İstanbul Üniv. Fen Fak. Mec. Seri A, 46-47 (1981-1982), 45-54

THERMAL STRESS DISTRIBUTION IN AN INFINITE ELASTIC SOLID CONTAINING AN ANNULAR CRACK UNDER TORSION

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In the present paper the thermal stress distribution in an infinite elastic solid containing an annular crack under torsion has been studied. Using integral transform technique, the problem is reduced to that of simultaneous linear algebraic equation which are numerically computed. Finally the distribution of physical quantities are plotted for different values of the inner radius of the crack treating outer radius as unity. Also, the variations of stress intensity factors with a/b are illustrated graphically.

1. INTRODUCTION

The problem of determining the distribution of thermal stress in the vicinity of a crack in an elastic body is of interest since one encounters mechanical structures subjected to high temperature and pressure etc.. During recent years considerable work has been done on calculating thermal stresses in infinite solids, thick slabs and infinite cylinders containing penny-shaped and annular cracks [1-8] . The solutions for the selected problems for these types of fractures have been given by Shibuya et al [9] and Lai [10]. Shibuya et al [9] and Lal [10] solved the axisymmetric stress distribution in elastic solid and thick elastic slab containing a flat annular crack under torsion. Here, in this paper the thermoelastic problem in an infinite elastic solid containing an annular crack under torsion has been considered. The problem is reduced to the solution of triple integral equation. The triple integral equation is further reduced to an infinite system of simultaneous linear algebraic equations which are numerically computed. Finally the quantities of physical interest have been illustrated graphically.

2. THE AXISYMMETRIC PROBLEM

Consider the cylindrical co-ordinate system (r, θ, z) , where z-axis is taken parallel to the axis of symmetry of the material. Suppose that a flat annular crack under torsion is located on $z = 0$ and $a \leq r \leq b$, where a and b are respectively the inner and outer radii of the crack. Since the problem is symmetrical with respect to the plane $z = 0$, it is enough to consider only the solution for the half space $z \geq 0$ for which the thermal and elastic conditions on the boundary $z = 0$ are

$$\frac{\partial \theta}{\partial z}(r, 0) = 0\ ;\ \ 0 \leq r < a\ ;\ r > b, \tag{2.1}$$

$$0\,(r, 0) = \tau,\ a \leq r \leq b\ , \tag{2.2}$$

and

$$\sigma_{zz}(r, 0) = 0,\ \ 0 \leq r \leq \infty\ , \tag{2.3}$$

$$\sigma_{rz}(r, 0) = -\,w_0\, r,\ \ a < r < b\ , \tag{2.4}$$

$$u\,(r, 0) = 0,\ \ 0 \leq r \leq a\ ,\ r \geq b\ , \tag{2.5}$$

where w_0 is an angle per unit length when the infinite solid is under torsion around z-axis. We further assume that the distribution is localized, i.e. the temperature and the components of stress and displacement all vanish as $(r^2 + z^2)^{1/2} \rightarrow \infty$.

3. THE HANKEL TRANSFORM SOLUTION

In the case of infinite solid free from disturbances at infinity, the solutions obtained by Florence and Goodier [2] are

$$\begin{aligned} \bar{u}\,(\xi, z) &= \int_0^\infty r\, u\,(r, z)\, J_1(\xi, r)\, dr\ , \\ \bar{w}\,(\xi, z) &= \int_0^\infty r\, w\,(r, z)\, J_0(\xi, r)\, dr\ , \\ \bar{\theta}\,(\xi, z) &= \int_0^\infty r\, \theta\,(r, z)\, J_0(\xi, r)\, dr\ , \end{aligned} \tag{3.1}$$

where the displacements u, w and the temperature 0 are of the form

$$\begin{aligned} \bar{\theta} &= \xi^{-1} A\, e^{-\xi z}\ , \\ \bar{u} &= (A_1 + \xi\, z\, B)\, e^{-\xi z}\ , \\ \bar{w} &= (A_2 + \xi\, z\, B)\, e^{-\xi z}\ . \end{aligned} \tag{3.2}$$

These follow from the differential equations satisfied by $u, w,$ 0 in the linear static thermoelastic axisymmetric problem. The negative exponentials correspond to zero disturbance at infinity in the half space $z \geq 0$. The coefficients $A, B,$ A_1 , A_2 are the functions of ξ only, with the relations

$$A_1 - A_2 + (3 - 4\,\eta)\, B = 2\,(1 + \eta)\,\alpha\,\xi^{-1} A\ , \tag{3.3}$$

where η is Poisson's ratio and the coefficient of expansion. The inversion theorem for the transforms can be written from (3.1) by interchanging ξ and r, and transforming the bar over u, w and 0 to the right hand sides.

