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On Almost Strongly θ-m-Continuous Functions

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Abstract

We introduce the notion of almost strongly θ-m-continuous functions as functions from a set satisfying some minimal conditions into a topological space. We obtain several characterizations and properties of such functions. The functions enable us to formulate a unified theory of almost strong θ-continuity [26] and almost strong θ-semi-continuity [5].

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1 Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of modifications of continuity. Fomin [13] introduced the notion of θ-continuous functions. Noiri [23] introduced the notion of strongly θ-continuous functions. Noiri and Kang [26] introduced and studied almost strongly θ-continuous functions. Some of properties of almost strongly θ-continuous functions are studied in [15] and [44]. In 1994, Beceren et al. [5] introduced and studied almost strongly θ-semi-continuous functions. In 1997, Dube and Chauhan [11] introduced strongly closure semi-continuous functions which are equivalent to almost strongly θ-semi-continuous functions. These classes of functions have properties similar to the class of θ-continuous functions. In [35], the present authors introduced and investigated m-continuous functions. The notion of strongly θ-m-continuous functions is introduced in [29]. In this paper, in order to unify several characterizations of almost strongly θ-continuous functions and almost strongly θ-semi-continuous functions, we introduce a new notion of almost strongly θ-m-continuous functions which are functions from a set satisfying some minimal conditions into a topological space.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$). A subset A is said to be *δ-open* [43] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$.

The θ-closure of A, denoted by $\mathrm{Cl}_\theta(V)$, is defined as the set of all $x \in X$ such that $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be *θ-closed* [43]. The complement of a θ-closed set is said to be *θ-open*. It is shown in [43] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of X and $\mathrm{Cl}_\theta(S)$ is closed in X for every subset S of X.

Definition 2.1 Let (X, τ) be a topological space. A subset A of X is said to be

(1) *semi-open* [16] (resp. *preopen* [18], *α-open* [20], *β-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$),

(2) *δ-preopen* [39] (resp. *δ-semi-open* [33]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$).

Definition 2.2 The complement of a semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) set is said to be *semi-closed* [7] (resp. *preclosed* [18], *α-closed* [19], *β-closed* [1] or *semi-preclosed* [3], *δ-preclosed* [39], *δ-semi-closed* [33]).

Definition 2.3 The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, δ-preclosed, δ-semi-closed) sets of X containing A is called the *semi-closure* [7] (resp. *preclosure* [12], *α-closure* [19], *β-closure* [2] or *semi-preclosure* [3], *δ-preclosure* [39], *δ-semi-closure* [33]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$, $\mathrm{pCl}_\delta(A)$, $\mathrm{sCl}_\delta(A)$).

Definition 2.4 The union of all semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*, *δ-preinterior*, *δ-semi-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$, $\mathrm{pInt}_\delta(A)$, $\mathrm{sInt}_\delta(A)$).

Throughout the present paper, (X,τ) and (Y,σ) denote topological spaces and $f:(X,\tau)\to(Y,\sigma)$ presents a (single valued) function.

Definition 2.5 A function $f:(X,\tau)\to(Y,\sigma)$ is said to be

(1) *θ-continuous* [13] (resp. *strongly θ-continuous* [23], *almost strongly θ-continuous* [26]) at $x\in X$ if for each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\mathrm{Cl}(U))\subset\mathrm{Cl}(V)$ (resp. $f(\mathrm{Cl}(U))\subset V$, $f(\mathrm{Cl}(U))\subset\mathrm{sCl}(V)$),

(2) *almost strongly θ-semi-continuous* [5] or *strongly closure-semi-continuous* [11]) at $x\in X$ if for each open set V of Y containing $f(x)$, there exists a semi-open set U of X containing x such that $f(\mathrm{sCl}(U))\subset\mathrm{sCl}(V)$,

(3) *θ-continuous* (resp. *strongly θ-continuous*, *almost strongly θ-continuous*, *almost strongly θ-semi-continuous*) if it has this property at each $x\in X$.

3 Almost strongly θ-m-continuous functions

Definition 3.1 A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [34] on X if $\emptyset\in m_X$ and $X\in m_X$. By (X,m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Remark 3.1 Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, $\beta(X)$, δPO(X) and δSO(X) are all m-structures on X.

Definition 3.2 Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [17] as follows:

(1) $\mathrm{mCl}(A) = \bigcap\{F : A \subset F, X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \bigcup\{U : U \subset A, U \in m_X\}$.

Remark 3.2 Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$, δPO(X), δSO(X)), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{pCl}_\delta(A)$, $\mathrm{sCl}_\delta(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{pInt}_\delta(A)$, $\mathrm{sInt}_\delta(A)$).

Lemma 3.1 (Maki et al. [17]) *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) If $(X - A) \in m$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 3.3 A function $f : (X, m_X) \to (Y, \sigma)$, where (X, m_X) is an m-space and (Y, σ) is a topological space, is said to be *m-continuous* [35] (resp. *almost m-continuous* [37], *weakly m-continuous* [36]) at $x \in X$ if for each open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$ (resp. $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$, $f(U) \subset \mathrm{Cl}(V)$). A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *m-continuous* (resp. *almost m-continuous*, *weakly m-continuous*) if it has the property at each point $x \in X$.

Lemma 3.2 (Popa and Noiri [35]) *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is m-continuous;

(2) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ *for every open set V of Y;*

(3) $\mathrm{mCl}(f^{-1}(K)) = f^{-1}(K)$ *for every closed set K of Y.*

Lemma 3.3 (Popa and Noiri [36]) *For a function* $f : (X, m_X) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) f *is weakly m-continuous;*

(2) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* B *of* Y.

Definition 3.4 A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *almost strongly* θ*-m-continuous* (resp. *strongly* θ*-m-continuous* [29], θ*-m-continuous* [27]) at $x \in X$ if for each open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(V)$ (resp. $f(\mathrm{mCl}(U)) \subset V$, $f(\mathrm{mCl}(U)) \subset \mathrm{Cl}(V)$). A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *almost strongly* θ*-m-continuous*, *strongly* θ*-m-continuous* or θ*-m-continuous* if it has the property at each point $x \in X$.

Remark 3.3 (1) Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$) and $f : (X, m_X) \to (Y, \sigma)$ is almost strongly θ-m-continuous, then f is almost strongly θ-continuous (resp. almost strongly θ-semi-continuous).

(2) The following implications hold:

strong θ-m-continuity $\Rightarrow$ almost strong θ-m-continuity $\Rightarrow$ θ-m-continuity,

where none of the implications is reversible. In Example 2.2 of [26], there is an almost strongly θ-continuous function which is not strongly θ-continuous. In Example 2.1 of [11], there is a θ-semi-continuous function which is not almost strongly θ-semi-continuous.

