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On the decomposition of curvature tensor in recurrent conformal Finsler space

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M.S. Knebelman [1]1) has developed conformal geometry of generalised metric spaces. The projective tensor and curvature tensors in conformal Finsler spaces were discussed by R.B.Misra [3,4]. M.Gama [6] has decomposed recurrent curvature tensor in an areal space of submetric class. The decomposition of recurrent curvature tensor in Finsler space was studied by B.B.Sinha and S.P.Singh [5]. The purpose of the present paper is to decompose the recurrent conformal curvature tensor and study the properties of conformal decomposition tensors.

1. Introduction

Let us consider an n-dimensional Finsler space F_n in which two distinct metric functions are represented by $F(x,\dot{x})$ and $\overline{F}(x,\dot{x})$. Then the corresponding metric tensors $g_{ij}(x,\dot{x})$ and $\overline{g}_{ij}(x,\dot{x})$ are called conformal if they are proportional to each other. These quantities are related as under:

(1.1) (a) $\overline{F}(x,\dot{x}) = e^{\sigma} F(x,\dot{x})$ (b) $\overline{g}_{ij}(x,\dot{x}) = e^{2\sigma} g_{ij}(x,\dot{x})$

(c) $\overline{g}^{ij}(x,\dot{x}) = e^{-2\sigma} g^{ij}(x,\dot{x})$,

where $\sigma = \sigma(x)$ is atmost a point function as shown by Knebelman [1]. The space equiped with quantities $\overline{F}(x,\dot{x}), \overline{g}(x,\dot{x})$ etc is called a conformal Finsler space usually denoted by $\overline{F}_n$

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1) The numbers in brackets refer to the references given at the end of the paper

2) $\partial_j = \partial / \partial x^j$ and $\dot{\partial}_j = \partial / \partial \dot{x}^j$

In a Finsler space F_n with entities $F(x,\dot{x}), g_{ij}(x,\dot{x})$ etc, the covariant derivatives of a vector $T^i(x,\dot{x})$ with respect to x^j in the sense of Cartan and Berwald are given by [2].

(1.2) $$T^i_{|j} \stackrel{def}{=} \partial_j T^i - (\dot{\partial}_m T^i) G^m_j + T^m \Gamma^{*i}_{mj} {}^{2)}$$

and

(1.3) $$T^i_{(j)} \stackrel{def}{=} \partial_j T^i - (\dot{\partial}_m T^i) G^m_j + T^m G^i_{mj},$$

where

(1.4) $$G^i_j(x,\dot{x}) = G^i_{mj}(x,\dot{x})\dot{x}^m = \Gamma^{*i}_{mj}(x,\dot{x})\dot{x}^m.$$

The connection coefficients are homogeneous functions of degree zero in $\dot{x}^i$. Considering quantities

(1.5) $$\sigma_m(x) \stackrel{def}{=} \partial_m \sigma, \qquad B^{ij}(x,\dot{x}) \stackrel{def}{=} \frac{1}{2} F^2 g^{ij} - \dot{x}^i \dot{x}^j ,$$

$$U^i_{jk}(x,\dot{x}) \stackrel{def}{=} 2\sigma_{(j}\delta^i_{k)} - \sigma_m \left\{ g^{im} g_{jk} - 2C^i_{r(j}\dot{\partial}_{k)} B^{rm} + g^{ir} C_{jks} \dot{\partial}_r B^{sm} \right\},$$

where $C^i_{mj}(x,\dot{x}) = g^{ih} C_{mhj}(x,\dot{x})$, one can have conformal connection coefficients :

(1.6) $$\overline{\Gamma}^{*i}_{jk}(x,\dot{x}) = \Gamma^{*i}_{jk}(x,\dot{x}) + U^i_{jk}(x,\dot{x}) ,$$

(1.7) $$\overline{G}^i_{jk}(x,\dot{x}) = G^i_{jk}(x,\dot{x}) - \sigma_m \dot{\partial}_j \dot{\partial}_k B^{im},$$

where the functions $B^{im}(x,\dot{x})$ are homogeneous of degree two in $\dot{x}^i$.

