RATE OF CONVERGENCE BY SZASZ-MIRAKYAN-BASKAKOV TYPE OPERATORS

VIJAY GUPTA AND PRABHAKAR GUPTA*
DEPARTMENT OF MATHEMATICS
INSTITUTE OF ENGINEERING & TECHNOLOGY
M.J.P. ROHILKHAND UNIVERSITY
BAREILLY-243006 (U.P.) INDIA

ABSTRACT: In the present paper, we study a new kind of Szasz-Mirakyan-Baskakov type operators. We establish a Voronovskja type asymptotic formula and an estimate of error is simultaneous approximation for the linear combination of these operators.

1. Recently Gupta and Srivastava [3] defined a new family of linear positive operators by combining the well known Szasz-Mirakyan operators and Baskakov operators as

\[ M_n(f,x) = (n-1) \sum_{\nu=0}^{\infty} p_{n,v}(x) \int_{0}^{x} q_{n,v}(t) f(t) dt, \quad x \in \mathbb{R}^+, \]

where

\[ p_{n,v}(x) = e^{-v} \frac{(nx)^v}{v!} \quad \text{and} \quad q_{n,v}(t) = \frac{(n-v-1)!}{v!(n-1)!} \frac{t^v}{(1+t)^{n+v}}. \]

Such type of other operators are defined and studied by some researchers (see e.g. [1],[2],[4] and [6] etc.)

We denote by \( L \) the class of all Lebesgue measurable functions \( f \) and \( R^+ \), satisfying

\[ \int_{0}^{\infty} \frac{|f(t)|}{(1+t)^n} dt < \infty \]

for some positive integer \( n \).

Obviously this class of functions is bigger than the class of all Lebesgue integrable functions on \( R^+ \).

For a fixed natural number \( k \), the linear combination \( M_n(f,k,x) \) of \( M_{a,n}(f,x) \) is defined as

\[ M_n(f,k,x) = \sum_{j=0}^{k} c(j,k) M_{a,n}(f,x), \]

where

\[ c(j,k) = \prod_{i=0}^{k} \frac{d_i}{d_i - d_j}, \quad k \neq 0; \quad c(0,0) = 1 \]

and \( d_0, d_1, \ldots, d_k \) are \((k+1)\) arbitrary but fixed distinct positive integers. It turns out from [3], that even in simultaneous approximation the order of approximation for the operators (1.1) is \( O(n^{-1}) \).

AMS Subject Classification (1985):41A30,41A36.

*Department of Mathematics S.R.M.S. College of Engineering and Technology, Bareilly-243001 (U.P.) INDIA. E.Mail:PRAMA63@HOTMAIL.COM
By using the above linear combination (1.2) we can improve the order of approximation. The object of the present paper is to obtain direct results in simultaneous approximation for the linear combinations $M_n(f, k, x)$ of the operators (1.1).

2. In this section, we shall mention certain results which are necessary for the proof of main results.

**Lemma 2.1.** For $m \in N \setminus \{0\}$, if we define $\mu_{n,m}(x) = \sum_{r=0}^{\infty} p_{n,r}(x) \frac{v}{n} (x - x)^m$, then there holds the recurrence relation $n \mu_{n,m+1}(x) = x \{ \mu_{n,m}(x) + m \mu_{n,m-1}(x) \}$.

Consequently

(i) $\mu_{n,m}(x)$ is a polynomial in $x$ of degree $\leq m$.

(ii) $\mu_{n,m}(x) = O(n^{-\lfloor \frac{m+1}{2} \rfloor})$, where $[\alpha]$ denotes the integral part of $\alpha$.

**Lemma 2.2.** Let the $m$-th order moment be defined by

$$T_{r,n,m}(x) = (n - r - 1) \sum_{r=0}^{\infty} p_{n,r}(x) \int_0^x q_{n,r+1}(t)(t - x)^m dt.$$ 

Then

$$T_{r,n,m}(x) = 1, \quad T_{r,n+1}(x) = \frac{(r+1) + x(r+2)}{(n-r-2)}, \quad n > r + 2$$

and there holds the recurrence relation

$$(n - r - m - 2) T_{r,n,m+1}(x) = T_{r,n,m}(x) \{ r(1 + x) + (m + 1)(1 + 2x) \} + x T_{r,n,m}(x) + mx(2 + x) T_{r,n,m-1}(x),$$

where $n > r + m + 2$.

