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AFFINE MOTION IN RECURRENT AREAL SPACES OF SUBMETRIC CLASS

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ABSTRACT

We consider affine motion in recurrent areal space of submetric class , denoted by $A_n^{*(m)}$. The necessary and sufficient conditions are obtained when affine motions admit contra –fields . Other special cases are also discussed .

In a set of papers K. Takano ([1] ,[2]) has studied affine motion in non-Riemannian K^*-spaces . Affine motions in recurrent Finsler was studied by R.S.Sinha [5] . T. Igarashi[4] has introduced the theory of Lie-derivative in an areal space of submetric class . The concept of deformed areal spaces and the homothetic transformations in areal space of submetric class was developed by O.P.Singh ([6],[7]) . Recently the present author [8] discussed the motion with contra field in a symmetric areal space of submetric class. In this paper the author wishes to study affine motion in recurrent areal space of submetric class.

1 INTRODUCTION

Let us consider an n-dimensional areal space $A_n^{(m)}$ of submetric class equipped with the fundamental function $F(x^i, p_\alpha^i)$; $p_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$, the normalized metric tensor $g_{ij}(x,p)$ and symmetric connection parameter $\Gamma_{jk}^{*i}(x,p)$ [3] . Throughout this paper the indices $i,j,k,l...$ run from 1 to n while the indices $\alpha,\beta,\lambda,\mu...$vary from 1 to m, $(1 \le m \le (n-1))$. An areal space $A_n^{(m)}$, in which the curvature tensor R_{jkl}^i satisfies the relation

$$R_{jkl|m}^i = K_m R_{jkl}^i \quad , \tag{1.1}$$

where the suffix after the bar denotes covariant differentiation and K_m a non-zero covariant vector, is called recurrent areal space of submetric class. We denote such space by $A_n^{*(m)}$.

In an areal space of submetric class $A_n^{*(m)}$, we consider an infinitesimal point transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta t \ , \tag{1.2}$$

where $\xi^i(x)$ is a contravariant vector field of class C^2 and is only a point function and δt is a constant.

The Lie-derivative of a mixed tensor T_j^i with respect to (1.2) is given by [4]

$$L_\xi T_j^i = T_{j|h}^i \xi^h - T_j^h \xi_{|h}^i + T_h^i \xi_{|j}^h + T_{j,r}^{i,\lambda} \xi_{|h}^r p_\lambda^h \quad , \tag{1.3}$$

where the symbol L_ξ denotes the operator of Lie-differentiation.

Some important formulae concerning the operators of Lie-differentiation and covariant differentiation are expressed as follows :

(1.4)

$$L_\xi\left(T_{j|k}^i\right) - \left(L_\xi T_j^i\right)_{|k} = T_j^h \ L_\xi \Gamma_{hk}^{*i} - T_h^i \ L_\xi \Gamma_{jk}^{*h} - T_j^i{}_{;r}^{\lambda} \left(L_\xi \Gamma_{sk}^{*r}\right) p_\lambda^s \ ,$$

$$L_\xi\left(T_j^i{}_{;r}^{\lambda}\right) - \left(L_\xi T_j^i\right)_{;r}^{\lambda} = 0 \tag{1.5}$$

and

$$\left(L_\xi \Gamma_{jk}^{*i}\right)_{|l} - \left(L_\xi \Gamma_{jl}^{*i}\right)_{|k} = L_\xi R_{jkl}^i + \Gamma_{jk}^{*i}{}_{;r}^{\lambda} \left(L_\xi \Gamma_{sl}^{*i}\right) p_\lambda^s - \Gamma_{jk}^{*i}{}_{;r}^{\lambda} \left(L_\xi \Gamma_{sk}^{*i}\right) p_\lambda^s \tag{1.6}$$

2. AFFINE MOTION IN RECURRENT AREAL SPACE $A_n^{*(m)}$

In an areal space of submetric class if the original space and the deformed space with connection parameter $\Gamma_{jk}^{*i} + L_\xi \Gamma_{jk}^{*i} \delta t$ have the same connection, the transformation (1.2) is called affine motion of the space $A_n^{(m)}$. In such case, it is necessary and sufficient that we have

$$L_\xi \Gamma^{*i}_{jk} = 0 \ . \tag{2.1}$$

Under an affine motion in view of (1.6) and (2.1), we necessarily obtain

$$L_\xi R^i_{jkl} = 0 \ . \tag{2.2}$$

Applying (1.4) for the curvature tensor R^i_{jkl} and using (2.1), we get

$$L_\xi R^i_{jkl|m} = 0 \ . \tag{2.3}$$

In view of (2.2) and (2.3), the equation (1.1) yields

$$L_\xi K_m = 0 \tag{2.4}$$

since $A_n^{*(m)}$ is a non-flat space.
Thus we state

Theorem 2.1 If a recurrent areal space of submetric class $A_n^{*(m)}$ admits the affine motion (1.2) then the recurrence vector K_m is Lie–invariant.