The displacement transforms in (3.2) give the shear stress transform

$$\bar{\sigma}_{rz}(\xi, z) = \int_0^\infty r\,\sigma_{rz}(r, z)\,J_1(\xi r)\,dr$$

$$= -\mu\,\xi\,(A_1 - B + A_2 + 2\,B\,\xi\,z)\,e^{-\xi z}, \qquad (3.4)$$

and the normal stress transform

$$\bar{\sigma}_{zz}(\xi, z) = -\frac{2\mu}{(1-2\eta)}\,[(1-\eta)(A_2 - B) - \eta\,A_1 + (1+\eta)\,\alpha\,\xi^{-1}A +$$

$$+ (1-2\eta)\,B\,\xi\,z]\,e^{-\xi z}, \qquad (3.5)$$

where μ is the shear modulus.

The condition (2.3), with the help of (3.5) gives

$$(1-\eta)(A_2 - B) - \eta\,A_1 + (1+\eta)\,\alpha\,\xi^{-1}A = 0\,. \qquad (3.6)$$

By inversion of (3.4) we have in general

$$\sigma_{rz}(r, z) = -\mu\int_0^\infty \xi^2\,(A_1 - B + A_2 + 2\,B\,z)\,e^{-\xi z}\,J_1(\xi r)\,d\xi\,, \qquad (3.7)$$

using the elastic conditions (2.4) and (2.5), the equations (3.6) and (3.2) reduce to

$$\mu\int_0^\infty \xi^2\,(A_1 - B + A_2)\,J_1(\xi r)\,d\xi = w_0\,r\ ;\ a < r < b\,, \qquad (3.8)$$

$$\int_0^\infty \xi\,A_1\,J_1(\xi r)\,d\xi = 0,\ \ 0 \leq r \leq a\,,\ r \geq b\,. \qquad (3.9)$$

4. THE HEAT CONDUCTION PROBLEM

A suitable representation of the temperature field satisfying the Laplace's equation and vanishing at infinity is taken as in (3.2)

$$\theta(r, z) = \int_0^\infty A(\xi)\,e^{-\xi z}\,d\xi\,,$$

where $A(\xi)$ is an unknown function to be determined from the conditions on $z = 0$.

Imposition of conditons (2.1) and (2.2) leads to the derivation of the triple integral equations

$$\int_0^\infty \xi\, A(\xi)\, J_0(\xi r)\, d\xi = 0\,,\ 0 \le r < a\,,\ r > b, \tag{4.2}$$

$$\int_0^\infty A(\xi)\, J_0(\xi r)\, d\xi = \tau\,,\ \ a \le r \le b\,. \tag{4.3}$$

Therefore, the present problem is reduced to one of finding a solution of the triple integral equations (4.2) and (4.3). However it is very difficult to solve these equations directly. We use a similar method to that of Shibuya [9] . By putting

$$\begin{aligned} a &= c - d,\ \ b = c + d \\ r^2 &= c^2 + d^2 - 2\,cd\cos\varphi\,, \end{aligned} \tag{4.4}$$

where c and $2d$ are respectively the average radius and width of the crack and the variable r in $a \le r \le b$ will be replaced with a new one φ in $0 \le \varphi \le \pi$, when $r = a$ corresponds to $\varphi = 0$ and $r = b$ to $\varphi = \pi$. Then assuming

$$A(\xi) = \tau \sum_{n=0}^{\infty} a_n\, J_n(\xi c)\, J_n(\xi d) \tag{4.5}$$

and using the result of [11]

$$\int_0^\infty \xi\, J_0(\xi r)\, J_n(\xi c)\, J_n(\xi d)\, d\xi = \begin{cases} 0, \ 0 \le r < a\,,\ r > b \\ \dfrac{1}{\pi cd}\cdot\dfrac{\cos(n\varphi)}{\sin\varphi}\,,\ a < r < b. \end{cases} \tag{4.6}$$

The equation (4.2) is automatically satisfied by using the equations (4.5) and (4.6).

