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NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY IN DISCRETE INCLUSIONS DESCRIBED BY CONVEX MULTIVALUED MAPPINGS AND DUALITY

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Abstract

Necessary and sufficient conditions of optimality for convex case are deduced for the considered optimization problem (P_M) with discrete inclusions on the basis of the apparatus of locally conjugate mappings for convex compositions and the cones of tangent directions. Then duality problem (P_D) is formulated for the considered problem (P_M) and duality theorem is proved.

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1. Introduction

It is known that optimization problems with discrete $t = 0, 1, ..., T-1$ time in euclidean n-dimensional spaces can be turn to the minimization problem of a function in nT-dimension spaces on an intersection of finitely number sets and one of the theorems of mathematics programming is used. Note that the locally adjoint mappings for convex compositions is given by B.N. Pshenichnyi [1]. The same results for superlinear mappings are obtained by A.M.Rubinov [2]. Different optimization problems and duality for discrete and differential inclusions are derived by E.N. Mahmudov (see [3]-[6]). In the presented work, optimality conditions for a multivalued mapping constraint problem, which is equivalent to the problem with discrete inclusions, have been obtained.

2. Necessary information and problem statement

Lets consider the following convex discrete time problem:

$$
(P_M) \qquad \begin{array}{ll} minimize & g(x_T) \\ subject\ to & x_{t+1} \in a_t(x_t),\ t = 0, 1, ..., T-1 \\ & x_0 \in N,\ x_T \in M \end{array} \qquad \begin{array}{r} (1) \\ (2) \\ (3) \end{array}
$$

where a_t is a convex multivalued mapping for each $t = 0, 1, ..., T-1$ such that $a_t : X^t \to X^{t+1}$, all X^t are an euclidean n-dimensional space, g is a convex function, $N \subseteq X^0$ and $M \subseteq X^T$ are convex sets.

The set of vectors $x_0, x_1, ..., x_T$ which hold (1)-(3) is called optimal trajectory and denoted by $\{x_t\}_{t=0}^T$. Let us show the composition of mappings $a_0, a_1, ..., a_{T-1}$ by a^T, i.e. $a^T \equiv a_{T-1} \circ a_{T-2} \circ ... \circ a_0$. It is obvious that $a^T : X^0 \to X^T$. On the other hand, the composition for multivalued mappings is given by

$$
\begin{aligned} (a_t \circ a_{t-1})(x_{t-1}) &= \{x_{t+1} : x_{t+1} \in a_t(x_t),\ x_t \in a_{t-1}(x_{t-1})\} \\ &= \bigcup_{x_t \in a_{t-1}(x_{t-1})} a_t(x_t) \end{aligned}
$$

After all these, the problem (1)-(3) is equal to problem below

$$
(P_V) \qquad \begin{array}{ll} minimize & g(x_T) \\ subject\ to & x_T \in a^T(x_0), \\ & x_0 \in N,\ x_T \in M \end{array} \qquad \begin{array}{r} (4) \\ (5) \\ (6) \end{array}
$$

The graph (gph) of composition of multivalued mappings can be defined by using the definition of graph of the multivalued mappings like below:

$$
gph(a_t \circ a_{t-1}) = \{(x_{t-1}, x_{t+1}) : (x_{t-1}, x_t) \in gph\, a_{t-1},\ (x_t, x_{t+1}) \in gph\, a_t\}
$$

Now let us give some necessary definitions.

Definition 1. $K_a(x, y) = \{(\bar{x}, \bar{y}) : (x_0, y_0) + \lambda(\bar{x}, \bar{y}) \in gph\ a, \lambda > 0\}$.