Definition 3.5 Let S be a subset of an m-space (X, m_X). A point $x \in X$ is called

(1) an m_θ*-adherent* point of S if $\mathrm{mCl}(U) \cap S \neq \emptyset$ for every $U \in m_X$ containing x,

(2) an m_θ*-interior* point of S if $\mathrm{mCl}(U) \subset S$ for some $U \in m_X$ containing x.

The set of all m_θ-adherent points of S is called the m_θ*-closure* of S and is denoted by $\mathrm{mCl}_\theta(S)$. If $A = \mathrm{mCl}_\theta(A)$, then A is called m_θ*-closed.* The complement of an m_θ-closed set is said to be m_θ*-open.* The set of all m_θ-interior points of S is called the m_θ*-interior* of S and is denoted by $\mathrm{mInt}_\theta(S)$.

Remark 3.4 Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$), $\beta(X)$), then $\mathrm{mCl}_\theta(S) = \mathrm{Cl}_\theta(S)$ [43] (resp. $\mathrm{sCl}_\theta(S)$ [8], $\mathrm{pCl}_\theta(S)$ [31], $\mathrm{spCl}_\theta(S)$ [25]).

Lemma 3.4 (Noiri and Popa [27]) *Let A and B be subsets of (X, m_X). Then the following properties hold:*

(1) $X - \mathrm{mCl}_\theta(A) = \mathrm{mInt}_\theta(X - A)$ *and* $X - \mathrm{mInt}_\theta(A) = \mathrm{mCl}_\theta(X - A)$,

(2) A is m-θ-open if and only if $A = \mathrm{mInt}_\theta(A)$,

(3) $A \subset \mathrm{mCl}(A) \subset \mathrm{mCl}_\theta(A)$ *and* $\mathrm{mInt}_\theta(A) \subset \mathrm{mInt}(A) \subset A$,

(4) If $A \subset B$, then $\mathrm{mCl}_\theta(A) \subset \mathrm{mCl}_\theta(B)$ *and* $\mathrm{mInt}_\theta(A) \subset \mathrm{mInt}_\theta(B)$,

(5) If A is m_X-open, then $\mathrm{mCl}(A) = \mathrm{mCl}_\theta(A)$.

Theorem 3.1 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost strongly θ-m-continuous;

(2) $f^{-1}(V)$ is m_θ-open for every regular open set V of Y;

(3) $f^{-1}(F)$ is m_θ-closed for every regular closed set F of Y;

(4) For each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset V$;

(5) $f^{-1}(V)$ is m_θ-open for every δ-open set V of Y;

(6) $f^{-1}(F)$ is m_θ-closed for every δ-closed set K of Y;

(7) $f(\mathrm{mCl}_\theta(A)) \subset \mathrm{Cl}_\delta(f(A))$ *for every subset A of X;*

(8) $\mathrm{mCl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\delta(B))$ *for every subset B of Y;*

(9) $f^{-1}(\mathrm{Int}_\delta(B)) \subset \mathrm{mInt}_\theta(f^{-1}(B))$ *for every subset B of Y;*

(10) $f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{sCl}(V)))$ *for every open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any regular open set of Y and $x \in f^{-1}(V)$. Since f is almost strongly θ-m-continuous, there exists $U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(V) = V$. Thus $x \in U \subset \mathrm{mCl}(U) \subset f^{-1}(V)$ which implies that $x \in \mathrm{mInt}_\theta(f^{-1}(V))$. Therefore, $f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(V))$. By Lemma 3.4, we obtain $f^{-1}(V) = \mathrm{mInt}_\theta(f^{-1}(V))$ and hence $f^{-1}(V)$ is m_θ-open.

(2) $\Rightarrow$ (3): Let F be any regular closed set of Y. By (2), we have $f^{-1}(F) = X - f^{-1}(Y - F) = X - \mathrm{mInt}_\theta(f^{-1}(Y - F)) = X - \mathrm{mInt}_\theta(X - f^{-1}(F)) = \mathrm{mCl}_\theta(f^{-1}(F))$. This shows that $f^{-1}(F)$ is m_θ-closed.

(3) $\Rightarrow$ (4): Let $x \in X$ and V be any regular open set of Y containing $f(x)$. By (3), we have $X - f^{-1}(V) = f^{-1}(Y - V) = \mathrm{mCl}_\theta(f^{-1}(Y - V)) = X - \mathrm{mInt}_\theta(f^{-1}(V))$. This implies that $f^{-1}(V) = \mathrm{mInt}_\theta(f^{-1}(V))$. Therefore, there exists $U \in m_X$ containing x such that $\mathrm{mCl}(U) \subset f^{-1}(V)$; hence $f(\mathrm{mCl}(U)) \subset V$.

(4) $\Rightarrow$ (5): Let V be any δ-open set of Y and $x \in f^{-1}(V)$. There exists a regular open set G of Y such that $f(x) \in G \subset V$. By (4), there exists

$U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset G$. Therefore, we obtain $x \in U \subset \mathrm{mCl}(U) \subset f^{-1}(V)$ which implies that $x \in \mathrm{mInt}_\theta(f^{-1}(V))$. Hence $f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(V))$. By Lemma 3.4, $f^{-1}(V) = \mathrm{mInt}_\theta(f^{-1}(V))$ and hence by Lemma 3.4 $f^{-1}(V)$ is m_θ-open.

(5) $\Rightarrow$ (6): Let K be any δ-closed set of Y. By (5) we have $f^{-1}(K) = X - f^{-1}(Y-K) = X - \mathrm{mInt}_\theta(f^{-1}(Y-K)) = \mathrm{mCl}_\theta(f^{-1}(K))$. Therefore, $f^{-1}(K) = \mathrm{mCl}_\theta(f^{-1}(K))$. This shows that $f^{-1}(K)$ is m_θ-closed.

(6) $\Rightarrow$ (7): Let A be a subset of X. Since $\mathrm{Cl}_\delta(f(A))$ is δ-closed in Y, by (6) we have $f^{-1}(\mathrm{Cl}_\delta(f(A))) = \mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}_\delta(f(A))))$. Let $x \notin f^{-1}(\mathrm{Cl}_\delta(f(A)))$. Then there exists $U \in m_X$ containing x such that $\mathrm{mCl}(U) \cap f^{-1}(\mathrm{Cl}_\delta(f(A))) = \emptyset$ and hence $\mathrm{mCl}(U) \cap A = \emptyset$. Hence $x \notin \mathrm{mCl}_\theta(A)$. Therefore, we obtain $f(\mathrm{mCl}_\theta(A)) \subset \mathrm{Cl}_\delta(f(A))$.

