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Stability Estimates of First Order Linear and Nonlinear Neutral Delay Differential Equations

Agamali AGAMALIYEV and Ali Fuat YENİÇERİOĞLU

Abstract

The problem of stabilizing a linear and nonlinear differential equation with neutral delay by means of linear feedback without neutral delay is considered. A sufficient condition on the open loop system is presented for the possibility of actualizing the arbitrarily prescribed degree of stability in the closed loop system.

Keywords: Neutral, delay, feedback control, stability, w-periodic.
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1. INTRODUCTION

Consider the nonlinear neutral delay differential equation

$$x'(t) = -a_0(t)x(t) + a_1(t)x(t-w) + a_2(t)x'(t-w) + h(t, x(t-w), x'(t-w))$$
$$-b_0(t)u(t) + b_1(t)u(t-w) + b_2(t)u'(t-w) \quad , \qquad t \geq 0 \qquad (1)$$

$$x(t) = g(t) \quad , \quad -w \leq t \leq 0 \quad (w > 0)$$

where $t \geq 0$ represents time, $x(t)$, $u(t) \in IR$, $a_j(t)$ and $b_j(t)$, $(j = 0,1,2)$ are w-periodic continuous functions for $t \geq 0$, $g(t)$ is a continuously differentiable given function on the segment $[-w,0]$. h has domain $IR \times IR \times IR$ and h has continuous in all variables at each point of $IR \times IR \times IR$ and thare exists a positive number w such that for all $(t, x_1, x_2) \in IR \times IR \times IR$

$$h(t+w,x_1,x_2)=h(t,x_1,x_2). \qquad (2)$$

We introduce a state feedback controller $u(t)$ is given

$$u(t)=k(t)x(t) \qquad (3)$$

where $k(t)$ is a w-periodic continuously differentiable function for all $t\in IR$.

This note considers the problem of stabilizing a nonlinear system with neutral delay by means of linear feedback without neutral delay. The object is to find a sufficient condition on the open loop system for the possibility of actualizing an arbitrarily prescribed degree of stability in the closed loop system. Such stabilization problems have been studied extensively for delay system [3-5].

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations [9-12]. The delay appears in complicated systems with logical and computing devices, where certain time for information processing in needed.

The theory of linear and nonlinear ordinary and delay differential equations and control theory has been developed in the fundamental monographs [8-13]. The behavior of solutions of neutral and delay differential equations have been investigated in [1], [2], [6].

The aim of this paper is to suggest an alternative approach to the stability analysis of solutions of (1). We obtained the stability estimates for the solution of problem (1) in dependence on the coefficients.

2. STABILITY ESTIMATES

Theorem 2.1. Assume that $h(t,x_1,x_2)\equiv 0$, $a(t)>0$ and

$$|c(t)|+\left|\frac{b(t)}{a(t)}-c(t)\right|\leq 1 \qquad (4)$$

where

$$a(t)=a_0(t)+k(t)b_0(t)\ ,\quad c(t)=a_2(t)+k(t-w)b_2(t)$$

and (5)

$$b(t) = a_1(t) + k(t-w)b_1(t) + k'(t-w)b_2(t)$$

for $t \in [0,\infty)$. Then for the solution of (1) the estimate

$$|x(t)| \quad , \quad \frac{|b(t)x(t) + c(t)x'(t)|}{a(t)} \le F_1 \tag{6}$$

holds $\forall t \in [0,\infty)$, where

$$F_1 = \max\left\{ \max_{-w \le t \le 0} |g(t)| \quad , \quad \max_{-w \le t \le 0} \frac{|b(t+w)g(t) + c(t+w)g'(t)|}{a(t+w)} \right\} \tag{7}$$

Proof. Substitution of (3) into (1) yields the closed loop time-neutral delay equation, as

$$x'(t) = -\left(a_0(t) + k(t)b_0(t)\right)x(t) + \left(a_1(t) + k(t-w)b_1(t) + k'(t-w)b_2(t)\right)x(t-w)$$

$$+ \left(a_2(t) + k(t-w)b_2(t)\right)x'(t-w) \quad , \quad t \ge 0$$

or

$$x'(t) = -a(t)x(t) + b(t)x(t-w) + c(t)x'(t-w) \quad , \quad t \ge 0 \; . \tag{8}$$

