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Transient Waves in Laminated Elastic Half-Space

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The study is devoted to analytic investigation of propagation of longitudinal transient waves in laminated elastic composite half-space. The problem is solved by using Laplace integral transform method. The recurrence relation between the powers of reflection coefficients for neighbouring layers are obtained. This makes easy to find the inverse Laplace transforms. The solutions obtained are analysed for finite value of time. The solution for periodically layered half-spase is obtained as particular case.

1. INTRODUCTION

Dynamics of elastic solids has important application in seismology and in many branches of tecnology. Moreower, it is well known that wave propagation is a powerful method of investigation in determining physical and mechanical properties of material systems. Longitudinal and transversal waves propagating in the materials, is sensitive to the elastic properties of the material [1-12]. Wave propagation in laminated composite materials is very important and very difficult [2-12]. A new effective analytic method has been proposed in [3] to solve the transient wave problems in laminated elastic composite with any non-proportional hereditary proreties of components. Propagation of SH waves through laminated composite materials was studed in [12] by using the transfer matrix method.

The paper is devoted to analytic investigation of propagation of longitudinal transient waves in laminated elastic composite half-space consisting of the homogeneous isotropic layers lying on the homogeneous and isotropic elastic half-space. The problem is solved by using Laplace integral transform method. The recurrence relation between the powers of reflection coefficients depending on the parameter of the Laplace transform for neighbouring layers are obtained. This makes easy to find the inverse Laplace transforms. These are analysed for finite values of time. The solution for periodically layered half-spase is obtained as particular case.

2. FORMULATION OF THE PROBLEM

The work is devoted to studying one-dimensional plane waves in laminated half-space with simpliest structure consisting of the homogeneous isotropic layers with plane-paralell bounds, occupying the domain

$$H_{m-1} \equiv \sum_{k=0}^{m-1} h_k \le x \le \sum_{k=0}^{m} h_k \equiv H_m \quad (m=1,2,...,N), \quad h_0 = H_0 = 0 \tag{1}$$

and lying on the homogeneous and isotropic elastic half-space $H_N \le x \le \infty$ $(H_{N+1} = \infty, h_{N+1} = \infty)$. Each layer of m medium and half-space is caracterized with its density

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and elastic modulus ρ_m, λ_m, μ_m $(m=1,2,...,N+1)$ and its state is defined with the field of small elastic displacements $u_m(x,t)$ satisfying the equation of the motion

$$\frac{\partial^2 u_m}{\partial t^2} = c_m^2 \frac{\partial^2 u_m}{\partial x^2}, \quad H_{m-1} < x < H_m, \quad t > 0 \tag{2}$$

where $c_m = \sqrt{(\lambda_m + 2\mu_m)/\rho_m}$ is the velocity of longitudinal waves. At instant $t=0$ the media are in their natural nonperturbed state. It is required to construct the solution of equation (2) in each domain of m medium satisfying the initial conditions

$$u_m = 0, \quad \frac{\partial u_m}{\partial t} = 0 \quad t=0, \quad H_{m-1} < x < H_m \tag{3}$$

and the boundary conditions expressing the continuity of displacement and equality of stress across the bounds $x = H_m$

$$u_m = u_{m+1}, \quad \sigma_m = \sigma_{m+1}, \quad x = H_m, \quad t \geq 0 \tag{4}$$

and the given condition on the exterior bound of medium

$$\sigma_1(0,t) = -f(t), \quad t > 0. \tag{5}$$

Besides, the conditions indicated there must sayisfy the boundness of the solution $u_{N+1}(x,t)$ for $x \to \infty$. Here $\sigma_m(x,t) = (\lambda_m + 2\mu_m)\partial u_m/\partial x$ is normal stresses.

Note the problem of obtaining the displacement fields $u_m(x,t)$ is correct, i.e. there is a unique solution which continuously depends on the date.

