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Spin 1 Spinor Construction with Clifford Algera and Dirac Spin 1/2 Spinors

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ABSTRACT: A compatible spin 1 spinor representation with Clifford algebra (1,3) (or $Cl_{1,3}$) is derived for both $(1/2,1/2)$ and $(1,0)\oplus(0,1)$ Lorentz group representations with spin 1/2 particles Dirac spinors in $Cl_{1,3}$. The relation between the two different representations of spin 1 spinors is analogous to the relation between the electromagnetic vector potential field A^{μ} and the electromagnetic field strength tensor $F^{\mu\nu}$. From this relationship, the two representations are combined by the formula $u(p,\lambda)=\not{\epsilon}(p,\lambda)\wedge\not{p}/m$. We also note that the Grassmann basis provides more convenient basis for spin 1 spinors especially in chiral representations of $(1,0)\oplus(0,1)$, even though the Clifford basis is more fitting for spin 1/2 and $(1/2,1/2)$ spinor representations for both helicity and handedness.

Keywords: Clifford Algebra, Spinors

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INTRODUCTION

The first mathematical term spinors is discovered by Elie Cartan (Cartan, 1938). Later Paul Ehrenfest put the term “spinors” with his work on quantum physics (Tomonaga, 1998). It was Wolfgang Pauli who used the first spinors in mathematical physics in 1927 by his Pauli matrices (Pauli, 1927). In 1930 G. Juvet (Juvet, 1930) and Fritz (Sauter, 1930) found that they could use left ideals of a matrix algebra to represent spinors. In this algebra, the left column of matrices could be used as vectors and left minimal ideals are spinor space. The usage of the minimal left ideal of Clifford algebras began with Marcel Riesz in 1947 (Riesz, 1947).

The paper begins with the definition and properties of Clifford algebra. We continue with spinors in Clifford algebra. First, minimal left ideal or projection method of Dirac spinors similar to Hestenes’ description (Hestenes, 1975) in projective spin-1/2 representation group $(1/2,0)\oplus(0,1/2)$. Then, we use the vector definition of $Cl_{1,3}$ to find out the polarization vectors which are $(1/2,1/2)$ spinors in terms of Clifford numbers. Moreover, these spinors can be shown in terms of spin 1/2 spinors. It is discovered that $(1,0)\oplus(0,1)$ spin 1 spinors can be expressed as $u(p,\lambda)=\not{\epsilon}(p,\lambda)\wedge \not{p}/m$ similar to the electromagnetic field strength tensor $F_{\mu\nu}$ and vector potential field A_μ are related to each other. In the next section, we argue that the Grassmann basis or Witt basis (Pavšič, 2010) is a more fitting choice of basis for the spinors in chiral representation and in spherical harmonics form of coordinates when comparing with the Clifford basis. We also pointed out how this basis is in harmony with natural way of occurring of particles like as in light-front form of dynamics. The last section before conclution is about the how local transformations can express spin 1 spinors in terms of fibers in gauge fields.

MATERIALS AND METHODS

Clifford Algebra (1,3)

Clifford algebra (1,3) shortly $Cl_{1,3}$, is a Clifford algebra with Minkowski space metric which is $g_{\mu\nu}=diag\{1,-1,-1,-1\}$ in quadratic form $x^\mu g_{\mu\nu}x^\nu=x^2$, where generators are γ_μ with $\gamma_\mu\gamma_\nu=-\gamma_\nu\gamma_\mu$. The Clifford algebra $Cl_{1,3}$ consists of different grades as $(1,\gamma_\mu,\gamma_{\mu\nu},I\gamma_\mu,I)$ as scalar, vectors, bivectors, trivectors, and volume element which is $I=\gamma_0\gamma_1\gamma_2\gamma_3$. It has total 16 dimensions.