**Lemma 2.3.** Let $f$ be $r$ times differentiable on $\mathbb{R}$ such that $f^{(r-1)}(t) = o(t^{\alpha})$ for some $\alpha > 0$ as $t \to \infty$ then for $r = 1, 2, \ldots$ and $n > \alpha + r$, we have

$$M_{n,m}^{(r)}(f, x) = \frac{n^r(n - r - 1)!}{(n-2)!} \sum_{r=0}^{\infty} p_{n,r}(x) \int_0^x q_{n,r+1}(t)f^{(r)}(t) dt.$$ 

The proof of the above lemma easily follows by using Leibnitz theorem as in [6].

**Lemma 2.4.** There exist polynomials $\phi_{r,1,r}(x)$ independent of $n$ and $n$ such that

$$x^r \frac{d^r}{dx^r}[e^{-m}(nx)] = \sum_{2r \leq l \leq r} \binom{n}{l} (n - nx)^l \phi_{r,1,r}(x)[e^{-m}(nx)^{l-1}].$$

**Lemma 2.5.** If $C(j, k)$, $j = 0, 1, \ldots, k$ are defined as in (1.2) then

$$\sum_{j=0}^{k} C(j, k) d_j^{m} = \binom{1, m=0}{0, m=1, 2, k}.$$
3 In this section we shall prove the main results.

**THEOREM 3.1.** Let \( f \in L \) be bounded on every finite subinterval of \( \mathbb{R}^1 \) and \( f^{(2k+r+2)}(x) \) exist at a fixed point \( x \in (0, \infty) \). Let \( f(t) = O(t^\alpha) \) as \( t \to \infty \) for some \( \alpha > 0 \), then we have

\[
\lim_{n \to \infty} n^k |\alpha(n,r,k) M_n^{(r)}(f(x),x) - f^{(r)}(x)| = \sum_{i=r+1}^{2k+r+2} Q(i,k,r,x) f^{(r)}(x),
\]

where

\[
\alpha(n,r,k) = \left[ \sum_{j=0}^k \frac{C(j,k)(d_j n)^r (d_j n - r - 2)!}{(d_j n - 1)!} \right]^r,
\]

and \( Q(i,k,r,x) \) are certain polynomials in \( x \) of degree at most \( i \).

**PROOF.** By Taylor’s expansion we have

\[
f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t,x)(t-x)^{2k+r+2},
\]

where \( \varepsilon(t,x) \to 0 \) as \( t \to x \) and \( \varepsilon(t,x) = O((t-x)^\beta) \) for some \( \beta > 0 \). Making use of Lemma 2.3, we have

\[
n^{k+1} |\alpha(n,r,k) M_n^{(r)}(f(x),x) - f^{(r)}(x)|
\]

\[
= n^{k+1} |\alpha(n,r,k) \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^k C(j,k)(d_j n)^r (d_j n - r - 1)! \sum_{v=0}^\infty p_{d_j n,v} (x) \int_0^\infty q_{d_j n,v} (t) e(t,x)(t-x)^{2k+r+2} dt
\]

\[
+ \alpha(n,r,k) n^{k+1} \sum_{j=0}^k C(j,k)(d_j n - 1) \sum_{v=0}^\infty p_{d_j n,v}^{(r)}(x) \int_0^\infty q_{d_j n,v} (t) e(t,x)(t-x)^{2k+r+2} dt
\]

\[
= \sum_{i=r+1}^{2k+r+2} Q(i,k,r,x) f^{(r)}(x) + E_{n,r,k}(x) + o(1)
\]

By Lemma 2.2, we have

\[
E_{n,r,k}(x) = \alpha(n,r,k) n^{k+1} \sum_{j=0}^k C(j,k)(d_j n - 1) \sum_{v=0}^\infty p_{d_j n,v}^{(r)}(x) \int_0^\infty q_{d_j n,v} (t) e(t,x)(t-x)^{2k+r+2} dt
\]

In order to complete the proof of (3.1), it suffices to show that

\[
I = n^{k+1} (n-1) \sum_{v=0}^\infty p_{d_j n,v}^{(r)}(x) \int_0^\infty q_{d_j n,v} (t) e(t,x)(t-x)^{2k+r+2} dt
\]

tends to zero as \( n \to \infty \).

