İstanbul Üniv. Fen Fak. Mat. Dergisi **57-58** (1998-1999), 71-78

RATE OF CONVERGENCE BY SZASZ-MIRAKYAN-BASKAKOV TYPE OPERATORS

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ABSTRACT: In the present paper, we study a new kind of Szasz-Mirakyan-Baskakov type operators. We establish a Voronovskja type asymptotic formula and an estimate of error is simultaneous approximation for the linear combination of these operators.

1.Recently Gupta and Srivastava [3] defined a new family of linear positive operators by combining the well known Szasz-Mirakyan operators and Baskakov operators as

$$(1.1)\qquad M_n(f,x)=(n-1)\sum_{v=0}^{\infty}p_{n,v}(x)\int_0^{\infty}q_{n,v}(t)f(t)dt\quad ,x\in R^+,$$

where

$$p_{n,v}(x)=e^{-nx}\frac{(nx)^v}{v!}\quad\text{and}\quad q_{n,v}(t)=\frac{(n-v-1)!}{v!(n-1)!}\frac{t^v}{(1+t)^{n+v}}.$$

Such type of other operators are defined and studied by some researchers (see e.g. [1],[2],[4] and [6] etc.)

We denote by L the class of all Lebesgue measurable functions f and R^+, satisfying

$$\int_0^{\infty}\frac{|f(t)|}{(1+t)^n}dt<\infty \text{ for some positive integer n.}$$

Obviously this class of functions is bigger than the class of all Lebesgue integrable functions on R^+

For a fixed natural number k, the linear combination $M_n(f,k,x)$ of $M_{d_jn}(f,x)$ is defined as

$$(1.2)\qquad M_n(f,k,x)=\sum_{j=0}^{k}C(j,k)\,M_{d_jn}(f,x),$$

where

$$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i},k\neq 0;C(0,0)=1$$

and $d_0,d_1,\dots d_k$ are (k+1) arbitrary but fixed distinct positive integers. It turns out from [3], that even in simultaneous approximation the order of approximation for the operators (1.1) is $O(n^{-1})$.

AMS Subject Classification (1985):41A30,41A36.
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By using the above linear combination (1.2) we can improve the order of approximation. The object of the present paper is to obtain direct results in simultaneous approximation for the linear combinations $M_n(f,k,x)$ of the operators (1.1).

2. In this section, we shall mention certain results which are necessary for the proof of main results.

LEMMA 2.1. For $m \in N \cup \{0\}$, if we define $\mu_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x)(\frac{v}{n} - x)^m$,

then there holds the recurrence relation $n\mu_{n,m+1}(x) = x[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)]$.

Consequently

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree $\leq m$.

(i) $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integral part of α.

LEMMA 2.2. Let the m-th order moment be defined by

$$T_{r,n,m}(x) = (n-r-1)\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n-r,v+r}(t)(t-x)^m dt.$$

Then

$$T_{r,n,0}(x) = 1,\ T_{r,n,1}(x) = \frac{[(r+1)+x(r+2)]}{(n-r-2)},\ n > r+2$$

and there holds the recurrence relation

$$(n-r-m-2)T_{r,n,m+1}(x) = T_{r,n,m}(x)[r(1+x)+(m+1)(1+2x)] + xT'_{r,n,m}(x) + mx(2+x)T_{r,n,m-1}(x),$$

where n>r+m+2.

LEMMA2.3. Let f be r times differentiable on R^+ such that $f^{(r-1)}(t) = O(t^{\alpha})$ for some $\alpha > 0$ as $t \to \infty$ then for r=1,2,... and $n > \alpha + r$, we have

$$M_n^{(r)}(f,x) = \frac{n^r(n-r-1)!}{(n-2)!}\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n-r,v+r}(t)f^{(r)}(t)dt$$

The proof of the above Lemma easily follows by using Leibnitz theorem as in [6].

