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# On $m$-$D$-Separation Axioms

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**Abstract**

We introduce the notions of $m$-$D$-sets and some lower separation axioms $m$-$D_i$ $(i = 0, 1, 2)$ on $m$-structures, which are weaker than topological structures, and obtain a unified theory of separation axioms $D_i$, $s$-$D_i$, $p$-$D_i$, $\theta$-$D_i$, $\delta$-$semiD_i$, $\delta$-$preD_i$ $(i = 0, 1, 2)$ in topological spaces.

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## 1 Introduction

In 1982, Tong [28] introduced the notion of $D$-sets and used these sets to introduce a separation axiom $D_1$ which is strictly between $T_0$ and $T_1$. In 1975, Maheshwari and Prasad [19] introduced new separation axioms semi-$T_0$, semi-$T_1$ and semi-$T_2$ by using semi-open sets due to Levine [17]. Borşan [4] and Caldas [5] introduced the notions of $s$-$D$-sets and a separation axiom $s$-$D_1$ which is strictly between semi-$T_0$ and semi-$T_1$. In 1990, Kar and Bhattachryya [16] introduced new separation axioms pre-$T_0$, pre-$T_1$ and pre-$T_2$ by using preopen sets due to Mashhour et al. [21]. Recently, Caldas [6] and Jafari [15] introduced independently the notions of $p$-$D$-sets and a separation axiom $p$-$D_1$ which is strictly between pre-$T_0$ and pre-$T_1$. Quite recently, Caldas, Fukutake, Georgiou, Jafari and Noiri introduced the notions of $\theta$-$D$-sets and a separation axiom $\theta$-$D_1$ [8], $\delta$-semi$D$-sets and a separation axiom $\delta$-semi$D_1$ [9], and $\delta$-pre$D$-sets and a separation axiom $\delta$-pre$D_1$ [7].

In this paper, we define an $m$-space $(X, m)$, where $X$ is a nonempty set and $m$ is a subfamily of the power set of $X$ and also satisfies the following conditions: $\emptyset, X \in m$. By using the $m$-spaces, we define the notions of $m$-$D$-sets and separation axioms $m$-$D_i$ ($i = 0, 1, 2$) and establish a unified theory of separation axioms $D_i$, $s$-$D_i$, $p$-$D_i$, $\theta$-$D_i$, $\delta$-*semi*$D_i$, $\delta$-*pre*$D_i$ ($i = 0, 1, 2$) in topological spaces.

# 2 Preliminaries

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ denote topological spaces. Let $A$ be a subset of $X$. We denote the closure and the interior of $A$ by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. The $\theta$-closure (resp. $\delta$-closure) of $A$, $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$), is defined by the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ (resp. $A \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \emptyset$) for every open set $U$ containing $x$. A subset $A$ is said to be *$\theta$-closed* (resp. *$\delta$-closed*) [29] if $A = \mathrm{Cl}_\theta(A)$ (resp. $A = \mathrm{Cl}_\delta(A)$). The complement of a $\theta$-closed (resp. $\delta$-closed) set is said to be *$\theta$-open* (resp. *$\delta$-open*).

**Definition 2.1** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be
(1) *semi-open* [17] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(2) *preopen* [21] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(3) *$\alpha$-open* [24] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(4) *$\delta$-semiopen* [25] if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$,
(5) *$\delta$-preopen* [27] if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$.

The family of all semi-open (resp. preopen, $\alpha$-open, $\delta$-semiopen, $\delta$-preopen, $\theta$-open, $\delta$-open) sets in a topological space $(X, \tau)$ is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$ or $\tau^\alpha$, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$, $\tau_\theta$, $\tau_\delta$).

**Definition 2.2** The complement of a semi-open (resp. preopen, $\alpha$-open, $\delta$-semiopen, $\delta$-preopen) set is said to be *semi-closed* [11] (resp. *preclosed* [21], *$\alpha$-closed* [23]), *$\delta$-semiclosed* [25], *$\delta$-preclosed* [27]).

**Definition 2.3** The intersection of all semi-closed (resp. preclosed, $\alpha$-closed, $\delta$-semiclosed, $\delta$-preclosed) sets of $(X, \tau)$ containing a subset $A$ is called the *semi-closure* [11] (resp. *preclosure* [14], *$\alpha$-closure* [23], *$\delta$-semiclosure* [25], *$\delta$-preclosure* [27]) and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{sCl}_\delta(A)$, $\mathrm{pCl}_\delta(A)$).