Substituting $A(\xi)$ from (4.5) into (4.3), we get

$$\sum_{n=0}^{\infty} a_n \int_0^\infty J_n(\xi c)\, J_n(\xi d)\, J_0(\xi r)\, d\xi = 1,\ \ a \le r \le b\,. \tag{4.7}$$

In consideration of the formula

$$J_0(\xi r) = J_0(\xi c)\, J_0(\xi d) + 2 \sum_{m=1}^{\infty} J_m(\xi c)\, J_m(\xi d)\cos(m\varphi),\ \ a \le r \le b,$$

equation (4.7) can be written as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n \int_0^\infty J_n(\xi c)\, J_n(\xi d)\, \{\, J_0(\xi c)\, J_0(\xi d) + \\ &+ 2 \sum_{m=1}^{\infty} J_m(\xi c)\, J_m(\xi d)\cos(m\varphi)\,\}\, d\xi = 1\,. \end{aligned} \tag{4.8}$$

Since equation (4.8) must hold for any arbitrary value of φ, we find that the equation is reduced to the following system of simulataneous equations for the determination of the coefficients a_n :

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} J_n(\xi c) J_n(\xi d) J_m(\xi c) J_m(\xi d) d\xi = \delta_{0m}, \quad m = 0, 1, 2, \dots, \tag{4.9}$$

where δ_{0m} is the Kronecker's delta. Consequently, the present mixed boundary value problem is reduced to the solution of the simultaneous equation (4.9).

In particular the temperature on the surface is

$$\frac{\theta(r, 0)}{\tau} = \sum_{n=0}^{\infty} a_n I_0^n, \tag{4.10}$$

where

$$I_0^n = \int_0^{\infty} J_n(\xi r) J_n(\xi c) J_n(\xi d) d\xi, \tag{4.11}$$

which is well defined by Shibuya et al. [9] .

5. THE THERMOELASTIC PROBLEM

Equations (3.3) and (3.6) permit the expressions of A_1, A_2 in terms of A,B:

$$A_1 = (1 + \eta) \alpha \xi^{-1} A - 2(1 + \eta) B, \quad A_2 = -(1 + \eta) \alpha \xi^{-1} A + (1 - 2\eta) B. \tag{5.1}$$

These convert (3.8) and (3.9) to triple integral equations

$$2\mu \int_0^{\infty} \xi^2 B(\xi) J_1(\xi r) d\xi = w_0 r; \quad a < r < b, \tag{5.2}$$

$$\int_0^{\infty} \xi B(\xi) J_1(\xi r) d\xi = \alpha\beta \int_0^{\infty} A(\xi) J_1(\xi r) d\xi; \quad 0 \le r \le a, \ r \ge b. \tag{5.3}$$

To solve these equations, we use the integral formula [11]

$$\int_0^{\infty} J_1(\xi r) J_n(\xi c) J_n(\xi d) d\xi = \begin{cases} 0, & 0 \le r \le a, \ r \ge b \\ \dfrac{\sin(n\varphi)}{n \pi r}, & a \le r \le b. \end{cases} \tag{5.4}$$

Putting the value of $A(\xi)$ from (4.5) into (5.3) and using the result (5.4), the equation (5.3) reduces to

$$\int_0^{\infty} \xi B(\xi) J_1(\xi r) d\xi = 0, \quad 0 \le r \le a, \ r \ge b. \tag{5.5}$$

To solve the triple integral equations (5.2) and (5.5), we use the same method as above. Let us assume

$$\xi B(\xi) = \frac{c\, d\, w_0}{4\mu} \sum_{n=1}^{\infty} b_n J_n(\xi c) J_n(\xi d) . \tag{5.6}$$

Using the result of (5.4), the equation (5.6) satisfies the equation (5.5). Substituting equation (5.6) into (5.2), we get

$$\frac{cd}{2} \sum_{n=1}^{\infty} b_n \int_0^{\infty} \xi J_n(\xi c) J_n(\xi d) J_1(\xi r)\, d\xi = r, \quad a < r < b . \tag{5.7}$$