LEMMA 2.4. There exist polynomials $\phi_{i,j,r}(x)$ independent of n and v such that

$$x^r \frac{d^r}{dx^r}[e^{-nx}(nx)] = \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i (v-nx)^j \phi_{i,j,r}(x)[e^{-nx}(nx)^v]$$

LEMMA 2.5. [5] If C(j, k), j=0,1,...k are defined as in (1.2) then

$$\sum_{j=0}^{k} C(j,k)d_j^{-m} = \begin{cases} 1, m=0 \\ 0, m=1,2,..k \end{cases}$$

3 In this section we shall prove the main results.

THEOREM 3.1. Let $f \in L$ be bounded on every finite subinterval of R^1 and $f^{(2k+r+2)}(x)$ exist at a fixed point $x \in (0,\infty)$. Let $f(t) = O(t^{\alpha})$ as $t \to \infty$ for some $\alpha > 0$, then we have

$$(3.1)\quad \lim_{n\to\infty} n^{k+1}[\alpha(n,r,k)\, M^{(r)}{}_n(f,k,x) - f^{(r)}(x)] = \sum_{i=r+1}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x).$$

where

$$\alpha(n,r,k) = [\sum_{j=0}^{k} \frac{C(j,k)(d_j n)^r (d_j n - r - 2)!}{(d_j n - 1)!}]^{-1}$$

and Q(i, k, r ,x) are certain polynomials in x of degree at most i.

PROOF. By Taylors's expansion we have

$$f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+r+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O((t-x)^{\beta})$ for some $\beta > 0$. Making use of Lemma 2.3, we have

$$n^{k+1}[\alpha(n,r,k)\, M_n^{(r)}(f,k,x) - f^{(r)}(x)]$$

$$= n^{k+1}[\alpha(n,r,k) \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{k} C(j,k)(d_j n)^r \frac{(d_j n - r - 1)!}{(d_j n - 2)!} . \sum_{v=0}^{\infty} p_{d_j n,v}(x) \int_0^{\infty} q_{d_j n - r, v+r}(t) \frac{d^r}{dx^r}(t-x)^i - f^{(r)}(x)]$$

$$+ \alpha(n,r,k) n^{k+1} \sum_{j=0}^{k} C(j,k)(d_j n - 1) \sum_{v=0}^{\infty} p^{(r)}_{d_j n,v}(x) \int_0^{\infty} q_{d_j n,v}(t)\varepsilon(t,x)(t-x)^{2k+r+2}\, dt$$

$$= \sum_{i=r+1}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x) + E_{n,r,k}(x) + o(1)$$

By Lemma 2.2 , we have

$$E_{n,r,k}(x) = \alpha(n,r,k) n^{k+1} \sum_{j=0}^{k} C(j,k)(d_j n - 1). \sum_{v=0}^{\infty} p^{(r)}_{d_j n,v}(x) \int_0^{\infty} q_{d_j n,v}(t)\varepsilon(t,x)(t-x)^{2k+r+2}\, dt$$

In order to complete the proof of (3.1) , it suffices to show that

$$I = n^{k+1}(n-1) \sum_{v=0}^{\infty} p^{(r)}_{d_j n,v}(x) \int_0^{\infty} q_{n,v}(t)\varepsilon(t,x)(t-x)^{2k+r+2}\, dt$$

tends to zero as $n \to \infty$.
Now by Lemma 2.4 we have

$$|I| \le n^{k+1}(n-1)\sum_{v=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i |v-nx|^j \left|\phi_{i,j,r}(x)\right| p_{n,v}(x)\int_0^\infty q_{n,v}\left|\varepsilon(t,x)\right|\left|t-x\right|^{2k+r+2}dt$$

$$\le n^{k+1}M(x)(n-1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j . \int_0^\infty q_{n,v}(t)\left|\varepsilon(t,x)\right|\left|t-x\right|^{2k+r+2}dt$$

$$\le n^{k+1}M(x)(n-1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left(\sum_{v=0}^{\infty} p_{n,v}(x)(v-nx)^{2j}\right)^{1/2}\left(\sum_{v=0}^{\infty} p_{n,v}(x)\left[\int_0^\infty q_{n,v}(t)\left|\varepsilon(t,x)\right|\left|t-x\right|^{2k+r+2}dt\right]^2\right)^{1/2}$$

where $M(x) = \sup\limits_{\substack{2i+j\le r\\ i,j\ge 0}}\left|\phi_{i,j,r}(x)\right|.$

