ON THE (p, q)-ORDER AND (p, q)-TYPE OF ENTIRE FUNCTIONS

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In this paper, we have considered a unified mean $P_{\delta,k}(r)$ for an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| = r$ and have obtained certain growth relations on (p, q)-order and (p, q)-type of $f(z)$. We have also studied the results pertaining to the means $I_\delta(r)$ and $P_{\delta,k}(r)$ for the n th derivative $f^{(n)}(z)$ of an entire function $f(z)$. It will be assumed throughout that all entire functions under consideration are of same index pair (p, q).

1. Introduction. The (p, q)-order $\rho(p, q)$ of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, having an index pair (p, q), $p \geqq q \geqq 1$ is given by [1]:

$$\mathrm{hm} \sup_{r \to \infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \rho(p, q) \equiv \rho \,, \tag{1.1}$$

and the function $f(z)$ having (p, q)-order ρ $(b < \rho < \infty)$ is said to be of (p, q)-type $T(p, q)$ [2], if

$$\mathrm{iim} \sup_{r \to \infty} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^\rho} = T(p, q) \equiv T \,, \tag{1.2}$$

where $M(r) = \max_{|z|=r} |f(z)|$, $\log^{[0]} x = x$, $\log^{[n]} x = \log(\log^{[n-1]} x)$ for $0 < \log^{[n-1]} x < \infty$, $b = 1$ if $p = q$ and $b = 0$ if $p > q$.

Let us define

$$I_\delta(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \,, \quad 0 < \delta < \infty \tag{1.3}$$

and

$$P^m_{\delta,k}(r, f) = P_{\delta,k}(r, f) \equiv P_{\delta,k}(r) =$$

$$= \frac{k+1}{2\pi (\log^{[m]} r)^{k+1}} \int_c^r \int_0^{2\pi} \frac{|f(xe^{i\theta})|^\delta (\log^{[m]} x)^k}{V_{[m-1]}(x)} \, dx \, d\theta \,, \tag{1.4}$$

where $-1 < k < \infty$; $m = 0, 1, 2, \ldots$; c is a constant depending on m and

$$V_{[m]}(x) = \prod_{l=0}^{m} \log^{[l]} x .$$

Our aim in this paper is to investigate certain growth relations of the means $I_\delta(r)$ and $P_{\delta,k}(r)$ for an entire function of (p, q)-order ρ and (p, q)-type T. It will be assumed throughout that all entire functions under consideration are of same index pair (p, q). For the definition of index pair etc. see Juneja et al. [1], [2].

2. **Theorem 1.** If $f_1(z)$ and $f_2(z)$ are two entire functions of (p, q)-orders ρ_1 and ρ_2 $(0 \leqq \rho_1 \leqq \infty,\ 0 \leqq \rho_2 \leqq \infty)$, then a sufficient condition for $\rho_1 = \rho_2$ is that

$$\limsup_{r\to\infty} (P_{\delta,k}(r, f_2) - P_{\delta,k}(r, f_1)) \tag{2.1}$$

exists and is finite. The condition is also necessary if $0 \leqq \rho_1 < \infty$ and $0 \leqq \rho_2 < \infty$.

Proof. We suppose the superior limit in (2.1) exists and is equal to β, that is

$$\limsup_{r\to\infty} (P_{\delta,k}(r, f_2) - P_{\delta,k}(r, f_1)) = \beta .$$

Then for any $\varepsilon > 0$ and sufficiently large r,

$$P_{\delta,k}(r, f_2) - P_{\delta,k}(r, f_1) < \beta + \varepsilon,$$

or,

$$\frac{P_{\delta,k}(r, f_2)}{P_{\delta,k}(r, f_1)} - 1 < \frac{\beta + \varepsilon}{P_{\delta,k}(r, f_1)} .$$

Hence,

$$\lim_{r\to\infty} \left\{ \frac{P_{\delta,k}(r, f_2)}{P_{\delta,k}(r, f_1)} - 1 \right\} = 0,$$

since $P_{\delta,k}(r)$ increases with r. Therefore, as $r \to \infty$

$$P_{\delta,k}(r, f_2) \sim P_{\delta,k}(r, f_1) , \tag{2.2}$$

or [4],

$$\rho_2 = \limsup_{r\to\infty} \frac{\log^{[p]} P_{\delta,k}(r, f_2)}{\log^{[q]} r} = \limsup_{r\to\infty} \frac{\log^{[p]} P_{\delta,k}(r, f_1)}{\log^{[q]} r} = \rho_1 ,$$

showing that the condition (2.1) is sufficient.

