İstanbul Üniv. Fen Fak. Mec. Seri A, 46-47 (1981-1982), 61-66 

# W-THIRD ORDER RECURRENT FINSLER SPACE

H.D. PANDEY - S.D. TRIPATHI

In this paper some results on the third order recurrent Finsler spaces have been obtained.

## 1. INTRODUCTION

Let an $n$-dimensional Finsler space $F_n$ [1] be equipped with a positively homogeneous function $F(x, \dot{x})$ whose metric tensor $g_{ij}(x, \dot{x}) \stackrel{\text{def.}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2 (x, \dot{x})$ is symmetric in its lower indices and positively homogeneous of degree zero in $\dot{x}^i$.

The covariant derivative of tensor $T^i_j (x, \dot{x})$ in the sense of Berwald is given by

$$T^i_{j(h)} = \partial_h T^i_j - (\dot{\partial}_m T^i_j) G^m_h + T^k_j G^i_{kh} - T^i_k G^k_{jh} \left( \partial_i = \frac{\partial}{\partial x^i} , \dot{\partial}_i = \frac{\partial}{\partial \dot{x}^i} \right), \quad (1.1)$$

where $G^i (x, \dot{x})$ is positively homogeneous of degree two in its directional argument and given by

$$G^i (x, \dot{x}) \stackrel{\text{def.}}{=} \frac{1}{4} g^{ih} \{ 2 \partial_{(j} g_{k)h} - \partial_h g_{jk} \} \dot{x}^j \dot{x}^k \quad (2 T_{(hk)} = T_{hk} + T_{kh}). \quad (1.1)\text{a}$$

The $G^i_{hk} (x, \dot{x})$ are Berwald's connection coefficients, given by $G^i_{jk} \stackrel{\text{def.}}{=} \frac{\partial^2 G^i}{\partial \dot{x}^j \partial \dot{x}^k}$.

The Berwald's curvature tensor $H^i_{jhk} (x, \dot{x})$, projective curvature tensor $W^i_{jhk} (x, \dot{x})$ and projective deviation tensor $W^i_{jh} (x, \dot{x})$ , $W^i_j (x, \dot{x})$ are given as follows :

$$H^i_{jhk} (x, \dot{x}) = 2 \{ \partial_{[k} G^i_{h]j} - G^i_{rj[h} G^r_{k]} + G^r_{j[h} G^i_{k]r} \} \; (2T_{[hk]} = T_{hk} - T_{kh}), \quad (1.2)$$

$$W^i_{jhk} = H^i_{jhk} + \frac{\delta^i_j}{n+1}(H_{hk} - H_{kh}) + \frac{\dot{x}^i}{n+1}(\dot{\partial}_j H_{hk} - \dot{\partial}_j H_{kh}) +$$

$$+ \frac{\delta^i_h}{n^2-1}(n H_{jk} + H_{kj} + \dot{x}^r \dot{\partial}_j H_{kr}) -$$

$$- \frac{\delta^i_k}{n^2-1}(n H_{jh} + H_{hj} + \dot{x}^r \dot{\partial}_j H_{hr}), \tag{1.3}$$

$$W^i_j = H^i_j - H \delta^i_j - \frac{1}{n+1}(\dot{\partial}_m H^m_j - \dot{\partial}_j H)\dot{x}^i \tag{1.4}$$

and

$$W^i_{jh} = H^i_{jh} + \frac{\dot{x}^i}{n+1}(H_{hk} - H_{kh}) +$$

$$+ \frac{\delta^i_j}{n^2-1}(n H_h + \dot{x}^r H_{hr}) - \frac{\delta^i_h}{n^2-1}(n H_j + \dot{x}^r H_{jr}), \tag{1.5}$$

where $\delta^i_j$ are Kronecker delta.