We solve problem (1.1)-(1.4) using the Laplace integral method. Removing the non-difficult procedures, we write the final solutions in the Laplace transform (here the bar above the function denotes its Laplace transform with the parameter p)

$$\bar{u}_1(x,p) = \frac{\bar{f}(p)}{\rho_1 c_1 p}\left[e^{-\frac{px_1}{c_1}} + \sum_{k_1=1}^{\infty} \theta_1^{k_1}\left(e^{-p\frac{2h_1k_1+x_1}{c_1}} + e^{-p\frac{2h_1k_1-x_1}{c_1}}\right)\right], \tag{6}$$

$$\bar{u}_m(x,p) = \bar{u}_{m-1}(H_{m-1},p)\left[e^{-\frac{px_m}{c_m}} + \sum_{k_1=1}^{\infty}(-\theta_m)^{k_m}\left(e^{-p\frac{2h_mk_m+x_m}{c_m}} + e^{-p\frac{2h_mk_m-x_m}{c_m}}\right)\right], \quad (m=1,2,...,N+1)$$

where $x_m = x - H_{m-1}$, $0 \leq x_m \leq h_m$ and

$$\theta_m = \frac{\theta_{0m} + \theta_{m+1}e^{-2ph_{m+1}/c_{m+1}}}{1 + \theta_{0m}\theta_{m+1}e^{-2ph_{m+1}/c_{m+1}}}, \quad \theta_{0m} = \frac{\rho_m c_m - \rho_{m+1}c_{m+1}}{\rho_m c_m + \rho_{m+1}c_{m+1}} \tag{7}$$

are the reflection coefficients. The series in (6) represents the geometric series and convergent absolutely and uniformly as $|\theta_m| < 1$. According to (6) we find the Laplace transforms of stresses

$$\bar{\sigma}_1(x,p) = -\bar{f}(p)\left[e^{-\frac{px_1}{c_1}} + \sum_{k_1=1}^{\infty}\theta_1^{k_1}\left(e^{-p\frac{2h_1k_1+x_1}{c_1}} - e^{-p\frac{2h_1k_1-x_1}{c_1}}\right)\right],$$

$$\bar{\sigma}_m(x,p) = \bar{\sigma}_{m-1}(H_{m-1},p)\left[e^{-\frac{px_m}{c_m}} + \sum_{k_1=1}^{\infty}\theta_m^{k_m}\left(e^{-p\frac{2h_mk_m+x_m}{c_m}} - e^{-p\frac{2h_mk_m-x_m}{c_m}}\right)\right]. \tag{8}$$

For obtaining the inverse Laplace transforms of the solution, let us simplify the expression of θ_m

$$\theta_m = \theta_{0m} + \left(1-\theta_{0m}^2\right)\theta_{m+1}\frac{e^{-2ph_{m+1}/c_{m+1}}}{1+\theta_{0m}\theta_{m+1}e^{-2ph_{m+1}/c_{m+1}}}$$

Here we find

$$\theta_m^{k_m} = \theta_{0m}^{k_m} + \sum_{j=1}^{k_m} C_{k_m}^j \theta_{0m}^{k_m-j}\left(1-\theta_{0m}^2\right)^j D^j, \quad D^j = \theta_{m+1}^j \frac{e^{-2pjh_{m+1}/c_{m+1}}}{\left(1+\theta_{0m}\theta_{m+1}e^{-2ph_{m+1}/c_{m+1}}\right)^j}.$$

Using the inequality $\left|\theta_{m+1}\exp\left(-2ph_{m+1}/c_{m+1}\right)\right|<1$, the function D^j may be represented in the form

$$D^j = \sum_{r=1}^{\infty}(-1)^r \theta_{0m}^r \theta_{0m+1}^{r+j}\frac{(r+j-1)!}{(j-1)!r!}e^{-\frac{2h_{m+1}(r+j)p}{c_{m+1}}}$$

Noting $r+j=k_{m+1}$ and denoting that $(k_{m+1}-1)!\left[(m-1)!(k_{m+1}-j)!\right]^{-1} = C_{k-1}^{m-1}$ is the binomial coefficient, after substituing D^j into $\theta_m^{k_m}$ and grouping the similar terms, we find the following recurrence relation connecting $\theta_m^{k_m}$ with the powers of θ_{m+1}

$$\theta_m^{k_m} = \theta_{0m}^{k_m} + \sum_{k_{m+1}=1}^{\infty} A_{k_m}^{k_{m+1}}\theta_{m+1}^{k_m+1}e^{-\frac{2h_{m+1}k_{m+1}p}{c_{m+1}}}, \tag{9}$$

$$A_{k_m}^{k_{m+1}}\left(\theta_{0m}\right) = \sum_{j=1}^{k_m}(-1)^{k_{m+1}-j} C_{k_m}^j C_{k_{m+1}}^{j-1}\theta_{0m}^{k_m+k_{m+1}-2j}\left(1-\theta_{0m}^2\right)^j. \tag{10}$$