(7) $\Rightarrow$ (8): Let B be any subset of Y. Then by (7) we have $f(\mathrm{mCl}_\theta(f^{-1}(B))) \subset \mathrm{Cl}_\delta(B)$ and hence $\mathrm{mCl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\delta(B))$.

(8) $\Rightarrow$ (9): Let B be any subset of Y. Let $x \in f^{-1}(\mathrm{Int}_\delta(B))$. Then $f(x) \in \mathrm{Int}_\delta(B)$ and $f(x) \notin Y - \mathrm{Int}_\delta(B) = \mathrm{Cl}_\delta(Y-B)$. Hence $x \notin f^{-1}(\mathrm{Cl}_\delta(Y-B))$. By (8) we have $x \notin \mathrm{mCl}_\theta(f^{-1}(Y-B))$. Therefore, there exists $U \in m_X$ containing x such that $x \in U \subset \mathrm{mCl}(U) \subset f^{-1}(B)$. Hence $x \in \mathrm{mInt}_\theta(f^{-1}(B))$. Therefore, $f^{-1}(\mathrm{Int}_\delta(B)) \subset \mathrm{mInt}_\theta(f^{-1}(B))$.

(9) $\Rightarrow$ (10): Let V be any open set of Y. Then $V \subset \mathrm{Int}(\mathrm{Cl}(V)) \subset \mathrm{Int}_\delta(\mathrm{sCl}(V))$ and by (9) $f^{-1}(V) \subset f^{-1}(\mathrm{Int}_\delta(\mathrm{sCl}(V))) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{sCl}(V)))$.

(10) $\Rightarrow$ (1): Let V be any open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{sCl}(V)))$. Hence, there exists $U \in m_X$ containing x such that $x \in U \subset \mathrm{mCl}(U) \subset f^{-1}(\mathrm{sCl}(V))$ which implies that $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(V)$. Therefore, f is almost strongly θ-m-continuous.

Remark 3.5 Let $f : (X,\tau) \to (Y,\sigma)$ be a function. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$) and $f : (X, m_X) \to (Y,\sigma)$ is almost strongly θ-m-continuous, then by Theorem 3.1 we obtain the characterizations established in Theorem 3.1 of [26] (resp. Theorem 2.1 of [5], Theorem 2.6 of [11] and Theorem 3.1 of [14]).

Corollary 3.1 *If $f^{-1}(\mathrm{Cl}_\delta(B))$ is m_θ-closed for every subset B of Y, then f is almost strongly θ-m-continuous.*

Proof. Let B be any subset of Y. Since $f^{-1}(\mathrm{Cl}_\delta(B))$ is m_θ-closed, $\mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}_\delta(B))) = f^{-1}(\mathrm{Cl}_\delta(B))$. Then $\mathrm{mCl}_\theta(f^{-1}(B)) \subset \mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}_\delta(B))) = f^{-1}(\mathrm{Cl}_\delta(B))$. Therefore, $\mathrm{mCl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\delta(B))$ and by Theorem 3.1 f is almost strongly θ-m-continuous.

Theorem 3.2 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost strongly θ-m-continuous;

(2) $\mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))) \subset f^{-1}(F)$ for every closed set F of Y;

(3) $\mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset B of Y;

(4) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let F be any closed set of Y. Then $Y - F$ is open in Y. By Theorem 3.1 and Lemma 3.4, we have

$$X - f^{-1}(F) = f^{-1}(Y - F) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(Y - F)))) = \mathrm{mInt}_\theta(X - f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))) = X - \mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))).$$

Therefore, $\mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))) \subset f^{-1}(F)$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then $\mathrm{Cl}(B)$ is closed in Y and by (2) we have $\mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset f^{-1}(\mathrm{Cl}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then we have $f^{-1}(\mathrm{Int}(B)) = X - f^{-1}(\mathrm{Cl}(Y - B)) \subset X - \mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(Y - B))))) = \mathrm{mInt}_\theta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$. Therefore, we obtain $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$.

(4) $\Rightarrow$ (1): Let V be any regular open set of Y. By (4) $f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(V))$ and hence $f^{-1}(V) = \mathrm{mInt}_\theta(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is m_θ-open and by Theorem 3.1 f is almost strongly θ-m-continuous.

Theorem 3.3 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost strongly θ-m-continuous;

(2) $\mathrm{mCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every $G \in \beta(Y)$;

(3) $\mathrm{mCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every $G \in \mathrm{SO}(Y)$;

(4) $f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))$ for every $V \in \mathrm{PO}(Y)$.

Proof. (1) $\Rightarrow$ (2): Let V be any β-open set of Y. It follows from Theorem 2.4 of [3] that $\mathrm{Cl}(V)$ is regular closed. Since f is almost strongly θ-m-continuous, by Theorem 3.2 we have $\mathrm{mCl}_\theta(f^{-1}(V)) \subset \mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset f^{-1}(\mathrm{Cl}(V))$. Therefore, we obtain $\mathrm{mCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): This is obvious since $\mathrm{SO}(Y) \subset \beta(Y)$.

(3) $\Rightarrow$ (4): Let V be any preopen set of Y. Then $Y - V$ is preclosed in Y and hence $\mathrm{Cl}(\mathrm{Int}(Y - V)) \subset Y - V$. Since $\mathrm{Cl}(\mathrm{Int}(Y - V))$ is regular closed, it is semi-open in Y. By (3), we have $\mathrm{mCl}_\theta(f^{-1}(\mathrm{Cl}(\mathrm{Int}(Y - V)))) \subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(Y - V))) \subset f^{-1}(Y - V)$. Therefore, we obtain $f^{-1}(V) \subset X -$

$\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(Y-V)))) = X - \text{mCl}_\theta(X - f^{-1}(\text{Int}(\text{Cl}(V)))) = \text{mInt}_\theta(f^{-1}(\text{Int}(\text{Cl}(V))))$.

(4) $\Rightarrow$ (1): Let V be any regular open set of Y. Then V is preopen and $f^{-1}(V) \subset \text{mInt}_\theta(f^{-1}(\text{Int}(\text{Cl}(V)))) = \text{mInt}_\theta(f^{-1}(V))$. By Lemma 3.4, $f^{-1}(V) = \text{mInt}_\theta(f^{-1}(V))$ and $f^{-1}(V)$ is m_θ-open in X. It follows from Theorem 3.1 that f is almost strongly θ-m-continuous.

Lemma 3.5 (Noiri [24]) *For a subset V of a topological space (Y, σ), the following properties hold:*

(1) $\alpha\text{Cl}(V) = \text{Cl}(V)$ *for every* $V \in \beta(Y)$,

(2) $\text{pCl}(V) = \text{Cl}(V)$ *for every* $V \in \text{SO}(Y)$,

(3) $\text{sCl}(V) = \text{Int}(\text{Cl}(V))$ *for every* $V \in \text{PO}(Y)$.