**Definition 2.4** The union of all semi-open (resp. preopen, $\alpha$-open, $\delta$-semiopen, $\delta$-preopen) sets of $X$ contained in $A$ is called the *semi-interior* (resp. *preinterior*, *$\alpha$-interior*, *$\delta$-semiinterior*, *$\delta$-preinterior*) of $A$ and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{sInt}_\delta(A)$, $\mathrm{pInt}_\delta(A)$).

If we replace open sets in the usual definition of $T_i$ ($i = 0, 1, 2$) with semi-open (resp. preopen, $\theta$-open, $\delta$-semiopen, $\delta$-preopen) sets, we obtain separation axioms *s-$T_i$* [19] (resp. *p-$T_i$* [16], *$\theta$-$T_i$* [10], *$\delta$-semi$T_i$* [9], $(\delta, p)$-$T_i$ [7]) ($i = 0, 1, 2$).

**Definition 2.5** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is called a *D-set* [28] (resp. *s-D-set* [4], [5], *p-D-set* [15], [6], *$\theta$-D-set* [10], *$\delta$-semiD-set*[9], *$\delta$-preD-set* [7]) if there exist two open (resp. semi-open, preopen, $\theta$-open, $\delta$-semiopen, $\delta$-preopen) sets $U, V$ in $X$ such that $U \neq X$ and $A = U - V$.

If we replace open sets in the usual definitions of $T_0, T_1, T_2$ with $D$-sets (resp. $s$-$D$-sets, $p$-$D$-sets, $\theta$-$D$-sets, $\delta$-semi$D$-sets, $\delta$-pre$D$-sets), we obtain the definitions of separation axioms $D_i$ [28] (resp. $s$-$D_i$ [4], [5], $p$-$D_i$ [15], [6], $\theta$-$D_i$ [10], *$\delta$-semi$D_i$* [9], $(\delta, p)$-$D_i$ [7]) for $i = 0, 1, 2$.

We shall begin with the definition of $m$-spaces in order to establish a unified theory of separation axioms $D_i$, $s$-$D_i$, $p$-$D_i$, $\theta$-$D_i$, *$\delta$-semi$D_i$*, $(\delta, p)$-$D_i$ for $i = 0, 1, 2$.

**Definition 2.6** A subfamily $m$ of the power set $\mathcal{P}(X)$ of a nonempty set $X$ is called an *minimal structure* (briefly *$m$-structure*) on $X$ if $m$ satisfies the following properties: $\emptyset \in m$ and $X \in m$.

We call the pair $(X, m)$ an $m$-space. Each member of $m$ is said to be *$m$-open* and the complement of an $m$-open set is said to be *$m$-closed*.

**Definition 2.7** An minimal structure $m$ on a nonempty set $X$ is said to have property $(\mathcal{B})$ [20] if the union of any family of subsets belonging to $m$ belongs to $m$.

**Remark 2.1** An $m$-structure with the property $(\mathcal{B})$ is called a *generalized topology* by Lugojan [18]. Császár [12] called a family $m$ a *generalized topology* if it satisfies $\emptyset \in m$ and has the property $(\mathcal{B})$. Mashhour et al.

[22] called a family $m$ *supratopology* if it satisfies $X \in m$ and has the property $(\mathcal{B})$. In the present paper, we do not always assume the property $(\mathcal{B})$ on $m$-structures.

**Remark 2.2** Let $(X, \tau)$ be a topological space. Then the families $\tau_\theta, \tau_\delta, \tau$, SO$(X)$ PO$(X)$, $\alpha(X)$, $\delta$SO$(X)$, $\delta$PO$(X)$ are all $m$-structures on $X$ with the property $(\mathcal{B})$. It is well-known that $\tau_\theta, \tau_\delta, \alpha(X)$ are topologies for $X$ and the other are not topologies.

**Definition 2.8** Let $(X, m)$ be an $m$-space and $A$ a subset of $X$. The *$m$-closure* mCl$(A)$ of $A$ and *$m$-interior* mInt$(A)$ of $A$ are defined in [20] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F \text{ and } X - F \in m\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A \text{ and } U \in m\}$.