Using the relation

$$\xi J_i(\xi r) = \frac{2r}{cd \sin \varphi} \sum_{m=1}^{\infty} J_m(\xi c) J_m(\xi d) \sin(m\varphi), \quad a < r < b , \tag{5.8}$$

we can rewrite equation (5.7) as follows :

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \sin(m\varphi) \int_0^{\infty} J_n(\xi c) J_n(\xi d) J_m(\xi c) J_m(\xi d)\, d\xi = \sin\varphi , \quad a < r < b. \tag{5.9}$$

Since equation (5.9) must hold for arbitrary values of φ, we find that the equation is reduced to the following infinite system of simultaneous equations for determination of the coefficients b_n :

$$\sum_{n=1}^{\infty} b_n \int_0^{\infty} J_n(\xi c) J_n(\xi d) J_m(\xi c) J_m(\xi d)\, d\xi = \delta_{1m} , \quad m = 1, 2, 3, \ldots, \tag{5.10}$$

where δ_{1m} is the Kronecker's delta. Consequently the mixed boundary value problem is reduced to a solution of equation (5.10).

6. DISPLACEMENT AND STRESS COMPONENTS ON $z = 0$ AND STRESS INTENSITY FACTORS

The displacement and stress components on $z = 0$ are

$$u(r, 0) = \begin{cases} 0, \quad 0 \leq r \leq a, \quad r \geq b , \\ \displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n \pi r} \left(\frac{c\, d\, w_0}{4\mu} b_n - \alpha \beta \tau a_n \right) , \quad a \leq r \leq b , \end{cases} \tag{6.1}$$

$$\sigma_{rz}(r, 0) = \begin{cases} 0, \quad a < r < b , \\ \displaystyle -\frac{c\, d\, w_0}{2} \left\{ \frac{r}{cd} + \frac{1}{2} \sum_{n=1}^{\infty} b_n \frac{\partial I_0^{\,n}}{\partial r} \right\} ; \quad 0 \leq r < a, \ r > b. \end{cases} \tag{6.2}$$

Using equation (6.2), we find that the stress intensity factors N_a and N_b at the inner and the outer tips of the crack are, respectively, defined by the following equations:

$$N_a = \lim_{r \to a-0} \sqrt{2(a-r)}\; \sigma_{rz}(r, 0)\,,$$
$$N_b = \lim_{r \to b+0} \sqrt{2(b-r)}\; \sigma_{rz}(r, 0)\,. \tag{6.3}$$

The stress intensity factors N_a and N_b depend upon $0 \leq r < a$ or $r > b$ in accordance with the magnitude of r.

(a) **The case of** $0 \leq r < a$: Since $r < c$, in this case

$$\frac{N_a}{w_0} = -\frac{d\sqrt{c}}{2} \sum_{n=1}^{\infty} B_n b_n \left[\frac{\sin(2\varphi_a)}{n+1} F\left(\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\varphi_a\right) \times \right.$$
$$\times F\left(\frac{1}{2}, n+\frac{1}{2}; 1; \sin^2\Psi_a\right) + \sin(2\Psi_a) F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\varphi_a\right) \times$$
$$\left. \times F\left(\frac{3}{2}, n+\frac{3}{2}; 2; \sin^2\Psi_a\right)\right], \tag{6.4}$$

where

$$\left\{\begin{matrix}\varphi_a\\ \Psi_a\end{matrix}\right\} = \frac{1}{2}\left[\frac{\pi}{2} \pm \sin^{-1}\left\{\frac{a-d}{c}\right\}\right] \text{ and } B_n = \frac{\Gamma(n+\frac{3}{2})}{4\Gamma(n+1)\Gamma(\frac{1}{2})} \cdot \frac{1}{c}\left(\frac{d}{c}\right)^n.$$

