İstanbul Üniv. Fen. Fak. Mat. Der. 60 (2001), 41-54

Stability of one parameter C_0-semigroups on hereditarily indecomposable Banach spaces

Sânziana Caraman

Abstract

The main result is the following stability theorem: *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a hereditarily indecomposable Banach space X, with the infinitesimal generator A and domain $D(A)$, if we denote by A^* the adjoint of A and by $\sigma_p(A^*)$ the point spectrum of A^*, then $\mathcal{T}$ is strongly stable (which means $\lim_{t\to\infty} \|T(t)x\| = 0$, $\forall x \in X$), if and only if $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$.*

1 Introduction

The asymptotic behaviour of solutions of a differential equation $u'(t) = Au(t)$, $t \geq 0$ is frequently related to spectral properties of the operator A. In contrast to the case of finite dimensional space, where a classical theorem, due to Liapunov states that stability is equivalent to negativeness of the real parts of the eigenvalues of A, there is no simple characterization of strong stability for C_0-semigroups on Banach, or Hilbert spaces.

There have been obtained sufficient conditions for stability and we shall mention a theorem of Nagy and Foiaş [1], that *if $\mathcal{T}$ is a completely non-unitary contraction semigroup in a Hilbert space, such that $m(i\sigma(A) \cap \mathbb{R}) = 0$ then $\mathcal{T}$ is strongly stable* (where m denotes the Lebesgue measure on $\mathbb{R}$). For semigroups on Banach spaces, the most powerful result is due to Arendt–Batty [2] and the independently proof of Lyubich–Phóng [3] and is known as the ABLP theorem: *if $\mathcal{T} = (T(t))_{t\geq 0}$ is a C_0-semigroup on a Banach space X, with generator A, so that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$, then $\mathcal{T}$ is strongly stable.*

In the last decade, the conditions of the ABLP theorem were refined and there have been obtained interesting generalizations of this theorem (see the survey paper [4]), but a spectral characterization for stability is still an open question.

However, sufficient and necessary conditions for strong stability have been found for particular classes of operators on particular spaces. Such an example is a theorem of Huang–Räbiger [5] where the countability of $\sigma(A)\cap i\mathbb{R}$ is characterized by means of compactness property of the ultrapowers of $\mathcal{T}$, i.e.: *let* $\mathcal{T} = (T(t))_{t\geq 0}$ *be a bounded* C_0*–semigroup with generator* A *on a superreflexive Banach space* X. *Then* $\sigma(A)\cap i\mathbb{R}$ *is countable if and only if* $\mathcal{T}$ *is superstable.*

In the present paper we deal with hereditarily indemcomposable Banach spaces (briefly, H.I.) and state a criterion for strong stability of C_0–semigroups on such spaces.

The study of H.I. spaces is almost recent and appeared in a natural way as an answer to some mathematical enigma. One of these was revealed by Lindenstrauss [6] and asked whether every infinite–dimensional Banach space X was decomposable, that is could be written as a topological direct sum $X = Y \oplus Z$, with Y and Z infinite dimensional subspaces. The answer turned to be negative and in [7], Gowers and Maurey built a Banach space X which is not only not decomposable, but does not have a decomposable subspace, i.e. a H.I. space.

Equivalently this can be expressed as it follows: *if* Y *and* Z *are two infinite–dimensional subspaces of* X *and* $\varepsilon > 0$, *then there exist* $y \in Y$ *and* $z \in Z$ *such that* $\|y\| = \|z\| = 1$ *and* $\|y - z\| < \varepsilon$. Or in other words: *whenever* Y *and* Z *are closed infinite–dimensional subspaces of* X, *satisfying* $Y \cap Z = \{0\}$, *then* $Y + Z$ *is non–closed.*

But the property of X seems to be a key to another unsolved problem. That is: *does every space contain an unconditional basic sequence?"* For a long time a major problem was whether every separable Banach space had a basis which was answered negatively by Enflo in 1973 [8]. On the other hand, every space contains a basic sequence (that is there exists an infinite sequence which is a basis for its closed linear span). The problem was if under any permutation of the basis it still remained a basis and this was called unconditional basis. In 1991, Gowers found a contraexample and shortly afterwards it was established that *an H.I. space cannot contain an unconditional basic sequence.*

The H.I. spaces proved also to be the answer to a question of Banach known as "the hyperplane problem", that is whether *there exist spaces that fail to be isomorphic to a subspace of codimension one.*

In this paper we are interested in the behaviour of a C_0–semigroup in an H.I. Banach space.

