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Research Article

Zero-divisor graphs of Catalan monoid

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Abstract

Let $\mathcal{C}_n$ be the Catalan monoid on $X_n = \{1, \dots, n\}$ under its natural order. In this paper, we describe the sets of left zero-divisors, right zero-divisors and two sided zero-divisors of $\mathcal{C}_n$; and their numbers. For $n \geq 4$, we define an undirected graph $\Gamma(\mathcal{C}_n)$ associated with $\mathcal{C}_n$ whose vertices are the two sided zero-divisors of $\mathcal{C}_n$ excluding the zero element θ of $\mathcal{C}_n$ with distinct two vertices α and β joined by an edge in case $\alpha\beta = \theta = \beta\alpha$. Then we first prove that $\Gamma(\mathcal{C}_n)$ is a connected graph, and then we find the diameter, radius, girth, domination number, clique number and chromatic numbers and the degrees of all vertices of $\Gamma(\mathcal{C}_n)$. Moreover, we prove that $\Gamma(\mathcal{C}_n)$ is a chordal graph, and so a perfect graph.

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1. Introduction

The zero-divisor graph was introduced by Beck on commutative rings in [3]. In Beck's definition zero element is a vertex in the graph too, later the standard definition of zero-divisor graphs on commutative rings was given by Anderson and Livingston in [1]. Let R be commutative ring, let $Z(R)$ be the set of the zero-divisors of R. The zero-divisor graph of R is an undirected graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, where distinct vertices x and y of $\Gamma(R)$ are adjacent if and only if $xy = 0$. Demeyer et. all have considered this definition for semigroups and they defined and found some basic properties of the zero-divisor graph of a commutative semigroup with zero in [5, 6]. In particular, they proved that the zero-divisor graph of a commutative semigroup with zero is connected. Since then, the zero-divisor graphs of some special classes of commutative semigroups with zero have been researched (see [4, 17]). For non-commutative rings, a directed zero-divisor graph and some undirected zero-divisor graphs were defined by Redmond in [14]. For a ring R let $Z_R(T)$ be the set of all two sided zero-divisor elements of R. Then Redmond defines an undirected zero-divisor graph $\Gamma(R)$ with vertices $Z_R(T) \setminus \{0\}$, where distinct vertices x and y are adjacent with a single edge if and only if $xy = 0 = yx$ (see [14, Definition 3.4.]). If R is a non-commutative ring, then $\Gamma(R)$ need not to be connected (for an example see [14, Figure 9.]) and if R is a commutative ring then $\Gamma(R)$ coincide with standard zero-divisor graph of R in [14]. As Demeyer et. all, we can consider this definition for non-commutative semigroups which have zero.

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Let $X_n = \{1, \ldots, n\}$ finite set with its natural order. Let T_n be the full transformation semigroup on X_n. We call a transformation $\alpha : X_n \to X_n$ order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and order-decreasing if $x\alpha \leq x$ for all $x \in X_n$. Some properties of the semigroup $\mathcal{C}_n$, which consists of all order-preserving and order-decreasing transformations have been investigated over the last forty years (see, for example [2,10,12]). Moreover, $\mathcal{C}_n$ is called Catalan monoid too. Since $1\alpha = 1$ for every $\alpha \in \mathcal{C}_n$, if we take and fix $\theta \in \mathcal{C}_n$ as the unique constant map, then $\theta\alpha = \theta = \alpha\theta$ for every $\alpha \in \mathcal{C}_n$, and so θ is the zero element of $\mathcal{C}_n$. Moreover, it is clear that $\mathcal{C}_n$ is a non-commutative semigroup for $n \geq 3$.

For $n \geq 2$ let $\mathcal{C}_n^* = \mathcal{C}_n \setminus \{\theta\}$. Then we define the following sets

$$\begin{aligned}
L = L(\mathcal{C}_n) &= \{\alpha \in \mathcal{C}_n \mid \alpha\beta = \theta \text{ for some } \beta \in \mathcal{C}_n^*\}, \\
R = R(\mathcal{C}_n) &= \{\alpha \in \mathcal{C}_n \mid \gamma\alpha = \theta \text{ for some } \gamma \in \mathcal{C}_n^*\} \text{ and} \\
T = T(\mathcal{C}_n) &= \{\alpha \in \mathcal{C}_n \mid \alpha\beta = \theta = \gamma\alpha \text{ for some } \beta, \gamma \in \mathcal{C}_n^*\} = L \cap R
\end{aligned}$$

which are called the set of left zero-divisors, right zero-divisors and (two sided) zero-divisors of $\mathcal{C}_n$, respectively. It is known that the cardinality of $\mathcal{C}_n$ is $\frac{1}{n+1}\binom{2n}{n}$ which is called n-th Catalan number (see, for example [10]). In this paper we find the left zero-divisors, right zero-divisors and zero-divisors of $\mathcal{C}_n$, and then we find their numbers.

