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Motion with contra field in Finsler spaces II

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An affine motion of special type in non-Riemannian K*-spaces was developed by Takano [6]* which he called an affine motion with contra field. Later on , the same concept was extended to Finsler spaces by Misra and Meher[2]. The same authors [3] have discussed CA-motion in projective symmetric Finsler space.The present author and Gatoto[5] have studied projective motion in Finsler spaces. The object of the present paper is to define CA and CP motions in Finsler space F_n. The notations of Hiramatu[1] and Singh[4] are used in the sequel.

1. INTRODUCTION

In an n-dimentional Finsler space F_n,we consider an infinitesimal transformation[7]

(1.1) $$\bar{x}^i = x^i + \xi^i(x)dt,$$

where $\xi^i(x)$ is a contravariant vector field and dt is an infinitesimal constant.

The covariant derivatives of a scalar $f(x,\dot{x})$ and a tensor $T^i_{jk}(x,\dot{x})$ ** are given by

(1.2) $$f_{;l} = f_{,l} - f|_a \Gamma^{*a}_{bl} \dot{x}^b$$

and

(1.3) $$T^i_{jk;h} = T^i_{jk,h} - T^i_{jk}|_a \Gamma^{*a}_{bh} \dot{x}^b + \Gamma^{*a}_{jk} \Gamma^{*i}_{ah} - T^i_{ak} \Gamma^{*a}_{jh} - T^i_{ja} \Gamma^{*a}_{kh}$$

respectively ,where a comma and a vertical bar denote the partial derivatives of a function with respect to x^i and $\dot{x}^i$ respectively.

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* Number in the bracket refers to the reference at the end.

** Indices i,j,k, ... run over the natural numbers I,2,3,..., n.

The Lie-derivatives, with respect to the transformation (1.1), of the mixed tensor T^i_{jk} and the connection coefficient Γ^{*i}_{jk} are expressed as

$$(1.4)\qquad \pounds T^i_{jk} = T^i_{jk;a}\xi^a + T^i_{jk}\big|_a \xi^a_{;b}\dot{x}^b - T^a_{jk}\xi^i_{;a} + T^i_{ak}\xi^a_{;j} + T^i_{ja}\xi^a_{;k}$$

and

$$(1.5)\qquad \pounds\Gamma^{*i}_{jk} = \xi^i_{;j;k} + R^{*i}_{jkl}\xi^l + \Gamma^{*i}_{jk}\big|_a \xi^a_{;b}\dot{x}^b$$

respectively, where

$$(1.6)\qquad R^{*i}_{jkl} = \left(\Gamma^{*i}_{jk,l} - \Gamma^{*i}_{jk}\big|_a \Gamma^{*a}_{bl}\dot{x}^b\right) - \left(\Gamma^{*i}_{jl,k} - \Gamma^{*i}_{jl}\big|_a \Gamma^{*a}_{bk}\dot{x}^b\right) + \Gamma^{*a}_{jk}\Gamma^{*i}_{al} - \Gamma^{*i}_{ak}\Gamma^{*a}_{jl}$$

is the corresponding curvature tensor field.

The Lie-derivative of the curvature tensor field, in terms of the connection coefficient Γ^{*i}_{jk}, is defined as

$$(1.7)\qquad \left(\pounds\Gamma^{*i}_{jk}\right)_{;l} - \left(\pounds\Gamma^{*i}_{jl}\right)_{;k} = \pounds R^{*i}_{jkl} + \Gamma^{*i}_{jk}\big|_a \pounds\Gamma^{*a}_{bl}\dot{x}^b - \Gamma^{*i}_{jl}\big|_a \pounds\Gamma^{*a}_{bk}\dot{x}^b .$$

2 CA-MOTION IN A SYMMETRIC FINSLER SPACE

Definition 2.1: A Finsler space is called a symmetric space when its curvature tensor field R^{*i}_{jkh} satisfies the relation

$$(2.1)\qquad R^{*i}_{jkh;l} = 0.$$

The infinitesimal transformation (1.1) is said to be affine motion if it satisfies the condition $\pounds g_{ij} = 0$. In such case we necessarily have

$$(2.2)\qquad \pounds\Gamma^{*i}_{jk} = \xi^i_{;j;k} + R^{*i}_{jkl}\xi^l + \Gamma^{*i}_{jk}\big|_a \xi^a_{;b}\dot{x}^b = 0 .$$

In view of (1.7) and (2.2), we immediately get

$$(2.3)\qquad \pounds R^{*i}_{jkh} = 0 ,$$

which implies that if the Finsler space F_n admits an affine motion then the curvature tensor R^{*i}_{jkh} is Lie-invariant.

