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# SEMI-STRONG U NUMBERS IN THE P-ADIC FIELD $\mathbf{Q}_p$ (*)

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**Abstract.** *In this paper, by adjusting Alnıaçik's article [1] to the p-adic numbers field, $Q_p$, we have obtained some uncountable subfields of $Q_p$.*

**Introduction.** For the convenience of the reader we shall briefly recall Koksma's well known classification [2] for the p-adic numbers, which was introduced by Schlikewei [3].

For an algebraic number $\alpha$, define the height $H(\alpha)$ as the height of the minimal polynomial of $\alpha$, say $P(x) \in Z[x]$, where the $P$ is supposed to be normalized, such that, its coefficients are relatively prime.

For a p-adic number $\xi$ in $Q_p$ and a natural number $n$ put

$$w_n^*(H,\xi) = \min_{\substack{\deg \alpha \leq n \\ H(\alpha) \leq H \\ \alpha \neq \xi}} |\xi - \alpha|_p$$

furthermore

$$w_n^*(\xi) = \lim_{H\to\infty}\sup\left(-\frac{\log w_n^*(H,\xi)}{\log H}\right)$$

and

$$w^*(\xi) = \lim_{n\to\infty}\sup\frac{w_n^*(\xi)}{n}.$$

Define $\mu^*(\xi)$ as being the smallest $n$, such that $w_n^*(\xi) = \infty$, if such an $n$ exists. Otherwise put $\mu^*(\xi) = \infty$. Now call a p-adic number $\xi$ a

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$$S^* - number \text{ if } 0 < w^*(\xi) < \infty \text{ and } \mu^*(\xi) = \infty,$$
$$T^* - number \text{ if } \quad w^*(\xi) = \infty \quad \text{and } \mu^*(\xi) = \infty,$$
$$U^* - number \text{ if } \quad w^*(\xi) = \infty \quad \text{and } \mu^*(\xi) < \infty.$$

Every $S^*-, T^*, U^*-number$ is a $S-, T-, U-$, number respectively. Moreover in [1] Long has proved that Mahler's subclasses $U_m$ are equal to the Koksma's subclasses $U_m^*$.

For the proof of the main results we shall need the following lemmas.

**Lemma 1.** Let $\alpha, \beta$ be two p-adic algebraic numbers with different minimal polynomials. Then, for $|\alpha|_p = p^{-h}$ and $r = \min\{0, h\}$,

$$|\alpha - \beta|_p > \frac{c_1}{H(\alpha)^{M-1} H(\beta)^M},$$

where $M > \max\{\deg \alpha, dg e \beta\}$ and $c_1 = \frac{p^{(M-1)r - M(|h|+1)}}{(2M!)}$ (Schlikewei [3] ).

**Lemma 2.** Let $P(x) = a_n x^n + \ldots + a_0 \in Z[x]$. If $\alpha$ is a root of $P$ then

$$|\alpha - \beta|_p > \frac{1}{H(P)} \text{ (Morrison [5] ).}$$

**Lemma 3.** Let $\alpha_1, \ldots \alpha_k$ $(k \geq 1)$ be algebraic numbers in $Q_p$ with $Q[(\alpha_1, \ldots \alpha_k); Q] = q$ and let $F(y, x_1, \ldots x_k)$ be a polynomial with integral coefficients, whose degree in $y$ is at least one. If $\eta$ is an algebraic number such that $F(\eta, \alpha_1, \ldots \alpha_k) = 0$, then the degree of $\eta \leq dq$ and

$$h_\eta \leq 3^{2dq+(l_1+\ldots+l_k)q} . h_{\alpha_1}^{l_1 q} \ldots h_{\alpha_k}^{l_k q}$$

where $h_\eta$ is the height of $\eta$, $h_{\alpha_i}$ is the height of $\alpha_i$ $(i = 1, \ldots, k)$. $H$ is the maximum of the absolute values of the coefficients of $F$, $l_i$ is the degree of $F$ in $x_i$ $(i = 1, \ldots, k)$ and $d$ is the degree of $F$ in $y$.( O.Ş.İÇEN [6] )

The first main result of one of this paper is the following theorem.