For a semigroup S with zero 0 if $T(S) \setminus \{0\} \neq \emptyset$ where $T(S) = \{z \in S \mid zx = 0 = yz \text{ for } x, y \in S \setminus \{0\}\}$, then we similarly define the (undirected) zero-divisor graph $\Gamma(S)$ associated with S whose the set of vertices is $T(S) \setminus \{0\}$ with distinct two vertices joined by an edge in case $xy = 0 = yx$ for some $x, y \in T(S) \setminus \{0\}$. Notice that $\theta \in T(\mathcal{C}_n)$ for all $n \geq 2$, but $T(\mathcal{C}_n) \setminus \{\theta\} \neq \emptyset$ if $n \geq 3$. Moreover, $\Gamma(\mathcal{C}_3)$ is a graph with exactly one vertex and no edge. In this paper, we prove that $\Gamma(\mathcal{C}_n)$ is a connected graph and then we find the diameter, radius, girth, domination number, clique and chromatic numbers and the degrees of all vertices of $\Gamma(\mathcal{C}_n)$ for $n \geq 4$. Moreover, we prove that $\Gamma(\mathcal{C}_n)$ is a chordal graph, and so a perfect graph for $n \geq 4$.

For semigroup terminology see [9,11] and for graph theoretical terminology see [16].

2. Primaries

A lattice path L in $\mathbb{Z}^d$ of lenght k with steps in S is a sequence $v_0, v_1, \ldots, v_k$ in $\mathbb{Z}^d$ such that each consecutive difference $v_i - v_{i-1}$ lies in S for every $i = 1, \ldots, k$. In the two-dimensional space $\mathbb{Z}^2$, let (x_1, y_1) and (x_2, y_2) be two points with $x_1 \leq x_2$ and $y_1 \leq y_2$. Then a North-East (NE) lattice path from (x_1, y_1) to (x_2, y_2) is a lattice path in $\mathbb{Z}^2$ with steps in $S = \{\ (0,1),\ (1,0)\ \}$. $(0,1)$ steps are called North steps, and $(1,0)$ steps are called East steps. For $n \in \mathbb{Z}^+$ a Dyck path is a NE lattice path from $(0,0)$ to (n,n) that lies below but may touch the diagonal $y = x$ (see, for example [13,15]). It is a well-known fact that the number of all the Dyck paths from $(0,0)$ to (n,n) is

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

the n-th Catalan number. (Andre's) reflection principle is a tool to prove this fact (see [8]). Moreover, Higgins proved that the cardinality of $\mathcal{C}_n$ is equal to C_n in [10]. Thus there are bijections from $\mathcal{C}_n$ to DP_n which is the set of Dyck paths from $(0,0)$ to (n,n). Since bijections from $\mathcal{C}_n$ to DP_n are useful throughout this paper, we state and prove this well-known fact with defining a bijection from $\mathcal{C}_n$ to DP_n.

Proposition 2.1. *Let DP_n be the set of Dyck paths from $(0,0)$ to (n,n). Then there is a bijection from $\mathcal{C}_n$ to DP_n.*

Proof. We define a function $f : \mathcal{C}_n \to DP_n$ as follows: For the zero element θ, let $\theta f = P_\theta$ where

$$P_\theta : (0,0) - (1,0) - \cdots - (n,0) - (n,1) - \cdots - (n,n),$$

which is clearly a Dyck path. For any $\alpha \in \mathcal{C}_n \setminus \{\theta\}$ and for any $i \in X_n$ let $h_i(\alpha)$ be the horizontal line from $(i-1, i\alpha - 1)$ to $(i, i\alpha - 1)$. For every $i \in X_n$, since $i\alpha \leq i$, $h_i(\alpha)$ does not cross the diagonal. Notice that if $i\alpha = (i+1)\alpha$ for any $1 \leq \ i \leq n-1$, then $h_i(\alpha)$ and $h_{i+1}(\alpha)$ are adjacent lines, and that if $i\alpha \neq (i+1)\alpha$ then $h_i(\alpha)$ lower than $h_{i+1}(\alpha)$. Since $\alpha \neq \theta$ there exists at least one $i \in X_{n-1}$ such that $h_i(\alpha)$ lower than $h_{i+1}(\alpha)$. Suppose that there exist k many lower horizontal lines, say $h_{i_1}(\alpha), \ldots, h_{i_k}(\alpha)$ with $i_1 \ < \cdots < i_k < n$. Notice that $i_1\alpha = 1$. Then let $\alpha f = P_\alpha$ where

$$\begin{aligned} P_\alpha \ \ : \ \ & (0,0) - \cdots - (i_1 - 1, 0) - (i_1, 0) - (i_1, 1) - \cdots - (i_1, i_2\alpha - 1) \\ & -(i_1 + 1, i_2\alpha - 1) - \cdots - (i_2 - 1, i_2\alpha - 1) - (i_2, i_2\alpha - 1) \\ & -(i_2, i_2\alpha) - \cdots - (i_2, i_3\alpha - 1) - (i_2 + 1, i_3\alpha - 1) - \cdots \\ & -(i_{k-1}, i_k\alpha - 1) - (i_{k-1} + 1, i_k\alpha - 1) - \cdots \\ & -(i_k - 1, i_k\alpha - 1) - (i_k, i_k\alpha - 1) - (i_k, i_k\alpha) - \cdots - (i_k, n\alpha - 1) \\ & -(i_k + 1, n\alpha - 1) - \cdots - (n, n\alpha - 1) - (n, n\alpha) - \cdots - (n,n), \end{aligned}$$

which is clearly a Dyck path.

Let α and β be distinct two elements in $\mathcal{C}_n$. Then there exists at least one $2 \leq i \leq n$ such that $i\alpha \neq i\beta$, and so the horizontal lines $h_i(\alpha)$ and $h_i(\beta)$ are different. Thus f is injective.