Now by Lemma 2.4 we have
\[ |f| \leq n^{k+1} (n-1) \sum_{v=0}^{\infty} \sum_{t \geq 0} n^{|v-nx|} |\phi_{v,z}(x)| p_{n,v}(x) \int_0^t q_{n,v}(s) |e(t,x)||t-x|^{2k+r+2} \, dt \]

\[ \leq n^{k+1} M(x)(n-1) \sum_{t \geq 0} q_{n,v}(x)|v-nx|^{1/2} \left( \int_0^t q_{n,v}(s) |e(t,x)||t-x|^{2k+r+2} \, dt \right)^{1/2} \]

\[ \leq n^{k+1} M(x)(n-1) \sum_{t \geq 0} n^{1/2} \left( \int_0^t q_{n,v}(s) |e(t,x)||t-x|^{2k+r+2} \, dt \right)^{1/2} \]

where \( M(x) = \sup_{n \geq 0} |\phi_{v,z}(x)|. \)

For a given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |e(t,x)| < \varepsilon \) whenever \( 0 < |t-x| < \delta. \) For \( |t-x| \geq \delta, \) we have \( |e(t-x)| < K|t-x|^{r+1}. \) Thus

\[ \left| \int_0^t q_{n,v}(s) e(t,x) |t-x|^{2k+r+2} \, dt \right|^2 \leq \left( \int_0^t q_{n,v}(s) \, ds \right)^2 (t-x)^{1+k+2r+4} \, dt \]

\[ = \frac{1}{(n-1)} \left( \int_{|t-x| < \delta} + \int_{|t-x| > \delta} q_{n,v}(s) e(t,x) |t-x|^{2k+r+4} \, dt \right)^2 \]

\[ \leq \frac{1}{(n-1)} \int_{|t-x| < \delta} q_{n,v}(s) e(t,x) |t-x|^{2k+r+4} \, dt + \int_{|t-x| > \delta} q_{n,v}(s) K^2 (t-x)^{1+k+2r+4} \, dt \]

Applying Lemma 2.2, we get

\[ \sum_{t \geq 0} q_{n,v}(x) \left( \int_0^t q_{n,v}(s) e(t,x) |t-x|^{2k+r+2} \, dt \right)^{1/2} \leq \frac{1}{(n-1)} \sum_{t \geq 0} q_{n,v}(x) \left( \int_0^t q_{n,v}(s) e(t,x) |t-x|^{2k+r+4} \, dt \right)^{1/2} \]

\[ + \frac{K^2}{(n-1)} \sum_{t \geq 0} q_{n,v}(x) \int_{|t-x| > \delta} (t-x)^{1+k+2r+4} \, dt \]

\[ = \varepsilon^2 O(n^{-(2k+r+4)}) + \frac{K^2}{(n-1)} \sum_{t \geq 0} q_{n,v}(x) \left( \frac{(t-x)}{\delta^{2s+1+2r+2r+4}} \right)^{1/s} \, dt \]

\[ \leq \varepsilon^2 O(n^{-(2k+r+4)}) + O(n^{-(3s+2)}) \]

Thus, we get

\[ |f| \leq n^{k+1} M(x)(n-1) \sum_{t \geq 0} \left( n^{r+1} O(n^{-(3s+2)}) + n^{-(2k+r+4)} \right) \]

\[ = \varepsilon^2 O(n^{-(3s+2)}) + O(n^{-(2k+r+4)}) \]

choosing \( s> r+2 \) and \( n \) sufficiently large this in turn implies \( l \to 0 \) as \( n \to \infty. \) This completes the proof.

**Theorem 3.2.** Let \( 1 \leq p \leq 2k + 2 \) and \( f \in L \) be bounded on every finite sub interval of
Also let \( f(t) = O(t^n) \) as \( t \to \infty \) for some \( \alpha > 0 \). If \( f^{(p+r)} \) exists and is continuous on \( (a - \eta, b + \eta) \), \( \eta > 0 \) having the modulus of continuity \( \omega_{f^{(p+r)}}(\delta) \) on \( (a - \eta, b + \eta) \). Then for \( n \) sufficiently large

\[
\|\alpha(n, r, k) M_n^{(r)}(f, k) - f^{(r)}\|_{(a, b)} \leq \max\{C_1 n^{-\alpha} \omega_{f^{(p+r)}}(n^{-\frac{1}{2}}), C_2 \eta^{-(k+1)}\}
\]

where \( C_1 = C_1(k, p, r), C_2 = C_2(k, p, r, f) \) and \( \alpha(n, r, k) \)

is as defined Theorem 3.1.

