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ON AN EINSTEIN PROJECTIVE SASAKIAN MANIFOLD

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Abstract

In this paper , We have proved that a projectively flat Sasakian manifold is an Einstein manifold. Also, if an Einstein Sasakian manifold is projectively flat then it is locally isometric with a unit sphere S^n (1). It has also been proved that if in an Einsten Sasakian manifold the relation K(X,Y).P= 0 holds, then it is locally isometric with a unit sphere S^n (1).

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1. Introduction. Let (M^n, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, a (1-1) tensor field Ø and the associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a K - contact Riemannian manifold ([1], [2]). A K-contact Riemannian manifold is called a Sasakian manifold [2] if

$$(D_X \varnothing)(Y) = g(X, Y)\ \xi - \eta(Y) X \tag{1.1}$$

holds, where D denotes the operator of covariant differentiation with respect to g. We deal with a type of Sasakian manifold in which

$$K(X, Y).\ P = 0, \tag{1.2}$$

where P is the projective curvature tensor (see [5], [6]) defined by

$$P(X, Y)Z = K(X,Y)Z - \frac{1}{n-1}[\,Ric(Y,Z)X - Ric(X,Z)Y\,], \tag{1.3}$$

K is the Riemannian curvature tensor, Ric is the Ricci tensor of type (0,2) and K (X,Y) is considered as derivation of the tensor algebra at each point of the manifold for tangent vectors X,Y. In this connection we mention the works of K. Sekigawa [3] and Z.L. Szabo [4] who studied Riemannian manifolds satisfying the conditions similar to it. It is easy to see that K (X,Y). K = 0 implies K (X,Y). P = 0. So it is meaningful to undertake the study of manifolds satisfying the condition (1.2).

Let R and r denote the Ricci tensor of type (1,1) and the scalar curvature of M^n respectively. It is known that in a Sasakian manifold M^n, besides the relation (1.1), the following relations also hold (see [1], [2]):

$$\varnothing(\xi)=0 \tag{1.4}$$

$$\eta(\xi)=1 \tag{1.5}$$

$$g(\xi, X)=\eta(X) \tag{1.6}$$

$$Ric(X, \xi)=(n-1)\,\eta(X) \tag{1.7}$$

$$g(K(\xi,X)Y, \xi)=g(X,Y)-\eta(X)\,\eta(Y) \tag{1.8}$$

$$K(\xi, X)\,\xi=-X+\eta(X)\,\xi \tag{1.9}$$

$$g(\varnothing X, \varnothing Y)=g(X,Y)-\eta(X)\,\eta(Y) \tag{1.10}$$

$$K(\xi, X)\,Y=g(X,Y)-\eta(Y)\,X \tag{1.11}$$

and

$$\eta(\varnothing X)=0 \tag{1.12}$$

for any vector fields X, Y.

The above results will be used in the next section.

2. Sasakian manifold satisfying P (X,Y)Z=0. Let us suppose that in a Sasakian manifold

$$P(X,Y)Z=0 \tag{2.1}$$

Then it follows from (1.3) that

$$K(X,Y)Z=\frac{1}{n-1}\,[\,Ric(Y,Z)\,X-Ric(X,Z)\,Y\,] \tag{2.2}$$

or,

$$g(K(X,Y)Z, U)=\frac{1}{n-1}\,Ric(Y,Z)\,g(X,U)-Ric(X,Z)\,g(Y,U)\,]. \tag{2.3}$$

Taking $X=U=\xi$ in (2.3) and then using (1.5), (1.6), (1.7) and (1.8), we get

$$g(Y, Z) - \eta(Y)\eta(Z) = \frac{1}{n-1} \quad [\,Ric(Y,Z) - (n-1)\eta(Y)\eta(Z)\,] \tag{2.4}$$

Consequently,

$$Ric(Y,Z) = k\,g(Y,Z), \tag{2.5}$$

where k = (n-1). Thus we have the following result:

THEOREM 2.1. A projectively flat Sasakian manifold is an Einstein manifold.

Next, we prove the following:

THEOREM 2.2. The scalar curvature r of a projectively flat Sasakian manifold M^n is constant.

PROOF. From (2.5), we have

$$R(Y) = (n-1)\,Y, \tag{2.6}$$

where Ric (Y,Z) = g (R(Y),Z). Contracting (2.6) with respect to 'Y', we have r = n (n-1), which proves the result.

THEOREM 2.3. A projectively flat Einstein Sasakian manifold M^n ($n \geq 2$) is locally isometric with a unit sphere $S^n(1)$.

PROOF. Let the Riemannian manifold be Einstein, i.e.

$$Ric(X,Y) = kg(X,Y),$$

where k is constant. Then (2.2) reduces to

$$K(X,Y)Z = \frac{k}{n-1} \quad [g(Y,Z)X - g(X,Z)Y] \tag{2.7}$$

or,

$$g(K(X,Y)Z,V) = \frac{k}{n-1} \quad [\,g(Y,Z)g(X,V) - g(X,Z)\,g(Y,V)\,]. \tag{2.8}$$

Taking X = V= ξ in (2.8) and then using (1.5), (1.6) and (1.8), we get

$$g(Y,Z) - \eta(Y)\eta(Z) = \frac{k}{n-1} \quad g(Y,Z) - \eta(Z)\eta(Y)\,]$$

or,

$$[\ \frac{k}{n-1} \ -1]\,[\,g(Y,Z) - \eta(Y)\eta(Z)\,] = 0.$$

This shows that either k = n-1 or g (Y,Z) = η (Y) η(Z). Now, if g (Y,Z) = η (Y) η (Z), then from (1.10), we get g (∅Y, ∅Z) = 0, which is not possible. Therefore, k = n-1 and putting this value of k in (2.7) we get the result.