Corollary 3.2 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost strongly θ-m-continuous;

(2) $\text{mCl}_\theta(f^{-1}(V)) \subset f^{-1}(\alpha\text{Cl}(V))$ *for every* $V \in \beta(Y)$;

(3) $\text{mCl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{pCl}(V))$ *for every* $V \in \text{SO}(Y)$;

(4) $f^{-1}(V) \subset \text{mInt}_\theta(f^{-1}(\text{sCl}(V)))$ *for every* $V \in \text{PO}(Y)$.

Theorem 3.4 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost strongly θ-m-continuous;

(2) $\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}_\delta(B))))) \subset f^{-1}(\text{Cl}_\delta(B))$ *for every subset B of Y;*

(3) $\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Ci}(B))))) \subset f^{-1}(\text{Cl}_\delta(B))$ *for every subset B of Y;*

(4) $\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(V))))) \subset f^{-1}(\text{Cl}(V))$ *for every open set V of Y;*

(5) $\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(V))))) \subset f^{-1}(\text{Cl}(V))$ *for every preopen set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then $\text{Cl}_\delta(B)$ is closed in Y. By Theorem 3.2, $\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}_\delta(B))))) \subset f^{-1}(\text{Cl}_\delta(B))$.

(2) $\Rightarrow$ (3): This is obvious since $\text{Cl}(B) \subset \text{Cl}_\delta(B)$ for every subset B.

(3) $\Rightarrow$ (4): This is obvious since $\text{Cl}(V) = \text{Cl}_\delta(V)$ for every open set V (Lemma 2 of [43]).

(4) $\Rightarrow$ (5): Let V be any preopen set of Y. Then we have $V \subset \text{Int}(\text{Cl}(V))$ and $\text{Cl}(V) = \text{Cl}(\text{Int}(\text{Cl}(V)))$. Now, set $G = \text{Int}(\text{Cl}(V))$, then G is open in Y and $\text{Cl}(G) = \text{Cl}(V)$. Then by (4) we obtain $\text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(V))))) \subset f^{-1}(\text{Ci}(V))$.

F.6

(5) $\Rightarrow$ (1): Let K be any regular closed set of Y. Then we have $\text{Int}(K) \in \text{PO}(Y)$ and by (5) $\text{mCl}_\theta(f^{-1}(K)) = \text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(K)))) = \text{mCl}_\theta(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(\text{Int}(K)))))) \subset f^{-1}(\text{Cl}(\text{Int}(K))) = f^{-1}(K)$. By Lemma 3.4, $\text{mCl}_\theta(f^{-1}(K)) = f^{-1}(K)$ and hence $f^{-1}(K)$ is m_θ-closed in X. By Theorem 3.1, f is almost strongly θ-m-continuous.

Remark 3.6 By Theorems 3.2, 3.3 and 3.4, we obtain new characterizations of almost strongly θ-continuous functions and almost strongly θ-semi-continuous functions.

4 Relationships with other forms of m-continuity

Lemma 4.1 (Popa and Noiri [34]) *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \text{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Theorem 4.1 *Let (Y, σ) be a regular space. For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are eqivalent:*
(1) f is m-continuous;
(2) f is strongly θ-m-continuous;
(3) f is almost strongly θ-m-continuous.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and V be any open set of Y containing $f(x)$. Since Y is regular, there exists an open set W such that $f(x) \in W \subset \text{Cl}(W) \subset V$. Since f is m-continuous, there exists $U \in m_X$ containing x such that $f(U) \subset W$. We shall show that $f(\text{mCl}(U)) \subset \text{Cl}(W)$. Suppose that $y \notin \text{Cl}(W))$. There exists an open set G containing y such that $G \cap W = \emptyset$. Since f is m-continuous, by Lemma 3.2 $f^{-1}(G) = \text{mInt}(f^{-1}(G))$ and $f^{-1}(G) \cap U = \emptyset$ which implies that $f^{-1}(G) \cap \text{mCl}(U) = \emptyset$. Because if $f^{-1}(G) \cap \text{mCl}(U) \neq \emptyset$, then $\text{mInt}(f^{-1}(G)) \cap \text{mCl}(U) \neq \emptyset$. Let $z \in \text{mInt}(f^{-1}(G) \cap \text{mCl}(U)$. Then $z \in \text{mInt}(f^{-1}(G))$ and $z \in \text{mCl}(U)$. There exists $V \in m_X$ containing z such that $V \subset f^{-1}(G)$. Since $z \in \text{mCl}(U)$, by Lemma 4.1 $V \cap U \neq \emptyset$ which implies that $f^{-1}(G) \cap U \neq \emptyset$. This is a contradiction. Therefore, $f^{-1}(G) \cap \text{mCl}(U) = \emptyset$. Therefore, we have $G \cap f(\text{mCl}(U)) = \emptyset$ and hence $y \notin f(\text{mCl}(U))$. Consequently, we obtain $f(\text{mCl}(U)) \subset \text{Cl}(W) \subset V$. This shows that f is strongly θ-m-continuous.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y containing $f(x)$. Since

Y is regular, there exists an open set W such that $f(x) \in W \subset \mathrm{sCl}(W) \subset \mathrm{Cl}(W) \subset V$. Since f is almost strongly θ-m-continuous, there exists $U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(W) \subset V$; hence $f(U) \subset V$. This shows that f is m-continuous.

Definition 4.1 A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *faintly m-continuous* [28] if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

Lemma 4.2 (Noiri and Popa [28]) *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is faintly m-continuous;

(2) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ for every θ-open set V of Y;

(3) $f^{-1}(F) = \mathrm{mCl}(f^{-1}(F))$ for every θ-closed set F of Y.

Corollary 4.1 *Let (Y, σ) be a regular space. For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are eqivalent: strong θ-continuity, almost strong θ-continuity, m-continuity, almost m-continuity, weak m-continuity and faint m-continuuity.*

Proof. It is pointed out in Remark 4.2 of [28] that m-continuity, almost m-continuity, weak m-continuity and faint m-continuity are equivalent of one another. Therefore, this is an immediate consequence of Theorem 4.1.

Remark 4.1 (1) For a function $f : (X, \tau) \to (Y, \sigma)$, let $m_X = \tau$. Then by Theorem 4.1 we obtain the results established in Theorem 4.2 of [26] and Corollary 3.8 of [44].

(2) The results of Theorem 4.1 are true if Y is Hausdroff and rim-compact because it is shown in Theorem 4 of [21] that every Hausdroff and rim-compact space is regular.

