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On the generalized type and approximation of an entire harmonic function in $\mathbb{R}^3$ having index pair (p,q)

Mushtaq Shaker Abdul-Hussein and G.S. Srivastava

ABSTRACT. Let H_R, $0 < R < \infty$, be the class of all harmonic functions H in $\mathbb{R}^3$ that are regular in the open ball D_R of radius R centered at the origin and continuous on the closure $\overline{D_R}$ of D_R. For $H \in H_R$, set

$$E_n(H,R) = \inf_{g \in \Pi_n} \left\{ \max_{(x_1,x_2,x_3)\in D_R} \left| H(x_1,x_2,x_3) - g(x_1,x_2,x_3) \right| \right\},$$

where Π_n consists of all harmonic polynomials of degree at most n. In the present paper, we have introduced the concept of (p,q)- type of an entire harmonic function H with respect to the proximate order with index pair (p,q) and obtained its coefficient characterization in term of the approximation errors $E_n(H,R)$.

1. INTRODUCTION.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be a non-constant entire function where $\lambda_0 = 0$, and $\{\lambda_n\}_1^{\infty}$ is a strictly increasing sequence of positive integers such that no element of the sequence $\{a_n\}_1^\infty$ is zero. We denote by $\phi(r) = \phi(r,f) = \max_{|z|=r}|f(z)|$, the maximum modulus of $f(z)$ for $|z| = r$. It is well known that $\log\phi(r)$ is indefinitely increasing convex functions of $\log r$.

To estimate the growth of a non-constant entire function $f(z)$, concepts of order and type are used. Thus $f(z)$ is said to be of order ρ $(0 \leq \rho \leq \infty)$ if

(1.1) $$\rho = \limsup_{r\to\infty} \frac{\log \log \phi(r)}{\log r}.$$

Further, if the order ρ of $f(z)$ satisfies $0 < \rho < \infty$, then the type T of $f(z)$ is defined as

(1.2) $$T = \limsup_{r\to\infty} \frac{\log \phi(r)}{r^\rho} .$$

To further classify the entire functions of slow growth $(\rho = 0)$ or fast growth $(\rho = \infty)$ the concepts of (p,q)-order and (p,q)-type were introduced by Juneja et al. ([1], [2]). Thus for integers $p \geq q \geq 0$, we set

(1.3) $$\limsup_{r\to\infty} \frac{\log^{[p]} \phi(r)}{\log^{[q]} r} = \rho(p,q) = \rho .$$

Here $\log^{[p]} x = \log(\log,\ldots,\log x)$, where logarithm is taken p times. An entire function $f(z)$ is said to be of index pair (p,q), $p \geq q \geq 1$ if $b < \rho(p,q) < \infty$ and $\rho(p-1,q-1)$ is not a non- zero finite number, where

$b=1$ if $p=q$ and $b=0$ if $p>q$. If $f(z)$ has index pair (p,q), then $\rho=\rho(p,q)$ is called the (p,q) order of $f(z)$. If $f(z)$ is of (p,q)-order ρ, the (p,q)-type of $f(z)$ is defined by

$$(1.4) \qquad T=T(p,q)=\limsup_{r\to\infty}\frac{\log^{[p-1]}\phi(r)}{(\log^{[q-1]}r)^{\rho}}, \qquad 0\le T\le\infty.$$

For further definitions and notations, we shall refer to [1] and [2].

The concept of proximate order for entire functions of index pair (p,q) was introduced by Nandan et al. [4]. A real valued function ρ defined on $(0,\infty)$ is said to be a proximate order of an entire function with index pair (p,q) if it satisfies the properties

(i) $\lim_{r\to\infty}\rho(r)=\rho, \qquad b<\rho<\infty,$

(ii) $\lim_{r\to\infty}\Lambda_{[q]}(r)\,\rho'(r)=0$, where $\Lambda_{[q]}(r)=\prod_{i=0}^{q}\log^{[i]}r.$

The formula for the generalized (p,q) type and generalized lower (p,q) type of an entire function with respect to the proximate order with index pair (p,q) in terms of the coefficients $\{a_n\}$ in Taylor series expansion of $f(z)$ has been obtained by Nandan et al. [5].

