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ON THE GEOMETRIC MEAN AND THE ZEROS OF AN INTEGRAL FUNCTION

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In this paper it has been investigated the geometric mean and the zeros of an integral function and has been found some inequalities, which are best possible, in terms of exponent of convergence and its lower orders.

1. Let $f(z)$ be an integral function and let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^+ n(r)}{\log r} = \begin{matrix}\rho_1\\ \lambda_1\end{matrix} \tag{1.1}$$

$n(r)$ being the number of zeros of $|f(z)|$ in $|z| \le r$, let $G(r)$ and $g_\delta(r)$ denote the geometric means of $|f(z)|$, defined as

$$G(r) = \exp \frac{1}{2\pi} \int_0^{2\pi} \log |f(r e^{i\theta})| \, d\theta \tag{1.2}$$

and

$$g_\delta(r) = \exp \frac{(\delta + 1)}{2\pi r^{\delta+1}} \int_0^r \int_0^{2\pi} \log |f(x e^{i\theta})| \, x^\delta \, dx \, d\theta \, .$$

Various authors have studied the properties of geometric mean values. Shah [1] has extended the results of Polyà and Szegö [2]. Srivastava [3] and Bose and Srivastava [4] have also studied some of the properties of these mean values. Kuldip Kumar [5] has generalized the result of Shah [1]. In this paper we have derived some inequalities, which are best possible, in terms of exponent of convergence and its lower orders. We have also generalized the Result of Titchmarsh [7].

2. Theorem 1. If $f(z)$ is an integral function, then

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log \left\{ \dfrac{G(r)}{g_\delta(r)} \right\}}{\log r} = \begin{matrix}\rho_1\\ \lambda_1\end{matrix} . \tag{2.1}$$

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Proof. Let $\{Z_n\}^\infty$ denote the zeros of $|f(z)|$, $|Z_n| = r_n$ and suppose
$$0 = r_1 = \dots = r_q < r_{q+1} \leq r_{q+2} \dots (q \geq 0).$$
Let $n > q+1$, $r_n \leq r < r_{n+1}$, $f(z) = z^q F(z)$,
then
$$G(r,f) = r^q G(r,F)$$
and
$$g_\delta(r,f) = r^q \frac{-q}{e^{\delta+1}} g(r,F).$$
Therefore
$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{\frac{1}{n(r)}} = \exp\left\{-\frac{1}{\delta+1} + \frac{1}{(\delta+1)n(r)} \sum_{r_\nu \leq r} \left(\frac{r_\nu}{r}\right)^{\delta+1}\right\},$$
further
$$\sum (r_\nu)^{\delta+1} = \int_0^r x^{\delta+1}\, d\,n(x)$$
$$= r^{\delta+1} n(r) - (\delta+1) \int_0^r x^\delta n(x)\, dx.$$
Hence
$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{\frac{1}{n(r)}} = \exp\left\{-\frac{1}{r^{\delta+1} n(r)} \int_0^r x^\delta n(x)\, dx\right\}$$
(See [5, p. 40]) and for $\delta = 1$ see [1]).

Therefore
$$\frac{G(r)}{g_\delta(r)} = e^{\frac{1}{r^{\delta+1}} \int_0^r x^\delta n(x)\, dx} \tag{2.2}$$
$$\leq e^{\frac{n(r)}{\delta+1}}$$
and also
$$\frac{G(2r)}{g_\delta(2r)} = e^{\frac{1}{(2r)^{\delta+1}} \int_0^{2r} x^\delta n(x)\, dx} \tag{2.3}$$
$$> e^{\frac{1}{(2r)^{\delta+1}} \int_r^{2r} x^\delta n(x)\, dx}$$
$$\geq e^{\frac{n(r)}{2^{\delta+1}}\left(\frac{2^{\delta+1}-1}{\delta+1}\right)}.$$
Taking limits in (2.2), (2.3) and using (1.1), we get

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \left\{\frac{G(r)}{g_\delta(\gamma)}\right\}}{\log r} = \begin{matrix}\rho_1 \\ \lambda_1\end{matrix} .$$

Corollary. We have

$$e^{r^{\lambda_1-\varepsilon}} g_\delta(r) < G(r) < e^{r^{\rho_1+\varepsilon}} g_\delta(r) ,$$

for $r > r_0$.