Now we wish to examine the possibility of the existence of an affine motion of the form

$$\bar{x}^i = x^i + \xi^i(x)\delta t \ , \quad \xi^i_{|j} = \Phi(x,p)\delta^i_j \tag{2.5}$$

in the recurrent areal space of submetric class.
Taking lie-derivative of the curvature tensor R^i_{jkl}, we get

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(2.6)

$$L_\xi R^i_{jkl} = R^i_{jkl|h}\xi^h - R^h_{jkl}\xi^i_{|h} + R^i_{hkl}\xi^h_{|j} + R^i_{jhl}\xi^h_{|k} + R^i_{jkh}\xi^h_{|l} + R^{i\ \ \cdot\lambda}_{jkl\ ,r}\xi^r_{|h}p^h_\lambda = 0$$

by virtue of (2.2) .
Introducing latter of (2.5) in (2.6) , we obtain

$$(2.7)\qquad R^i_{jkl|h}\xi^h + 2\Phi R^i_{jkl} + \Phi R^{i\ \ \cdot\lambda}_{jkl\ ,h} p^h_\lambda = 0 \quad .$$

Noting (1.1) in (2.7) , it becomes

$$(2.8)\qquad \left(K_h\xi^h + 2\Phi\right)R^i_{jkl} + \Phi R^{i\ \ \cdot\lambda}_{jkl\ ,h} p^h_\lambda = 0 \quad .$$

Let us assume that $R^i_{jkl};^\lambda_h\ p^h_\lambda = 0$, then (2.8) reduces to

$$(2.9)\qquad \Phi = -\frac{1}{2}K_h\xi^h$$

since the space $A_n^{*(m)}$ is non-flat .

Conversely , if the relation (2.9) is true , the equation (2.8) takes the form

$$(2.10)\qquad K_m\xi^m R^i_{jkl};^\lambda_h\ p^h_\lambda = 0 \quad .$$

Since $K_m\xi^m \neq 0$, the equation (2.10) yields

$$(2.11)\qquad R^i_{jkl};^\lambda_h\ p^h_\lambda = 0 \ .$$

Accordingly we state

Theorem 2.2 If $A_n^{*(m)}$ admits an affine motion of the form (2.5), the necessary and sufficient condition for $\Phi(x,p)$ to be expressed in the form

$$\Phi = -\frac{1}{2}K_h\xi^h$$

is that the condition $R^i_{jkl};^{\lambda}_h \, p^h_{\lambda} = 0$ holds.

3. FURTHER DISCUSSION

In this section , we shall deal with two special cases of the affine motion in a recurrent areal space of submetric class .

(a) **Contra Field** In an areal space of submetric class , if the vector $\xi^i(x)$ satisfies the relation

$$\xi^i_{|j} = 0 \tag{3.1}$$

the vector field $\xi^i(x)$ is called a contra field .
In this case we consider a special affine motion of the form

$$\bar{x}^i = x^i + \xi^i(x)\delta t\,,\ \ \xi^i_{|j} = 0 \tag{3.2}$$

In view of (3.2) , the equation (2.1) yields

$$L_{\xi}\Gamma^{*i}_{jk} = R^i_{jkh}\xi^h = 0\,. \tag{3.3}$$

Applying the latter of (3.2) and (2.2) in the equation (2.6), we get

$$R^i_{jkl|h}\xi^h = 0 \tag{3.4}$$

In view of (1.1) , it becomes

$$R^i_{jkl}K_h\xi^h = 0\,. \tag{3.5}$$

But $A^{*(m)}_n$ is a non-flat space , that is , $R^i_{jkl} \neq 0$, therefore it is obvious that

$$K_h\xi^h = 0 \tag{3.6}$$

From the equations (3.3) and (3.6), we conclude that for $A_n^{*(m)}$ to admit an affine motion of the form (3.2), it is necessary that we have

$$(3.7) \qquad K_h\xi^h = 0 \ , \ R^i_{jkh}\xi^h = 0 \ .$$

Conversely, if (3.7) is true, then from the identity $R^i_{jkl} + R^i_{khj} + R^i_{hjk} = 0$ and the latter of (3.7), we get

$$(3.8) \qquad R^i_{hjk}\xi^h = 0 \ .$$

But by using (3.8) in the Ricci identity

$$(3.9) \qquad \xi^i_{|j|k} - \xi^i_{|k|j} = R^i_{hjk}\xi^h \ ,$$

we obtain (3.1) and hence $\xi^i(x)$ is contra field in $A_n^{*(m)}$.

In such case in view (3.6), the equation (2.6) immediately implies that $L_v R^i_{jkh} = 0$, which is integrability condition of $L_\xi \Gamma^{*i}_{jk} = 0$. Thus (3.7) is also a sufficient condition for $A_n^{*(m)}$ admitting (3.2).
Hence we state

Theorem 3.1 When a recurrent areal space of submetric class admits an affine motion in order that the vector $\xi^i(x)$ spans a contra field, it is necessary and sufficient that the conditions $K_h\xi^h = 0$ and $R^i_{jkh}\xi^h = 0$ be valid.

(b) Concurrent Field In an areal space of submetric class ,if the vector $\xi^i(x)$ satisfies the relation

$$(3.10) \qquad \xi^i_{|j} = K\delta^i_j \ ,$$

where K is a constant, then the vector field $\xi^i(x)$ is called a concurrent vector field.

Here we consider the affine motion

$$\bar{x}^i = x^i + \xi^i(x)\delta t \quad , \quad \xi^i_{|j} = K\delta^i_j \ . \tag{3.11}$$

From the latter of (3.11) , we find

$$\xi^i_{|j|k} - \xi^i_{|k|j} = 0 \ . \tag{3.12}$$

Application of (3.12) in the Ricci identity (3.9) yields $R^i_{hjk}\xi^h = 0$.
Taking covariant differentiation of the above equation and noting (1.1) and (3.10), we obtain

$$KR^i_{mjk} = 0 \ . \tag{3.13}$$

Since K is non zero constant , the equation (3.13) implies

$$R^i_{mjk} = 0 \ , \tag{3.14}$$

which contradicts our assumption that the space $A_n^{*(m)}$ is non flat .
Accordingly ,we state

Theorem **3.2** The general recurrent areal space of submetric class $A_n^{*(m)}$ does not admit the affine motion (3.11) .

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