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Almost Paracontact Structures and Time Dependent Lagrangians

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Let M be a smooth manifold and TM its tangent bundle. We associate to a nonlinear connection in the vector bundle $\xi:\Re\times TM\to\Re\times M$ an almost paracontact structure on $\Re\times TM$. As it is well-known ([1],[3]) that any regular time dependent Lagrangian $L:\Re\times TM\to\Re$ defines a nonlinear connection in the vector bundle ξ, an almost paracontact structure depending on L is obtained. Several properties of it are pointed out.

1. NONLINEAR CONNECTIONS IN THE VECTOR BUNDLE ξ

Let M be a smooth manifold of dimension n. We take $(t,x^i)\equiv(t,x)$ as local coordinates on $\Re\times M$. The indices i,j,k,... will run from 1 to n and the Einstein convention on summation will be used. The coordinates in the fibres of ξ will be denoted by (y^i) and so $\Re\times TM$ is coordinate by $(t,x^i,y^i)\equiv(t,x,y)$. A change of local coordinates $(t,x,y)\to(\bar{t},\bar{x},\bar{y})$ has the form

(1.1) $\bar{t}=t,\ \bar{x}^i=\bar{x}^i(x^1,...,x^n),\ \bar{y}^i=\dfrac{\partial\bar{x}^i}{\partial x^j}(x)y^j$

with $rank\left(\dfrac{\partial\bar{x}^i}{\partial x^j}\right)=n.$

If we set $V_uE=Ker\xi_{*,u}$, where ξ_* is the differential of ξ, then $u\to V_uE$, $u\in E=\Re\times TM$ is a distribution on E (vertical), locally spanned by $\left(\dfrac{\partial}{\partial y^i}:=\dot{\partial}_i\right)$.

One may check that setting

(1.2) $J\left(\dfrac{\partial}{\partial t}\right)=0,\ J\left(\dfrac{\partial}{\partial x^i}\right)=\dot{\partial}_i,\ J\left(\dot{\partial}_i\right)=0$

one obtains a well-defined (1,1)-tensor field on E. It satisfies $J^2=0$ and the Nijenhuis tensor associated to it vanishes.

In the following it will be convenient to put $t=x^0$ and to use the Greek indices $\alpha,\beta,\gamma...$ ranging over 0,1,2,...,n.

A nonlinear connection in the vector bundle ξ is a distribution $u\to H_uE, u\in E$ that is supplementary to the vertical distribution.

Such a distribution is completely determined by (n+1) linear independent local vector fields. We take these vector fields in the form

(1.3) $\delta_\alpha=\partial_\alpha-N_\alpha^i(t,x,y)\dot{\partial}_i,$

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where $\partial_\alpha := \dfrac{\partial}{\partial x^\alpha}$ and the minus sign is taken for convenience. We note also that $\xi_*(\delta_\alpha) = \dfrac{\partial}{\partial x^\alpha}$. The functions (N^i_α) are called the local coefficient of the said nonlinear connection and they obey the following low of transformation:

(1.4) $\overline{N}^i_\alpha \dfrac{\partial \overline{x}^\alpha}{\partial x^\beta} = \dfrac{\partial \overline{x}^i}{\partial x^k} N^k_\beta - \dfrac{\partial^2 \overline{x}^i}{\partial x^\beta \partial x^k} y^k$.

If we decompose (1.3) in the form

(1.5) $\delta_0 = \dfrac{\partial}{\partial t} - N^i_0(t,x,y)\dot{\partial}_i, \quad \delta_i = \partial_i - N^k_i(t,x,y)\dot{\partial}_k$

it comes out that (N^k_0) behave like the components of a vector field on M and $(N^k_i(t,x,y))$ behave like the local coefficients of a nonlinear connection in the tangent bundle (see [3]).