2 Preliminaries

In the next section we need some basic knowledge from semigroup theory and also some important spectral results applied to a H.I. Banach space.

Let's first consider X a Banach space.

Definition 2.1. A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a *(one–parameter) semigroup* if

$$\begin{aligned} &T(t+s) = T(t)T(s) \text{ for all } t, s \geq 0 \\ &T(0) = I. \end{aligned}$$

Definition 2.2. A semigroup $(T(t))_{t\geq 0}$ is called *strongly continuous* (or *C_0–semigroup*) if

$$\lim_{t\searrow 0} T(t)x = x \text{ for all } x \in X.$$

An important feature of the C_0 semigroup is

Proposition 2.3. *For every strongly continuous semigroup $(T(t))_{t\geq 0}$ there exists constants $w \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{wt} \text{ for all } t \geq 0.$$

Moreover a semigroup is called *bounded* if we can take $w = 0$,

$$\|T(t)\| \leq M, \text{ for all } t \geq 0.$$

Definition 2.4. The infinitesimal generator $A : D(A) \subset X \to X$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X is the operator

$$Ax = \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x)$$

defined for every x in its domain

$$D(A) = \{x \in X \mid \text{there exists } \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x)\}.$$

The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely and has the following properties:

(i) If $x \in D(A)$, then $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x, \text{ for all } t \geq 0$$

(ii) For every $t \geq 0$ and $x \in X$ one has

$$\int_0^t T(s)x\,ds \in D(A)$$

(iii) For every $t \geq 0$ one has

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)x\,ds \quad \text{if } x \in X \\ &= \int_0^t T(s)Ax\,ds \quad \text{if } x \in D(A). \end{aligned}$$

And now let's recall some results from spectral theory [9].

Assume that X is a Banach complex space and $T \in L(X)$, where $L(X)$ is the set of all linear bounded operators defined on X with values in X. Then $T \in L(X)$ *is called strictly singular if for every infinite dimensional subspace* Y *of* X*, the restriction* T/Y *of* T *to* Y *is not isomorphism.* And $\lambda \in \mathbb{C}$ *is infinitely singular for* T *if* $T - \lambda I$ *is strictly singular*, in other words, *for every* $\varepsilon > 0$ *there exists an infinite dimensional subspace* Y_ε *of* X*, such that the restriction of* $T - \lambda I$ *to* Y_ε *has norm at most* ε.

Let's denote by α_T the set of all $\lambda \in \mathbb{C}$ so that $T - \lambda I$ is an isomorphism on some finite codimensional subspace of X. And denote by β_T the set of all $\lambda \in \mathbb{C}$ so that λ is infinitely singular for T or $\beta_T = \mathbb{C} \setminus \alpha_T$.

We resume some of the needed properties of α_T in the following theorem (see for instance [7])

Theorem 2.5. *Let* X *be a Banach space and* $T \in L(X)$ *and* α_T*, as defined above, then:*

(1) α_T *is an open set of* $\mathbb{C}$

(2) $\alpha_T \neq \mathbb{C}$

(3) $\mathrm{Ker}(T - \lambda I)$ *is finite dimensional, when* $\lambda \in \alpha_T$:

(4) *If* $\lambda \in \alpha_T$ *and* $(x_n)_{n \in N}$ *is a bounded sequence in* X*, such that* $(T - \lambda I)x_n$ *is norm-convergent, then* x_n *has a subsequence which is norm-convergent and the image of* $(T - \lambda I)$ *of any closed subspace of* X *is closed.*

The next theorem is the key result in studying the spectrum of a bounded operator defined on a H.I. Banach space. The proof can be found in Fredholm theory, or more elementary, based on the previous properties of α_T in [7].

