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On the Scattering Data of Sturm-Liouville Problem with a Spectral Parameter in the Boundary Condition

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Abstract

In this paper, it is given the definition of the scattering data and the many properties of scattering data are investigated for the differential equation

$$-y'' + q(x)y = \lambda^2 y$$

on half line containing a spectral parameter in the boundary condition

$$y'(0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)y(0) = 0.$$

Key Words: Sturm-Liouville operator, inverse problem of scattering theory, scattering function, scattering data.

1. INTRODUCTION

On the semiaxis $0 \leq x < \infty$, we consider the boundary-value problem generated by the differential expression

$$-y'' + q(x)y = \lambda^2 y \tag{1}$$

and the boundary condition

$$y'(0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)y(0) = 0 \tag{2}$$

where λ is a spectral parameter and the coefficient $q(x)$ is real-valued function satisfying the condition

$$\int_0^\infty (1+x)\,|q(x)|\,dx < \infty; \tag{3}$$

here the numbers $\alpha_0, \alpha_1$and α_2 are non-negative real numbers.

On the semiaxis, the inverse problem of scattering theory (IPST) of the equation (1) with boundary condition $y(0) = 0$ was solved in [1] and [2]. On the semiaxis, the same problem with boundary condition

$$y'(0) = hy(0)$$

was investigated in [3]. In that, h was any real number and the function $q(x)$ was real-valued function satisfying the condition (3).

The boundary problems with spectral parameter dependent boundary condition for the equations (1) are interesting with their physical applications [10]. Many problems in wave theory of mathematical physics, geophysics and seismology can be reducible to such this problems [4]. The inverse problems of spectral analysis for such that boundary problems with different cases were investigated in [5] and [6]. The inverse problem with respect to the spectral function on semiaxis was solved in [7].

It is well known (see [2]) that the boundary value problem of (1), (2) has bounded solutions $u(\lambda, x)$ for $-\infty < \lambda < \infty$ and $\lambda = i\lambda_k$ $(k = \overline{1,n})$, moreover, as $x \to \infty$

$$u(\lambda, x) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1), \ (-\infty < \lambda < \infty)$$

and

$$u(i\lambda_k, x) = m_k e^{-\lambda_k x}(1 + o(1)), \ (k = \overline{1,n}),$$

where the numbers λ_k, m_k and the function $S(\lambda)$ will be defined later. From the formulas above it follows that the behavior of radial wave functions $u(\lambda, x)$ at infinity is determined by the collection

$$\left\{S(\lambda) \ (-\infty < \lambda < \infty); \lambda_k; \ m_k \ (k = \overline{1,n}) \ \right\}.$$

The collection

$$\left\{S(\lambda) \ (-\infty < \lambda < \infty); \lambda_k; \ m_k \ (k = \overline{1,n}) \ \right\}$$

is called the scattering data of the boundary problem (1), (2). The inverse problem of scattering theory for the boundary problem (1), (2) is uniquely construction the boundary problem (1), (2) with respect to the scattering data, and the theorem of uniqueness was shown in [8]. The continuity of the scattering function on the real axis and Levinson's type formula were obtained in [9].

In this paper it is investigated the properties of scattering data. Put

$$\sigma(x) \equiv \int_x^\infty |q(t)|\, dt, \quad \sigma_1(x) \equiv \int_x^\infty \sigma(t)dt.$$

It can be easily shown(see [2]) that, for all λ from the closed upper half-plane the equation (1) has the solution $e(\lambda, x)$ that can be expressed as

$$e(\lambda, x) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt. \tag{4}$$

Moreover, the kernel function $K(x,t)$ satisfies the relations

$$|K(x,t)| \leq \frac{1}{2}\sigma(\frac{x+t}{2}) \exp\left\{\sigma_1(x) - \sigma_1(\frac{x+t}{2})\right\}, \tag{5}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)dt, \tag{6}$$

and $K(x,t)$ has partial derivatives with respect to both of variables satisfying the inequality

$$\left|\frac{\partial K(x_1,x_2)}{\partial x_j} + \frac{1}{4}q(\frac{x_1+x_2}{2})\right| \leq \frac{1}{2}\sigma(x_1)\sigma(\frac{x_1+x_2}{2}) \exp\left\{\sigma_1(x_1) - \sigma_1(\frac{x_1+x_2}{2})\right\}. \tag{7}$$