Now, we establish the necessary part of the theorem by showing that if $\rho_1 \neq \rho_2$, then (2.1) is not finite. We suppose that $\rho_2 > \rho_1$, then

$$\limsup_{r\to\infty} \frac{\log^{[p]} P_{\delta,k}(r, f_2)}{\log^{[q]} r} > \limsup_{r\to\infty} \frac{\log^{[p]} P_{\delta,k}(r, f_1)}{\log^{[q]} r} ,$$

or,

$$\limsup_{r\to\infty} \frac{\log^{[p]} P_{\delta,k}(r, f_2)}{\log^{[q]} r} - \limsup_{r\to\infty} \frac{\log^{[p]} P_{\delta,k}(r, f_1)}{\log^{[q]} r} = a > 0 .$$

This gives

$$\limsup_{r\to\infty} \log \left\{ \frac{\log^{[p-1]} P_{\delta,k}(r, f_2)}{\log^{[p-1]} P_{\delta,k}(r, f_1)} \right\} = \infty .$$

Hence,

$$\limsup_{r\to\infty} \left\{ \frac{\log^{[p-1]} P_{\delta,k}(r, f_2)}{\log^{[p-1]} P_{\delta,k}(r, f_1)} - 1 \right\} = \infty$$

from which it follows that (2.1) is not finite.

Theorem 2. If $f_1(z)$ and $f_2(z)$ are two entire functions of same (p, q)-order ρ $(b < \rho < \infty)$ and perfectly regular (p, q)-growth of (p, q)-types T_1 $(0 \leqq T_1 < \infty)$ and T_2 $(0 \leqq T_2 < \infty)$, respectively, then, as $r \to \infty$,

$$\log \left\{ \frac{\log^{[p-2]} P_{\delta,k}(r, f_1)}{\log^{[p-2]} P_{\delta,k}(r, f_2)} \right\} = \begin{cases} O\,(\log^{[q-1]} r)^{\rho} \text{ if, and only if } T_1 \neq T_2 \\ o\,(\log^{[q-1]} r)^{\rho} \text{ if, and only if } T_1 = T_2 . \end{cases} \tag{2.3}$$

Proof. For every entire function $f(z)$ of (p, q)-order ρ $(b < \rho < \infty)$ and perfectly regular (p, q)-growth of type T, it follows [4] that

$$\lim_{r\to\infty} \frac{\log^{[p-1]} P_{\delta,k}(r)}{(\log^{[q-1]} r)^{\rho}} = T. \tag{2.4}$$

Making use of (2.4) for the entire functions $f_1(z)$ and $f_2(z)$, and subtracting the resulting expressions, we find

$$\lim_{r\to\infty} \frac{\log \left\{ \dfrac{\log^{[p-2]} P_{\delta,k}(r, f_1)}{\log^{[p-2]} P_{\delta,k}(r, f_2)} \right\}}{(\log^{[q-1]} r)^{\rho}} = T_1 - T_2 ,$$

from which the result in (2.3) is immediate.

3. In this section we shall study results pertaining to the means $I_\delta(r)$ and $P_{\delta,k}(r)$ for the n th derivative $f^{(n)}(z)$ of an entire function $f(z)$. The function $I_\delta(r)$ is defined as follows:

$$I_\delta(r, f^{(n)}) = \frac{1}{2\pi} \int_0^{2\pi} |f^{(n)}(r\, e^{i\theta})|^{\delta}\, d\theta\, , \; 0 < \delta < \infty . \tag{3.1}$$

Theorem 3. If $I_\delta(r, f^{(n)})$ and $P_{\delta,k}(r, f^{(n)})$ are the means of the n th derivative $f^{(n)}(z)$ of an entire function $f(z)$, then, for any k $(-1 < k < \infty)$ and $0 < r_1 < r_2 < \infty$,

$$I_\delta(r_1, f^{(n)}) \leqq \frac{(\log^{[m]} r_2)^{k+1} P_{\delta,k}(r_2, f^{(n)}) - (\log^{[m]} r_1)^{k+1} P_{\delta,k}(r_1, f^{(n)})}{(\log^{[m]} r_2)^{k+1} - (\log^{[m]} r_1)^{k+1}} \leqq$$
$$\leqq I_\delta(r_2, f^{(n)}) . \tag{3.2}$$

Proof. From (1.4), we have

$$P_{\delta,k}(r, f^{(n)}) = \frac{k+1}{2\pi (\log^{[m]} r)^{k+1}} \int_c^r \int_0^{2\pi} \frac{|f^{(n)}(x e^{i\theta})|^\delta (\log^{[m]} x)^k}{V_{[m-1]}(x)} dx\, d\theta$$

$$= \frac{k+1}{(\log^{[m]} r)^{k+1}} \int_c^r \frac{I_\delta(x, f^{(n)}) (\log^{[m]} x)^k}{V_{[m-1]}(x)} dx .$$