The entire grades or multivector structure of $Cl_{1,3}$ is

$$Cl_{1,3}=\mathbb{R}\oplus\mathbb{R}^4\oplus\Lambda^2\mathbb{R}^4\oplus\Lambda^3\mathbb{R}^4\oplus\Lambda^4\mathbb{R}^4$$

The Poincare isomorphism to $Cl_{1,3}$ can be expressed as $M^{\mu\nu}\simeq i\gamma^\mu\wedge\gamma^\nu/2$ where the wedge product is $\gamma^\mu\wedge\gamma^\nu=\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)/2$ in shortly as $\gamma^{\mu\nu}$. In Clifford algebra, we define the boost and rotation operators by rotors R. For boosts, the rotor is $R=e^{-\gamma_{0i}\phi^i/2}$, $i=1,2,3$. For rotations, it is $R=e^{\gamma_{ij}\phi^k/2}$, $i,j=1,2,3$ with $i\neq j$. The rotor transforms a vector v as $v'=RvR^{-1}$, this transformation is also the same for any multivector A as $A'=RAR^{-1}$.

Spinors

We begin with the definition of Dirac spinors from (Hestenes, 1975) which are also left minimal ideals. Then, since Clifford algebra generators are already part of spin-1 group of $(1/2,1/2)$, polarization vectors $\varepsilon(p,\lambda)$ are associated with Clifford numbers as vectors and expressed with the spherical harmonics in order to get -1,0,1 spin states. First, the rest frame spinors of $(1/2,1/2)$ can be defined as vectors of $Cl_{1,3}$ with Cartesian representation of their spins and then the derivation of the $(1,0)\oplus(0,1)$ spinors from the polarization vectors holds similarity between field strength tensor and vector potential as $u(p,\lambda)=\not\varepsilon(p,\lambda)\wedge \not p/m$.

Hestenes' Projection Method and Spin 1/2 Spinors

Describing spinors as left minimal ideals in Clifford algebra is well known method (Pavšič, 2010; Lounesto, 1997; Hestenes, 1986). In this approach, column representation is used because of isomorphism between Clifford numbers and matrices. We take Hestenes' convention of spinors which is $U=1/4(1+\gamma_0)(1+\sigma_3)$ projection for spinors as minimal left ideals. Here the factors $1/2(1+\gamma_0)$ and $1/2(1+\sigma_3)$ are energy and spin projection operators and we make a slight notation as we use our spin operator as $i\gamma_{12}$ instead of σ_3. Then, the projective spin spinor representation of $(1/2,0)\oplus(0,1/2)$ is given by

$$u=\frac{1}{2}(1+\gamma_0)(1+i\gamma_{12}). \tag{7}$$

u is given as positive energy and positive helicity spinor so in order to get other spinors, some operators are needed applied on spinor u given by (7). Two operators are needed to get other spinors and these are charge and spin, raising and lowering operators:

$$Q^{\mp}=\frac{1}{2}(\pm\gamma_1-i\gamma_2), \tag{8}$$

$$S^{\mp}=\frac{1}{2}(i\gamma_{23}\pm\gamma_{31}). \tag{9}$$

They become $Q^{\mp}=RQ^{\mp}(0)R^{-1}$ and $S^{\mp}=RS^{\mp}(0)R^{-1}$ in moving frame. These two operators are the change of two projections in spinors: spin and energy (handedness). Since we only have two states: spin up or down or particle or anti-particle, one of the $\pm$ get rid of one state and change the other state and they can be simplified for spin 1/2 case as

$$Q=Q^{+}+Q^{-}, \qquad S=S^{+}+S^{-}. \tag{10}$$

The Dirac spinor representations can derived from u via these operator as

$$u^1=u, \quad u^2=Su, \quad v^1=SQu, \quad v^2=Qu, \tag{11}$$

where they are spin up and down for particles and anti-particles and these spinors in moving frame are expressed as

$$u^{(i)}(p)=Ru^{(i)}(0). \tag{12}$$

The rotor is chosen as a single boost in any direction (Dirac boost) $R = e^{-\gamma_{0i}\phi^i/2}$

$$R = \cosh\frac{\phi}{2} - \gamma_{0i}\frac{\phi^i}{\phi}\sinh\frac{\phi}{2}, \tag{13}$$

where $\cosh\frac{\phi}{2} = \sqrt{\frac{E+m}{2m}}$ and $\frac{\phi^i}{\phi}\sinh\frac{\phi}{2} = \frac{p^i}{\sqrt{2m(E+m)}}$ with $\phi = \sqrt{(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2}$.