Definition 2. Subdifferential of the function $g(x_T)$ at a point x_T is given by the formula

$$
\partial g(x_T) = \{x_T^* :\ g(x_T) - g(\bar{x}_T) \geq < x_T^*, x_T - \bar{x}_T >,\ \forall \bar{x}_T \in X^T\}.
$$

Definition 3. $(a^T)^*(x_T^*, (\tilde{x}_0, \tilde{x}_T)) = \{x_0^* : (-x_0^*, x_T^*) \in K_{a^T}^*(\tilde{x}_0, \tilde{x}_T)\}$.

Theorem 1. *Let the function $g(x_T)$ has the minimum value at the point $\tilde{x}_T$. Furthermore, let $K_{a^T}(\tilde{x}_0, \tilde{x}_T)$, $K_N(\tilde{x}_0)$, $K_M(\tilde{x}_T)$ be cones of tangent directions for the sets gph a^T, N, M, respectively, where $\tilde{x}_0 \in$ dom a^T. Then there exist a number $\lambda \geq 0$ and vectors $x_T^* \in \partial g(\tilde{x}_T)$, $x_0^* \in K_N^*(\tilde{x}_0)$, $x_e^* \in K_M^*(\tilde{x}_T)$ which are not equal simultaneously to zero such that*

$$x_0^* \in (a^T)^*(x_T^*; (\tilde{x}_0, \tilde{x}_T)),\ x_T^* + x_e^* \in \lambda \partial g(\tilde{x}_T),\ x_e^* \in K_M^*(\tilde{x}_T).$$

In addition, if $\lambda = 1$ then these conditions are also sufficient for optimality.

Proof. It can be see easily that the solution of the problem (4)-(6) is equivalent to the minimization problem of the function $f(z) = g(x_T)$, $z = (x_0, x_T)$ on the intersection set $gph\ a^T \cap N \times X^T \cap X^0 \times M \subseteq Z = X^0 \times X^T$. Furthermore subdifferential of $f(z)$ at point $\tilde{z} = (\tilde{x}_0, \tilde{x}_T)$ has the form

$$\partial f(\tilde{z}) = \{0\} \times \partial g(\tilde{x}_T)\ . \tag{7}$$

Then,

$$\partial f(\tilde{z}) = \{z^* = (\tilde{x}_0^*, \tilde{x}_T^*) : \tilde{x}_0^* = 0,\ \tilde{x}_T^* \in \partial g(\tilde{x}_T)\}$$

According to conditions in theorem $K_{a^T}(\tilde{z})$, $K_N(\tilde{x}_0) \times X^T$, $X^0 \times K_M(\tilde{x}_T)$ are cones of tangent directions for the sets $gph\ a^T, N \times X^T, X^0 \times M$ at the point $\tilde{z}$, respectively. If we calculate their dual cones we find

$$(K_N(\tilde{x}_0) \times X^T)^* = K_N^*(\tilde{x}_0) \times \{0\}, \tag{8}$$

$$(X^0 \times K_M(\tilde{x}_T))^* = \{0\} \times K_M^*(\tilde{x}_T)\ . \tag{9}$$

Now, using Theorem IV.2.4 in [1] and the formulas (7)-(9) there exist a number $\lambda \geq 0$ and vectors x_0^*, x_e^*, (x_0^{1*}, x_T^{1*}) not equal simultaneously to zero such that

$$\lambda(0, \tilde{x}_T^*) = (x_0^{1*}, x_T^{1*}) + (x_0^*, 0) + (0, x_e^*), \tag{10}$$

$$\tilde{x}_T^* \in \partial g(\tilde{x}_T),\ (x_0^{1*}, x_T^{1*}) \in K_{a^T}^*(\tilde{z}), \tag{11}$$

$$x_0^* \in K_N^*(\tilde{x}_0),\ x_e^* \in K_M^*(\tilde{x}_T)\ . \tag{12}$$

We have from (10) that $\lambda \tilde{x}_T^* = x_T^{1*} + x_e^*$, $x_0^{1*} + x_0^* = 0$ or $x_0^{1*} = -x_0^*$, $x_T^{1*} = \lambda \tilde{x}_T^* - x_e^*$. Consequently