Let $0 \le t \le w$. Then

$$x(t) = x(t,0)g(0) + \int_0^t x(t,s)\left(b(s)g(s-w) + c(s)g'(s-w)\right)ds \; ,$$

where

$$x(t,s) = \exp\left\{ -\int_s^t a(\tau)d\tau \right\}.$$

Using

$$\int_\tau^t x(t,s)a(s)ds = 1 - x(t,\tau), \tag{9}$$

we have that

$$| x(t) | \le x(t,0) \, | g(0) | + \int_0^t x(t,s) a(s) \frac{| b(s) g(s-w) + c(s) g'(s-w) |}{a(s)} ds$$

$$\le x(t,0) \max_{-w \le t \le 0} | g(t) | + \max_{0 \le t \le w} \frac{| b(t) g(t-w) + c(t) g'(t-w) |}{a(t)} \int_0^t x(t,s) a(s) ds$$

$$\le \max \left\{ \max_{-w \le t \le 0} | g(t) | \quad , \quad \max_{-w \le t \le 0} \frac{| b(t+w) g(t) + c(t+w) g'(t) |}{a(t+w)} \right\} \left(x(t,0) + 1 - x(t,0) \right)$$

$$= F_1 ,$$

then we can write

$$| x(t) | \le F_1 \tag{10}$$

for $t \in [0, w]$.

Now, we will obtain an estimate for $\dfrac{| b(t) x(t) + c(t) x'(t) |}{a(t)}$.

From (8) it follows that

$$b(t) x(t) + c(t) x'(t) = -\left(c(t) a(t) - b(t) \right) x(t) + c(t) \left(b(t) g(t-w) + c(t) g'(t-w) \right)$$

.

Let condition (4) be satisfied. Then, using (10) we can write

$$\frac{| b(t) x(t) + c(t) x'(t) |}{a(t)} \le \left| \frac{b(t)}{a(t)} - c(t) \right| | x(t) | + | c(t) | \frac{| b(t) g(t-w) + c(t) g'(t-w) |}{a(t)}$$

$$\le \left(\left| \frac{b(t)}{a(t)} - c(t) \right| + | c(t) | \right) F_1 \le F_1 .$$

Thus the estimate

$$\frac{| b(t) x(t) + c(t) x'(t) |}{a(t)} \le F_1 \tag{11}$$

holds for any t, $t \in [0, w]$.

From (10) and (11) there follows estimate (6) for any $t \in [0, w]$. Suppose that estimate (6) holds for $t \in [(n-1)w, nw]$.

If $nw \le t \le (n+1)w$, then

$$x(t) = x(t, nw)x(nw) + \int_{nw}^{t} x(t,s)\big(b(s)x(s-w) + c(s)x'(s-w)\big)ds.$$

Using the periodicity of $b(t)$, $c(t)$ and $a(t)$, we have that

$$| x(t) | \le x(t,nw) \, | x(nw) | + \int_{nw}^{t} x(t,s)a(s) \frac{| b(s)x(s-w) + c(s)x'(s-w) |}{a(s)} ds$$

$$\le x(t,nw) \, | x(nw) | + \big(1 - x(t,nw)\big) \max_{nw \le t \le (n+1)w} \frac{| b(t)x(t-w) + c(t)x'(t-w) |}{a(t)}$$

$$\le \max\left\{ \max_{(n-1)w \le t \le nw} | x(t) | \ , \ \max_{(n-1)w \le t \le nw} \frac{| b(t)x(t) + c(t)x'(t) |}{a(t)} \right\} \big(x(t,nw) + 1 - x(t,nw)\big)$$

$$\le F_1.$$

Thus, the estimate

$$| x(t) | \le F_1 \tag{12}$$

follows for any t, $t \in [nw, (n+1)w]$.