$$R = \sqrt{\frac{E+m}{2m}}\left(1 + \gamma_{0i}\frac{p^i}{E+m}\right). \tag{14}$$

and its conjugate is given by $\bar{u}^1 = u^1 R^{-1}$. Spinors of anti-particles are charge conjugation of positive energy spinors so that they are given by $v^1 = -i\gamma_2 u^{1*}$ and $v^2 = -i\gamma_2 u^{2*}$ or negative energy solutions (Feynman-Stueckelberg interpretation) $v^1(p) = u^4(-p)$, $v^2(p) = u^3(-p)$ instead of u^3 and u^4 and in terms of Clifford numbers $v^1 = -iu^4\gamma_2$ and $v^2 = -iu^3\gamma_2$. However, the conjugate spinors can be found also with charge lowering operator in (8) to apply to spinor as shown in (11).

The Polarization Vectors

By definition Clifford numbers are also vectors as $v = \gamma_\mu x^\mu$ and we can begin with defining the 0 helicity as the z direction vector as $\varepsilon(0,0) = \gamma_3$ for $\varepsilon(p,\lambda)$ in $Cl_{1,3}$. Using described spin raising and lowering operators from (9), S^+ and S^- on $\varepsilon(0,0) = \gamma_3$, we can also define spin $+1$ and -1 polarization vectors so polarization vectors at rest frame becomes

$$\varepsilon(0,+) = -(\gamma_1 + i\gamma_2)/\sqrt{2}, \tag{15}$$

$$\varepsilon(0,0) = \gamma_3, \tag{16}$$

$$\varepsilon(0,-) = -(-\gamma_1 + i\gamma_2)/\sqrt{2}. \tag{17}$$

These spinors also confirms a relationship between the polarization vectors and spherical harmonics as $\varepsilon(0,+) \sim Y_1^1$, $\varepsilon(0,0) \sim Y_1^0$, $\varepsilon(0,-) \sim Y_1^{-1}$.

As in Lorentz transformations are done by two-sided rotors for spin 1 spinors contrary to spin-1/2 spinors. The polarization vectors in momentum space boosted in any arbitrary direction by $R = e^{-\gamma_{0i}\phi^i/2}$ as the same Dirac spinors used for spin-1/2 case.

In comparison of these polarization vectors with the classical expressions in $SO(1,3)$ group presented, one-to-one correspondence of each component is found by $\varepsilon^\mu(p,\lambda) = \gamma^\mu.\varepsilon(p,\lambda)$.

Similar to the notation from (Ashdown et al., 1998), spin 1 states from spin 1/2 states can be constructed as

$$A_{m-m'} = \psi_m \psi^*_{m'}\Gamma. \tag{18}$$

The same connection can acquired for the polarization vectors as

$$\varepsilon(p,m-m') = u^{m}(p)\bar{u}^{m'}(p)\Gamma(p) - v^{m}(p)\bar{v}^{m'}(p)\Gamma(p). \quad (19)$$

Where, $\Gamma(p) = RSQR^{-1}$ is defined with spin operator S and charge operator Q from (10) and we can construct the polarization vectors from the Dirac spinors in $Cl_{1,3}$ as

$$\varepsilon(p,+) = (u^{1}(p)\bar{v}^{2}(p) - v^{1}(p)\bar{u}^{2}(p))/\sqrt{2} = R(-\gamma_1 - i\gamma_2)R^{-1}/\sqrt{2}, \quad (20)$$

$$\varepsilon(p,0) = (u^{1}(p)\bar{v}^{1}(p) + u^{2}(p)\bar{v}^{2}(p) - v^{1}(p)\bar{u}^{1}(p) - v^{2}(p)\bar{u}^{2}(p))/2 = R\gamma_3 R^{-1}, \quad (21)$$

$$\varepsilon(p,-) = (u^{2}(p)\bar{v}^{1}(p) - v^{2}(p)\bar{u}^{1}(p))/\sqrt{2} = R(\gamma_1 - i\gamma_2)R^{-1}/\sqrt{2}. \quad (22)$$

As we expected, it shows a direct relation between change of spin of the spin 1/2 Dirac spinors and the polarization vectors.