**PROOF.** By Taylor’s expansion of \( f \), for each \( t \in [0, \infty) \) and \( x \in [a, b] \), we have

\[
f(t) = \sum_{r=0}^{p+r} \frac{f^{(r)}(x)}{r!} (t - x)^r + \frac{1}{(p+r)!} \left( f^{(p+r)}(\xi) - f^{(p+r)}(x) \right) (t - x)^{p+r} \chi(t) + h(t, x)(1 - \chi(t))
\]

where \( \xi \) lies between \( t \) and \( x \) and \( \chi(t) \) is the characteristic function of \( (a - \eta, b + \eta) \). For

\[
t \in [0, \infty] \setminus (a - \eta, b + \eta) \text{ and } x \in [a, b],
\]

we define

\[
h(t, x) = f(t) - \sum_{r=0}^{p+r} \frac{f^{(r)}(x)}{r!} (t - x)^r
\]

For \( t \in (a - \eta, b + \eta) \) and \( x \in [a, b] \), we have

\[
f(t) = \sum_{r=0}^{p+r} \frac{f^{(r)}(x)}{r!} (t - x)^r + \frac{1}{(p+r)!} \left( f^{(p+r)}(\xi) - f^{(p+r)}(x) \right) (t - x)^{p+r}
\]

Now using (3.2) and Lemma 2.3, we get

\[
\alpha(n, r, k) M_n^{(r)}(f, k, x) - f^{(r)}(x) = \alpha(n, r, k) \left[ \sum_{r=0}^{p+r} \frac{f^{(r)}(x)}{r!} \sum_{j=0}^{k} C(j, k)(d, n)^r \frac{(d, n - r - 1)!}{(d, n - 2)!} \right.
\]

\[
\sum_{r=0}^{\infty} p_{d, n, x}(x) \int d_{d, n, r, x, r}(t) \frac{d^r}{dt^r} (t - x)^r dt - f^{(r)}(x)
\]

\[
\left. + \sum_{r=0}^{\infty} C(j, k)(d, n - 1) \sum_{r=0}^{\infty} p_{d, n, x}(x) \int d_{d, n, r, x, r}(t) \frac{1}{(p+r)!} \left( f^{(p+r)}(\xi) - f^{(p+r)}(x) \right) (t - x)^{p+r} \chi(t) dt
\]

\[
+ \sum_{r=0}^{\infty} C(j, k)(d, n - 1) \sum_{r=0}^{\infty} p_{d, n, x}(x) \int d_{d, n, r, x, r}(t) h(t, x)(1 - \chi(t)) dt \right)
\]

= \( I_1 + \alpha(n, r, k)(I_2 + I_3) \), say

Applying Lemma 2.2 and Lemma 2.5, we obtain

\( I_1 \leq K_1 n^{-(k+1)} \), uniformly for \( x \in [a, b] \), where \( K_1 = K_1(k, p, r, f) \)

Now we obtain \( I_2 \), let

\( I_4 = (n - 1) \sum_{r=0}^{\infty} p_{n, x}(x) \int d_{n, x}(t) \frac{1}{(p+r)!} \left( f^{(p+r)}(\xi) - f^{(p+r)}(x) \right) (t - x)^{p+r} \chi(t) dt \)

Using Lemma 2.4, we have
\[ |l_4| \leq (n-1) \sum_{v=0}^{\infty} \sum_{r=2v+1}^{\infty} n'|v-nx| \int p_{n,v}(x) \frac{|\phi_{1,v,r}(x)|}{x^r} \int_0^\infty q_{n,v}(t) \left| f^{(r+1)}(t) - f^{(r)}(x) \right| |t-x|^{r+1} \chi(t) dt \]

\[ \leq \frac{(n-1)}{(p+r)!} K_2 \sum_{r=2v+1}^{\infty} \sum_{v=0}^{\infty} n'|v-nx| \int p_{n,v}(x) \frac{|\phi_{1,v,r}(x)|}{x^r} \int_0^\infty q_{n,v}(t) |t-x|^{r+1} \chi(t) dt \]

\[ = \frac{(n-1)}{(p+r)!} K_2 \omega_r (p+r)(\delta) \sum_{r=2v+1}^{\infty} \sum_{v=0}^{\infty} n'|v-nx| \int p_{n,v}(x) \frac{|\phi_{1,v,r}(x)|}{x^r} \int_0^\infty q_{n,v}(t) |t-x|^{r+1} \chi(t) dt \]

where

\[ K_2 = \sup_{x \in [a,b]} \sup_{\xi \in [x]} \frac{|\phi_{1,v,r}(x)|}{x^r} \]