For any $P \in DP_n$ there are n many horizontal lines of length 1 in P, say $h_1, h_2, \ldots, h_n$ from left to right. Let $y_1, y_2, \ldots, y_n$ be the ordinates of the horizontal lines, respectively. Notice that $y_1 \leq y_2 \leq \cdots \leq y_n$ and that $y_1 = 0$, $y_i \leq i-1$ for $2 \leq i \leq n$. Then we consider the transformation $\alpha : X_n \to X_n$ defined by $i\alpha = y_i + 1$ for each $i \in X_n$. Then it is clear that $\alpha \in \mathcal{C}_n$ and $\alpha f = P$, and so f is onto. Therefore, f is a bijection, as required. $\square$

For an example, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 5 \end{pmatrix} \in \mathcal{C}_5,$$

and let f be the function defined in the proof of Proposition 2.1. Then the horizontal lines are

$$\begin{aligned} & h_1(\alpha) = (0,0) - (1,0), \quad h_2(\alpha) = (1,0) - (2,0), \quad h_3(\alpha) = (2,2) - (3,2), \\ & h_4(\alpha) = (3,2) - (4,2), \quad h_5(\alpha) = (4,4) - (5,4). \end{aligned}$$

Moreover, the Dyck path associated with α is

$$\begin{aligned} \alpha f \ \ : \ \ & (0,0) - (1,0) - (2,0) - (2,1) - (2,2) - (3,2) - (4,2) - (4,3) \\ & -(4,4) - (5,4) - (5,5). \end{aligned}$$

3. Zero-divisors of $\mathcal{C}_n$

In this section, we find the left zero-divisors, right zero-divisors and two-sided zero-divisors of $\mathcal{C}_n$ and their numbers.

Lemma 3.1. *For $n \geq 2$, let L be the set of left zero-divisors and R be the set of right zero-divisors of $\mathcal{C}_n$. Then we have*

$$L = \{\alpha \in \mathcal{C}_n \mid n\alpha < n\} \text{ and } R = \{\alpha \in \mathcal{C}_n \mid 2\alpha = 1\}.$$

Proof. Let $n \geq 2$, and let $\alpha \in \mathcal{C}_n$ such that $n\alpha < n$. If we consider the transformation which defined by

$$i\beta = \begin{cases} 1 & i \leq n\alpha \\ 2 & i > n\alpha, \end{cases}$$

then it is clear that $\beta \in \mathcal{C}_n^* = \mathcal{C}_n \setminus \{\theta\}$ and $\alpha\beta = \theta$.

Conversely, let α be a left zero-divisor of $\mathcal{C}_n$. Then there exists $\gamma \in \mathcal{C}_n^*$ such that $\alpha\gamma = \theta$. If we assume that $n\alpha = n$, then we have

$$n\gamma = (n\alpha)\gamma = n\theta = 1,$$

and so $\gamma = \theta$, which is a contradiction. Therefore, the set of all the left-zero divisors of $\mathcal{C}_n$ is L.

Let $\alpha \in \mathcal{C}_n$ such that $2\alpha = 1$. If we consider the transformation which defined by

$$i\lambda = \begin{cases} 1 & i = 1 \\ 2 & i \neq 1, \end{cases}$$

then it is clear that $\lambda \in \mathcal{C}_n^* = \mathcal{C}_n \setminus \{\theta\}$ and $\lambda\alpha = \theta$.

Conversely, let α be a right zero-divisor of $\mathcal{C}_n$. Then there exists $\mu \in \mathcal{C}_n^*$ such that $\mu\alpha = \theta$. If we assume that $2\alpha \neq 1$, then we have $2\alpha = 2$ and $i\alpha \geq 2$ for every $2 \leq i \leq n$. Since the equation $(i\mu)\alpha = i\theta = 1$, we must have $i\mu = 1$ for every $1 \leq i \leq n$, and so $\mu = \theta$, which is a contradiction. Therefore, the set of all the right-zero divisors of $\mathcal{C}_n$ is R. □

Lemma 3.2. *For $n \geq 2$ let L be the set of left zero-divisors and R be the set of right zero-divisors of $\mathcal{C}_n$. Then we have*

$$|L| = |R| = \frac{3}{n+1}\binom{2n-2}{n}.$$

Proof. For $n \geq 2$ let $A = \{\alpha \in \mathcal{C}_n \mid n\alpha = n\}$. If we consider the function $f : A \to \mathcal{C}_{n-1}$ defined by $\alpha f = \alpha_{|X_{n-1}}$ for every $\alpha \in A$, then it is clear that f is a bijection, and so $|A| = |\mathcal{C}_{n-1}|$. Then it follows from Lemma 3.1 that $L = \mathcal{C}_n \setminus A$, and so

$$|L| = \frac{1}{n+1}\binom{2n}{n} - \frac{1}{n}\binom{2n-2}{n-1} = \frac{3}{n+1}\binom{2n-2}{n}.$$

Let $B = \{\alpha \in \mathcal{C}_n \mid 2\alpha = 2\}$. For every $\alpha \in B$ if we consider the transformation $\hat{\alpha} : X_{n-1} \to X_{n-1}$ defined by $i\hat{\alpha} = (i+1)\alpha - 1$, then it is clear that $\hat{\alpha} \in \mathcal{C}_{n-1}$. Moreover, if we consider the function $g : B \to \mathcal{C}_{n-1}$ defined by $\alpha g = \hat{\alpha}$ for every $\alpha \in B$, then it is also clear that g is a bijection, and so $|B| = |\mathcal{C}_{n-1}|$. Similarly, it follows from Lemma 3.1 that $|R| = \frac{3}{n+1}\binom{2n-2}{n}$, as required. □

If T is the set of all two sided zero-divisors of $\mathcal{C}_n$, then it is clear that

$$T = L \cap R = \{\alpha \in \mathcal{C}_n \mid 2\alpha = 1 \text{ and } n\alpha < n\}.$$

Thus, if $n = 2$ then $T = \{\theta\}$, and if $n \geq 3$ then $T \setminus \{\theta\} \neq \emptyset$.