$(1,0)\oplus(0,1)$ Lorentz Group Spinors

We use the relation similar to the electromagnetic field strength tensor related with the four potential as $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ since the vector potential is $A^{\mu} = \varepsilon^{\mu}e^{ip.x}$. Here, the field tensor becomes $F = i\varepsilon(p)\wedge pe^{ip.x}$ where $\varepsilon(p) = \gamma_{\mu}\varepsilon^{\mu}(p)$ and $p = \gamma_{\mu}p^{\mu}$. Due to the similarity with $F^{\mu\nu}$, the spinor is given by $u(p) \square\, \varepsilon_i \wedge p/m$. For the rest frame, $u(0) \square\, \gamma_i \wedge \gamma_0$ and our rest frame spinors on $(1,0)\oplus(0,1)$ representation are thus given by

$$u(0,+) = -\gamma_{01}(1 - i\gamma_{12})/\sqrt{2}, \quad (23)$$

$$u(0,-) = \gamma_{01}(1 + i\gamma_{12})/\sqrt{2}, \quad (24)$$

$$u(0,0) = \gamma_{03}. \quad (25)$$

We can observe that the spinors of $(1/2,1/2)$ are associated with vectors while spin $(1,0)\oplus(0,1)$ group are with bivectors.

There exist two $SU(2)$ subalgebras in bivectors of $Cl_{1,3}$ which can be separated into $A = J + iK$ and $B = J - iK$ to project right-handed and left-handed spinor parts in chiral representation in terms of rotation (J) and boost (K) operators. In Clifford algebra, we can write down A and B such that $A = (1+\gamma_5)J$ and $B = (1-\gamma_5)J$ since $\gamma_5 J = iK$. Thus, we have the same operator as spin 1/2 spinor to project into right-handed spinors $(1,0)$ and left-handed spinors $(0,1)$ as shown in Table. The electric and magnetic field can be expressed similarly as $E + iB \in (1,0)$ and $E - iB \in (0,1)$ since these parts are related to the bivectors as $\gamma_{\mu\nu}F^{\mu\nu} = \nabla \wedge A$ with $\nabla = \gamma^{\mu}\partial_{\mu}$ and $A = \gamma_{\mu}A^{\mu}$.

Table. Relation of $\{x, y, z\}$ directions in $(1/2,1/2)$ and in $(1,0)\oplus(0,1)$ representations for each right-handed $(1+\gamma_5)$ and left-handed $(1-\gamma_5)$ cases.

$(1/2,1/2)$	$(1,0)$	$(0,1)$
γ_1	$(\gamma_{01}-i\gamma_{23})$	$(\gamma_{01}+i\gamma_{23})$
γ_2	$(\gamma_{02}-i\gamma_{31})$	$(\gamma_{02}+i\gamma_{31})$
γ_3	$(\gamma_{03}-i\gamma_{12})$	$(\gamma_{03}+i\gamma_{12})$

In order to find components of spinors, we will introduce a new basis Grassmann or Witt basis in the next section. In this basis, we compare the $(1,0)\oplus(0,1)$ spinors with the spinors defined in terms of spin $\pm$ states rather than in terms of the Cartesian states (x, y, z) and also in the Chiral basis.

RESULTS AND DISCUSSION

Grassmann Algebra and Clifford Spinors

In constructing spinors, we can use Grassmann or Witt basis from (Winnberg, 1977) instead of Clifford basis.