3. Let us set

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \left\{\frac{G(r)}{g_\delta(r)}\right\}}{\phi(r)\, r^{\rho_1}} = \begin{matrix}p \\ q\end{matrix} ; \tag{3.1}$$

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{n(r)}{\phi(r)\, r^{\rho_1}} = \begin{matrix}c \\ d\end{matrix} ,$$

where $\phi(r)$ is a positive, continuous and indefinitely increasing function of r and $\phi(lr) \sim \phi(r)$ as $r \to \infty$ for every constant $l > 0$. We now prove the following :

Theorem 2. If $f(z)$ is an integral function then,

(*i*) $$\frac{d}{(\delta+\rho_1+1)} \leq q \leq p \leq \frac{c}{(\delta+\rho_1+1)}$$

(*ii*) $$\left(\frac{c}{d}\right)^{\frac{\delta+1}{\rho_1}} q \leq \frac{d}{(\delta+\rho_1+1)} + d \left\{\frac{\left(\frac{c}{d}\right)^{\delta+1/\rho_1} - 1}{\delta+1}\right\}$$

(*iii*) $$\left\{\frac{(\delta+1)(c-d)+c\,\rho_1}{c\,\rho_1}\right\}^{\rho_1/\delta+2} \geq \frac{c}{(\delta+\rho_1+1)} .$$

Proof. From (2.2), we have for $h > 0$

$$\{(1+h)r\}^{\delta+1} \log \frac{G\{r(1+h)\}}{g_\delta\{r(1+h)\}} = \int_0^{r_0} x^\delta n(x)dx + \int_{r_0}^{r} x^\delta n(x)dx + \int_r^{r(1+h)} x^\delta n(x)\, dx ,$$

$$O(1) + (c+\varepsilon) \int_{r_0}^{r} \phi(x)\, x^{\delta+\rho_1} dx + n\,[r(1+h)] \left[\frac{\{r(1+h)\}^{\delta+1} - r^{\delta+1}}{\delta+1}\right]$$

$$\sim (c+\varepsilon)\, \phi(r) \frac{r^{\delta+\rho_1+1}}{(\delta+\rho_1+1)} + n\,\{r(i+h)\} \left\{\frac{(1+h)^{\delta+1}-1}{\delta+1}\right\} r^{\delta+1} .$$

(Using [6, Lemma 5])

Taking limits, we get

$$(1+h)^{\delta+\rho_1+1}\, p \leq \frac{c}{(\delta+\rho_1+1)} + c(1+h)^{\rho_1}\left\{\frac{(1+h)^{\delta+1}-1}{\delta+1}\right\} \tag{3.2}$$

and

$$(1+h)^{\delta+\rho_1+1}\, q \leq \frac{c}{(\delta+\rho_1+1)} + d(1+h)^{\rho_1}\left\{\frac{(1+h)^{\delta+1}-1}{\delta+1}\right\}. \tag{3.3}$$

Similarly we obtain

$$(1+h)^{\delta+\rho_1+1}\, p \geq \frac{d}{(\delta+\rho_1+1)} + c\left\{\frac{(1+h)^{\delta+1}-1}{\delta+1}\right\} \tag{3.4}$$

and

$$(1+h)^{\delta+\rho_1+1}\, q \geq \frac{d}{(\delta+\rho_1+1)} + d\left\{\frac{(1+h)^{\delta+1}-1}{\rho_1+1}\right\}. \tag{3.5}$$

It can be seen that the minimum of the right hand expressions of (3.2) and (3.3) occur at $h=0$ and $(1+h)^{\rho_1}=\frac{c}{d}$. Substituting $h=0$ in (3.2) and $(1+h)^{\rho_1}=\frac{c}{d}$ in (3.3), we get 2nd parts of (*i*) and (*ii*) respectively. Taking $(1+h)^{\delta+1}=\frac{(\delta+1)(c-d)+c\rho_1}{c\,\rho_1}$ in (3.4) and $h=0$ in (3.5) we get (*iii*) and first part of (*i*) respectively.

Theorem 3. If $f(z)$ is an integral function, then

$$e^{\frac{d}{c(\delta+\rho_1+1)}} \leq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{G(r)}{g_\delta(r)}\right\}^{\frac{1}{n(r)}} \leq e^{\frac{c}{d(\delta+\rho_1+1)}}.$$

Proof. From (3.1), we have for $\varepsilon > 0$

$$(q-\varepsilon)\, r^{\rho_1}\phi(r) < \log\left\{\frac{G(r)}{g_\delta(r)}\right\} < (p+\varepsilon)\, r^{\rho_1}\phi(r) \tag{4.1}$$

and

$$(d-\varepsilon) r^{\rho_1}\phi(r) < n(r) < (c+\varepsilon) r^{\rho_1}\phi(r). \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\frac{(q-\varepsilon)}{(c+\varepsilon)} < \frac{1}{n(r)}\log\left\{\frac{G(r)}{g_\delta(r)}\right\} < \frac{(p+\varepsilon)}{(d-\varepsilon)}$$

for $r > r_0$.

