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NOTES ON BOUNDARIES

N. ERGUN

Some certain properties of boundary sets in ordinary topological spaces are established in this note. Sample results are: 1) $\partial(A \cup B) = \partial A \cup \partial B$ iff $\partial(A \cap B) \cup (\partial A \cap \partial B) \subseteq \partial(A \cup B)$, 2) $\partial\partial(A \cup B) \subseteq \partial A \cup \partial\partial B$, 3) $\partial A \cap \partial B \subseteq \partial(A \cup B)$ implies $\partial\partial(A \cup B) \subseteq \partial\partial\, A \cup \partial\partial B$.

Introduction

Some certain properties of boundary sets in ordinary topological spaces are established in this note. X denotes the fixed topological space with no specific property or separation axiom. $\overset{\circ}{A} = \text{int}\ A$, $\overline{A} = \text{cl}\ A$ are the interior and the closure of the subset A in X as usually. The boundary of A is $\partial A = \overline{A} \cap \overline{X - A} = \overline{A} - \overset{\circ}{A}$. The boundary of ∂A will be written by $\partial\partial A$. It is well known that if A is open or closed then $\partial\partial A = \partial A$ holds, i.e. ∂A is nowhere dense.

A more general satement could also be proved : Nowhere dense subsets are preciesely the subsets of those special boundary sets with form $\partial(A \cup \partial B)$ where A is semi-open or semi-closed after remembering the basic inclusion $\partial(A \cup \partial B) \subseteq \partial A \cup \partial\partial B$.

Recall that a subset A is called semi-open (resp. semi-closed) iff there exists an open G (resp. a closed K) with $G \subseteq A \subseteq \overline{G}$ (resp. $\overset{\circ}{K} \subseteq A \subseteq K$) [3]. Hence boundaries of semi-open or semi-closed subsets are also nowhere dense since $\partial A \subseteq \partial G$ or $\partial A \subseteq \partial K$ hold respectively. In fact boundaries of semi-open (resp. semi-closed) sets are precisely boundaries of open sets since $\partial A = \partial\partial A = \partial(X - \partial A)$ hold if A is semi-open or semi-closed. Notice also that in any space with a dense subset D with empty interior, closed subsets are nothing but boundaries since $\partial((\overset{\circ}{K} \cap D) \cup \partial K) = K$ holds for any closed K in such spaces. Thus closed subsets are boundaries in Euclidean spaces, see 3B of [4].

The following basic facts will be used frequently throughout the note without any explicit mentioning :

$$G \subseteq X \text{ is open iff } \text{cl}\,(G \cap \overline{A}) = \text{cl}\,(G \cap A) \text{ for all } A \subseteq X \qquad (1)$$

$$K \subseteq X \text{ is closed iff } \operatorname{int}(K \cup \mathring{A}) = \operatorname{int}(K \cup A) \text{ for all } A \subseteq X \qquad (2)$$

$$\mathring{A} \cap \overline{B} \subseteq \operatorname{cl}(\mathring{A} \cap B) \quad , \quad \operatorname{int}(A \cup B) \subseteq \mathring{A} \cup \overline{B} \qquad (3)$$

$$A \subseteq \partial B \text{ iff } A \subseteq \overline{B} \text{ and } A \cap \mathring{B} = \phi \qquad (4)$$

$$\partial(A \cap B) \cup \partial(A \cup B) \subseteq \partial A \cup \partial B\,. \qquad (5)$$

Results

Any result at the sequel stated without any additional condition or hypothesis is true for all subsets of X.

Proposition 1. We have the inclusions:

$$(\partial A - \overline{B}) \cup (\partial B - \overline{A}) \subseteq \partial(A \cup B) \subseteq (\partial A - \overline{B}) \cup (\partial B - \overline{A}) \cup (\partial A \cap \partial B)\,,$$

$$(\partial A \cup \mathring{B}) \cup (\partial B \cap \mathring{A}) \subseteq \partial(A \cap B) \subseteq (\partial A \cap \mathring{B}) \cup (\partial B \cap \mathring{A}) \cup (\partial A \cap \partial B)\,,$$

$$\partial(A \cup \overline{B}) \cup \partial(B \cup \overline{A}) \subseteq \partial(A \cup B) \subseteq \partial(A \cup \overline{B}) \cup \partial(B \cup \overline{A}) \cup (\partial A \cap \partial B)\,,$$

$$\partial(A \cap \mathring{B}) \cup \partial(B \cap \mathring{A}) \subseteq \partial(A \cap B) \subseteq \partial(A \cap \mathring{B}) \cup \partial(B \cap \mathring{A}) \cup (\partial A \cap \partial B)\,.$$

