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SOME REPRESENTATIONS OF CONJUGATE AND LOCALLY CONJUGATE MAPPINGS

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Abstract

Duality relations are obtained for quasisuperlinear functions. Relation between conjugate and locally conjugate mappings is investigated.

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1. Introduction

A great number of optimization problems arising in mathematical programming, classical optimal control problems, differential games, economic dynamics and so on, can be expressed in terms of differential inclusions. And the main object in studying optimization problems for differential inclusions consists of obtaining necessary and sufficient conditions for optimality. One of the central directions in optimal control theory to obtain necessary and sufficient conditions is by using conjugate and locally conjugate mappings. We refer the reader to the survey papers [1],[6-8],[14],[18-20]. So in this paper we deal with conjugate and locally conjugate mappings that facilitate deriving optimality conditions. In Section 2, under some conditions, locally conjugate mappings are given by the subgradient of the "support function" which is important in necessary and sufficient conditions.(Theorem 1.1)

By the Lemma 1.2 in Section 2 are expressed essential duality relations which can be applied furthemore in different optimization problems.

In Section 3 we investigate the relation between conjugate mapping and locally conjugate mapping in Theorem 1.5.

2.Necessary concepts and duality relation

A multivalued mapping $a : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is convex if its graph $gfa = \{(x, y) : y \in a(x)\}$ is a convex subset of $\mathbb{R}^{2n}$. a is convex-valued if $a(x)$ is a convex set for each $x \in doma$ ($doma = \{x : a(x) \neq \emptyset\}$). a is closed if gfa is a closed set in $\mathbb{R}^{2n}$. a is bounded if there exists a constant c such that $\|a(x)\| \leq c(1+\|x\|)$.

Let us define

$$W_a(x, y^*) = \begin{cases} \inf\{< y, y^* >: y \in a(x)\}, & a(x) \neq \emptyset \\ +\infty, & a(x) = \emptyset, \end{cases}$$

$a(x, y^*) = \{y \in a(x) :< y, y^* >= W_a(x, y^*)\}$ and

for any set M

$W_M(x^*) = \inf_{y \in M} < x^*, y >$.

Definition 1.1: The cone $K_M(z_0)$ of tangent directions of the set M at a point $z_0 \in M$ is called a local tent if for each $\bar{z}_0 \in riK_M(z_0)$(the relative interior of the set $K_M(z_0)$) there exist a convex cone $K \subseteq K_M(z_0)$ and a continuous function $\Psi(\bar{z})$, defined in a neighborhood of the origin, such that

i)$\bar{z}_0 \in riK$, $LinK = LinK_M(z_0)$, where $LinK$ is the linear span of K,

ii) $\Psi(\bar{z}) = \bar{z} + r(\bar{z})$, $\|\bar{z}\|^{-1} r(\bar{z}) \to 0$, as $\bar{z} \to o$

iii) $z_0 + \Psi(\bar{z}) \in M, \bar{z} \in K \cap B_\varepsilon(0)$ for some $\varepsilon > 0$, there $B_\varepsilon(0)$ is a ball centered in the origin with radius ε.

For a convex mapping a at a point $z_0 = (x_0, y_0) \in gfa$, the set

$$K_{gfa}(x_0, y_0) = cone(gfa - (x_0, y_0))$$
$$= \{\bar{z} = (\bar{x}, \bar{y}) : \bar{x} = \lambda(x - x_0), \bar{y} = \lambda(y - y_0), \lambda > 0, \forall (x, y) \in gfa\}$$

is a local tent of a.

Definition 1.2: 1)$h(\bar{x}, x)$ is called the upper convex approximation(UCA) of a function $g(x)$ at a point $x \in domg = \{x : |g(x)| < +\infty\}$ [1] if:

$$\text{i)} h(\bar{x}, x) \geq F(\bar{x}, x) = \sup_{\tau(.)} \limsup_{\lambda \downarrow 0} \tfrac{1}{\lambda}(g(x + \lambda \bar{x} + \tau(\lambda)) - g(x))$$

$\lambda^{-1}\tau(\lambda) \to 0, \quad \lambda \downarrow 0 \quad$ for all $\quad \bar{x} \neq 0$.

ii)$h(\bar{x}, x)$ is a convex closed (lower semicontinuous) positive homogeneous function of $\bar{x}$.