Lemma 3.3. *For $n \geq 3$ and T be the two sided zero-divisors set of $\mathcal{C}_n$. If $n = 3$ then $|T| = 2$ and if $n \geq 4$ then*

$$|T| = \frac{3}{n+1}\binom{2n-2}{n} - \frac{3}{n-3}\binom{2n-4}{n}.$$

Proof. For $n = 3$ it is clear that $|T| = 2$. For $n \geq 4$ let $A = \{\alpha \in \mathcal{C}_n \mid n\alpha = n\}$ and $B = \{\alpha \in \mathcal{C}_n \mid 2\alpha = 2\}$. Then since $T = L \cap R$, it follows from Lemma 3.1 that

$$T = (\mathcal{C}_n \setminus A) \cap (\mathcal{C}_n \setminus B) = \mathcal{C}_n \setminus (A \cup B).$$

Since $A \cap B = \{\alpha \in \mathcal{C}_n \mid 2\alpha = 2 \text{ and } n\alpha = n\}$, we similarly define a bijection from $A \cap B$ to $\mathcal{C}_{n-2}$ so that the cardinality of $A \cap B$ is Catalan number C_{n-2}. Therefore,

$$\begin{aligned} |T| &= |\mathcal{C}_n| - |A| - |B| + |A \cap B| = C_n - 2C_{n-1} + C_{n-2} \\ &= \frac{3}{n+1}\binom{2n-2}{n} - \frac{3}{n-3}\binom{2n-4}{n}, \end{aligned}$$

as required. □

4. Zero-divisor graph of $\mathcal{C}_n$

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an undirected graph, $V(\Gamma)$ denotes the vertex set of Γ and $E(\Gamma)$ denotes the edge set of Γ. If Γ contains no loops or multiple edges then Γ is called a simple graph. In this section we shall assume that Γ is a simple graph. For distinct two vertices $u, v \in V(\Gamma)$ if there exist distinct vertices $v_0, v_1, \ldots, v_n \in V(\Gamma)$ such that $v_0 = u$, $v_n = v$ and $v_{i-1} - v_i$ is an edge in $E(\Gamma)$ for each $1 \leq i \leq n$, then $u - v_1 - \cdots - v_{n-1} - v$ is called a path from u to v of length n in Γ. For every distinct two vertices $u, v \in V(\Gamma)$ if there exits a path from u to v, then Γ is called a connected graph. Let $u, v \in V(\Gamma)$ and let u, v be different vertices, then the length of the shortest path between u and v in Γ is denoted by $d_\Gamma(u, v)$. The eccentricity of a vertex v in a connected simple graph Γ is denoted by $\mathrm{ecc}(v)$ and

$$\mathrm{ecc}(v) = \max\{d_\Gamma(u, v) \mid u \in V(\Gamma)\}.$$

The diameter $\mathrm{diam}(\Gamma)$, the radius $\mathrm{rad}(\Gamma)$ and the central vertex set $C(\Gamma)$ of Γ defined by

$$\begin{aligned}
\mathrm{diam}(\Gamma) &= \max\{\mathrm{ecc}(v) \mid v \in V(\Gamma)\}, \\
\mathrm{rad}(\Gamma) &= \min\{\mathrm{ecc}(v) \mid v \in V(\Gamma)\} \text{ and} \\
C(\Gamma) &= \{v \in V(\Gamma) \mid \mathrm{ecc}(v) = \mathrm{rad}(\Gamma)\},
\end{aligned}$$

respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $\deg_\Gamma(v)$ and it is the number of adjacent vertices to v in Γ. Moreover $\Delta(\Gamma)$ shows that the maximum degree and $\delta(\Gamma)$ shows that minimum degree among all the degrees in Γ.

Let D be a non-empty subset of the vertex set $V(\Gamma)$ of Γ. For each vertices of Γ, if the vertex in D or the vertex is adjacent to D then D is called a dominating set for Γ. The domination number of Γ is

$$\min\{|D| \mid D \text{ is a dominating set of } \Gamma\}$$

and this number is denoted by $\gamma(\Gamma)$. In Γ the length of shortest cycle is called girth of Γ and it is denoted by $\mathrm{gr}(\Gamma)$, moreover if Γ does not contain any cycles, then its girth is defined to be infinity.

Let C be the non-empty subset of $V(\Gamma)$, if u and v are adjacent vertices in Γ for every $u, v \in C$, then C is called a clique. Number of all the vertices in any maximal clique of Γ is called clique number of Γ and it is denoted by $\omega(\Gamma)$. If we colour all the vertices in Γ with the rule of no two adjacent vertices have the same colour, then the minimum number of colours needed to colour of Γ is called chromatic number of Γ, it is denoted by $\chi(\Gamma)$.