The variation in $G^i_{jkh}(x,\dot{x})$ under the conformal change (1.1) is given by

(1.8) $$\overline{G}^i_{jkh}(x,\dot{x}) = G^i_{jkh}(x,\dot{x}) - \sigma_m \dot{\partial}_j \dot{\partial}_k \dot{\partial}_h B^{im} ,$$

where
$$\overline{G}^i_{jkh}(x,\dot{x}) \stackrel{def}{=} \dot{\partial}_h G^i_{jk} .$$

The curvature tensors K^i_{ljk}, H^i_{ljk} and H^i_{kh} transform under the conformal change (1.1) as

(1.9) $$\overline{K}^i_{rjk} = K^i_{rjk} + 2U^i_{r[j|k]} + 2\left[\sigma_m \left\{\dot{\partial}_s \left(\Gamma^{*i}_{r[j} + U^i_{r[j}\right)\right\}\dot{\partial}_{k]} B^{sm} + U^i_{s[k} U^s_{j]r}\right]$$

$$(1.10)\qquad \overline{H}^i_{rjk} = H^i_{rjk} + 2\sigma_{m[(j)}\dot{\partial}_{k]}\dot{\partial}_r B^{im} - 2\sigma_m\{\dot{\partial}_r(\dot{\partial}_{[j}B^{im})_{(k)]} - (\dot{\partial}_{[j}B^{is})G^m_{k]rs}\} + 2\sigma_m\sigma_s\dot{\partial}_r\{(\dot{\partial}_{[j}B^{pm})\dot{\partial}_{k]}\dot{\partial}_p B^{is}\}$$

and

$$(1.11)\qquad \overline{H}^i_{kh} = H^i_{kh} - 2\sigma_m\{\dot{\partial}_{[k}B^{im}\}_{(h)]} + 2\sigma_{m[(k)}\dot{\partial}_{h]}B^{im} + 2\sigma_m\sigma_r(\dot{\partial}_{[k}B^{sm})\dot{\partial}_{h]}\dot{\partial}_s B^{ir}$$

respectively.

The Bianchi identities satisfied by $\overline{H}^i_{jk}, \overline{H}^i_{rjk}$ and $\overline{K}^i_{rjk}$ are [4]

$$(1.12)\qquad \overline{H}^i_{[jk(\bar{h})]} = 0\ ,$$

$$(1.13)\quad H^i_{r[jk(\bar{h})]} = \{H^p_{[kj} - \sigma_{q[(j)}\dot{\partial}_k B^{pq} + \sigma_{q[(k)}\dot{\partial}_j B^{pq}\}G^i_{h]pr} + \sigma_m[\{(\dot{\partial}_{[j}B^{pm})_{(k)} - (\dot{\partial}_{[k}B^{pm})_{(j)}\}G^i_{h]pr} + \{H^p_{[jk} + \sigma_{q[(j)}\dot{\partial}_k B^{pq} - \sigma_{q[(k)}\dot{\partial}_j B^{pq}\}\dot{\partial}_{h]}\dot{\partial}_b\dot{\partial}_r B^{im}\ ] - \sigma_m\sigma_t[G^i_{pr[j}\{(\dot{\partial}_k B^{st})\dot{\partial}_{h]}B^{pm} - (\dot{\partial}_h B^{st})\dot{\partial}_{k]}\dot{\partial}_s B^{pm}\ \} + (\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{it})\{(\dot{\partial}_k B^{pm})_{(h)]} - (\dot{\partial}_h B^{pm})_{(k)]}\}\ ] + \sigma_m\sigma_q\sigma_t(\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{iq})\{(\dot{\partial}_{k]}B^{st})\dot{\partial}_h\dot{\partial}_s B^{pm} - (\dot{\partial}_h B^{st})\dot{\partial}_{k]}\dot{\partial}_s B^{pm}\}.$$