For a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|\varepsilon(t,x)\right| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \ge \delta$, we have $\left|\varepsilon(t-x)\right| \le K|t-x|^{\alpha}$. Thus

$$\left[\int_0^\infty q_{n,v}(t)\left|\varepsilon(t,x)\right|\left|t-x\right|^{2k+r+2}dt\right]^2 \le \left(\int_0^\infty q_{n,v}(t)dt\right)\left(\int_0^\infty q_{n,v}(t)\right)^2(t-x)^{4k+2r+4}dt$$

$$= \frac{1}{(n-1)}\left(\int_{|t-x|<\delta} + \int_{|t-x|\ge\delta} q_{n,v}(t)\{\varepsilon(t,x)\}^2(t-x)^{4k+2r+4}dt\right.$$

$$\le \frac{1}{(n-1)}\left\{\int_{|t-x|<\delta} q_{n,v}(t)\varepsilon^2(t-x)^{4k+2r+4}dt + \int_{|t-x|\ge\delta} q_{n,v}(t)K^2(t-x)^{4k+2r+4}dt\right\}$$

Applying Lemma 2.2, we get

$$\sum_{v=0}^{\infty} p_{n,v}(x)\left\{\int_0^\infty q_{n,v}(t)\left|\varepsilon(t,x)\right|\left|t-x\right|^{2k+r+2}dt\right\} \le \frac{1}{(n-1)}\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^\infty q_{n,v}(t)\varepsilon^2(t-x)^{2k+2r+4}dt$$

$$+\frac{K^2}{(n-1)}\sum_{v=0}^{\infty} p_{n,v}(x)\int_{|t-x|\ge\delta} q_{n,v}(t)(t-x)^{4k+2r+2\alpha+4}dt$$

$$= \varepsilon^2 O(n^{-(2k+r+4)}) + \frac{K^2}{(n-1)}\sum_{v=0}^{\infty} p_{n,v}(x)\int_{|t-x|\ge\delta} q_{n,v}(t)\frac{(t-x)^{2s}}{\delta^{2s-(4k+2r+2\alpha+4)}}dt$$

$$\le \varepsilon^2 O(n^{-(2k+r+4)}) + O(n^{-(s+2)})$$

Thus, we get

$$|I| \le n^{k+1}M(x)(n-1)\sum_{\substack{2i+j\le r\\ i,j\ge r}} n^{i+1}O(n^{-j/2})\{\varepsilon^2 O(n^{-(2k+r+4)}) + O(n^{-(s+2)})\}^{1/2}$$

$$= \varepsilon^2\{O(1) + O(n^{-r+4-s-2})\}^{1/2}$$

$$= \varepsilon\, O(1),$$

choosing $s>r+2$ and n sufficiently large this in turn implies $I\to 0$ as $n\to\infty$. This completes the proof.

THEOREM 3.2. Let $1 \le p \le 2k+2$ and $f \in L$ be bounded on every finite sub interval of

$[0,\infty)$. Also let $f(t)=O(t^{\alpha})$ as $t\to\infty$ for same $\alpha>0$. If $f^{(p+r)}$ exists and is continuous on $(a-\eta,b+\eta),\eta>0$ having the modulus of continuity $\omega_{f^{(p+r)}}(\delta)$ on $(a-\eta,b+\eta)$. Then for n sufficiently large

$$\left\|\alpha(n,r,k)\,M_n^{(r)}(f,k,\cdot)-f^{(r)}\right\|_{C[a,b]}\le Max\{C_1n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}),C_2n^{-(k+1)}\}$$

where $C_1=C_1(k,p,r)$, $C_2=C_2(k,p,r,f)$ *and* $\alpha(n,r,k)$
is as defined Theorem 3.1.