Let $V' \subseteq V(\Gamma)$. The (vertex) induced subgraph $\Gamma' = (V', E')$ is a subgraph of Γ and its vertex set is V', moreover its edge set consists of all of the edges in $E(\Gamma)$ that have both endpoints in V'. If $\chi(\Lambda) = \omega(\Lambda)$ for each induced subgraph Λ of Γ, in this case Γ is called a perfect graph. A chordal graph is a simple graph, it does not contain an induced cycle of length 4 or more. Thus in chordal graphs every induced cycle has exactly three vertices.

In this section we prove that $\Gamma(\mathcal{C}_n)$ is a connected graph and then we find the diameter, radius, girth, domination number, clique number and chromatic numbers and the degrees of all vertices of $\Gamma(\mathcal{C}_n)$ for $n \geq 4$. Moreover, we prove that $\Gamma(\mathcal{C}_n)$ is a chordal graph, and so a perfect graph for $n \geq 4$.

For $n \geq 2$ and $\alpha \in \mathcal{C}_n^*$, let k_α be the element of X_{n-1} such that $k_\alpha \alpha = 1$ and $(k_\alpha + 1)\alpha \neq 1$, and moreover, let $t_\alpha = n\alpha$. For $n \geq 3$ and for $\alpha \in T^* = T \setminus \{\theta\}$ observe that $2 \leq k_\alpha, t_\alpha \leq n - 1$.

Lemma 4.1. *Let $n \geq 3$ and $\alpha, \beta \in T^*$. Then $\alpha\beta = \theta$ if and only if $t_\alpha \leq k_\beta$. In particular, $\alpha^2 = \theta$ if and only if $t_\alpha \leq k_\alpha$.*

Proof. ($\Rightarrow$) For $n \geq 3$ and for $\alpha, \beta \in T^*$ let $\alpha\beta = \theta$. Assume that $t_\alpha > k_\beta$. Then $1 = n(\alpha\beta) = (n\alpha)\beta = t_\alpha\beta \geq (k_\beta + 1)\beta \geq 2$, which is a contradiction. Thus $t_\alpha \leq k_\beta$.

($\Leftarrow$) For $n \geq 3$ and for $\alpha, \beta \in T^*$ let $t_\alpha \leq k_\beta$. Since $n(\alpha\beta) = (n\alpha)\beta = t_\alpha\beta = 1$, $\alpha\beta = \theta$, as required. $\square$

As a result, for $n \geq 3$ and for $\alpha, \beta \in T^*$, $\alpha\beta = \theta = \beta\alpha$ if and only if $t_\alpha \leq k_\beta$ and $t_\beta \leq k_\alpha$.

Since $\Gamma(\mathcal{C}_3)$ is a graph which has exactly only one vertex and no edge, from now on we only consider the case $n \geq 4$. Moreover, for convenience, we use Γ instead of $\Gamma(\mathcal{C}_n)$. From Lemma 3.3 we have the following immediate corollary.

Corollary 4.2. *For $n \geq 4$, $|V(\Gamma)| = \frac{3}{n+1}\binom{2n-2}{n} - \frac{3}{n-3}\binom{2n-4}{n} - 1$.*

For $n \geq 4$ we fix the following zero-divisor of $\mathcal{C}_n$:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}. \tag{4.1}$$

Lemma 4.3. *Γ is a connected graph for each $n \geq 4$. In fact $\gamma(\Gamma) = 1$ for $n \geq 4$.*

Proof. For $n \geq 4$ consider π defined in (4.1). For each $\alpha \in T^* \setminus \{\pi\}$ it follows from Lemma 4.1 that $\alpha\pi = \theta = \pi\alpha$ since $t_\alpha \leq n-1$ and $2 \leq k_\alpha$. Thus every element in $T^* \setminus \{\pi\}$ is adjacent to π in Γ, so Γ is connected and $\gamma(\Gamma) = 1$. $\square$

Lemma 4.4. *$\mathrm{diam}(\Gamma) = 2$ and $\mathrm{rad}(\Gamma) = 1$ for $n \geq 4$. Moreover, $C(\Gamma) = \{\pi\}$.*

Proof. For $n \geq 4$ if we consider π defined in (4.1), then it is clear that $\mathrm{diam}(\Gamma) \leq 2$ and $\mathrm{rad}(\Gamma) = 1$. To show that $\mathrm{diam}(\Gamma) = 2$ consider two elements α and β in $T^* \setminus \{\pi\}$ such that $2 \leq k_\alpha \leq n-2$ and $t_\beta = n-1$. Now it follows from Lemma 4.1 that $\beta\alpha \neq \theta$, and so α and β are not adjacent vertices in Γ. Thus $ecc(\alpha) = 2$, and so $\mathrm{diam}(\Gamma) = 2$.

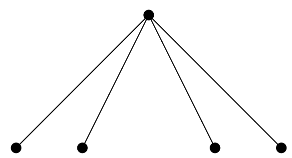
In addition consider two elements α and β in $T^* \setminus \{\pi\}$ such that $3 \leq t_\alpha \leq n-1$ and $k_\beta = 2$. Similarly, from Lemma 4.1, $\alpha\beta \neq \theta$, and so α and β are not adjacent vertices in Γ. Thus $\mathrm{ecc}(\alpha) = 2$, and so

$$C(\Gamma) = \{\alpha \in T^* \mid k_\alpha = n-1 \text{ and } t_\alpha = 2\} = \{\pi\},$$

as required. $\square$

Theorem 4.5. $gr(\Gamma) = \begin{cases} \infty & \text{if } n = 4 \\ 3 & \text{if } n \geq 5. \end{cases}$