and

$$(1.14)\quad \overline{K}^i_{r[jk|\bar{h}]} + (\dot{\partial}_p\Gamma^{*i}_{r[j})H^p_{kh]} + (\dot{\partial}_p U^i_{r[j})H^p_{kh]} + \{\dot{\partial}_p(\Gamma^{*i}_{r[j} + U^i_{r[j})\}[\{(\dot{\partial}_h B^{pm})_{(k)]} - (\dot{\partial}_k B^{pm})_{(h)]}\}\sigma_m) + \{(\dot{\partial}_k B^{sm})\dot{\partial}_{h]}\dot{\partial}_s B^{pt} - (\dot{\partial}_h B^{sm})\dot{\partial}_{k]}\dot{\partial}_s B^{pt}\}\sigma_m\sigma_t + \sigma_{|m|(k)}\dot{\partial}_{h]}B^{pm} - \sigma_{|m|(h)}\dot{\partial}_{k]}B^{pm}(\ ] = 0$$

respectively . The notations $(\bar{k})$ and $|\bar{h}$ represent covariant differentiation in the sense of Berwald and Cartan respectively in conformal Finsler space $\overline{F}_n$.

2. Decomposition of conformal curvature tensor $\overline{H}^i_{rjk}$

The recurrent conformal curvature tensor $\overline{H}^i_{rjk}$ is characterised by the condition

$$(2.1)\qquad \overline{H}^i_{rjk(\bar{m})} = \overline{V}_m\overline{H}^i_{rjk}\ , \qquad \overline{H}^i_{rjk} \neq 0\ ,$$

where the barred index enclosed within the round bracket denotes the covariant derivative with respect to x's for the connection parameters

$\overline{G}^i_{jk}(x,\dot{x})$. The covariant vector $\overline{V}_m(x,\dot{x})$ is called the conformal recurrence vector. The conformal space equipped with such curvature tensor is called recurrent conformal Finsler space and we denote it by $\overline{F}^*_n$.

We consider the decomposition of the recurrent conformal curvature tensor in the form

(2.2) $$\overline{H}^i_{rjk} = \overline{X}^i \overline{\Phi}_{rjk},$$

where $\overline{\Phi}_{rjk}$ is a homogeneous conformal decomposition tensor and $\overline{X}^i$ is a non-zero conformal vector such that

(2.3) $$\overline{X}^i \overline{V}_i = 1$$

Sinha and Singh [5] have decomposed the recurrent curvature tensor H^i_{rjk} in similar manner

(2.4)(a) $$H^i_{rjk} = X^i \Phi_{rjk},$$

where the decomposition vector X^i also satisfies the condition

(2.5) $$X^i V_i = 1.$$

Transvecting (2.,4)(a) by $\dot{x}^r$ and noting $H^i_{rjk}\dot{x}^r = H^i_{jk}$ [2], We obtain

(2.4)(b) $$H^i_{jk} = X^i \Phi_{jk},$$

where $\Phi_{jk} = \Phi_{rjk}\dot{x}^r$.

The decomposition tensor Φ_{rjk} satisfies the identities

(2.6) $$\Phi_{rjk} = -\Phi_{rkj}$$

and

(2.7) $$\Phi_{rjk} + \Phi_{jkr} + \Phi_{krj} = 0.$$

We notice that the decomposition vector X^i and the recurrence vector V_i are transformed conformally as under :

(2.8) $$\overline{X}^i = e^{-\sigma} X^i$$

and

(2.9) $$\overline{V}_i = e^{\sigma} V_i$$

respectively.