In particular, the function $e(\lambda, x)$ is an analytic function of λ on the upper half-plane and is continuous on the real axis. In the upper half-plane $\operatorname{Im}\lambda \geq 0$, it is satisfied the following inequality

$$|e(\lambda,x)| \leq \exp\left\{-\operatorname{Im}\lambda x + \sigma_1(x)\right\}, \tag{8}$$

$$\left|e(\lambda,x) - e^{i\lambda x}\right| \leq \left\{\sigma_1(x) - \sigma_1(x + \frac{1}{|\lambda|})\right\} \exp\left\{-\operatorname{Im}\lambda x + \sigma_1(x)\right\}, \tag{9}$$

$$\left|e'(\lambda,x) - i\lambda e^{i\lambda x}\right| \leq \sigma(x)\exp\left\{-\operatorname{Im}\lambda x + \sigma_1(x)\right\}. \tag{10}$$

The functions $e(\lambda, x)$ and $e(-\lambda, x)$ are form a fundamental system of solutions of equation (1) for any real $\lambda \neq 0$, and their Wronskian equals to $2i\lambda$:

$$W\left\{e(\lambda,x), e(-\lambda,x)\right\} = e'(\lambda,x)e(-\lambda,x) - e(\lambda,x)e'(-\lambda,x) = 2i\lambda. \tag{11}$$

2. MAIN RESULTS

By $\omega(\lambda, x)$ we denote the solution of the equation (1) satisfying the initial data

$$\omega(0, x) = 1, \quad \omega'_x(\lambda, 0) = -(\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2).$$

Theorem 1 *For all real numbers of $\lambda \neq 0$, the following identity is valid:*

$$\frac{2i\lambda\omega(\lambda, x)}{e'(\lambda, 0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda, 0)} = \overline{e(\lambda, x)} - S(\lambda)e(\lambda, x), \tag{12}$$

where

$$S(\lambda) = \frac{\overline{e'(\lambda, 0)} + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)\overline{e(\lambda, 0)}}{e'(\lambda, 0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda, 0)},$$

with

$$\overline{S(\lambda)} = S(-\lambda).$$

Proof. Since two function $e(-\lambda, x)$ and $e(\lambda, x)$ form a fundamental system of solutions to equation (1) for all $\lambda \neq 0$, we can write

$$\omega(\lambda, x) = c_1(\lambda)e(\lambda, x) + c_2(\lambda)\overline{e(\lambda, x)}$$

where

$$c_1(\lambda) = \frac{\overline{e'(\lambda, 0)} + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)\overline{e(\lambda, 0)}}{-2i\lambda},$$

and

$$c_2(\lambda) = \frac{e'(\lambda, 0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda, 0)}{2i\lambda}.$$

Then

$$\begin{aligned} \omega(\lambda, x) &= \frac{\overline{e'(\lambda, 0)} + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)\overline{e(\lambda, 0)}}{-2i\lambda} e(\lambda, x) + \\ &+ \frac{e'(\lambda, 0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda, 0)}{2i\lambda} \overline{e(\lambda, x)}. \end{aligned} \tag{13}$$

Let $E(\lambda,0) = e'(\lambda,0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda,0)$. Since $q(x)$ is real, it follows that $e(-\lambda,0) = \overline{e(\lambda,0)}$, and hence that $E(\lambda,0) \neq 0$ for all real $\lambda \neq 0$. To prove this we assume that there is a non-zero $\lambda_0 \in (-\infty,\infty)$ such that

$$E(\lambda_0,0) = e'(\lambda_0,0) + (\alpha_0 + i\alpha_1\lambda_0 + \alpha_2\lambda_0^2)e(\lambda_0,0) = 0$$

or

$$e'(\lambda_0,0) = -(\alpha_0 + i\alpha_1\lambda_0 + \alpha_2\lambda_0^2)e(\lambda_0,0).$$

From the formula (10) we get

$$e'(\lambda_0,0)\overline{e(\lambda_0,0)} - e(\lambda_0,0)\overline{e'(\lambda_0,0)} = 2i\lambda_0$$

or

$$-2i\alpha_1\lambda_0 \left|e(\lambda_0,0)\right|^2 = 2i\lambda_0.$$

Since $\alpha_1 \geq 0$ we have a contradiction, hence we obtain that $E(\lambda,0) \neq 0$ for all real $\lambda \neq 0$.