Theorem 2.6. *If X is a Banach space, $T \in L(X)$ and λ belongs to α_T and to the boundary of the spectrum of T, then λ is an eigenvalue for T and it is also an isolated point for $\sigma(T)$.*

Now, let's consider X a hereditarily indecomposable Banach space as defined in the introduction. We are interested in the spectrum of a bounded operator on such spaces and this is precisely the content of the next theorem.

Theorem 2.7. *Let X be an H.I. Banach space and $T \in L(X)$ then there exists a unique $\lambda_T \in \sigma(T)$ so that $T = \lambda_T I + S$, where S is strictly singular and the spectrum $\sigma(T)$ is either finite or consists of a sequence $(\lambda_n)_{n=1}^{\infty}$ of eigenvalues, converging to λ_T.*

3 Characterization theorem for stability of C_0–semigroups on a H.I. Banach space

Let X be a Banach space with dual X^*. We denote by $< x^*, x >$ the value of $x^* \in X^*$ at $x \in X$. Let A be a linear operator with dense domain $D(A)$ in X. Recall that the adjoint A^* of A is a linear operator from $D(A^*) \subset X^*$ defined as follows $D(A^*) = \{x^* \in X^* \mid$ for which it exists $y^* \in X^*$ such that $< x^*, Ax > = < y^*, x >$ for all $x \in D(A)\}$ and if $x^* \in D(A)$ then $y^* = A^* x^*$.

Let, also $\{T(t)\}_{t \geq 0}$ be a C_0 semigroup on X with generator $A : D(A) \subset X \to X$. For $t \geq 0$ let $T^*(t)$ be the adjoint operator on $T(t)$. Obviously $\{T^*(t)\}_{t\geq 0}$ satisfies the semigroup property and, therefore, is called the adjoint semigroup of $T(t)$. However, $T^*(t)$ does not need to be a C_0 semigroup on X^* since the mapping $T(t) \to T^*(t)$ does not necessarily conserve the strong continuity of $T(t)$. But if we denote by Y^* the closure $D(A^*)$ in X^* and by $T^+(t)$ the restriction of $T^*(t)$ to Y^* then we have that $T^+(t)$ is a C_0–semigroup on Y^*.

In the following we also need the notion of point spectrum of A denoted by $\sigma_p(A)$ and defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not one-to-one}\}$$

and residual spectrum

$$\sigma_r(A) = \{\lambda \in \mathbb{C} \mid \text{range}(\lambda I - A) \text{ is not dense in } X\}.$$

Before stating our main result let us give an auxiliary useful result [10].

Lemma 3.1. *Let $T(t)$ be a C_0-semigroup and let A be its infinitesimal generator. If*

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x\,ds$$

then

$$(\lambda I - A)B_\lambda(t)x = e^{\lambda t}x - T(t)x \text{ for every } x \in X$$

and

$$B_\lambda(t)(\lambda I - A)x = e^{\lambda t}x - T(t)x \text{ for every } x \in D(A).$$

Theorem 3.2. *Let X be an H.I. Banach space and $\mathcal{T} = \{T(t)\}_{t\geq 0}$ a bounded C_0-semigroup with infinitesimal generator A, then $\mathcal{T}$ is strongly stable if and only if $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$.*

Proof. *The necessity.* Let's assume there exists $\lambda \in \mathbb{R}$ such that $i\lambda \in \sigma_r(A)$. As the range of $(\lambda I - A)$ is not dense in X, by Hahn–Banach theorem it results that there exists $x^* \in X^*$, $x^* \not\equiv 0$ so that $< x^*, (i\lambda I - A)x >= 0$ for all $x \in D(A)$.

By Lemma 3.1 we have

$$< x^*, e^{i\lambda t}x - T(t)x >= 0 \text{ for all } x \in D(A)$$

or

$$< e^{i\lambda t}x^*, x >=< T^*(t), x^*, x >, \ x \in D(A).$$

Thus, there exists $x^* \in X^*$ so that $T^*(t)x^* = e^{i\lambda t}x^*$, $t \geq 0$. Let $x \in X$, be with $< x^*, x >= 1$ then $< T(t)x, x^* >= e^{i\lambda t}$ $(t \geq 0)$. Hence $T(t)$ is not stable.