Applying equations (2.2) and (2.4)(a) in equation (1.10), we obtain

(2.10) $$\overline{X}^i \overline{\Phi}_{rjk} = X^i \Phi_{rjk} + 2\sigma_{m[(j)}\dot{\partial}_{k]}\dot{\partial}_r B^{im} - 2\sigma_m \{\dot{\partial}_r (\dot{\partial}_{[j} B^{im})_{(k)]} - (\dot{\partial}_{[j} B^{is}) G^m_{k]rs}\} + 2\sigma_m \sigma_s \dot{\partial}_r \{(\dot{\partial}_{[j} B^{pm}) \dot{\partial}_{k]} \dot{\partial}_p B^{is}\}.$$

Transvecting (2.10) by $\overline{V}_i$ and using the relations (2.3), (2.5), and (2.9), it yields

(2.11) $$\overline{\Phi}_{rjk} = e^{\sigma}[\Phi_{rjk} + 2\sigma_{m[(j)}\dot{\partial}_{k]}\dot{\partial}_r B^{im} V_i - 2\sigma_m \{\dot{\partial}_r (\dot{\partial}_{[j} B^{im})_{(k)]} - (\dot{\partial}_{[j} B^{is}) G^m_{k]rs}\} V_i + \sigma_m \sigma_s \dot{\partial}_r \{(\dot{\partial}_{[j} B^{pm}) \dot{\partial}_{k]} \dot{\partial}_p B^{is}\} V_i \;],$$

which represents the conformal transformation of the decomposition tensor Φ_{rjk} under the change (1.1)

Thus we state

Theorem 2.1 : Under the decomposition (2.2), the conformal decomposition tensor Φ_{rjk} is expressed in the form (2.11).

Interchanging the indices j and k in the equation (2.11), we get

(2.12) $$\overline{\Phi}_{rjk} = -\overline{\Phi}_{rkj}$$

in view of (2.6).

The cyclic permutation of the indices r ,j ,k in the equation (2.11) yields the identity

(2.13) $$\overline{\Phi}_{[rjk]} = e^{\sigma} V_i [\sigma_m \dot{\partial}_{[r} (\dot{\partial}_k B^{im})_{(j)]} - \sigma_m \dot{\partial}_{[r} (\dot{\partial}_j B^{im})_{(k)]} + \sigma_m \sigma_s \dot{\partial}_{[r} \{(\dot{\partial}_j B^{pm}) \dot{\partial}_{k]} \dot{\partial}_p B^{is}\} - \sigma_m \sigma_s \dot{\partial}_{[r} \{(\dot{\partial}_k B^{pm}) \dot{\partial}_{j]} \dot{\partial}_p B^{is}\} \;]$$ [3)]

by applying (2.7) and the symmetry property of G^i_{rjk} .

Hence we state

Theorem 2.2 : Under the decomposition (2.2) ,the conformal decomposition tensor $\overline{\Phi}_{rjk}$ satisfies the identities (2.12) and (2.13) .

Transvecting the equation (2 .2) by $\dot{x}^r$, we have

(2.14) $$\overline{H}^i_{jk} = \overline{X}^i \overline{\Phi}_{jk},$$

where

(2.15) $$\overline{\Phi}_{jk} = \overline{\Phi}_{rjk} \dot{x}^r$$

and $\overline{H}^i_{jk}$ satisfies the relation (1.11).

Applying the equation (2.5),(2.9) and (2.14) in the equation (1.11), it assumes the form

(2.16) $$\overline{\Phi}_{kh} = e^{\sigma} V_i \left[H^i_{kh} - 2\sigma_m \{ \dot{\partial}_{[k} B^{im} \}_{(h)]} + 2\sigma_{m[(k)} \dot{\partial}_{h]} B^{im} + 2\sigma_m \sigma_r \left(\dot{\partial}_{[k} B^{sm} \right) \dot{\partial}_{h]} \dot{\partial}_s B^{ir} \right].$$

Interchanging the indices k and h in the equation (2.16) ,we get

(2.17) $$\overline{\Phi}_{kh} = -\overline{\Phi}_{hk}$$

by virtue of the relation $H^i_{kh} = -H^i_{hk}$ [2].