**Theorem 1.** Let $(\alpha_i)_{i \in N}$ be a sequence of p-adic algebraic numbers with
(1) $\deg \alpha_i = m_i \leq l$, $\lim_{i \to \infty} H(\alpha_i) = \infty$, ( $k > 0$ constant),
(2) $0 < |\alpha_{i+1} - \alpha_i|_p = \frac{1}{H(\alpha_i)^{w_i}}$, where $\lim_{i \to \infty} w_i = \infty$.
(3) $|\alpha_{i+1} - \alpha_i|_p < \frac{1}{H(\alpha_i)^k}$ for some $\delta > 0$.
Then $\lim_{i \to \infty} \alpha_i \in U_m^*$, where $m = \liminf_{i \to \infty} m_i$.

*Proof.* It follows from lemma 2 and hypothesis (2) that consecutively $\alpha_i'$s cannot be conjugates and if $i$ is sufficiently large, $|\alpha_{i+1}|_p = |\alpha_i|_p$. Hence by putting $|\alpha_i|_p = p^{-h}$ $(h \in Z)$ and $t = \min\{0, h\}$ and using lemma 1, we get

(4) $\frac{c_0}{H(\alpha_i)^k H(\alpha_{i+1})^{k-1}} < |\alpha_{i+1} - \alpha_i|_p$,

where $c_0 = \frac{p^{(k-1)t-k(|h|+1)}}{(2k!)^k}$.

Since $\lim_{i\to\infty} |\alpha_{i+1} - \alpha_i|_p = 0$, the sequence $(\alpha_i)_{i\in N}$ is a Cauchy sequence in $Q_p$ and so $\lim_{i\to\infty} \alpha_i = \xi$ exists. Let's show that $\xi \in U_m^*$. First we shall prove that, for sufficiently large $i$ and $s > i$, $|\alpha_s - \alpha_i|_p = |\alpha_{i+1} - \alpha_i|_p$. Indeed, combining (4) and (2) and using both the facts that $H(\alpha_i) \to \infty$ and $w_i \to \infty$ we obtain

(5) $H(\alpha_i)^{\frac{w_i}{2}} < H(\alpha_{i+1})^k$

and from this

(6) $H(\alpha_i)^{w_i} < H(\alpha_i)^{w_{i+1}}$ ($i$ large enough).

Combining the relations (2) and (6) we get

$|\alpha_{i+1} - \alpha_i|_p > |\alpha_{i+t} - \alpha_{i+t+1}|_p$, for each $t = 2, \ldots s - i$.

Hence

(7) $|\alpha_s - \alpha_i|_p = |\alpha_{i+1} - \alpha_i|_p = \frac{1}{H(\alpha_i)^{w_i}}$ ( $i$ large and $s > i$)

Since $\alpha_i \to \xi$, for sufficiently large $i$ there is a $s > i$ such that

$|\xi - \alpha_s|_p < H(\alpha_i)^{-w_{i+1}}$.

Therefore a combination of (6), (7) and (8) together with the equalities,

$|\xi - \alpha_i|_p = \max\{|\xi - \alpha_s|_p, |\alpha_s - \alpha_i|_p\} = |\alpha_i - \alpha_s|_p = H(\alpha_i)^{-w_i}$,

gives us

(9) $|\xi - \alpha_i|_p = H(\alpha_i)^{-w_i}$, (for $i$ large).

On the other hand, since $\liminf_{i\to\infty} m_i = m$, we have a subsequence $(\alpha_{i_k})_{k\in N}$ of $(\alpha_i)_{i\in N}$ such that $\liminf_{k\to\infty} \alpha_{i_k} = m$. Hence for sufficiently large $k$, $\deg \alpha_{i_k} = m$. Hence, using (9) we get $|\xi - \alpha_{i_k}|_p = H(\alpha_{i_k})^{-w_{i_k}}$, which shows that $\mu^*(\xi) \leq m$.