The curvature tensor $H^i_{jhk}(x, \dot{x})$ satisfies the following identities:

$$H^i_{jhk} + H^i_{hkj} + H^i_{kjh} = 0, \tag{1.6}$$

$$H^i_{rjk(l)} + H^i_{rhl(j)} + H^i_{rlj(k)} = 0 \tag{1.7}$$

and

$$H = \frac{1}{n-1} H^i_i. \tag{1.8}$$

The followings are the commutation formulas for tensors of second order:

$$T_{ij(h)(k)} - T_{ij(k)(h)} = \dot{\partial}_r T_{ij} H^r_{hk} - T_{rj} H^r_{ihk} - T_{ir} H^r_{jhk}, \tag{1.9}$$

$$T^i_{(j)(h)(k)} - T^i_{(j)(k)(h)} = - \dot{\partial}_m T^i_{(j)} H^m_{hk} - T^i_{(m)} H^m_{jhk} + H^i_{mhk} T^m_{(j)}. \tag{1.10}$$

Recurrent Finsler space of first and second order for non-zero curvature tensor $W^i_{jhk}(x, \dot{x})$ are given by [2,3]

$$W^i_{jhk(l)} = \lambda_l W^i_{jhk}, \tag{1.11}$$

$$W^i_{jhk(l)(m)} = a_{lm} W^i_{jhk}, \tag{1.12}$$

where $\lambda_l$ is non-zero recurrence vector field and $a_{lm}(x, \dot{x})$ is recurrence tensor of order two.

## 2. RECURRENT PROJECTIVE CURVATURE

**Definition 2.1.** An $n$-dimensional Finsler space is said to be third order recurrent Finsler space if the projective curvature tensor field satisfies the relation

$$W^i_{hjk(l)(m)(n)} = b_{lmn} W^i_{hjk} , \tag{2.1}$$

where $b_{lmn}$ is a non-zero recurrence tensor field of third order. We denote such space by $3RF_n$.

**Theorem 2.1.** The Wely's tensor fields $W^i_{jk}$ and $W^i_k$ are necessarily third order recurrent in $3RF_n$.

**Proof.** Transvecting (2.1) by $\dot{x}^h$ and using $W^i_{jk} = W^i_{hjk} \dot{x}^h$, we get

$$W^i_{jk(l)(m)(n)} = b_{lmn} W^i_{jk} , \tag{2.2}$$

where we have used the fact $\dot{x}^i_{(k)} = 0$.

In a similar way, from the above relation, we can deduce

$$W^i_{k(l)(m)(n)} = b_{lmn} W^i_k . \tag{2.3}$$

From equations (2.2) and (2.3) we have the Theorem 2.1.

**Theorem 2.2.** In a $3RF_n$ the recurrence tensor field $b_{lmn}$ satisfies the following identities :

$$b_{[lm]n} = a_{[lm]n} + a_{[lm]} \lambda_n , \tag{2.4}$$

$$b_{[lm]n} = \lambda_{[l(m)](n)} + \lambda_{[l(m)]} \lambda_n + \lambda_{[l<(n)>} \lambda_{m]} + \lambda_{[l} \lambda_{m](n)} , \tag{2.5}$$

where the indices in $< >$ are free from symmetric and skew symmetric operations.

**Proof.** Differentiating (1.11) covariantly with respect to $x^m$ and $x^n$ in the sense of Berwald and remembering the definition (2.1), we get

$$b_{lmn} = \lambda_{l(m)(n)} + \lambda_{l(m)} \lambda_n + \lambda_{l(n)} \lambda_m + \lambda_l \lambda_m \lambda_n + \lambda_l \lambda_{m(n)} . \tag{2.6}$$

Interchanging the indices $l, m$ and substracting thus obtained equation from (2.6), we get (2.5).

The commutation formula (1.10) for projective curvature tensor $W^i_{hjk}$ is given by

$$W^i_{hjk(l)(m)(n)} - W^i_{hjk(l)(n)(m)} = -\dot{\partial}_p W^i_{hjk(l)} H^p_{mn} + W^p_{hjk(l)} H^i_{pmn} - \tag{2.7}$$
$$- W^i_{pjk(l)} H^p_{hmn} - W^i_{hpk(l)} H^p_{jmn} -$$
$$- W^i_{hjk(l)} H^p_{kmn} - W^i_{hjk(p)} H^p_{lmn} .$$

With the help of (1.11), (1.12) and (1.9), we get

$$(b_{lnm} - b_{lnm}) = \lambda_l (a_{mn} - a_{nm}) - \dot{\partial}_p \lambda_l H^p_{mn} - \lambda_p H^p_{lmn} , \tag{2.8}$$

because the curvature tensor $W^i_{hjk}$ is non-zero.