Definition 4.2 A topological space (X, τ) is said to be

(1) *almost regular* [41] if for any regular closed set F and any point $x \in X - F$ there exist disjoint open sets U and V such that $x \in U$ and $\dot{F} \subset V$,

(2) *semi-regular* if for each open set U of X and each point $x \in U$ there exists a regular open set G of X such that $x \in G \subset U$.

Theorem 4.2 *If a function $f : (X, m_X) \to (Y, \sigma)$ is θ-m-continuous and (Y, σ) is almost-regular, then f is almost strongly θ-m-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since (Y, σ) is almost-regular, by Theorem 2.2 of [41], there exists a regular open set G of Y such that $f(x) \in G \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Since f is θ-m-continuous, there exists $U \in m$ containing x such that $f(\mathrm{mCl}(U)) \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{sCl}(V)$. Therefore, f is almost strongly θ-m-continuous.

Remark 4.2 Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $m = \tau$. Then by Theorem 4.2 we obtain a result established in Theorem 4.2 of [26].

Theorem 4.3 *If a function $f : (X, m_X) \to (Y, \sigma)$ is almost strongly θ-m-continuous and (Y, σ) is semi-regular, then f is strongly θ-m-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. By semi-regualrity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost strongly θ-m-continuous, by Theorem 3.1 there exists $U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset G \subset V$ and hence f is strongly θ-m-continuous.

Remark 4.3 Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $m = \tau$. Then by Theorem 4.3 we obtain a result established in Theorem 4.2 of [26].

Definition 4.3 An m-space $(X, m_X$ is said to be *m-regular* [27] if for each m_X-closed set F and each $x \notin F$, there exist disjoint m_X-open sets U and V such that $x \in U$ and $F \subset V$.

Remark 4.4 Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$). Then m-regularity coincides with regularity (resp. semi-regularity [10], pre-regularity [31], semi-pre-regularity [25]).

Definition 4.4 A minimal structure m on a nonempty set X is said to have *property* $(\mathcal{B})$ [17] if the union of any families of subsets belonging to m belongs to m.

Lemma 4.3 (Noiri and Popa [27]) *Let X be a nonempty set with an m-structure m_X satisfying the property $\mathcal{B}$. Then (X, m_X) is m-regular if and only if for each $x \in X$ and each m-open set U containing x, there exists an m-open set V such that $x \in V \subset \mathrm{mCl}(V) \subset U$.*

Theorem 4.4 *Let (X, m_X) be m-regular and m_X satisfy the property $(\mathcal{B})$. Then, a function $f : (X, m_X) \to (Y, \sigma)$ is almost strongly θ-m-continuous if and only if it is almost m-continuous.*

Proof. It is obvious that every almost strongly θ-m-continuous function is almost m-continuous. Suppose that f is almost m-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. By the almost m-continuity of f, there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{sCl}(V)$; hence $x \in U \subset f^{-1}(\mathrm{sCl}(V))$. Since (X, m_X) is m-regular, by Lemma 4.3 there exists $G \in m_X$ such that $x \in G \subset \mathrm{mCl}(G) \subset f^{-1}(\mathrm{sCl}(V))$; hence $f(\mathrm{mCl}(G)) \subset \mathrm{sCl}(V)$. This shows that f is almost strongly θ-m-continuous.

Remark 4.5 Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $m = \mathrm{SO}(X)$. Then by Theorem 4.4 we obtain results established in Theorem 2.5 of [11] and Theorem 2.9 of [5].

Definition 4.5 A subset K of an m-space (X, m_X) is said to be *m-closed relative to* (X, m_X) [27] if for any cover $\{V_\alpha : \alpha \in \Delta\}$ of K by m-open sets of (X, m_X), there exists a finite subset Δ_0 of Δ such that $K \subset \bigcup\{\mathrm{mCl}(V_\alpha) : \alpha \in \Delta_0\}$. If X is m-closed relative to (X, m_X), then (X, m_X) is said to be *m-closed.*

Remark 4.6 Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\delta\mathrm{PO}(X)$). The definition of "m-closed" gives the one of *quasi H-closed* [38] (resp. *s-closed* [8], *p-closed* [9], *δ_p-closed* [40]).

Theorem 4.5 *If $f : (X, m_X) \to (Y, \sigma)$ is a θ-m-continuous function from an m-closed space (X, m_X) onto a Urysohn space (Y, σ), then f is almost strongly θ-m-continuous.*

Proof. First, we shall show that (Y, σ) is quasi-H-closed. Let $\{V_\alpha : \alpha \in \Delta\}$ be any open cover of Y. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ-m-continuous, there exists an m_X-open set $U(x)$ containing x such that $f(\mathrm{mCl}(U(x))) \subset \mathrm{Cl}(V_{\alpha(x)})$. The family $\{U(x) : x \in K\}$ is a cover of X by m_X-open sets of X. Since (X, m_X) is m-closed, there exist a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X \subset \bigcup\{\mathrm{mCl}(U(x_k)) : x_k \in X, 1 \leq k \leq n\}$. Therefore, we obtain

$$\begin{aligned} Y = f(X) &\subset \bigcup\{f(\mathrm{mCl}(U(x_k))) : x_k \in X, 1 \leq k \leq n\} \\ &\subset \bigcup\{\mathrm{Cl}(V_{\alpha(x_k)}) : x_k \in X, 1 \leq k \leq n\}. \end{aligned}$$

This shows that (Y, σ) is quasi-H-closed. Every quasi-H-closed Urysohn space is almost-regular [32]. By Theorem 4.2, f is almost strongly θ-m-continuous.

Definition 4.6 A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *m-irresolute* [30] at $x \in X$ if for each open set V of (Y, σ) containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. A function f is said to be m-irresolute if it has this property at each $x \in X$.

Lemma 4.4 (Noiri and Popa [30]) *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

(1) f is m-irresolute if and only if $f(\mathrm{mCl}(A)) \subset \mathrm{sCl}(f(A))$ for every subset A of X,

(2) Let m satisfy the property ($\mathcal{B}$). Then f is m-irresolute if and only if $f^{-1}(V)$ is m-open for every semi-open set V of Y.

Theorem 4.6 *If a function $f : (X, m_X) \to (Y, \sigma)$ m-irresolute and m_X has the property ($\mathcal{B}$), then f is almost strongly θ-m-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. By Lemma 4.4, $f(\mathrm{mCl}(f^{-1}(V))) \subset \mathrm{sCl}(f(f^{-1}(V))) \subset \mathrm{sCl}(V)$. Let $U = f^{-1}(V)$. By Lemma 4.4, $x \in U \in m_X$ and $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(V)$. Hence f is almost strongly θ-m-continuous.

Remark 4.7 Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $m = \mathrm{SO}(X)$. If $f : (X, m_X) \to (Y, \sigma)$ is m-irresolute, then by Theorem 4.6 we obtain a result established in Theorem 2.2 of [11].