According to the formula (13) we obtain

$$\frac{2i\lambda\omega(\lambda,x)}{e'(\lambda,0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda,0)} = \overline{e(\lambda,x)} - S(\lambda)e(\lambda,x)$$

where

$$S(\lambda) = \frac{\overline{e'(\lambda,0)} + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)\overline{e(\lambda,0)}}{e'(\lambda,0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e(\lambda,0)}$$

and

$$\overline{E(\lambda,0)} = e'(-\lambda,0) + (\alpha_0 - i\alpha_1\lambda + \alpha_2\lambda^2)e(-\lambda,0) = E(-\lambda,0),$$

$$S(\lambda) = \overline{S(-\lambda)}.$$

This proves the theorem. ∎

The function $S(\lambda)$ is called the scattering function of the boundary problem (1), (2).

Theorem 2 *The function $E(\lambda,0)$ may have only a finite number of zeros on the half plane* $\operatorname{Im}\lambda > 0$, *they are all simple and lie on the imaginary axis. The function $\lambda\left[E(\lambda,0)\right]^{-1}$ is bounded in any neighborhood of the point $\lambda = 0$.*

Proof. The proof of the first part of the theorem was proven in [8]. Now, we shall prove that the function $\lambda\left[E(\lambda,0)\right]^{-1}$ is bounded in $D_\rho = \{\lambda \mid |\lambda| \leq \rho, \operatorname{Im}\lambda \geq 0\}$ for sufficiently small numbers ρ. If $E(0,0) = e'(0,0) + \alpha_0 e(0,0) \neq 0$, it is clearly that this function is bounded in $D\rho$.

We assume that $E(0,0) = e'(0,0) + \alpha_0 e(0,0) = 0$. We let δ denote the infimum of the distance between two neighboring zeros of the function $E(\lambda,0)$, and show next that $\delta > 0$. Otherwise, we could exhibit a sequence of zeros, $\left\{i\widetilde{\lambda}_k\right\}$ and $\{i\lambda_k\}$ of the zeros the function $E(\lambda,0)$, such that $\lim\limits_{k\to\infty}(\widetilde{\lambda}_k - \lambda_k) = 0$, $\widetilde{\lambda}_k > \lambda_k \geq 0$, and $\max\limits_k \widetilde{\lambda}_k < M$. Then it follows from the estimate (9) that, for A large enough, the inequality $e(i\lambda,x) > \frac{1}{2}e^{-\lambda x}$ holds uniformly with respect to $x \in [A,\infty)$ and $\lambda \in [0,\infty)$, whence

$$\int\limits_A^\infty e(i\widetilde{\lambda}_k,x)\overline{e(i\lambda_k,x)}dx > \frac{e^{-2AM}}{8M}. \tag{14}$$

On the other hand, according to the formula (11) in [8] we have

$$\begin{aligned} 0 &= (\lambda_k - \widetilde{\lambda}_k)\int\limits_0^\infty e(i\widetilde{\lambda}_k,x)\overline{e(i\lambda_k,x)}dx + \alpha_2((\lambda_k - \widetilde{\lambda}_k)\cdot \\ &\quad \cdot e(i\widetilde{\lambda}_k,0)\overline{e(i\lambda_k,0)} + \alpha_1 e(i\widetilde{\lambda}_k,0)\overline{e(i\lambda_k,0)}. \end{aligned} \tag{15}$$

