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ON THE GAP SERIES AND LIOUVILLE NUMBERS

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ABSTRACT. In this paper it is proved that the values of some gap series with rational coefficients are either a Liouville number or a rational number for the arguments from the set of Liouville numbers under certain conditions. In this work the method which is used in [4] is extended to the gap series.

INTRODUCTION

Mahler [3] divided in 1932 the complex numbers into four classes A, S, T, U as follows.

Let $P(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. The number $H(P) = \max\{|a_n|, \ldots, |a_0|\}$ is called the height of $P(x)$. Let ξ be a complex number and

$$\omega_n(H,\xi) = \min\{|P(\xi)| \ : \ \text{degree of } P \leq n,\ H(P) \leq H,\ P(\xi) \neq 0\} ,$$

where n and H are natural numbers. Let

$$\omega_n(\xi) = \limsup_{H\to\infty} \frac{-\log \omega_n(H,\xi)}{\log H} ,$$

and

$$\omega(\xi) = \limsup_{n\to\infty} \frac{\omega_n(\xi)}{n} .$$

The inequalities $0 \leq \omega_n(\xi) \leq \infty$ and $0 \leq \omega(\xi) \leq \infty$ hold. From $\omega_{n+1}(H,\xi) \leq \omega_n(H,\xi)$ we get $\omega_{n+1}(\xi) \geq \omega_n(\xi)$. If for an index $\omega_n(\xi) = +\infty$, then $\mu(\xi)$ is defined as the smallest of them; otherwise $\mu(\xi) = +\infty$. So μ is uniquely determined and both of $\mu(\xi)$ and $\omega(\xi)$ cannot be finite. Therefore there are the following four possibilities for ξ. ξ is called

$$\begin{array}{ll} A-\text{number if} & \omega(\xi)=0\ ,\ \mu(\xi)=\infty, \\ S-\text{number if} & 0<\omega(\xi)<\infty\ ,\ \mu(\xi)=\infty, \\ T-\text{number if} & \omega(\xi)=\infty\ ,\ \mu(\xi)=\infty, \\ U-\text{number if} & \omega(\xi)=\infty\ ,\ \mu(\xi)<\infty. \end{array}$$

The class A is composed of all algebraic numbers. The transcendental numbers are divided into the classes S, T, U. ξ is called a U-number of degree m $(1 \leq m)$ if $\mu(\xi) = m$. U_m denotes the set of U-numbers of degree m. The elements of the subclass U_1 are called Liouville numbers. A real number ξ is called a Liouville number if and only if for every integer $n > 0$ there exists integers p_n, q_n $(q_n > 1)$ with

$$0 < \left| \xi - \frac{p_n}{q_n} \right| < q_n^{-n} .$$

Koksma [1] set up in 1939 another classification of complex numbers. He divided them into four classes A^*, S^*, T^*, U^*. Let ξ be a complex number and

$$\omega_n^*(H, \xi) = \min\{|\xi - \alpha| \; : \; \text{degree of } \alpha \leq n \, , \, H(\alpha) \leq H \, , \, \alpha \neq \xi\} \; ,$$

where α is an algebraic number. Let

$$\omega_n^*(\xi) = \limsup_{H \to \infty} \frac{-\log(H \omega_n^*(H, \xi))}{\log H} \; ,$$

and

$$\omega^*(\xi) = \limsup_{n \to \infty} \frac{\omega_n^*(\xi)}{n} \; .$$

We have $0 \leq \omega_n^*(\xi) \leq \infty$ and $0 \leq \omega^*(\xi) \leq \infty$. If for an index $\omega_n^*(\xi) = +\infty$, then $\mu^*(\xi)$ is defined as the smallest of them; otherwise $\mu^*(\xi) = +\infty$. So μ^* is uniquely determined and both of $\mu^*(\xi)$ and $\omega^*(\xi)$ cannot be finite. There are the following four possibilities for ξ. ξ is called

$$\begin{aligned} A^* - \text{number if} \quad & \omega^*(\xi) = 0 \, , \; \mu^*(\xi) = \infty, \\ S^* - \text{number if} \quad & 0 < \omega^*(\xi) < \infty \, , \; \mu^*(\xi) = \infty, \\ T^* - \text{number if} \quad & \omega^*(\xi) = \infty \, , \; \mu^*(\xi) = \infty, \\ U^* - \text{number if} \quad & \omega^*(\xi) = \infty \, , \; \mu^*(\xi) < \infty. \end{aligned}$$

ξ is called a U^*-number of degree m $(1 \leq m)$ if $\mu^*(\xi) = m$. The set of U^*-numbers of degree m is denoted by U_m^*.

Wirsing [5] proved that both classifications are equivalent, i.e. A-, S-, T-, U-numbers are as same as A^*-, S^*-, T^*-, U^*-numbers. Moreover every U-number of degree m is also a U^*-number of degree m and conversely.

LeVeque [2] proved that the subclass U_m is not empty.

