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SEPARATION OF DIFFERENT FORMS OF THE DIRAC OPERATOR IN HILBERT SPACES

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ABSTRACT

In this paper we study the separation property of two different forms of the Dirac differential operator

(i) $Lu(x) = i^{-1}\alpha.grad\ u(x) + \beta(x)u(x), x \in R^3$ in the Hilbert space $H_1 = L_2(R^3)^4$ and

(ii) $Gu(x) = i^{-1}B\frac{d}{dx}u(x) + v(x)u(x), x \in R$ in the Hilbert space $H_2 = L_2(R\ )^\ell$.

1 - INTRODUCTION

The term "separation" and many results of the separation of differential expressions are due to Everitt and Giertz [5-8]. They obtained the separation property of the Sturm-Liouville differantial operator $P(y) = -y''(x) + q(x)y(x)$ in the space $L_2(R)$ where $q(x)$ is a real valued function, they studied the following question:

If $y(x) \in L_2(R)$ and $-y''(x) + q(x)y(x) \in L_2(R)$ then, what are the conditions imposed on q(x) must satisfy that imply $y''(x) \in L_2(R)$ and $q(x)y(x) \in L_2(R)$?

The problems of separativity have been attracted by many mathematicians such as Biomatov [2-4], Zettel [15] and Mohamed [10 - 14]. Separation for differential expressions with matrix coefficient was first examined by Bergbaev [1]. He has obtained the conditions on q (x) in order that the Schrodinger operator

$$S[u] = -\Delta u_\ell(x) + q(x)u(x), x \in R^n$$

be separated in the space $L_p(\Omega)^\ell, \Omega \subseteq R$ where Δ is the Laplace operator in R^n and q (x) is an m × m positive Hermitian matrix .

The Hilbert space $H = L_2(R^n)^\ell$ denotes the space of all vector functions

$u(x) = (u_1(x), u_2(x), \ldots\ldots, u_\ell(x))$, $x \in R^n$ that equipped with the norm

$$\|u\| = \left(\sum_{j=1}^{\ell} \int_{R^n} |u_j(x)|^2 dx \right)^{\frac{1}{2}}, \; x \in R^n$$

and the symbol $<u, v>$ where $<u,v> = \sum_{j=1}^{\ell} \int_{R^n} u_j(x)\overline{v_j(x)}\,dx$, $u, v \in H$

denotes the inner product in H.

The space $W_2^1(R^n)^\ell$ is the space of all vector functions $u(x) = (u_1(x), \ldots, u_\ell(x))$, $x \in R^n$ which has generalized derivative $Du(x)$ such that $u(x)$ and its derivative $Du(x)$ belong to $L_2(R^n)^\ell$, we say that the vector function $u(x) = (u_1(x), u_2(x), \ldots\ldots, u_\ell(x))$, $x \in R^n$ belong to $W_{2,loc}^1(R^n)^\ell$ if for all function $\phi(x) \in C_0^\infty(R^n)$ the vector function

$$\phi(x)u(x) \in W_2^1(R^n)^\ell.$$

The objective of this paper is to study two problems:

In the first problem we study the conditions must be imposed on the potential $\beta(x)$ in order that Dirac operator

$$Lu(x) = i^{-1}\alpha.grad\; u(x) + \beta(x)u(x), x \in R^3 \qquad (1)$$

be separated in the Hilbert space $H_1 = L_2(R^3)^4$ (H_1 is the space H with n=3 and $\ell = 4$).

In the development of the theoretical physics, the Dirac operator finds its natural place in the attempt to obtain a relativistic wave equation for the electron. The operator of the form (1) discribs moving of a particle in the space R^3 this operator acts on a vector valued (or spinor-valued) function $u(x) = (u_1(x), \ldots\ldots\ldots, u_4(x))$ with 4-components of the space variable $x = (x_1, x_2, x_3)$.We denote by C^4 the 4-dimentional complex vector space in which the values of $u(x)$ lie. α is a 3-components vector $\alpha = (\alpha_1 \; \alpha_2; \alpha_3)$ with components α_j which are operator in C^4 and may be identified with their representation by 4×4 matrices.Similarly $\beta(x)$ is a 4 x 4 positive Hermitian matrix.Thus $Lu = V = (v_1, v_2, v_3, v_4)$ has components

