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PROPERTIES OF FINITE SYSTEMS OF CONVEX CURVES

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This paper is concerned with properties of a plane closed curves which contains the class of all ovals. We introduce the width of a pair of curves in our class. All pairs of constant width are found and the counterpart of Barbier theorem is proved. Moreover the extensions of some global theorems onto finite system of curves of the considered class are given.

1. Introduction

In this paper we will consider the class *AI* of plane closed curves of the following form

$$z(\theta) = \int_0^\theta r(u)\, e^{iu}\, du \quad \text{for } 0 \leq \theta \leq 2\pi\,, \tag{1}$$

where r is a continuous, 2π-periodic and positive function. If r is a differentiable function, then $\frac{1}{r}$ is the curvature and the class *AI* contains the class of ovals (see [6], [5]). A continuity and periodicity of the function r imply that the Fourier series expansion of r

$$\frac{1}{2} a_0 + \sum_{j=1}^{\infty} (a_j \cos ju + b_j \sin ju) \tag{2}$$

converges uniformly to r (see [4]).

We note that a curve of the form (1) is closed if and only if

$$a_1 = b_1 = 0\,. \tag{3}$$

We introduce the width of a pair of curves of the class *AI*. This notion is a natural generalization of the width of a convex curve. We determine all pairs of constant width. Then we prove the counterpart of Barbier theorem (see [5]) and we give the necessary and sufficient condition to be a pair of constant width.

Moreover, some global theorems we extend onto a finite system of curves of the class *AI*.

2. Pairs of curves

Let two curves C_1, C_2 of the class AI be given,

$$z_m(\theta) = w_m + \int_0^{\theta} r_m(u)\, e^{iu}\, du \quad \text{for } 0 \leq \theta \leq 2\pi, \tag{4}$$

where w_m are fixed complex numbers.

We introduce the following vector and functions:

$$\left.\begin{aligned} p(\theta) &= z_1(\theta) - z_2(\theta + \pi) \\ \delta(\theta) &= [p(\theta), e^{i\theta}] \\ \Delta(\theta) &= [p(\theta), ie^{i\theta}], \end{aligned}\right\} \tag{5}$$

where $[a + ib, c + id] = ad - bc$.

Definition 1. The function δ will be called the width of the pair (C_1, C_2).

We note that if $C_1 = C_2$, then δ is a usual width function of C_1.

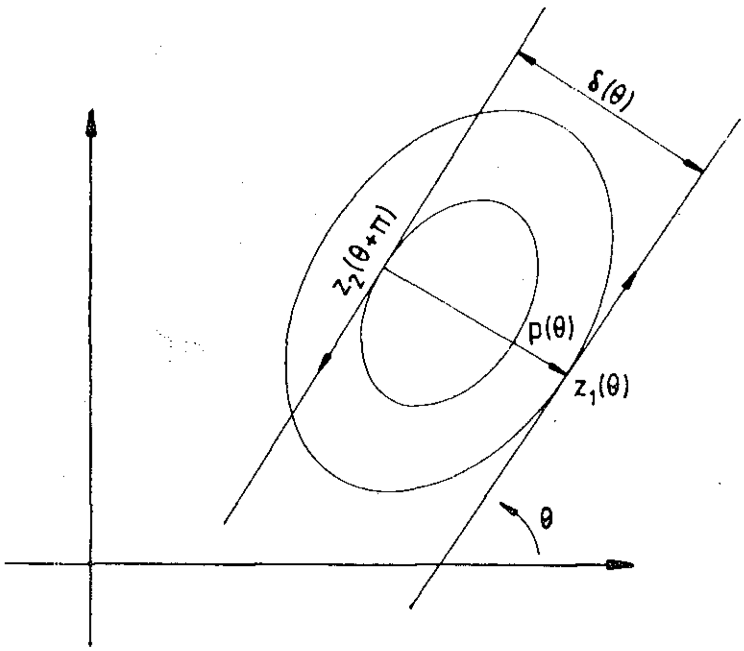

Fig. 1

Definition 2. A pair (C_1, C_2) such that $\delta \equiv$ const. will be called of constant width.

Theorem 1. If a pair (C_1, C_2) is of constant width δ, then

$$L_1 + L_2 = 2\pi\delta, \tag{6}$$

where L_m denotes the perimeter of C_m.