Noting the fact that the coefficient of the reflection $\theta_N = \theta_{0N}$ does not depend on the parameter p (thus $h_{N+1}=\infty$), by the successive application of formula (1.9) we find

$$\theta_m^{k_m} = \theta_{0m}^{k_m} + \sum_{k_{m+1}=1}^{\infty} A_{k_m}^{k_{m+1}}\theta_{0m+1}^{k_{m+1}}e^{-\frac{2h_{m+1}k_{m+1}p}{c_{m+1}}} + \sum_{k_{m+1},k_{m+2}=1}^{\infty} A_{k_m}^{k_{m+1}}A_{k_{m+1}}^{k_{m+2}}\theta_{0m+2}^{k_{m+3}}\exp\left(-\frac{2h_{m+1}k_{m+1}p}{c_{m+1}}-\frac{2h_{m+2}k_{m+2}p}{c_{m+2}}\right)+\dots$$
$$+\sum_{k_{m+1},k_{m+2},\dots,k_N=1}^{\infty} A_{k_m}^{k_{m+1}}A_{k_{m+1}}^{k_{m+2}}\dots A_{k_{N-1}}^{k_N}\theta_{0N}^{k_N}\exp\left(-\frac{2h_{m+1}k_{m+1}p}{c_{m+1}}-\frac{2h_{m+2}k_{m+2}p}{c_{m+2}}-\dots-\frac{2h_Nk_Np}{c_N}\right). \tag{11}$$

Now from formulas (5) using (11) we find

$$u_m(x,t) = u_{m-1}\left(H_{m-1},t-\frac{x_m}{c_m}\right)+\sum_{k_m=1}^{\infty}\left(-\theta_{0m}\right)^{k_m}u_{m-1}\left(H_{m-1},t-\frac{2h_mk_m}{c_m};x_m\right)$$
$$+\sum_{k_m,k_{m+1}=1}^{\infty}(-1)^{k_m}A_{k_m}^{k_{m+1}}\theta_{0m+1}^{k_{m+1}}u_{m-1}\left(H_{m-1},t-\frac{2h_mk_m}{c_m}-\frac{2h_{m+1}k_{m+1}}{c_{m+1}};x_m\right)+\dots$$
$$+\sum_{k_m,k_{m+1},\dots k_N=1}^{\infty}(-1)^{k_m}A_{k_m}^{k_{m+1}}A_{k_{m+1}}^{k_{m+2}}\dots A_{k_{N-1}}^{k_N}\theta_{0N}^{k_N}u_{m-1}\left(H_{m-1},t-\frac{2h_mk_m}{c_m}-\frac{2h_{m+1}k_{m+1}}{c_{m+1}}-\dots-\frac{2h_Nk_N}{c_N};x_m\right), \tag{12}$$

$$\sigma_m(x,t) = \sigma_{m-1}\left(H_{m-1},t-\frac{x_m}{c_m}\right)+\sum_{k_m=1}^{\infty}\theta_{0m}{}^{k_m}\sigma_{m-1}\left(H_{m-1},t-\frac{2h_mk_m}{c_m};x_m\right)$$

$$+\sum_{k_m,k_{m+1}=1}^{\infty} A_{k_m}^{k_{m+1}}\theta_{0m+1}^{k_{m+1}}\sigma_{m-1}\left(H_{m-1},t-\frac{2h_m k_m}{c_m}-\frac{2h_{m+1}k_{m+1}}{c_{m+1}};x_m\right)+\ldots$$

$$+\sum_{k_m,k_{m+1},\ldots k_N=1}^{\infty} A_{k_m}^{k_{m+1}}A_{k_{m+1}}^{k_{m+2}}\ldots A_{k_{N-1}}^{k_{k_N}}\theta_{0N}^{k_N}\sigma_{m-1}\left(H_{m-1},t-\frac{2h_m k_m}{c_m}-\frac{2h_{m+1}k_{m+1}}{c_{m+1}}-\ldots-\frac{2h_N k_N}{c_N};x_m\right),$$