The harmonic functions in R^3 are the solutions of Laplace equation

$$(1.5) \qquad \frac{\partial^2 H}{\partial x_1^2}+\frac{\partial^2 H}{\partial x_2^2}+\frac{\partial^2 H}{\partial x_3^2}=0.$$

A harmonic function H regular about the origin can be expanded as

(1.6) $$H \equiv H(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(a_{nm}^{(1)} \cos m\phi + a_{nm}^{(2)} \sin m\phi \right) r^n \, p_n^m(\cos\theta),$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\phi$, $x_3 = r\sin\theta\sin\phi$ and $P_n^m(t)$ are associated Legendre functions of first kind, of degree m and order n. A harmonic polynomial of degree k is a polynomial of degree k in x_1, x_2 and x_3 which satisfies (1.5). A harmonic function H is said to be regular in $D_R = \{(x_1,x_2,x_3) : x_1^2 + x_2^2 + x_3^2 < R^2,\ 0 < R \le \infty\}$, if the series (1.6) converges uniformly on every compact subset of D_R. The harmonic function H is called entire if it is regular in D_∞.

We denote by

$$M(r,H) = \max_{x_1^2+x_2^2+x_3^2=r^2} |H(x_1,x_2,x_3)|,$$

then $M(r,H)$ is called the maximum modulus of H. It is well known that $\log M(r,H)$ is an indefinitely increasing convex functions of $\log r$.

To study the growth of a non-constant entire harmonic function H, concepts of order and type are used following the classical definitions of these parameters for entire functions. Thus $f(z)$ is said to be of order ρ $(0 \le \rho \le \infty)$ if

(1.7) $$\rho \equiv \rho(H) = \limsup_{r\to\infty} \frac{\log\log M(r,H)}{\log r}.$$

Further , if order ρ of H satisfies $0<\rho<\infty$, then the of type of H is defined as

(1.8) $$T \equiv T(H)=\limsup_{r\to\infty}\frac{\log M(r,H)}{r^{\rho}}.$$

As in case of entire functions, the concepts of index pair, (p,q)-order and (p,q)-type of $H(z)$ can be given. The entire harmonic function H is said to be of (p,q)-order ρ, where

(1.9) $$\rho=\limsup_{r\to\infty}\frac{\log^{[p]} M(r,H)}{\log^{[q]} r}.$$

For $b<\rho<\infty$, the (p,q)-type T of H is defined as

(1.10) $$T=\limsup_{r\to\infty}\frac{\log^{[p-1]} M(r,H)}{\left(\log^{[q-1]} r\right)^{\rho}}$$

where $b=1$ if $p=q$ and $b=0$ if $p>q$ and $(0\le t\le T\le\infty)$.

Let $H\in \mathrm{H}_R$, $(0<R<\infty)$ and let $E_n(H,R)$ denote the error in approximating the function H by harmonic polynomials of degree at most n in uniform norm, where

$$E_n(H,R)=\inf_{g\in\Pi_n}\|H-g\|_n$$

and Π_n consists of all harmonic polynomials of degree at most n and

$$\|H-g\|_n=\sup_{(x_1,x_2,x_3)\in\bar{D}_R}|H(x_1,x_2,x_3)-g(x_1,x_2,x_3)|.$$

G.P. Kapoor and A. Nautiyal [3] obtained necessary and sufficient conditions on the rate of decrease of $E_n(H,R)$ as $n \to \infty$ so that $H \in \mathrm{H}_R$ has analytic continuation as an entire function of order ρ $(0<\rho<\infty)$ and type T. In this paper, using the concept of proximate order, we have defined the generalized type of H and obtained the coefficient characterization of generalized type. Our results generalize the results mentioned above.

2. AUXILIARY RESULTS.

In this section we give some results that have been used in proving our main result.