$$(-x_0^*, \lambda \tilde{x}_T^* - x_e^*) \in K_{a^T}^*(\tilde{z})$$

Then, using the Definition 3 of the locally conjugate mapping we have

$$x_0^* \in (a^T)^*(\lambda \tilde{x}_T^* - x_e^*, \tilde{z})$$

where the vectors λ, x_0^*, x_e^* are not equal simultaneously to zero. Let $x_T^* = \lambda \tilde{x}_T^* - x_e^*$, $x_e^* \in K_M^*(\tilde{x}_T)$. It follows from [1,Theorem IV.2.4] that

$$x_T^* + x_e^* = \lambda \tilde{x}_T^*\ ,\ x_e^* \in K_M^*(\tilde{x}_T)$$

and so finally

$$x_T^* + x_e^* \in \lambda \partial g(\tilde{x}_T)\ ,\ x_e^* \in K_M^*(\tilde{x}_T)$$

where the vectors λ, x_0^*, x_e^* are not equal simultaneously to zero.

The relationship between $(a^T)^*$ with a_t^*, $t = 0, 1, 2, ..., T-1$, are given in the following Theorem 2.

Theorem 2. *Let $K_{a_t}(\tilde{x}_t, \tilde{x}_{t+1})$ be cones of tangent directions for the multivalued mappings a_t, $t = 0, 1, ..., T-1$, at the point $(\tilde{x}_t, \tilde{x}_{t+1}) \in gph\ a_t$. In addition, we suppose that there exist the points $\bar{x}_t^0 \in X^t$ for which hold one of the conditions below*

(a) $(\bar{x}_t^0, \bar{x}_{t+1}^0) \in ri\ gph\ a_t\ ,\ t = 0, 1, 2, ..., T-1$

(b) $(\bar{x}_t^0, \bar{x}_{t+1}^0) \in int\ gph\ a_t\ ,\ t = 0, 1, 2, ..., T-2$

$(\bar{x}_{T-1}^0, \bar{x}_T^0) \in gph\ a_{T-1}\ .$

Then the following equality is verified

$(a_{T-1} \circ a_{T-2} \circ \cdots \circ a_0)^*(x_T^*; (\tilde{x}_0, \tilde{x}_T))$

$$= \{x_0^* : x_0^* \in a_0^*(x_1^*, (\tilde{x}_0, \tilde{x}_1)), \cdots, x_{T-1}^* \in a_{T-1}^*(x_T^*, (\tilde{x}_{T-1}, \tilde{x}_T))\} \tag{13}$$

or shortly

$$(a^T)^*(\cdot, (\tilde{x}_0, \tilde{x}_T)) = a_0^*(\cdot, (\tilde{x}_0, \tilde{x}_1)) \circ \cdots \circ a_{T-1}^*(\cdot, (\tilde{x}_{T-1}, \tilde{x}_T)) .$$

Proof. Let us prove the theorem for the composition $a_t \circ a_{t-1}$. The following statements are equivalent to each other by the definition of locally conjugate mapping:

$$x_{t-1}^* \in (a_t \circ a_{t-1})^*(x_{t+1}^*, (\tilde{x}_{t-1}, \tilde{x}_{t+1}))$$

and

$$- < \bar{x}_{t-1}, x_{t-1}^* > + < \bar{x}_{t+1}, x_{t+1}^* > \geq 0$$

$$(\bar{x}_{t-1}, \bar{x}_{t+1}) \in K_{a_t \circ a_{t-1}}(\tilde{x}_{t-1}, \tilde{x}_{t+1}) \tag{14}$$

where $K_{a_t \circ a_{t-1}}(\tilde{x}_{t-1}, \tilde{x}_{t+1})$ are cones of tangent directions at point $(\tilde{x}_{t-1}, \tilde{x}_{t+1}) \in gph(a_t \circ a_{t-1})$. On other hand $(\tilde{x}_{t-1}, \tilde{x}_{t+1}) \in gph(a_t \circ a_{t-1})$ implies that for some $\tilde{x}_t \in X^t$ $(x_{t-1}, \tilde{x}_t) \in gph\ a_{t-1}$, $(x_t, \tilde{x}_{t+1}) \in gph\ a_t$.

