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A NOTE ON EUCLIDEAN BALLS

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In this note we prove that by using only some of the basic facts of general topology, the closure of any open ball of *any subset* in an Euclidean space is the corresponding closed ball. This statement is therefore true in any locally compact normed space.

It is known that in any pseudometric space (X, d) the balls

$$B_d(A, \varepsilon) = \{x : d(x, A) < \varepsilon\}, \quad B_d[A, \varepsilon] = \{x : d(x, A) \leq \varepsilon\}$$

of any subset $A \subseteq X$ are open and closed respectively and furthermore the following inclusions

$$\overline{B_d(A, \varepsilon)} \subseteq B_d[A, \varepsilon],$$
$$B_d(A, \varepsilon) \subseteq \operatorname{int} B_d[A, \varepsilon],$$
$$\partial B_d(A, \varepsilon) \subseteq \{x : d(x, A) = \varepsilon\}$$

hold. Here as usually the conventional notations $\overline{A}$, ∂A and int A denote respectively the closure, the boundary and the interior of A in the mentioned metric space. We prove in this note that in Euclidean spaces the first one is an equality but the last two ones are not necessarily equalities. Terminology is consistent with Dugundji [1]. In the sequel d_n denotes the metric of the n-dimensional Euclidean space E^n ; Z^n denotes the set of all n-tuples with integer coordinates, i.e. the set of all knot points in E^n ; $x \wedge y$ denotes the minimum of the real numbers x and y and A will always denote a non-empty subset. Balls of empty set are conventionally accepted empty.

For the sake of completeness we firstly state the following two well known lemmas :

Lemma 1. Let $(B_\alpha)_{\alpha \in I}$ be any family of nonempty subsets of a metric space. Then

$$d\Big(A, \bigcup_{\alpha \in I} B_\alpha\Big) = \inf_{\alpha \in I} d(A, B_\alpha).$$

Proof. Write $B_0 = \bigcup_\alpha B_\alpha$ for convenience. Then $d(A, B_0) \leq \inf_\alpha d(A, B_\alpha)$ is clear. The assumption $d(A, B_0) < \inf_\alpha d(A, B_\alpha)$ yields firstly the existence of a positive ε_0 and consequently a point $x_0 \in B_0$ such that

$$d(A, x_0) < d(A, B_0) + \varepsilon_0 < \inf_\alpha d(A, B_\alpha)$$

which gives the following contradiction

$$\inf_\alpha d(A, B_\alpha) \leq d(A, B_{\alpha_0}) \leq d(A, x_0) < \inf_\alpha d(A, B_\alpha)$$

where $x_0 \in B_{\alpha_0}$ is assumed for an appropriate $\alpha_0 \in I$.

Lemma 2. If A is a singleton then all the inclusions at the first page are equalities in E^n.

Proof. For any $y \in B_{d_n}[x, \varepsilon]$ satisfying $d_n(x, y) = \varepsilon$ we have $y \in \overline{B_{d_n}(x, \varepsilon)}$ since

$$\alpha y + (1 - \alpha) x \in B_{d_n}(y, \delta) \cap B_{d_n}(x, \varepsilon)$$

holds for any $\delta < \varepsilon$ where the positive α is chosen so that $\varepsilon - \delta < \varepsilon \alpha < \varepsilon$.

Therefore

$$\overline{B_{d_n}(x, \varepsilon)} = B_{d_n}[x, \varepsilon]$$

is obtained. Now take any $y \in \operatorname{int} B_{d_n}[x, \varepsilon]$. So there exists a positive δ_y with $B_{d_n}(y, \delta_y) \subseteq B_{d_n}[x, \varepsilon]$. The supposition $d_n(x, y) = \varepsilon$ yields easily the contradiction

$$z = y + \frac{\delta_y}{2\varepsilon}(y - x) \in B_{d_n}(y, \delta_y) - B_{d_n}[x, \varepsilon] = \phi .$$

Hence the equalities

$$\operatorname{int} B_{d_n}[x, \varepsilon] = B_{d_n}(x, \varepsilon), \quad \partial B_{d_n}(x, \varepsilon) = \partial B_{d_n}[x, \varepsilon] = \{y : d_n(x, y) = \varepsilon\}$$

are easily derived.