Proof. From (5) we can derive the following system of differential equations

$$\left.\begin{aligned} \delta'(\theta) &= \Delta(\theta), \\ \Delta'(\theta) &= -\delta(\theta) + r_1(\theta) + r_2(\theta+\pi). \end{aligned}\right\} \tag{7}$$

We have

$$L_m = \int_0^{2\pi} r_m(u)\, du.$$

Thus integrating the second relation of (7) we immediately obtain (6).

It is easy to see that the well-known theorem of Barbier (see [5]) is a special case of Th. 1.

Moreover, from (7) we get

Proposition 1. If a pair (C_1, C_2) is of constant width δ, then we have

$$r_1(\theta) + r_2(\theta + \pi) \equiv \delta \quad \text{for } 0 \leq \theta \leq 2\pi. \tag{8}$$

Let us consider a pair (C_1, C_2) of constant width. Let C_m, $m=1, 2$ be given by (4), $w_m = x_m + iy_m$ and

$$r_m(u) = \frac{1}{2} a_0^m + \sum_{j=2}^{\infty} (a_j^m \cos ju + b_j^m \sin ju). \tag{9}$$

We give some characterization of the pair (C_1, C_2) with the help of the coefficients of the Fourier series expansion. We have

$$\delta = [p(\theta), e^{i\theta}] = [w_1 - w_2, e^{i\theta}] +$$

$$+ \int_0^{\theta} (r_1(u) - r_2(u))\, [e^{i\theta}, e^{i\theta}]\, du - \int_{\theta}^{\theta+\pi} r_2(u)\, [e^{iu}, e^{i\theta}]\, du =$$

$$= (x_1 - x_2) \sin\theta - (y_1 - y_2) \cos\theta -$$

$$- \int_0^{\theta} (r_1(u) - r_2(u)) \sin(u - \theta)\, du + \int_{\theta}^{\theta+\pi} r_2(u) \sin(u - \theta)\, du =$$

$$= (x_1 - x_2)\sin\theta + (y_2 - y_1)\cos\theta - \frac{1}{2}(a_0^1 - a_0^2)\int_0^\theta \sin(u-\theta)\,du -$$

$$-\sum_{j=2}^{\infty}\Big\{(a_j^1 - a_j^2)\int_0^\theta \cos ju\sin(u-\theta)\,du + (b_j^1 - b_j^2)\int_0^\theta \sin ju\sin(u-\theta)\,du\Big\} +$$

$$+\frac{1}{2}a_0^2\int_\theta^{\theta+\pi}\sin(u-\theta)\,du + \sum_{j=2}^{\infty}\Big\{a_j^2\int_\theta^{\theta+\pi}\cos ju\sin(u-\theta)\,du +$$

$$+ b_j^2\int_0^{\theta+\pi}\sin ju\sin(u-\theta)\,du\Big\} =$$

$$= (x_1 - x_2)\sin\theta + (y_2 - y_1)\cos\theta + \frac{1}{2}(a_0^1 + a_0^2) + \frac{1}{2}(a_0^2 - a_0^1)\cos\theta +$$

$$+\frac{1}{2}\sum_{j=2}^{\infty}\Big\{(a_j^1 - a_j^2)\left[\frac{1}{j+1}(\cos j\theta - \cos\theta) - \frac{1}{j-1}(\cos j\theta - \cos\theta)\right] +$$

$$-(b_j^1 - b_j^2)\left[\frac{1}{j+1}(\sin j\theta + \sin\theta) - \frac{1}{j-1}(\sin j\theta - \sin\theta)\right]\Big\} +$$

$$+\frac{1}{2}\sum_{j=2}^{\infty}\Big\{a_j^2\left[\frac{-1}{j+1}(\cos(j\theta+(j+1)\pi)-\cos j\theta)+\frac{1}{j-1}(\cos(j\theta+(j-1)\pi)-\cos j\theta)\right]+$$

$$+b_j^2\left[\frac{1}{j+1}(\sin(j\theta+(j+1)\pi)-\sin j\theta)-\frac{1}{j-1}(\sin(j\theta+(j-1)\pi)-\sin j\theta)\right]\Big\} =$$