We can redefine our Clifford basis with a slight notation difference as

$$\theta_1=(\gamma_0+\gamma_3)/\sqrt{2}, \qquad \theta_2=(-\gamma_1-i\gamma_2)/\sqrt{2}, \tag{26}$$

$$\bar{\theta}_1=(\gamma_0-\gamma_3)/\sqrt{2}, \qquad \bar{\theta}_2=(\gamma_1-i\gamma_2)/\sqrt{2}, \tag{27}$$

where they satisfy the properties of Grassmann algebra as

$$(\theta_1)^2=(\bar{\theta}_1)^2=(\theta_2)^2=(\bar{\theta}_2)^2=0 \quad \text{and} \quad \theta_i\theta_j=-\theta_j\theta_i \quad \text{for} \quad i\neq j \tag{28}$$

The polarization vectors can be rewritten with this new basis as

$$\varepsilon(p,\lambda)=\gamma_\mu\varepsilon^\mu=\theta_1\varepsilon^+ +\bar{\theta}_1\varepsilon^- -\theta_2\varepsilon^L+\bar{\theta}_2\varepsilon^R, \qquad (41)$$

here $\varepsilon^+=(\varepsilon^0+\varepsilon^3)/\sqrt{2}$, $\varepsilon^-=(\varepsilon^0-\varepsilon^3)/\sqrt{2}$, $\varepsilon^R=(\varepsilon^1+i\varepsilon^2)/\sqrt{2}$, $\varepsilon^L=(\varepsilon^1-i\varepsilon^2)/\sqrt{2}$.

Similarly $(1,0)\oplus(0,1)$ Lorentz group spinors in chiral representation:

$$u(p,\lambda)=\theta_1\theta_2u^1+(\bar{\theta}_1\theta_1-\theta_1\bar{\theta}_1+\bar{\theta}_2\theta_2-\theta_2\bar{\theta}_2)u^2+\bar{\theta}_1\bar{\theta}_2u^3+\bar{\theta}_1\theta_2u^4+(\bar{\theta}_1\theta_1-\theta_1\bar{\theta}_1-\bar{\theta}_2\theta_2+\theta_2\bar{\theta}_2)u^5 \; +\theta_1\bar{\theta}_2u^6. \tag{29}$$

When the Grassmann basis and Clifford basis are compared, one may see that the previous expression looks much nicer than the one in Clifford basis in previous section. We also see how they are related in such a way that θ_2 and $\bar{\theta}_2$ defines circular polarizations (spin-down or spin-up in spin 1/2 spinors) and θ_1 and $\bar{\theta}_1$ defines helicity $\pm$ states and each components of $u(p,\lambda)$ in $Cl_{1,3}$.

The u spinor can be expressed with the polarization vector as

$$u(p,\lambda)=\not\varepsilon(p,\lambda)\wedge \not p/m, \tag{30}$$

where $\not p$ as $\not p=\theta_1 p^{+}-\theta_2 p^{L}+\bar{\theta}_2 p^{R}+\bar{\theta}_1 p^{-}$.

Now, it is possible to write down the u spinors components as

$$u^1(p,\lambda)=(p^L\varepsilon^{+}-p^{+}\varepsilon^{L})/m,$$

$$u^2(p,\lambda)=(p^{-}\varepsilon^{+}-p^{+}\varepsilon^{-}-p^L\varepsilon^R+p^R\varepsilon^L)/\sqrt{2}m,$$

$$u^3(p,\lambda)=(-p^R\varepsilon^{-}+p^{-}\varepsilon^R)/m,$$

$$u^4(p,\lambda)=(p^L\varepsilon^{-}-p^{-}\varepsilon^L)/m,$$

$$u^5(p,\lambda)=-(p^{+}\varepsilon^{-}-p^{-}\varepsilon^{+}+p^L\varepsilon^R-p^R\varepsilon^L)/\sqrt{2}m,$$

$$u^6(p,\lambda)=(p^{+}\varepsilon^R-p^R\varepsilon^{+})/m. \tag{31}$$

This conversion works for any polarization vectors and spinors in any frame as long as the u spinor of spin-1 is expressed in chiral representation. We compared the polarization vectors from the previous works presented in (Ji et al., 2015; Li et al., 2015) where spin $(n,0)\oplus(0,n)$ spinors from $n=1/2$ to $n=2$ including $n=1$ are presented in the form interpolating between the instant form dynamics and the front form dynamics, now known as the light-front dynamics. Regardless of the forms of the relativistic dynamics, we confirmed the validity of the above relation given by (30).

