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ISOPTICS OF OPEN, CONVEX CURVES AND CROFTON-TYPE FORMULAS

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ABSTRACT. In this paper we introduce the isoptics for some plane, open, convex curves. We derive the sine theorem and a Crofton-type formula for such curves.

1. INTRODUCTION

Let C be a plane, open, convex curve without asymptotics. The curve divides plane $\mathbb{R}^2$ into two sets - a convex and a concave one. The convex set will be called an interior of the curve C and the second set its exterior. We assume that the coordinate system Oxy is chosen in such a way that the point O belongs to the interior of C and that the axis Oy meets the curve C once.

We will show that exactly two tangent lines to C pass through any exterior point. Let us fix a point (x_0, y_0) and note that a vertical line $x = x_0$ intersects the curve C transversally. Next, we rotate this vertical line in negative direction until it intersects C transversally. Let the obtained line has an equation $y = cx + d$. Let $\mathcal{A}$ be a set of slopes of the lines passing through (x_0, y_0) and disjoint with the curve C. This means that if $p \in \mathcal{A}$ then $p < c$. Let $a_0 = \sup \mathcal{A}$ and consider a line $y = a_0 x + b_0$ containing the point (x_0, y_0). We claim that this line is tangent to C. This line can not intersect C transversally since it is in contradiction with the choose of the slope a_0 (in this case we would have $\sup \mathcal{A} < a_0$). If the line $y = a_0 x + b_0$ does not intersect the curve C transversally then either it is a tangent line for C or it does not intersect C. Assume that this line does not intersect C. This line can not be an assymptote. Now, let us consider the parallel lines $y = a_0 x + e$ which are disjoint with C. The set all such coefficients e will be denoted by $\mathcal{E}$. This set $\mathcal{E}$ is bounded for above, thus its supremum $e_0 = \sup \mathcal{E}$ exists. The limit line $y = a_0 x + e_0$ neither is an assymptote nor intersect C transversally. Thus it has to be the tangent line to C. But this situation is contradictory with the choose of a_0.

Thus we proved that the line $y = a_0 x + b_0$ is tangent to C. Similarly we show the existence of the second tangent.

2. ISOPTICS

Let us fix a point $z \in C$. A tangent line l at this point we will call a support line of C at the point z. Let a point A be an orthogonal projection of the origin O on the line l.

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Let t be a measure of an angle marked on Fig 2.1.

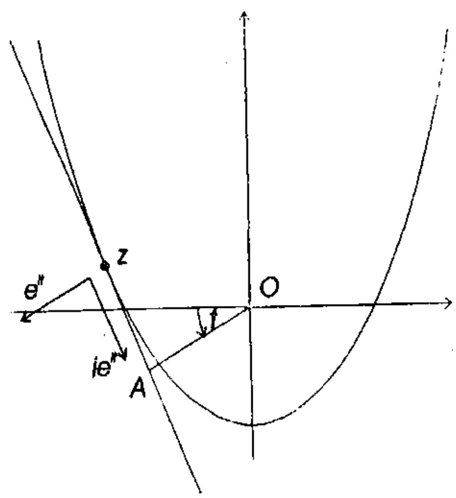

Fig. 2.1

Let $e^{it} = [\cos t, \sin t]$, $ie^{it} = [-\sin t, \cos t]$ and $p(t) = |OA|$. Then

$$z(t) = -p(t)e^{it} - \dot{p}(t)ie^{it}$$

is a parametric equation of the curve C. Let

$$\begin{aligned} q(t) &= z(t) - z(t+\alpha), \\ b(t) &= \left\{q(t), e^{it}\right\} \\ B(t) &= \left\{q(t), ie^{it}\right\}, \end{aligned}$$

where $\{u, w\}$ for $u = [u_1, u_2]$, $w = [w_1, w_2]$. Then

$$\begin{aligned} b(t) &= \dot{p}(t) - p(t+\alpha)\sin\alpha - \dot{p}(t+\alpha)\cos\alpha \\ B(t) &= -\,p(t) + p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha \end{aligned}$$

Let us take a point $z(t) \in C$ and consider the tangent line to C at $z(t)$. Let $z(t')$, be a point on C such that the angle between the tangent lines at $z(t)$ and $z(t')$ is equal to $\pi - \alpha$.

