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A Study of Outbreak of Measles Epidemic in Greater Kolkata

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ABSTRACT

The SIR (Susceptible-Infected-Removal) type of epidemic model for the outbreak of measles is considered here with the time dependent recovery rate in greater Kolkata. The infectivity curves of the disease has been computed both from the model and from the data available upto the year 1995. These have been extrapolated upto 2005 years. It has been seen that the number of the infectives for the forthcoming year is almost steady.
Keywords : SIR, Epidemic, Recovery

1. Introduction

The disease like measles break-out is almost a regular event in epidemic form in greater Kolkata. It is observed that in the context of a classical Susceptible-Infected-Removal (SIR) model [1,2] for the infectious disease like measles the epidemic can only persist when the susceptibles are being supplied steadily, for example, through birth and immigration. We consider the case which occurs in the greater part of Kolkata and its neighbourhood. So it becomes important to study such a mathematical model applicable for this epidemic in this area. Here both the discrete and continuous models are considered. The data of this measles epidemic is available for the period from the year 1979 to 1995. The census report of the population for Kolkata can be found for the period upto the year 1991 [Census of India, 1981, 1991] [3,4] and also for the period upto 2001 very recently. In this model, it is assumed that the recovery rate is a function of time, of course, a very slowly varying function of it. As the data are presently available after the year 1995, we can only extrapolate for the subsequent years upto the year 2005 with a rate of population as calculated from the growth rate of it upto the year 2001.

2. The Model

The continuous SIR model is given below :

$$\frac{dS}{dt} = -\beta SI + \mu N$$

$$\frac{dI}{dt} = \beta SI - \gamma I = \beta I\left(S - \frac{\gamma}{\beta}\right) = \beta I (S - \rho)$$

where $\rho = \dfrac{\gamma}{\beta}$

and $\dfrac{dR}{dt} = \gamma I$

with initial conditions $S(0) = S_0 > 0$, $I(0) = I_0 > 0$ and $R(0) = R_0 = 0$

and where, β = susceptible rate

μ = specific growth rate

$\rho = \gamma/\beta$ is called effective removal rate i.e. the ratio of the rate at which individuals are removed from the infected category to the rate at which they are added to the same category.

This continuous model can be converted into following discrete model.

$$\left.\begin{aligned}
& S_{t+1} - S_t = -\beta S_t I_t + \mu N_t \\
\text{or,}\quad & S_{t+1} = S_t (1 - \beta I_t) + \mu N_t \\
& I_{t+1} - I_t = \beta S_t I_t - \gamma I_t \\
& \text{(I)} \\
\text{or,}\quad & I_{t+1} = I_t (\beta S_t - \gamma + 1) \\
& R_{t+1} - R_t = \gamma I_t \\
& R_{t+1} = \gamma I_t + R_t
\end{aligned}\right\}$$

For very small number of infectives I_t compared to the population N_t

$$S_t \simeq N_t$$

Now, the population of Kolkata and the infectives for the measles are cited in the following Tables :

Table I

Years	Population
1978	40,46,613
1979	40,73,002
1980	40,99,563
1981	41,26,297
1982	41,52,097
1983	41,78,059
1984	42,04,183
1985	42,30,417
1986	42,56,922
1987	42,83,540
1988	43,10,323
1989	43,37,275
1990	43,64,394
1991	43,91,684

Human population of Kolkata (Data from Census Report 1981) [3]

Table II

Year	Infectives	Year	Infectives
1979	138	1988	335
1980	266	1989	112
1981	144	1990	257
1982	162	1991	115
1983	92	1992	213
1984	66	1993	154
1985	55	1994	146
1986	209	1995	128
1987	115		

Yearwise infectives of measles in Kolkata (Source I.D. Hospital). [5]

The infectivity curve is plotted against the data of Table **II** which is depicted in Figure 1.