$$V_K(x) = i^{-1} \sum_{j=1}^{3} \sum_{h=1}^{4} (\alpha_j)_{kh} \frac{\partial u_h(x)}{\partial x_j} + \sum_{h=1}^{4} \beta_{kh}(x) u_h(x),$$

i.e $V(x) = i^{-1}\sum_{j=1}^{3}\alpha_j \frac{\partial u(x)}{\partial x_j} + \beta(x)u(x)$.

The matrices α_j and $\beta(x)$ are Hermitian symmetric and satisfy the relations :

$\alpha_j\beta(x) + \beta(x)\alpha_j = 0$ and $\alpha_j^2 = I$ (the identity matrix) $\forall j = 1,2,3$.

Since L is a formal differential operator, we can construct from L various operators in the basic Hilbert space $H_1 = L_2(R^3)^4$ consisting of all C^4 – valued functions such that

$\|u\|^2 = \sum_{j=1}^{4} \int_{R^3} |u_j(x)|^2 dx$, $x \in R^3$, and the associated inner product is

$$< u, v >= \sum_{j=1}^{4} \int_{R^3} u_j(x)\overline{v_j(x)}\, dx .$$

See Kato, T. [9] .

In the second problem we study the separation property of the Dirac differential operator

$$Gu\,(x) = i^{-1} B \frac{d}{dx} u(x) + v(x)u(x) \qquad , x \in R \qquad (2)$$

in the Hilbert space $H_2 = L_2(R\ )^{\ell}$ (H_2 is the space H with n =1), where $u(x)$ = ($u_1(x)$, $u_2(x)$,........ , $u_{\ell}(x)$) and B is an $\ell \times \ell$ Hermatian matrix, whose elements are independent on x, with $B^2 = I$ (the identity matrix) . The potential $v(x) \in$ L(H_2) is a bounded linear operator on H_2.

2- Statement of the results

In the following, we study the separation property of the Dirac differential operator in the form (1) in the Hilbert space $H_1 = L_2(R^3)^4$.

Def. 2-1.[5] The differential operator L of the form (1) is said to be separated in H_1 if the following statement holds:

If $u(x) \in H_1 \cap W^1_{2,loc}(R^3)^4$ *and* $L\, u(x) \in H_1$ implies that $\alpha.grad\, u(x)$ *and* $\beta(x)u(x) \in H_1$

i.e $\sum_{j=1}^{3}\alpha_j \frac{\partial u}{\partial x_j}$ and $\beta(x)u(x) \in H_1$

which is equivalent to the following coercive estimate

$$\|L_o u\| + \|\beta u\| \leq M_1(\|Lu\| + \|u\|), \quad (3)$$

where $Lu = L_0 u + \beta u$, $L_0 u = i^{-1}\sum_{j=1}^{3}\alpha_j \frac{\partial u}{\partial x_j}$ and M_l is a constant independent on u.

Theorem 2-2 The Dirac differential operator of the form (1) is separated in the Hilbert space H_1 if the following condition is satisfied

$$\sum_{j=1}^{3}\left\|\frac{\partial \beta}{\partial x_j}\right\|\|u\|^2 \leq \delta\|\beta u\|^2 \quad \text{, where } \delta \in]0,2[\quad (4)$$

Proof. Since $<Lu, \beta u> = < L_0 u + \beta u, \beta u>$

$$= < L_0 u, \beta u> + <\beta u, \beta u>, \quad (5)$$

$$<L_0 u, \beta u> = < i^{-1}\sum_{j=1}^{3}\alpha_j \frac{\partial u}{\partial x_j}, \beta u> = \sum_{j=1}^{3}< i^{-1}\alpha_j \frac{\partial u}{\partial x_j}, \beta u>.$$

From the definition of the scalar product in H and by integrating by parts, we obtain $<\frac{du}{dx}, v> = -<u, \frac{dv}{dx}>$ for all $u, v \in C_0^\infty(R^3)^4$.

