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DIFFERENCE SCHEME FOR WAVE EQUATION WITH STRONG DISSIPATIVE TERM

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ABSTRACT

In this study, for the periodic problem with respect to time of wave theory, three-level difference scheme is presented. The difference scheme is constructed by the method of integral identities with the use of linear basis functions and interpolating quadrature rules with the remainder terms in integral form. Error of difference solution is estimated.

AMS Subject Classifications. 65L10, 35J35

1. Introduction

We discuss the following boundary value problem

$$Lu \equiv \frac{\partial^2 u}{\partial t^2} + L_1\left[\frac{\partial u}{\partial t}\right] + L_0 u = f(x,t), \tag{1.1}$$

$$(x,t) \in D = (0,\bullet) x (0,T],$$

$$u(0,t) = u(\bullet,t) = 0, \tag{1.2}$$

$$u(x,0) = u(x,T), \tag{1.3}$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T), \tag{1.4}$$

where

$$L_1\left[\frac{\partial u}{\partial t}\right] = -\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial t \partial x}\right) + b(x,t)\frac{\partial u}{\partial t},$$

$$L_0 u = -\frac{\partial}{\partial x}\left(c(x,t)\frac{\partial u}{\partial x}\right) + d(x,t)u$$

and a,b,c,d,f are sufficiently smooth functions in $\overline{D}$.

The equations of this type arise in many areas of mathemetical physics and fluid mechanics. These are used for studies about communication lines, electron plasm waves in plasmas, ion acoustics waves and other physical models. (Bullough 1980, Lonngren 1978, Ikezi 1978).

We presented three-level difference scheme for this problem. For wave equation with strong dissipative term, difference schemes is constructed, mathematical researches are done and approximate error is presented that the convergence is $0\left(h^2 + \tau^2\right)$, has been proved.

Existence, uniqueness and stability of exact solution of this type problems were investigated by several mathematician. (Lagnese 1972, Leopold 1985, Webb 1980, Lebedev 1957).

And also, the numerical solutions of this type equations in simpler models are researched. (Amiraliyev 1988).

2. Establish of Difference Scheme

We suppose that $0<\alpha\le a(x,t)\le a^*$, $0\le b_*\le b(x,t)\le b^*$, $0<c_*\le c(x,t)\le c^*$, $0\le d_*\le d(x,t)\le d^*$ and $\bar{a}_* \le \frac{\partial a(x,t)}{\partial t} \le \bar{a}^*$, $\bar{b}_* \le \frac{\partial b(x,t)}{\partial t} \le \bar{b}^*$, $\bar{c}_* \le \frac{\partial c(x,t)}{\partial t} \le \bar{c}^*$, $\bar{d}_* \le \frac{\partial d(x,t)}{\partial t} \le \bar{d}^*$ in problem (1.1)-(1.4).

Now, let us establish the mesh $\omega_{h\tau} = \omega_h \times \omega_\tau$ in domain D, such that

$$\omega_h = \left\{x_i = ih,\ i = 1,2,\dots,N-1,\ h = \lambda / N\right\},$$

$$\omega_\tau = \left\{t_j = j\tau,\ j = 1,2,\dots,M-1,\ \tau = T/M\right\}$$

and $\bar{\omega}_h = \omega_h \cup \{x = 0, \lambda\}$, $\omega_\tau^+ = \omega_\tau \cup \{t = T\}$.

We establish the scheme in two stage. Firstly, if the basis functions

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \dfrac{x - x_{i-1}}{h} & , \quad x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) \equiv \dfrac{x_{i+1} - x}{h} & , \quad x_i < x < x_{i+1}, \\ 0 & , \quad x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

$$i = 1,2,\dots,N-1$$

is applied, the following semi-discrete difference relation can be obtained

$$\lambda_h u_i(t) \equiv \frac{\partial^2 u}{\partial t^2}(x_i,t) - \left(a(x_{i-0.5},t)\left(\frac{\partial u}{\partial t}\right)_{\bar{x}}\right)_{x,i} + b(x_i,t)\frac{\partial u}{\partial i}(x_i,t) - \left(c(x_{i-0.5},t)u_{\bar{x}}\right)_{x,i} +$$

