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ON THE MODULUS CONTINUITY OF RANDOM VECTOR FIELDS

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We give a number of results with respect to the modulus of continuity of random vector fields.

1. INTRODUCTION

Modulus continuity of random vector fields has not yet been sufficiently investigated.

In this paper, an attempt has been made to investigate modulus continuity of random vector fields. Let $X(x)$ be a seperable random n vector field on $[0,1]^n$ (See for instance [1], [2]). We consider $t = (l_1, l_2, \dots, l_i)^{-1}$, where $\{l_s\}$ is a sequence of integers and $l_s \geq 1$. We may take $X(x)$ to be a column vector and X' the corresponding row vector.

2. MAIN RESULTS

Theorem 1. Let $X(x)$ be a seperable random n vector field on $[0,1]^n$ and let

$$\sup_{|\tilde{t}_i| \leq t} P\{ \| X(x+\tilde{t}) - X(x) \| \geq f(t) \} \leq h(t), \tag{1}$$

where $f(t)$ is a positive nondecreasing even function on $(0, +\infty)$ such that

$$\sum_{i=1}^{\infty} f(t_i) < \infty \tag{2}$$

and $h(t)$ is a nondecreasing even function on $(0, +\infty)$ such that

$$\sum_{i=1}^{\infty} t_{i+1}^{-n} h(t_i) < +\infty \tag{3}$$

then with probability one, there exists a random variable $T(w)$ such that

$$P\{T(w) > 0\} = 1$$

and

$$\| X(x+\tilde{t}) - X(x) \| \leq h(t), \tag{4}$$

for $|\tilde{t}_i| \leq t < T(w)$, where

$$h(t) = 2 \sum_{l=N}^{\infty} f(t_l)$$

and N is determined from the relation

$$t_N \leq t \leq t_{N-1} \, . \tag{5}$$

Proof. By (1) the random vector field is stochastically continuous and therefore any dense set which is countable everywhere may be regarded as a separability set. For the separability set, we choose $Q = \bigcup_{i=1}^{\infty} Q_i^n$, where

$$Q_i = \{ K.t_i : K = 0,1,\ldots,t_i^{-1} \}$$

$$Q_i^n = Q_i \,\ldots\ldots\ldots\ldots\ldots Q_i \, .$$

We will also consider the sets

$$S_{i,s} = \{ x : S_m \, . \, t_i \leq x_m < (S_m + 1) \, . \, t_i \, , m = \overline{1,n} \} \, ,$$

$$M_{i+1,\bar{s}} = Q_{i+1}^n \cap S_{i,\bar{s}} \, ,$$

where $\bar{S} = (S_1 , \ldots , S_n)$, $S_m = \overline{0, t_i^{-1} - 1}$. An easy calculation shows that the number of points in the set $M_{i+1,\bar{s}}$ is equal to l_{i+1}^n. We introduce the sequence of random events

$$B_i = \{ \omega : \sup || X(S_1 \, . \, t_i ,\ldots, S_n \, . \, t_i) - X(x) || > f(t_i) \}$$
$$0 \leq S_m < t_i^{-1}$$
$$x \in M_{i+1,\bar{s}}$$
$$m = \overline{1,n} \, .$$

We observe that by (1)

$$P(B_i) \leq \sum_{s,x} P \{ || X(S_1 \, .t_i ,\ldots,S_n \, . \, t_i) - X(x) || > f(t_i) \} \leq t_{i+1}^{-n} \, . \, h(t_i)$$

and by the Borel-Cantelli lemma (see for example [3]), it follows from (3) that there exists an event B such that $P(B) = 1$ and for $\omega \in B$ we may find a number $N_1(\omega)$ such that the inequalities

$$|| X(S_1 \, . \, t_i ,\ldots, S_n \, . \, t_i) - X(x) || \leq f(t_i) \tag{6}$$

hold for all

$$i > N_1(\omega), \bar{S} = (S_1 \, .t_i ,\ldots,S_n \, . \, t_i) \, , 0 \leq S_m < t_i^{-1} \text{ and } x \in M_{i+1,\bar{s}} \, .$$

By the seperability of $X(x)$,

$$\sup_{\substack{|X'_m - x''_m| \le t \\ m=\overline{1,n}}} || X(x') - X(x'') || = \sup_{\substack{|x'_m - x''_m| \le t \\ m=\overline{1,n}}} || X(x') - X(x'') ||$$

with probability one, where $x', x'' \in Q$.