The value of μ can be calculated from the average values of μ_1, μ_2, ..., obtained from the Table **I** by using the formula –

$$\mu_i = \frac{1}{t} \ln \frac{N_i}{N_{i-1}} \qquad (i = 1, 2, \ldots)$$

(with t =1, time unit being one year).

Table III

The values of μ_i (i = 1, 2, ...) is given below

Human Population	$\mu_i = \ln \dfrac{N_i}{N_{i-1}}$
40,46,613	--
40,73,002	.006
40,99,563	.006
41,26,297	.006
41,52,097	.006
41,78,059	.006
42,04,183	.006
42,30,417	.006
42,56,992	.006
42,83,540	.006
43,10,323	.006
43,37,275	.006
43,64,394	.006
43,91,684	.006

The average value of μ is .006. This value is taken as the value of μ in the equation (1).
Then the number of susceptibles is computed from the Tables **I** and II which is given below.

Table IV

Year	Infectives	Year	Infectives
1979	40,72,864	1988	43,09,998
1980	40,99,297	1989	43,37,163
1981	41,26,153	1990	43,64,137
1982	41,51,935	1991	43,91,576
1983	41,77,967	1992	44,17,900
1984	42,04,117	1993	44,44,547
1985	42,30,416	1994	44,71,303
1986	42,56,713	1995	44,98,231
1988	42,83,425		

Yearwise calculated susceptible of measles.

Using the average value of $\mu = .006$ we calculate the value of β from the Tables I, II, III and IV. Then the average value of β has been calculated from the first equations of (I). Using the average value of β in the second equations of (I) the value of γ can be found by using the Tables II and IV. Then the average value of γ is calculated. This average value of γ has found to be 2.9613. The time-dependent recovery rate taken as to be of the following form is estimated with this discrete model :

$$\gamma(t) = \gamma_0 e^{a_0 + a_1 t + a_2 t^2}$$

where γ_0 is taken to be the average value of γ as calculated above. The constants a_0, a_1, a_2 has been computed by the method of least-square fittings [6]. The normal equations for this method are—

$$v_1 = a_0 + a_1 + a_2 + .314547$$
$$v_2 = a_0 + 2a_1 + 4a_2 - .186170$$
$$v_3 = a_0 + 3a_1 + 9a_2 - .014465$$

where, $V_t = a_0 + a_1 t + a_2 t^2 - \ln(\gamma_t / \gamma_0)$ $\quad (t = 0, 1, 2, \ldots, 10)$

The Residual equations are—

$15a_0 + 120a_1 + 1240a_2 + 1.85766 = 0$
$120a_0 + 1240a_1 + 14400a_2 + 8.58536 = 0$
and $1240\, a_0 + 14400a_1 + 178312a_2 + 13.42527 = 0$
Solving these above equations we get the values of
$a_0 = .20556 \qquad a_1 = -\ .001199 \qquad a_2 = .000087.$
Therefore $\gamma(t) = 2.9613 \exp\,(.20556 - .00119t + .000087t^2)$

A SIR Epidemic Model

It is clear from the continuous model that—

$$\frac{d}{dt}(S + 1 + R) = \mu N$$

or, $$\frac{dN}{dt} = \mu N \qquad \text{as } S + 1 + R = N(t)$$

The solution is $N = N_0\, e^{\mu t}$

N_0 be the value of N at $t = 0$.

It is also evident from the discrete model that—

$$S_{t+1} - S_t = -\ \beta S_t I_t + \mu N_t$$

$$I_{t+1} - I_t = \beta S_t I_t - \gamma I_t$$

and $R_{t+1} - R_t = \gamma I_t$ and by adding these three equations we have

$$S_{t+1} + I_{t+1} + R_{t+1} - (S_t + 1_t + R_t) = \mu N_t$$

$$N_{t+1} - N_t = \mu N_t$$

$$N_{t+1} = (1 + \mu)\, N_t$$

The value of μ as calculated above can also be used here. Also it is evident that from the Tables I and II $N_t \sim S_t$ as I_t is very small compared to N_t.