We shall complete the proof by showing the opposite inequality $\mu^*(\xi) \geq m$. For this we shall distinguish two cases according to $m = 1$ or $m > 1$. In the case $m = 1$, by definition of $\mu^*(\xi)$, we have $\mu^*(\xi) \geq 1$. So together with $\mu^*(\xi) \leq m = 1$, we obtain $\mu^*(\xi) = 1$.

Now suppose that $m > 1$. Let $\beta$ be a p-adic number of degree $< m$. Since $\liminf_{i\to\infty} m_i = m$, $\deg \alpha_i \geq m$ for sufficiently large $i$. Applying lemma 1, we get

(10) $| \beta - \alpha_i |_p > c_1 H(\alpha_i)^{-(k-1)} . H(\beta)^{-k}$ ($i$ large).

On the other hand, as $w_i \to \infty$, for sufficiently large $i$, we have

(11) $w_i > \frac{2k(k+\delta)}{\delta}$

Now suppose that the p-adic algebraic number $\beta$ satisfies the condition.

(12) $H(\beta) > \max\{H(\alpha_{i_0}), \frac{1}{c_1}\}$.

where $i_0$ is a sufficiently large, fixed index. It is clear that there exists a natural number $i \geq i_0$ such that, for every p-adic algebraic number satisfying (12), we have

(13) $H(\alpha_i) \leq H(\beta) \leq H(\alpha_{i+1})$.

Taking into account (5), (11) and (13) we can have only one of the following cases:

$$\textbf{(I)}\ H(\alpha_i) \leq H(\beta) \leq H(\alpha_{i+1})^{\frac{\delta}{2k}}$$

(14) or

$$\textbf{(II)}\ H(\alpha_{i+1})^{\frac{\delta}{2k}} \leq H(\beta) \leq H(\alpha_{i+1})$$

Suppose that the first relation in(14) holds. Then a combination of(2), (3), (9), (10), (12) and (14) gives us

(15) $| \beta - \alpha_i |_p > H(\beta)^{-2k} > H(\alpha_{i+1})^{-\delta} > H(\alpha_i)^{-w_i} = | \xi - \alpha_i |_p$.

Furthermore, using the equation $| \xi - \beta |_p = \max\{| \xi - \alpha_i |_p, | \beta - \alpha_i |_p\}$,we get

(16) $| \xi - \beta_i |_p > H(\alpha_i)^{-2k}$ (for large $i$).

If he second relation in(14) holds, then using the relations (9), (10), (11), (12), we get

$$| \xi - \alpha_{i+1} |_p = H(\alpha_{i+1})^{-w_{i+1}} < H(\beta)^{-\frac{2k(k+\delta)}{\delta}} < H(\beta)^{-\frac{2k}{\delta} - k^2} < | \alpha_{i+1} - \beta |_p$$

so that

(17) $| \xi - \beta |_p = | \alpha_{i+1} - \beta |_p > H(\beta)^{-\frac{2k^2}{\delta} - k}$.

As the exponent of $H(\beta)$ on the right hand side of (17) is greater than that of (16), (17) is verified for all p-adic algebraic numbers of degree at most $m-1$ and height greater than $\max\{H(\alpha_{i_0}), \frac{1}{c_1}\}$. This shows us that $\mu^*(\xi) \geq m$. This result together with the inequality $\mu^*(\xi) \leq m$ imply that $\mu^*(\xi) = m$ also in case $m > 1$, as well. Hence $\xi \in U_m$, and this completes the proof.

**Definition 1.** Given a $U$ number $\xi$ in $Q_p$. If there is a sequence $(\alpha_i)$ of p-adic algebraic numbers satisfying the conditions (1), (2) and 3 of Theorem 1, then we say that "$\xi = \lim_{i\to\infty} \alpha_i$ is an irregular semi-strong $U - number$."

In Theorem 1 we have seen that if $\liminf_{i\to\infty} m_i = m$, then $\xi \in U_m$.

In the sequel $U^{is}$ and $U_m^{is}$ will denote the set of all irregular semi-strong $U-, U_m-$ numbers.