$$= (x_1 - x_2)\sin\theta + (y_2 - y_1)\cos\theta + \frac{1}{2}(a_0^1 + a_0^2) + \frac{1}{2}(a_0^2 - a_0^1)\cos\theta +$$

$$+\frac{1}{2}\sum_{j=2}^{\infty}\Big\{(a_j^1 - a_j^2)\frac{-2}{j^2-1}(\cos j\theta - \cos\theta) - (b_j^1 - b_j^2)\left[\frac{-2}{j^2-1}\sin j\theta + \frac{2j}{j^2-1}\sin\theta\right]\Big\} +$$

$$+\frac{1}{2}\sum_{j=2}^{\infty}\Big\{a_j^2\left[\frac{-1}{j+1}((-1)^{j+1}-1)\cos j\theta + \frac{1}{j-1}((-1)^{j+1}-i)\cos j\theta\right] +$$

$$+ b_j^2\left[\frac{1}{j+1}((-1)^{j+1}-1)\sin j\theta - \frac{1}{j-1}((-1)^{j-1}-1)\sin j\theta\right]\Big\} =$$

$$= \frac{1}{2}(a_0^1 + a_0^2) + \left(x_1 - x_2 - \sum_{j=2}^{\infty} \frac{j}{j^2-1}(b_j^1 - b_j^2) \right) \sin 0 +$$

$$+ \left(y_2 - y_1 - \frac{1}{2}(a_0^1 - a_0^2) + \sum_{j=2}^{\infty} \frac{1}{j^2-1}(a_j^1 - a_j^2) \right) \cos \theta +$$

$$+ \sum_{j=2}^{\infty} \frac{1}{j^2-1}[a_j^2 - a_j^1 + ((-1)^{j+1} - 1) a_j^2] \cos j\theta +$$

$$+ \sum_{j=2}^{\infty} \frac{1}{j^2-1}[b_j^1 - b_j^2 - ((-1)^{j+1} - 1) b_j^2] \sin j\theta =$$

$$= \frac{1}{2}(a_0^1 + a_0^2) + \left(x_1 - x_2 - \sum_{j=2}^{\infty} \frac{j}{j^2-1}(b_j^1 - b_j^2) \right) \sin \theta +$$

$$+ \left(y_2 - y_1 - \frac{1}{2}(a_0^1 - a_0^2) + \sum_{j=2}^{\infty} \frac{1}{j^2-1}(a_j^1 - a_j^2) \right) \cos \theta +$$

$$+ \sum_{j=2}^{\infty} \frac{1}{j^2-1}[(-1)^{j+1} a_j^2 - a_j^1] \cos j\theta + \sum_{j=2}^{\infty} \frac{1}{j^2-1}[b_j^1 - (-1)^{j+1} b_j^2] \sin j\theta .$$

Hence we obtain the following theorem :

Theorem 2. A pair (C_1, C_2) is of constant width if and only if the Fourier coefficients of r_m satisfy the conditions

$$a_0^1 + a_0^2 = 2 ,$$

$$\left\{ \begin{array}{l} a_j^1 + a_j^2 = 0 \\ b_j^1 + b_j^2 = 0 \end{array} \right. \quad \text{for even } j ,$$

$$\left\{ \begin{array}{l} a_j^1 - a_j^2 = 0 \\ b_j^1 - b_j^2 = 0 \end{array} \right. \quad \text{for odd } j , \tag{10}$$

$$x_1 - x_2 = \sum_{j=2,4,\dots}^{\infty} \frac{j}{j^2-1}(b_j^1 - b_j^2) ,$$

$$y_1 - y_2 = \sum_{j=2,4,\dots}^{\infty} \frac{1}{j^2-1}(a_j^1 - a_j^2) .$$

We note that if $C_1 = C_2$ and C_1 is of constant width, then we obtain the characterization of curves of constant width given by Tennison [6].

3. Parallel curves

Let us consider two curves C_1, C_2 of the class AI given by (4).