Spin 1 Spinors in Gauge Theory

Gauge field is a local transformation and spin 1 fields are related with transformation between two spin 1/2 fields. Spin 1 spinors requires two sided transformation because they are considered to be mediators of spin 1/2 spinors and similar studies exist about two-sided equivalence in (Chisholm and Farwell, 1991). Fibers are a set of internal dimension and similar to operators given in (10) spin and charge and spin 1 spinors are local transformation of left minimal ideal spinors or projection spinors. The space fiber of left minimal ideal is the polarization vectors as $q\gamma^{\mu}A_{\mu}^{i}=mR\gamma_i R^{-1}$.

We can write down the fibers in the wave function as phase transformation

$$\psi'(x)=e^{ie\int A_{\mu}dx^{\mu}}e^{ipx}. \tag{32}$$

Then we observe them in local transformation between two point of space-time so we can describe a transition between x and dx as

$$\psi'(x+dx)=\psi'(x)+\gamma^{\mu}\partial_{\mu}\psi'(x)dx+\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu}\psi'(x)\frac{dx^2}{2}+\ldots \tag{33}$$

and when we take derivatives, the expansion of the wave function become

$$\psi'(x+dx)=\psi'(x)+i\gamma^{\mu}(p_{\mu}+eA_{\mu})\psi'(x)dx$$

$$-\left\{p^{2}+\gamma^{\mu\nu}(i(\partial_{\mu}p_{\nu})+ie^{2}(\partial_{\mu}A_{\nu})+e(A_{\mu}p_{\nu}+A_{\nu}p_{\mu}))\right\}\psi'(x)\frac{dx^{2}}{2}+\ldots \qquad (34)$$

It can be noticed that while the polarization vectors appear in first order of the expansion, u spinors emerge in the second order in the transformation of the wave function.

CONCLUSION

We provide an easy and universal way to write down $(1,0)\oplus(0,1)$ spinors from polarization vectors ($(1/2,1/2)$ spinors) as shown in (44) by constructing spin-1 spinor in $Cl_{1,3}$. While we find a simple expression for their relation as $u=\not{\varepsilon}\wedge\not{p}/m$ from $F=\nabla\wedge A$ for $Cl_{1,3}$ spin 1 spinors, we need to define the basis with which the u spinor is expressed. We introduce Grassmann or Witt basis in terms of Clifford numbers to express u spinors in chiral representation since each Grassmann number is related with spin and helicity. The Clifford basis is more convenient for polarization vectors as $\varepsilon^{\mu}(p,\lambda)=\gamma^{\mu}.\varepsilon(p,\lambda)$ since $(1/2,1/2)$ Lorentz group can be correlated with $e_{\mu}x^{\mu}$ components as $\gamma_{\mu}x^{\mu}$. However for chiral representation, the Grassmann basis is more suitable since we could express the helicity with $\{\theta_1,\bar{\theta}_1\}$ and the spin with $\{\theta_2,\bar{\theta}_2\}$ and the u spinor components determined by both the spin states $\{1,0,-1\}$ and the handedness $\{u_R,u_L\}$ in $(1,0)\oplus(0,1)$ Lorentz group spinor. Moreover the spin 1/2 Dirac spinor are connected with the Grassmann basis with $\theta_2\bar{\theta}_2$ and $\bar{\theta}_2\theta_2$ since they represent positive and negative spin projection respectively. Nevertheless, $\theta_1\bar{\theta}_1$ and $\bar{\theta}_1\theta_1$ represent helicity projection because it shows relation between z-direction of momentum and energy not energy projection alone but they are still connected.

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