Taking the limit and using (*i*) of Theorem 2, we get the result.

Corollary. If $c = d$, in the above theorem, then

$$\lim_{r\to\infty} \left\{\frac{G(r)}{g_\delta(r)}\right\}^{\frac{1}{n(r)}} = e^{\frac{1}{(\delta+\rho_1+1)}} .$$

5. Theorem 4. If $f(0) \neq 0$ and $\log \dfrac{G(r)}{g_\delta(r)} \sim p\, r^{\rho_1} \phi(r)$, then

$$n(r) \sim (\delta + \rho_1 + 1)\, p\, r^{\rho_1} \phi(r)$$

and conversely.

Proof. From (i) of the Theorem 2, if $c = d$, $p = q = \dfrac{c}{\delta + \rho_1 + 1}$. Suppose now $p = q$, we shall show that $c = d$.

If $0 < \eta < 1$, we have from (2.2)

$$n(r)\,\eta < \frac{1}{r^{\delta+1}} \int_r^{r+\eta r} x^\delta\, n(x)\, dx$$

$$= (1+\eta)^{\delta+1} \log \left\{\frac{G(r+\eta r)}{g_\delta(r+\eta r)}\right\} - \log \frac{G(r)}{g_\delta(r)}$$

$$= p(1+\eta)^{\delta+p_1+1} r^{\rho_1} \phi(r+\eta r) - p\, r^{\rho_1} \phi(r) + 0\,(r^{\rho_1} \phi(r)) .$$

Hence

$$\lim_{r\to\infty} \sup \frac{n(r)}{r^{\rho_1}\phi(r)} \leq p\,(\delta + \rho_1 + 1) + \beta\,\eta ,$$

where β is a constant. Since η is arbitrary, we get

$$\lim_{r\to\infty} \sup \frac{n(r)}{r^{\rho_1}\phi(r)} \leq p\,(\delta + \rho_1 + 1) .$$

By considering the integral

$$\log \left\{\frac{G(r)}{g_\delta(r)}\right\} - (1-\eta)^{\delta+1} \log \frac{G(r-\eta r)}{g_\delta(r-\eta r)} ,$$

we get

$$\lim_{r\to\infty} \inf \frac{n(r)}{r^{\rho_1}\phi(r)} \geq p\,(\delta + \rho_1 + 1)$$

and hence the result.

6. Theorem 5. If $f(z)$ is an integral function of order ρ and

$$\lim_{t\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(t)}{t^\rho} = \begin{matrix}\lambda_1\\ \lambda_2\end{matrix} \tag{6.1}$$

then

$$\pi\lambda_2 \operatorname{cosec} \pi\rho \leq \lim_{x\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log f(x)}{x^\rho} \leq \pi\lambda_1 \operatorname{cosec} \pi\rho \ .$$

Proof. From (6.1) we have, for any $\varepsilon > 0$,

$$(\lambda_2 - \varepsilon)\, t^\rho < n(t) < (\lambda_1 + \varepsilon)\, t^\rho \tag{6.2}$$

for $t > t_0$.

$$\text{Now } f(z) = \sum_{n=1}^{\infty} \log\left(1 + \frac{z}{z_n}\right)$$

$$= \sum_{1}^{\infty} n \left[\log\left(1 + \frac{z}{z_n}\right) - \log\left(1 + \frac{z}{z_{n+1}}\right)\right]$$

$$= \sum_{1}^{\infty} n \int_{z_n}^{z_{n+1}} \frac{z\, dt}{t(z+t)}$$

$$= z \int_{z_1}^{\infty} \frac{n(t)}{t(z+t)}\, dt$$

$$= z \int_{0}^{\infty} \frac{n(t)}{t(z+t)}\, dt \ .$$

If z is a real number then,

$$\log f(x) = x \int_0^\infty \frac{n(t)}{t(x+t)}\, dt$$

$$= x \int_0^{t_0} \frac{n(t)}{t(x+t)}\, dt + \int_{t_0}^{\infty} \frac{n(t)}{t(x+t)}\, dt$$