Proof. Since $\Gamma(\mathcal{C}_4)$ is isomorphic to the following graph



$\mathrm{gr}(\Gamma(\mathcal{C}_4)) = \infty$. For $n \geq 5$ if we consider α and β in $T^* \setminus \{\pi\}$ such that $k_\alpha = 3$, $t_\alpha = 2$, $k_\beta = 2$ and $t_\beta = 3$. Notice that $\alpha \neq \pi$ since $n \geq 5$. Then it follows from Lemma 4.1 that α and β are adjacent vertices in Γ. Thus $\pi - \alpha - \beta - \pi$ is a cycle of length 3 in Γ. Therefore, $gr(\Gamma) = 3$ since Γ is simple. $\square$

Theorem 4.6. *For $n \geq 4$ and for $\alpha \in T^* = V(\Gamma)$ let $k = k_\alpha$ and $t = t_\alpha$. Then*

$$deg_\Gamma(\alpha) = \begin{cases} \binom{n-t+k-1}{k-1} - 1 & \text{if } k < t \\ \binom{n-t+k-1}{k-1} - 2 & \text{if } k = t \text{ or } k = t+1 \\ \binom{n-t+k-1}{k-1} - \binom{n-t+k-1}{k-t-2} - 2 & \text{if } k > t+1. \end{cases}$$

Proof. For $n \geq 4$ and for $\alpha \in T^*$ let $k = k_\alpha$ and $t = t_\alpha$. For any $\beta \in T^*$ suppose that $\alpha\beta = \theta = \beta\alpha$. It follows from Lemma 4.1 that $t_\beta \leq k$ and $t \leq k_\beta$. Let

$$A_\alpha = \{\beta \in T^* \mid t_\beta \leq k \text{ and } t \leq k_\beta\}.$$

Thus if $\alpha^2 \neq \theta$, then the degree of α is $|A_\alpha|$, and if $\alpha^2 = \theta$, then the degree of α is $|A_\alpha| - 1$.

Consider the elements of DP_n, the set of all the Dyck paths from $(0,0)$ to (n,n), which have the following form

$$(0,0) - (1,0) - \cdots - (t,0) - \cdots - (n,k-1) - (n,k) - \cdots - (n,n),$$

and consider the function $f : \mathcal{C}_n \to DP_n$ defined in the proof of Proposition 2.1. Denote the set of all elements in DP_n of the above form by $DP_{n,\alpha}$. For any $P \in DP_{n,\alpha}$ it is clear that $Pf^{-1} \in T$ since $2 \leq t \leq n-1$ and $2 \leq k \leq n-1$. If $Q \in DP_{n,\alpha}$ is the path

$$(0,0) - (1,0) - \cdots - (t,0) - \cdots - (n,0) - (n,1) - \cdots - (n,k) - (n,k+1) - \cdots - (n,n)$$

then $Qf^{-1} = \theta$, and so we have $|A_\alpha| = |DP_{n,\alpha}| - 1$.

Suppose that $k \leq t+1$. Then the NE lattice paths from $(t,0)$ to $(n,k-1)$ do not cross the diagonal, and so $|DP_{n,\alpha}| = \binom{n-t+k-1}{k-1}$. If $k < t$ then $\alpha^2 \neq \theta$, and so $deg_\Gamma(\alpha) = \binom{n-t+k-1}{k-1} - 1$. If $k = t$ or $k = t+1$ then $\alpha^2 = \theta$, and so $deg_\Gamma(\alpha) = \binom{n-t+k-1}{k-1} - 2$.

Suppose that $k > t+1$. Then $\alpha^2 = \theta$ and some of the NE lattice paths from $(t,0)$ to $(n,k-1)$ do cross the diagonal. Let us find the number of NE lattice paths which crossing the diagonal. If we use the reflection principle, then $(n,k-1)$ reflects to $(k-2,n+1)$ according to the line $y = x+1$. Thus the number of those paths are equal to the number of all NE lattice paths from $(t,0)$ to $(k-2,n+1)$ is $\binom{n-t+k-1}{k-t-2}$. Therefore, $deg_\Gamma(\alpha) = \binom{n-t+k-1}{k-1} - \binom{n-t+k-1}{k-t-2} - 2$, as required. □

For $n \geq 4$ if we consider π defined in (4.1) and simplicity of Γ, then it is clear that $\Delta(\Gamma) = |T^*| - 1$. Moreover, if we consider α in $V(\Gamma)$ such that $t_\alpha = n-1$ and $k_\alpha = 2$, then it follows from Theorem 4.6 that $deg_\Gamma(\alpha) = 1$. Thus we have the following immediate corollary.

Corollary 4.7. $\Delta(\Gamma) = \frac{3}{n+1}\binom{2n-2}{n} - \frac{3}{n-3}\binom{2n-4}{n} - 2$ *and* $\delta(\Gamma) = 1$ *for* $n \geq 4$.

Theorem 4.8. *For $n \geq 4$ Γ is a chordal, and so a perfect graph.*

Proof. For $n \geq 4$ assume that there exists an induced subgraph of Γ which is an m-cycle with $m \geq 4$. Let $v_1 - v_2 - \cdots - v_m - v_1$ be an m-cycle in Γ with $m \geq 4$. Let $k_i = k_{v_i}$ and $t_i = t_{v_i}$ for each $1 \leq i \leq m$. Moreover, let $k = \min\{k_i \mid 1 \leq i \leq m\}$. Without losing generality assume that $k = k_1$. Then $t_2 \leq k$ and $t_m \leq k$ since v_1 is adjacent to both v_2 and v_m. Since $t_2 \leq k \leq k_m$ and $t_m \leq k \leq k_2$, it follows that v_2 and v_m are adjacent vertices, which is a contradiction.