(b) **The case of** $r > b$: Since $c < r$, in this case

$$\frac{N_b}{w_0} = \frac{dc}{2\sqrt{b}} \sum_{n=1}^{\infty} B_n a_n \left[\sin(2\varphi_a) F\left(n+\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\varphi_b\right) \times \right.$$
$$\times F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\Psi_b\right) + \sin(2\Psi_b) F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\varphi_b\right) \times$$
$$\left. \times F\left(n+\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\Psi_b\right)\right], \tag{6.5}$$

where

$$\left\{\begin{matrix}\Psi_b\\ \varphi_b\end{matrix}\right\} = \frac{1}{2}\left[\frac{\pi}{2} \pm \sin^{-1}\left(\frac{a}{b}\right)\right],$$

and

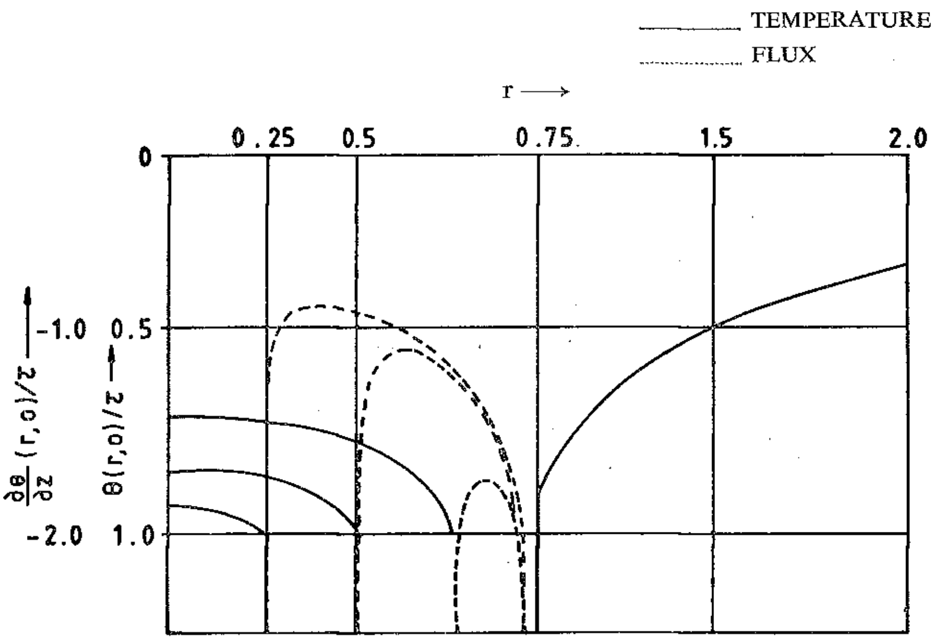
$$B_n = \frac{(-1)^{n+1}}{2\pi}\left[\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^2 \frac{1}{b(n+1)}\left(\frac{cd}{b^2}\right)^n.$$

7. NUMERICAL CALCULATIONS

The values of a_n and b_n are computed from equations (4.9) and (5.10). These equations are integrated numerically by means of Simpson's second rule by assuming the larger upper limit, i.e. 500 and the interval in numerical integrals is 0.2. On the other hand, the asymptotic formula of the Bessel function is used for large values of ξ. The integral sine and cosine functions si(x) and ci(x) are computed with good accuracy by using approximate formulae [12] . The approximate solutions of the set of simultaneous equations (4.9) are calculated and are independent of the material properties of the material. On the other hand, for finding the approximate solution of the equation (5.10), the Poisson's ratio is taken as 0.3.

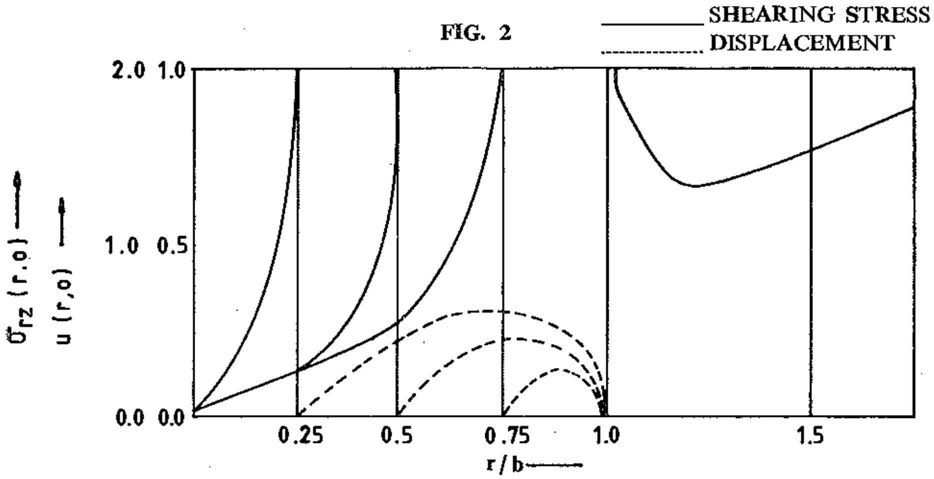
Figure 1 shows the radial distribution of temperature and flux for different values of inner radius $a = 0.25$, 0.5, 0.75 treating the outer radius of the crack $b = 1$. Similarly figure 2 shows the distribution of displacement and shearing stress for the same values of a and b as above.