5 Some properties

Definition 5.1 An m-space (X, m_X) is said to be *m-Urysohn* [27] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ containing x and y, respectively, such that $\mathrm{mCl}(U) \cap \mathrm{mCl}(V) = \emptyset$.

Theorem 5.1 *If a function $f : (X, m_X) \to (Y, \sigma)$ is an almost strongly θ-m-continuous injection and (Y, σ) is Hausdorrf, then (X, m_X) is m-Urysohn.*

Proof. Let x_1, x_2 be any distinct points of X. Then, $f(x_1) \neq f(x_2)$. Since (Y, σ) is Hausdorrf, there exist open sets $V_i (i = 1, 2)$ such that $f(x_i) \in V_i$ and $V_1 \cap V_2 = \emptyset$; hence $\mathrm{sCl}(V_1) \cap \mathrm{sCl}(V_2) = \emptyset$. Since f is almost strongly θ-m-continuous, there exists $U_i \in m_X$ containing x_i such that $f(\mathrm{mCl}(U_i)) \subset \mathrm{sCl}(V_i)$ for $i = 1, 2$. This implies that $\mathrm{mCl}(U_1) \cap \mathrm{mCl}(U_2) = \emptyset$. Hence (X, m_X) is m-Urysohn.

Remark 5.1 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m = \tau$ (resp. SO(X)) and $f : (X, m_X) \to (Y, \sigma)$ is almost strongly θ-m-continuous, then by Theorem 5.1 we obtain the results established in Theorem 4.6 of [26] (resp. Theorem 2.5 of [5]).

Theorem 5.2 *Let (X, m_X) be an m-space. If for any distinct points $x_1, x_2 \in X$, there exists a function f of (X, m_X) onto a Hausdorff space (Y, σ) such that*

(1) $f(x_1) \neq f(x_2)$,

(2) f is θ-m-continuous at x_1, and

(3) f is almost strongly θ-m-continuous at x_2,

then (X, m_X) is m-Urysohn.

Proof. Let x_1, x_2 be any distinct points of X. Then, by the hypothesis there exists a function $f : (X, m_X) \to (Y, \sigma)$, where (Y, σ) is Hausdorff, which satisfies three conditions. Now let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since (Y, σ) is Hausdorff, there exist open sets $V_i, i = 1, 2$ such that $y_i \in V_i$ and $V_1 \cap V_2 = \emptyset$. This implies that $\mathrm{Cl}(V_1) \cap \mathrm{sCl}(V_2) = \emptyset$. Since f is θ-m-continuous at x_1, there exists $U_1 \in m_X$ containing x_1 such that $f(\mathrm{mCl}(U_1)) \subset \mathrm{Cl}(V_1)$. Since f is almost strongly θ-m-continuous at x_2, there exists $U_2 \in m_X$ containing x_2 such that $f(\mathrm{mCl}(U_2)) \subset \mathrm{sCl}(V_2)$. This implies that $\mathrm{mCl}(U_1) \cap \mathrm{mCl}(U_2) = \emptyset$. This shows that (X, m_X) is m-Urysohn.

Theorem 5.3 *Let X be a nonempty set with two minimal structures m_1, m_2 such that $U \cap V \in m_1$ whenever $U \in m_1$ and $V \in m_2$ and (Y, σ) a Hausdorff space. If a function $f : (X, m_1) \to (Y, \sigma)$ is almost strongly θ-m-continuous and a function $g : (X, m_2) \to (Y, \sigma)$ is θ-m-continuous, then $A = \{x \in X : f(x) = g(x)\}$ is m_1-θ-closed.*

Proof. Let $x \in X - A$, then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V and W of Y such that $f(x) \in V, g(x) \in W$ and $V \cap W = \emptyset$;hence $\mathrm{sCl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since f is almost strongly θ-m-continuous, there exists $G \in m_1$ containing x such that $f(\mathrm{mCl}(G)) \subset \mathrm{sCl}(V)$. Since g is θ-m-continuous, there exists $H \in m_2$ containing x such that $g(\mathrm{mCl}(W)) \subset \mathrm{Cl}(W)$. Now put $U = G \cap H$, then $U \in m_1, x \in U$ and $f(\mathrm{mCl}(U)) \cap g(\mathrm{mCl}(U)) = \emptyset$. Therefore, we obtain $\mathrm{mCl}(U) \cap A = \emptyset$ and $x \in X - m_1Cl_\theta(A)$. This shows that $\mathrm{m_1Cl_\theta}(A) \subset A$. By Lemma 3.4, $A = \mathrm{m_1Cl_\theta}(A)$ and hence A is m_1-θ-closed.

Remark 5.2 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_1 = m_2 = \tau$. Then by Theorem 5.3 we obtain the result established in Theorem 5.3 of [26].

Definition 5.2 A function $f : (X, m_X) \to (Y, \sigma)$ is said to have a *strongly θ-m-closed graph* if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist an m_X-open set U containing x and an open set V of Y containing y such that $[\mathrm{mCl}(U) \times \mathrm{sCl}(V)] \cap \mathrm{G}(f) = \emptyset$.

Remark 5.3 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$) and $f : (X, m_X) \to (Y, \sigma)$ is a function, then the almost strongly θ-m-closed graph is said to be *strongly scl-closed* in [15] (resp. *almost semi-θ-closed* in [5]).

Lemma 5.1 *A function $f : (X, m_X) \to (Y, \sigma)$ has an almost strongly θ-m-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m_X-open set U containing x and an open set V of Y containing y such that $f(\mathrm{mCl}(U)) \cap \mathrm{sCl}(V) = \emptyset$.*

Theorem 5.4 *If $f : (X, m_X) \to (Y, \sigma)$ is an almost strongly θ-m-continuous function and (Y, σ) is Hausdorff, then $\mathrm{G}(f)$ is almost strongly θ-m-closed.*

Proof. Suppose that $(x, y) \in (X \times Y) - \mathrm{G}(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $V \cap W = \emptyset$; hence $\mathrm{sCl}(V) \cap \mathrm{sCl}(W) = \emptyset$. Since f is almost strongly θ-m-continuous, there exists an m_X-open set U containing x such that $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(W)$. This implies that $f(\mathrm{mCl}(U)) \cap \mathrm{sCl}(V) = \emptyset$ and by Lemma 5.1 $\mathrm{G}(f)$ is almost strongly θ-m-closed.

Remark 5.4 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$) and $f : (X, m_X) \to (Y, \sigma)$ is almot strongly θ-m-continuous, then by Theorem 5.4 we obtain the result established in Theorem 4.3 in [15] (resp. Theorem 2.7 of [5]).