The sufficiency. Let $A : D(A) \subset X \to X$, as A is the generator of a C_0-semigroup it results that $\{\lambda \in \mathbb{C} \mid Re\,\lambda > \omega\} \subseteq \rho(T)$ for some $\omega \in \mathbb{R}$ [10].

Therefore $\rho(A) \neq 0$ and $(A, D(A))$ is closed. If $D(A) = X$, then A is bounded by the closed graph theorem and by Theorem 2.7, $\sigma(A)$ is finite or consists of a sequence of eigenvalues converging to λ_A (where λ_A is the infinitely singular point of A). If $D(A) \neq X$ then let's take $\alpha \in \rho(A)$ and define the resolvent of A, $R(\alpha, A) = (\alpha I - A)^{-1}$. As $R(\alpha, A)$ is a bounded linear operator it results, by the same theorem, that the spectrum $\sigma(R(\alpha, A)) = \{\alpha_n\}_{n=1}^{\infty} \cup \{\alpha_R\}$, where α_n is a sequence of eigenvalues of $R(\alpha, A)$ and α_R is the infinitely singular value of $R(\alpha, A)$. But, by the spectral mapping theorem, [11] it results that $\sigma(R(\alpha, A)) = \{0\} \cup \{(\alpha - \lambda)^{-1} \mid \lambda \in \sigma(A)\}$. α_R must be 0, otherwise 0 is an eigenvalue for $R(\alpha, A)$ which is invertible, so that $R(\alpha, A)x = 0$ with $x \neq 0$ is impossible. Since eigenvalues of $R(\alpha, A)$ correspond to eigenvalues of A it results that $\alpha_n = (\alpha - \lambda_n)^{-1}$ and when $\alpha_n \to 0$ it implies $\lambda_n \to \infty$.

In conclusion, $\sigma(A)$ is finite (possibly empty) or consists of a sequence of eigenvalues $\{\lambda_n\}_{n \in N}$ that either converges to $\lambda_A \in \mathbb{C}$ or is unbounded. Using, now, a result from spectral theory (see also [2]), which is $\sigma_p(A) \cap i\mathbb{R} \subset \sigma_p(A^*) \cap i\mathbb{R}$ and taking into account the hypothesis it results that $\sigma_p(A) \cap i\mathbb{R} = \emptyset$. Therefore the intersection of the spectrum of A with the imaginary axis is either empty or is λ_A.

Now, in order to prove the strong stability of $\mathcal{T}$, we can use Arendt–Batty technique [2], but the proof is much easier as we don't need transfinite induction.

Let's first analyse the case when $\sigma(A) \cap i\mathbb{R} = \{\lambda_A\}$, $\lambda_A = i\lambda$ with $\lambda \in \mathbb{R}$. Rescaling $\mathcal{T}$ if necessary, we can assume that $\lambda \neq 0$.

Let $x \in X$ and denote by $F(t) = T(t)\left[T\left(\frac{2\pi}{|\lambda|}\right) - I\right]x$, $t \geq 0$ and by $f(z) = \int_0^{\infty} e^{-tz}F(t)dt$, $Re\, z > 0$ and $f_t(z) = \int_0^t e^{-sz}F(s)ds$, $Re\, z > 0$. Observe that if $T\left(\frac{2\pi}{|\lambda|} - I\right)x = y$ then $f(z) = \int_0^{\infty} e^{-tz}T(t)y\, dt = R(z, A)y$, $Re\, z > 0$, hence, the singular set of f is contained in $\{i\lambda\}$ and $f(0) = -A^{-1}y$. On the other hand, $\int_0^t F(s)ds = T(t)A^{-1}y - A^{-1}y$ and finally we have that $\|f_t(0) - f(0)\| = \|T(t)A^{-1}y\|$ for any y and $t \geq 0$.