Letting $k \to \infty$, we get

$$\alpha_1 \left|e(i\lambda_k,0)\right|^2 = 0.$$

This relation gives a contradiction if $\alpha_1 > 0$. If $\alpha_1 = 0$, from the formula (15) we write

$$\begin{aligned} 0 &= \int\limits_0^\infty e(i\widetilde{\lambda}_k,x)\overline{e(i\lambda_k,x)}dx + e(i\widetilde{\lambda}_k,0)e(i\lambda_k,0) = \\ &= \int\limits_0^A e(i\widetilde{\lambda}_k,x)\left[\overline{e(i\lambda_k,x) - e(i\widetilde{\lambda}_k,x)}\right]dx + \int\limits_0^A e(i\widetilde{\lambda}_k,x)\overline{e(i\widetilde{\lambda}_k,x)}dx + \\ &+ \int\limits_A^\infty e(i\widetilde{\lambda}_k,x)\overline{e(i\lambda_k,x)}dx + e(i\widetilde{\lambda}_k,0)e(i\lambda_k,0). \end{aligned}$$

Taking limit as $k \to \infty$, we get

$$0 \geq \varliminf_{k\to\infty} \int_A^\infty e(i\widetilde{\lambda}_k, x)\overline{e(i\lambda_k, x)}dx.$$

But this contradict with the inequality (14). Thus, the assumption is not true, so that $\delta > 0$.

If $\rho < \frac{1}{2}\delta$, the function $E(\lambda, 0)$ has not other zeros in the half-disc D_ρ. So, the function

$$\phi(\lambda, x) = \frac{\lambda\omega(\lambda, x)}{E(\lambda, 0)}$$

is regular in the interior of the half-disc D_ρ and is bounded in the half-circle $\{\lambda \mid |\lambda| = \rho, \operatorname{Im}\lambda \geq 0\}$. According to the properties of the function $\phi(\lambda, x)$ and formula (12), we have the function $\phi(\lambda, x)$ is uniformly bounded for $-\rho \leq \lambda \leq \rho$. However, since it is not shown that the function $\phi(\lambda, x)$ is continuous as $\lambda \to \infty$, we can't use the maximum principle to prove this function is bounded in D_ρ. For this we consider the equations family

$$-y'' + q_\beta(x)y = \lambda^2 y$$

where

$$q_\beta(x) = \begin{cases} q(x) & ,\text{if } x < \beta \\ 0 & ,\text{if } x \geq \beta \end{cases},$$

and the functions $e_\beta(x, x)$, $\omega_\beta(x)$ are corresponding solutions to $e(\lambda, x)$, and $\omega(\lambda, x)$, respectively, and $E_\beta(\lambda, 0)$ is corresponding quantity of them.

If $x < \beta$, it follows that

$$\omega_\beta(\lambda, x) = \omega(\lambda, x),$$

$$\phi_\beta(\lambda, x) = \frac{\lambda\omega_\beta(\lambda, x)}{E_\beta(\lambda, 0)} = \frac{\lambda\omega(\lambda, x)}{e'_\beta(\lambda, 0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)e_\beta(\lambda, 0)}$$

where

$$e_\beta(\lambda, x) = e^{i\lambda x} + \int_x^\infty K_\beta(x, t)e^{i\lambda t}dt,$$

$$e'_\beta(\lambda, x) = i\lambda e^{i\lambda x} - K_\beta(x, \lambda)e^{i\lambda t} + \int_x^\infty \frac{\partial K_\beta(x, t)}{\partial x}e^{i\lambda t}dt.$$

In particular we get $K_\beta(x,t) = 0$ for $x+t > 2\beta$ and

$$\lim_{\beta\to\infty} K_\beta(x,t) = K(x,t),$$

$$\lim_{\beta\to\infty} \frac{\partial K_\beta(x,t)}{\partial x} = \frac{\partial K(x,t)}{\partial x},$$

and from (5), (7), we prove that as uniformly for β

$$|K_\beta(x,t)| \le \frac{1}{2}\sigma(\frac{x+t}{2}) \exp\left\{\sigma_1(x) - \sigma_1(\frac{x+t}{2})\right\},$$

$$\left|\frac{\partial K_\beta(x,t)}{\partial x} + \frac{1}{4}q_\beta(\frac{x+t}{2})\right| \le \frac{1}{2}\sigma(x)\sigma(\frac{x+t}{2}) \exp\left\{\sigma_1(x) - \sigma_1(\frac{x+t}{2})\right\}.$$

Hence, the function $e_\beta(\lambda, x)$ is an exact function of λ and is uniformly $\lim_{\beta\to\infty} e_\beta(\lambda,t) = e(\lambda,x)$ in the upper half-plane $\operatorname{Im}\lambda \ge 0$.