2) The set

$\partial h(0, x) = \{x^* \in \mathbb{R}^n : h(\bar{x}, x) \geq < \bar{x}, x^* >, \bar{x} \in \mathbb{R}^n\}$,

is called a subdifferential of the function g at the point x and is denoted by $\partial g(x)$, there symbol $< .,. >$ denotes scalar product. It is known that when $g(x)$ is convex, the given definition coincides with the usual definition of the subdifferential.(see [1])

A function g is said to be proper if it does not take the value $-\infty$ and is not identically $+\infty$.

3)The mapping

$a^*(y^*; z) = \{x^* : (-x^*, y^*) \in K_a^*(z)\}$

is called a locally conjugate mapping (LCM) to the convex mapping a at the point z.

Theorem 1.1: Let $a : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be convex-valued closed bounded continuous mapping such that the function $W_a(x, y^*) = \inf_{y \in a(x)} < y, y^* >$ is continuous differentiable on x. Let us suppose that the vector $\bar{z}_1 = (\bar{x}_1, \bar{y}_1)$ satisfies the inequality

$$< \bar{x}_1, \frac{\partial W_a(x_0, y^*)}{\partial x} > - < \bar{y}_1, y^* >< 0.$$

Then the following statements are true for a point $z_0 = (x_0, y_0), y_0 \in a(x_0, y^*)$:

i)The cone

$$K_a(z_0) = \left\{ \bar{z} :< \bar{x}, \frac{\partial W_a(x_0, y^*)}{\partial x} > - < \bar{y}, y^* >< 0 \right\}$$

is the smooth local tent, which is the cone of tangent directions to $gf.a$(graph of a) at the point z_0.

ii) LCM a^* corresponding to the cone $K_a(z_0)$ may be given by the formula

$$a^*(y^*; z_0) = \left\{ \frac{\partial W_a(x_0, y^*)}{\partial x} \right\}$$

Proof. If $S_a(x, y^*)$, $y^* \in \mathbb{R}^n$, is the support function to $a(x)$, then by the theory of convex analysis it is known that $y \in a(x)$ if and only if $< y, y^* > \leq S_a(x, y^*)$ for all $y^* \in \mathbb{R}^n$. Since $S_a(x, y^*) = -W_a(x, -y^*)$, the preceding inequality means that $< y, y^* > \geq W_a(x, y^*)$. Thus $a(x)$ is given by

$$a(x) = \{y : W_a(x, y^*) - < y, y^* > \leq 0\}, \quad y^* \in \mathbb{R}^n. \tag{1}$$

Suppose

$$f_{y^*}(z) = W_a(x, y^*) - < y, y^* >, \tag{2}$$

then by the Lemma 3.1[1,p.225], $f_{y^*}(z)$ is continuous on y^* and is continuous differentiable on z. By the Theorem 2.2[1,p.211], UCA(upper convex approximation) $h_{y^*}(\bar{z}, z)$ of the function $f_{y^*}(z)$ is

$$h_{y^*}(\bar{z}, z) = < \bar{z}, \frac{\partial W_a(x, y^*)}{\partial x} \times \{-y^*\} > . \qquad (3)$$

Furthermore $f_{y^*}(z_0) = 0$ and $f_{y^*}(z)$ has an UCA $h_{y^*}(\bar{z}, z_0)$, which is continuous on $\bar{z}$ and by the condition on the vector $\bar{z_1}$, we have $h_{y^*}(\bar{z}_1, z_0) < 0$. Then applying Theorem 3.3[1,p.234] by (7) we see that i) of the theorem follows. Since in this case

$$-con\partial f_{y^*}(z_0) = con \left\{ -\frac{\partial W_a(x_0, y^*)}{\partial x}, y^* \right\},$$

then by the same Theorem 3.3[1,p.234] the equality

$$a^*(y^*; z_0) = \left\{ \frac{\partial W_a(x_0, y^*)}{\partial x} \right\}$$

holds. This, in turn, implies that ii) is correct. The Theorem is proved.