3. Computation of Infectives for the Measles

With the value of $\mu = .006$ the susceptible rate β can be computed from the first equation of (1) by using the number of susceptibles. With these values one can calculate the number of infectives I_t for each years from 1979 and onwards upto 2005. This is depicted in the following Table.

Table V

Year	Infective	Year	Infective
1979	138.00	1993	185.03
1980	231.46	1994	175.01
1981	164.04	1995	153.17
1982	180.42	1996	151.53
1983	113.52	1997	151.23
1984	48.08	1998	151.216
1985	42.66	1999	151.225
1986	189.62	2000	151.2149
1987	87.15	2001	151.2148
1988	273.34	2002	151.21479
1989	85.12	2003	151.21478
1990	223.14	2004	151.214788
1991	183.21	2005	151.21478
1992	256.15		

Yearwise calculated infectives of measles

The infectivity curve is plotted against the data of Table **V**, which shows in the Figure 2.

4. The Continuous Model

To find the pattern of the infectives it is interesting to study the continuous model also. In the continuous model we again assume that the number of susceptibles is almost equal to the number of population.

Therefore, $$\frac{dI}{dt} = \beta SI - \gamma I$$

we assume that $S(t) = N(t)\,K$ where K is << 1

Then, $$\frac{dI}{dt} = \beta I\,N(t)\,K - \gamma I$$

or, $$\frac{d}{dt}(\ln I) = \beta KN - \gamma \tag{2}$$

Integrating (2) we have,

$$I = A\ \exp\left[\frac{\beta KN_0 e^{\mu t}}{\mu} - \gamma t\right]$$

We take as the initial condition $I = I_0$ when $t = 0$.

Consequently, we have

$$I = I_0\ \exp\left\{\frac{\beta KN_0}{\mu}(e^{\mu t} - 1) - \gamma t\right\}$$

$$\cong I_0 \exp\,(\beta KN_0 - \gamma)\,t \tag{3}$$

The initial values of the population N_0 = 40,46,613 and the number of infectives I_0 = 138 are given in Table I and II respectively.

The computed values of β and γ are given by β = .000000864 and γ = 3.63 and we take the value of K=1.

On substitution, we have from (3) $I = 138\ e^{-.14t}$ (4)
where t = 0 and t = 20 correspond to the year 1979 and 2000 respectively. The infectivity curve for the equation (4) is depicted in the Figure 3 for the interval of time $0 \leq t \leq 20$.

5. Discussion and Conclusion

In this article the number of the infectives have been computed after the year 1995 by extrapolation [6]. It has been seen that the relative number of infectives decrease with the increase of the population in this period. This may be due to the general health awareness among the people as well as the measures taken by the authorities in the spreading epidemic. The infectivity curves obtained both from the data depicted in Fig.I and from the theoretical calculated number of infectives depicted in Fig.2 are in good agreement. The two curves are shown in Fig.3. In the discrete model the reproduction number $R_0 = S_0\beta/\gamma = 1.18$ which is greater than one. Thus, the epidemic can break out, and a locally asymptotically stable endemic state is possible. On the other hand, in the continuous model, the reproduction number $R_0 = 0.92$ which is less than one, so the disease dies out, i.e. in the course of time the disease is completely wiped out. For the case of continuous model the infectivity curve (Fig.3), shows an asymptotic decrease of the out-break of the disease. In Fig.3, a comparison discrete model with the data has been depicted. Also, a curve of the result obtained from the continuous model is also given there.

Acknowledgement

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LEGENDS OF FIGURES

Fig.1 The infectivity curve shows variation of infectives of measles with the time. The graph is plotted against data.

Fig.2 The infectivity curve shows the variation of infectives of measles upto 2005 years. The graph is obtained from discrete SIR model.

Fig.3 Comparison of the infectivity curves showing the variation of infectivity of measles with the time as obtained from data (dotted line) and from SIR discrete model (dashed line) and also from continuous model (solid line).