PROOF. By Taylor's expansion of f, for each $t\in[0,\infty)$ *and* $x\in[a,b]$, we have

$$(3.2)\quad f(t)=\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{\{f^{(p+r)}(\xi)-f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r}\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\eta,b+\eta)$. For $t\in[0,\infty]\setminus(a-\eta,b+\eta)$ *and* $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i$$

For $t\in(a-\eta,b+\eta)$ *and* $x\in[a,b]$, we have

$$f(t)=\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{\{f^{(p+r)}(\xi)-f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r}$$

Now using (3.2) and Lemma 2.3, we get

$$\alpha(n,r,k)\,M_n(f,k,x)-f^{(r)}(x)=\alpha(n,r,k)[\{\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{k}C(j,k)(d_jn)^r\frac{(d_jn-r-1)!}{(d_jn-2)!}$$

$$\sum_{v=0}^{\infty}p_{d_jn,v}(x)\int_0^{\infty}q_{d_jn-r,v+r}(t)\frac{d^r}{dx^r}(t-x)^i\,dt-f^{(r)}(x)\}]$$

$$+\sum_{j=0}^{k}C(j,k)(d_jn-1)\sum_{v=0}^{\infty}p_{d_jn,v}^{(r)}(x)\int_0^{\infty}q_{d_jn,v}(t).\frac{\{f^{(p+r)}(\xi)-f^{(p+r)}(x)\}(t-x)^{p+r}\chi(t)}{(p+r)!}dt$$

$$+\sum_{j=0}^{k}C(j,k)(d_jn-1)\sum_{v=0}^{\infty}p_{d_jn,v}^{(r)}(x)\int_0^{\infty}q_{d_jn,v}(t)h(t,x)(1-\chi(t))dt$$

$$=I_1+\alpha(n,r,k)(I_2+I_3),\quad say$$

Applying Lemma 2.2 and Lemma 2.5, we obtain

$I_1\le K_1n^{-(k+1)}$, uniformly for $x\in[a,b]$, where $K_1=K_1(k,p,r,f)$

Now we obtain I_2, let

$$I_4=(n-1)\sum_{v=0}^{\infty}p_{n,v}^{(r)}(x)\int_0^{\infty}q_{n,v}(t)\frac{\{f^{(p+r)}(\xi)-f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r}\chi(t)dt\text{ S}$$

Using Lemma 2.4, we have

$$\left|I_4\right| \le (n-1)\sum_{v=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left|v-nx\right|^j \frac{\left|\phi_{i,j,r}(x)\right|}{x^r} p_{n,v}(x).\int_0^{\infty} q_{n,v}(t)\frac{\left|f^{(p+r)}(\xi)-f^{(p+r)}(x)\right|}{(p+r)!}\left|t-x\right|^{p+r}\chi(t)dt$$

$$\le \frac{(n-1)}{(p+r)!}K_2\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\sum_{v=0}^{\infty} p_{n,v}(x)\left|v-nx\right|^j\int_0^{\infty} q_{n,v}(t)[1+(t-x)/\delta]\,\omega_t(p+r)(\delta)\left|t-x\right|^{p+r}dt$$

$$= \frac{(n-1)}{(p+r)!}K_2\omega_t(p+r)(\delta)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\sum_{v=0}^{\infty} p_{n,v}(x)\left|v-nx\right|^j.\int_0^{\infty} q_{n,v}(t)[\left|t-x\right|^{p+r}+\frac{(t-x)^{p+r+1}}{\delta}]dt,$$

where

$$K_2 = \sup_{\substack{2i+j\le r\\ i,j\ge 0}}\ \sup_{x\in[a,b]}\frac{\left|\phi_{i,j,r}(x)\right|}{x^r}$$