**Remark 2.3** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m = \tau$ (resp. SO$(X)$, PO$(X)$, $\delta$SO$(X)$, $\delta$PO$(X)$, $\alpha(X)$, $\tau_\theta, \tau_\delta$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{sCl}_\delta(A)$, $\mathrm{pCl}_\delta(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{Cl}_\theta(A)$, $\mathrm{Cl}_\delta(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{sInt}_\delta(A)$, $\mathrm{pInt}_\delta(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{Int}_\theta(A)$, $\mathrm{Int}_\delta(A)$).

**Lemma 2.1** (Maki [20]) *Let $(X, m)$ be an $m$-space and $A$, $B$ subsets of $X$. Then the following properties hold:*

*(1)* $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

*(2)* $\mathrm{mCl}(\emptyset) = \emptyset, \mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

*(3) If $A \subset B$, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

*(4)* $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

*(5)* $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

**Lemma 2.2** (Popa and Noiri [26]) *Let $(X, m)$ be an $m$-space, $A$ a subset of $X$ and $x \in X$. Then $x \in$ mCl$(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing $x$.*

**Lemma 2.3** (Popa and Noiri [26]) *Let $(X, m)$ be an $m$-space and $m$ have the property $(\mathcal{B})$. Then for a subset $A$ of $X$, the following properties hold:*

*(1) $A \in m$ if and only if $A =$ mInt$(A)$,*

*(2) $A$ is $m$-closed if and only if $A =$ mCl$(A)$,*

*(3)* mCl$(A)$ *is $m$-closed and* mInt$(A)$ *is $m$-open.*

**Definition 2.9** A function $f : (X, m_X) \to (Y, m_Y)$, where $(X, m_X)$ and $(Y, m_Y)$ are $m$-spaces, is said to be *M-continuous* [26] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$.

**Lemma 2.4** (Popa and Noiri [26]) *Let $(X, m_X)$ be an m-space and $m$ have the property ($\mathcal{B}$). For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

*(1) $f$ is M-continuous;*

*(2) $f^{-1}(V) \in m_X$ for every $V \in m_Y$;*

*(3) $f^{-1}(K)$ is $m_X$-closed for every $m_Y$-closed set $K$ of $Y$.*

# 3 $m$-$D$-sets

**Definition 3.1** An $m$-space is said to be

(1) $m$-$T_0$ if for any pair of distinct points $x, y$ of $X$, there exists an $m$-open set of $X$ containing $x$ but not $y$ or an $m$-open set of $X$ containing $y$ but not $x$,

(2) $m$-$T_1$ if for any pair of distinct points $x, y$ of $X$, there exist an $m$-open set of $X$ containing $x$ but not $y$ and an $m$-open set of $X$ containing $y$ but not $x$,

(3) $m$-$T_2$ [26] if for any pair of distinct points $x, y$ of $X$, there exist $m$-open sets $U, V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Definition 3.2** A subset $A$ of an $m$-space $(X, m)$ is called an *$m$-$D$-set* if there exist two $m$-open sets $U$ and $V$ such that $U \neq X$ and $A = U - V$.

Every $m$-open set different from $X$ is an $m$-$D$-set since we can take as follows: $A = U$ and $V = \emptyset$.

**Definition 3.3** An $m$-space is said to be

(1) $m$-$D_0$ if for any pair of distinct points $x, y$ of $X$, there exists an $m$-$D$-set of $X$ containing $x$ but not $y$ or an $m$-$D$-set of $X$ containing $y$ but not $x$,

(2) $m$-$D_1$ if for any pair of distinct points $x, y$ of $X$, there exist an $m$-$D$-set of $X$ containing $x$ but not $y$ and an $m$-$D$-set of $X$ containing $y$ but not $x$,

(3) $m$-$D_2$ if for any pair of distinct points $x, y$ of $X$, there exist $m$-$D$-sets $U, V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Remark 3.1** Let $(X, \tau)$ be a topological space. If $m = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\tau_\theta$, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$), then we obtain the definitions of separation axioms $D_i$ [28] (resp. $s$-$D_i$ [4], [5], $p$-$D_i$ [15], [6], $\theta$-$D_i$ [10], $\delta$-$semiD_i$ [9], $(\delta, p)$-$D_i$ [7]) for $i = 0, 1, 2$.