Proof. The right side of the second formula is a consequence of the following

$$\partial(A \cap B) \subseteq (\partial A \cap \overline{B}) \cup (\partial B \cap \overline{A})$$

which is well known or easy to obtain and its left side follows by

$$\partial A \cap \mathring{B} \subseteq (\mathring{B} \cap \overline{A}) - \operatorname{int}(A \cap B)\,.$$

The left side of the fourth formula is straightforward after (4) and its right side could be obtained by using firstly the right and then the left side of the second formula. The first and the third formulas follow respectively by the second and the fourth. Notice that all the unions of the first two formulas are mutually disjoint and the second formula is a constiderable improvement of a formula by Bourbaki [1], page 118. The following also follows from the second:

$$\partial A - \overline{B} \subseteq \partial(A - \overline{B}) \subseteq \partial(A - B)\,.$$

Proposition 2.

$$\partial(\partial A \cap \partial B) = (\partial\partial A \cap \partial B) \cup (\partial A \cap \partial\partial B)\,,$$

$$\partial(\partial A \cup \partial B) \subseteq (\partial\partial A - \operatorname{int} \partial B) \cup (\partial\partial B - \operatorname{int} \partial A)\,,$$

$$\partial(\partial A \cup \partial B) \subseteq \partial(A \cup B) \cup (\overline{A} \cap \partial B) \cup (\overline{B} \cap \partial A)\,,$$

$$\partial(\partial A \cup \partial B) \subseteq \partial(A \cap B) \cup (\partial A - \mathring{B}) \cup (\partial B - \mathring{A})\,.$$

Proof. Notice that the inclusion

$$\partial(\partial A \cap \partial B) \subseteq (\partial\partial A \cap \partial B) \cup (\partial\partial B \cap \partial A)$$

has already been stated in the proof of the Proposition 1. The reverse inclusion follows easily by (4). The second formula is obtained by the right side of the first formula of Proposition 1 after noticing

$$\partial\partial A \cap \partial\partial B \subseteq \partial\partial A - \operatorname{int} \partial B .$$

Notice also that, one gets the following by the first formula of Proposition 1:

$$\partial(\partial A \cup \partial B) - (\overline{A} \cap \overline{B}) = ((\partial A \cup \partial B) - \overline{A}) \cup ((\partial A \cup \partial B) - \overline{B})$$
$$\subseteq (\partial B - \overline{A}) \cup (\partial A - \overline{B}) \subseteq \partial(A \cup B).$$

Hence, yielding the third formula of this proposition is not difficult. The fourth follows directly from the third.

Remark 1. If A (or B) is open or closed, then

$$\partial(\partial A \cup \partial B) = (\partial\partial A - \operatorname{int} \partial B) \cup (\partial\partial B - \operatorname{int} \partial A) .$$

Proof. Let A be an open or a closed subset. Then $\operatorname{int} \partial A = \phi$ and therefore the intersection of the right side of the second inclusion formula of the Proposition 2 with $\operatorname{int}(\partial A \cup \partial B)$ is equal to

$$(\partial\partial B \cup (\partial\partial A - \operatorname{int} \partial B)) \cap \operatorname{int}(\operatorname{int} \partial A \cap \partial B) = \phi$$

and so this inclusion becomes an equality by (4).

Corollary 1. $\partial(\partial\partial A \cup \partial\partial B) = \partial\partial A \cup \partial\partial B \supseteq \partial(\partial A \cup \partial B)$,

$$\partial(\partial\partial A \cap \partial\partial B) = \partial\partial A \cap \partial\partial B \subseteq \partial(\partial A \cap \partial B),$$
$$\partial(\partial A \cup \partial B) = \partial A \cup \partial B \Rightarrow \partial(\partial A \cap \partial B) = \partial A \cap \partial B .$$

Proposition 3. $\partial\overline{A} \cup \partial\overline{B} = \partial(\overline{A} \cap \overline{B}) \cup \partial(\overline{A} \cup \overline{B})$,

$$\partial\mathring{A} \cup \partial\mathring{B} = \partial(\mathring{A} \cap \mathring{B}) \cup \partial(\mathring{A} \cup \mathring{B}),$$
$$\partial\overline{A} \cup \partial\mathring{B} = \partial(\overline{A} - \mathring{B}) \cup \partial(\mathring{B} - \overline{A}) .$$

Proof. Note that the inclusions

$$(\partial\overline{A} \cup \partial\overline{B}) \cap (\overline{A} \cap \overline{B}) \subseteq \partial(\overline{A} \cap \overline{B}),$$
$$(\partial\overline{A} \cup \partial\overline{B}) - (\overline{A} \cap \overline{B}) \subseteq \partial(\overline{A} \cup \overline{B})$$

are derived respectively by (4) and the left side of the first formula of Proposition 1. Therefore the first equality follows. The others are consequences of the first.