Definition 5.3 A subset K of a topological space (Y, σ) is said to be *N-closed relative to* (Y, σ) [6] if for any cover $\{V_\alpha : \alpha \in \Delta\}$ of K by open sets of (Y, σ), there exists a finite subset Δ_0 of Δ such that $K \subset \bigcup\{\mathrm{sCl}(V_\alpha) : \alpha \in \Delta_0\}$. If Y is N-closed relative to (Y, σ), then (Y, σ) is said to be *nearly compact* [42].

Theorem 5.5 *If $f : (X, m_X) \to (Y, \sigma)$ is an almost strongly θ-m-continuous function and K is m-closed relative to (X, m_X), then $f(K)$ is N-closed relative to (Y, m_Y).*

Proof. Let K be m-closed relative to (X, m_X). Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(K)$ by open sets of (Y, σ). For each $x \in K$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is almost strongly θ-m-continuous, there exists an m_X-open set $U(x)$ containing x such that $f(\mathrm{mCl}(U(x))) \subset \mathrm{sCl}(V_{\alpha(x)})$. The family $\{U(x) : x \in K\}$ is a cover of K by m_X-open sets of X. Since K is m-closed relative to (X, m_X), there exist a finite number of points, say, $x_1, x_2, ..., x_n$ in K such that $K \subset \bigcup\{\mathrm{mCl}(U(x_k)) : x_k \in K, 1 \leq k \leq n\}$. Therefore, we obtain

$$\begin{aligned} f(K) &\subset \bigcup\{f(\mathrm{mCl}(U(x_k))) : x_k \in K, 1 \leq k \leq n\} \\ &\subset \bigcup\{\mathrm{sCl}(V_{\alpha(x_k)}) : x_k \in K, 1 \leq k \leq n\}. \end{aligned}$$

This shows that $f(K)$ is N-closed relative to (Y, σ).

Corollary 5.1 *If $f : (X, m_X) \to (Y, \sigma)$ is an almost strongly θ-m-continuous surjection and (X, m_X) is m-closed, then (Y, σ) is nearly-compact.*

Remark 5.5 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$) and $f : (X, m_X) \to (Y, \sigma)$ is almost strongly θ-m-continuous, then by Theorem 5.5 and Corollary 5.1 we obtain the result established in Theorem 5.1 of [26] (resp. Theorem 2.1 of [5]).

Definition 5.4 An m-space (X, m_X) is said to be m-hyperconnected if $\mathrm{mCl}(U) = X$ for every m_X-open set U of X.

Remark 5.6 Let (X, τ) be a topological space. If $m = \tau$ or $\mathrm{SO}(X)$, then the definition of hyperconnected spaces is obtained by [22].

Theorem 5.6 *If $f : (X, m_X) \to (Y, \sigma)$ is an almost strongly θ-m-continuous surjection and (X, m_X) is m-hyperconnected, then (Y, σ) is hyperconnected.*

Proof. Let V be a nonempty open set of Y. Since f is surjective, there exists $x \in f^{-1}(V)$ and $U \in m_X$ containing x such that $f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(V)$. Since (X, m_X) is m-hyperconnected, $\mathrm{mCl}(U) = X$ and hence $Y = f(\mathrm{mCl}(U)) \subset \mathrm{sCl}(V)$. Therefore, $Y = \mathrm{sCl}(V)$ and by Theorem 3.1 of [22] (Y, σ) is hyperconnected.

Remark 5.7 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \mathrm{SO}(X)$ and $f : (X, m_X) \to (Y, \sigma)$ is almost strongly θ-m-continuous, then by Theorem 5.6 we obtain the result established in Theorem 2.10 of [5].

Definition 5.5 Let (X, m_X) be an m-space and A a subset of X. The *m-θ-frontier* of A [27], $\mathrm{mFr}_\theta(A)$, is defined by $\mathrm{mFr}_\theta(A) = \mathrm{mCl}_\theta(A) \cap \mathrm{mCl}_\theta(X - A)$.

Theorem 5.7 *The set of all points $x \in X$ at which a function $f : (X, m_X) \to (Y, \sigma)$ is not almost strongly θ-m-continuous is identical with the union of the m-θ-frontiers of the inverse images of regular open sets containing $f(x)$.*

Proof. Suppose that f is not almost strongly θ-m-continuous at $x \in X$. Then there exists a regular open set V of Y containing $f(x)$ such that $f(\mathrm{mCl}(U))$ is not contained in $\mathrm{sCl}(V)$ for every m_X-open set U containing x. Then $\mathrm{mCl}(U) \cap (X - f^{-1}(V)) \neq \emptyset$ for every m_X-open set U containing x and hence $x \in \mathrm{mCl}_\theta(X - f^{-1}(V))$. On the other hand, we have $x \in f^{-1}(V) \subset \mathrm{mCl}_\theta(f^{-1}(V))$ and hence $x \in \mathrm{mFr}_\theta(f^{-1}(V))$.

Conversely, suppose that f is almost strongly θ-m-continuous at $x \in X$ and let V be any regular open set of Y containing $f(x)$. Then by Theorem 3.1 we have $x \in f^{-1}(V) \subset \mathrm{mInt}_\theta(f^{-1}(\mathrm{sCl}(V))) = \mathrm{mInt}_\theta(f^{-1}(V))$. Therefore, $x \notin \mathrm{mFr}_\theta(f^{-1}(V))$ for each regular open set V of Y containing $f(x)$. This completes the proof.

6 New forms of almost strong θ-m-continuity

First we recall the relationships among some generalizations of open sets. If a subset A of a topological space (X, τ) is semi-open and semi-closed, then it is said to be *semi-regular* [8]. It is shown in [8] that the semi-closure $\mathrm{sCl}(U)$ is semi-open and semi-regular for any semi-open set U of (X, τ). This property is very useful. The set of all semi-regular sets of (X, τ) is denoted by $\mathrm{SR}(X)$. For a subset A of a topological space (X, τ), we put $\mathrm{srCl}(A) = \cap\{F : A \subset F, F \in \mathrm{SR}(X)\}$.

Let A be a subset of a topological space (X, τ). A point x of X is called a *semi-θ-cluster point* of A if $\mathrm{sCl}(U) \cap A \neq \emptyset$ for every $U \in \mathrm{SO}(X)$ containing x. The set of all semi-θ-cluster points of A is called the *semi-θ-closure* [8] of A and is denoted by $\mathrm{sCl}_\theta(A)$. A subset A is said to be *semi-θ-closed* if $A = \mathrm{sCl}_\theta(A)$. The complement of a semi-θ-closed set is said to be *semi-θ-open*. The family of all semi-θ-open sets of (X, τ) is denoted by

θSO(X). A subset A of a topological space (X, τ) is said to be *b-open* [4] if $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$.

We have the following diagram in which the converses of implications need not be true as shown in [30].

