{1}{d^2}\right\}}\right)}{\log(wd)}\right) M < z < 2M$$

This refined characterization delineates the possible range of values for z based on the parameter M and the given variables.

3.1. The Case $5|d$

This section proves that Theorem 2.1 holds true under the condition $5|d$.

Lemma 3.4. If a positive integer solution (x, y, z) to (1.2) is considered under the assumption that d is congruent to 0 in modulo 5, then the only positive integer solution to (1.2) is $(x, y, z) = (1, 1, 2)$.

PROOF. Certainly, $(1, 1, 2)$ is the unique solution of (1.2) when $M = \max\{x, y\} = 1$. Assume that

$M > 1$. Applying Lemma 3.3 for $d \geq 5$, it follows that

$$1.68M < \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(25)}\right) < z \leq 2M$$

Thus, it follows that $z \geq 5$. Given that y is odd, as stated in Lemma 3.1,

$$\Lambda = {\alpha_1}^{s_1} - \alpha^{s_2}$$

is set up where $a_1 = 9d^2 + 1$, $a_2 = 1 - 16d^2$, $s_1 = x$, and $s_2 = y$.

Considering $p = 5$ and setting $g = 1$ satisfies the condition outlined before Proposition 2.4. Therefore, set $E = 2$ and apply Proposition 2.4 to obtain

$$2z \leq \frac{36.1}{8(\log 5)^4}\left(\max\left\{\log s' + \log(2\log 5) + 0.4, 12\log 5, 5\right\}\right)^2 \log\left(9d^2+1\right)\log\left(16d^2-1\right) \tag{3.1}$$

where

$$s' = \frac{x}{\log\left(16d^2-1\right)} + \frac{y}{\log\left(9d^2+1\right)}$$

Since $z \geq 5$, applying modulo d^4 to (1.2) yields $9x + 16y \equiv 0 \pmod{d^2}$. Then, $M \geq \frac{d^2}{25}$. As

$$z \geq \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) M$$

by Lemma 3.3, (3.1), and $r' \leq \frac{M}{\log 3d}$,

$$2\left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) M \leq \frac{36.1}{8(\log 5)^4}\left(\max\left\{\log\left(\frac{M}{\log 3d}\right) + \log(2\log 5) + 0.4, 12\log 5\right\}\right)^2 \log\left(9d^2+1\right)\log\left(16d^2-1\right) \tag{3.2}$$

is obtained. Let

$$k = \max\left\{\log\left(\frac{M}{\log 3d}\right) + \log(2\log 5) + 0.4, 12\log 5\right\}$$

Suppose

$$k = \log\left(\frac{M}{\log 3d}\right) + \log(2\log 5) + 0.4 \geq 12\log 5$$

The inequality

$$\log M \geq 12\log 5 - \log(2\log 5) - 0.4$$

leads to the conclusion that $M > 50841462$. However, from (3.2)

$$2M \leq (0.68)(\log M + 1.57)^2 \log(225M+1)\log(400M-1)$$

and this implies $M < 8128$. This discrepancy results in a contradiction. If $k = 12\log 5$, then (3.2) takes the form

$$\frac{2d^2}{25}\left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) \leq 251\log(9d^2+1)\log(16d^2-1)$$

This implies that $d \leq 629$. Hence,

$$M < \frac{251\log\left(9d^2+1\right)\log\left(16d^2-1\right)}{2\left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right)}$$

$$1.68x < \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(25)}\right)x < \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right)x < z \leq 2x \tag{3.3}$$

and

$$\begin{aligned} 1.84y < \left(2 - \frac{\log\left(\frac{26}{16}\right)}{\log(25)}\right)y < \left(2 - \frac{\log\left(\frac{26d^2-26}{16d^2-16}\right)}{\log(5d)}\right)y \\ < \left(2 - \frac{\log\left(\frac{25d^2}{16d^2-1}\right)}{\log(5d)}\right)y < z \leq 2y \end{aligned} \tag{3.4}$$