In view of the equations (2.4)(b) and (2.5), the equation (2.16) becomes

(2.18) $$\overline{\Phi}_{kh} = e^{\sigma} \left[\Phi_{kh} - 2\sigma_m \{ \dot{\partial}_{[k} B^{im} \}_{(h)]} V_i + 2\sigma_{m[(k)} \dot{\partial}_{h]} B^{im} V_i + 2\sigma_m \sigma_r \left(\dot{\partial}_{[k} B^{sm} \right) \dot{\partial}_{h]} \dot{\partial}_s B^{ir} V_i \right],$$

which gives the conformal transformation of the decomposition tensor Φ_{kh} under the change (1.1)

Accordingly, we have

Theorem 2.3 : Under the decomposition (2.2) and (2.14) the conformal decomposition tensor $\overline{\Phi}_{kh}$ is expressed in the form (2.18)

In view of the equations (2.2),(2.3),(2.4),(2.9) and (2.14), the identities (1.12) and (1.13) assume the forms

(2.19) $$\overline{\Phi}_{[jk(h)]} = 0$$

and

(2.20)
$$\overline{\Phi}_{r[jk(\overline{h})]} = e^{\sigma}V_i\{X^p\Phi_{[kj} - \sigma_{q[(j)}\dot{\partial}_k B^{pq} + \sigma_{q[(k)}\dot{\partial}_j B^{pq}\}G^i_{h]pr}$$
$$+ e^{\sigma}V_i\Big[\{(\dot{\partial}_{[j}B^{pm})_{(k)} - (\dot{\partial}_{[k}B^{pm})_{(j)}\}G^i_{h]pr} + \{X^p\Phi_{[jk} + \sigma_{q[(j)}\dot{\partial}_k B^{pq}$$
$$- \sigma_{q[(k)}\dot{\partial}_j B^{pq}\}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_r B^{im}\Big] - e^{\sigma}V_i\sigma_m\sigma_t\Big[G^i_{pr[j}\{(\dot{\partial}_k B^{st})\dot{\partial}_{h]}\dot{\partial}_s B^{pm}$$
$$- (\dot{\partial}_h B^{st})\dot{\partial}_{k]}\dot{\partial}_s B^{pm}\} + (\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{it})\{(\dot{\partial}_k B^{pm})_{(h)]} - (\dot{\partial}_h B^{pm})_{(k)]}\}\Big]$$
$$+ e^{\sigma}V_i\sigma_m\sigma_q\sigma_t(\dot{\partial}_p\dot{\partial}_r\partial_{[j}B^{iq})\{(\dot{\partial}_k B^{st})\dot{\partial}_{h]}\dot{\partial}_s B^{pm} - (\dot{\partial}_h B^{st})\dot{\partial}_{k]}\dot{\partial}_s B^{pm}\}$$

respectively . Hence we have

Theorem 2.4 : Under the decomposition (2.2) and (2.14), the conformal decomposition tensors $\overline{\Phi}_{jk}$ and $\overline{\Phi}_{rjk}$ satisfy the Bianchi identities (2.19) and (2.20) respectively .

Differentiating (2.2) covariantly with respect to x^h in the sense of Berwald, we get

(2.21)
$$\overline{H}^i_{rjk(\overline{h})} = \overline{X}_{(\overline{h})}\overline{\Phi}_{rjk} + \overline{X}^i\overline{\Phi}_{rjk(\overline{h})} .$$

Applying (2.1) and (2.2) in the above equation, we find

(2.22)
$$\overline{X}^i\overline{\Phi}_{rjk}\overline{V}_h = \overline{X}_{(\overline{h})}\overline{\Phi}_{rjk} + \overline{X}^i\overline{\Phi}_{rjk(\overline{h})} .$$

Let us assume that the conformal vector $\overline{X}^i$ is covariant constant, then (2.22) reduces to

(2.23)
$$\overline{\Phi}_{rjk(\overline{h})} = \overline{V}_h\overline{\Phi}_{rjk}$$

Conversely, if the above equation is true, the equation (2.22) yields

$$\overline{X}^i_{(\overline{h})}\overline{\Phi}_{rjk} = 0 .$$

Since $\overline{\Phi}_{rjk}$ is non-zero conformal decomposition tensor ,it implies

(2.25)
$$\overline{X}^i_{(\overline{h})} = 0 ,$$

which shows that $\overline{X}^i$ is covariant constant in the space $\overline{F}^*_n$.