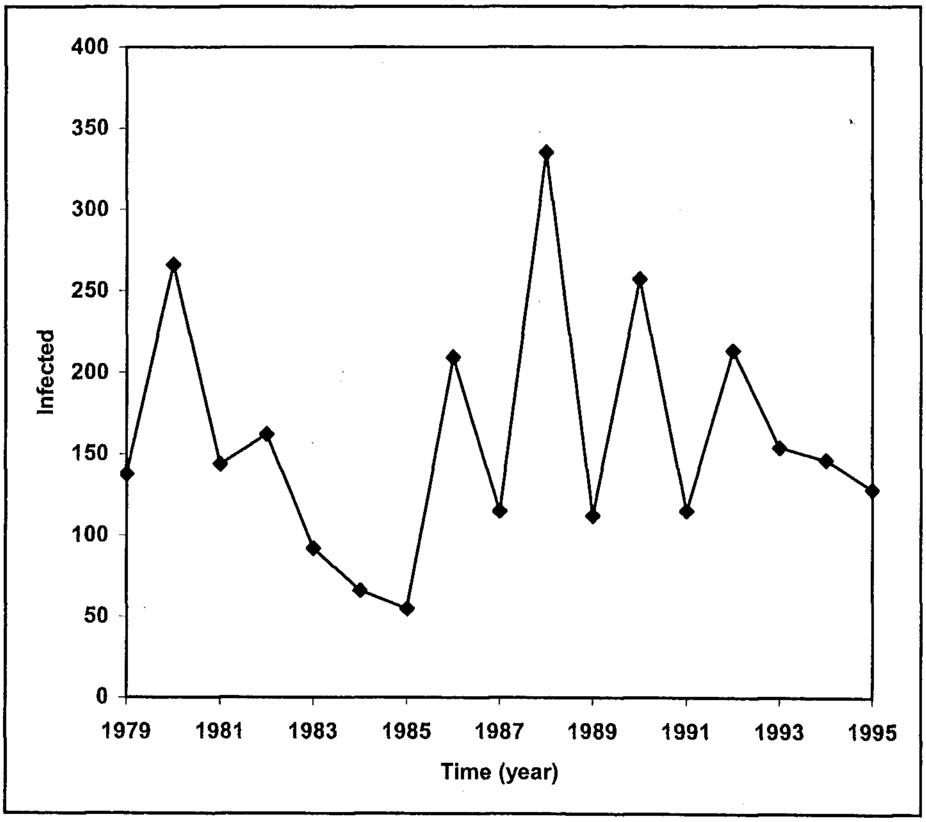

Figure 1.