Let $O^+(gfa)$ be the recession cone[2] to a convex function a in the space $Z = X \times Y$, i.e.

$$O^+(gfa) = \{\bar{z} : z + \lambda\bar{z} \in gfa, \lambda \geq 0, \forall z \in gfa\}. \qquad (4)$$

For such convex function a, let us define

$$\Omega_a(x^*, y^*) = \inf\{- < x, x^* > + < y, y^* >: (x, y) \in gfa\}. \qquad (5)$$

It is evident that

$$\Omega_a(x^*, y^*) = \inf_x \{- < x, x^* > + W_a(x, y^*)\}. \qquad (6)$$

Definition 1.2: The function

$$a^*(y^*) = \{x^* : (-x^*, y^*) \in (O^+ gfa)^*\}$$

is called conjugate function to a convex function a. It is clear that if mapping a is superlinear[5], i.e. gfa is a cone, then this definition coincides with the definition of B.H.Pshenichnyi [1].

Conjugate function can be used in different problems connected with duality theorems.

Definition 1.3: Multivalued mapping a is called quasisuperlinear if its graph is in the form of

$$gfa = M + K,$$

where M is a convex compactum, K is a closed convex cone.

Lemma 1.1: For a convex mapping a we have

$$dom\Omega_a = \{(-x^*, y^*) : \Omega_a(x^*, y^*) > -\infty\} \subseteq (O^+gfa)^*.$$

If a is a quasisuperlinear mapping then

$$dom\Omega_a = K^*.$$

Proof. Let us assume the contrary: let $(-x_0^*, y_0^*) \in dom\Omega_a$, but $(-x_0^*, y_0^*) \notin (O^+gfa)^*$. It means that there exists a pair $(\bar{x}_0, \bar{y}_0) \in O^+gfa$, for which

$$- < x_0^*, \bar{x}_0 > + < y_0^*, \bar{y}_0 >< 0.$$

By the definition of O^+gfa, we have

$$(x, y) + \lambda(\bar{x}_0, \bar{y}_0) \in gfa, \quad (x, y) \in gfa, \quad \lambda > 0.$$

Then

$$- < x + \lambda\bar{x}_0, x_0^* > + < y + \lambda\bar{y}_0, y_0^* >= - < x_0^*, x > + < y_0^*, y > +$$
$$+\lambda\{- < \bar{x}_0, x_0^* > + < \bar{y}_0, y_0^* >\} \to -\infty \quad \text{for} \quad \lambda \to +\infty,$$

which contradicts the fact that $(-x_0^*, y_0^*) \in dom\Omega_a$. This proves the first statement of the lemma. Furthermore, when a is a quasisuperlinear mapping, applying Result 9.1.2[2] and Lemma3.6.1[1],we get

$$(O^+gfa)^* = [O^+(M + K)]^* = (O^+M)^* \cap (O^+K)^* = \mathbb{R}^n \cap K^* = K^*.$$

On the other hand

$$dom\Omega_a = dom(\Omega_M + \Omega_K) = dom\Omega_M \cap dom\Omega_K = dom\Omega_K = K^*.$$

Hence

$$dom\Omega_a = K^*.$$

Lemma is proved.