2.A PARACONTACT STRUCTURE ON $E = \Re \times TM$

Assume that $E = \Re \times TM$ is endowed with a nonlinear connection. Thus we have a decomposition $TE = HE \oplus VE$ and a local frame $(\delta_0, \delta_i, \dot{\partial}_i)$ adapted to this decomposition.
We denote by $(dt, dx^i, \delta y^i)$ with $\delta y^i = dy^i + N^i_\alpha dx^\alpha$ the dual of this local frame. We consider the linear map $Q_u : T_u E \to T_u E, u \in E$ defined by

(2.1) $Q_u(\delta_0) = 0,\ Q_u(\delta_i) = \dot{\partial}_i,\ Q_u(\dot{\partial}_i) = \delta_i$.

This is well-defined since $\delta_i = \dfrac{\partial \overline{x}^j}{\partial x^i} \overline{\delta}_j$ and $\dot{\partial}_i = \dfrac{\partial \overline{x}^j}{\partial x^i} \dot{\partial}_j$.

The mapping $Q : u \to Q_u$ defines a (1,1)-tensor field on E which obviously is of rank 2n.
Moreover, a direct calculation gives

(2.2) $Q^3 - Q = 0$.

Thus Q defines a f(3,-1)-structure on E.
Furthermore, we have

Theorem 2.1. Let $E = \Re \times TM$ be endowed with a nonlinear connection N. Then E carries an almost paracontact structure (Q, δ_0, dt).

Proof. We have $dt(\delta_0) = 1$ and from $Q^2(\delta_0) = 0, \quad Q^2(\delta_i) = \delta_i, \quad Q^2(\dot{\partial}_i) = \dot{\partial}_i$ it follows that $Q^2 = I - \delta_0 \otimes \delta t$, q.e.d.

We notice that $dt \circ Q = 0$.

The tensor field that by its vanishing assures the normality of the almost paracontact structure (Q, δ_0, dt) reduces to the Nijenhuis tensor field N_Q associated to Q since dt is a closed 1-form (see [2]).

We recall that $N_Q(X,Y] = [QX,QY] + Q^2[X,Y] - Q[QX,Y] - Q[X,QY],\ X,Y \in \chi(E)$

Here $\chi(E)$ is the module of vector fields on E. In order to evaluate N_Q in the local frame $(\delta_0, \delta_i, \dot{\partial}_i)$ we firstly notice that $[\delta_\alpha, \delta_\beta] = R^i_{\alpha\beta}\dot{\partial}_i\ [\delta_\alpha, \dot{\partial}_i] = \dot{\partial}_i N^j_\alpha \dot{\partial}_j, [\dot{\partial}_i, \dot{\partial}_j] = 0$ where

$R^i_{\alpha\beta} = \partial_\beta N^i_\alpha - \partial_\alpha N^i_\beta + N^k_\alpha \dot{\partial}_k N^i_\beta - N^k_\beta \dot{\partial}_k N^i_\alpha$.

By a direct calculation one gets:

$N_Q(\delta_0, \delta_i) = -\dot{\partial}_i N^j_0 \delta_j + R^j_{oi} \dot{\partial}_j$

$N_Q(\delta_0, \dot{\partial}_i) = -R^j_{oi} \delta_j + \dot{\partial}_i N^j_0 \dot{\partial}_j$

$N_Q(\delta_i, \delta_j) = (\dot{\partial}_i N^k_j - \dot{\partial}_j N^k_i)\delta_k + R^k_{ij} \dot{\partial}_k$

$N_Q(\delta_i, \dot{\partial}_j) = -R^k_{ij} \delta_k + (\dot{\partial}_j N^k_i - \dot{\partial}_i N^k_j)\dot{\partial}_k$.

$N_Q(\dot{\partial}_i, \dot{\partial}_j) = (\dot{\partial}_i N^k_j - \dot{\partial}_j N^k_i)\delta_k + R^k_{ij} \dot{\partial}_k$

The almost paracontact structure (Q, δ_0, dt) is normal if $N_Q \equiv 0$. Thus we have

Theorem 2.1. The almost paracontact structure (Q, δ_0, dt) is normal if and only if the conditions:

(i) $\dot{\partial}_i N^j_0 = 0,$

(ii) $R^k_{0i} = 0,$

(iii) $\dot{\partial}_i N^j_0 = \dot{\partial}_j N^k_i,$

(iv) $R^k_{ij} = 0,$

hold good.

