The Classification of Frobenius Q - Groups

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Abstract. A finite group whose complex characters are rationally-valued is called a Q-group. In this paper, Frobenius Q-groups were completely classified.

Keywords: : Q-Groups, Frobenius Groups

AMS Subject Classification Number: 20C15

1. INTRODUCTION

***Q-Groups*.** A Q-group is a finite group all of whose irreducible complex characters are rationally-valued. For example, all of the symmetric groups and finite elemantary abelian 2-groups are Q-groups. Kletzing's lecture notes, [3] , presents a detailed investigation into the structure of Q-groups.General classification of Q-groups has not been able to be done up to now, but some special Q-groups have been classified. Frobenius Q-groups were classified by [1]. In this paper, we will classify such groups in a different way.

Notation. C the field of complex numbers ; Z the ring of rational integers ; $Gen(x)$ the set of generators of cyclic group $\langle x \rangle$; $[x]$ the conjugacy class of x ; $N_G(x)$ the normalizer of x in G ; $C_G(x)$ the centralizer of x in G ; $Aut(x)$ the group of automorphisms of $\langle x \rangle$; φ Euler's function ; $\times|$ semidirect product ; $\pi(H)$ the set of prime divisors of $|H|$; $C(H)$ the centre of H ; Q_8 the quaternion group of order 8 ; $E(p^n)$ the elemantary abelian p-group of order p^n ; $IN_0 = IN \cup \{0\}$.

Let G be a finite group of order n and let ξ be a primitive n-th root of unity in the field C. Then, all of the complex character values of G lie in the subring $Z[\xi]$ of C. Moreover, if G is a Q-group, these values lie in Z since $Z[\xi] \cap Q = Z$. Now, we say

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that a finite group is a Q-group if and only if the values of its all the irreducible complex characters lie in Z.

Another characterization of Q-groups is the following theorem.

Theorem 1. Let G be a finite group. Then, G is a Q-group if and only if for every $x \in G$, $Gen(x) \subseteq [x]$ i.e. for every $x \in G$, $N_G(x)/C_G(x) \cong Aut(x)$ [3].

Now, we can easily see the following corollary.

Corrollary 2. Let G be a Q-group. Then:

1) $[N_G(x):C_G(x)] = |Aut(x)| = \varphi(|x|)$, for every $x \in G$.

2) Let $x \in G$. If $L \le C_G(x) \le N_G(x) \le K$ where both L and K are subgroups of G, then $\varphi(|x|) \mid [K:L]$.

3) If $G \neq \{1\}$, $2 \mid |G|$.

4) If N is a normal subgroup of G, then G/N is a Q-group.

Frobenius Groups. A finite group G is a Frobenius group if and only if it contains a proper subgroup $H \neq \{1\}$, called a Frobenius complement, such that $H \cap H^x = \{1\}$ for all $x \notin H$. By Frobenius Theorem [2, p.63], a Frobenius group G with complement H has a normal subgroup K, called Frobenius kernel, such that $G = K \times\!| H$. Some basic properties of such groups are the following theorem.

Theorem 3. Let G be a Frobenius group with kernel K and complement H. Then:

1) If $2 < p \in \pi(H)$ and $P \in Syl_p(H)$, then P is a cyclic [2, p.184].
2) If $2 \mid |H|$, then

a) H contains only one involution. Thus, the involution is element of $C(H)$.

b) If u is the involution of H, then $t^u = t^{-1}$ for every $t \in K$.

c) K is an abelian group.

2. MAIN THEOREM

Frobenius Q-groups are completely classified in the following theorem.

Theorem. Let G be a Frobenius Q-group. Then, G is one of the following groups.

1) $E(3^n) \times| Z_2$, where $n \in IN$ and Z_2 acts on $E(3^n)$ by inverting every element.

2) $E(3^{2m}) \times| Q_8$, where $m \in IN$ and $E(3^{2m})$ is a direct sum of m copies of the 2-dimensional irreducible modules of group algebra Z_3Q_8.