**Remark 3.2** Let $(X, m)$ be an $m$-space. By Definitions 3.1 and 3.3, we have the following diagram:

$$\begin{array}{ccccc} m\text{-}T_2 & \Rightarrow & m\text{-}T_1 & \Rightarrow & m\text{-}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ m\text{-}D_2 & \Rightarrow & m\text{-}D_1 & \Rightarrow & m\text{-}D_0 \end{array}$$

First, we obtain characterizations of $m$-$T_0$-spaces and $m$-$T_1$-spaces.

**Theorem 3.1** *An $m$-space $(X, m)$ is $m$-$T_0$ if and only if for any pair of distinct points $x, y \in X$,* $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$.

**Proof.** *Necessity.* Let $x, y$ be distinct points of $X$. There exists (1) an $m$-open set $U$ such that $x \in U$ and $y \notin U$ or (2) an $m$-open set $V$ such that $y \in V$ and $x \notin V$. In case (1), $x \notin \mathrm{mCl}(\{y\})$ and hence $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$. In case (2), $y \notin \mathrm{mCl}(\{x\})$ and hence $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$.

*Sufficiency.* Suppose that $x, y \in X, x \neq y$ and $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$. Let $z$ be a point of $X$ such that $z \in \mathrm{mCl}(\{x\})$ and $z \notin \mathrm{mCl}(\{y\})$. Since $z \notin \mathrm{mCl}(\{y\})$, there exists $V \in m$ such that $z \in V$ and $y \notin V$. Since $z \in \mathrm{mCl}(\{x\})$, we have $x \in V$. In case that $z$ is a point of $X$ such that $z \in \mathrm{mCl}(\{y\})$ and $z \notin \mathrm{mCl}(\{x\})$, we obtain an $m$-open set U such that $x \notin U$ and $y \in U$. This shows that $(X, m)$ is $m$-$T_0$.

**Theorem 3.2** *Let $(X, m)$ be an $m$-space and $m$ have the property ($\mathcal{B}$). Then $(X, m)$ is $m$-$T_1$ if and only if for each point $x \in X$, the singleton $\{x\}$ is $m$-closed.*

**Proof.** *Necessity.* Suppose that $(X, m)$ is $m$-$T_1$ and $x$ be any point of $X$. For each point $y \in X - \{x\}$, there exists an $m$-open set $V_y$ such that $y \in V_y$ and $x \notin V_y$; hence $y \in V_y \subset X - \{x\}$. Therefore, we obtain $X - \{x\} = \bigcup_{y \in X - \{x\}} V_y$ and $\bigcup_{y \in X - \{x\}} V_y$ is an $m$-open set and hence $\{x\}$ is $m$-closed.

*Sufficiency.* Let $x, y$ be any distinct points of $X$. Then $y \in X - \{x\}$ and $X - \{x\}$ is an $m$-open set containing $y$. Similarly, $X - \{y\}$ is an $m$-open set containing $x$. This shows that $(X, m)$ is $m$-$T_1$.

**Definition 3.4** An $m$-space $(X, m)$ is said to be *$m$-symmetric* if for points $x, y$ of $X$, $x \in \mathrm{mCl}(\{y\})$ implies that $y \in \mathrm{mCl}(\{x\})$.

**Theorem 3.3** *Let $(X, m)$ be an $m$-space and $m$ have the property ($\mathcal{B}$). For an $m$-space $(X, m)$, the following properties are equivalent:*
*(1) $(X, m)$ is $m$-symmetric and $m$-$T_0$;*
*(2) $(X, m)$ is $m$-$T_1$.*

**Proof.** (2) $\Rightarrow$ (1): This is obvious by Theorem 3.2.
(1) $\Rightarrow$ (2): Let $x, y$ be distinct points of $X$. Since $(X, m)$ is $m$-$T_0$, we may assume that $x \in U$ and $y \notin U$ for some $U \in m$. Then $x \notin \mathrm{mCl}(\{y\})$ and hence $y \notin \mathrm{mCl}(\{x\})$. Therefore, there exists $V \in m$ such that $y \in V$ and $x \notin V$. This shows that $(X, m)$ is $m$-$T_1$.

**Theorem 3.4** *An $m$-space $(X, m)$ is $m$-$D_0$ if and only if it is $m$-$T_0$.*

**Proof.** By the definitions, it is obvious that $m$-$T_0$ implies $m$-$D_0$. We shall show that $m$-$D_0$ implies $m$-$T_0$. Suppose that $(X, m)$ is $m$-$D_0$ and $x$ and $y$ are distinct points of $X$. Then at least one of these points, say x, belongs to an $m$-$D$-open set $A$ and $y \notin A$. Let $A = U - V$, where $U \neq X$ and $V, U$ are $m$-open sets. Since $y \notin A$, we have two cases: (1) $y \notin U$; (2) $y \in U$ and $y \in V$.
In case (1), we have $x \in U$, $y \notin U$ and $U \in m$.
In case (2), we have $y \in V$, $x \notin V$ and $V \in m$.
This shows that $(X, m)$ is $m$-$T_0$.