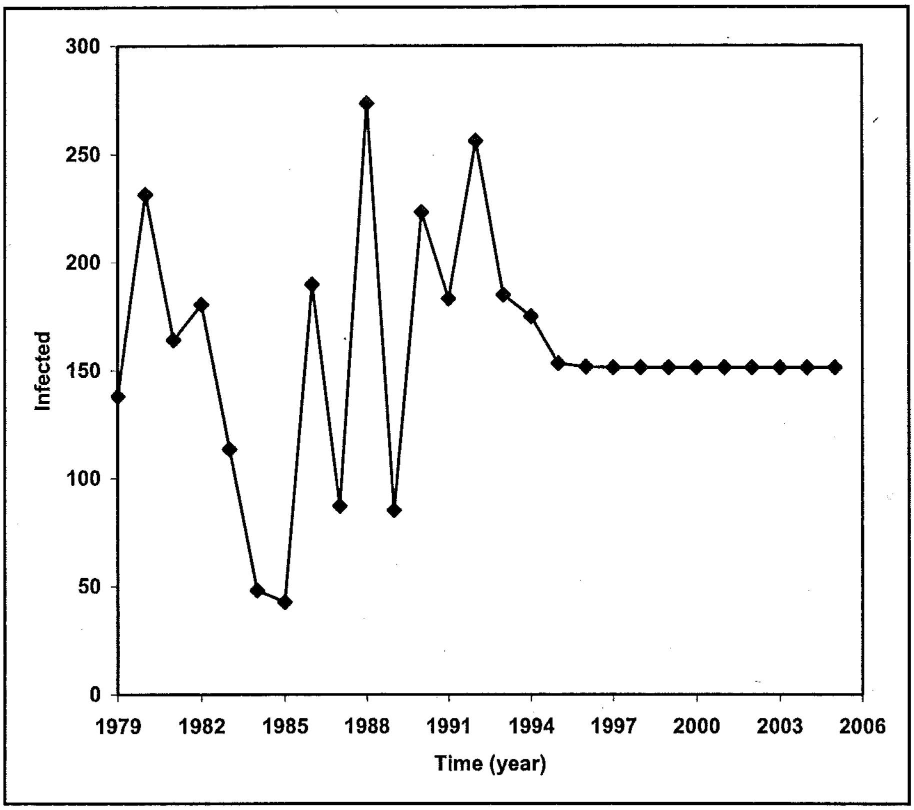
Infected
300
250
200
150
100
50
0
1979
1982
1985
1988
1991
1994
1997
2000
2003
2006
Time (year)

Figure 2.

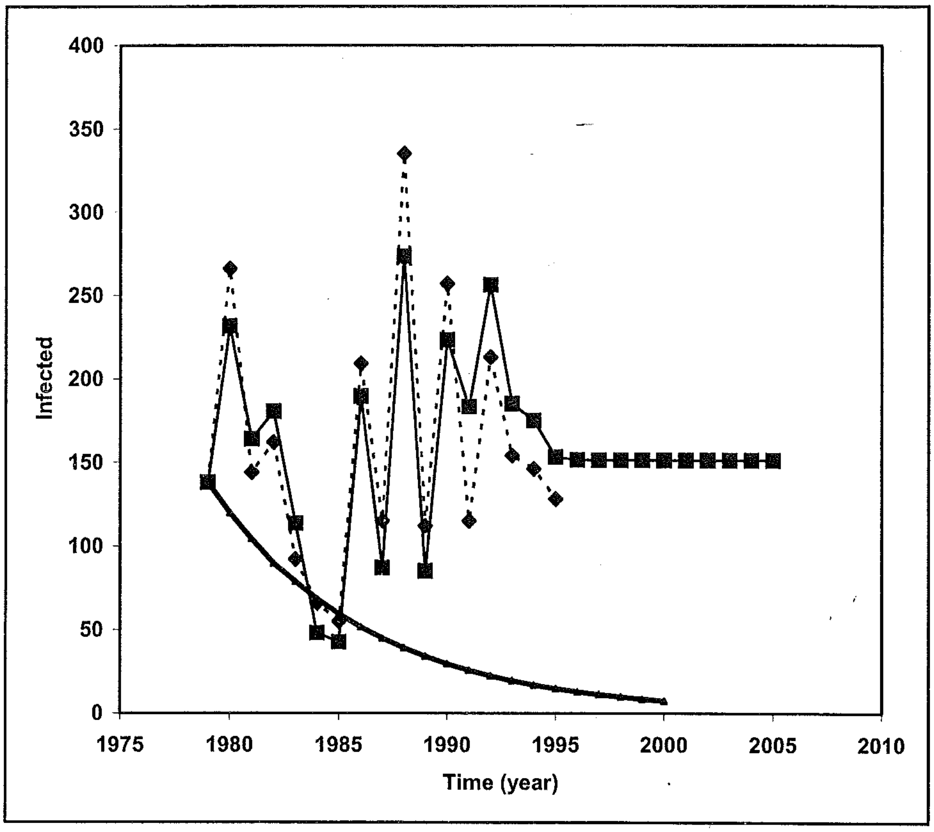

Figure 3.

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