(3.3) and (3.4) lead to the conclusion that there are no positive integer solutions for (1.2) when $z \leq 6$. Assuming $z > 6$, an analysis of (1.2) is performed by considering congruences modulo d^4, d^6, and d^8.

i. Applying modulo d^4 to (1.2) results in $9d^2x + 16d^2y \equiv 0 \pmod{d^4}$ which is further expressed as

$$9x + 16y \equiv 0 \pmod{d^2} \tag{3.5}$$

ii. Analysis of (1.2) yields a simplified expression

$$9x + 9^2d^2\frac{x(x-1)}{2} + 16y - 16^2d^2\frac{y(y-1)}{2} \equiv 0 \pmod{d^4} \tag{3.6}$$

iii. The analysis extends to modulo d^8 with a more complex expression

$$\begin{aligned} 9x + 9^2d^2\frac{x(x-1)}{2} + 9^3d^4\frac{x(x-1)(x-2)}{6} \\ + 16y - 16^2d^4\frac{y(y-1)}{2} + 16^3d^4\frac{y(y-1)(y-2)}{6} \equiv 0 \pmod{d^6} \end{aligned} \tag{3.7}$$

(3.5)-(3.7) summarize the congruence conditions derived from (1.2) modulo d^2, d^4, and d^6, respectively. These conditions lead to bounds on all the variables x, y, and z. Through an exhaustive search using a Maple program running for several hours, no additional positive integer solutions (d, x, y, z) were discovered for (1.2) beyond the solution $(x, y, z) = (1, 1, 2)$ when $5|d$. Hence, it is confirmed that there are no other positive integer solutions to (1.2). □

3.2. The Case $d \equiv \pm 2 \pmod 5$

This section proves that Theorem 2.1 holds true under the condition $d \equiv \pm 2 \pmod 5$.

Lemma 3.5. For a positive integer solution (x, y, z) to (1.2) where $d \equiv \pm 2 \pmod 5$, it is established that the sole positive integer solution is $(x, y, z) = (1, 1, 2)$.

Let d be even. Thus, x is also even from Lemma 3.2. Applying modulo 5 to (1.2) results in the equation

$$2^x + 3^y \equiv 0 \pmod 5$$

However, this is impossible when x is even and y is odd.

Proceed by first establishing Lemma 3.6 and Lemma 3.7, starting with the assumption that d is odd, which implies that x is also odd as indicated in Lemma 3.2.

Lemma 3.6. If d is odd and $d \equiv \pm 2 \pmod 5$, then $x = 1$ and y is odd.

Proof. With reference to Lemma 3.2, our focus is directed specifically towards the scenario where $d > 2$ is an odd number. Additionally, as implied by Lemma 3.1, it is established that y is an odd integer. Consequently,

$$\left(\frac{9d^2+1}{16d^2-1}\right)=\left(\frac{25d^2}{16d^2-1}\right)=1$$

and

$$\begin{aligned}\left(\frac{5d}{16d^2-1}\right)&=\left(\frac{5}{16d^2-1}\right)\left(\frac{d}{16d^2-1}\right)\\&=\left(\frac{16d^2-1}{5}\right)\left(\frac{16d^2-1}{d}\right)\\&=\left(\frac{3}{5}\right)\left(\frac{16d^2-1}{d}\right)(-1)^{\frac{16d^2-2}{2}\frac{d-1}{2}}\\&=(-1)(-1)^{\frac{d-1}{2}}(-1)^{(8d^2-1)\frac{d-1}{2}}\\&=(-1)(-1)^{\frac{d-1}{2}(8d^2-1+1)}\\&=-1\end{aligned}$$