Before studying the behaviour of $\|f_t(0) - f(0)\|$ as $t \to \infty$ by the help of the modified type of contour integral introduced by Newmann [12], let's remark an useful inequality:

$$\left\|\int_0^t e^{-i\lambda s}F(s)ds\right\| \leq \frac{4\pi}{|\lambda|}M\|x\|, \text{ where } M = \sup_{t \geq 0}\|T(t)\|. \tag{3.4}$$

It can be easily verified that:

$$\int_0^t e^{-i\lambda s}F(s)ds = \int_0^t e^{-i\lambda s}T\left(s+\frac{2\pi}{|\lambda|}\right)x\,ds - \int_0^t e^{-i\lambda s}T(s)x\,ds =$$
$$= \int_t^{t+\frac{2\pi}{|\lambda|}} e^{-i\lambda s}T(s)x\,ds - \int_0^{\frac{2\pi}{|\lambda|}} e^{-i\lambda s}T(s)x\,ds$$

and so

$$\left\|\int_t^{t+\frac{2\pi}{|\lambda|}} e^{-i\lambda s}T(s)x\,ds\right\| \leq \frac{2\pi}{|\lambda|}M\|x\|.$$

Hence we obtain (3.4).

Consider, now, the disjoint intervals $(-\infty,-R)$, $(\lambda-\varepsilon,\lambda+\varepsilon)$, $(R,+\infty)$ where R is any real number greater than 0 and $R-|\lambda|-\varepsilon>0$ and $|\lambda|-\varepsilon>0$.

And let's take a simply connected open set $D\supset\{z \mid Re\,z\geq 0,\ z\neq i\lambda\}$ such that f has a holomorphic extension on D and consider the following contour Γ in D given by $\Gamma = \bigcup_{i=1}^{4}\Gamma_i$ where $\Gamma_1 = \{|z| = R \mid Re\,z>0\}$, $\Gamma_2 = \{|z-i\lambda| = \varepsilon \mid Re\,z>0\}$ and Γ_3 is a smooth path joining iR to $i(\lambda+\varepsilon)$ and Γ_4 is a smooth path joining $i(\lambda-\varepsilon)$ to $-iR$, both Γ_3 and Γ_4 lying entirely (except the endpoints) within $D\cap\{Re\,z<0\}$ so that Γ is a closed simple contour with 0 in its interior.

To serve our purpose, we shall consider a holomorphic function in the interior of the domain delimitated by Γ which is $H(z) = h(z)e^{tz}(f(z)-f_t(z))$ where $h(z) = \left(1+\frac{z^2}{R^2}\right)\left(1+\frac{\varepsilon^2}{(z-i\lambda)^2}\right)\frac{\lambda^2}{\lambda^2-\varepsilon^2}$. And by Cauchy's theorem $f(0)-f_t(0) = H(0) = \frac{1}{2\pi i}\int_\Gamma \frac{H(z)}{z}dz$. It remains to estimate the integrand on different parts of Γ.

a) if $|z| = R$, $Re\,z>0$ that is $z = Re^{i\theta}$ with $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ we have:

(3.5)
$$\|(f_t(z)-f(z))e^{tz}\| = \left\|\int_t^\infty e^{-(s-t)z}F(s)ds\right\| =$$
$$= \left\|\int_0^\infty e^{-sz}F(s+t)ds\right\| \leq \frac{M}{R\cos\theta}\|y\|$$

And so,

(3.6)
$$\left|1+\frac{z^2}{R^2}\right| = 2\cos\theta$$

There is a unique $\alpha = \left[1 + \frac{\varepsilon^2}{(R - |\lambda|)^2}\right] \frac{\lambda^2}{\lambda^2 - \varepsilon^2}$ so that

$$|h(z)| \leq \left(1 + \frac{\varepsilon^2}{(R - |\lambda|)^2}\right) \frac{\lambda^2}{\lambda^2 - \varepsilon^2} \cdot 2\alpha \cos\theta = 2\alpha\cos\theta \tag{3.7}$$

And by (3.5), (3.6) and (3.7) we have that:

$$\left\| \int_{\substack{|z|=R; \\ Re\, z>0}} \frac{H(z)}{z} dz \right\| \leq \frac{2\pi M\alpha}{R} \|y\| \tag{3.8}$$

b) If $|z - i\lambda| = \varepsilon$, $Re\, z > 0$, $z = i\lambda + \varepsilon e^{i\theta}$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have:

$$\left\|(f(z) - f_t(z))e^{tz}\right\| = \left\|e^{tz} \int_t^\infty e^{-s\varepsilon e^{i\theta}} \left(e^{-i\lambda s} F(s)\right) ds\right\|.$$