Here for $m=N,N+1$ we define

$$u_{N+1}(x,t)=u_N\left(H_N,t-\frac{x_{N+1}}{c_{N+1}}\right),\quad \sigma_{N+1}(x,t)=\sigma_N\left(H_N,t-\frac{x_{N+1}}{c_{N+1}}\right),$$

$$u_N(x,t)=u_{N-1}\left(H_{N-1},t-\frac{x_N}{c_N}\right)+\sum_{k_N=1}^{\infty}(-\theta_{0N})^{k_N}u_{N-1}\left(H_{N-1},t-\frac{2h_N k_N}{c_N};x_N\right),\qquad (13)$$

$$\sigma_N(x,t)=\sigma_{N-1}\left(H_{N-1},t-\frac{x_N}{c_N}\right)+\sum_{k_N=1}^{\infty}\theta_{0N}{}^{k_N}\sigma_{N-1}\left(H_{N-1},t-\frac{2h_N k_N}{c_N};x_N\right)$$

Here the notation

$$f(z,y;x_m)=f\left(z,y-\frac{x_m}{c_m}\right)-f\left(z,y+\frac{x_m}{c_m}\right)$$

is defined. For the displacement field, in the first layer $u_1(x,t)$ we take

$$f(y;x_1)=f\left(y-\frac{x_1}{c_1}\right)+f\left(y+\frac{x_1}{c_1}\right)$$

Satisfaction by formula (1.12) the equation of motion, initial and boundary conditions are easily seen. It is seen from (1.13.) that if $\rho_N c_N=\rho_{N+1}c_{N+1}$ $(\theta_{0N}=0)$ the reflection waves on the plane $x=H_N$ is absent. For $\rho_{N+1}c_{N+1}\to 0$ we have $\theta_{0N}\to 1$, then the reflection takes place as in the case for the free boundary.,i.e. for the layered plate with the finite thickness H_N, but for $\rho_{N+1}c_{N+1}\to\infty$, $\theta_{0N}\to -1$ reflection takes place as it does from absolutely hard bound (N layered plate lies on the absolutely hard half-space).

The sum in formulas(1.2) for each concrete time consists of finite number of the terms because all functions included in it are equal to zero for negative values of their arguments. Each term in (1.12) describes the influence of the waves coming to the first layer after the reflection from the corresponding boundary. For calculating the amplitude of these waves it is necessary to define the coefficients $A_{k_m}^{k_{m+1}}(\theta_{0m})$ where k_m and k_{m+1} are the numbers of the reflections from the m and $m+1$ layer,respectively,which for given time are defined by the formulas

$$k_1=\left[\frac{c_1 t}{2h_1}\right],\quad k_2=\left[\frac{c_2 t-h_1 c_2/c_1}{2h_2}\right],\ldots,k_m=\left[\frac{t-h_1/c_1-\ldots-h_{m-1}/c_{m-1}}{2h_m/c_m}\right]$$

Here the bracket denotes the whole part of the number in it.

Some of coefficients $A_{k_m}^{k_{m+1}}(\theta_{0m})$ for $\theta_{0m}\in[0,1]$ are given in the Table. For the negative θ_{0m} the formula

$$A_{k_m}^{k_{m+1}}\left(-\left|\theta_{0m}\right|\right)=\left(-1\right)^{k_m+k_{m+1}} A_{k_m}^{k_{m+1}}\left(\left|\theta_{0m}\right|\right) \qquad (14)$$

will be used.

From the tables it is seen that $\left|A_{k_m}^{k_{m+1}}\right|<1$ which denotes the quick convergence of series in (1.12).

In the applications we often meet the medium with periodic structure. In this case formulas (1.12) are rapidly simplified. For example, for the twice component periodical medium

$\rho_1 c_1=\rho_3 c_3=\ldots=\rho_{2m-1}c_{2m-1}=\ldots,\quad \rho_2 c_2=\rho_4 c_4=\ldots=\rho_{2m}c_{2m}=\ldots$

Then

$\theta_{01}=-\theta_{02}=\theta_{03}=\ldots=\theta_{02m-1}=-\theta_{02m}=\ldots,\quad A_j^n\left(\theta_{01}\right)=A_j^n\left(\theta_{03}\right)=\ldots=A_j^n\left(\theta_{02m-1}\right)=\ldots,$

$A_j^n\left(\theta_{0m}\right)=A_j^n\left(\theta_{0m+1}\right)$ when $j+n=2k$; $\quad A_j^n\left(\theta_{0m}\right)=-A_j^n\left(\theta_{0m+1}\right)$ when

$j+n=2k+1,\quad k=1,2,\ldots$ (15)

Thus in this case, it is sufficient to know only $A_j^n\left(\theta_{01}\right)$.