Remark 2. If A and B are both open or both closed, then the basic formula (5) becomes equality by Proposition 3.

Proposition 4. We have the expansions :

$$\partial(A \cap B) = \overline{\partial A \cap \mathring{B}} \cup \overline{A \cap \partial_2 B} \cup \partial(\mathring{A} \cap \mathring{B}),$$

$$\partial(A \cup B) = \overline{\partial A - \bar{B}} \cup \overline{A \cap \partial_1 B} \cup \partial(\bar{A} \cup \bar{B}).$$

Proof. Note that the disjoint subsets $\partial_1 A = \bar{A} - A$ and $\partial_2 A = A - \mathring{A}$ have both empty interiors. They satisfy $\partial A = \partial_1 A \cup \partial_2 A$ for all $A \subseteq X$ and additionally the following equivalencies are clear :

$$A \text{ is open iff } \partial_2 A = \phi \text{ iff } \partial_1 A = \partial A,$$

$$A \text{ is closed iff } \partial_1 A = \phi \text{ iff } \partial_2 A = \partial A.$$

Furthermore

$$\begin{aligned} \partial(A \cap B) - \overline{A \cap \partial_2 B} &\subseteq \mathrm{cl}\,((A \cap B) - \overline{A \cap \partial_2 B}) \\ &\subseteq \mathrm{cl}\,((A \cap B) - ((A \cap B) - \mathring{B})) \\ &= \mathrm{cl}\,(A \cap \mathring{B}). \end{aligned}$$

Therefore one could get

$$\partial(A \cap B) = \partial(A \cap \mathring{B}) \cup \overline{A \cap \partial_2 B}.$$

Also note that

$$\begin{aligned} \partial(A \cap \mathring{B}) &= \mathrm{cl}\,(\bar{A} \cap \mathring{B}) - \mathrm{int}\,(A \cap \mathring{B}) \\ &= (\mathrm{cl}\,(\mathring{A} \cap \mathring{B}) \cup \overline{\partial A \cap \mathring{B}}) - \mathrm{int}\,(A \cap \mathring{B}) \\ &= \partial(\mathring{A} \cap \mathring{B}) \cup \overline{\partial A \cap \mathring{B}} \end{aligned}$$

since $\mathrm{cl}\,(\partial A \cap \mathring{B})$ is disjoint with $\mathrm{int}\, A$. Also note that

$$\partial_2(X - A) = \partial_1 A \quad , \quad \partial_2(X - A) = \partial_1 A.$$

Hence both of the expansions with respect to second set are now established.

Proposition 5. The following are equivalent :

(*i*) $\bar{A} \cap \bar{B} = \overline{A \cap B}$

(*ii*) $\partial A \cap \partial B \subseteq \partial(A \cap B)$

(*iii*) $\partial(A \cap B) = (\bar{A} \cap \partial B) \cup (\bar{B} \cap \partial A)$

(*iv*) $\partial(A \cap B) = \partial(A \cap \mathring{B}) \cup \partial(\mathring{A} \cap B) \cup (\partial A \cap \partial B)$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are straightforward. After noticing $\mathring{A} \cap \partial B \subseteq \partial(\mathring{A} \cap B)$ by the second formula of Proposition 1, the condition (iv) evidently implies

$$\overline{A} \cap \overline{B} = (\mathring{A} \cap \mathring{B}) \cup (\mathring{A} \cap \partial B) \cup (\mathring{B} \cap \partial A) \cup (\partial A \cap \partial B) \subseteq \overline{A \cap B}$$

and therefore (i) is obtained.

Proposition 6. The following are equivalent :

(i) $\operatorname{int}(A \cup B) = \mathring{A} \cup \mathring{B}$

(ii) $\partial A \cap \partial B \subseteq \partial(A \cup B)$

(iii) $\partial(A \cup B) = (\partial A - \mathring{B}) \cup (\partial B - \mathring{A})$

(iv) $\partial(A \cup B) = \partial(A \cup \overline{B}) \cup \partial(B \cup \overline{A}) \cup (\partial A \cap \partial B)$.

Proof. Use Proposition 5.

Proposition 7. The following are equivalent :

(i) $\partial(A \cup B) = \partial A \cup \partial B$

(ii) $(\partial A \cap \overline{B}) \cup (\partial B \cap \overline{A}) \subseteq \partial(A \cup B)$

(iii) $\operatorname{int}(A \cap B) = \mathring{A} \cup \mathring{B}$ and $(\partial A \cap \mathring{B}) \cup (\partial B \cap \mathring{A}) \subseteq \partial(A \cup B)$

Proof. The condition (iii) and (4) imply $\partial A \subseteq \partial(A \cup B)$ since $\partial A - \mathring{B}$ and $\operatorname{int}(A \cup B)$ are disjoint by (iii).