Theorem. For any subset $A \subseteq E^n$ and $0 < \varepsilon$

$$\overline{B_{d_n}(A, \varepsilon)} = B_{d_n}[A, \varepsilon], \quad \partial B_{d_n}[A, \varepsilon] \subseteq \partial B_{d_n}(A, \varepsilon).$$

Proof. Let $d_n(x, A) = \varepsilon$. If there is a point $a \in \bar{A}$ with $d_n(x, a) = \varepsilon$ then

$$x \in B_{d_n}[a, \varepsilon] = \overline{B_{d_n}(a, \varepsilon)} \subseteq \overline{B_{d_n}(\bar{A}, \varepsilon)} \,.\, \overline{B_{d_n}(A, \varepsilon)}$$

[This is the unique case indeed. In fact there exists a positive δ_0 so that $\varepsilon < \delta_0$ and $A \not\subseteq B_{d_n}[x, \delta_0]$ since the opposite supposition yields for each $a \in A$ that $d_n(x, a) = \varepsilon$ again]. Hence

$$\varepsilon = d_n(x, \bar{A}) \leq d_n(x, \bar{A} \cap B_{d_n}[x, \delta_0]),$$

$$\varepsilon < \delta_0 \leq d_n(x, E^n - B_{d_n}[x, \delta_0]) \leq d_n(x, \bar{A} - B_{d_n}[x, \delta_0])$$

and therefore

$$\varepsilon \leq d_n(x, \bar{A} \cap Bd_n[x, \delta_0]) \wedge d_n(x, \bar{A} - Bd_n[x, \delta_0]) = d_n(x, \bar{A}) = \varepsilon$$

are obtained by the Lemma 1. Hence we get

$$d_n(x, \bar{A} \cap Bd_n[x, \delta_0]) = \varepsilon .$$

Since $Bd_n[x, \delta_0] \cap \bar{A}$ is compact in E^n (Theorem XI.4.3. of [1]), there exists a point $y \in Bd_n[x, \delta_0] \cap \bar{A}$ such that $d_n(x, y) = \varepsilon$ (proof of the statement XI. 4.4. of [1]). Therefore

$$x \in Bd_n[y, \varepsilon] = \overline{Bd_n(y, \varepsilon)} \subseteq \overline{Bd_n(\bar{A}, \varepsilon)} = \overline{Bd_n(A, \varepsilon)}$$

and consequently

$$Bd_n[A, \varepsilon] \subseteq \overline{Bd_n(A, \varepsilon)}$$

follows. The other inclusion is now clear.

Remark 1. In E^n the following inclusions

$$\operatorname{int} Bd_n[A, \varepsilon] \subseteq Bd_n(A, \varepsilon) ,$$
$$\partial Bd_n(A, \varepsilon) \subseteq \partial Bd_n[A, \varepsilon] ,$$
$$\{x \in E^n : d_n(x, A) = \varepsilon\} \subseteq \partial Bd_n(A, \varepsilon)$$

are not necessarily true.

Proof. Take $A = Bd_n[x_0, \varepsilon + \delta] - Bd_n(x_0, \varepsilon)$ where $x_0 \in E^n$ is any fixed point. Then

$$\partial Bd_n(x_0, \varepsilon) \subseteq A ,$$
$$\varepsilon \leq d_n(x_0, E^n - Bd_n(x_0, \varepsilon)) \leq d_n(x_0, A) \leq d_n(x_0, \partial Bd_n(x_0, \varepsilon)) = \varepsilon$$

and therefore $x_0 \in Bd_n[A, \varepsilon]$ are obtained. Hence

$$Bd_n[A, \varepsilon] = Bd_n[x_0, 2\varepsilon + \delta] , \quad Bd_n(A, \varepsilon) = Bd_n(x_0, 2\varepsilon + \delta) - x_0$$

and consequently

$$x_0 \in \operatorname{int} Bd_n[A, \varepsilon] - Bd_n(A, \varepsilon) ,$$
$$\partial Bd_n(A, \varepsilon) = \{x_0\} \cup \partial Bd_n[A, \varepsilon]$$

are found. More generally define the subset $E^n - Bd_n(Z^n, \varepsilon)$ as A and let $0 < 3\varepsilon < 1$. Then

$$Bd_n(Z^n, \varepsilon) = \bigcup_{x \in Z^n} Bd_n(x, \varepsilon) \subseteq Bd_n[A, \varepsilon]$$

and therefore

$$Bd_n[A, \varepsilon] = E^n , \quad Bd_n(A, \varepsilon) = E^n - Z^n$$

are obtained. Hence it is easy to see that

$$Z^n \subseteq \text{int}\, B_{d_n}[A, \varepsilon] - B_{d_n}(A, \varepsilon),$$

$$B_{d_n}(A, \varepsilon) = Z^n \not\subset \partial B_{d_n}[A, \varepsilon] = \phi .$$

Remark 2. Lemma 2 and Theorem 1 are true in any locally compact normed linear space.

REFERENCE

[1] DUGUNDJI, J. : Topology, Ailyn and Bacon, Boston, 1966.

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ÖZET

Bu çalışmada, bir Öklid uzaymdaki herhangi bir alt cümlenin herhangi bir açık topunun kapanışının, onun kapalı topu olduğu ispat edilmektedir.