Since $\left(\log^{[q-1]} r\right)^{\rho(r)}$ is a monotonically increasing function of r for $0<r_{\circ}<r<\infty$, [6, Theorem 3] we denote by $\Omega(x)$ the single valued function which satisfies the condition

(2.1) $$x=\left(\log^{[q-1]} r\right)^{\rho(r)-A} \Leftrightarrow \log^{[q-1]} r=\Omega(x),$$

$$A=1 \text{ if } (p,q)=(2,2) \text{ and } A=0 \text{ if } (p,q)\neq(2,2)$$

We have [5,Theorem 1]

Lemma 1. The function $\Omega(x)$ defined above has following properties :

(2.2) (a) $$\lim_{x\to\infty}\frac{d\log\Omega(x)}{d\log x}=\frac{1}{\rho-A}$$

(2.3) (b) For every η, $0<\eta<\infty$

$$\lim_{x\to\infty} \frac{\Omega\,(\eta x)}{\Omega\,(x)} = \eta^{1/(\rho-A)} .$$

The following result is due to Kapoor and Nautiyal [3, Theorem 3].

Lemma 2. Let $H \in \mathrm{H}_R$. Then H has analytic continuation as an entire harmonic function of finite order ρ, $(0<\rho<\infty)$ and type $T(0<T<\infty)$ if and only if

$$\limsup_{n\to\infty} \left(n\ E_n(H,R)\right)^{\rho/n} = e\rho T R^{\rho} .$$

The following result was given by Kapoor and Nautiyal [3, Lemma 1].

Lemma 3. Associated Legendre functions $P_n^m(t)$ satisfy

$$\max_{-1\le t\le 1} \left|P_n^m(t)\right| \le K\ \left[(n+m)!/(n-m)!\right]^{1/2},$$

where K is a constant independent of n and m.

The following result was given by Kapoor and Nautiyal [3, Lemma 2].

Lemma 4. Let $H \in \mathrm{H}_R$ be entire and $r' > 1$. Then, for all $r > 2r'R$ and all sufficiently large values of n, we have

$$E_n(H,R) \le \bar{k}\, M(r,H)\ \left(r'R/r\right)^{n+1}, \text{ where } \bar{k} \text{ is a constant.}$$

3. MAIN RESULT.

Let $\rho(r)$ be proximate order satisfying the conditions (i) and (ii) as defined above. Let H be an entire harmonic function of (p,q) order ρ. The generalized type T^* of H is defined as

(3.1) $$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,H)}{\left(\log^{[q-1]} r\right)^{\rho(r)}} = T^* .$$

Now we prove

Theorem. Let H be an entire harmonic function of (p,q) -order ρ and $\rho(r)$ be a proximate order of index pair (p,q). Then the generalized type T^* of H is given by

(3.2) $$T^* = M \limsup_{n\to\infty}\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}(E_n(H,R))^{-1/n}}\right]^{\rho-A}$$

where

$$M = \begin{cases} (\rho-1)^{\rho-1}/\rho^{\rho} & \text{if } (p,q)=(2,2) \\ 1/e\rho\ R^{\rho} & \text{if } (p,q)=(2,1) \\ 1 & \text{for other index pairs } (p,q) \end{cases}.$$

Proof. From (3.1), for all $\varepsilon > 0$ and all $r > r_\circ$ $(0 < r_\circ = r_\circ(\varepsilon) < r < \infty)$, we have

$$\log^{[p-1]} M(r,H) < (T^* + \varepsilon)\left(\log^{[q-1]} r\right)^{\rho(r)}$$

or $$\log M(r,H) < \exp^{[p-2]}\left\{(T^* + \varepsilon)\left(\log^{[q-1]} r\right)^{\rho(r)}\right\}.$$

By using Lemma 4, we have

$$\log E_n(H,R) - \log \bar{k} - (n+1)\log\left(\frac{r'R}{r}\right) < \exp^{[p-2]}\left\{(T^* + \varepsilon)\left(\log^{[q-1]} r\right)^{\rho(r)}\right\}$$

(3.3) $\log E_n(H,R) < \exp^{[p-2]}\left\{(T^* + \varepsilon)\left(\log^{[q-1]} r\right)^{\rho(r)}\right\} + \log\bar{k} + (n+1)\log\left(\frac{r'R}{r}\right).$