$$+d(x_i,t)u(x_i,t) = f(x_i,t) - R_i(t). \tag{2.1}$$

$$i = 1,2,\dots,N-1,\ t\in(0,T],$$

$$R_i(t) = \left(R^{(0)}(t)\right)_{x,i} + R_i^{(1)}(t).$$

Now, as before, using the basis function $\{\varphi_i(t)\}$, exact discrete difference relation

$$\lambda u_i^j \equiv u_{\bar{t}t,i}^j - \left(a(x_{i-0.5},t_j)\left(u_{\overset{\circ}{t}\bar{x}}^j\right)\right)_{x,i} + b(x_i,t_j)u_{\overset{\circ}{t},i}^j - \left(c(x_{i-0.5},t_j)u_{\bar{x}}^j\right)_{x,i} + d(x_i,t_j)u(x_i,t_j)$$

$$= f(x_i,t_j) - R_i^j, \qquad (2.2)$$

$$i = 1,2,\ldots,N-1 \;;\; j = 1,2,\ldots,M-1.$$

$$R_i^j = \left(R^{(0)}\right)_{x,i}^j + \left(R^{(1)}\right)_i^j$$

can be written by means of appropriate approximation rules from (2.1). If the inital data in problem (1.1)-(1.4) is sufficiently smooth, the local erros of $R^{(0)}$ and $R^{(1)}$ will be in form $0\left(h^2+\tau^2\right)$.

Thus, we can take approximation of condition (1.4). In addition to the basis functions above, we take the following basis function

$$\varphi_0(t) \equiv \begin{cases} \varphi_0^{(1)}(t) \equiv \dfrac{t_1 - t}{\tau} & , t_0 < t < t_1, \\ \varphi_0^{(2)}(t) \equiv \dfrac{t - t_{M-1}}{\tau} & , t_{M-1} < t < t_M, \\ 0 & , t \notin (t_0,t_1) \cup (t_{M-1},t_M). \end{cases}$$

If relation (2.1) is multiplied by $\varphi_0(t)$ and integreted in interval $(0,T)$, then by means of condition (1.4) the following difference relation, which is appropriate to point $t=t_M$, can be obtained

$$u_{\bar{t}t}^M - \left(a(x_{i-0.5},t_M)u_{\overset{\circ}{t}\bar{x}}^M\right)_{x,i} + b(x_i,t_M)u_{\overset{\circ}{t}}^M - \left(c(x_{i-0.5},t_M)u_{\bar{x}}^M\right)_{x,i} + d(x_i,t_M)u(x_i,t_M)$$

$$= f(x_i,t_M) - R_i^M. \qquad (2.3)$$

$$R_i^M = \left(R^{M(0)}\right)_{x,i} + R_i^{M(1)}.$$

Under the sufficiently smoothness conditions, each of terms $R_i^{M(0)}, R_i^{M(1)}$ has form $0\left(h^2+\tau\right)$.

Finally, difference scheme for problem (1.1) - (1.4)

$$\lambda y \equiv y_{\bar{t}t} - \left(a(x-h/2,t)y_{\overset{\circ}{t}\bar{x}}\right)_x + b(x,t)y_{\overset{\circ}{t}} - \left(c(x-h/2,t)y_{\bar{x}}\right)_x +$$

$$+ d(x,t)y(x,t) = f(x,t), \qquad (2.4)$$

$$(x,t) \in \omega_h \times \omega_\tau^+,$$

$$y(0,t)=y(\bullet,t)=0 \quad , \ t\in\overline{\omega}_\tau ,$$

$$y(x,0)=y(x,T) \quad , \ x\in\overline{\omega}_h ,$$

$$y(x,\tau)=y(x,T+\tau) \quad , \ x\in\overline{\omega}_h ,$$

can be given by means of relations (2.2) and (2.3).