Now let us consider an affine motion with contra –field. In such case we assume a special infinitesimal transformation in the form

$$(2.4)\qquad \text{a) } \bar{x}^i = x^i + \xi^i(x)dt \qquad \text{b) } \xi^i_{;j} = \lambda\delta^i_j ,$$

where $\lambda(x,\dot{x})$ is some non-zero scalar function .

Definition 2.2 : A Finsler space F_n, which admits the transformation (2.4) defines an affine motion with contra field . Such a motion is called CA-motion.

In view of the definition of Lie-derivative, the equation (2.3) is expressed in the form

$$(2.5) \quad R^{*i}_{jkl;m}\xi^m - R^{*m}_{jkl}\xi^i_{;m} + R^{*i}_{mkl}\xi^m_{;j} + R^{*i}_{jml}\xi^m_{;k} + R^{*i}_{jkm}\xi^m_{;l} + R^{*i}_{jkl}\big|_m \xi^m_{;p}\dot{x}^p = 0$$

If the Finsler space under consideration is symmetric , the equation (2.5) becomes

$$(2.6) \qquad -R^{*m}_{jkl}\xi^i_{;m} + R^{*i}_{mkl}\xi^m_{;j} + R^{*i}_{jml}\xi^m_{;k} + R^{*i}_{jkm}\xi^m_{;l} + R^{*i}_{jkl}\big|_m \xi^m_{;p}\dot{x}^p = 0 ,$$

by virtue of (2.1) .

Since the Finsler space F_n admits a CA-motion , the equation (2.6) reduces to

$$(2.7) \qquad 2\lambda R^{*i}_{jkl} + \lambda R^{*i}_{jld}\big|_p \dot{x}^p = 0 .$$

Since λ is a non-zero scalar function , we express

$$(2.8) \qquad R^{*i}_{jkl} = -\tfrac{1}{2} R^{*i}_{jkl}\big|_p \dot{x}^p .$$

Accordingly we state

Theorem 2.1: A symmetric Finsler space F_n, which admits CA-motion ,the curvature tensor R^{*i}_{jkl} is expressed in the form (2.8) .

3. CHARACTERISTICS OF SCALAR FUNCTION

In a Finsler space F_n , which admits a CA-motion ,the non-zero scalar function $\lambda(x,\dot{x})$ exists and satisfies the relation (2.4)b. In such case the relation (2.2) assumes the form

$$(3.1) \qquad \lambda_k \delta^i_j + R^{*i}_{jkl}\xi^l + \lambda \Gamma^{*i}_{jk}\big|_b \dot{x}^b = 0 ,$$

where $\lambda_k = \lambda_{;k}$ is a gradient vector field .

Since the connection coefficient Γ^{*i}_{jk} is homogeneous of degree zero in $\dot{x}^i$,the equation (3.1) reduces to

(3.2) $$\lambda_k \delta^i_j + R^{*i}_{jkl}\xi^l = 0 \ .$$

Transvecting the above equation by ξ^k, it becomes

(3.3) $$\lambda_k \delta^i_j \xi^k + R^{*i}_{jkl}\xi^k \xi^l = 0 \ .$$

The curvature tensor R^{*i}_{jkl} is skew-symmetric with respect to the indices k, l, which causes vanishing of the second term and hence the equation (3.3) takes the form

(3.4) $$\lambda_k \xi^k = 0 \quad .$$

Thus we state

Theorem 3.1: In a Finsler space F_n, which admits CA-motion, the vector fields ξ^i and λ_i are orthogonal to each other .

The contravariant vector ξ^i is independent of $\dot{x}^i$ and the covariant δ-derivative is homogeneous of degree zero in the direction argument so is the function λ. Hence we find

(3.5) $$\lambda|_p \dot{x}^p = 0 \ .$$

The Lie-derivative of the scalar function λ is expressed as

(3.6) $$£\lambda = \lambda_l \xi^l + \lambda|_l \xi^l_{;m} \dot{x}^m .$$

In a Finsler space F_n, which admits CA-motion ,the relation (3.4) holds good and hence the equation (3.6) becomes

(3.7) $$£\lambda = \lambda|_l \xi^l_{;m} \dot{x}^m .$$

In view of the equation (2.4) b ,the equation (3.7) yields

(3.8) $$£\lambda = \lambda\lambda|_m \dot{x}^m \quad .$$

Applying (3.5) in the above equation, it reduces to

(3.9) $$£\lambda = 0 .$$

Hence we have

Theorem 3.2: In a Finsler space F_n, which admits CA-motion, the scalar function λ is Lie-invariant .