For $(p,q) = (2,1)$ we have

$$\log E_n(H,R) < \left\{(T^* + \varepsilon)\ r^{\rho(r)}\right\} + \log\bar{k} + (n+1)\log\left(\frac{r'R}{r}\right).$$

As the above inequality holds for all $r > r_\circ(\varepsilon)$, we choose r such that

$$r^{\rho(r)} = \frac{n+1}{\rho(T^* + \varepsilon)},$$

i.e. $$r = \Omega\left(\frac{n+1}{\rho(T^* + \varepsilon)}\right).$$

Substituting this value of r, we get on using Lemma 1

$$\log E_n(H,R) < \frac{n+1}{\rho} + \log\bar{k} - (n+1)\log\left[\frac{\Omega\left(n+1/\rho(T^* + \varepsilon)\right)}{r'R}\right]$$

$$< \frac{n+1}{\rho} + \log\bar{k} - (n+1)\log\left[\frac{\Omega(n+1)}{\rho^{1/\rho}(T^* + \varepsilon)^{1/\rho}\ r'R}\right]$$

Hence $$[E_n(H,R)]^{1/n} \le e^{1/\rho}\left[\frac{\Omega(n+1)}{\left\{\rho(T^* + \varepsilon)\right\}^{1/\rho}}\right]^{-1}\left(1 + o(1)\right)$$

or $$\left[\frac{\Omega(n)}{\left(E_n(H,R)\right)^{-1/n}}\right]^{\rho} \le e\,\rho\,(T^* + \varepsilon)\,(r'R)^{\rho} + o(1).$$

As $r' > 1$ is arbitrary, we get on proceeding to limits,

$$\text{(3.4)} \qquad \limsup_{n\to\infty}\left[\frac{\Omega(n)}{(E_n(H,R))^{-1/n}}\right]^{\rho} \le e\rho\, T^* R^{\rho} \ .$$

Now we prove the result for $(p,q)=(2,2)$. The inequality (3.3) takes the form

$$\log E_n(H,R) < (T^* + \varepsilon)(\log r)^{\rho(r)} + \log \bar{k} + (n+1)\ \log\left(\frac{r' R}{r}\right)$$

Again we choose r, such that

$$(\log r)^{\rho(r)-1} = \frac{n+1}{\rho(T^* + \varepsilon)} \qquad \text{or} \qquad \log r = \Omega\left(\frac{n+1}{\rho(T^* + \varepsilon)}\right).$$

Hence (3.3) can be written as

$$\log E_n(H,R) < (T^* + \varepsilon)(\log r)^{\rho(r)} + \log \bar{k} + \rho(T^* + \varepsilon)(\log r)^{\rho(r)-1}\ \log\left(\frac{r' R}{r}\right)$$

or $$\log E_n(H,R) < (n+1)\ \frac{(1-\rho)}{\rho}\ \Omega\left(\frac{n+1}{\rho\ (T^* + \varepsilon)}\right)\ \{1+o(1)\}$$

or $$\frac{\Omega(n+1)}{[\rho\ (T^* + \varepsilon)]^{1/\rho}} < \frac{\rho}{\rho-1}\ \log(E_n(H,R))^{-1/n}\ \{1+o(1)\}$$

or $$\left[\frac{\Omega(n)}{\log(E_n(H,R))^{-1/n}}\right]^{\rho} < \left(\frac{\rho}{\rho-1}\right)^{\rho}(\rho\ (T^* + \varepsilon))\{1+o(1)\},$$

for all sufficiently large n. Hence

$$(3.5) \qquad \limsup_{n\to\infty}\left[\frac{\Omega(n)}{\log(E_n(H,R))^{-1/n}}\right]^{\rho-1} \leq \frac{\rho^{\rho}}{(\rho-1)^{\rho-1}}\, T^{*}.$$

Lastly let $(p,q)\neq(2,1)$ and $(2,2)$. In (3.3) we choose

$$\left(\log^{[q-1]} r\right)^{\rho(r)} = \frac{1}{(T^{*}+\varepsilon)} \log^{[p-2]} \frac{n+1}{\rho}$$

or $$\log^{[q-1]} r = \Omega\left(\frac{1}{(T^{*}+\varepsilon)}\log^{[p-2]}\frac{n+1}{\rho}\right).$$