Transvecting (2.23) by $\dot{x}^r$ and using (2.15), we obtain

(2.26) $$\overline{\Phi}_{jk(\overline{h})} = \overline{V}_h \overline{\Phi}_{jk}$$

Thus we state

Theorem 2.5 : In a recurrent conformal Finsler space $\overline{F}_n^*$, the necessary and sufficient condition for the conformal decomposltion tensor field s $\overline{\Phi}_{rjk}$ and $\overline{\Phi}_{jk}$ to be recurrent is that the conformal vector field $\overline{X}^i$ is covariant constant in the sense of Berwald .

Decomposition of conformal curvature tensor $\overline{K}_{rjk}^i$

In this section we consider the decomposition of the conformai curvature tensor $\overline{K}_{rjk}^i$. In similar manner, the recurrent conformal curvature tensor $\overline{K}_{rjk}^i$ is characterised by

(3.1) $$\overline{K}_{rjk|\overline{h}}^i = \overline{V}_h \overline{K}_{rjk}^i, \quad \overline{K}_{rjk}^i \neq 0 ,$$

where the symbol $|\overline{h}$ denotes the covariant derivative with respect to x^i for the connection coefficients $\overline{\Gamma}_{jk}^{*i}(x,\dot{x})$. The non-zero covariant vector $\overline{V}_h(x,\dot{x})$ is called the conformal recurrence vector .

We decompose the recurrent conformal curvature tensor $\overline{K}_{rjk}^i$ in the following manner :

(3.2) $$\overline{K}_{rjk}^i = \overline{X}^i \overline{\Psi}_{rjk} ,$$

where $\overline{\Psi}_{rjk}$ is conformal decomposition tensor and $\overline{X}^i$ is conformal vector which satisfies the relation (2.3).

In a recurrent Finsler space, if the curvature tensor K_{rjk}^i is decomposed as

(3.3) $$K_{rjk}^i = X^i \Psi_{rjk} , \quad K_{rjk}^i \neq 0$$

where the decomposition vector X^i satisfies the relation (2.5), then the decomposition vector X^i and the recurrence vector V_i are transformed conformally in the form (2.8) and (2.9) respectively .

In view of the identities $K^i_{rjk} = -K^i_{rkj}$ and $K^i_{[rjk]} = 0$ [2], the decomposition tensor Ψ_{rjk} satisfies the identities

(3.4) $$\Psi_{rjk} = -\Psi_{rkj}\ ,$$

(3.5) $$\Psi_{[rjk]} = 0.$$

Using the equations (3.2) and (3.3) in the equation (1.9) , it assumes the form

(3.6) $$\overline{X}^i\overline{\Psi}_{rjk} = X^i\Psi_{rjk} + 2U^i_{r[j|k]} + 2\left[\sigma_m\left\{\dot{\partial}_s\left(\Gamma^{*i}_{r[j} + U^i_{r[j}\right)\dot{\partial}_{k]}B^{sm}\right\} + U^i_{s[k}U^s_{j]_r}\right].$$

Transvecting the above equation by $\overline{V}_i$ and applying the equations (2.3),(2.5) and (2.9), it becomes

(3.7) $$\overline{\Psi}_{rjk} = e^{\sigma}\Psi_{rjk} + 2e^{\sigma}U^i_{r[j|k]} + 2e^{\sigma}\left[\sigma_m\left\{\dot{\partial}_s\left(\Gamma^{*i}_{r[j} + U^i_{r[j}\right)\right\}\dot{\partial}_{k]}B^{sm} + U^i_{s[k}U^s_{jr]}\right],$$

which gives the conformal transformation of the decomposition tensor Ψ_{rjk} under the change (1.1) .

Accordingly, we have

Theorem 3.1: Under the decomposition (3.2), the decomposition tensor $\overline{\Psi}_{rjk}$ is expressed in the form (3.7).