Now, we will obtainn the estimate for $\frac{| b(t)x(t) + c(t)x'(t) |}{a(t)}$. From (8) it follows that

$$b(t)x(t) + c(t)x'(t) = -\big(c(t)a(t) - b(t)\big)x(t) + c(t)\big(b(t)x(t-w) + c(t)x'(t-w)\big).$$

Let condition (4) be satisfied. Then, using (12) we can write

$$\frac{| b(t)x(t) + c(t)x'(t) |}{a(t)} \le \left| \frac{b(t)}{a(t)} - c(t) \right| | x(t) | + | c(t) | \frac{| b(t)x(t-w) + c(t)x'(t-w) |}{a(t)}$$

$$\le \left(\left| \frac{b(t)}{a(t)} - c(t) \right| + | c(t) | \right) F_1 \le F_1.$$

Thus the estimate

$$\frac{|b(t)x(t)+c(t)x'(t)|}{a(t)} \le F_1 \qquad (13)$$

holds for any t, $t \in [nw,(n+1)w]$.

Using (12) and (13), we have estimate (6) for any t, $t \in [nw,(n+1)w]$. So, using mathematical induction we obtained estimate (6) for any t, $t \in [0,\infty)$. This proves Theorem 2.1.

Theorem 2.2. Suppose that:

i) $a(t) > 0$, $\forall t \in [0,\infty)$;

ii) in the domain $IR \times IR \times IR$, the function h satisfies $\left(h(t,x_1,x_2) \ne 0\right)$

$$|h(t,x_1(t),x_2(t))| \le m(t)\,|x_1(t)| \qquad (14)$$

where $m(t)$ is a continuous function; and

iii) $$\left|\frac{b(t)}{a(t)} - c(t)\right| + |c(t)| + \frac{m(t)}{a(t)} \le 1 \ , \ \forall t \in [0,\infty), \qquad (15)$$

where $a(t)$, $b(t)$ and $c(t)$ are defined in (5). Then for the solution of problem (1) the estimate

$$|x(t)| \ , \quad \frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \le F_2 \qquad (16)$$

holds $\forall t \in [0,\infty)$, where

$$F_2 = \max\left\{ \max_{-w\le t\le 0} |g(t)| \ , \ \max_{-w\le t\le 0} \frac{|b(t+w)g(t)+c(t+w)g'(t)+h(t+w,g(t),g'(t))|}{a(t+w)} \right\}. \qquad (17)$$

Proof. Let $0 \le t \le w$. Then

$$x(t) = x(t,0)g(0) + \int_0^t x(t,s)\left[b(s)g(s-w)+c(s)g'(s-w)+h(s,g(s-w),g'(s-w))\right]ds$$

where

$$x(t,s) = \exp\left\{-\int_s^t a(\tau)d\tau\right\}.$$

Using (9) we have that

$$|x(t)| \le x(t,0)\,|g(0)| + \int_0^t x(t,s)a(s)\frac{|b(s)g(s-w)+c(s)g'(s-w)+h(s,g(s-w),g'(s-w))|}{a(s)}ds$$

$$\le x(t,0)\,|g(0)| + \max_{0\le s\le w}\frac{|b(s)g(s-w)+c(s)g'(s-w)+h(s,g(s-w),g'(s-w))|}{a(s)}\int_0^t x(t,s)a(s)ds$$

$$\le \max\left\{\max_{-w\le t\le 0}|g(t)| \quad , \quad \max_{-w\le t\le 0}\frac{|b(t+w)g(t)+c(t+w)g'(t)+h(t+w,g(t),g'(t))|}{a(t+w)}\right\}$$

$$\times\left(x(t,0)+1-x(t,0)\right)$$

$$= F_2,$$

then we can write

$$|x(t)| \le F_2 \tag{18}$$

for $t\in[0,w]$.

Now, we will obtain an estimate for $\dfrac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)}$.

From (8) it follows that

$$b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t)) = -\left(c(t)a(t)-b(t)\right)x(t)$$

$$+c(t)\left[b(t)g(t-w)+c(t)g'(t-w)+h(t,g(t-w),g'(t-w))\right]+h(t,x(t),x'(t)).$$

Using condition (14), (15) and the estimate (18), we find

$$\frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \le \left|\frac{b(t)}{a(t)}-c(t)\right| |x(t)|$$

$$+|c(t)|\frac{|b(t)g(t-w)+c(t)g'(t-w)+h(t,g(t-w),g'(t-w)|}{a(t)}+\frac{m(t)\,|x(t)|}{a(t)}$$

$$\le \left(\left|\frac{b(t)}{a(t)}-c(t)\right|+|c(t)|+\frac{m(t)}{a(t)}\right)F_2 \le F_2.$$

Thus the estimate

$$\frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \le F_2 \tag{19}$$

holds for any t, $t \in [0,w]$.