DIAGRAM I

regular open →	δ-open →	open →	α-open →	preopen →	δ-preopen
↓	↓	|	↓	↓	|
semi-regular →	semi-θ-open →	δ-semi-open →	semi-open →	b-open →	semi-preopen

Remark 6.1 In the diagram above, the following are to be noted:

(1) It is shown in [33] that openness and δ-semi-openness are independent of each other,

(2) It is shown in [30] that δ-preopenness and semi-preopenness are independent of each other.

If we take as m_X the families of generalized open sets stated in the diagram, we can define new kinds of almost strongly θ-m-continuous functions. By the resullts established in Sections 3-5, we can obtain those properties. We investigate the relationships among these functions.

Lemma 6.1 *Let m_1 and m_2 be two m-structures on a nonempty set X. If $m_1 \subset m_2$ and a function $f : (X, m_1) \to (Y, \sigma)$ is almost strongly θ-m-continuous, then $f : (X, m_2) \to (Y, \sigma)$ is almost strongly θ-m-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since $f : (X, m_1) \to (Y, \sigma)$ is almost strongly θ-m-continuous, there exists $U \in m_1$ containing x such that $f(\mathrm{m_1Cl}(U)) \subset \mathrm{sCl}(V)$. Since $m_1 \subset m_2$, we have $x \in U \in m_2$ and $\mathrm{m_2Cl}(U) \subset \mathrm{m_1Cl}(U)$. Therefore, we obtain $f(\mathrm{m_2Cl}(U)) \subset \mathrm{sCl}(V)$. This shows that $f : (X, m_2) \to (Y, \sigma)$ is almost strongly θ-m-continuous.

Let RO(X) (resp. RC(X)) be the family of all regular open (resp. regular closed) sets of a topological space (X, τ). The family of all δ-open sets of

(X, τ) forms a topology for X which is weaker than τ. This topology has $\mathrm{RO}(X)$ as the base. We shall denote it by τ_δ although it is usualy denoted by τ_s. Then we have $\mathrm{RO}(X) \subset \tau_\delta \subset \tau \subset \tau^\alpha$, where $\tau^\alpha = \alpha(X)$. For a subset A of X, we set $\mathrm{rCl}(A) = \cap\{K : A \subset K \text{ and } K \in \mathrm{RC}(X)\}$.

Lemma 6.2 *Let (X, τ) be a topological space. Then $\alpha\mathrm{Cl}(U) = \mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))$ for every $U \in \alpha(X)$.*

Proof. Let U be any α-open set of (X, τ). Since $\mathrm{RO}(X) \subset \tau \subset \tau^\alpha$, we have $\alpha\mathrm{Cl}(U) \subset \mathrm{Cl}(U) \subset \mathrm{rCl}(U)$. Suppose that $x \notin \alpha\mathrm{Cl}(U)$. There exists $G \in \tau^\alpha$ containing x such that $G \cap U = \emptyset$. Hence we have $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \cap U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) = \emptyset$. Since $x \in G \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \in \mathrm{RO}(X)$, we have $x \notin \mathrm{rCl}(U)$. Therefore, we obtain $\mathrm{rCl}(U) \subset \alpha\mathrm{Cl}(U)$ and $\alpha\mathrm{Cl}(U) = \mathrm{Cl}(U) = \mathrm{rCl}(U)$ for every $U \in \alpha(X)$. Moreover, for every $U \in \alpha(X)$, we have $\mathrm{Cl}(U) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))) = \mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))$. Therefore, we obtain $\alpha\mathrm{Cl}(U) = \mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))$ for every $U \in \alpha(X)$.

Theorem 6.1 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) $f : (X, \mathrm{RO}(X)) \to (Y, \sigma)$ is almost strongly θ-m-continuous;
(2) $f : (X, \tau_\delta) \to (Y, \sigma)$ is almost strongly θ-m-continuous;
(3) $f : (X, \tau) \to (Y, \sigma)$ is almost strongly θ-m-continuous;
(4) $f : (X, \tau^\alpha) \to (Y, \sigma)$ is almost strongly θ-m-continuous.

Proof. Since $\mathrm{RO}(X) \subset \tau_\delta \subset \tau \subset \tau^\alpha$, by Lemma 6.1 we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y containing $f(x)$. There exists an α-open set U containing x such that $f(\alpha\mathrm{Cl}(U)) \subset \mathrm{sCl}(V)$. Since $U \in \tau^\alpha$, we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \in \mathrm{RO}(X)$. By Lemma 6.2, we have $f(\mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))) = f(\alpha\mathrm{Cl}(U)) \subset \mathrm{sCl}(V)$. This shows that $f : (X, \mathrm{RO}(X)) \to (Y, \sigma)$ is almost strongly θ-m-continuous.

Corollary 6.1 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) $f : (X, \tau) \to (Y, \sigma)$ is almost strongly θ-continuous;
(2) $f : (X, \tau_\delta) \to (Y, \sigma)$ is almost strongly θ-continuous;
(3) $f : (X, \tau^\alpha) \to (Y, \sigma)$ is almost strongly θ-continuous.

Theorem 6.2 *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) $f : (X, \mathrm{SR}(X)) \to (Y,\sigma)$ is almost strongly θ-m-continuous;
(2) $f : (X, \theta\mathrm{SO}(X)) \to (Y,\sigma)$ is almost strongly θ-m-continuous;
(3) $f : (X, \delta\mathrm{SO}(X)) \to (Y,\sigma)$ is almost strongly θ-m-continuous;
(4) $f : (X, \mathrm{SO}(X)) \to (Y,\sigma)$ is almost strongly θ-m-continuous.

Proof. Since $\mathrm{SR}(X) \subset \theta\mathrm{SO}(X) \subset \delta\mathrm{SO}(X) \subset \mathrm{SO}(X)$, by Lemma 6.1 we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1): Suppose that $f : (X, \mathrm{SO}(X)) \to (Y,\sigma)$ is almost strongly θ-m-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. There exists $U \in \mathrm{SO}(X)$ containing x such that $f(\mathrm{sCl}(U)) \subset \mathrm{sCl}(V)$. By Proposition 2.2 of [8], $\mathrm{sCl}(U) \in \mathrm{SR}(X)$ and we have $x \in \mathrm{sCl}(U)$. Moreover, we have $\mathrm{srCl}(\mathrm{sCl}(U)) = \mathrm{sCl}(U)$. Therefore, we obtain $f(\mathrm{srCl}(\mathrm{sCl}(U))) = f(\mathrm{sCl}(U)) \subset \mathrm{sCl}(V)$. This shows that $f : (X, \mathrm{SR}(X)) \to (Y,\sigma)$ is almost strongly θ-m-continuous.

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