Remark 2.1. The condition (i) says that the covector (N^j_0) does not depend on the directional variables (y^i). The tensor field $t^k_{ij} = \dot{\partial}_i N^k_j - \dot{\partial}_j N^k_i$ is called the torsion of N.

Thus (iii) says that N is without torsion.

The condition (ii) and (iv) are equivalent with $R^i_{\alpha\beta} = 0$. This means that N is without curvature. Equivalently, the horizontal distribution is integrable.

3. AN ALMOST PARACONTACT STRUCTURE ASSOCIATED TO A TIME DEPENDENT LAGRANGIAN

A smooth function $L : \Re \times TM \to \Re, (t, v) \to L(t, v)$ is called a time dependent Lagrangian.

It is said that L is regular if the matrix $g_{ij}(t, x, y) := \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$ is of rank n on E.

Theorem 3.1. [1] The functions $N_L = (N^k_0(t, x, y)), N^k_i(t, x, y))$, where

$$N^k_0(t, x, y) = \frac{1}{2} g^{ki} \frac{\partial^2 L}{\partial t \partial y^i}, \quad N^k_i(t, x, y) = \dot{\partial}_i G^k(t, x, y),$$

$$G^k(t, x, y) = \frac{1}{4} g^{ki} \left(\frac{\partial^2 L}{\partial y^i \partial x^h} y^h - \frac{\partial L}{\partial x^i} \right)$$

are the local coefficients of a nonlinear connection on E which is completely determined by the regular Lagrangian L. Having N_L we can construct an almost paracontact structure as in Section 2. This will be completely determined by L. We denote it also by (Q, δ_0, dt).

Let us consider a particular case as follows.

Assume that $\Re \times M$ is endowed with a Riemannian metric of local coefficients $\gamma_{ij}(t,x)$ and set $L(t,x,y)=\gamma_{ij}(t,x,y)y^i y^j$.

Thus we get a regular Lagrangian with $g_{ij}=\gamma_{ij}$.

By a direct calculation we find

$$N_0^k(t,x,y)=\gamma^{ki}\frac{\partial \gamma_{ij}}{\partial t}y^j$$

$$N_i^k(t,x,y)=\gamma_{ij}^k(t,x)y^j$$

where $\gamma^k{}_{ij}(t,x)$ are the Christoffel symbols constructed with $\gamma_{ij}(t,x)$. By Theorem 2.1, the almost paracontact structure (Q,δ_0,dt) associated to $L(t,x,y)=\gamma_{ij}(t,x,y)y^i y^j$ is normal if and only if the metric $\gamma=(\gamma_{ij})$ does not depend on t (because of (i)) and it is locally flat (because of (ii) and (iv)). We notice that condition (iv) is satisfied whenever N is derived from a regular Lagrangian. Given a regular Lagrangian, we have a regular matrix $(g_{ij}(t,x,y))$. Its entries behave like the coefficients of a (0,2)-tensor field on M, that is they define a d-tensor field on E. For general notion of d-tensor field we refer to [3]. Assuming that the matrix $(g_{ij}(t,x,y))$ is positive definite we can obtain a Riemannian metric on E as follows:

$$G(t,x,y)=dt\otimes dt+g_{ij}dx^i\otimes dx^j+g_{ij}\delta y^i\otimes \delta y^j.$$

Differently saying

$$G(\delta_0,\delta_0)=1, G(\delta_i,\delta_j)=g_{ij}, G(\dot{\partial}_i,\dot{\partial}_j)=g_{ij}.$$

We have

Theorem 3.2. The Riemannian metric G satisfies the following equations

$$G(QX,QY)=G(X,Y)-dt(X)dt(Y),$$

$$dt(X)=G(\delta_0,X)$$

The proof is immediate. The Theorem 3.2 says that (Q,δ_0,dt,G) is a Riemannian almost paracontact structure associated to a regular Lagrangian.

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