**Example.** If $p$ is a prime number and $\alpha$ is a p-adic algebraic number of degree $m$, then

$\kappa = \alpha + \sum_{i=1}^{\infty} p^{n!}$ is in $U_m^{is}$.

By defining $\alpha_n = \alpha + p^{1!} + ... + p^{n!}$, one can show that $(\alpha_n)_{n\in N}$ satisfies the conditions of Theorem 1.

**Definition 2.** Let $(x_i)_{i\in N}$ be a strictly increasing sequence in natural numbers satisfying $\lim_{i\to\infty} \frac{\log x_{i+1}}{\log x_i} = \infty$ and $(a_i)_{i\in N}$ be another natural numbers sequence. We say that "the sequence $(a_i)_{i\in N}$ is comparable with $(x_i)_{i\in N}$" if there exists a subsequence $(x_{n_i})_{i\in N}$ of $(x_i)_{i\in N}$ and positive real numbers $k_1, k_2$ such that

$x_{n_i}^{k_1} < a_i \leq x_{n_i}^{k_2}$ ( $i = 1, 2, ...$).

The main result of the paper is the following theorem.

**Theorem 2.** Let $(x_i)_{i\in N}$ be as above definition. Then the set,

$$F = \left\{\xi \in U_m^{is} : (H(\alpha_i))_{i\in N} \text{ is comparable with } (x_i)_{i\in N} \text{, where } \lim_{i\to\infty} \alpha_i = \xi,\ m \in N\right\} \cup A$$

is a subfield of continuum cardinality of $Q_p$. Here $A$ denotes the set of p-adic algebraic numbers.

*Proof.* Let $y_1, y_2 \in F$. Assume that $y_1 \in U_r^{ps}, y_2 \in U_t^{ps}$. Then there are positive numbers $s_1, s_2, s_3, s_4, \rho_1, \rho_2$ and sequences of algebraic numbers $(\alpha_i)_{i\in N}, (\beta_i)_{i\in N}$ ( $\deg \alpha_i \leq l, \deg \beta_i \leq l$ where $l \geq \max\{r, t\}$) such that

$(A)$ $0 < | \alpha_{i+1} - \alpha_i |_p = \frac{1}{H(\alpha_i)^{w_i}} < \frac{1}{H(\alpha_{i+1})^{\rho_1}}$, where $\lim_{i\to\infty} w_i = \infty$, $\lim_{i\to\infty} H(\alpha_i) = \infty$

$(B)$ $0 < | \beta_{i+1} - \beta_i |_p = \frac{1}{H(\beta_i)^{\nu_i}} < \frac{1}{H(\beta_{i+1})^{\rho_2}}$, where $\lim_{i\to\infty} \nu_i = \infty$, $\lim_{i\to\infty} H(\beta_i) = \infty$

and the subsequences $(x_{n_i})_{i\in N}, (x_{m_i})_{i\in N}$ of $(x_i)_{i\in N}$ satisfying

$(C)$ $x_{n_i}^{s_1} < H(\alpha_i) \leq x_{n_i}^{s_2}$ ( $i = 1, 2, \ldots$).

$(D)$ $x_{m_i}^{s_3} < H(\beta_i) \leq x_{m_i}^{s_4}$ ( $i = 1, 2, \ldots$).

Let $(x_{r_i})_{i\in N}$ denote the monotonic union sequence formed from $(x_{n_i})_{i\in N}, (x_{m_i})_{i\in N}$. Assume that $x_{r_{i_0}} > \max\{H(\alpha_i), H(\beta_i)\}$. We shall introduce positive integers $j(i), l(i)$ and then p-adic algebraic numbers $\delta_i$ as follows. For $i \geq i_0$,

$$n_{j(i)} = \max\{ n_\nu : n_\nu \leq r_i\}$$

(18)

$$m_{l(i)} = \max\{ m_\nu : m_\nu \leq r_i\}$$

$$\delta_i = \alpha_{j(i)} + \beta_{l(i)}.$$

Now consider the set $B = \{\delta_i : i \geq i_0\}$. If $B$ contains only finitely many p-adic algebraic numbers then $\lim_{i\to\infty} \delta_i = y_1 + y_2 \in B$. Hence $y_1 + y_2$ is a p-adic algebraic numbers i.e.e $y_1 + y_2 \in F$.