Let

$$\left.\begin{aligned} q(\theta) &= z_1(\theta) - z_2(\theta) \\ \omega(\theta) &= [q(\theta), e^{i\theta}] \\ \Omega(\theta) &= [q(\theta), ie^{i\theta}]. \end{aligned}\right\} \tag{11}$$

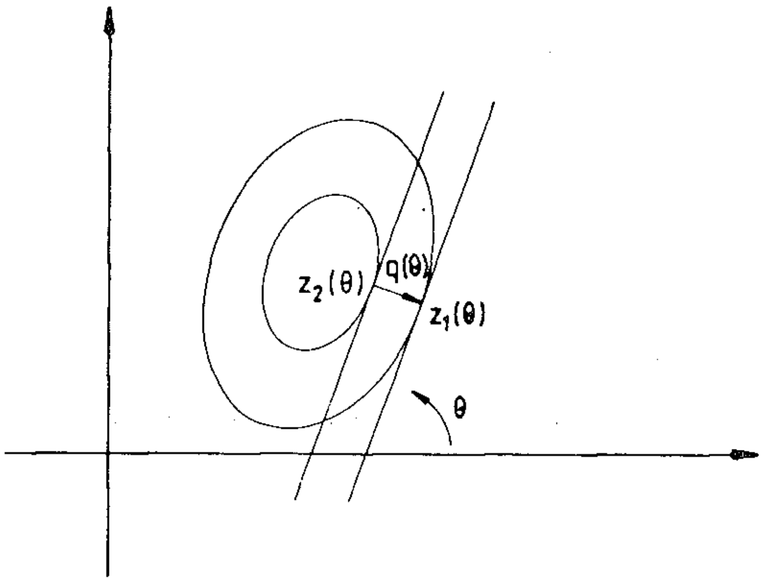

Fig. 2

We note that the curve C_2 lies in the interior of the set bounded by C_1 if and only if

$$\omega(\theta) > 0 \quad \text{for} \quad 0 \leqslant \theta \leqslant 2\pi. \tag{12}$$

We prove the following theorem:

Theorem 3. If $\omega \equiv \text{const.} > 0$, then a pair (C_1, C_2) forms a parallel curves.

Proof. Making use of the results of the previous paragraph we may write

$$\omega = [q(\theta), e^{i\theta}] = [w_1 - w_2, e^{i\theta}] - \int_0^\theta (r_1(u) - r_2(u)) \sin(u - \theta)\, du =$$

$$= (x_1 - x_2) \sin\theta + (y_2 - y_1) \cos\theta + \frac{1}{2}(a_0^1 - a_0^2)(1 - \cos\theta) +$$

$$+ \sum_{j=2}^{\infty} \left\{ (a_j^1 - a_j^2)(\cos\theta - \cos j\theta)\frac{1}{j^2-1} + (b_j^1 - b_j^2)\frac{1}{j^2-1}(\sin j\theta - j\sin\theta) \right\} =$$

$$= \frac{1}{2}(a_0^1 - a_0^2) + \left(x_1 - x_2 - \sum_{j=2}^{\infty} \frac{j}{j^2-1}(b_j^1 - b_j^2) \right) \sin\theta +$$

$$+ \left(y_2 - y_1 - \frac{1}{2}(a_0^1 - a_0^2) + \sum_{j=2}^{\infty} \frac{1}{j^2-1}(a_j^1 - a_j^2) \right) \cos\theta -$$

$$- \sum_{j=2}^{\infty} \frac{1}{j^2-1}(a_j^1 - a_j^2) \cos j\theta + \sum_{j=2}^{\infty} \frac{1}{j^2-1}(b_j^1 - b_j^2) \sin j\theta .$$

Hence we get

$$a_j^1 = a_j^2 \quad \text{and} \quad b_j^1 = b_j^2 \quad \text{for all} \quad j \geq 2 ,$$

$$\omega = \frac{1}{2}(a_0^1 - a_0^2) , x_1 = x_2 , y_2 - y_1 = \omega ,$$

or equivalently

$$r_2 = \omega + r_1 \quad \text{and} \quad w_2 = w_1 + i\omega .$$

Thus we have

$$q(\theta) = \omega i e^{i\theta} \quad \text{and} \quad |q(\theta)| = \omega , \ \Omega \equiv 0 .$$

It means that the vectors q and the tangent vector to C_1 are orthogonal. It ends the proof.

4. Finite sequences of curves

Let us consider an arbitrary finite sequence of curves $C_1, C_2, \ldots, C_M$ belonging to AI. Let C_m, $m = 1,\ldots,M$ be given by (4) and

$$f(\theta) = \sum_{m=1}^{M} \int_{0}^{\theta+\pi} r_m(u)\, du \quad \text{for} \quad \theta \leq \theta \leq 2\pi . \tag{13}$$

The value $f(\theta)$ is equal to the sum of all oriented arcs contained between the points $z_m(\theta)$ and $z_m(\theta + \pi)$.