Now let us consider the following two cones at point $(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1})$ in space $X^{t-1} \times X^t \times X^{t+1}$:

$$K_{t-1}(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1}) = \{(\bar{x}_{t-1}, \bar{x}_t, \bar{x}_{t+1}) : (\tilde{x}_{t-1}, \tilde{x}_t) \in K_{a_{t-1}}(\tilde{x}_{t-1}, \tilde{x}_t)\}$$

$$= K_{a_{t-1}}(\tilde{x}_{t-1}, \tilde{x}_t) \times X^{t+1} ,$$

$$K_t(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1}) = \{(\bar{x}_{t-1}, \bar{x}_t, \bar{x}_{t+1}) : (\bar{x}_t, \bar{x}_{t+1}) \in K_{a_t}(\tilde{x}_t, \tilde{x}_{t+1})\}$$

$$= X^{t-1} \times K_{a_t}(\tilde{x}_t, \tilde{x}_{t+1})$$

and let us write (14) as follow

$$- < \bar{x}_{t-1}, x_{t-1}^* > + < \bar{x}_t, 0 > + < \bar{x}_{t+1}, x_{t+1}^* > \geq 0 ,$$

$$(\bar{x}_{t-1}, \bar{x}_t) \in K_{a_{t-1}}(\tilde{x}_{t-1}, \tilde{x}_t) , \quad (\bar{x}_t, \bar{x}_{t+1}) \in K_{a_t}(\tilde{x}_t, \tilde{x}_{t+1}) \tag{15}$$

Then using Theorem 1.3.2 and Theorem II.3.10 in [1] we have obviously

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$$(-x_{t-1}^*, 0, x_{t+1}^*) \in [K_{t-1}(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1}) \cap K_t(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1})]^*$$

$$= K_{t-1}^*(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1}) + K_t^*(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1})$$

$$= K_{a_{t-1}}^*(\tilde{x}_{t-1}, \tilde{x}_t) \times \{0\} + \{0\} \times K_{a_t}^*(\tilde{x}_t, \tilde{x}_{t+1})$$

After all these the vector $(-x_{t-1}^*, 0, x_{t+1}^*)$ can be represented in the form:

$$-x_{t-1}^* = -x_{t-1}^{1*}, \quad 0 = x_t^{1*} - x_t^{2*}, \quad x_{t+1}^* = x_{t+1}^{2*},$$

$$(-x_{t-1}^{1*}, x_t^{1*}) \in K_{a_{t-1}}^*(\tilde{x}_{t-1}, \tilde{x}_t), \quad (-x_t^{2*}, x_{t+1}^{2*}) \in K_{a_t}^*(\tilde{x}_t, \tilde{x}_{t+1}) \qquad (16)$$

The statements (16) are equivalent to the following statements

$$x_{t-1}^* = x_{t-1}^{1*}, \quad x_{t-1}^{1*} \in o_{t-1}^*(x_t^{1*}, (\tilde{x}_{t-1}, \tilde{x}_t)), \quad x_t^{1*} \in a_t^*(x_{t+1}^*, (\tilde{x}_t, \tilde{x}_{t+1}))$$

Let us denote $a_{t-1} \circ a_t$ by a_t and $a_{t+2} \circ a_{t+1}$ by a_{t+1} etc. Then the proof ends using these recurrent statements.

Theorem 3. *The formula (15) is valid for polyhedral multivalued mappings a_t, $t = 0, 1, ..., T-1.$, without conditions (a),(b) of the theorem.*

Proof. Using Theorem 1.4.14 [1] instead of Theorem I.3.2 and Theorem II.3.10 [1] is sufficient in polyhedral case.