**Remark 3.3** (1) If $(X, \tau)$ is a topological space and $m = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$), then the result established in Theorem 1 of [28] (resp. Theorem 2.1 of [4] and Theorem 2.5 of [5], Theorem 2.2 of [6] and Theorem 3.1 of [15]) is obtained by Theorem 3.4.
(2) It follows from examples in [28], [4] and [6] that the separation axiom $m$-$D_1$ is strictly between $m$-$T_0$ and $m$-$T_1$.

**Theorem 3.5** *Let $(X, m)$ be an $m$-space and $m$ have the property ($\mathcal{B}$). Then $(X, m)$ is $m$-$D_1$ if and only if it is $m$-$D_2$.*

**Proof.** *Necessity.* Suppose that $(X, m)$ is $m$-$D_1$ and $x$ and $y$ are distinct points of $X$. There exist two $m$-$D$-sets $A_1$ and $A_2$ such that

$x \in A_1, y \notin A_1$ and $y \in A_2, x \notin A_2$. Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$, where $U_i, V_i \in m$ for $i = 1, 2$, $U_1 \neq X$ and $U_2 \neq X$. Since $x \notin A_2$, there are two cases: (1) $x \notin U_2$ or (2) $x \in U_2$ and $x \in V_2$.

(1) $x \notin U_2$.

Since $y \notin A_1$, there are two cases: $y \notin U_1$ or $y \in U_1$ and $y \in V_1$.

($1_a$) $y \notin U_1$.

Since $x \in A_1 = U_1 - V_1$ and $x \notin U_2$, we have $x \in U_1 - (V_1 \cup U_2)$. Since $y \notin U_1$ and $y \in A_2 = U_2 - V_2$, we have $y \in U_2 - (V_2 \cup U_1)$. Since $m$ has the property ($\mathcal{B}$), $V_1 \cup U_2$ and $V_2 \cup U_1$ are $m$-open sets. Moreover, $U_1 \neq X$ and $U_2 \neq X$ and hence $U_1 - (V_1 \cup U_2)$ and $U_2 - (V_2 \cup U_1)$ are $m$-$D$-sets and they are disjoint.

($1_b$) $y \in U_1$ and $y \in V_1$.

We have $x \in A_1 = U_1 - V_1, y \in V_1$ and $(U_1 - V_1) \cap V_1 = \emptyset$. Moreover, $(U_1 - V_1)$ and $V_1$ are $m$-$D$-sets.

(2) $x \in U_2$ and $x \in V_2$.

Then we have $x \in V_2, y \in A_2 = U_2 - V_2$. Moreover, $V_2$ and $U_2 - V_2$ are disjoint $m$-$D$-sets. Consequently, $(X, m)$ is $m$-$D_2$.

*Sufficiency.* This is obvious by the definitions.

**Remark 3.4** If $(X, \tau)$ is a topological space and $m = \tau$ (resp. SO($X$), PO($X$)), then the result established in Theorem 2 of [28] (resp. Theorem 3.3 of [4] and Theorem 2.5 of [5], Theorem 2.2 of [6] and Theorem 3.1 of [15]) is obtained by Theorem 3.5.

**Corollary 3.1** *Let $(X, m)$ be a $m$-symmetric $m$-space and $m$ have the property ($\mathcal{B}$). For an $m$-space $(X, m)$, the following axioms are equivalent: $m$-$T_1$, $m$-$T_0$, $m$-$D_2$, $m$-$D_1$ and $m$-$D_0$.*

**Proof.** This is an immediate consequence of Theorems 3.3, 3.4 and 3.5.

# 4 Properties of axiom $m$-$D_1$

**Definition 4.1** Let $(X, m)$ be an $m$-space. A point $x$ of $X$ is called an *mcc* point if $X$ is the unique $m$-open set that contains $x$.