Suppose that $\omega \in B$ and $T(\omega) = t_{N_1}(\omega)$. Let $t < T(\omega)$ and determine the number N from the conditions $t_N \le t \le t_{N-1}$ and $| x'_m - x''_m \le | t$ for all m where $x', x'' \in Q$. With these assumptions, we may write the points x' and x'' in the form

$$x' = (S'_i \,.\, t_{N+l}, \ldots, S'_n \,.\, t_{N+l}),$$
$$x'' = (S''_1 \,.\, t_{N+l}, \ldots, S''_n \,.\, t_{N+l})$$

where $l > 0$.

We now estimate the norm $|| X(x') - X(x'') ||$. In the case where $x', x'' \in S_{N+l-1,s}$,

$$|| X(x') - X(x'') || \le 2 \,.\, f(t_{N+l-1}).$$

Otherwise we construct a sequence of cubes of the form $S_{i,\bar{s}}$ in the following way: Let $x'' = S''_m \,.\, t_{N+l}$ and $\hat{x}'_m = S'_m \,.\, t_{N+l}$. If $S'_m < S''_m$, we choose the number of the form $\hat{x}''_{m,1} = S''_{m,i}\, t_{N+l-1}$ which is nearest to $\hat{x}''_m$ on the left. We then choose the number of the form $\hat{x}''_{m,2} = S''_{m,2} \,.\, t_{N+l-2}$ which is nearest to $\hat{x}''_{m,1}$, and so on.

If $S'_m > S''_m$, we proceed similarly, moving to the right. We carry-out the procedure just described for every m ($m = \overline{1,n}$), continuing to refine the partitions until the point x' belongs for the first time to some cube $S_{N+l-1,\bar{s}}$ with a vertex at the point $x''_r = (S''_{1,r} \,.\, t_{N+l-r}, \ldots, S''_{n,r} \,.\, t_{N+l-r})$. Note that the intermediate vertices of the cubes just constructed were at the points $x''_k = (S''_{1,k} \,.\, t_{N+l-k}, \ldots, S''_{n,k} \,.\, t_{n+l-k})$, $K = \overline{1, r-1}$. If $x' \neq x''_r$ we construct an increasing sequence of cubes $S'_{i,\bar{s}}$ containing the point x' and the last term of this sequence must contain the point x''_r.

Let the vertices of the cubes of the sequence be denoted by x_j'; then the inequality

$$|| X(x'') - X(x') || \le || X(x'') - X(x''_1) || + \ldots + || X(x''_{r-1}) - X(x''_r) || + \ldots$$
$$+ || X(x_1') - X(x') || \le 2 \sum_{i=N}^{\infty} f(t_i) = g(t) \qquad (7)$$

completes the proof of the Theorem 1.

Theorem 2. Let $X(x)$ be a seperable random n vector field on $[0,1]^n$ with independent components, satisfying the conditions of Theorem 1. Then

$$P \sup_{\substack{|x'_m - x''_m| \leq t \\ m=\overline{1,n}}} \{ || X(x') - X(x'') || > g(t) \} \leq \lambda(t) \tag{8}$$

where

$$g(t) = 2 \sum_{i=N}^{\infty} f(t_i), \tag{9}$$

$$\lambda(t) = \sum_{i=N}^{\infty} t_{i+n}^{-n} \cdot h(t_i) \tag{10}$$

and N is determined from (5) .