Theorem 4. *Let the trajectory $\{\tilde{x}_t\}_{t=0}^T$ be the optimal trajectory for the problem (1)-(3) and $K_{a_t}(\tilde{x}_t, \tilde{x}_{t+1})$, $K_N(\tilde{x}_0)$, $K_M(\tilde{x}_T)$ be cones of tangent directions for the convex multivalued mappings a_t, $t = 0, 1, ..., T-1$, the sets N and M respectively. Then under the conditions of Theorem 2 for the trajectory $\{\tilde{x}_t\}_{t=0}^T$ to be an optimal of the problem (1)-(3)it is necessary and sufficient that there exist some vectors x_e^*, x_t^*, $t = 0, ..., T$, not equal to zero simultaneously, such that*

$$x_t^* \in a_t^*(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})), \ t = 0, ..., T-1 \qquad (17)$$

$$x_T^* + x_e^* \in \partial_x g(\tilde{x}_T), \quad x_e^* \in K_M^*(\tilde{x}_T) \ ,$$

$$x_0^* \in K_N^*(\tilde{x}_0)$$

Proof. The proof is obtained from Theorem 1 and 2 taking into account that the inclusions $x_0^* \in (a^T)^*(x_T^*; (\tilde{x}_0, \tilde{x}_T))$ and $x_t^* \in a_t^*(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1}))$, $t = 0, ..., T-1$ are equivalent to each other.

Theorem 5. *Let the conditions of the Theorem 4 be valid and for each t the multivalued mapping $a_t(x)$ be a closed set for all $x_t \in X^t$. Then for the trajectory $\{\tilde{x}_t\}_{t=0}^T$ to be an optimal of the problem (1)-(3) it is necessary and sufficient that there exist some vectors x_e^*, x_t^*, $t = 0, 1, ..., T$ and the number $\lambda = 0, 1$, not equal to zero simultaneously, such that*

$$\tilde{x}_{t+1} \in \partial_{y^*} W_{a_t}(\tilde{x}_t, x_{t+1}^*) \quad , x_t^* \in \partial_x W_{a_t}(\tilde{x}_t, x_{t+1}^*) \ , \ t = 0, 1, ..., T-1$$

$$x_T^* + x_e^* \in \partial_x g(\tilde{x}_T), \ x_e^* \in K_M^*(\tilde{x}_T) \ ,$$

$$x_0^* \in K_N^*(\tilde{x}_0).$$

Proof. It is known that

$$W_a(x, y^*) = \inf_y \{< y, y^* >: y \in a(x)\}$$

$$a(x; y^*) = \{y \in a(x) :< y, y^* >= W_a(x, y^*)\}$$

From Theorem III.2.1 in [1] if $y \in a(x, y^*), z = (x, y)$ then $a^*(y^*; z) = \partial_x W_a(x, y^*)$. On the other hand From (17) we have $a_t^*(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})) \neq \emptyset$. Thus we can write

$$a_t^*(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})) = \partial_x W_{a_t}(\tilde{x}_t, x_{t+1}^*) \tag{18}$$

$$\tilde{x}_{t+1} \in a_t(\tilde{x}_t, x_{t+1}^*) \tag{19}$$

It is clear that $-W_a(x, y^*) = \sup_y \{- < y, y^* >: y \in a(x)\}$. Let for each t, the multivalued mapping $a_t(x)$ be a closed set for all $x \in X^t$. Then $-W_a(x, y^*)$ is a convex function with respect to y^* and using Theorem II.3.11 in [1] we have

$$\partial_{y^*}(-W_a(x, y^*)) = -a(x; y^*)$$

Then $W_a(x,\cdot)$ is concave because of $-W_a(x,\cdot)$ is convex for all fixed x. From the definition of subdifferential of the concave functions we have

$$\partial_{y^*}(W_a(x,y^*)) = -\partial_{y^*}(-W_a(x,y^*))$$

and thus $\partial_{y^*}(W_a(x,y^*)) = a(x;y^*)$. Consequently, (19) can be written in the form

$$\tilde{x}_{t+1} \in \partial_{y^*} W_{a_t}(\tilde{x}_t, x^*_{t+1}) \tag{20}$$

Finally using the formulae (18) and (20) the proof is completed.