Theorem. Let

$$f(z) = \sum_{i=0}^{\infty} c_{n_i} z^{n_i}$$

be a gap series with non-zero rational coefficients $c_{n_i} = \frac{b_{n_i}}{a_{n_i}}$ (a_{n_i}, b_{n_i} integers; $b_{n_i} \neq 0$ and $a_{n_i} \geq 1$) satisfying the following conditions

$$(1) \qquad \lim_{i\to\infty} \frac{n_{i+1}}{n_i} = +\infty$$

and

$$(2) \qquad \limsup_{i\to\infty} \frac{\log A_{n_i}}{n_i} < +\infty$$

where $A_{n_i} = [a_{n_0}, \ldots, a_{n_i}]$.

Furthermore let ξ be a Liouville number for which the following property holds:

ξ has an approximation with rational numbers p_{n_i}/q_{n_i} ($q_{n_i} > 1$) so that the following inequality holds for sufficiently large i

$$(3) \qquad \left|\xi - \frac{p_{n_i}}{q_{n_i}}\right| < q_{n_i}^{-n_i\omega(n_i)} \quad \left(\lim_{i\to\infty} \omega(n_i) = \lim_{i\to\infty} \frac{n_{i+1}}{n_i \log q_{n_i}} = +\infty\right).$$

We assume that the radius of convergence R_f of the gap series is positive and the inequality $0 < |\xi| < R_f$ holds. Then $f(\xi)$ is either a Liouville number or a rational number.

Proof. From (2), we have the inequality which we will use later

$$(4) \qquad A_{n_i} \leq A^{n_i}$$

for sufficiently large i where $A > 0$ is a suitable constant.

Now we consider the polynomials

$$f_k(z) = \sum_{i=0}^{k} c_{n_i} z^{n_i} \qquad (k = 1, 2, 3, \ldots).$$

Since

$$\left|f(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \leq |f(\xi) - f_k(\xi)| + \left|f_k(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right|$$

we can determine an upper bound for $|f(\xi) - f_k(\xi)|$ and $\left|f_k(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right|$. The following equality holds.

$$(5) \qquad f_k(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right) = \sum_{i=0}^{k} c_{n_i}\left(\xi - \frac{p_{n_k}}{q_{n_k}}\right) \cdot \left(\xi^{n_i-1} + \xi^{n_i-2}\frac{p_{n_k}}{q_{n_k}} + \cdots + \left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_i-1}\right)$$

From (3) it follows that, for sufficiently large k

(6)
$$\left|\frac{p_{n_k}}{q_{n_k}}\right| \leq |\xi| + 1.$$

According to the Cauchy inequality, we have

(7)
$$|c_{n_i}| \leq \frac{M}{\rho^{n_i}} \qquad (i = 0, 1, 2, \ldots)$$

($|\xi| < \rho < R_f$, $\rho = \frac{|\xi|+R_f}{2}$, M denotes the maximum value of $|f(z)|$ on the $|z| = \rho$. If $R_f = +\infty$, ρ is to be chosen as $\rho > |\xi|$.)

It follows from (7) that

(8)
$$|c_{n_i}| \leq MB^{n_k} \leq (M_1B)^{n_k} = M_2^{n_k}$$

where $\max(1, \frac{1}{\rho}) = B$, $\max(1, M) = M_1$, $M_2 = M_1B$. Thus, using (3), (5), (6) and (8) we get for sufficiently large k

$$\begin{aligned} \left|f_k(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| &\leq q_{n_k}^{-n_k\omega(n_k)} \sum_{i=0}^{k} |c_{n_i}| \left|\xi^{n_i-1} + \xi^{n_i-2}\frac{p_{n_k}}{q_{n_k}} + \cdots + \left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_i-1}\right| \\ &\leq q_{n_k}^{-n_k\omega(n_k)}(k+1)M_2^{n_k} n_k(|\xi|+1)^{n_k-1} \\ &\leq q_{n_k}^{-\frac{n_{k+1}}{\log q_{n_k}}} n_k^2 M_2^{n_k}(|\xi|+1)^{n_k}. \end{aligned}$$

We have from (1) $\lim_{k\to\infty} n_k = +\infty$ and so it follows that for sufficiently large k

$$n_k^2 \leq c^{n_k},$$

where $c > 1$ is a suitable constant. Therefore, we get

$$\left|f_k(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \leq q_{n_k}^{-\frac{n_{k+1}}{\log q_{n_k}}} c_1^{n_k},$$

where $c_1 = cM_2(|\xi| + 1)$. Using (3) and (4) we deduce that

(9)
$$\begin{aligned} \left|f_k(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| &\leq \frac{1}{2}(q_{n_k}^{n_k} A^{n_k})^{-\omega(n_k)} \\ &\leq \frac{1}{2}(q_{n_k}^{n_k} A_{n_k})^{-\omega(n_k)} \end{aligned}$$

for sufficiently large k.