FIG. 1



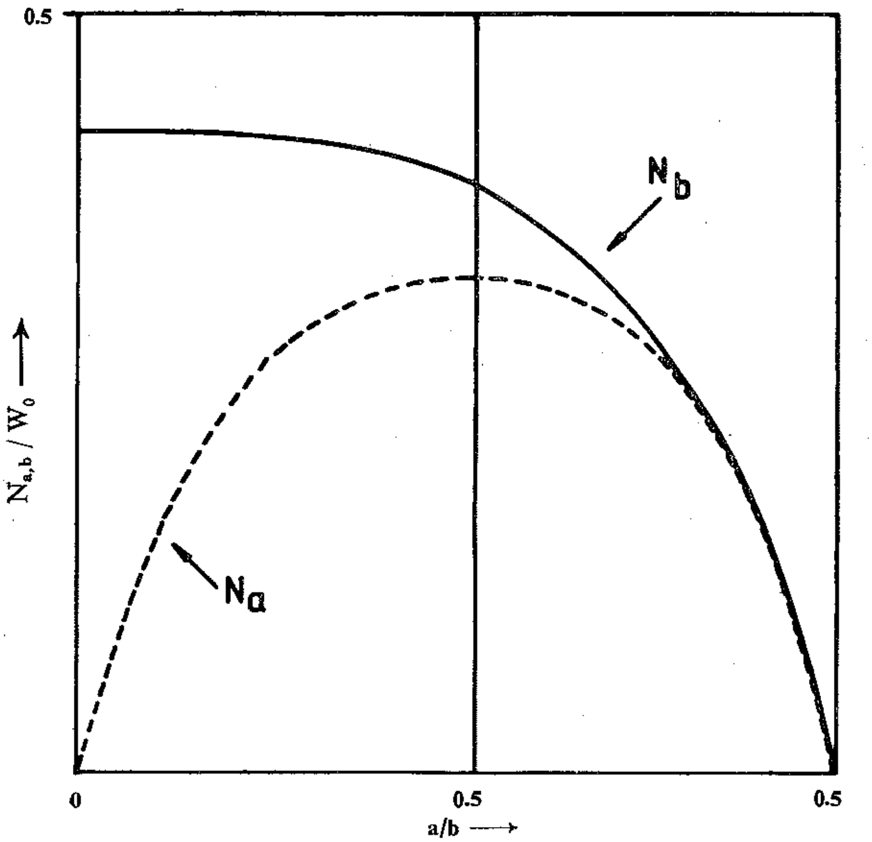
THE DISTRIBUTION OF TEMPERATURE AND FLUX
FOR $b = 1.0$ AND $a = 0.25$, 0.5, 0.75

FIG. 2



THE DISTRIBUTION OF u(r, 0) AND σ_{rz} (r, 0) FOR b = 1.0 AND a = 0.25, 0.5, 0.75

FIG. 3



THE VARIATION OF STRESS INTENSITY FACTORS WITH a/b

In fig. 3 we show the variations of stress intensity factors with a/b. The stress intensity factor N_b at the outer tip of the crack is always greater than N_a at the inner tip. With increasing a/b, N_b decreases monotonously and tends to zero as $a/b \rightarrow 1$. N_a is zero at $a/b = 0$ and increases as the value a/b gets large and becomes maximum at $a/b = 0.5$, where $a/b \rightarrow 1.0$, N_a decreases monotonously and approaches N_b.

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ÖZET

Bu çalışmada, torsiyon durumunda bir yuvarlak çatlak içeren bir sonsuz elastik cisimdeki ısısal gerilim dağılımı incelenmektedir.