3. Duality

For problem (P_V) with convex structure dual problem are constructed using the theorems of duality of operations of addition and infimal convolution of convex function. This duality problem consist of the following:

$$(P_D) \qquad \sup\{-g^*(z^*_T) + \Omega_{a^T}(x^*_0, z^*_T - x^*_e) + W_M(x^*_e) + W_N(x^*_0)\}$$

Theorem 6. If the solutions $\tilde{x}_T$ and $\{x^*_0, z^*_T, x^*_e\}$, $z^*_T \in \partial g(\tilde{x}_T)$ satisfy the condition of the Theorem 1, then they are solutions of the direct (P_V) and dual (P_D) problems, respectively and their values are equal to each other.
Proof. The fact that $\tilde{x}_T$ is a solution of the direct problem (P_V) was proved in the Theorem 1. Study the remaining assertions. By the definition of a LCM the condition $x^*_0 \in (a^T)^*(x^*_T; (\tilde{x}_0, \tilde{x}_T))$ of Theorem 1 is equivalent to the inequality

$$-< x^*_0, x_0 - \tilde{x}_0 > + < x^*_T, x_T - \tilde{x}_T > \geq 0, \ \ \forall (x_0, x_T) \in gph\ a^T$$

This means that

$$(x^*_0, x^*_T) \in dom\ \Omega_a \tag{21}$$

where $dom\ \Omega_a = \{(x^*_0, x^*_T) : \Omega_a(x^*_0, x^*_T) > -\infty\}$. Further, since [1],[3], $\partial g(x_T) \subset dom\ g^*$ it is clear, that

$$z^*_T \in dom\ g^* \tag{22}$$

Consequently it follows from (21) and (22) that $\{x_0^*, z_T^*, x_e^*\}$ is an admissible solution. It remains to show that $\{x_0^*, z_T^*, x_e^*\}$ is optimal solution. By the Lemma III.2.2. in [1]

$$\Omega_{aT}(x_0^*, z_T^* - x_e^*) + < x_0^*, \tilde{x}_0 >= W_{aT}(\tilde{x}_0, z_T^* - x_e^*)$$

Here $W_{aT}(\tilde{x}_0, z_T^* - x_e^*) =< \tilde{x}_T, z_T^* - x_e^* >$ and so

$$\Omega_{aT}(x_0^*, z_T^* - x_e^*) =< \tilde{x}_T, z_T^* - x_e^* > - < x_0^*, \tilde{x}_0 > \qquad (23)$$

On the other hand the inclusions $x_0^* \in K_N^*(\tilde{x}_0)$, $x_e^* \in K_M^*(\tilde{x}_T)$ imply that

$$W_N(x_0^*) =< x_0^*, \tilde{x}_0 >, \;\; W_M(x_e^*) =< x_e^*, \tilde{x}_T, > \qquad (24)$$

respectively. Now, Note that $z_T^* \in \partial g(\tilde{x}_T)$ is equivalent with the inequality

$$g^*(z_T^*) =< \tilde{x}_T, z_T^* > -g(\tilde{x}_T) \qquad (25)$$

Then taking into account (21)-(25) it is not hard to see that

$$-g^*(z_T^*) + \Omega_{aT}(x_0^*, z_T^* - x_e^*) + W_M(x_e^*) + W_N(x_0^*) = g(\tilde{x}_T).$$

The proof is completed.

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