Using the Jacobi symbol notation $\left(\frac{*}{*}\right)$ deduce that z is an even integer. Suppose that $x\geq 3$. Applying modulo 8 to (1.2)

$$2^x+(-1)^y\equiv 1 \pmod 8$$

and thus

$$2^x\equiv 2 \pmod 8$$

This implies that x must be equal to 1. □

Consequently, (1.2) transforms into the following

$$9d^2+1+\left(16d^2-1\right)^y=(5d)^z \tag{3.8}$$

Lemma 3.7. $y\geq \frac{1}{16}\left(d^2-9\right)$

PROOF. As $y\geq 3$ and $x=1$, (1.2) leads to

$$(5d)^z\geq 9d^2+1+\left(16d^2-1\right)^3>(5d)^3$$

Applying modulo d^4 to equation (3.8) yields

$$9d^2+1+16d^2y-1\equiv 0 \pmod{d^4}$$

and thus

$$9+16y\equiv 0 \pmod{d^2}$$

Having established this claim, the subsequent step involves deriving an upper bound for y. □

Lemma 3.8. $y<2521\log 5d$

PROOF. Consider

$$\left(9d^2+1\right)^x+\left(16d^2-1\right)^y=(5d)^z \tag{3.9}$$

If $y=1$, then clearly $z=2$. Assume that $y\geq 3$. Then, $z>2$ from (3.9). For simplicity set the following notation $p=9d^2+1$, $q=16d^2-1$, and $r=5d$ and consider the linear form of two logarithms

$$\Lambda=z\log r-y\log q$$

Since

$$0<\Lambda<e^{\Lambda}-1=\frac{r^z}{q^y}-1=\frac{p}{q^y} \tag{3.10}$$

then

$$\log\Lambda<\log p-y\log q \tag{3.11}$$

From Proposition 2.3,

$$\log\Lambda\geq -25.2D^4\left(\max\left\{\log t^{'}+0.38,10\right\}\right)^2\log q\log r \tag{3.12}$$

where

$$t^{'}=\frac{y}{\log r}+\frac{z}{\log q}$$

and

$$q^{y+1}-r^z=qq^x-r^z=q\left(r^z-p\right)-r^z=(q-1)r^z-pq>\left(16d^2-2\right)25d^2-\left(9d^2+1\right)\left(16d^2-1\right)>0$$

Since $z>2$, then $q^{y+1}>r^z$. Therefore, $t^{'}<\frac{2y+1}{\log r}$. Write $M=\frac{y}{\log r}$, and thus

$$t^{'}<2M+\frac{1}{\log r}$$

Combining (3.11) and (3.12),

$$y\log q<\log p+25.2\left(\max\left\{\log\left(2M+\frac{1}{\log r}\right)+0.38,10\right\}\right)^2\log q\log r$$

Since $\frac{\log p}{\log q\log r}<1$ and $\log r=\log 5d>2$, for $d\geq 3$, the inequality can be expressed as follows:

$$M<1+25.2\left(\max\{\log(2M+0.5)+0.38,10\}\right)^2$$

If $\log(2M+0.5)+0.38>10$, then $M\geq 7532$. However, the inequality

$$M<1+25.2\left(\log(2M+0.5)+0.38\right)^2$$

implies that $M\leq 1867$. Thus, $\max\{\log(2M+0.5)+0.38,10\}=10$ implies $M<2521$. Hence, $x<2521\log 5d$. By combining Lemma 3.7 and Lemma 3.8,

$$\frac{1}{16}\left(d^2-9\right)<2521\log 5d$$

This implies $d\leq 566$. From (3.10),

$$\frac{z}{y}-\frac{\log q}{\log r}<\frac{p}{yq^y\log r}$$

Thus,

$$\left|\frac{\log q}{\log r}-\frac{z}{y}\right|<\frac{p}{yq^y\log r}$$

which further implies

$$\left|\frac{\log q}{\log r}-\frac{z}{y}\right|<\frac{1}{2y^2}$$

Thereby, $\frac{z}{y}$ is a convergent in the simple continued fraction expansion to $\frac{\log q}{\log r}$. Consider $\frac{z}{y}=\frac{a_n}{b_n}$ where $\frac{a_n}{b_n}$ represents the n-th convergent of the simple continued fraction expansion of $\frac{\log q}{\log r}$. Since $\gcd(a_n,b_n)=1$, it follows that $b_n\leq y$. Hence, an upper bound for b_n is given by $b_n<2521\log 5d$ according to Lemma 3.8. Any such convergent $\frac{a_n}{b_n}$ satisfies

$$\frac{1}{b_n\left(b_n+b_{n+1}\right)}<\left|\frac{\log q}{\log r}-\frac{a_n}{b_n}\right|$$