Adding two more expressions obtained by the cyclic interchange of the indices $l$, $m$, $n$ to (2.8) and using (1.6), we have

$$\{b_{l[mn]} - \lambda_l a_{[mn]}\} + \{b_{m[nl]} - \lambda_m a_{[nl]}\} + \{b_{n[lm]} - \lambda_n a_{[lm]}\} +$$
$$+ \frac{1}{2} \{\dot{\partial}_p \lambda_l H^p_{mn} + \dot{\partial}_p \lambda_m H^p_{nl} + \dot{\partial}_p \lambda_n H^p_{lm}\} = 0. \tag{2.9}$$

If the recurrence vector $\lambda_l$ is independent of directional argument the above relation (2.9) reduces to

$$b_{l[mn]} - \lambda_l a_{[mn]} + b_{m[nl]} - \lambda_l a_{[nl]} + b_{n[lm]} - \lambda_n a_{[lm]} = 0 . \tag{2.10}$$

Thus we have the following theorems:

**Theorem 2.3.** In $3RF_n$ the recurrence tensor $b_{lmn}$ satisfies (2.9).

Theorem 2.4. In $3RF_n$, if the recurrence vector is independent of $\dot{x}^i$ then (2.10) holds.

**Theorem 2.5.** In $3RF_n$ the Bianchi identity satisfied by the projective tensor field takes the form

$$b_{lmn} W^j_{hk} + b_{hmn} W^j_{kl} + b_{kmn} W^j_{lh} =$$
$$= \frac{\dot{x}^j}{n+1} \{ (H_{hk(l)(m)(n)} + H_{kl(h)(m)(n)} + H_{lh(k)(m)(n)}) -$$
$$- (H_{kh(l)(m)(n)} + H_{lk(h)(m)(n)} + H_{hl(k)(m)(n)}) \} +$$
$$+ \frac{\delta^j_h}{n^2-1} \{n (H_{k(l)(m)(n)} - H_{l(k)(m)(n)}) +$$
$$+ \dot{x}^r (H_{kr(l)(m)(n)} - H_{lr(k)(m)(n)})\} +$$

$$+ \frac{\delta_k^j}{n^2 - 1} \{ n ( H_{l(h)(m)(n)} - H_{h(l)(m)(n)} ) +$$

$$+ \dot{x}^r ( H_{lr(h)(m)(n)} - H_{hr(l)(m)(n)} ) \} +$$

$$+ \frac{\delta_l^j}{n^2 - t} \{ n ( H_{h(k)(m)(n)} - H_{k(h)(m)(n)} ) +$$

$$+ \dot{x}^r ( H_{hr(k)(m)(n)} - H_{kr(h)(m)(n)} ) \} . \tag{2.11}$$

**Proof.** Differentiating (1.5) covariantly with respect to $x^l$, we have

$$W^j_{hk(l)} = H^j_{hk(l)} + \frac{\dot{x}^j}{n+1} \{ H_{hk(l)} - H_{kh(l)} \} +$$

$$+ \frac{\delta_h^j}{n^2 - 1} \{ n H_{k(l)} + \dot{x}^r H_{kr(l)} \} -$$

$$- \frac{\delta_k^j}{n^2 - 1} \{ n H_{h(l)} + \dot{x}^r H_{hr(l)} \} . \tag{2.12}$$