Now , we shall show that for s=0,1,2...

$$(n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\left|v-nx\right|^j\int_0^{\infty} q_{n,v}(t)\left|t-x\right|^s dt$$

We have

$$(n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\left|v-nx\right|^j\int_0^{\infty} q_{n,v}(t)\left|t-x\right|^s dt = O(n^{(j-s)/2})$$

$$\le (n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\left|v-nx\right|^j[(\int_0^{\infty} q_{n,v}(t)dt)^{1/2}(\int_0^{\infty} q_{n,v}(t)(t-x)^{2s}dt)^{1/2}]$$

$$\le \frac{(n-1)}{\sqrt{(n-1)}}\sum_{v=0}^{\infty} p_{n,v}(x)\left|v-nx\right|^j[\int_0^{\infty} q_{n,v}(t)(t-x)^{2s}dt]^{1/2}$$

$$\le [\sum_{v=0}^{\infty} p_{n,v}(x)(v-nx)^{2j}]^{1/2}[(n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^{2s}dt]^{1/2}$$

$$= O(n^{j/2}).O(n^{-s/2}) = O(n^{(j-s)/2})$$

Uniformly in x , by Lemma 2.1 and Lemma 2.2. Therefore

$$\left|I_4\right| \le \frac{K_3\omega_t(p+r)(\delta)}{(p+r)!}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i[O(n^{(j-p-r)/2})+\frac{1}{\delta}O(n^{(j-p-r-1)/2})]$$

Choosing $\delta = n^{-1/2}$, we get

$$\left|I_4\right| \le K_4 n^{-p/2}\omega_t(p+r)(n^{-1/2}),$$

Hence

$$\left|I_2\right| \le K_5 n^{-p/2}\omega_t(p+r)(n^{-1/2}),\ \ where\ K_5 = K_5(k,p,r)$$

Finally, we estimate I_3. Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose a $\delta>0$ in such a way that $\left|t-x\right|\ge\delta\ \ for\ all\ x\in[a,b]$

Let

$$I_5 = (n-1)\sum_{\nu=0}^{\infty} p_{n,\nu}^{(r)}(x)\int_0^{\infty} q_{n,\nu}(t)h(t,x)\{1-\chi(t)\}dt$$

Using Lemma 2.4, we get

$$|I_5| \le (n-1)\sum_{\nu=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i|\nu-nx|^j \frac{|\phi_{i,j,r}(x)|}{x^r} p_{n,\nu}(x)\int_{|t-x|\ge\delta} q_{n,\nu}(t)|h(t,x)|dt$$

If β is any integer $\ge \max\{\alpha, 2k+r+2\}$, then we can find a constant K_6 *such that* $|h(t,x)| \le K_6|t-x|^{\beta}$, *for* $|t-x| \ge \delta$.

Making use of Cauchy-Schwarz inequality, we have

$$|I_5| \le K_2(n-1)\sum_{\nu=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i|\nu-nx|^j p_{n,\nu}(x)\int_{|t-x|\ge\delta} q_{n,\nu}(t)K_6|t-x|^{\beta}dt$$

$$\le K_7(n-1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\sum_{\nu=0}^{\infty}|\nu-nx|^j p_{n,\nu}(x)\int_{|t-x|\ge\delta} q_{n,\nu}(t)\frac{(t-x)^{2s}}{\delta^{2s-\beta}}dt$$

$$= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2}).O(n^{-s}) = O(n^{(r-2s)/2}) = O(n^{-(k+1)})$$

uniformly on [a, b] where s is a natural number bigger than $\beta/2$.

Hence $|I_3| \le K_8 n^{-(k+1)}$.

Combining the estimate of I_1, I_2 and I_3 we get the required result.

Acknowledgement: The first author is thankful to U.G.C. for research support under unassigned grant, group D, Minor Research Project 1996-97.

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