By setting $b_{n+1} = u_{n+1}b_n + b_{n-1}$,

$$\frac{1}{(b_n)^2(b_n+b_{n+1})} < \left|\frac{\log q}{\log r} - \frac{a_n}{b_n}\right| < \frac{p}{yq^y \log r} < \frac{p}{b_n q^{b_n} \log r}$$

where u_n is the n-th partial quotient of the simple continued fraction expansion of $\frac{\log q}{\log r}$ refer to [24]. Therefore, b_n and u_{n+1} satisfy

$$u_{n+1} + 2 > \frac{q^{b_n} \log r}{pb_n} \tag{3.13}$$

As a final step, a short computer program in Maple was utilized to verify that no convergents $\frac{a_n}{b_n}$ of $\frac{\log q}{\log r}$ satisfy equation (3.13) when $b_n < 2521 \log(5d)$, for $1 < d \leq 566$. This process took only a few seconds to complete, concluding the proof. Therefore, Lemma 3.5 is also proven. □

3.3. The Case $d \equiv \pm 1 \pmod 5$

This section proves that Theorem 2.1 holds true under the condition $d \equiv \pm 1 \pmod 5$.

Lemma 3.9. (1.2), with d being a positive integer such that $d \equiv \pm 1 \pmod 5$, possesses a unique positive integer solution $(x, y, z) = (1, 1, 2)$.

Proof. Consider the positive integers k_1 and k_2 and a positive integer d satisfying $d \equiv \pm 1 \pmod 5$. (1.2) is expressed as follows:

$$9d^2 + 1 = 5^{k_1}A, \qquad \left(9d^2+1\right)^x = 5^{k_1 x}A^x \tag{3.14}$$

$$16d^2 - 1 = 5^{k_2}B, \qquad \left(16d^2-1\right)^y = 5^{k_2 y}B^y \tag{3.15}$$

where A and B are nonzero integers not congruent to 0 modulo 5. Then, (1.2) can be rewritten as

$$5^{k_1 x}A^x + 5^{k_2 y}B^y = (5m)^z \tag{3.16}$$

Firstly, consider the case $k_1 x > k_2 y$. This implies

$$5^{k_2 y}\left(5^{k_1 x - k_2 y}A^x + B^y\right) = 5^z m^z$$

which leads to

$$k_2 y = z \tag{3.17}$$

Substituting (3.17) back into (3.16),

$$\left(9d^2+1\right)^x = \left((5d)^{k_2}\right)^y - \left(16d^2-1\right)^y \tag{3.18}$$

Applying Proposition 2.5 [22], $y = 1$ is found. Therefore, (3.15) simplifies to

$$\left(16d^2-1\right)^y = 5^{k_2 y}B^y = 5^{k_2}B \tag{3.19}$$

Substituting (3.17) into (3.19) with $y = 1$,

$$16d^2 = 5^z B + 1 \tag{3.20}$$

Delve into the case $z = 3$, for (1.2). This transforms into

$$\left(9d^2+1\right)^x + 16d^2 - 1 = (5d)^3$$

However, when $x \geq 2$, it leads to

$$(5d)^3 > \left(9d^2+1\right)^x \geq \left(9d^2+1\right)^2 > 9^2 d^4$$

which results in the contradiction $5^3 > 9^2 d$ since $d > 1$.

Indeed, when $y = 1$ is set and $x = 1$ in (1.2), it simplifies to

$$9d^2 + 1 + 16d^2 - 1 = (5d)^3$$

However, this results in a contradiction under the condition $d \equiv \pm 1 \pmod 5$.