Now we can determine an upper bound for $|f(\xi) - f_k(\xi)|$. From (7) and $|\xi| < \rho < R_f$, it follows that

$$\begin{aligned} |f(\xi) - f_k(\xi)| &\leq \sum_{i=k+1}^{\infty} |c_{n_i}||\xi|^{n_i} \\ &\leq \frac{M}{\rho^{n_{k+1}}}|\xi|^{n_{k+1}}\left(1 + \frac{|\xi|}{\rho} + \frac{|\xi|^2}{\rho^2} + \cdots\right) \\ &= \left(\frac{|\xi|}{\rho}\right)^{n_{k+1}} \frac{M}{1 - \frac{|\xi|}{\rho}} = \frac{c_2}{c_3^{n_{k+1}}} \end{aligned}$$

where $c_2 = \frac{M}{1-\frac{|\xi|}{\rho}} > 0$, $c_3 = \frac{\rho}{|\xi|} > 1$.

Thus, using (3) and (4) we have for sufficiently large k

$$\begin{aligned} |f(\xi) - f_k(\xi)| &\leq \frac{1}{2}(A^{n_k} q_{n_k}^{n_k})^{-\omega(n_k)\lambda} \\ &\leq \frac{1}{2}(A_{n_k} q_{n_k}^{n_k})^{-\omega(n_k)\lambda} \end{aligned} \tag{10}$$

where λ is to be chosen as $0 < \lambda < \min(\log c_3, 1)$ and so it follows from (9) and (10)

$$\left| f(\xi) - f_k\left(\frac{p_{n_k}}{q_{n_k}}\right) \right| \leq (q_{n_k}^{n_k} A_{n_k})^{-\omega(n_k)\lambda} \tag{11}$$

for sufficiently large k. For $\omega(n_k) \to +\infty$ we have $\omega(n_k)\lambda \to +\infty$. Moreover

$$f_k\left(\frac{p_{n_k}}{q_{n_k}}\right) = \sum_{i=0}^{k} c_{n_i}\left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_i} = \frac{h_{n_k}}{A_{n_k} q_{n_k}^{n_k}} \qquad (k = 1, 2, 3, \ldots)$$

are rational numbers with h_{n_k} integers. It follows from (11) that

$$\lim_{k \to \infty} f_k\left(\frac{p_{n_k}}{q_{n_k}}\right) = f(\xi).$$

If the sequence $\left\{f_k\left(\frac{p_{n_k}}{q_{n_k}}\right)\right\}$ is constant then $f(\xi)$ is a rational number. Otherwise $f(\xi)$ is a Liouville number.

Example. We consider the number

$$\xi = \sum_{i=0}^{\infty} \frac{1}{a^{n_i}}$$

with $n_i = (i!)^{i!}$, $a > 10$ an integer. Because of Theorem 1 in [6] we know that ξ is a Liouville number.

The conditions of Theorem are satisfied for the following coefficients: $b_{n_i} = 1$, $a_{n_i} = a^{n_i}$ with $n_i = (i!)^{i!}$ $(i = 0, 1, 2, \ldots)$, $a > 10$ an integer. For these coefficients we obtain

$$\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = +\infty, \quad \lim_{i\to\infty} \frac{\log A_{n_i}}{n_i} < +\infty$$

where $A_{n_i} = [a_{n_0}, a_{n_1}, \ldots, a_{n_i}]$ and so we have the conditions (1) and (2). Furthermore it follows for sufficiently large i

$$\left|\xi - \frac{p_{n_i}}{q_{n_i}}\right| < q_{n_i}^{-n_i\omega(n_i)} \quad \left(\lim_{i\to\infty} \omega(n_i) = \lim_{i\to\infty} \frac{n_{i+1}}{n_i \log q_{n_i}} = +\infty\right)$$

where $q_{n_i} = a^{n_i}$, $p_{n_i} = q_{n_i} \sum_{\nu=0}^{i} a^{-n_\nu}$. This is the condition (3). For this ξ the inequality $0 < |\xi| < R_f$ is satisfied. Thus the conditions of Theorem are satisfied for the ξ and

$$f(z) = \sum_{i=0}^{\infty} \frac{1}{a^{n_i}} z^{n_i}.$$

Therefore $f(\xi)$ is either a Liouville number or a rational number.

References

[1] **KOKSMA, J.F.** : *Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen*, Monatsh. Math. Physik, **48** (1939), 176-189.

[2] **LeVEQUE, W.** : *On Mahler's U-Numbers*, London Math. Soc., **28** (1953), 220-229.

[3] **MAHLER, K.** : *Zur Approximation der Exponentialfunktion und des Logarithmus I*, J. reine angew. Math., **166** (1932), 118-136.

[4] **ORYAN, M.H.** : *On the Power Series and Liouville Numbers*, Doğa Tr. J. of Mathematics, **14** (1990), 79-90.

[5] **WIRSING, E.** : *Approximation mit algebraischen Zahlen beschränkten Grades*, J. Reine Angew. Math., **206** (1961), 67-77.

[6] **ZEREN, B.M.** : *Über einige komplexe und p-adische Lückenreihen mit Werten aus der Mahlerschen Unterklassen U_m*, İst. Üniv. Fen Fak. Mec. Seri A, **45** (1980), 89-130.

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