3. Error Estimates of Approximate Solution

The following difference problem for the error can be written, while $z=y-u$,

$$\bullet z=R \ , \qquad (x,t)\in\omega_h\times\omega_\tau' $$
$$z(0,t)=z(\bullet,t)=0 \quad , \ t\in\overline{\omega}_\tau ,$$
$$z(x,0)=z(x,T) \quad , \ x\in\overline{\omega}_h ,$$
$$z(x,\tau)=z(x,T+\tau) \quad , \ x\in\overline{\omega}_h .$$

Lemma 3.1. When condition

$$\lambda_0\left(c_*+\frac{\lambda^2}{8}d_*-\frac{1}{4}\aleph_0-\frac{\lambda^2}{32}\aleph_1\right)-\frac{1}{2}\left(\bar{c}^*+\bar{d}^*\right)>0, \tag{3.1}$$

held, the following estimation for error of difference problem

$$\|z_t\|^2+\|z_{\bar{x}}\|^2+\|z\|^2\le C\tau\sum_{i=1}^{M}\left(\left\|R_i^{(0)}\right\|^2+\left\|R_i^{(1)}\right\|^2\right)\exp\left(-C_1 t_{M-i}\right),t\in\omega_\tau^+,$$

(3.2)

is true, such that

$$\lambda_0<\frac{4}{\bullet}\alpha+\frac{b_*}{2} \ \text{ and } \ \aleph_0=\max\left(\bar{a}^*,3\,\bar{a}^*\right),\ \aleph_1=\max\left(\bar{b}^*,3\,\bar{b}^*\right),$$

where C and C_1 , which are independent of h and τ , are positive constants.

Proof. We begin to prove by taking equivalence,

$$\left(\lambda z, z_t+\lambda z\right)=\left(\left(R_i^{(0)}\right)_x+R_i^{(1)}, z_t+\lambda z\right) \tag{3.3}$$

$\lambda>0$, which will be chosen later, is real parameter.

The relation (3.3) leads after some transformations to the inequality of the form

$$\delta_t \le -C_1 \delta + \rho$$

where,

$$\delta(t) = \left(\left\{ \frac{1}{2} + \frac{3\tau}{4} b - \frac{\lambda}{4}\tau^2 b - \frac{\tau^2}{2} d - \mu_1 \tau \right\} z_t, z_t \right) + \left(\left\{ \frac{3\tau}{4} a - \frac{\lambda}{4}\tau^2 a - \frac{\tau^2}{2} c - \mu_4 \tau \right\} z_{\bar{x}t}, z_{\bar{x}t} \right) +$$

$$+\lambda(z_t, z) + \left(\left\{ \frac{1}{2} c + \frac{\lambda}{4} a \right\} \bar{z}_{\bar{x}}, \bar{z}_{\bar{x}} \right) + \left(\left[\frac{\lambda a}{4} - \frac{\tau}{2} \left\{ \frac{1}{2} c_t + \frac{\lambda}{4} a_t - \lambda c + \lambda \mu_4 \right\} \right] z_{\bar{x}}, z_{\bar{x}} \right) + \left(\left\{ \frac{d}{2} + \frac{\lambda}{4} b \right\} \bar{z}, \bar{z} \right) +$$

$$+ \left(\left[\frac{\lambda b}{4} - \frac{\tau}{2} \left\{ \frac{1}{2} d_{\bar{t}} + \frac{\lambda}{4} b_{\bar{t}} - \lambda d + \lambda \mu_2 \right\} \right] z, z \right)$$

is a mesh function.

$$C_1 = \min\left\{ \frac{D_1}{B_1}, \frac{D_2}{B_2}, \frac{D_3}{B_3} \right\} > 0 \quad \text{and} \quad \rho = \left(\frac{1}{4\mu_1} + \frac{\lambda}{4\mu_2} \right) \left\| R_i^{(1)} \right\|^2 + \frac{1+\lambda}{4\mu_4} \left\| R_i^{(0)} \right\|^2 ,$$

$$D_1 = \frac{\lambda^2}{8} \min\left(0, b_* + \frac{\tau}{4} \bar{b}_* - \frac{\tau}{2} d^* - \lambda + \frac{\lambda}{4} \tau^2 \bar{b}_* - \mu_1 \right) + \alpha + \frac{\tau}{4} \bar{a}_* - \frac{\tau}{2} c^* + \frac{\lambda}{4} \tau^2 \bar{a}_* - \mu_4 \ ,$$

$$D_2 = \frac{\lambda^2}{8} \min\left(0, \frac{1}{2} \left\{ -\frac{1}{2} \bar{d}^* - \frac{\lambda}{4} \bar{b}^* + \lambda d_* - \lambda \mu_2 \right\} \right) + \frac{1}{2} \left\{ -\frac{1}{2} \bar{c}^* - \frac{\lambda}{4} \bar{a}^* + \lambda c_* - \lambda \mu_4 \right\} \ ,$$