Proof. Let

$$B_{i,\bar{s},x} = \{ \omega : || X(S_1 \cdot t_i, \ldots, S_n \cdot t_i) - X(x) || \leq f(t_i) \}$$

and

$$B_N = \bigcap_{i=N}^{\infty} \bigcap_{S} \bigcap_{x \in M_{i+1,s}} B_{i,\bar{s},x} .$$

It follows from the proof of Theorem 1 that if $\omega \in B_N$, then

$$|| X(x') - X(x'') || \leq g(t) = 2 \sum_{i=N}^{\infty} f(t_i),$$

$| x'_m - x''_m | \leq t_N, m = \overline{1,n}$. Therefore

$$\{ \omega : \sup_{\substack{|x'_m - x''_m| \leq t,\, m=\overline{1,n} \\ x', x'' \in Q}} || X(x') - X(x'') || > g(t) \} \subset \overline{B}_N . \tag{11}$$

Since

$$P(\overline{B}_N) \leq \sum_{m=N}^{\infty} \sum_{\bar{s}} \sum_{x} P(\overline{B}_{m,\bar{s},x}) \leq \sum_{j=N}^{\infty} t_{j+1}^{-n} \cdot h(t_j) ,$$

(8) is implied by (11).

Now let

$$Z_X(t) = \sup_{\substack{|t_m| < t,\, x \in [0,1]^n \\ m = \overline{1,n}}} || X(x + \tilde{t}) - X(x) || .$$

Using Theorem 1 and 2, we have obtained estimates of $Z_X(t)$, valid with probability one, and estimates of the form

$$P\{Z_X(t) > \overline{g(t)}\} \leq \overline{\lambda(t)},$$

$$P\{Z_X(t) > \nu\} < \alpha(t, \nu),$$

under the assumption that

$$\sup_{\substack{|t_m| \leq t, \\ m = \overline{1,n},\, x}} E|| X(x + \tilde{t}) - X(x) ||^{\alpha} \leq \gamma_{\alpha}^{2}(t),$$

where E denotes expectation.

In the table on p. 12 we give the results of calculations of the functions $\overline{\gamma(t)}$, $\overline{\lambda(t)}$ and $\alpha(t, \nu)$ for several functions $g_{\alpha}(t)$.

For example, let us state and prove the following assertion, implied by Theorem 1.

Corollary 1. If $X(x)$ is a Gaussian random n vector field on $[0,1]^n$ an if

$$\sup_{\substack{|t_m| \leq t,\, m = \overline{1,n} \\ x \in [0,1]^n}} \{E|| X(x + \tilde{t}) - X(x) ||^2\} \leq \frac{1}{|\ln t|^{1+\varepsilon}} \cdot C = \gamma_{\alpha}^{2}(t),$$

where $\varepsilon > 0$ and C is a positive constant and E denotes the expectation, then

$$|| X(x + \tilde{t}) - X(x) || \leq \frac{1}{|\ln t|^{\varepsilon/2}} \cdot a$$

with probability one, where a is a positive constant.