4. CP-MOTION IN A FINSLER SPACE

The infintesimal transformation (1.1) defines a projective motion if it transforms the system of geodesics into that of geodesics. The necessary and sufficient condition for (1.1) to be a projective motion is that the Lie-derivative of Γ^{*i}_{jk} must satisfy

(4.1) $$£\Gamma^{*i}_{jk} = 2\delta^i_{(j}p^*_{k)} + \dot{x}^i p^*_{jk} ,$$

where

(4.2) $$p^*_k = \dot{\partial}_k p^* \quad , \quad p^*_{jk} = \dot{\partial}_j p^*_k$$

for some homogeneous scalar function $p^*(x,\dot{x})$ of degree one in $\dot{x}^i$.For the homogeneity property of p^*_j and p^*_{jk}, they also satisfy

(4.3) $$p^*_j \dot{x}^j = p^* \quad , \quad p^*_{jk}\dot{x}^k = 0 .$$

Let us assume that there exists a projective motion with contra field . In such case the infinitesimal transformation(1.1) will also satisfy the condition (2.4) .

Definition 4.1: When a Finsler space F_n admitting the projective motion also satisfied the condition (2.4), then the motion is called projective motion with contra field. We denote such motion as CP-motion.

In view of equations (1.5) and (4.1), we obtain

(4.4) $$\xi^i_{;j;k} + R^{*i}_{jkl}\xi^l + \Gamma^{*i}_{jk}\big|_a \xi^a_{;b}\dot{x}^b = 2\delta^i_{(j}p^*_{k)} + \dot{x}^i p^*_{jk} .$$

In case of CP-motion, the equation (4.4) takes the form

(4.5) $$\lambda_k\delta^i_j + R^{*i}_{jkl}\xi^l = 2\delta^i_{(j}p^*_{k)} + \dot{x}^i p^*_{jk} ,$$

since Γ^{*i}_{jk} is homogeneous function of degree zero in $\dot{x}^i$ and $\lambda_k = \lambda_{;k}$ is a gradient vector field .

Transvecting the equation (4.5) by ξ^k, we have

(4.6) $$\lambda_k\xi^k\delta^i_j + R^{*i}_{jkl}\xi^k\xi^l = \delta^i_j p^*_k\xi^k + p^*_j\xi^i + \dot{x}^i p^*_{jk}\xi^k .$$

We notice that the skew-symmetry property of R^{*i}_{jkl} with respect to the indices k and l causes vanishing of the second term in the left hand side of the equation (4.6) and hence we have

(4.7) $$\lambda_k \xi^k \delta^i_j = \delta^i_j p^*_k \xi^k + p^*_j \xi^k + \dot{x}^i p^*_{jk} \xi^k .$$

Contracting the indices i and j and using the latter part of (4.3), the above equation reduces to

(4.8) $$\lambda_k \xi^k = \left(1+\frac{1}{n}\right) p^*_k \xi^k .$$

Accordingly we state

Theorem 4.1: In a Finsler space F_n, which admits CP-motion ,the scalar functions λ_k and p^*_k satisfy the relation (4.8) .

Let us assume that the projective scalar function p^*_k is orthogonal to ξ^k ,that is

(4.9) $$p^*_k \xi^k = 0 .$$

Then in view of (4.9), the equation (4.8) reduces to (3.4), which implies that the motion is CA-motion from Theorem 3.1.

Conversely, if the relation (3.4) is true, the equation (4.7) takes the form

(4.10) $$\delta^i_j p^*_k \xi^k + p^*_j \xi^i + \dot{x}^i p^*_{jk} \xi^k = 0 .$$

Contraction of the indices i and j in the equation (4.10) yields

(4.11) $$(n+1) p^*_k \xi^k = 0 ,$$

since p^*_j is homogeneous of degree zero in $\dot{x}^i$.

From the above equation we conclude that the projective scalar function p^*_k is orthogonal to the contravariant vector ξ^k. Thus we have

Theorem 4.2: In a Finsler space F_n, the necessary and sufficient condition for CP-motion to be a CA-motion is that the projective scalar function p^*_k is orthogonal to the vector function ξ^k.

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