3) $E(5^2) \times| Q_8$, where $E(5^2)$ is the 2-dimensional irreducible module of group algebra Z_5Q_8.

3. FROBENIUS *Q*-GROUPS

Now, we shall investigate the structure of Frobenius Q-groups.

Theorem 4. Let G be a Frobenius Q-group with kernel K and complement H. Then:

1) H is a Q-group.

2) $2 \nmid |K|$

3) If P is a 2-sylow subgroup of G, then $P \cong Z_2$ or $P \cong Q_8$.

Proof. 1) H is a Q-group since G is a Q-group, $G/K \cong H$ and G/K is also Q-group by Corollary 2.

2) We have $2 \,\big|\, |H|$ by Corollary 2.3) since $\{1\} \neq H$ is a Q-group. Then, H contains only one involution by Theorem 3. Let u be the involution of H. Now, we assume that $2 \,\big|\, |K|$. Then, K contains an involution x by Cauchy Theorem. Therefore, by Theorem 3 we have $x^u = x$, which is not possible since Frobenius complement is always a regular group of automorphism of the Frobenius kernel.

3) Let P be a 2-sylow subgroup of H. Then, P is a 2-sylow subgroup of G since $G = K \times| H$ and $2 \nmid |K|$. Moreover, P is a cyclic or generalized quaternion group since H contains only one involution [2, p.96]. Let $|H| = 2^{s+1}.n$ where $n \in IN$, $(2,n) = 1$, $s \in IN_0$.

Then,

i) If $P = \langle x \rangle$ where $x \in H$, $|x| = 2^{s+1}$, $s \in IN_0$, we have $\varphi(|x|) \,\big|\, \underbrace{[H : \langle x \rangle]}_{=n}$ since H is a Q-group and $\langle x \rangle \leq C_H(x) \leq N_H(x) \leq H$. Then, $P \cong Z_2$ since $2^s | n$ and $(2,n) = 1$.

ii) If $P = \left\langle a, b \;\middle|\; a^{2^s} = 1,\, a^{2^{s-1}} = b^2,\, b.a.b^{-1} = a^{-1} \right\rangle$ where $2 \leq s \in IN$, we have $\varphi(|a|) \,\big|\, \underbrace{[H : \langle a \rangle]}_{=2n}$ since $\langle a \rangle \leq C_H(a) \leq N_H(a) \leq H$. Then, we find $s = 2$ since $2 \leq s$ $2^{s-1} \,\big|\, 2n$ and $(2, n) = 1$, so $P \cong Q_8$.

Thus, we know that $P \cong Z_2$ or $P \cong Q_8$.

Lemma 5. Let G be a Frobenius Q-group with complement H. If $a \in H$ and $4 \,\big|\, |a|$, then $|a| = 4$.

Proof. Let P be a 2-sylow subgroup of H. Then, by Theorem 4. we have $P \cong Q_8$ since H has an element whose order is divided by 4. Now, we know that $8 \nmid |a|$ since $\exp P = \exp Q_8 = 4$. Let $|H| = 8m$ and $|a| = 4k$ where $m, k \in IN$, $(2, m) = 1 = (2, k)$.

Then, we have $[N_H(a):C_H(a)] = \varphi(|a|) = \varphi(4.k) = \varphi(4).\varphi(k) = 2.\varphi(k)$ since H is a Q-group by Theorem 4. Moreover, we have $2.\varphi(k) \mid [H:\langle a\rangle]$ since $\langle a\rangle \le C_H(a) \le N_H(a) \le H$ and Corollary 2.2) . Also, $[H:\langle a\rangle] = 2.t$ where $t \in IN$, $(2,t)=1$ since $[H:\langle a\rangle] = \dfrac{8.m}{4.k} = \dfrac{2.m}{k}$ and $(2,k)=1$. Thus, we find $\varphi(k)=1$ since $\varphi(k) \mid t$ and $(2,t)=1$. This implies that $k=1$ since $(2,k)=1$. Therefore, the lemma is proved now.

Theorem 6. Let G be a Frobenius Q-group with complement H. Then, $H \cong Z_2$ or $H \cong Q_8$.