$$D_3 = \frac{\lambda^2}{8} \min\left(0, \frac{1}{2} \left\{ -\frac{1}{2} \bar{d}^* - \frac{3\lambda}{4} \bar{b}^* + \lambda d_* - \lambda \mu_2 \right\} \right) + \frac{1}{2} \left\{ -\frac{1}{2} \bar{c}^* - \frac{3\lambda}{4} \bar{a}^* + \lambda c_* - \lambda \mu_4 \right\},$$

$$B_1 = \frac{\lambda^2}{8} \max\left(0, \frac{1}{2} + \frac{3\tau}{4} b^* - \frac{\lambda}{4} \tau^2 b_* - \frac{\tau^2}{2} d_* - \mu_1 \tau + \lambda \mu_3 \right) + \frac{3\tau}{4} a^* - \frac{\lambda}{4} \tau^2 \alpha - \frac{\tau^2}{2} c_* - \mu_4 \tau ,$$

$$B_2 = \frac{\lambda^2}{8} \max\left(0, \frac{d^*}{2} + \frac{\lambda b^*}{4} \right) + \frac{c_*}{2} + \frac{\lambda}{4} a^* \ ,$$

$$B_3 = \frac{\lambda^2}{8} \max\left(0, \left[\frac{\lambda b^*}{4} + \frac{\lambda}{4\mu_3} - \frac{\tau}{2} \left\{ \frac{\bar{d}_*}{2} + \frac{\lambda}{4} \bar{b}_* - \lambda d^* + \lambda \mu_2 \right\} \right] \right) + \frac{\lambda a^*}{4} - \frac{\tau}{2} \left\{ \frac{\bar{c}_*}{2} + \frac{\lambda}{4} \bar{a}_* - \lambda c^* + \lambda \mu_4 \right\}.$$

It is easily to get that, under the condition (3.1) for sufficiently small μ_i, $i = \overline{1,4}$,

$$\delta(t) \ge C \left(\|z_t\|^2 + \|\bar{z}_{\bar{x}}\|^2 + \|z\|^2 \right) \ , \quad C>0 \ . \tag{3.4}$$

If we apply the difference analogue of differential inequality, which have periodic condition, we have inequality

$$\delta_K \le \left[1-\exp(-C_1T)\right]^{-1}\left\{\tau\sum_{i=1}^{M}|\rho_i|\exp(-C_1t_{M-i})\right\}\exp(-C_1t_K)+\tau\sum_{i=1}^{K}|\rho_i|\exp(-C_1t_{K-i}).$$

$$K=1,2,\ldots,M$$

Proof of the lemma is completed by taking into account (3.4).

Thus the following theorem can be given.

Theorem 3.1. Let u be an element of $C^5(\overline{D})$ and conditions of Lemma 3.1 are hold, then the solution of difference problem (2.4) converges to solution of (1.4) in $\omega_h \times \omega_\tau^+$ and also estimation for convergence rate

$$\|z_t\|+\|\overset{\circ}{z}_{\overline{x}}\|+\|z\| \le C\left(h^2+\tau^2\right), \quad t\in\omega_\tau^+, \tag{3.5}$$

is true.

Proof. The proof can be directly taken from (3.2) and local errors which have been obtained. Really we can write the following inequality for right hand of (3.2)

$$\tau\sum_{j=1}^{M}\left(\|R_j^{(0)}\|+\|R_j^{(1)}\|\right)\exp(-C_1t_{M-j}) \le \tau\sum_{j=1}^{M-1}\left(\|R_j^{(0)}\|+\|R_j^{(1)}\|\right)+\tau\left(\|R^{M(0)}\|+\|R^{M(1)}\|\right).$$

This inequality and local errors

$$\left|R^{(0)}\right|,\left|R^{(1)}\right| \le C\left(h^2+\tau^2\right), \quad (x,t)\in\omega_h\times\omega_\tau,$$

$$\left|R^{M(0)}\right|,\left|R^{M(1)}\right| \le C\left(h^2+\tau\right), \quad x\in\omega_h,$$

show that thruthfulness of the theorem.

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