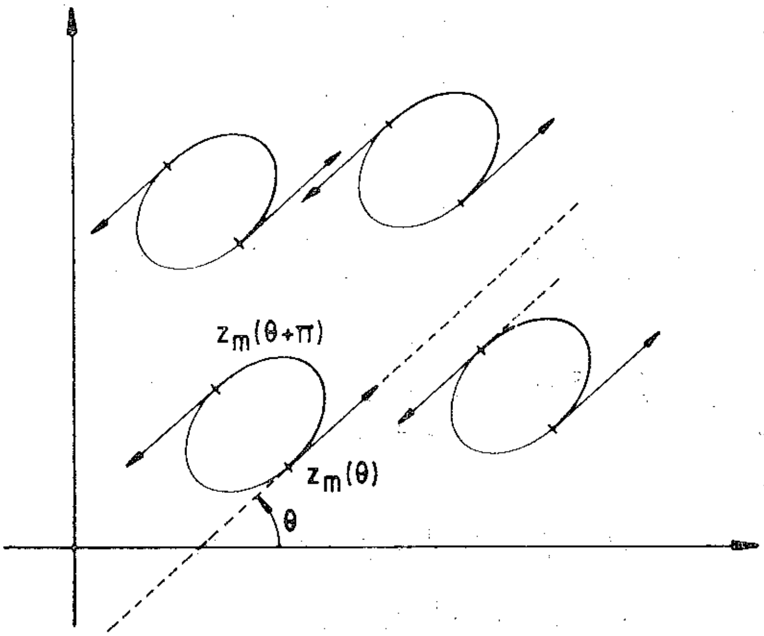

Fig. 3

By L_m we denote the perimeter of C_m.

We consider the function defined by the formula

$$g(\theta) = \sum_{m=1}^{M} \left(\int_{0}^{\theta+\pi} r_m(u)\, du - \frac{1}{2} L_m \right). \tag{14}$$

We note that, if $M = 1$, then the zero of g determines an arc pair (see [2]). Now, we prove the generalization of Th. 4 [2].

Theorem 4. There exist at least three zeros of the function g.

Proof. We have

$$g(\theta+\pi)=\sum_{m=1}^{M}\Big(\int_{\theta+\pi}^{\theta+2\pi} r_m(u)\,du-\frac{1}{2}L_m\Big)=$$

$$=\sum_{m=1}^{M}\Big(\int_{\theta}^{\theta+2\pi} r_m(u)\,du-\int_{\theta}^{\theta+\pi} r_m(u)\,du-\frac{1}{2}L_m\Big)=$$

$$=\sum_{m=1}^{M}L_m-\sum_{m=1}^{M}\int_{\theta}^{\theta+\pi} r_m(u)\,du-\frac{1}{2}\sum_{m=1}^{M}L_m=-g(\theta)\,.$$

Thus there exists θ_0 such that $g(\theta_0)=g(\theta_0+\pi)=0$. We may assume that $0_0=0$. Moreover, we have

$$-\int_0^{\pi} g(t)\sin t\,dt=g(t)\cos t\,\Big|_0^{\pi}-\int_0^{\pi}\Big(\sum_{m=1}^{M}(r_m(t+\pi)-r_m(t))\Big)\cos t\,dt=$$

$$=-\sum_{m=1}^{M}\Big(\int_0^{2\pi} r_m(t+\pi)\cos t\,dt-\int_0^{\pi} r_m(t)\cos t\,dt\Big)=\sum_{m=1}^{M}\int_0^{2\pi} r_m(t)\cos t\,dt=0$$

since each C_m is a closed curve.

The same considerations as in the proof of Blaschke-Süss Theorem (see [5], p. 202) guarantee the existence of two further zeros of g. It completes our proof.

Theorem 5. The function f reaches at least three extrema.

Proof. It suffices to apply the same considerations as in Th. 4.

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ÖZET

Bu çalışmada bütün ovallerin sınıfını içeren kapalı düzlem eğrilerinin özellikleri incelenmektedir.