The following example shows that the inverse inclusion generally is not true. In fact, let $a : X \to 2^Y (X, Y$ one-dimensional axises) is given as:

$$a(x) = \{y : y \geq x^2\} \quad , \quad gfa = \{(x, y) : y \geq x^2\}.$$

Check that $O^+gfa = \{0\} \times Y^+$, where Y^+ is the positive y-axis. Therefore $(O^+gfa)^* = \{(-x^*, y^*) : x^* \in X, y^* \in Y^+\}$. Then it is clear that $(-x_0^*, y_0^*) \in (O^+gfa)^*$, $\quad x_0^* = 1, y_0^* = 0$, but $(-x_0^*, y_0^*) \notin dom\Omega_a$.

Lemma 1.2: Let a be a quasisuperlinear mapping and $W_a(., y^*)$ be proper closed function. Then the relation

$$\sup_{x^* \in a^*(y^*)} \{< x, x^* > + \Omega_M(x^*, y^*)\} = \inf_{y \in a(x)} < y, y^* >$$

holds.

Proof. From Lemma 1.1, we have

$$dom\Omega_a = (O^+gfa)^* = K^*.$$

Therefore with regard to Theorem 4.1.III[1] we find the relation

$$\sup_{x^*}\{\Omega_a(x^*, y^*) + < x, x^* >\} =$$

$$\sup_{x^*}\{< x, x^* > + \Omega_M(x^*, y^*) : x^* \in a^*(y^*)\} = W_a(x, y^*).$$

Remark 1.2.1: If $M = \{0\}$, then $\Omega_M = 0$ and so the result of the above lemma coincides with the result of the Theorem 4.5.III[1,p.129].

Lemma 1.3: Let a be a convex mapping. Then the point x_0 is a solution of the problem

$$\inf_x\{- < x, x^* > + W_a(x, y^*)\}, \quad x^*, y^* \in \mathbb{R}^n$$

if and only if

$$x^* \in a^*(y^*, z_0), \quad y_0 \in a(x_0, y^*).$$

Proof. By the Theorem 2.1.IV[1], x_0 is a minimum point of the convex function

$$- < x, x^* > + W_a(x, y^*)$$

if and only if

$$0 \in \partial_x[- < x_0, x^* > + W_a(x_0, y^*)],$$

i.e.

$$x^* \in \partial_x W_a(x_0, y^*).$$

And, therefore by the definition of Ω_a it is evident that $y_0 \in a(x_0, y^*)$. Then by the Theorem 2.1.III[1], we find the required result.

Theorem 1.2: Let a be a convex-valued closed bounded continuous mapping, satisfying the Lipschitz condition, and let the function $W_{a_z}(\bar{x}, y^*)$ be closed, where

$$a_z(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_a(z)\}.$$

Then for arbitrary $y \in a(x, y^*), z = (x, y) \in gfa$, the function $W_{a_z}(., y^*)$ is an UCA for $W_a(., y^*)$ and, besides,

$$a^*(y^*; z) = \partial_x W_a(x, y^*).$$

Proof. If $\bar{z} = (\bar{x}, \bar{y}) \in K_a(z), z = (x, y), y \in a(x)$, then by the definition of the cone of tangent directions, there is a function $\tau(\lambda)$, $\lambda^{-1}\tau(\lambda) \to 0$, $\lambda \downarrow 0$ ($\tau(\lambda) \in Z = X \times Y$.) such that $z + \lambda\bar{z} + \tau(\lambda) \in gfa$ for a sufficiently small $\lambda \geq 0$. That means

$$y + \lambda\bar{y} + \tau_y(\lambda) \in a(x + \lambda\bar{x} + \tau_x(\lambda)), \tau = (\tau_x, \tau_y), \tau_x(\lambda) \in X, \tau_y(\lambda) \in Y.$$

Since a satisfies the Lipschitz condition, $W_a(x, y^*)$ also satisfies the same condition by Lemma 3.2.V[1,p.226]. For such functions we have

$$F(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(W_a(x + \lambda\bar{x}, y^*) - W_a(x, y^*)).$$