And letting $G(s) = \int_0^s e^{-i\lambda u} F(u) du$, we obtain:

$$\begin{aligned} \|(f(z) - f_t(z))e^{tz}\| &= \left\|e^{tz}\left(-e^{-t\varepsilon e^{i\theta}} - \varepsilon e^{i\theta} \int_t^\infty e^{-s\varepsilon e^{i\theta}} G(s) ds\right)\right\| \leq \\ &\leq \frac{4\pi}{|\lambda|} M\|x\| \left(1 + \varepsilon \int_t^\infty e^{(t-s)\varepsilon\cos\theta} ds\right) \leq \frac{8\pi M}{|\lambda|\cos\theta} \|x\| \end{aligned} \tag{3.9}$$

And since

$$|h(z)| \leq \frac{4\lambda^2}{\lambda^2 - \varepsilon^2} \cdot \cos\theta \tag{3.10}$$

and

$$\frac{1}{|z|} \leq \frac{1}{|\lambda| - \varepsilon} \tag{3.11}$$

we obtain by (3.9), (3.10) and (3.11) that:

$$\left| \int_{\substack{|z-i\lambda|=\varepsilon \\ Re\, z>0}} \frac{H(z)}{z} \right| \leq \varepsilon \frac{32M|\lambda|\pi}{(\lambda^2 - \varepsilon^2)(|\lambda| - \varepsilon)} \|x\| = 32\pi\varepsilon M\beta\|x\|, \tag{3.12}$$

where $\beta = \frac{|\lambda|}{(\lambda^2 - \varepsilon^2)(|\lambda| - \varepsilon)}$.

c) Let's consider, now, the contour $\Gamma_3 \cup \Gamma_4$. As $0 \notin \Gamma_3 \cup \Gamma_4$ the function $\dfrac{h(z)f(z)}{z}$ is bounded for $z \in \Gamma_3 \cup \Gamma_4$ and also, as Γ_3 and Γ_4 are lying entirely within $D \cap \{\mathrm{Re}\, z < 0\}$ it follows that, for $z \in \Gamma_3 \cup \Gamma_4$, $\lim_{t\to\infty} e^{tz} = 0$ and, finally, by the bounded convergence theorem we have, $\lim_{t\to\infty} \int_{\Gamma_i} \dfrac{h(z)f(z)e^{tz}}{z} dz = 0$, $i = 1, 2$. But f_t is an entire function therefore

(3.13)

$$\int_{\Gamma_3 \cup \Gamma_4} \frac{h(z)f_t(z)e^{tz}}{z} dz = \int_{\substack{|z|=R \\ Re\, z<0}} \frac{h(z)f_t(z)e^{tz}}{z} dz + \int_{\substack{|z-i\lambda|=\varepsilon \\ Re\, z<0}} \frac{h(z)f_t(z)e^{tz}}{z} dz$$

For $z = Re^{i\theta}$ with $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ we have, as in (a), that:

$$\left\| f_t(z)e^{tz} \right\| \leq \frac{M}{R|\cos\theta|} \|y\|$$

And it results that:

$$\text{(3.14)} \qquad \left\| \int_{\substack{|z|=R \\ Re\, z<0}} \frac{h(z)f_t(z)e^{tz}}{z} dz \right\| \leq \frac{2\pi M\alpha}{R} \|y\|$$

For $|z - i\lambda| = \varepsilon$, $Re\, z < 0$, that is $z = i\lambda + \varepsilon e^{i\theta}$ and $\theta \in \left(\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ we have similarly to (b) that:

$$\text{(3.15)} \qquad \left\| \int_{\substack{|z-i\lambda|=\varepsilon \\ Re\, z<0}} \frac{h(z)f_t(z)e^{tz}}{z} dz \right\| \leq 32\pi\varepsilon M\beta\|x\|$$

Therefore, taking into account (3.8), (3.12), (3.13), (3.14) and (3.15) we obtain:

$$\text{(3.16)} \qquad \|f(0) - f_t(0)\| \leq \frac{1}{2\pi} \cdot 2 \left[\frac{2\pi M a}{R} \|y\| + 32\pi\varepsilon M\beta\|x\| \right]$$