for $t > t_0(\varepsilon)$, applying (6.2) we get

$$\log f(x) = x \int_0^{t_0} \frac{n(t)}{t(x+t)}\, dt + x \int_{t_0}^{\infty} \frac{(\lambda_1 + \varepsilon)}{t(x+t)}\, t^\rho\, dt$$

$$= x \int_0^{t_0} \frac{n(t) - (\lambda_1 + \varepsilon)\, t^\rho}{t(x+t)}\, dt + x \int_0^{\infty} \frac{(\lambda_1 + \varepsilon)}{t(x+t)}\, t^\rho\, dt$$

$$= O(1) + x \int_0^{\infty} \frac{(\lambda_1 + \varepsilon)}{t(x+t)}\, dt ,$$

putting $t = xu$ in the 2nd term, we get

$$\log f(x) < O(1) + x^{\rho} (\lambda_1 + \varepsilon) \pi \operatorname{cosec} \pi\rho ,$$

proceeding to limit, we get

$$\lim_{x\to\infty} \sup \frac{f(x)}{x^{\rho}} \leq \pi\lambda_1 \operatorname{cosec} \pi\rho . \tag{6.3}$$

On the other hand, for $t > t_0(\varepsilon)$

$$\text{iog} f(x) > x \int_0^{t_0} \frac{n(t)}{t(x+t)} + x \int_{t_0}^{\infty} \frac{(\lambda_2 - \varepsilon)}{t(x+t)} \, t \, dt$$

$$= x \int_0^{t_0} \frac{n(t) - (\lambda_2 - \varepsilon) t^{\rho}}{t(x+t)} + x \int_0^{\infty} \frac{(\lambda_2 - \varepsilon)}{t(x+t)} \, {}^{\rho} \, dt$$

$$= O(1) + x \int_0^{\infty} \frac{(\lambda_2 - \varepsilon)}{t(x+t)} \, dt .$$

Again putting $t = xu$ in the second term and integrating we get

$$\log f(x) > O(1) + x^{\rho} (\lambda_2 - \varepsilon) \pi \operatorname{cosec} \pi \rho ,$$

proceeding to limit we get ,

$$\lim_{x\to\infty} \inf \frac{\log f(x)}{x^{\rho}} \geq \pi\lambda_2 \operatorname{cosec} \pi \rho . \tag{6.4}$$

Combining the results (6.3) and (6.4), we get

$$\pi\lambda_2 \operatorname{cosec} \pi\rho \leq \lim_{x\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log f(x)}{x^{\rho}} \leq \pi\lambda_1 \operatorname{cosec} \pi\rho . \tag{6.5}$$

Corollary. If in the above theorem $\lim_{t\to\infty} \frac{n(t)}{t^{\rho}} = \lambda$ holds,

then (6.5) leads to conclude $\lambda_1 = \lambda_2 = \lambda$

$$\log f(x) \sim \pi \lambda x^{\rho} \operatorname{cosec} \pi\rho$$

which is the result ([7, p. 185]) .

REFERENCES

[1] SHAH, S.M. : *A note on means of entire functions*, Publ. Math. 2, Fasc. 2, Debrecen (1951), 95-99.

[2] POLYA, G. and SZEGÖ, G. : Aufgaben und Lehrsätze aus der Analysis II.

[3] SRIVASTAVA, S.N. : *On the geometric means of integral functions*, Publ. Math. **13**, Fasc. 1-4, Debrecen (1966), 73-76.

[4] BOSE, S.K. and SRIVASTAVA, S.N. : *On the geometric mean of an integral function*, Ganita **22** (1971), 1-7.

[5] KUMAR, K. : *On the geometric means of an integral function*, Monatsh. Math. **70** (1966), 39-46.

[6] ROGOSINSKI, W.W. and HARDY, G.H. : *Notes on Fourier Series (III)*, Quart. J. Math. Oxford **16** (1945), 49-58.

[7] TITCHMARSH, E.C. : *On integral functions with real negative zeros*, Proc. London Math. Soc. (2), **26** (1927), 185-200.

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ÖZET

Bu çalışmada bir tam fonksiyonun geometrik ortalaması ve sıfır yerleri incelenmekte, yakınsaklık eksponenti ve onun alt mertebeleri cinsinden bazı, mümkün olan en iyi eşitsizlikler elde edilmektedir.