Let δ_β be the infimum distance of two neighboring zeros of the function $E_\beta(\lambda, 0)$. It can be seen that $\inf \delta_\beta = \delta_0 > 0$. The function $E_\beta(\lambda, 0)$ has at most one root for any number β in the half-disc $D_{\rho_0}(\ \rho_0 = 0,5\delta_0)$. We denote this root by $i\lambda_\beta$, if it hasn't any root we assume that $\lambda_\beta = 0$.

Now we construct the function $\phi_\beta(\lambda,x)\frac{\lambda - i\lambda_\beta}{\lambda + i\lambda_\beta}$. This function is meromorphic in the whole plane and is regular in the interior of the half-disc D_{ρ_0}. In the upper closed half-plane we get

$$\left|\frac{\lambda - i\lambda_\beta}{\lambda + i\lambda_\beta}\right| \le 1$$

and the function $E_\beta(\lambda, 0)$ does not vanish on the curve $\{\lambda \mid |\lambda| = \rho_0, \operatorname{Im}\lambda \ge 0\}$,but it is uniformly bounded on this curve for any β. By the formula (8) these functions are uniformly bounded on the interval $-\rho_0 \le \lambda \le \rho_0$. Hence, these functions which is beginning with a number β are regular on D_{ρ_0} and ∂D_{ρ_0}, and it holds

$$\sup_{\lambda\in\partial D_{\rho_0}} \left|\phi_\beta(\lambda,x)\frac{\lambda - i\lambda_\beta}{\lambda + i\lambda_\beta}\right| = c(x) < \infty.$$

So, according the Maximum Theorem on the modulo of regular functions we have

$$\sup_{\lambda\in D_{\rho_0}} \left|\phi_\beta(\lambda,x)\frac{\lambda - i\lambda_\beta}{\lambda + i\lambda_\beta}\right| = c(x) < \infty.$$

Since $\lim\limits_{\beta\to\infty} \lambda_\beta = 0$ and $\lim\limits_{\beta\to\infty} \phi_\beta(\lambda, x) = \phi(\lambda, x)$, taking limit in the last inequality we obtain

$$\sup_{\lambda\in D_{\rho_0}} |\phi(\lambda, x)| \leq c(x).$$

But, for sufficiently small $x = x_0$ we have

$$\inf_{\lambda\in D_{\rho_0}} |\omega(\lambda, x)| > \frac{1}{2}x_0$$

and so,

$$\sup_{\lambda\in D_{\rho_0}} \left|\lambda \left[E(\lambda, 0)\right]^{-1}\right| \leq \frac{2c(x_0)}{x_0} < \infty.$$

The theorem is proved. ■

Theorem 3 *The function $1 - S(\lambda)$ is the Fourier transform of a function $F_S(x)$*

$$1 - S(\lambda) = \int_{-\infty}^{\infty} F_S(t)e^{-\lambda t}dt$$

and $F_S(x)$ of the form

$$F_S(x) = F_S^{(1)}(x) + F_S^{(2)}(x),$$

where $F_S^{(1)}(x) \in L_1(-\infty, \infty)$, whereas $F_S^{(2)}(x) \in L_2(-\infty, \infty)$ and $F_S^{(2)}(x)$ is bounded in whole axis.

Proof. From the formula (4) it follows that

$$e(\lambda, 0) = 1 + \int_0^{\infty} K(0, t)e^{i\lambda t}dt,$$

$$e'(\lambda, 0) = i\lambda - K(0, 0) + \int_0^{\infty} K_x(0, t)e^{i\lambda t}dt.$$

We shall use the following notations for shortly:

$$q_0 = K(0, 0); \quad \varphi_0(\lambda) = \alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2 - q_0;$$

$$K_1(t) = K_x(0,t) + \alpha_0 K(0,t); \quad K_2(t) = \alpha_1 K(0,t) + \alpha_2 K_t(0,t);$$
$$K_3(t) = \alpha_1 K(0,t) - \alpha_2 K_t(0,t)$$

and

$$\widetilde{K}_j(-\lambda) = \int_0^\infty K_j(t) e^{i\lambda t} dt, \quad j = 1,2,3.$$