Adding the expressions obtained by the cyclic interchange of the indices $h, k, l$ in (2.12), we obtain

$$W^j_{hk(l)} + W^j_{kl(h)} + W^j_{lh(k)} = H^j_{hk(l)} + H^j_{kl(h)} + H^j_{lh(k)} +$$

$$+ \frac{\dot{x}^j}{n+1} \{ H_{hk(l)} + H_{kl(h)} + H_{lh(k)} - H_{kh(l)} - H_{lk(h)} - H_{hl(k)} \} +$$

$$+ \frac{\delta_h^j}{n^2 - 1} \{ n H_{k(l)} - n H_{l(k)} + \dot{x}^r H_{kr(l)} - \dot{x}^r H_{lr(k)} \} +$$

$$+ \frac{\delta_k^j}{n^2 - 1} \{ n H_{l(h)} - n H_{h(l)} + \dot{x}^r H_{lr(h)} - \dot{x}^r H_{hr(l)} \} +$$

$$+ \frac{\delta_l^j}{n^2 - 1} \{ n H_{h(k)} - n H_{k(l)} + \dot{x}^r H_{hr(k)} - \dot{x}^r H_{kr(h)} \} . \tag{2.13}$$

Using (1.7) in (2.13) and differentiating covariantly with respect to $x^m$, $x^n$ successively, we have

$$W^j_{hk(l)(m)(n)} + W^j_{kl(h)(m)(n)} + W^j_{lh(k)(m)(n)} =$$

$$= \frac{\dot{x}^j}{n+1} \{H_{hk(l)(m)(n)} + H_{kl(h)(m)(n)} + H_{lh(k)(m)(n)} - \tag{2.14}$$

$$- H_{kh(l)(m)(n)} - H_{lk(h)(m)(n)} - H_{hl(k)(m)(n)}\} +$$

$$+ \frac{\delta^j_h}{n^2-1} \{n (H_{k(l)(m)(n)} - H_{l(k)(m)(n)}) + \dot{x}^r (H_{kr(l)(m)(n)} - H_{lr(k)(m)(n)})\} +$$

$$+ \frac{\delta^j_k}{n^2-1} \{n (H_{l(h)(m)(n)} - H_{h(l)(m)(n)}) + \dot{x}^r (H_{lr(h)(m)(n)} - H_{hr(l)(m)(n)})\} +$$

$$+ \frac{\delta^j_l}{n^2-1} \{n (H_{hl(k)(m)(n)} - H_{kl(h)(m)(n)}) + \dot{x}^r (H_{hr(k)(m)(n)} - H_{kr(h)(m)(n)})\}.$$

Using (2.1) in (2.14) we get the required result.

## REFERENCES


[1] RUND, H. : **The Differential Geometry of Finsler Space**, Springer-Verlag, Berlin, 1959.

[2] MISHRA, R.S. and PANDEY, H.D. : *Recurrent Finsler Space*, J. Indian Math. Soc. 32 (1968), 17-22.

[3] PANDEY, H.D. and PANDE, J.P. : *A W-Recurrent Finsler Space of the Second Order*, Bulletin de l'Académie Serbe des Sciences et des Arts, Classe des sciences mathématiques et naturelles, sciences mathématiques LV (1976), No.9, 65-71.

[4] PANDEY, H.D. : *The Recurrent Finsler Space of Third Order*, Indian Journal of Pure and Applied Mathematics (to appear).

[5] SINHA, B.B. and SINGH, S.P. : *On Recurrent Space of Second Order in Finsler Spaces*, The Yokohama Mathematical Journal **XVIII** (1970), 27-32.

[6] WALHKER, A.G. : *On Ruse Spaces of Recurrent Curvature*, Proc. London Math. Soc , **50** (1950), 36-64.

[7] MOOR, A. : *Untersuchungen über Finslerräume von rekurrenter Krümmung*, Tensor, N S., **13** (1963), 1-18.

[8] RAY, A.K. : *On generalized 2-recurrent tensors in Riemannian spaces*, Acad. Roy. Belg. Bull. Cl. Sei. **5** (58) (1972), 220-228.



DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JABALPUR
JABALPUR, INDIA

DEPARTMENT OF MATHEMATICS
KISAN INTER COLLEGE
BASTI, 272001 (U.P.), INDIA


## ÖZET


Bu çalışmada, 3. mertebeden tekrarlı Finsler uzayları hakkında bazı sonuçlar elde edilmektedir.