Hence we suppose that $B$ contains infinitely many p-adic algebraic numbers. In this case, we define a subsequence $(\delta_{i_k})_{k\in N}$ of $(\delta_i)_{i\in N}$ as follows.

(19) If $\delta_{i_{k+1}} = \delta_{i_k+s}$ then $\delta_{i_k} = \delta_{i_k+1} = \ldots = \delta_{i_k+s-1}$ ( $s = 1, 2, \ldots, i_{k+1} - i_k - 1, i_{k+1} - i_k$)

Now by using lemma 3,(18), $(C)$ and $(D)$, we have

$H(\delta_{i_k}) < 3^{4l^2} x_{r_{i_k}}^{l^2(s_2+s_4)}$. Since $x_{r_{i_k}} \to \infty$, for large $k$ we get

$(20)_k \quad H(\delta_{i_k}) < x_{r_{i_k}}^{s_5}$ where $s_5 = l^2(s_2 + s_4)$.

On the other hand, from the definitions of $j(i)$and $t(i)$, we see that

$(21)\ 0 \le j(i_{k+1}) - j(i_{k+1} - 1) \le 1$ and $0 \le t(i_{k+1}) - t(i_{k+1} - 1) \le 1$.

Moreover, from the definition of $(\delta_{i_k})_{k\in N}$, one can see easily that the numbers in (21) cannot be zero at the same time. iff the numbers are both different from zero, then a combination of $(A)$, $(B)$, $(C)$, (19) and (21) gives us

$(22)\ |\delta_{i_{k+1}} - \delta_{i_k}|_p = |\delta_{i_{k+1}} - \delta_{i_{k+1}-1}|_p \le \max\{|\alpha_{j(i_{k+1})} - \alpha_{j(i_{k+1}-1)}|_p, |\beta_{t(i_{k+1})} - \beta_{t(i_{k+1}-1)}|_p\} < \max\{(H(\alpha_{j(i_{k+1}-1)+1}))^{-\rho_1}, (H(\beta_{t(i_{k+1}-1)+1}))^{-\rho_2}\} = \frac{1}{x_{r_{i_{k+1}}}^{\rho}}$.

where $\rho = \min\{s_1\rho_1, s_3\rho_2\}$. Hence using the relations $(20)_{k+1}$ and (22) and putting $\nu_{i_k} = \frac{\rho \log x_{r_{i_{k+1}}}}{s_5 \log x_{r_{i_k}}}$, we get

$(23)\ |\delta_{i_{k+1}} - \delta_{i_k}|_p < H(\delta_{i_k})^{-\nu_{i_k}}$ ($k$ large).

Let's show that $H(\delta_{i_k}) \to \infty$ as $k \to \infty$. Indeed, by using lemma 1,(22) and the hypothesis $\frac{\log x_{r_{i_{k+1}}}}{\log x_{r_{i_k}}} \to \infty$, for sufficiently large $k$, we have $x_{r_{i_k}}^{\frac{5\rho}{8l^2}} < H(\delta_{i_{k+1}})$. Thus, since the limit of the left side of the last relation is infinity, we get $H(\delta_{i_k}) \to \infty$. Hence by Theorem 1, we have $\lim_{k\to\infty} \delta_{i_k} = y_1 + y_2 \in U_m^*$ ( for some $m \le l^2$).

Now we show that $(H(\delta_{i_k}))_{k\in N}$ is comparable with $(x_{r_{i_k}})_{k\in N}$. To this end, first remark that, for sufficiently large $k$, the sequence $(H(\delta_{i_k}))_{k\in N}$ is strictly increasing and then by using lemma 1 and (22) we get

$(24)\ c_2\ H(\delta_{i_k})^{-l^2} H(\delta_{i_{k+1}})^{-l^2} < |\delta_{i_{k+1}} - \delta_{i_k}|_p < x_{r_{i_{k+1}}}^{-\rho}$.