Now, we shall show that for \( s=0,1,2 \ldots \)

\[ (n-1) \sum_{v=0}^{\infty} p_{n,v}(x) |v-nx| \int q_{n,v}(t) |t-x|^s dt \]

We have

\[ (n-1) \sum_{v=0}^{\infty} p_{n,v}(x) |v-nx| \int q_{n,v}(t) |t-x|^s dt = O(n^{s+2}) \]

\[ \leq (n-1) \sum_{v=0}^{\infty} p_{n,v}(x) |v-nx| \left[ \left( \int q_{n,v}(t) dt \right)^2 \left( \int q_{n,v}(t) (t-x)^2 dt \right)^2 \right] \]

\[ \leq \frac{(n-1)}{\sqrt{(n-1)}} \sum_{v=0}^{\infty} p_{n,v}(x) |v-nx| \left[ \int q_{n,v}(t) (t-x)^2 dt \right]^2 \]

\[ \leq \left( \sum_{v=0}^{\infty} p_{n,v}(x) (v-nx)^2 \right)^{1/2} \left[ (n-1) \sum_{v=0}^{\infty} p_{n,v}(x) \int q_{n,v}(t) (t-x)^2 dt \right]^{1/2} \]

\[ = O(n^{1/2}) \cdot O(n^{1/2}) = O(n^{s+1}) \]

Uniformly in \( x \), by Lemma 2.1 and Lemma 2.2. Therefore

\[ |l_4| \leq K_4 \omega_r (p+r)(\delta) \sum_{r=2v+1}^{\infty} \sum_{v=0}^{\infty} n'|v-nx| \left[ O(n^{s+r+2}) + \frac{1}{\delta} O(n^{s+r+1}) \right] \]

Choosing \( \delta = n^{-1/2} \), we get

\[ |l_4| \leq K_4 n^{-r/2} \omega_r (p+r)(n^{-1/2}), \]

Hence

\[ |l_4| \leq K_4 n^{-r/2} \omega_r (p+r)(n^{-1/2}), \text{ where } K_5 = K_3(k,p,r) \]

Finally, we estimate \( l_3 \). Since \( t \in [0,\infty) \setminus (a-\eta,b+\eta) \), we can choose a \( \delta > 0 \) in such a way that \( |t-x| \geq \delta \) for all \( x \in [a,b] \)
Using Lemma 2.4, we get

\[
I_3 = (n-1) \sum_{r=0}^{n-2} \sum_{i=0}^{r} n^r |v - nx| \left( \frac{\phi_{r,i}(x)}{x^r} \right) p_{n,r}(x) \int q_{n,r}(t) |h(t,x)| dt
\]

If \( \beta \) is any integer \( \geq \max \{ \alpha, 2k + r + 2 \} \), then we can find a constant \( K_\beta \) such that \( |h(t,x)| \leq K_\beta |x|^\beta \) for \( |t - x| \geq \delta \).

Making use of Cauchy-Schwarz inequality, we have

\[
|I_3| \leq K_\beta (n-1) \sum_{r=0}^{n-2} \sum_{i=0}^{r} n^r |v - nx| \left( \frac{\phi_{r,i}(x)}{x^r} \right) p_{n,r}(x) \int q_{n,r}(t) |h(t,x)| dt
\]

\[
= K_\beta (n-1) \sum_{r=0}^{n-2} \sum_{i=0}^{r} n^r |v - nx| \left( \frac{\phi_{r,i}(x)}{x^r} \right) p_{n,r}(x) \int q_{n,r}(t) \left( \frac{|t - x|^\beta}{8^2 \beta} \right) dt
\]

uniformly on \([a, b]\) where \( s \) is a natural number bigger than \( \beta / 2 \).

Hence \( |I_3| \leq K_\beta n^{(k+1)} \).

Combining the estimate of \( I_1, I_2 \) and \( I_3 \) we get the required result.

Acknowledgement: The first author is thankful to U.G.C. for research support under unassigned grant, group D, Minor Research Project 1996-97.

REFERENCES