Interchanging the indices j and k in the equation (3.7) and noting (3.4), we get

(3.8) $$\overline{\Psi}_{rjk} = -\overline{\Psi}_{rkj}\ .$$

Also the cyclic permutation of the indices r , j ,k in the equation (3.7) yields

(3.9) $$\overline{\Psi}_{[rjk]} = 0$$

in view of (3.5) .

Thus we have

Theorem 3.2: Under the decomposition (3.2), the conformal decomposition tensor $\overline{\Psi}_{rjk}$ satisfies the identities (3.8) and (3.9).

Applying (2.3),(2.4)(b),(2.5),(2.9) and (3.2) in the Bianchi identity (1.14), it assumes the form

(3.10)
$$\overline{\Psi}_{r[jk|\bar{h}]}+e^{\sigma}V_i\left(\dot{\partial}_p\Gamma^{*i}_{r[j}\right)X^p\Phi_{kh]}+e^{\sigma}V_i\left(\dot{\partial}_pU^i_{r[j}\right)X^p\Phi_{kh]}$$
$$+e^{\sigma}V_i\left\{\dot{\partial}_p\left(\Gamma^{*i}_{r[j}+U^i_{r[j}\right)\right\}\left[\left\{\left(\dot{\partial}_hB^{pm}\right)_{(k)]}-\left(\dot{\partial}_kB^{pm}\right)_{(h)]}\right\}\sigma_m\right.$$
$$+\left\{\left(\dot{\partial}_kB^{sm}\right)\dot{\partial}_{h]}\dot{\partial}_sB^{pt}-\left(\dot{\partial}_hB^{sm}\right)\dot{\partial}_{k]}\dot{\partial}_sB^{pt}\right\}\sigma_m\sigma_t$$
$$\left.+\sigma_{|m|(k)}\dot{\partial}_{h]}B^{pm}-\sigma_{|m|(h)}\dot{\partial}_{k]}B^{pm}\right]=0\ .$$

Hence we state

Theorem 3.3 : Under the decomposition (3.2) the conformal decomposition tensor $\overline{\Psi}_{rjk}$ satisfies the Bianchi identity (3.10).

Taking covariant differentiation of (3.2) with respect to x^h in the sense of Cartan ,it yields

(3.11)
$$\overline{K}^i_{rjk|\bar{h}}=\overline{X}^i_{|\bar{h}}\overline{\Psi}_{rjk}+\overline{X}^i\overline{\Psi}_{rjk|\bar{h}}\ .$$

In view of the equation (3.1), it becomes

(3.12)
$$\overline{V}_h\overline{K}^i_{rjk}=\overline{X}^i_{|\bar{h}}\overline{\Psi}_{rjk}+\overline{X}^i\overline{\Psi}_{rjk|\bar{h}}\ .$$

Using the decomposition (3.2) in the above equation, we get

(3.13)
$$\overline{X}^i\overline{V}_h\overline{\Psi}_{rjk}=\overline{X}^i_{|\bar{h}}\overline{\Psi}_{rjk}+\overline{X}^i\overline{\Psi}_{rjk|\bar{h}}\ .$$

Let us assume that the conformal vector $\overline{X}^i$ is covariant constant, that is, $\overline{X}^i_{|\bar{h}}=0$. Then the equation (3.13) reduces to

(3.14)
$$\overline{\Psi}_{rjk|\bar{h}}=\overline{V}_h\overline{\Psi}_{rjk}\ .$$

Conversely , if the above relation is true ,then from the equation (3.13) , we find

(3.15)
$$\overline{X}^i_{|\bar{h}}\ \overline{\Psi}_{rjk}=0\ .$$

Since $\overline{\Psi}_{rjk}$ is non-zero, it implies

(3.16)
$$\overline{X}^i_{|\bar{h}}=0,$$

which implies that the conformal vector $\overline{X}^i$ is covariant constant.
We have

Theorem 2.4 : Under the decomposition (3.2), the necessary and sufficient condition for the conformai decomposition tensor field $\overline{\overline{\Psi}}_{rjk}$ to be recurrent is that the conformai vector $\overline{X}^i$ is covariant constant in the sense of Cartan.

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