Hence (3.3) is written as

$$\log E_n(H,R) \leq \exp^{[p-2]}\left\{(T^{*}+\varepsilon)\left(\log^{[q-1]} r\right)^{\rho(r)}\right\} + \log\bar{k} + (n+1)\log\left(\frac{r'R}{r}\right)$$

or $$\log E_n(H,R) \leq \exp^{[p-2]}\left\{\log^{[p-2]}\frac{n+1}{\rho}\right\} + \log\bar{k} + \log\left(\frac{r'R}{r}\right)^{(n+1)}$$

or $$\log E_n(H,R) \leq \frac{n+1}{\rho} + \log\bar{k}\left(\frac{r'R}{r}\right)^{(n+1)}$$

or $$\log\left(\frac{r}{r'R}\right) < \frac{1}{\rho} + \log\left[E_n(H,R)\right]^{-1/n} + o(1)$$

As $q>2$, we get

$$\log^{[q-1]}(r) < \log^{[q-1]}\left[E_n(H,R)\right]^{-1/n}$$

or $$\Omega\left[\frac{1}{T^{*}+\varepsilon}\log^{[p-2]}\left(\frac{n+1}{\rho}\right)\right] < \log^{[q-1]}\left[E_n(H,R)\right]^{-1/n}$$

using Lemma1, we get

$$\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}}\right] \le \left(T^* + \varepsilon\right)^{1/\rho},$$

for all large values of n. Proceeding to limits, we obtain,

$$\limsup_{n\to\infty}\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}}\right]^{\rho} \le T^*. \tag{3.6}$$

Equations (3.4), (3.5) and (3.6) combine into

$$\limsup_{n\to\infty}\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}}\right]^{\rho-A} \le T^*/M \tag{3.7}$$

To prove the reverse inequality, let

$$\limsup_{n\to\infty}\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}}\right]^{\rho-A} = T/M, \qquad T < T^*.$$

Choose $\varepsilon > 0$ such that $\alpha = T + \varepsilon < T^*$. Then for all $n > n_\circ = n_\circ(\varepsilon)$,

$$\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}} < \left(\frac{\alpha}{M}\right)^{1/\rho-A}$$

or $$\left(\frac{M}{\alpha}\right)^{1/\rho-A}\Omega\left(\log^{[p-2]} n\right) < \log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}$$

or $$\Omega\left(\frac{M}{\alpha}\log^{[p-2]} n\right) < \log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}$$

or $$E_n(H,R) < \exp\left[-n \exp^{[q-2]} \Omega\left(\frac{M}{\alpha}\log^{[p-2]} n\right)\right]$$

To obtain the reverse inequality, we consider the following inequality see ([3, p.1028] and [6, Lemma 3]):

$$M(r,H) \le \left| a_{00}^{(1)} \right| + KK_{\circ}\sum_{n=1}^{\infty}(2n+1)^2 E_{n-1}(H,R)\,(r/R_*)^n ,$$

where $R_* < R$. Since H is an entire harmonic function, we have from [3, Lemma 2]

$$E_n(H,R) \le \bar{k}\, M(r,H)\,(r'R/r)^{n+1},$$

for all $r > 2r'R$ and all sufficiently large values of n. Hence

$$\limsup_{n\to\infty}[E_n(H,R)]^{1/n} \le r'R/r, \quad r \text{ being arbitrary},$$

or $$\limsup_{n\to\infty}[E_n(H,R)]^{1/n} = 0.$$

Consider the function

$$h(z) = \sum_{n=0}^{\infty}(2n+1)^2 E_{n-1}(H,R)\,(z/R_*)^n.$$

Then

$$\limsup_{n\to\infty}\left\{\frac{(2n+1)^2}{(R^*)^n}(E_{n-1}(H,R))\right\}^{1/n} = 0.$$