**Remark 4.1** If $(X, \tau)$ is a topological space and $m = \tau$ (resp. SO($X$), PO($X$)), then an *mcc* point is called a c.c. point [28] (resp. s-c.c point [4] and sc.c point [5], pcc point [15] and pc.c point [6]).

**Theorem 4.1** *If an m-space $(X, m)$ is m-$T_0$, then there exists at most one mcc point.*

**Proof.** Suppose that $(X, m)$ is m-$T_0$ and distinct points $x, y$ are *mcc* points. Since $(X, m)$ is m-$T_0$, there exists an $m$-open set $U$ such that $U$ contains one but not contains the other, say, $x \in U$ and $y \notin U$. Then $y \notin U$ and $U \neq X$. This shows that $x$ is not an *mcc* point.

**Theorem 4.2** *An m-$T_0$ m-space $(X, m)$ is m-$D_1$ if and only if it does not have any mcc point.*

**Proof.** *Necessity.* Suppose that $(X, m)$ is $m$-$D_1$. Each point $x \in X$ belongs to an $m$-$D$-set $A = U - V$, where $U$ is an $m$-open set and $U \neq X$. Therefore, $x$ is not an *mcc* point.

*Sufficiency.* Suppose that $(X, m)$ is $m$-$T_0$ and it does not have any *mcc* point. Let $x, y$ be any pair of distinct points. Then we may assume that there exists an $m$-open set $U$ such that $x \in U$ and $y \notin U$. Since $y$ is not an *mcc* point, there exists an $m$-open set $V$ such that $y \in V$ and $V \neq X$. Now set $B = V - U$, then $B$ is an $m$-$D$- set such that $y \in B$ and $x \notin B$. Therefore, $(X, m)$ is m-$D_1$.

**Remark 4.2** If $(X, \tau)$ is a topological space and $m = \tau$ (resp. SO$(X)$, PO$(X)$), then the result established in Theorem 4 of [28] (resp. Theorem 3.2 of [4] and Theorem 2.10 of [5], Theorem 2.5 of [6] and Theorem 3.2 of [15]) is obtained by Theorem 4.2.

**Theorem 4.3** *Let $f : (X, m_X) \to (Y, m_Y)$ be an M-continuous surjection and $m_X$ satisfy the property ($\mathcal{B}$). If $B$ is an m-D-set of $(Y, m_Y)$, then $f^{-1}(B)$ is an m-D-set of $(X, m_X)$.*

**Proof.** Let $B$ be an $m$-$D$-set of $(Y, m_Y)$. Then there exist $m_Y$-open sets $U$ and $V$ such that $B = U - V$ and $U \neq Y$. Since $f$ is $M$-continuous, by Lemma 2.4 $f^{-1}(U)$ and $f^{-1}(V)$ are $m_X$-open sets. Since $f$ is surjective, $f^{-1}(U) \neq X$ and $f^{-1}(B) = f^{-1}(U) - f^{-1}(V)$. Therefore, $f^{-1}(B)$ is an $m$-$D$-set of $(X, m_X)$.

**Remark 4.3** If $(X, \tau)$ is a topological space and $m =$ SO$(X)$ (resp. PO$(X)$), then the result established in Theorem 2.18 of [5] (resp. Theorem 2.11 of [6] and Theorem 5.1 of [15]) is obtained by Theorem 4.3.

**Theorem 4.4** *Let $f : (X, m_X) \to (Y, m_Y)$ be an M-continuous bijection and $m_X$ satisfy the property ($\mathcal{B}$). If $(Y, m_Y)$ is $m$-$D_1$, then $(X, m_X)$ is $m$-$D_1$.*

**Proof.** Suppose that $(Y, m_Y)$ is $m$-$D_1$. Let $x$ and $y$ be any pair of distinct points of $X$. Since $f$ is injective and $(Y, m_Y)$ is $m$-$D_1$, there exist $m$-$D$-sets $B_x$ and $B_y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin B_x$ and $f(x) \notin B_y$. By Theorem 4.3, $f^{-1}(B_x)$ and $f^{-1}(B_y)$ are $m$-$D$-sets containing $x$ and $y$, respectively, such that $y \notin f^{-1}(B_x)$ and $x \notin f^{-1}(B_y)$. This shows that $(X, m_X)$ is $m$-$D_1$.

**Remark 4.4** If $(X, \tau)$ is a topological space and $m = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$), then the result established in Theorem 2.19 of [5] (resp. Theorem 2.12 of [6] and Theorem 5.3 of [15]) is obtained by Theorem 4.4.