When $z \geq 4$, investigating (1.2) in modulo d^4 results in the inference that $y = 1$. This deduction is made by employing Proposition 2.5 in [22]. This simplifies the equation to

$$9d^2x + 16d^2 \equiv 0 \pmod{d^4}$$

and thus

$$9x + 16 \equiv 0 \pmod{d^2}$$

It can be observed that

$$d^2 \leq 9x + 16 \tag{3.21}$$

Substituting (3.20) into (3.21),

$$5^z B \leq 144x + 255 \tag{3.22}$$

Since $x < z$, (3.22) turns into (3.23):

$$5^z B \leq 144z + 255 \tag{3.23}$$

As a result, there are no positive integer solutions, for $z > 4$, and $z = 4$, the equation does not have any positive integer solutions for appropriate values of x and y. Similarly, by employing analogous procedures when $k_2y > k_1x$, it can be deduced that there exist no positive integer solutions for $z \geq 3$.

Finally, investigate the scenario $k_1x = k_2y$. Summing up (3.14) and (3.15),

$$25d^2 = 5^{k_1}A + 5^{k_2}B$$

Analyze this equation based on the positive integers k_1 and k_2:

i. $k_1 = 2$ and $k_2 \geq 3$

If $k_1 = 2$, then it is observed that k_2 must be even while y is odd. Thus,

$$2x = k_2y \tag{3.24}$$

and there is a positive integer such that k_3 satisfies $2k_3 = k_2$. Putting it into the (3.24), $x = k_3y$ is acquired. Then, (1.2) becomes

$$\left(\left(9d^2+1\right)^{k_3}\right)^y + \left(16d^2-1\right)^y = (5d)^z$$

Apply Proposition 2.5 [22], $y = 1$ is seen. Consequently, there are no solutions for $x > 2$.

ii. $k_1 \geq 3$ and $k_2 = 2$

$$\frac{k_1}{k_2} = \frac{y}{x}$$

since $k_1x = k_2y$. Note that $\gcd(x, y) = 1$. Indeed, if there exists an odd prime $p \geq 1$ such that $p|x$ and $p|y$, then by Zsigmondy Theorem [22] there is no solution, for x and y. Hence, it is clear that $x = 2$ and $k_2 = 2$ where y is odd. Therefore,

$$y = k_1 \geq 3 \qquad \text{and} \qquad x = k_2 = 2$$

(3.16) becomes

$$5^{k_1x}A^x + 5^{k_2y}B^y = (5d)^z$$

and thus

$$5^{2y}\left(A^2+B^y\right)=(5d)^z$$

If $5 \nmid (A^2+B^y)$, then $2y=z$. Then, (1.2) becomes

$$\left(9d^2+1\right)^x=\left((5d)^2\right)^y-\left(16d^2-1\right)^y$$

Applying Proposition 2.5 from Zsigmondy's theorem, it follows that $y=1$. However, this leads to a contradiction. Therefore, there exist no positive integer solutions, for x and y. Thus, $z \leq 2$.

If $5 | (A^2+B^y)$, by (3.17) and (3.19),

$$16d^2-1=5^{k_2}B=25B$$

and thus

$$9d^2+1=5^{k_1}A$$

If add the above equations side by side, then

$$25^2=5^{k_1}+25B \tag{3.25}$$

When taking (3.25) modulo 5,

$$1 \equiv B \pmod 5$$

In conclusion, no positive integer A can be found that satisfies the condition $5 \mid (A^2+B^y)$.

□

4. Conclusion

This research investigates the equation (1.2) with specific parameters $(p,u,w)=(9,16,5)$ and determines the unique solution $(x,y,z)=(1,1,2)$ when $d>1$. Particularly, it addresses an unexplored area in the literature by considering the case where u is a positive even integer and p is an odd integer in the equation

$$\left(pd^2+1\right)^x+\left(ud^2-1\right)^y=(wd)^z$$

In doing so, it guides future research in solving equations where the coefficient u is a positive even integer and contributes to the existing knowledge in this field. The aim is to take a step towards finding and generalizing many equations, leading to a generalized equation.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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