Proof. Let P be a 2-sylow subgroup of H. Then, we know that $P \cong Z_2$ or $P \cong Q_8$ by Theorem 4.

1) If $P \cong Z_2$, then $P \trianglelefteq H$ since H contains only one involution by Theorem 3. Therefore, we have $H = P \cong Z_2$ since H/P is a Q-group of odd order and Corollary 2.3).
2) If $P \cong Q_8$, then $2^3 \mid |H|$ and $2^4 \nmid |H|$. Let $|H| = 8m$ where $m \in IN$, $(2,m)=1$. We assume that $m > 1$. Then, there is $2 < p \in \pi(H)$ such that $p|m$. Let S be a p-Sylow subgroup of H such that $|S| = p^r$ where $r \in IN$. Since S is a cyclic by Theorem 3. , there is an element x of H such that $S = \langle x\rangle$, $|x| = p^r$. Then, we have $[N_H(x):C_H(x)] = \varphi(|x|) = p^{r-1}.(p-1)$ since H is a Q-group. Moreover, $p \nmid [N_H(x):C_H(x)]$ since $\langle x\rangle \le C_H(x) \le N_H(x)$ and Corollary 2.2). Thus, we find $p^{r-1}=1$ so $r=1$. Therefore, $[N_H(x):C_H(x)] = p-1$, $S = \langle x\rangle$ and $|x| = p$. Since $N_H(x)/C_H(x) \cong Aut(x)$ is cyclic, there is an element y of $N_H(x)$ such that $N_H(x)/C_H(x) = \langle \bar{y}\rangle$ where $\bar{y} = yC_H(x)$, $|\bar{y}| = p-1$. Then, we have $2 \mid |y|$ since $2 \mid p-1$, $|\bar{y}| = p-1$, $|\bar{y}| \mid |y|$. Let $|y| = 2^s r$ where $r,s \in IN$, $(2,r)=1$. Then, $s=1$ or $s=2$ since $\exp P = \exp Q_8 = 4$.

At first, we assume that $s=1$. Then, $|y|=2r$ where $r\in IN$, $(2,r)=1$. This implies that there are elements z,v of $\langle y\rangle$ such that $y=z.v=v.z$; $|z|=2$, $|v|=r$. Then, we have $z\in C_H(x)$ so $\bar{z}=\bar{1}$ since H contains only one involution by Theorem 3. Thus, $2\,\big|\,|\bar{v}|$ since $\bar{y}=\overline{z.v}=\bar{z}.\bar{v}=\bar{v}$, $|\bar{y}|=p-1$, $2\,|\,p-1$. Then, we have a contradiction since $2|r$.

Now, we assume that $s=2$. Then, we have $|y|=4r$ where $r\in IN$, $(2,r)=1$.This implies that $|y|=4$, by Lemma 5. Then, $\bar{y}^2=\overline{y^2}=\bar{1}$ since y^2 is a unique involution in H, so $|\bar{y}|\,\big|\,2$. Therefore, we find $p=3$ since $2\le p-1=|\bar{y}|$, $|\bar{y}|\,\big|\,2$. Then, we have $|H|=2^3.3=24$. As $y\in N_H(x)$, we have a semi-direct product $\langle x\rangle\times\!|\,\langle y\rangle$. Now let $M:=\langle x\rangle\times\!|\,\langle y\rangle$. We know that $|M|=12$ since $|x|=p=3$ and $|y|=4$. Then, M is a normal subgroup of H since $[H:M]=2$. Thus, we have $\langle x\rangle\lhd H$ since $M\underset{2}{\lhd}H$ and $\langle x\rangle$ is a characteristic subgroup of M, so $N_H(x)=H$ .Then, we find $|C_H(x)|=12$ since $[H:C_H(x)]=[N_H(x):C_H(x)]=\varphi(|x|)=\varphi(3)=2$ and $|H|=24$. Now, let R be a 2-Sylow subgroup of $C_H(x)$, then $|R|=4$. Moreover, R is a cyclic since H contains only one involution. Then, there is an element z of $C_H(x)$ such that $R=\langle z\rangle$ where $|z|=4$. Thus, we have $|xz|=12$. This is a contradiction by lemma 5.