Then $$< L_o u, \beta u> = -\sum_{j=1}^{3}< i^{-1}\alpha_j u, \frac{\partial \beta}{\partial x_j}u> - \sum_{j=1}^{3}< i^{-1}\alpha_j u, \beta\frac{\partial u}{\partial x_j}>. \quad (6)$$

In the following, we write $\beta'_{x_j} = \frac{\partial \beta}{\partial x_j}$, $u'_{x_j} = \frac{\partial u}{\partial x_j}$, j =1,2,3

Since α_j is a Hermitian matrix, we have

$$< i^{-1}\alpha_j u, \beta u'_{x_j}> = \overline{<\beta u'_{x_j}, i^{-1}\alpha_j u>} = \overline{<\alpha_j \beta u'_{x_j}, i^{-1}u>}.$$

Since $\alpha_j\beta(x) + \beta(x)\alpha_j = 0$ *and* β is a Hermitian matrix, we get

$$< i^{-1}\alpha_j u, \beta u'_{x_j}> = -\overline{<\beta\alpha_j u'_{x_j}, i^{-1}u>} = -\overline{<\alpha_j u'_{x_j}, i^{-1}\beta u>}.$$

Then $$\sum_{j=1}^{3}< i^{-1}\alpha_j u, \beta u'_{x_j}> = \sum_{j=1}^{3}\overline{< i^{-1}\alpha_j u'_{x_j}, \beta u>} = \overline{< L_o u, \beta u>}. \quad (7)$$

From (7) the equation (6) takes the form

$$< L_o u, \beta u >= -\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u > - \overline{< L_o u, \beta u >}.$$

Since $2Re\ Z = Z + \overline{Z}$, thus $2\operatorname{Re} < L_o u, \beta u >= -\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u >$. (8)

Equating the real parts of both sides of the equation (5) and using (8), we get

$$\operatorname{Re} < Lu, \beta u >=< \beta u, \beta u > -\frac{1}{2}\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u >,$$

which can be written in the form

$$\|\beta u\|^2 = \operatorname{Re} < Lu, \beta u > + \frac{1}{2}\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u >. \tag{9}$$

By using the Cauchy-Schwartz inequality, we get

$$\operatorname{Re} < Lu, \beta u > \leq |< Lu , \beta u >| \leq \|Lu\|.\|\beta u\|, \tag{10}$$

$$\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u > \leq \left|\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u >\right| \leq \sum_{j=1}^{3} \left|< i^{-1}\alpha_j u, \beta'_{x_j} u >\right| \leq \sum_{j=1}^{3} \|\alpha_j u\| \|\beta'_{x_j} u\|$$

.

Since α_j is a Hermitian matrix and $\alpha_j^2 = I$, hence

$$\|\alpha_j u\|^2 =< \alpha_j u, \alpha_j u >=< u, u >= \|u\|^2.$$

Thus $\sum_{j=1}^{3} < i^{-1}\alpha_j u, \beta'_{x_j} u > \leq \sum_{j=1}^{3} \|u\|^2.\|\beta'_{x_j}\|$.

From the condition (4), we get

$$\sum_{j=1}^{3} < i^{-1}\alpha_j u \beta'_{x_j}, u > \leq \delta \|\beta u\|^2 . \tag{11}$$

Substituting from (10) and (11) into (9), we get

$$\|\beta u\|^2 \leq \|Lu\| \|\beta u\| + \frac{\delta}{2}\|\beta u\|^2,$$

then $\|\beta u\| \leq \frac{1}{1-\delta/2}\|Lu\|$. (12)

Since $Lu = L_0 u + \beta u$.

Thus $\|L_0 u\| \leq \|Lu\| + \|\beta u\|$. (13)

From (12) the inequality (13) becomes

$$\|L_0 u\| \leq \|Lu\| + \frac{1}{1-\delta/2}\|Lu\|,$$

thus $\|L_0 u\| \leq \dfrac{2-\delta/2}{1-\delta/2}\|Lu\|$. (14)

By adding (12) and (14), we get

$$\|L_0 u\| + \|\beta u\| \leq \frac{3-\delta/2}{1-\delta/2}\|Lu\| \leq M_1\|Lu\|,$$

where $M_1 = \dfrac{3-\delta/2}{1-\delta/2}$, and $\delta < 2$ is a natural number .