Definition 2.1. *The cut locus C_α of the intersection points of the above defined pairs of tangent lines is said to be an α-isoptic of C.*

Now, we give a parametrization z_α of the isoptic C_α of C. We have that $z_\alpha = z(t) + \lambda(t)ie^{it} = z(t+\alpha) - \mu(t)ie^{it}$, $\lambda, \mu > 0$.

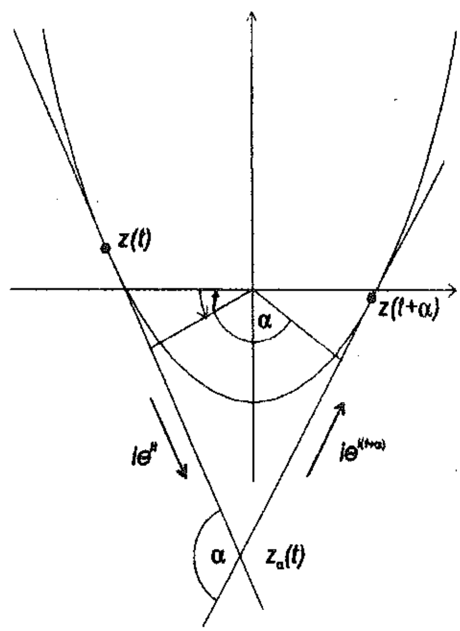

Fig. 2.2

Reasoning similarly as in [MM] we get

$$\lambda = b - \cot\alpha$$
$$\mu = \frac{B}{\sin\alpha}.$$

It follows that

$$z_\alpha(t) = -p(t)e^{it} + \frac{1}{\sin\alpha}\left(-p(t+\alpha) + p(t)\cos\alpha\right) ie^{it}.$$

Of course, $t + \alpha < \pi$ thus our isoptic curve is parameterized only for $t \in (0, \pi - \alpha)$.

3. Sine theorem for isoptics of open, convex curves

In the paper [CMM1] we have proved the sine theorem for ovals and in the paper [MM] we introduced the isoptics of rosettes. We note that S. Góźdź gave in [G] a sufficient condition for a Jordan curve to be an isoptic of an oval. In [GMM] we extended this result to the class of rosettes. In this section we will give the sine theorem for the open, convex, plane curves without isoptics. The inverse theorem remains at the moment an open problem.

Assume the notations as on Fig. 3.1, where $l_{t,\alpha}$ denote a tangent line to the isoptic $z_\alpha(t)$ at the point t.

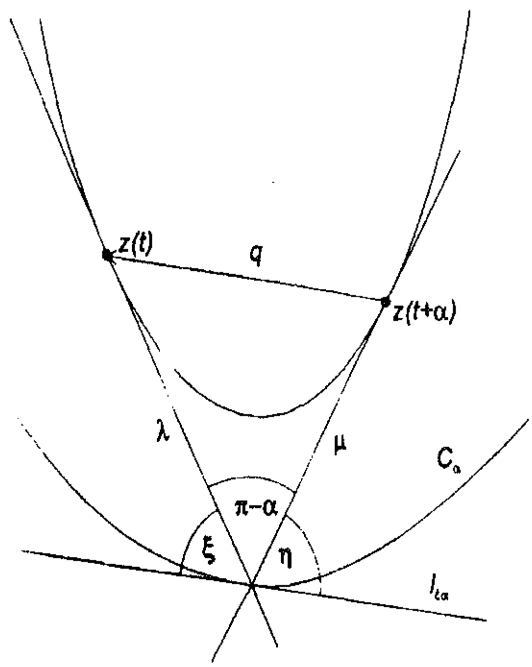

Fig. 3.1

It is easy matter to check that we have the following formulas for the derivatives of the functions b and B:

$$\dot{b}(t) = R(t) - R(t+\alpha)\cos\alpha + B(t)$$
$$\dot{B}(t) = -R(t+\alpha)\sin\alpha - b(t),$$

where $R(t) = p(t) + \ddot{p}(t)$ is a curvature radius of the curve C.