Since $H(\delta_{i_k})$ is strictly increasing and $H(\delta_{i_k}) \to \infty$. It follows from (24),

$$x_{r_{i_{k+1}}}^{\frac{\rho}{3l^2}} < H(\delta_{i_{k+1}}).$$

Hence a combining the last relation above and $(20)_{k+1}$, for large $k$, gives us

$$x_{r_{i_{k+1}}}^{\frac{\rho}{3l^2}} < H(\delta_{i_{k+1}}) < x_{r_{i_{k+1}}}^{s_5}.$$

Hence $y_1 + y_2 \in F$.

Now we show that the product $y_1y_2$ is in $F$. To show this we shall approximate $y_1y_2$ by algebraic numbers $\delta_i'$ as defined as

(25) $\delta_i^| = \alpha_{j(i)}\beta_{l(i)} \quad ( i \geq i_0)$.

If $B = \{ : i \geq i_0\}$contains only finitely many p-adic algebraic numbers, then it is closed. Therefore by (25) we have $y_1y_2 \in A \subset F$. If $B$ is not finite, we can choose a subsequence $( \delta_{i_k}^| )_{k\in N}$ of $( \delta_i^| )_{i\in N}$ as follows.

If $\delta_{i_{k+1}}^| = \delta_{i_k+s}^|$ then $\delta_{i_k}^| = \delta_{i_k+1}^| = ... = \delta_{i_k+s-1}^|$ ( $s = 1, 2..., i_{k+1} - i_k - 1, i_{k+1} - i_k$)

Now using (18) , $(C)$ and (22) we obtain

(26) $| \delta_{i_{k+1}}^| - \delta_{i_k}^| |_p = | \delta_{i_{k+1}}^| - \delta_{i_{k+1}-1}^| |_p \leq \max\{| \alpha_{j(i_{k+1})} - \alpha_{j(i_{k+1}-1)} |_p \cdot | \beta_{l(i_{k+1})} |_p, | \beta_{l(i_{k+1})} - \beta_{l(i_{k+1}-1)} |_p \cdot | \alpha_{j(i_{k+1}-1)} |_p\} \leq M \max\{| \alpha_{j(i_{k+1})} - \alpha_{j(i_{k+1}-1)} |_p, | \beta_{l(i_{k+1})} - \beta_{l(i_{k+1}-1)} |_p\}$,

where $M = \max\{1, | y_1 |_p, | y_2 |_p\}$. On the other hand, using an argument similar the one used in the previous steps, we obtain

(27) $H(\delta_{i_k}^|) < x_{i_k}^{3l^2} \quad$ ( for $k$ large).

Hence a combination of (26) and (27) gives us

$$| \delta_{i_{k+1}}^| - \delta_{i_k}^| |_p < H(\delta_{i_{k+1}}^|)^{\frac{-\lambda}{6l^2}} \quad \text{( for k large).}$$

Moreover, using the same arguments that we have used to get (23), we obtain

$$| \delta_{i_{k+1}}^| - \delta_{i_k}^| |_p < H(\delta_{i_k}^|)^{\frac{-\nu_{i_k}}{2}} \quad \text{( for } k \text{ large),}$$

which shows that $y_1y_2$ is in $U_m^{is}$ for some $m \leq l^2$. Next, as we have shown for $(H(\delta_{i_k}))_{k\in N}$, one can easily show that respectively, $H(\delta_{i_k}^|) \to \infty$ and ( $H(\delta_{i_k}^|))_{k\in N}$ is comparable with $(x_{r_{i_k}})_{k\in N}$ and so $y_1y_2 \in F$.

Finally let $\alpha \in A$ and $y_1 \in F - A$. Then using an similar argument to the one used to prove that $\alpha + y_1 \in F$ and approximating $\alpha y_1, \alpha + y_1$ by $(\alpha\alpha_i)_{i\in N}, (\alpha + \alpha_i)_{i\in N}$ respectively, one shows that $\alpha y_1, \alpha + y_1 \in F$.

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