İstanbul Üniv. Fen Fak. Mat. Der 48 (1987-1989), 109-111

A NOTE ON COMPACT PRESERVING FUNCTIONS

N. ERGUN

The following theorem is proved in this note : A function from a first countable space into a Hausdorff space is continuous iff it is compact preserving and all its fibers are closed.

Introduction

The following theorems characterizing continuity of certain compact preserving functions were all proved earlier.

Theorem A. [5] A function from a metrizable space X into a metrizable space Y is continuous iff it is compact preserving and all its fibers are closed.

Theorem B. [2] A function from a Hausdorff k_1-space X into a Hausdorff space Y is continuous iff it is compact preserving and all its fibers are closed.

Theorem C. [1] A function from a Hausdorff k_2-space X into a Hausdorff space Y is continuous iff it is compact preserving and all its fibers are closed.

In above and throughout the note the term *fiber* is used for an inverse set of a singleton under a function which is not necessarily a surjection. As is well known a function is called *compact preserving* iff images of all compact subsets are compact. A topological space X is called *k_1-space* iff $x \in A'$ necessarily yields the existence of a compact subset B_A of $A \cup \{x\}$ such that $x \in B_A'$ holds where A' denotes the set of all limit points of A in X as usually. X is called *k_2-space* iff $x \in A'$ necessarily yields the existence of a compact subset K_A and a subset $B_A \subseteq A$ such that $B_A \cup \{x\} \subseteq K_A$ and $x \in B_A'$ hold. These two concepts were defined respectively by Halfar [2] and Fuller [1]. Notice that these two conditions are trivially satisfied for those subsets A containing only finite number of elements. Every first countable space is trivially a k_1-space. Every locally compact and therefore every compact space and also every k_1-space is evidently, a k_2-space. Fuller has proved that the product space $X = [0, 1]^{[0,1]}$ is an example of locally connected continua which is not a k_1-space, see example 4.1 of Fuller [1]. Therefore Theorem C is generalizing both of the Theorem A and Theorem B. Cofinite topologies on infinite sets are examples of non-first countable k_1-spaces. They are compact, connected, locally connected non-Hausdorff spaces. But the space $X = \bigcup_{n=-\infty}^{\infty} S_n$ defined by Halfar [2] is an example of a σ-compact, connected, locally connected, Hausdorff, non-first countable k_1-space.

Result

By the notices given at the end of introduction the following theorem is independent of Theorem B or Theorem C. In it we explicitly use a proving technique of Hamlett [4] for being self contained.

Theorem D. A function f from a first countable X into a Hausdorff Y is continuous iff f is compact preserving and all its fibers are closed.

Proof. Only the sufficiency requires a proof. Let $\mathscr{B}_x = \{B_{nx} : n \in \mathbf{N}\}$ be the decreasing local basis of x in X. Take any basic neighborhood $W_{f(x)}$ of $f(x)$. If $B_{nx} - f^{-1}(W_{f(x)})$ is infinite for each n under the sufficiency hypothesis then by choosing the points

$$x_n \in (B_{nx} - f^{-1}(W_{f(x)})) - \{x_1, x_2, \ldots, x_{n-1}\}$$

one can define the compact subsets $K_n = \{x, x_n, x_{n+1}, \ldots\} \subseteq X$ and $f(K_n) - W_{f(x)} \subseteq Y$ each has infinite number of elements. Notice that the assumption of having a finite number of elements of an appropriate $f(K_{n_0}) - W_{f(x)}$ would yield the following contradiction :

$$\exists\, B_{mx} \in \mathscr{B}_x ; \quad \{x_k : k > m + n_0\} \subseteq B_{mx} \cap f^{-1}(f(K_{n_0}) - W_{f(x)}) = \phi$$

since fibers are closed by the hypothesis. Thus the family $\{f(K_n) - W_{f(x)} : n \in \mathbf{N}\}$ of nonempty compact-closed subsets with the finite intersection property would have an empty intersection in the compact Hausdorff subspace $f(K_1) - W_{f(x)}$, a contradiction. Therefore an appropriate difference set $B_{mx} - f^{-1}(W_{f(x)})$ and its image $f(B_{mx}) - W_{f(x)}$ must be finite which yields the existence of a suitable $B_{nx} \in \mathscr{B}_x$ disjoint with this difference set since

$$\overline{B_{mx} - f^{-1}(W_{f(x)})} \subseteq f^{-1}(f(B_{mx}) - W_{f(x)}) \subseteq X - \{x\} .$$

So $f(B_{nx}) \subseteq W_{f(x)}$ i.e. the local characterization of continuity at $x \in X$ is derived by supposing $m < n$.

This result is a generalization of Theorem A. The following special known results could also be deduced from the last theorem. As is well known the subset of all couples $(x, f(x))$ of $X \times Y$ is called as graph of f.

Corollary. A function from a first countable X into a Hausdorff Y is continuous iff f is a compact preserving function with closed graph.

Proof. Sufficiency follows from the well known fact that fibers of any function f with closed graph are all closed.

Corollary. A function from a first countable X into a compact Hausdorff Y is continuous iff its graph is closed.

Proof. It is well known that images of compact subsets under any function with closed graph are closed, see [1] for instance. This corollary could simply be derived from the Theorem 3.4. of [1] even without mentioning and using the first countableness of X. See also Theorem 3 of [6].

Corollary. A closed function from a first countable and regular X into a compact Hausdorff Y is continuous iff all its fibers are closed.

Proof. It is known that [1] the following properties are equivalent for any closed function f defined on a regular space: i) Fibers of f are closed, ii) Graph of f is closed. However, this last corollary is a special case of the following result due to Halfar [3].

Theorem E. A closed function from a regular space into a compact T_1 space is continuous iff all its fibers are closed.

REFERENCES

[1] FULLER, R.V. : Pacific J. Math. **25** (1968), 495-509.
[2] HALFAR, E. : Proc. Amer. Math. Soc. **11** (1960), 688-691.
[3] HALFAR, E. : Proc. Amer. Math. Soc. **8** (1957), 828-830.
[4] HAMLETT, T.R. : J. London Math. Soc. **10** (1975), 25-26.
[5] KLEE, V.L. and UTZ, W.R. : Proc. Amer. Math. Soc. **5** (1954), 182-184.
[6] NOIRI, T. : Casopis Pest. Math. **101** (1976), 379-382.

ÖZET

Bu kısa notta birinci sayılabilir bir uzaydan bir Hausdorff uzayına tanımlanan fonksiyonun sürekliliği için tüm lifleri kapalı bir tıkız koruyan fonksiyon olmasının gerek ve yeter koşullar olduğu gösterilmiştir.