Therefore,

$$P_{\delta,k}(r_1, f^{(n)}) = \frac{k+1}{(\log^{[m]} r_1)^{k+1}} \int_c^{r_1} \frac{I_\delta(x, f^{(n)}) (\log^{[m]} x)^k}{V_{[m-1]}(x)} dx \tag{3.3}$$

and

$$P_{\delta,k}(r_2, f^{(n)}) = \frac{k+1}{(\log^{[m]} r_2)^{k+1}} \int_c^{r_2} \frac{I_\delta(x, f^{(n)}) (\log^{[m]} x)^k}{V_{[m-1]}(x)} dx. \tag{3.4}$$

From (3.3) and (3.4), we find

$$(\log^{[m]} r_2)^{k+1} P_{\delta,k}(r_2, f^{(n)}) - (\log^{[m]} r_1)^{k+1} P_{\delta,k}(r_1, f^{(n)}) = \tag{3.5}$$

$$= (k+1) \int_{r_1}^{r_2} \frac{I_\delta(x, f^{(n)}) (\log^{[m]} x)^k}{V_{[m-1]}(x)} dx \leqq$$

$$\leqq I_\delta(r_2, f^{(n)}) \left((\log^{[m]} r_2)^{k+1} - (\log^{[m]} r_1)^{k+1} \right)$$

and

$$(\log^{[m]} r_2)^{k+1} P_{\delta,k}(r_2, f^{(n)}) - (\log^{[m]} r_1)^{k+1} P_{\delta,k}(r_1, f^{(n)}) \geqq \tag{3.6}$$
$$\geqq I_\delta(r_1, f^{(n)}) \left((\log^{[m]} r_2)^{k+1} - (\log^{[m]} r_1)^{k+1} \right) .$$

(3.5) and (3.6) give the desired result.

Theorem 4. If $I_\delta(r, f^{(n)})$ and $P_{\delta,k}(r, f^{(n)})$ are the means of the n th derivative $f^{(n)}(z)$ of an entire function $f(z)$ and $M(r, f^{(n)})$ is the maximum of $|f^{(n)}(z)|$, $|z| = r$, then for any $k > -1$,

$$\limsup_{r\to\infty} \frac{P_{\delta,k}(r, f^{(n)})}{(M(r, f^{(n)}))^\delta} \leqq \limsup_{r\to\infty} \frac{P_{\delta,k}(r, f^{(n)})}{I_\delta(r, f^{(n)})} \leqq 1 . \tag{3.7}$$

Proof. We have

$$P_{\delta,k}(r,f^{(n)}) = \frac{k+1}{(\log^{[m]}r)^{k+1}} \int_c^r \frac{I_\delta(x,f^{(n)})\,(\log^{[m]}x)^k}{V_{[m-1]}(x)}\,dx$$

$$= I_\delta(r,f^{(n)})\left[1 - \left\{\frac{\log^{[m]}c}{\log^{[m]}r}\right\}^{(k+1)}\right],$$

therefore,

$$\limsup_{r\to\infty} \frac{P_{\delta,k}(r,f^{(n)})}{I_\delta(r,f^{(n)})} \leqq 1\,. \tag{3.8}$$

From (3.1), we get

$$I_\delta(r,f^{(n)}) \leqq \{M(r,f^{(n)})\}^\delta,$$

it follows that

$$\frac{P_{\delta,k}(r,f^{(n)})}{I_\delta(r,f^{(n)})} \geqq \frac{P_{\delta,k}(r,f^{(n)})}{(M(r,f^{(n)}))^\delta}, \tag{3.9}$$

whence, in view of (3.8)

$$\limsup_{r\to\infty} \frac{P_{\delta,k}(r,f^{(n)})}{(M(r,f^{(n)}))^\delta} \leqq \limsup_{r\to\infty} \frac{P_{\delta,k}(r,f^{(n)})}{I_\delta(r,f^{(n)})} \leqq 1\,.$$

Theorem 5. For a class of entire functions for which $\log I_\delta(r)$ is an increasing convex function of $r \log r$, we have, for every arbitrary small $\varepsilon > 0$ and $r > r_0$,

$$\frac{P_{\delta,k}(r,f^{(1)})}{P_{\delta,k}(r)} > (1-\varepsilon)\,\frac{I_\delta(r-\alpha,f^{(1)})}{I_\delta(r-\alpha)}\,,$$

where α is fixed and > 0.