From (18) and (19) there follows estimate (16) for any $t \in [0,w]$. Suppose that estimate (16) holds for $t \in [(n-1)w, nw]$.

If $nw \le t \le (n+1)w$, then

$$x(t) = x(t,nw)x(nw) + \int_{nw}^{t} x(t,s)\big[b(s)x(s-w)+c(s)x'(s-w)+h(s,x(s-w),x'(s-w))\big]ds$$

Using (9), (2) and the periodicity of $b(t)$, $c(t)$ and $a(t)$, we have that

$$|x(t)| \le x(t,nw)|x(nw)| + \int_{nw}^{t} x(t,s)a(s)\frac{|b(s)x(s-w)+c(s)x'(s-w)+h(s,x(s-w),x'(s-w))|}{a(s)}ds$$

$$\le x(t,nw) + \big(1 - x(t,nw)\big)$$

$$\times \max\left\{ \max_{(n-1)w\le t\le nw} |x(t)| \quad , \quad \max_{(n-1)w\le t\le nw} \frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \right\}$$

$$\le F_2 .$$

Thus, the estimate

$$|x(t)| \le F_2 \tag{20}$$

follows for any t, $t \in [nw,(n+1)w]$. Now, we will obtain the estimate for $\frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)}$. From (8) it follows that

$$b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t)) = -\big(c(t)a(t)-b(t)\big)x(t)$$

$$+ c(t)\big[b(t)x(t-w)+c(t)x'(t-w)+h(t,x(t-w),x'(t-w))\big] + h(t,x(t),x'(t)) .$$

Using condition (14), (15) and the estimate (20), we can write

$$\frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \le \left|\frac{b(t)}{a(t)}-c(t)\right| |x(t)|$$

$$+|c(t)|\frac{|b(t)x(t-w)+c(t)x'(t-w)+h(t,x(t-w),x'(t-w)|}{a(t)}+\frac{m(t)|x(t)|}{a(t)}$$

$$\le \left(\left|\frac{b(t)}{a(t)}-c(t)\right|+|c(t)|+\frac{m(t)}{a(t)}\right)F_2 \le F_2 .$$

Thus the estimate

$$\frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \le F_2 \qquad (21)$$

holds for any t, $t\in[nw,(n+1)w]$.

Using (20) and (21), we have estimate (16) for any t, $t\in[nw,(n+1)w]$. So, using mathematical induction we obtained estimate (16) for any t, $t\in[0,\infty)$. This proves Theorem 2.2.

Theorem 2.3. Suppose that:

i) $a(t)>0$, $\forall t\in[0,\infty)$;

ii) in the domain $IR\times IR\times IR$, the function h satisfies $(h(t,x_1,x_2)\ne 0)$

$$|h(t,x_1(t),x_2(t))| \le k_1|x_1(t)|+k_2$$

where k_1 and k_2 are nonnegative constants; and

iii) $|b(t)-c(t)a(t)|+a(t)|c(t)|+k_1+\dfrac{k_2}{F_2}\le a(t)$, $\forall t\in[0,\infty)$

where F_2 is defined in (6); $a(t)$, $b(t)$ and $c(t)$ are defined in (5). Then for the solution of problem (1) the estimate

$$|x(t)| \ , \quad \frac{|b(t)x(t)+c(t)x'(t)+h(t,x(t),x'(t))|}{a(t)} \le F_2$$

holds $\forall t\in[0,\infty)$.

The proof of Theorem 2.3 is similar to the proof of Theorem 2.2.

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Department of Mathematics, Faculty of Science,
Istanbul University, Vezneciler 34459, İstanbul, Turkey
agamali@istanbul.edu.tr

Department of Mathematics, Faculty of Science,
Süleyman Demirel University, 32260 Isparta, Turkey
afuat@fef.sdu.edu.tr