**Theorem 4.5** *An m-space $(X, m_X)$, where $m_X$ satisfies the property ($\mathcal{B}$), is $m$-$D_1$ if and only if, for each pair of distinct points $x$, $y$ of $X$, there exists an M-continuous surjection $f$ of $(X, m_X)$ onto an $m$-$D_1$ m-space $(Y, m_Y)$ such that $f(x) \neq f(y)$.*

**Proof.** *Necessity.* For every pair of distinct points of $X$, it suffices to take the identity function on $X$.

*Sufficiency.* Let $x$ and $y$ be any pair of distinct points of $X$. By hypothesis, there exists an $M$-continuous surjection $f$ of $(X, m_X)$ onto an $m$-$D_1$ $m$-space $(Y, m_Y)$ such that $f(x) \neq f(y)$. Since $(Y, m_Y)$ is $m$-$D_1$, there exist $m$-$D$-sets $B_x$ and $B_y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin B_x$ and $f(x) \notin B_y$. By Theorem 4.3, $f^{-1}(B_x)$ and $f^{-1}(B_y)$ are $m$-$D$-sets containing $x$ and $y$, respectively, such that $y \notin f^{-1}(B_x)$ and $x \notin f^{-1}(B_y)$. This shows that $(X, m_X)$ is $m$-$D_1$.

**Remark 4.5** If $(X, \tau)$ is a topological space and $m = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$), then the result established in Theorem 2.20 of [5] (resp. Theorem 2.13 of [6] and Theorem 5.4 of [15]) is obtained by Theorem 4.5.

# 5 Applications

We recall some modified open sets: semi-$\theta$-open, $b$-open, $\beta$-open and semi-preopen.

Let $A$ be a subset of a topological space $(X, \tau)$. The semi-$\theta$-closure of $A$, $\mathrm{sCl}_\theta(A)$, is defined by the set of the points such that $A \cap \mathrm{sCl}(U) \neq \emptyset$ for every semi-open set $U$ containing $x$. A subset $A$ is said to be *semi-$\theta$-closed* [13] if $A = \mathrm{sCl}_\theta(A)$ . The complement of a semi-$\theta$-closed set is said to be *semi-$\theta$-open*.

**Definition 5.1** Let $(X, \tau)$ be a topological space. A subset $A$ is said to be

(1) *$\beta$-open* [1] or *semi-preopen* [2] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(2) *$b$-open* [3] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all $\beta$-open (resp. $b$-open, semi-$\theta$-open) sets in a topological space $(X, \tau)$ is denoted by $\beta(X)$ (resp. $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$).

**Definition 5.2** The complement of $\beta$-open (resp. $b$-open) sets in a topological space $(X, \tau)$ is said to be *$\beta$-closed* [1] (resp. *$b$-closed* [3], ).

**Definition 5.3** The intersection of all $\beta$-closed (resp. $b$-closed) sets of $(X, \tau)$ containing a subset $A$ is called the *$\beta$-closure* [1] (resp. *$b$-closure* [3]) and is denoted by $\beta\mathrm{Cl}(A)$ (resp. $\mathrm{bCl}(A)$).

**Definition 5.4** The union of all $\beta$-open (resp. $b$-open, semi-$\theta$-open) sets contained in a subset $A$ of a topological space $(X, \tau)$ is called the *$\beta$-interior* (resp. *$b$-interior*, *semi-$\theta$-interior*) of $A$ and is denoted by $\beta\mathrm{Int}(A)$ (resp. $\mathrm{bInt}(A)$, $\mathrm{sInt}_\theta(A)$).

Let $(X, \tau)$ be a topological space. The families $\beta(X)$, $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$, $\tau_\delta, \tau^\alpha$ are all $m$-structures with the property $(\mathcal{B})$. If we put $m = \beta(X)$ (resp. $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$, $\tau_\delta, \tau^\alpha$) then the pair $(X, m)$ is an $m$-space, where $m$ has the property $(\mathcal{B})$. Therefore, for each family $\beta(X)$, $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$, $\tau_\delta, \tau^\alpha$, we can apply the results established in Sections 3-5.

**Remark 5.1** For $\tau_\theta$, $\delta\mathrm{PO}(X)$ and $\delta\mathrm{SO}(X)$, the analogous results to ones established in Sections 3-5 are obtained in [10], [7] and [9], respectively.

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