It is well-known that every chordal graph is a perfect graph (see, for example [7,16]). □

Lemma 4.9. *For $n \geq 2$ let $A = \{\alpha \in \mathcal{C}_n^* \mid k_\alpha = k$ and $t_\alpha = t\}$. Then*

$$|A| = \begin{cases} 1 & \text{if } t = 2 \\ \binom{n-k+t-3}{t-2} & \text{if } 2 < t \leq k+1 \\ \binom{n-k+t-3}{t-2} - \binom{n-k+t-3}{t-k-2} & \text{if } t > k+1. \end{cases}$$

Proof. For $n \geq 2$ let $A = \{\alpha \in \mathcal{C}_n^* \mid k_\alpha = k \text{ and } t_\alpha = t\}$. Consider the elements of DP_n, the set of Dyck paths from $(0,0)$ to (n,n), which have the following form

$$(0,0) - \cdots - (k,0) - (k,1) - \cdots - (n-1,t-1) - (n,t-1) - (n,t) - \cdots - (n,n).$$

Denote the set of all elements in DP_n of the above form by $DP_{n,k,t}$. Thus it follows from Proposition 2.1 and its proof that $|A| = |DP_{n,k,t}|$.

For $t = 2$ there is only one Dyck path, namely

$$(0,0) - \cdots - (k,0) - (k,1) - \cdots - (n-1,1) - (n,1) - \cdots - (n,n),$$

and so $|A| = 1$.

Suppose that $2 < t \le k+1$. Then it is clear that all the NE lattice paths from $(k,1)$ to $(n-1,t-1)$ do not cross the diagonal, and so $|A| = \binom{n-k+t-3}{t-2}$.

Suppose that $t > k+1$. Then some of the NE lattice paths from $(k,1)$ to $(n-1,t-1)$ cross the diagonal. Let us find the number of NE lattice paths which cross the diagonal. If we use the reflection principle, then $(n-1,t-1)$ reflects to $(t-2,n)$ according to the line $y = x+1$. Thus the number of those paths are equal to the number of all NE lattice paths from $(k,1)$ to $(t-2,n)$ which is $\binom{n-k+t-3}{t-k-2}$. Therefore, $|A| = \binom{n-k+t-3}{t-2} - \binom{n-k+t-3}{t-k-2}$. □

Notice that in Lemma 4.9 for $n \ge 3$ if $2 \le k \le n-1$ and $2 \le t \le n-1$, then $A \subseteq T^*$. Let $A(n,k,t) = \{\alpha \in \mathcal{C}_n^* \mid k_\alpha = k \text{ and } t_\alpha = t\}$. Then we have a partition of T^*, namely

$$T^* = \bigcup_{k=2}^{n-1} \Big(\bigcup_{t=2}^{n-1} A(n,k,t) \Big)$$

for $n \ge 3$. Thus we have the following immediate corollary.

Corollary 4.10. *For $n \ge 5$*

$$n-2+\sum_{r=0}^{n-4}\Big(\sum_{k=2+r}^{n-1}\binom{n-k+r}{r+1}\Big)+\sum_{s=1}^{n-4}\Big(\sum_{k=2}^{s+1}\Big(\binom{n-k+s}{s+1}-\binom{n-k+s}{s+1-k}\Big)\Big)$$
$$=\frac{3}{n+1}\binom{2n-2}{n}-\frac{3}{n-3}\binom{2n-4}{n}-1.$$

Lemma 4.11. *Let K be a complete subgraph of Γ and $V(K) = \{v_1, v_2, \ldots, v_m\}$. Let $k_i = k_{v_i}$ and $t_i = t_{v_i}$ for $1 \le i \le m$. Then there is at most one $1 \le i \le m$ such that $t_i > k_i$.*

Proof. Assume that there are two distinct i and j such that $t_i > k_i$ and $t_j > k_j$. Without loss of generality suppose that $k_i \le k_j$. Since v_i and v_j are adjacent vertices in K, we have $t_j \le k_i$. Thus $t_j \le k_i \le k_j$, which is a contradiction. □

Theorem 4.12. *For $n \ge 4$*

$$\chi(\Gamma) = \omega(\Gamma) = \max\Big\{ \sum_{k=i}^{n-1}\sum_{t=2}^{i} |A(n,k,t)| \;\Big|\; 2 \le i \le n-1 \Big\}.$$

Proof. Let $n \ge 4$ and K be a maximal complete subgraph of Γ with the vertices set $V(K) = \{v_1, \ldots, v_m\}$, and let $k_i = k_{v_i}$, $t_i = t_{v_i}$ for $1 \le i \le m$. From Lemma 4.11, since K is complete, without loss of generality either $t_1 > k_1$ and $t_i \le k_i$ for all $2 \le i \le m$ or $t_i \le k_i$ for all $1 \le i \le m$. Notice that it is possible $t_x = t_y$ or $k_x = k_y$ for some $1 \le x \ne y \le m$.