3. An Einstein Sasakian manifold satisfying K (X,Y). P=0. Let the Riemannian manifold M be an Einstein manifold, then (1.3) gives

$$P(X,Y)Z = K(X,Y)Z - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y]. \tag{3.1}$$

We have,

$$\eta(P(X,Y)Z) = g(P(X,Y)Z, \xi)$$
$$= g(K(X,Y)Z - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y], \xi)$$
$$= \eta(X)g(Z,Y) - \eta(Y)g(Z,X)$$
$$- \frac{k}{n-1}[\eta(X)g(Z,Y) - \eta(Y)g(Z,X)]$$

or,

$$\eta(P(X,Y)Z) = [\frac{k}{n-1} - 1][\eta(Y)g(Z,X) - \eta(X)g(Z,Y)]. \tag{3.2}$$

Taking $X = \xi$ in (3.2) and then using (1.5) and (1.6), we get

$$\eta(P(\xi, Y)Z) = [\frac{k}{n-1} - 1][\eta(Y)\eta(Z) - g(Z,Y)]. \tag{3.3}$$

Again, taking $Z = \xi$ in (3.2) and then using (1.5) and (1.6), we get

$$\eta(P(X,Y)\xi) = 0. \tag{3.4}$$

Now,

$$(K(X,Y)P)(U,V)W = K(X,Y)P(U,V)W - P(K(X,Y)U,V)W$$
$$- P(U,K(X,Y)V)W - P(U,V)K(X,Y)W.$$

In view of (1.2) , we get

$K(X,Y)P(U,V)W - P(K(X,Y)U,V)W - P(U,K(X,Y)V)W$

$$- P(U,V)\,K(X,Y)W=0 \tag{3.5}$$

Therefore,

$$g[K(\xi,Y)P(U,V)W,\xi] - g[P(K(\xi,Y)U,V)W,\xi]$$
$$- g[P(U,K(\xi,Y)V)W,\xi] - g[P(U,V)K(\xi,Y)W,\xi] = 0.$$

From this it follows that

$$'P(U,V,W,Y) - \eta(Y)\eta(P(U,V)W) + \eta(U)\eta(P(Y,V)W)$$
$$+\eta(V)\eta(P(U,Y)W) + \eta(W)\eta(P(U,V)Y) - g(Y,U)\eta(P(\xi,V)W)$$
$$- g(Y,V)\eta(P(U,\xi)W) - g(Y,W)\eta(P(U,V)\xi) = 0, \tag{3.6}$$

where $g(P(U,V)W,Y) = {}'P(U,V,W,Y)$.

Putting $Y = U$ in (3.6), we get

$$'P(U,V,W,U) - \eta(U)\eta(P(U,V,W) + \eta(U)\eta(P(U,V)W)$$
$$+ \eta(V)\eta(PU,U)W) + \eta(W)\eta(P(U,V)U) - g(U,U)\eta(P(\xi,V)W)$$
$$- g(U,V)\eta(P(U,\xi)W) - g(U,W)\eta(P(U,V)\xi) = 0. \tag{3.7}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (3.7) for $U = e_i$ gives

$$\eta(P(\xi,V)W) = \frac{1}{n-1}\left[Ric(V,W) - \frac{r}{n}g(V,W)\right.$$
$$\left. + \left(\frac{k}{n(n-1)} - 1\right)(n-1)\eta(W)\eta(V)\right]. \tag{3.8}$$

Using (3.2) and (3.8), it follows from (3.6) that

$$'P(U,V,W,Y) + \frac{k}{n(n-1)}(Y,U)g(V,W) - \frac{k}{n(n-1)}g(U,W)g(Y,V)$$
$$+ \frac{1}{n-1}[Ric(U,W)g(Y,V) - Ric(V,W)g(Y,U)] = 0. \tag{3.9}$$

From (3.3) and (3.8), we get

$$\left(\frac{k}{n-1}-1\right)[\eta(V)\eta(W)-g(W,V)]=\frac{1}{n-1}\Big[Ric(V,W)-\frac{r}{r}g(V,W)$$
$$+\left(\frac{k}{n(n-1)}-1\right)(n-1)\eta(V)\eta(W)\Big].$$

For $r = nk$, we have

$$Ric(W,V) = (n-1)\, g(W,V). \tag{3.10}$$

Using (3.10) and takeing $r = nk$, the relation (3.9) reduces to

$$'P(U,V,W,Y) = \left(\frac{k}{n-1}\right)[g(Y,V)\, g(U,W) - g(Y,U)\, g(V,W)]. \tag{3.11}$$

From (3.1) and (3.11), we get

$$'K(U,V,W,Y) = [g(Y,U)\, g(V,W) - g(Y,V)\, g(U,W)], \tag{3.12}$$

where $'K(U,V,W,Y) = g(K(U,V)W, Y)$.

Thus we have the following:

THEOREM 3.1. If in an Einstein Sasakian manifold , the relation $K(X,Y).\, P = 0$ holds, then it is locally isometric with a unit sphere $S^n(1)$.

For a projectively symmetric Riemannian manifold we have $DP = 0$. Hence for such a manifold $K(X,Y).\, P=0$ holds. Thus we have the following corollary of the above theorem:

COROLLARY 3.2. A projectively symmetric Sasakian manifold M^n $(n\geq 2)$ is locally isometric with a unit sphere $S^n(1)$.

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