Then, we have

$$1 - S(\lambda) = \frac{2i\lambda(1+\alpha_2 q_0) + \widetilde{K}_1(-\lambda) + i\lambda\widetilde{K}_2(-\lambda) - \widetilde{K}_1(\lambda) - i\lambda\widetilde{K}_3(\lambda)}{\varphi_0(\lambda) + i\lambda(1+\alpha_2 q_0) + \widetilde{K}_1(-\lambda) + i\lambda\widetilde{K}_2(-\lambda)}. \quad (16)$$

Every one of the functions

$$\widetilde{f}_1(\lambda) = \frac{i\lambda(1+\alpha_2 q_0)}{\varphi_0(\lambda)}, \quad \widetilde{f}_2(\lambda) = \frac{\widetilde{K}_1(-\lambda) + i\lambda\widetilde{K}_2(-\lambda)}{\varphi_0(\lambda)},$$
$$\widetilde{f}_3(\lambda) = \frac{\widetilde{K}_1(\lambda) + i\lambda\widetilde{K}_3(\lambda)}{\varphi_0(\lambda)}.$$

is the Fourier transformation of a summable function. By the simple transformations, it follows that the right side of the inequality (16) equals to the assertion

$$\frac{2\widetilde{f}_1(\lambda) + \widetilde{f}_2(\lambda) - \widetilde{f}_3(\lambda)}{1 + \widetilde{f}_1(\lambda) + \widetilde{f}_2(\lambda)}.$$

Hence we have

$$1 - S(\lambda) = \frac{\widetilde{f}(\lambda)}{1 + \widetilde{K}(-\lambda)}$$

where the functions

$$\widetilde{f}(\lambda) = 2\widetilde{f}_1(\lambda) + \widetilde{f}_2(\lambda) - \widetilde{f}_3(\lambda),$$
$$\widetilde{K}(-\lambda) = \widetilde{f}_1(\lambda) + \widetilde{f}_2(\lambda)$$

are the Fourier transformation of the summable functions.

We can rewrite the formula (16) as form

$$1 - S(\lambda) = \widetilde{f}(\lambda)\left[\left\{1 + (1 - \widetilde{h}(\lambda N^{-1}))\widetilde{K}(-\lambda)\right\}^{-1} - 1\right] + +\widetilde{f}(\lambda) -$$
$$-\widetilde{f}(\lambda)\left\{\frac{1}{1 + \left\{1 - \widetilde{h}(\lambda N^{-1})\right\}\widetilde{K}(-\lambda)} - \frac{1}{1 + \widetilde{K}(-\lambda)}\right\} \quad (17)$$

where

$$\widetilde{h}(\lambda) = \begin{cases} 1, & if \ |\lambda| < 1 \\ 2 - |\lambda|, & if \ 1 \le \lambda \le 2 \\ 0, & if \ |\lambda| > 2 \end{cases}$$

is the Fourier transform of the function $h(x) \in L_1(-\infty, \infty)$. Also, $\widetilde{h}(\lambda N^{-1})$ is the Fourier transform of the function $h_N(x) = Nh(xN)$, and for all $f(x) \in L_1(-\infty, \infty)$ it holds

$$\lim_{N \to \infty} \|f(x) - h_N * f(x)\|_{L_1} = 0 \tag{18}$$

where $h_N * f(x)$ is the convolution of functions $h_N(x)$ and $f(x)$ from $L_1(-\infty, \infty)$. Note that the convolution $h_N * f(x)$ of functions $h_N(x)$ and $f(x)$ from $L_1(-\infty, \infty)$ is defined as

$$h_N * f(x) = \int_{-\infty}^{\infty} h_N(x-t) f(t) dt.$$

In general, recall that the Fourier transform of the convolution

$$f * g(x) = \int_{-\infty}^{\infty} f(x-t) g(t) dt$$

of two functions from $L_1(-\infty, \infty)$ equals the product $\widetilde{f}(\lambda)\widetilde{g}(\lambda)$ of their Fourier transforms, and the norm of the convolution does not exceed the product of norms:

$$\|f * g\|_{L_1} \le \|f\|_{L_1} \|g\|_{L_1}.$$

Consequently, if $\|f\|_{L_1} < 1$, then the series

$$-f(x) + f * f(x) - f * f * f(x) + ...$$

converges in the metric of $L_1(-\infty, \infty)$, its sum belongs to this space and its Fourier transform is equal to

$$-\widetilde{f}(\lambda) + \left\{\widetilde{f}(\lambda)\right\}^2 - \left\{\widetilde{f}(\lambda)\right\}^3 + ... = \left\{1 + \widetilde{f}(\lambda)\right\}^{-1} - 1.$$

We conclude, from (18) and the previous argument that for N large enough, the function

$$\left[1 + \left\{1 - \widetilde{h}(\lambda N^{-1})\right\} \widetilde{K}(-\lambda)\right]^{-1} - 1$$

is the Fourier transform of function from $L_1(-\infty,\infty)$. It follows that the sum of the first two terms in the right-hand side of (17) is also the Fourier transform of a summable function $F_S^{(1)}(x) \in L_1(-\infty,\infty)$. Finally, since $\widetilde{h}(\lambda N^{-1}) = 0$ for $|\lambda| > 2N$, the third term in the same formula vanishes for $|\lambda| > 2N$ and is bounded. So, it is the Fourier transform of a bounded function $F_S^{(2)}(x) \in L_2(-\infty,\infty)$, and the theorem is proved. ■

Next using the identity (12), Theorem2 and Theorem3, we obtain (see [8]) Gelfand-Levitan-Marcenko's basic equation which has important role in the solution of IPST

$$F(x+y) + K(x,y) + \int_x^\infty K(x,t)F(t+y)dt = 0, \quad (x < y < \infty) \tag{19}$$

where

$$F(x) = \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}(1 - S(\lambda))e^{i\lambda x}dx \tag{20}$$

and

$$m_k^{-2} = \int_0^\infty |e(x,i\lambda_k)|^2\,dx + \frac{\alpha_1 + 2\alpha_2\lambda_k}{2\lambda_k}\,|e(0,i\lambda_k)|^2 .$$

The numbers m_k $(k = \overline{1,n})$ are called the norming constants of the boundary problem (1), (2). The collection $\{S(\lambda)\ (-\infty < \lambda < \infty); \lambda_k;\ m_k\ (k = \overline{1,n})\ \}$ which is defined above is called the scattering data of the boundary problem (1), (2). From the formula (20) we can determine the function $F(x)$ with respect to the scattering data $\{S(\lambda)\ (-\infty < \lambda < \infty); \lambda_k;\ m_k\ (k = \overline{1,n})\ \}$, and we obtain the basic equation (19). As it was shown in [8], the basic equation (19) has uniquely solution for every $x \geq 0$. Solving this equation we can find the kernel function $K(x,y)$ of the solution (4) and by formula (6) we can construct the potential $q(x)$.

Using the basic formula (19) we can get the some properties of the function $F(x)$. According to continuity of the kernel function $K(x,y)$, the function $F(x)$ is continuous for every $x \in [0,\infty)$, and hence the basic equation(19) holds for $y = x$, too. As in [2], it can be shown that, for all $x \geq 0$, the function $F(x)$ is a derivative function and holds the inequalities

$$|F(2x)| \leq \frac{1}{2}\sigma(x)ch\sigma_1(x)e^{\sigma_1(x)},$$

$$\left| F'(2x) - \frac{1}{4}q(x) + \frac{1}{4}\left\{ \int_x^\infty q(x)dt \right\}^2 \right| \le \sigma^2(x)e^{\sigma_1(x)} sh\sigma_1(x). \qquad (21)$$

Since the functions $(1+x)\,|q(x)|$ and $\sigma(x)$ are summable on $[0,\infty)$, $\sup(1+x)\sigma(x) < \infty$, and from the inequality (21) we have $(1+x)\,|F'(x)| \in L_1\,[0,\infty)$.

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