Finally, we obtain $\|L_0 u\| + \|\beta u\| \leq M_1\|Lu\|$,

where M_1 is a constant independent on u .

Now the coercive estimate (3) is valid and so the Dirac operator of the form (1) is separated in the Hilbert space $H_1 = L_2(R^3)^4$ under the condition (4) .

Theorem 2-3 Suppose that the conditions of theorem 2-2 are valid, moreover if the condition

$$\alpha_j\alpha_h + \alpha_h\alpha_j = 2\delta_{jh} I \ , \qquad \text{where } \delta_{jh} = \{ {}^{1}_{0} \quad {}^{j=h}_{j\neq h} . \tag{15}$$

is satisfied for $j, h = 1,2,3$ then

$\|u'\| \leq M_2\|Lu\|$, M_2 is a constant independent on u.

Proof. Since $L_0 u = i^{-1}\sum_{j=1}^{3}\alpha_j \dfrac{\partial u}{\partial x_j}$.

Thus $< L_0 u, L_0 u > = < \sum_{j=1}^{3}\alpha_j \dfrac{\partial u}{\partial x_j}, \sum_{h=1}^{3}\alpha_h \dfrac{\partial u}{\partial x_h} > = \sum_{j,h=1}^{3} < \alpha_j \dfrac{\partial u}{\partial x_j}, \alpha_h \dfrac{\partial u}{\partial x_h} >$

$$= \sum_{j,h=1}^{3} < \frac{\partial u}{\partial x_j}, \alpha_j\alpha_h \frac{\partial u}{\partial x_h} > .$$

By using the condition (15), we have

$$< L_0 u, L_0 u >= \sum_{j,h=1}^{3} < \frac{\partial u}{\partial x_j}, 2\delta_{jh} I - \alpha_h \alpha_j \frac{\partial u}{\partial x_h} >$$

$$= \sum_{j,h=1}^{3} < \frac{\partial u}{\partial x_j}, 2\delta_{jh} \frac{\partial u}{\partial x_h} > - \sum_{j,h=1}^{3} < \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} >$$

$$= \sum_{j=h=1}^{3} < \frac{\partial u}{\partial x_j}, 2\frac{\partial u}{\partial x_h} > - \sum_{j=h=1}^{3} < \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_h} > - \sum_{j\neq h=1}^{3} < \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} >,$$

thus $< L_0 u, L_0 u >= \sum_{j=h=1}^{3} < \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_h} > - \sum_{j\neq h=1}^{3} < \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} >$.

Equating the real parts in the both sides, we get

$$< L_0 u, L_0 u >= \sum_{j=h=1}^{3} < \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_h} > - \mathrm{Re} \sum_{j\neq h=1}^{3} < \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} > \qquad (16)$$

Suppose that

$$Z = < \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} > = - < u, \alpha_h \alpha_j \frac{\partial^2 u}{\partial x_j \partial x_h} > = - \overline{< \alpha_h \alpha_j \frac{\partial^2 u}{\partial x_j \partial x_h}, u >}.$$

Since α_j and α_h are Hermitian matrices, thus

$$Z = - \overline{< \frac{\partial^2 u}{\partial x_j \partial x_h}, \alpha_j \alpha_h u >} = \overline{< \frac{\partial^2 u}{\partial x_j \partial x_h}, \alpha_h \alpha_j u >}$$

$$= - \overline{< \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} >} = -\overline{Z}.$$

Thus $Z = -\overline{Z}$ and so $\mathrm{Re}\, Z = 0$,

$$\text{i.e } \mathrm{Re} \sum_{j\neq h=1}^{3} < \frac{\partial u}{\partial x_j}, \alpha_h \alpha_j \frac{\partial u}{\partial x_h} > = 0 \qquad (17)$$

From (17) the equation (16) becomes

$$< L_0 u, L_0 u >= \sum_{j=h=1}^{3} < \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_h} > = \sum_{j=1}^{3} < \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j} >.$$

Therefor $\|L_0 u\| = \|u'\|$.