Hence the function $h(z)$ of complex variable z is entire and

(3.8) $$M(r,H) \le 0(1) + M(r,h) \, . \, KK_{\circ},$$

where $M(r,h) := \sup_{|z|\le r} |h(z)|$. By [3], the (p,q) order ρ_1 of $h(z)$ is given by

$$\rho_1(p,q) = \alpha + \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} \left\{(2n+1)^2 \, E_{n-1}(H,R)/R_*^n\right\}^{1/n}},$$

where $\alpha = 1$ if $p = q = 2$ and $\alpha = 0$ otherwise. Now

$$\log\left[(2n+1)^2 \, E_n(H,R)/R_*^n\right]^{1/n} = \log\left[(2n+1)^{2/n} \, (E_n(H,R))^{1/n}/R_*\right].$$

Since $q \ge 1$, therefore

$$\rho_1(p,q) = \alpha + \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} \{E_{n-1}(H,R)\}^{1/n}}.$$

Combining the above result with the coefficient formula for (p,q) order of H [6,Theorem 1], it follows that the entire harmonic function H and the entire function $h(z)$ have same (p,q) order. Hence the proximate order for H and $h(z)$ are also same.

If T_1 denotes the generalized type of $h(z)$ with respect to the proximate order $\rho(r)$ of index pair (p,q) then by Theorem 3, [5], we have

$$T_1 = M_1 \limsup_{n\to\infty} \left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-2]}\left\{-(1/n)\log|b_n|\right\}}\right]^{\rho - A}$$

where A is as defined before, $b_n = (2n+1)^2 \, E_{n-1}(H,R)/R_*^n$ and

$$M_1 = \begin{cases} (\rho-1)^{\rho-1}/\rho^{\rho} & \text{if } (p,q)=(2,2) \\ 1/e\rho & \text{if } (p,q)=(2,1) \\ 1 & \text{if } (p,q)\neq(2,1) \text{ and } (2,2) \end{cases}.$$

For $(p,q)=(2,1)$, we have

$$|b_n|^{-1/n} = (2n+1)^{-2/n}\,[E_n(H,R)]^{-1/n}\,R_*,$$

hence $T_1 = \dfrac{1}{e\rho\,R_*^{\rho}} \limsup\limits_{n\to\infty} \left[\Omega(n)\{E_n(H,R)\}^{1/n}\right]^{\rho}$.

As $R_* < R$ was arbitrary, and $\Omega(n) = n^{1/\rho}$, we have

$$T_1 = \frac{1}{e\rho\,R^{\rho}} \limsup_{n\to\infty} \left[n\,\{E_n(H,R)\}^{1/n}\right]^{\rho}.$$

For the index pair (p,q), $q>1$, we obtain

$\log|b_n|^{-1/n} \cong \log[E_n(H,R)]^{-1/n}$, and we have

$$T_1 = M_1 \limsup_{n\to\infty}\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-2]}\left\{\log\left(E_n(H,R)\right)^{-1/n}\right\}}\right]^{\rho-A}.$$

Combining the two cases, we obtain

$$(3.9) \qquad T_1 = M\,.\limsup_{n\to\infty}\left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}}\right]^{\rho-A}$$

where M is as defined before. From (3.8) it is evident that $T^* \le T_1$. Now combining (3.7) and (3.9), we obtain

$$\frac{T^*}{M} = \limsup_{n\to\infty} \left[\frac{\Omega\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left(E_n(H,R)\right)^{-1/n}} \right]^{\rho-A}.$$

This complete the proof of the Theorem.

Corollary 1. Choosing $(p,q) = (2,1)$ and $\rho(r) \equiv \rho$, we get

$$T^* = \left(e\rho R^{\rho}\right)^{-1}. \limsup_{n\to\infty} n\left[E_n(H,R)\right]^{\rho/n}$$

and Theorem 2 of [3] follows.

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Mushtaq Shaker
Department of Mathematics,
Indian Institute of Technology Roorkee
Roorkee-247 667, India

G.S. Srivastava
Department of Mathematics,
Indian Institute of Technology Roorkee
Roorkee-247 667, India
E-mail address: girssfma@iitr.ernet.in