To prove this theorem we need the following lemma:

Lemma 1 [3]. For $r > r_0$, we have

$$I_\delta(r,f^{(1)}) \geqq I_\delta(r)\left\{\frac{\log I_\delta(r)}{\delta\, r \log r}\right\}^\delta.$$

Proof of Theorem 5. From (1.4), we have

$$P_{\delta,k}(r,f^{(1)}) > \frac{k+1}{(\log^{[m]}r)^{k+1}} \int_{r-\alpha}^r \frac{I_\delta(x,f^{(1)})\,(\log^{[m]}x)^k}{V_{[m-1]}(x)}\,dx \geqq$$

$$\geqq \frac{I_\delta(r-\alpha,f^{(1)})}{I_\delta(r-\alpha)}\left[\frac{k+1}{(\log^{[m]}r)^{k+1}} \int_{r-\alpha}^r \frac{I_\delta(x)\,(\log^{[m]}x)^k}{V_{[m-1]}(x)}\,dx\right],$$

since, by Lemma 1, $I_\delta(r, f^{(1)}) / I_\delta(r)$ increases with r. Hence

$$P_{\delta,k}(r, f^{(1)}) > \frac{I_\delta(r-\alpha, f^{(1)})}{I_\delta(r-\alpha)}\left[\frac{k+1}{(\log^{[m]} r)^{k+1}}\left\{\int_c^r - \int_c^{r-\alpha}\right\}\frac{I_\delta(x)(\log^{[m]} x)^k}{V_{[m-1]}(x)}\,dx\right]$$

$$= \frac{I_\delta(r-\alpha, f^{(1)})}{I_\delta(r-\alpha)}\left[P_{\delta,k}(r) - \left\{\frac{\log^{[m]}(r-\alpha)}{\log^{[m]} r}\right\}^{k+1} P_{\delta,k}(r-\alpha)\right].$$

Now, from the definitions of $I_\delta(r)$ and $P_{\delta,k}(r)$, we get

$$\log\left[\left\{\frac{\log^{[m]} r}{\log^{[m]}(r-\alpha)}\right\}^{k+1}\frac{P_{\delta,k}(r)}{P_{\delta,k}(r-\alpha)}\right] = \int_{r-\alpha}^{r}\frac{I_\delta(x)}{P_{\delta,k}(x)\,V_{[m]}(x)}\,dx.$$

Hence,

$$\exp\left[\frac{I_\delta(r-\alpha)}{P_{\delta,k}(r-\alpha)}(\log^{[m+1]} r - \log^{[m+1]}(r-\alpha))\right] \leqq$$

$$\leqq \left\{\frac{\log^{[m]} r}{\log^{[m]}(r-\alpha)}\right\}^{k+1}\frac{P_{\delta,k}(r)}{P_{\delta,k}(r-\alpha)} \leqq$$

$$\leqq \exp\left[\frac{I_\delta(r)}{P_{\delta,k}(r)}(\log^{[m+1]} r - \log^{[m+1]}(r-\alpha))\right].$$

Therefore,

$$P_{\delta,k}(r-\alpha) = o(P_{\delta,k}(r)),$$

and so, we find, for $r > r_0$,

$$P_{\delta,k}(r, f^{(1)}) > (1-\varepsilon)\,P_{\delta,k}(r)\,I_\delta(r-\alpha, f^{(1)})/I_\delta(r-\alpha).$$

Finally we show that :

Theorem 6. For a class of entire functions for which $\log I_\delta(r)$ is an increasing function of $r \log r$, we find

$$\frac{P_{\delta,k}(r, f^{(1)})}{P_{\delta,k}(r)} \leqq \frac{I_\delta(r, f^{(1)})}{I_\delta(r)}.$$

Proof. We have

$$P_{\delta,k}(r, f^{(1)}) = \frac{k+1}{(\log^{[m]} r)^{k+1}} \int_c^r \frac{I_\delta(x, f^{(1)})}{I_\delta(x)} \frac{I_\delta(x)(\log^{[m]} x)^k}{V_{[m-1]}(x)} dx \leqslant$$

$$\leqq \frac{I_\delta(r, f^{(1)})}{I_\delta(r)} \left\{ \frac{k+1}{(\log^{[m]} r)^{k+1}} \int_c^r \frac{I_\delta(x)(\log^{[m]} x)^k}{V_{[m-1]}(x)} dx \right\} =$$

$$= \frac{I_\delta(r, f^{(1)})}{I_\delta(r)} P_{\delta,k}(r),$$

since $I_\delta(x, f^{(1)})/I_\delta(x)$ is an increasing function and this proves the result.

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ÖZET

Bu çalışmada bir $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| = r$, tam fonksiyonuna ait bir $P_{\delta,k}(r)$ birleştirilmiş ortalaması gözönüne alınmakta ve $f(z)$ in (p, q) mertebe ve (p, q) tipine ilişkin bazı büyüme bağıntıları elde edilmektedir.