Finally, we find $m=1$ that is, $|H|=8$.

Theorem 7. Let G be a Frobenius Q-group with kernel K and complement H. Then, $H\cong Z_2$ or $H\cong Q_8$ and K is an elemantary abelian p-group where $p=3$ or $p=5$.

Proof. By Theorem 6, $H\cong Z_2$ or $H\cong Q_8$. Then, K is an abelian group since $2\,\big|\,|H|$ and Theorem 3. Moreover, we have $2\nmid |K|$ by Theorem 4. Now, we assume that x is an element of K such that $|x|=p^e$ where $e\in IN$, $2<p\in\pi(K)$. Then, we have $\varphi(|x|)\,\big|\,[G:K]$ so $p^{e-1}.(p-1)\,\big|\,[G:K]$ since $K\le C_G(x)\le N_G(x)\le G$ and

Corollary 2.2). This implies that $p^{e-1}.(p-1) \mid |H|$ since $[G:K]=|H|$. Then, we find $e=1$ since $p \nmid |H|$ and $2 < p \in IP$. Therefore, we have $|x|=p$ and $(p-1) \mid |H|$.

1) If $H \cong Z_2$, then $p-1 \in \{1,2\}$. Thus, $p=3$ since $2 < p \in \pi(K)$. Therefore, K is an elemantary abelian 3-group.
2) If $H \cong Q_8$, then $p-1 \in \{1,2,4,8\}$. Thus, we find $p=3$ or $p=5$ since $2 < p \in \pi(K)$. Now, let $|K|=3^a.5^b$ where $a,b \in IN_0$. We assume that $|K|=3^a.5^b$ where $a,b \in IN$. Then, K has an element of order 15, say z, since K is an abelian group. We have $\varphi(|z|) \mid [N_G(z):K]$ so $8 \mid [N_G(z):K]$ since $K \le C_G(z) \le N_G(z)$ and Corollary 2.2). Thus, $G = N_G(z)$ since $[G:K]=|H|=8$ by Frobenius Theorem. Moreover, we have $K = C_G(z)$ since $[G:K] = 8 = \varphi(|z|) = [N_G(z):C_G(z)] = [G:C_G(z)]$ and $K \le C_G(z)$. Therefore, we have $Q_8 \cong G/K = N_G(z)/C_G(z) \cong Aut(z)$. This is a contradiction because $Aut(z)$ is an abelian but Q_8 is not.

Conclusion. Let G be a Frobenius Q-group with kernel K and complement H.

1) If $H \cong Z_2$, then K is an elemantary abelian 3-group by Theorem 7. and for every $t \in K$, $t^u = t^{-1}$ where $1 \ne u \in H$.

2) If $H \cong Q_8$, then K is an elemantary abelian p-group where $p=3$ or $p=5$ by Theorem 7. Since $K \lhd G$, H acts on K by conjugation. Thus, K may be considered as a Z_pH-module, so K defines a representation of H over the field Z_p. Since $p \nmid |H|$, we can use ordinary representation theory, so K is a direct sum of some irreducible modules of group ring Z_pH by Maschke's Theorem and Wedderburn's Theorem. H has exactly five non-isomorphic irreducible module over the field Z_p and four of them are 1-dimensional so the other is 2-dimensional. In addition, for every 1-dimensional representation χ of H, $\chi(u) = 1 \in Z_p$ where u is the involusion of H. Thus, K must be a direct sum of copies of the 2-dimensional irreducible modules of group ring Z_pH since H is a regular group of automorphism of K. Also, if $p=5$, K is 2-dimensional irreducible module of Z_5H by [3, p.36, Corollary 36. 2) b)]. Finally, any Frobenius Q-group is one of the groups of our main theorem. Conversely, we can see

easily that all of the groups in our main theorem are indeed Frobenius groups. Moreover, the fact that these groups are Q-groups can be verified by using the [3, p.80, Corollary 80].

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