It is easily shown that

$$F(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(W_a(x + \lambda\bar{x} + \tau_x(\lambda), y^*) - W_a(x, y^*))$$

holds independently from the choice of $\tau(\lambda)$. From the definition of $W_a(x, y^*)$ and from the condition $y \in a(x, y^*)$ it follows that

$$\frac{1}{\lambda}(W_a(x+\lambda\bar{x}+\tau_x(\lambda), y^*)-W_a(x, y^*)) \leq \frac{1}{\lambda}(< y+\lambda\bar{y}+\tau_y(\lambda), y^* > - < y, y^* >) =$$

$$< \bar{y}, y^* > + < \frac{\tau_y(\lambda)}{\lambda}, y^* > .$$

Then we have

$$F(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(W_a(x + \lambda\bar{x} + \tau_x(\lambda), y^*) - W_a(x, y^*)) \leq$$

$$\limsup_{\lambda \downarrow 0}[< \bar{y}, y^* > + < \lambda^{-1}\tau_y(\lambda), y^* >] =< \bar{y}, y^* > .$$

It means that

$$F(\bar{x}, x) \leq \inf_{\bar{y}}\{< \bar{y}, y^* >: \quad \bar{y} \in a_z(\bar{x})\}.$$

In addition, given $\bar{x} \notin doma_z$ let us put $W_{a_z}(\bar{x}, y^*) = +\infty$. Then, by applying Lemma 1.2 to a_z, we get

$$W_{a_z}(\bar{x}, y^*) = \sup_{x^*}\{< \bar{x}, x^* >: \quad x^* \in a_z^*(y^*)\}.$$

But on the other hand, by the definition, $a^*(y^*; z) = a_z^*(y^*)$. Hence

$$F(\bar{x}, x) \leq W_{a_z}(\bar{x}, y^*) = \sup_{x^*}\{< \bar{x}, x^* >: \quad x^* \in a^*(y^*; z)\},$$

where $W_{a_z}(\bar{x}, y^*)$ is positive homogenous convex closed function of $\bar{x}$, i.e. $W_{a_z}(\bar{x}, y^*)$ is an UCA function of $W_a(., y^*)$ at the point x. Now to conclude the proof, it remains only to apply Theorem 3.2.II[1], thus we find

$$\partial W_a(x, y^*) = \partial h(0, x) = a^*(y^*; z).$$

3.Connection between conjugate and locally conjugate mappings

Let us investigate the relation between conjugate function and LCM(Locally Conjugate Mapping). We need the following two theorems.

Let $K_M(z)$ be the cone of tangent directions to a convex set $M \subseteq Z =$

$X \times Y$ at a point $z \in M$, i.e.

$$K_M(z) = con(M - z) = \{\bar{z} : \bar{z} = \lambda(z_1 - z), \lambda > 0, z_1 \in M\}. \tag{7}$$

Theorem 1.3: Let O^+M be the recession cone of a convex closed set $M \subset Z$. Then we have

$$\bigcap_{z \in M} K_M(z) = O^+M.$$

Proof. Let us show that

$$M = \bigcap_{z \in M} (z + K_M(z)). \tag{8}$$

In fact, let $z_0 \in M$ be an arbitrary fixed point. It is evident that all vectors as $\bar{z} = z_0 - z$ (in definition (11) they corresponds to $\lambda = 1$) belong to the cone $K_M(z)$, i.e. $z_0 \in z + K_M(z), z \in M$, then $z_0 \in \bigcap_{z \in M} (z + K_M(z))$. Conversely, if we have the last inclusion then $z_0 \in z + K_M(z)$ or there are such $z_1 \in M$ and a number $\gamma > 0$, that $z_0 - z = \gamma(z_1 - z) \in K_M(z)$. Hence $z_0 = \gamma z_1 + (1 - \gamma)z \in M$. Formula (12) follows.