Suppose that $t_1 > k_1$ and $t_i \le k_i$ for all $2 \le i \le m$. Let C be the set of all the different numbers among $t_2, t_3, \ldots, t_m$ and let D be the set of all the different numbers among $k_2, k_3, \ldots, k_m$. From Lemma 4.1, since $t_i \le k_1$ and $t_1 \le k_i$ for every $2 \le i \le m$, $\min(D) - \max(C) \ge 1$. Let $\min(D) = p$ (notice that $p \ge 2$), and let Ω be the induced subgraph of Γ with vertex set $V(\Omega) = (V(K) \setminus \{v_1\}) \cup A(n,p,p)$. Then it is clear that Ω is a complete subgraph, and that $|V(\Omega)| \ge |V(K)|$. Moreover, it is also clear that $t_v \le k_v$ for all v in Ω. Therefore, there exists at least one maximal complete subgraphs K of Γ such that $t_v \le k_v$ for all v in K.

Let K be a maximal complete subgraph of Γ with the vertices set $V(K) = \{v_1, \ldots, v_m\}$, and let $k_i = k_{v_i}$, $t_i = t_{v_i}$ and $t_i \le k_i$ for $1 \le i \le m$. For any $1 \le i \le m$ since K is a complete subgraph of Γ, $t_i \le k_j$ for all $j \in \{1, \ldots, m\} \setminus \{i\}$, and so $t_i \le k_j$ for all $1 \le j \le m$. Therefore, $\min(D) \ge \max(C)$ where C is the set of all the different numbers among $t_1, t_2, \dots, t_m$ and D is the set of all the different numbers among $k_1, k_2, \dots, k_m$. Now assume that $\min(D) \neq \max(C)$. Let $\max(C) = q$, and let Ω be the induced subgraph of Γ with vertex set $V(\Omega) = V(K) \cup A(n, q, q)$. Then it is clear that Ω is a complete subgraph, and that $|V(\Omega)| > |V(K)|$ which is a contradiction. Thus $\min(D) = \max(C) = r$. From the maximality of the complete subgraph of Γ we deduce that $C = \{2, 3, \dots, r\}$ and $D = \{r, r+1, \dots, n-1\}$. Thus,

$$V(K) = \bigcup\{A(n,k,t) \mid k \in D \text{ and } t \in C\},$$

and so $|V(K)| = \sum\limits_{k=r}^{n-1} \sum\limits_{t=2}^{r} |A(n,k,t)|$. Therefore,

$$\omega(\Gamma) = \max\Big\{ \sum_{k=i}^{n-1} \sum_{t=2}^{i} |A(n,k,t)| \ \mid \ 2 \leq i \leq n-1 \Big\}.$$

Since Γ is perfect graph, $\chi(\Gamma) = \omega(\Gamma)$, as required. $\square$

Example 4.13. If $\Gamma = \Gamma(\mathcal{C}_6)$ then $|V(\Gamma)| = 61$, $diam(\Gamma) = 2$, $rad(\Gamma) = 1$, $gr(\Gamma) = 3$, $\gamma(\Gamma) = 1$, $\Delta(\Gamma) = 60$, $\delta(\Gamma) = 1$, $\chi(\Gamma) = \omega(\Gamma) = 9$. Moreover, $deg_\Gamma(\alpha) = 8$ where $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 3 & 4 \end{pmatrix} \in V(\Gamma)$.

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References

[1] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (2), 434-447, 1999.

[2] G. Ayık, H. Ayık and M. Koç, *Combinatorial results for order-preserving and order-decreasing transformations*, Turkish J. Math. **35** (4), 617-625, 2011.

[3] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1), 208-226, 1988.

[4] K.C. Das, N. Akgüneş and A.S. Çevik, *On a graph of monogenic semigroup*, J. Inequal. Appl. **44**, 2013.

[5] F. DeMeyer and L. DeMeyer, *Zero divisor graphs of semigroups*, J. Algebra **283** (1), 190-198, 2005.

[6] F. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum **65** (2), 206-214, 2002.

[7] G.A. Dirac, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg **25** (1-2), 71-76, 1961.

[8] W. Feller, *An Introduction to Probability Theory and Its Applications Volume I*, New York, John Wiley & Sons Inc, 1957.

[9] O. Ganyushkin and V. Mazorchuk, *Classical Finite Transformation Semigroups: An Introduction*, Springer, 2009.

[10] P.M. Higgins, *Combinatorial results for semigroups of order-preserving mappings*, Math. Proc. Camb. Phil. Soc. **113** (2), 281-296, 1993.

[11] J.M. Howie, *Fundamentals of Semigroup Theory*, New York, NY, USA: Oxford University Press, 1995.

[12] A. Laradji and A. Umar, *On certain finite semigroups of order-decreasing transformations I*, Semigroup Forum, **69** (2), 184-200, 2004.

[13] S.G. Mohanty, *Lattice Path Counting and Applications*, Academic Press, New York, 1979.

[14] S.P. Redmond, *The zero-divisor graph of a non-commutative ring*, Int. J. Commut. Rings **1** (4), 203-211, 2002.

[15] R. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge University Press, 2001.

[16] K. Thulasiraman, S. Arumugam, A. Brandstädt, A. and T. Nishizeki, *Handbook of Graph Theory, Combinatorial Optimization, and Algorithms*, Chapman & Hall/CRC Computer and Information Science Series. CRC Press, Boca Raton, 2015.

[17] K. Toker, *On the zero-divisor graphs of finite free semilattices*, Turkish J. Math **40** (4), 824-831, 2016.