The Table

θ_{01}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
A_1^1	0.99	0.96	0.91	0.84	0.75	0.64	0.51	0.36	0.19
A_1^2	–0.099	–0.192	–0.273	–0.336	–0.375	–0.384	–0.357	–0.288	–0.171
A_1^3	0.0099	0.0384	0.0819	0.1344	0.1875	0.2304	0.25	0.2304	0.153
A_1^4	–0.001	–0.008	–0.024	–0.053	–0.093	–0.138	–0.174	–0.184	–0.138
A_1^5	0.0001	0.0015	0.0073	0.0215	0.047	0.083	0.1224	0.1474	0.124
A_2^1	0.188	0.384	0.541	0.672	0.75	0.768	0.714	0.576	0.342
A_2^2	–0.9603	–0.844	–0.664	–0.436	–0.1575	0.0512	0.2397	0.3312	0.2717
A_2^2	–0.194	–0.353	–0447	–0.456	–0.375	–0.215	–0.014	0.1613	0.212
A_2^4	0.029	0.107	0.209	0.2956	0.328	0.2765	0.134	–0.046	–0.161
A_2^2	–0.001	–0.006	–0.0179	–0.027	–0.023	0.011	0.082	0.169	0.198
A_3^1	0.029	0.115	0.245	0.402	0.562	0.691	0.749	0.691	0.459
A_3^2	0.201	0.529	0.610	0.684	0562	0.322	0.021	–0.244	–0.318
A_3^3	0.911	0.668	0.328	–0.020	–0.281	–0.373	–0.264	–0.008	0.205
A_3^4	–0.094	–0.155	-0.161	–0.110	–0.023	0.058	0.089	0.043	0.039
A_3^5	0.019	0.064	0.109	0.1208	0.082	0.006	–0.059	–0.058	0.017
A_4^5	0.0004	0.03	0.096	0.212	0.375	0.552	0.691	0.736	0.552
A_2^2	0.058	0.214	0.418	0.591	0.652	0.552	0.268	–0.092	–0.322
A_4^5	0.0125	0.206	0.214	0.148	0.0306	–0.078	–0.118	–0.058	–0.051
A_4^4	0.845	0.45	–0.008	–0.318	–0.328	–0.039	0.351	0.519	0.269
A_4^5	–0.036	–0.516	–0.380	–0.054	0.234	0.263	0.021	–0.197	–0.032

REFERENCES

1. Achenbach J. D., Wave propagation in elastic solids. North-Holland, Amsterdam, 1973.
2. Brekovskikh L. M., Waves in layered media. Academic Press, New York, 1960.
3. Ilyasov M., Memmedhasanov E., Stress waves in composite hereditary elastic rod, Collection of lectures: Actually Problems of Fundamental Sciences, Moskow 1991.
4. Achenbach J. D., Sun C. T., Herrman G., On the vibration of the laminated body, J. Appl. Mech. **35**, 1968, 467-475.
5. Barker L. M., A model for stress wave propagation in composite materials, J. Comp. Mater. **5**, 1971, 140-162.
6. Christensen R. M., Wave propagation in layered elastic media, J. Appl. Mech. 42, 1975, 153-158.
7. Hegemier G. A., Nayfeh A. H., A continuum theory for wave propagation in laminated composites, J. Appl. Mech. **40**, 1973, 503-510.
8. Peck J. C., Gurtman G. A., Dispertive pulse propagation parallel to the interfaces of laminated composite, J. Appl. Mach. 36, 1969, 479-484.
9. Ting T. C. T., Dynamical response of composites, Applied Mech. Reviews 33 (12), 1980.
10. Ting T. C. T., Mukunoki I., A theory of viscoelastic analogy for wave propagation normal to the layering of a layered medium, J. Appl. Mech. Reviews **46 (3)**, 1979, 329-336.
11. Ting T. C. T., Mukunoki I., Transient wave propagation normal to the layering of finite layered medium, Int. J. Solids Structures **16**, 1980, 239-251.
12. Louzar M., Lahrouni A., Chevalier Y., Propagation of SH wave in laminated elastic composite materials effect of frequency and of number of layers, Mech. Time-Depend. Mater. **10**, 2006, 165-171.