On the other hand, we easily show that

$$O^+[\bigcap_{z \in M} (z + K_M(z))] = \bigcap_{z \in M} [O^+(z + K_M(z))].$$

In fact if z is an arbitrary point of closed convex set $M = \bigcap_{z \in M} (z + K_M(z))$ then by the definition of the recession cone, it is evident that, directed ray $z + \lambda\bar{z}$, $\forall\lambda \geq 0$, is contained in any cone $z + K_M(z), z \in M$. But it means that

$$\bar{z} \in \bigcap_{z \in M} [O^+(z + K_M(z))].$$

Therefore

$$O^+M = O^+[\bigcap_{z \in M} (z + K_M(z))] = \bigcap_{z \in M} [O^+(z + K_M(z))] = \bigcap_{z \in M} K_M(z).$$

Theorem is proved.

Remark 1.3.1: In the statement of the above Theorem, the closedness of

M is essential.

Proof. Actually, let $M = \{(x,y) : x > 0, y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$. Clearly, $O^+M = M$. The set M contains points $(x_0, y_0) + \lambda(0, y_0)$, where $x_0 > 0$, $y_0 > 0$ are fixed. But $(0, y_0) \notin O^+M$.

Theorem **1.4:** Let M be a closed convex set and let $K_M^*(z)$ be the conjugate cone to the cone of tangent directions $K_M(z), z \in M$. Then

$$\overline{\bigcup_{z\in M} K_M^*(z)} = (O^+M)^*,$$

where the bar denotes closure.

Proof. It is sufficient to show that

$$\overline{\bigcup_{z\in M} K_M^*(z)} = (\bigcap_{z\in M} K_M(z))^*. \tag{9}$$

Get any fixed point $z_0^* \in \overline{\bigcup_{z_0\in M} K_M^*(z)}$. Then there exists a sequence $z_n^* \to z_0^*$, $z_n^* \in \bigcup_{z\in M} K_M^*(z)$. Let us define sequence $\{z_n\}$ by the relation $z_n^* \in K_M^*(z_n)$. Note that $z_n^* \in \bigcup_{z\in M} K_M^*(z)$ implies the existence of $z_n \in M$ such that $z_n^* \in K_M^*(z_n)$.

On the other hand, since $K_M(z_n) \supseteq \bigcap_{z\in M} K_M(z)$ it is evident that $K_M^*(z_n) \subseteq (\bigcap_{z\in M} K_M(z))^*$. So that $z_n^* \in (\bigcap_{z\in M} K_M(z))^*$, and therefore $z_0^* \in (\bigcap_{z\in M} K_M(z))^*$.

Let us prove the converse inclusion in (13). Let us $z_1^* \in (\bigcap_{z\in M} K_M(z))^*$ be arbitrary fixed point and let us assume the contrary i.e. let $z_1^* \notin \overline{\bigcup_{z\in M} K_M^*(z)}$. Then $z_1^* \notin K_M^*(z)$ for any $z \in M$. In other words, there exists a vector $\bar{z}_1 (\bar{z}_1 \neq 0)$ such that

$$< z_1^*, \bar{z}_1 >< 0, \quad \bar{z}_1 \in K_M(z), \quad \forall z \in M$$

or

$$< z_1^*, \bar{z}_1 >< 0, \quad \bar{z}_1 \in \bigcap_{z\in M} K_M(z), \quad \text{i.e.} \quad z_1^* \notin (\bigcap_{z\in M} K_M(z))^*.$$

This contradiction shows that

$$(\bigcap_{z\in M} K_M(z))^* \subseteq \bigcup_{z\in M} K_M^*(z).$$

The proof of the theorem is over now.

Theorem 1.5: Let a be a closed convex mapping. Then the conjugate function $a^*(y^*)$ and the LCM of a implies the following relation

$$a^*(y^*) = \overline{\bigcup_{z \in gfa} a^*(y^*; z)}, \quad y \in a(x, y^*).$$

Proof. Setting $M = gfa$ as in the previous theorem, we obtain

$$a^*(y^*) = \overline{\bigcup_{z \in gfa} a^*(y^*; z)}.$$

By the Theorem 2.1.III[1] $z = (x, y)$, $y \notin a(x, y^*)$, implies $a^*(y^*; z) = \emptyset$.

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