And finally,

$$\text{(3.17)} \qquad \|f(0) - f_t(0)\| \leq 2\frac{M\alpha}{R}\|y\| + 32\varepsilon M\beta\|x\|$$

But as $\|f_t(0) - f(0)\| = \left\|T(t)A^{-1}y\right\|$, $t \geq 0$ and since $R > 0$ can be chosen arbitrarily large and ε arbitrarily small, it results that $\lim_{t\to\infty} \left\|T(t)A^{-1}y\right\| = 0$

where $y = \left[T\left(\frac{2\pi}{|\lambda|}\right) - I\right] x$. It rermains to show that $\left[T\left(\frac{2\pi}{|\lambda|}\right) - I\right]$ has dense range in X.

If, by the contrary, we suppose that $1 \in \sigma_r\left(T\left(\frac{2\pi}{|\lambda|}\right)\right)$ then by the Spectral Mapping Theorem, it results that $\exists n \in N$ so that $\lambda_n = in|\lambda| \in \sigma_r(A)$.

But as $\sigma_p(A^*) = \sigma_r(A)$, then $\sigma_r(A) \cap i\mathbb{R} = \emptyset$ so that we arrived to a contradiction, that means that $\left[T\left(\frac{2\pi}{|\lambda|}\right) - I\right]$ is dense in X and we have: $\lim_{t\to\infty} \|T(t)\omega\| = 0$ for any $\omega \in D(A)$. Since $D(A)$ is dense in X it results that $\lim_{t\to\infty} \|T(t)x\| = 0$ for any $x \in X$.

In the case when $\sigma(A) \cap i\mathbb{R} = \emptyset$, things are much more simplier. It's enough to take: $F(t) = T(t)x$, for any $x \in X$ and $t \geq 0$. Observe that $\|F(t)\| \leq M\|x\|$, where $M = \sup_{t\geq 0} \|T(t)\| < \infty$.

And using the same notations as in the precedent case we obtain that:

$$\|T(t)y\| = \|f_t(0) - f(0)\| \text{ for any } y \in D(A) \text{ and } t \geq 0.$$

In order to study the behaviour of $\|f_t(0) - f(0)\|$ as $t \to \infty$ we estimate the integral $\int_\Gamma \frac{H(z)}{z} dz$, where H is a holomorphic function in the interior of a domain with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and is defined by $H(z) = \left(1 + \frac{z^2}{R^2}\right) e^{tz} [f_t(z) - f(z)]$. And Γ is a contour in an open set $D \supset \{z \mid Re\, z \geq 0\}$ where f has a holomorphic extension and consists of:

$$\Gamma_1 = \{|z| = R \mid Re\, z > 0\}, \; R \in \mathbb{R}_+^*$$

and Γ_2 a smooth path joining iR to $-iR$ and lying entirely in D except the endpoints.

By estimating the integrand on Γ_1 and respectively on Γ_2 we finally obtain that:

$$\|f_t(0) - f(0)\| \leq \frac{2M}{R} \|x\|. \tag{3.18}$$

It easily results that $\mathcal{T}$ is strongly stable.

4 An application to the stability theorem

In the following, let us consider a bounded C_0–semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A, on a reflexive Banach space X.

Then by Jacobs–Glicksberg–de Leeuw Theorem [13] the space X splits into two closed $T(t)$–invariant subspaces $X = X_0 \oplus X_r$ for all $t \geq 0$.

X_0 is called the irreversible part of X and is defined as:

$$X_0 = \{x \in X \mid 0 \text{ is a cluster point for the weak topology of } \{T(t)x\}_{t\geq 0}\}$$

X_r is the reversible part of X and is equal with

$$X_r = \overline{\text{span}}\{x \in D(A) \mid \exists i\lambda \in i\mathbb{R},\ Ax = i\lambda x\}$$

For each $t \geq 0$, let us denote by $T_0(t)$, the restriction of $T(t)$ to X_0 and by $T_r(t)$, the restriction of $T(t)$ to X_r. It is already proved that $(T_0(t))_{t\geq 0}$ and $(T_r(t))_{t\geq 0}$ are also bounded C_0–semigroups with generator A_0 and respectively A_r.