Now, we will compute a tangent vector to the isoptic C_α at the point t. We have $\dot{z}_\alpha(t) = \varrho(t)ie^{it} - \lambda(t)e^{it}$, where $\varrho(t) = B(t) + b(t)\cot\alpha$. Hence

$$\sin\xi = -\frac{\left[\dot{z}_\alpha(t), ie^{it}\right]}{|\dot{z}_\alpha(t)|} = \frac{\lambda(t)}{|\dot{z}_\alpha(t)|}.$$

On the other hand we have

$$|\dot{z}_\alpha(t)| = \sqrt{\varrho^2(t) + \lambda^2(t)} = \sqrt{\frac{b^2(t) + B^2(t)}{\sin^2\alpha}}.$$

But we have that $|q(t)| = \sqrt{b^2(t) + B^2(t)}$ thus $|\dot{z}_\alpha(t)| = \frac{|q(t)|}{\sin\alpha}$. Hence

$$\frac{|q(t)|}{\sin\alpha} = \frac{\lambda}{\sin\xi}.$$

Similarly

$$\sin\eta = \frac{\left[\dot{z}_\alpha(t), ie^{i(t+\alpha)}\right]}{|\dot{z}_\alpha(t)|} = \frac{\mu}{\sin\eta}.$$

Thus we proved

Theorem 3.1. *Under the above assumptions we have*

$$\frac{|q(t)|}{\sin\alpha} = \frac{\lambda}{\sin\xi} = \frac{\mu}{\sin\eta}.$$

4. Integral formula

In this section we derive some Crofton-type integral formula similar to those proved in [CMM1] and [CMM2]. Let $U \subset \mathbb{R}^2$ be defined by

$$U = \{(\alpha, t); \alpha \in (0, \pi) \wedge t \in (0, \pi - \alpha)\}$$

Let us consider a mapping

$$\begin{aligned} F : U &\longrightarrow \mathbb{R}^2 \\ (\alpha, t) &\longmapsto F(\alpha, t). \end{aligned}$$

Making use of the formulas

$$\begin{aligned} \frac{\partial b}{\partial \alpha} &= -R(t+\alpha)\cos\alpha, \\ \frac{\partial B}{\partial \alpha} &= -R(t+\alpha)\sin\alpha \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \frac{\mu}{\sin\alpha} ie^{it}, \\ \frac{\partial F}{\partial t} &= \dot{z}_\alpha = -\lambda e^{it} + \varrho ie^{it}. \end{aligned}$$

This means that the jacobian determinant $F' = \left\{\frac{\partial F}{\partial \alpha}, \frac{\partial F}{\partial t}\right\}$ is given by

$$F' = \frac{\lambda\mu}{\sin\alpha},$$

thus $F' > 0$ for any $(\alpha, t) \in U$. The mapping F is thus a diffeomorhism of the set U onto the exterior $\operatorname{ext} C$ of the curve C. Let $x \in \operatorname{ext} C$ and associate to x three numbers t_1, t_2, ω as on Fig. 4.1.

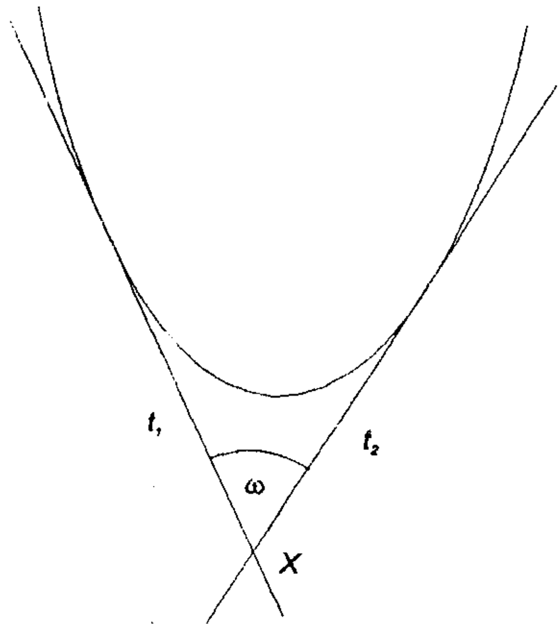

Fig. 4.1

This means that if $x \in C_\alpha$ then $t_1 = \lambda$, $t_2 = \mu$ and $\omega = \pi - \alpha$.

Theorem 4.1. *With the above notations we have*

$$\iint_{\text{ext}\, C} \frac{\sin\omega}{t_1 t_2} = \frac{\pi^2}{2}.$$

Proof. Using the mapping F we can easily calculate the integral

$$\iint_{\text{ext}\, C} \frac{\sin\omega}{t_1 t_2} = \int_0^\pi \left(\int_0^{\pi-\alpha} dt \right) d\alpha = \frac{\pi^2}{2}.$$

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