TABLE

$X(x)$	$\gamma_n^2(t)$	t_i	$f(t)$	$g(t)$	$\lambda(t)$	$\alpha(t, \nu)$
Gaussian r.v.f. on $[0,1]^n$	$\frac{C}{\lvert \ln t \rvert^{1+\varepsilon}}$	$\frac{1}{2^{2^i}}$	$\frac{2\sqrt{n.C}.C_1}{\lvert \ln t \rvert^{\varepsilon/2}}$ $C_1 \geq 1$ $C > 0$	$\frac{\sqrt{n.C}.2^{\frac{\varepsilon}{2}+2}}{2^{\varepsilon/2}-1} \times$ $\frac{C_1}{\lvert \ln t \rvert^{\varepsilon/2}}$	$\frac{1}{C_1} \frac{1}{2\sqrt{\pi n}} \times$ $\frac{1}{\sqrt{2}-1} \frac{1}{\sqrt{\lvert \ln t \rvert}}$	$K.\nu^{-1} \frac{1}{\lvert \ln t \rvert^{1+\frac{\varepsilon}{2}}}$, $K = \sqrt{\frac{C}{\pi}} \cdot \frac{2^{\frac{\varepsilon}{2}+1}}{2^{\frac{\varepsilon}{2}}-1} \times \frac{1}{\sqrt{2}-1}$
Gaussian r.v.f. on $[0,1]^n$	$\frac{C}{\lvert \ln t \rvert . \lvert \ln \lvert \ln \lvert t \rvert \rvert \rvert^{2+\varepsilon}}$	$\frac{1}{2^{2^i}}$	$\frac{C_1\sqrt{C}}{\lvert \ln \lvert \ln t \rvert \rvert^{1+\varepsilon/2}}$ $C_1^2 > 2n$	$\frac{2C_1\sqrt{C}}{\varepsilon \ln 2} \times$ $\frac{1}{\left\lvert \ln\left(\frac{1}{2} \ln t\right) \right\rvert^{\varepsilon/2}}$	$\frac{2}{\sqrt{2\pi}} \cdot \frac{1}{C_1} \times$ $\frac{1}{\sqrt{2}-1} \cdot \frac{1}{\lvert \ln t \rvert^{1/2}}$	$K.\nu^{-1} = \frac{1}{\lvert \ln t \rvert^{1/2}} \times$ $\frac{1}{\left\lvert \ln\left(\frac{1}{2} \lvert \ln t \rvert\right) \right\rvert^{\varepsilon/2}}$, $K = \frac{4\sqrt{C}}{\sqrt{2\pi}\,(\sqrt{2}-1)\,\varepsilon\,.\,(\ln 2)}$

Proof. Since the field is Gaussian, so

$$P\{\|X(x+\bar{t})-X(x)\|>f(t)\}\leq\frac{1}{\sqrt{2\pi}}\frac{\gamma(t)}{f(t)}.e^{-\frac{1}{2}\frac{f^2(t)}{\gamma^2(t)}}.$$

Let

$$f(t)=2C_1\sqrt{n}\,|\ln t|^{\frac{1}{2}}.\gamma(t),$$

where $C_1\geq 1$. Then

$$h(t)=(2C\sqrt{2\pi n})^{-1}t.(|\ln t|^{\frac{1}{2}})^{-1}.$$

We now calculate $\overline{g(t)}$ and $\overline{\lambda(t)}$, choosing $t_i=2^{-2^i}$.

We have

$$g(t)=2\sum_{i=N}^{\infty}2C_1\sqrt{n.C}\,|\ln t|^{\frac{1}{2}}\frac{1}{|\ln t|^{\frac{1+\varepsilon}{2}}}$$

$$=2^2C_1\sqrt{n.C}\sum_{i=N}^{\infty}\frac{1}{|\ln\frac{1}{2^{2^i}}|^{\varepsilon/2}}$$

$$=4C_1\sqrt{n.C}\sum_{i=N}^{\infty}\left(\frac{1}{2^{\varepsilon/2}}\right)^i$$

$$=\frac{C_1\sqrt{n.C}}{|\ln 2|^{\varepsilon/2}}.\frac{2^{\frac{\varepsilon}{2}+2}}{2^{\frac{\varepsilon}{2}}-1}..\frac{1}{2^{\varepsilon.\frac{N}{2}}}.$$

Taking account of (5), we get

$$g(t)=C_1\sqrt{n.C}\frac{2^{\frac{\varepsilon}{2}+2}}{2^{\frac{\varepsilon}{2}}-1}.\frac{1}{|\ln t|^{\varepsilon/2}}.$$

Similarly, we obtain

$$\overline{\lambda}(t)=(C_1)^{-1}.(2\sqrt{\pi n})^{-1}(\sqrt{2}-1)^{-1}(\sqrt{\ln|t|})^{-1}.$$

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Ö Z E T

Bu çalışmada, tesadüfi vektör alanlarının süreklilik modülü ile ilgili bir takım sonuçlar verilmektedir.