Theorem 4.1. *Let X be a reflexive H.I. Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a bounded C_0–semigroup with generator A. Then $\mathcal{T}_0 = (T_0(t))_{t\geq 0}$ is always strongly stable.*

Proof. First let us remark that relfexive H.I. Banach spaces exist (see, for instance, Gowers' example [7]).

By the definition of X_0, it results that A_0 has no eigenvalues on the imaginary axis, $\sigma_p(A_0) \cap i\mathbb{R} = \emptyset$.

Let us consider $\lambda \in \sigma_r(A_0)$. As $\overline{(\lambda I - A_0)X_0} \neq X_0$, by Hahn–Banach theorem, it results there exists $f \in X_0^*$ with $f \neq 0$ so that

$$< (\lambda I - A_0)x, f >= 0, \quad \text{for any } x \in X_0$$

and it follows that

$$< x, (\lambda I^* - A_0^*)f >= 0, \ x \in X$$

And finally

$$A_0^* f = \lambda f$$

Therefore, $\lambda \in \sigma_p(A_0^*)$ and so

(4.1) $$\sigma_r(A_0) \subseteq \sigma_p(A_0^*)$$

On the other hand, we have [2.3;2]

(4.2) $$\sigma_p(A_0) \cap i\mathbb{R} \subseteq \sigma_r(A_0)$$

By (4.1) and (4.2) it results

$$\sigma_p(A_0) \cap i\mathbb{R} \subseteq \sigma_p(A_0^*) \tag{4.3}$$

As X is reflexive, applying (4.3) to A_0^* we have

$$\sigma_p(A_0^*) \cap i\mathbb{R} \subseteq \sigma_p(A_0^{**}) = \sigma_p(A_0) \tag{4.4}$$

And (4.3), (4.4) imply

$$\sigma_p(A_0^*) \cap i\mathbb{R} = \sigma_p(A_0) \cap i\mathbb{R} = \emptyset$$

Suppose, now, that X_0 is an infinite subspace. As X is an H.I. space, it results that also X_0 is an H.I. space. Thus T_0 is a bounded C_0–semigroup defined on an H.I. Banach space and the adjoint of its generator has no eigenvalues on the imaginary axis. By Theorem 3.2 we may conclude that T_0 is strongly stable.

In the case X_0 a finite dimensional space, then it easily results the same conclusion.

Remark 4.1. If A_0 is unbounded then T is strongly stable on a finite codimension subspace of X.

References

[1] B.Sz. Nagy and C. Foiaş, *Harmonic analysis of operator on Hilbert Spaces*, North–Holland, Amsterdam (1970).

[2] W. Arendt and C.J.K. Batty, *Tauberian theorems and stability of one–parameter semigroup*, Trans. Amer. Math. Soc. 306 (1988), 837–852.

[3] Yu.I. Lyubich and Yũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. 88 (1988), 37–42.

[4] S. Caraman, *Stability of one–parameter semigroup of operators on Banach spaces*, Studii şi Cercetări, to appear.

[5] S. Huang and F. Räbiger, *Superstable C_0–semigroups on Banach spaces*, Tübinger Berichte zur Funk. Anal. Heft 1 (1991–92), 73–82.

[6] J. Lindenstrauss, *Some aspects of the theory of Banach spaces*, Adv. Math. 5 (1970), 159–180.

[7] W.T. Govers & B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. vol. 6, no. 4 (1993), 851-874.

[8] P. Enflo, *A counterexample to the approximation property in Banach spaces*, Acta Math. 130 (1973), 309–317.

[9] H.R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monograph no. 12, Academic Press, New York (1978).

[10] A. Pazy, *Semigroups of linear operators,* London Math., Soc. Monog. no. 12, Academic Press, New York, 1978.

[11] T. Kato, *Perturbation theory for linear operators*, Springer–Verlag, Berlin, Heidelberg, New York, 1976.

[12] D.J. Newman, *Simple analytic proof of the prime number theorem*, Amer. Math. Monthly 87, no.9 (1981), 636–696.

[13] U. Krengel, *Ergodic theorems*, De Gruyte, Berlin–New York, 1985.

Technical University "Gh. Asachi"
Department of Mathematics
Bd. Copou nr. 11
6600, Iaşi, Romania