From the inequality (14), we get

$\|u'\| \leq M_2\|Lu\|$, M_2 is a constant independent on u .

In the following, we study the separation of the Dirac differential operator of the form (2) with the potential $v(x) \in L(H_2)$ is a bounded linear operator on H_2 in the Hilbert space $H_2 = L_2(R)^\ell$.

Def. 2-4 The differential operator G of the form (2) is said to be separated in the Hilbert space $H_2 = L_2(R)^\ell$ if the following coercive estimate is valid:

$$\|Bu'\| + \|vu\| \leq M_3(\|Gu\| + \|u\|) \quad , \qquad (18)$$

where M_3 is a constant independent on u .

Theorem 2-5 The differential operator G of the form (2) is separated in the Hilbert space $H_2 = L_2(R)^\ell$ if the following condition

$$\|v'u\| + \|vu'\| \leq \gamma \|vu\| \quad , \gamma > 0 \qquad (19)$$

is satisfied for all $x \in R$.

Proof. From the equation (2), we get

$$< Gu, vu >=< i^{-1}Bu' + vu, vu >=< i^{-1}Bu', vu > + < vu, vu > .$$

Since $< \frac{du}{dx}, v >= - < u, \frac{dv}{dx} >$ for all $u, v \in C_0^\infty(R)^\ell$.

Thus $< Gu, vu >=< vu, vu > - < i^{-1}Bu, v'u > - < i^{-1}Bu, vu' >$.

Equating the real parts in the both sides, we get

$$\|vu\|^2 = \mathrm{Re} < Gu, vu > + \mathrm{Re} < i^{-1}Bu, v'u > + \mathrm{Re} < i^{-1}Bu, vu' > . \qquad (20)$$

By applying the Caushy-Schwartz inequality on the all terms of the R.H.S of the equation (20), we get

$$\mathrm{Re} < Gu, vu > \leq |< Gu, vu >| \leq \|Gu\|.\|vu\| \quad , \qquad (21)$$

$$\mathrm{Re} < i^{-1}Bu, v'u > \leq |< i^{-1}Bu, v'u >| \leq \|Bu\|.\|v'u\| \quad , \qquad (22)$$

$$\mathrm{Re} < i^{-1}Bu, vu' > \leq |< i^{-1}Bu, vu' >| \leq \|Bu\|.\|vu'\| . \qquad (23)$$

Since B is a hermitian matrix and B^2 is the identity matrix , hence

$$\|Bu\|^2 =< Bu, Bu >=< u, u >= \|u\|^2 .$$

Thus, the equations(22) and (23) take the forms

$$\mathrm{Re} < i^{-1}Bu, v'u > \leq \|u\|.\|v'u\| , \qquad (24)$$

$$\mathrm{Re} < i^{-1}Bu, vu' > \leq \|u\| . \|vu'\| . \qquad (25)$$

Substituting from (21), (24) and (25) into the equation (20), we get

$$\|vu\|^2 \leq \|Gu\| \|vu\| + \|u\| (\|v'u\| + \|vu'\|) .$$

By using the condition (19) , we obtain

$$\|vu\| \leq \|Gu\| + \gamma \|u\| \qquad (26)$$

Since $\quad Gu = i^{-1}Bu' + vu$.

Thus $\quad \|Bu'\| \leq \|Gu\| + \|vu\|$.

From the equation (26) the last equation becomes

$$\|Bu'\| \leq 2\|Gu\| + \gamma \|u\| . \qquad (27)$$

From the inequalities (26) and (27), we get

$$\|Bu'\| + \|vu\| \leq 3\|Gu\| + 2\gamma \|u\| .$$

Let $M_3 = \max\{2\gamma, 3\}$, we obtain

$$\|Bu'\| + \|vu\| \leq M_3 (\|Gu\| + \|u\|) .$$

Then, the coercive estimate (18) is valid and so the Dirac differential operator of the form (2) is separated in the Hilbert space $H_2 = L_2(R)^\ell$ under the condition (19) .

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