Corollary 2. $\partial(A \cap B) \cup (\partial A \cap \partial B) \subseteq \partial(A \cup B)$ iff $\partial(A \cup B) = \partial A \cup \partial B$.

Proof. See the second formula of Proposition 1, Proposition 6 and Proposition 7.

Corollary 3. $\partial(A \cap B) = \partial A \cap \partial B$ and $\operatorname{int}(A \cup B) = \mathring{A} \cup \mathring{B}$ imply $\partial(A \cup B) = \partial A \cup \partial B$.

Remark 3. The equality $\partial(A \cup B) = \partial A \cup \partial B$ deos not necessarily imply the condition $\partial(A \cap B) = \partial A \cap \partial B$. Just take the diadic rationals in $[0, 1]$ as A and all the triadic rationals in the same interval as B for a counter example in $\mathbf{R}^1$.

Proposition 8. a) $\partial(A\cap B)=\partial A\cap\partial B$ iff $\overline{A}\cap\overline{B}=\overline{A\cap B}$ and $\partial(A\cap B)\cap\overset{\circ}{A}=$ $=\partial(A\cap B)\cap\overset{\circ}{B}$.

b) $\partial(\partial A\cap\partial B)=\partial\partial A\cap\partial\partial B$ iff $\operatorname{int}\partial A\cap\partial B=\operatorname{int}\partial B\cap\partial A$.

Proof. a) Left to the reader.

b) Note that

$$\partial(\partial A\cap\partial B)\subseteq\partial\partial A \quad \text{iff} \quad \operatorname{int}\partial A\cap\partial B\subseteq\operatorname{int}(\partial A\cap\partial B)$$

by (4). Then Corollary 1 is used.

Proposition 9. The following are equivalent :

(*i*) $\partial(\partial A\cup\partial B)=\partial\partial A\cup\partial\partial B$.

(*ii*) $\partial A\cap\operatorname{int}(\partial A\cup\partial B)=\operatorname{int}\partial A$ and $\partial B\cap\operatorname{int}(\partial A\cup\partial B)=\operatorname{int}\partial B$.

(*iii*) $\partial A\cap\partial B\cap\operatorname{int}(\partial A\cup\partial B)=\operatorname{int}(\partial A\cap\partial B)$.

(*iv*) $\partial(\partial A\cap\partial B)=\partial\partial A\cap\partial\partial B$ and $\operatorname{int}(\partial A\cup\partial B)=\operatorname{int}\partial A\cup\operatorname{int}\partial B$.

(*v*) $\partial(\partial A\cap\partial B)\subseteq\partial\partial A\cap\partial\partial B\cap\partial(\partial A\cup\partial B)$.

Proof. (*i*) $\Leftrightarrow$ (*ii*): Note that the condition (*i*) implies $\partial A\cap\operatorname{int}(\partial A\cup\partial B)\subseteq$ $\subseteq\operatorname{int}\partial A$ and its dual one.

(*ii*) $\Leftrightarrow$ (*iii*): Necessity is clear. For the proof of sufficiency note that one gets the following by using the conditon (*iii*)

$$\begin{aligned}\partial A\cap\operatorname{int}(\partial A\cup\partial B)&=\operatorname{int}(\partial A\cap\partial B)\cup((\partial A-\partial B)\cap\operatorname{int}(\partial A\cup\partial B))\\&\subseteq\operatorname{int}\partial A\cup((\operatorname{int}\partial A\cup\partial B)-\partial B)\subseteq\operatorname{int}\partial A\,.\end{aligned}$$

(*iii*) $\Leftrightarrow$ (*iv*): Note that the condition (*iii*) $\equiv$ (*ii*) $\equiv$ (*i*) implies

$$\partial\partial A\cap\partial\partial B\subseteq\partial(\partial A\cup\partial B)\,,$$

$$\operatorname{int}\partial A\cap\partial B=\operatorname{int}\partial B\cap\partial A\,.$$

So the required implication follows by the Propositions 6 and 8b).

(*iv*) $\Rightarrow$ (*v*): Clear by Proposition 6 since $\partial\partial A\cap\partial\partial B\subseteq\partial(\partial A\cup\partial B)$ holds by (*iv*).

(*v*) $\Rightarrow$ (*i*): By using (*v*) at the last inclusion in the following

$$\begin{aligned}\partial\partial A\cap\partial\partial B\cap\operatorname{int}(\partial A\cup\partial B)&\subseteq\partial(\partial A\cap\partial B)\cap\operatorname{int}(\partial A\cup\partial B)\\&\subseteq\partial(\partial A\cup\partial B)\cap\operatorname{int}(\partial A\cup\partial B)=\phi\end{aligned}$$

one yields $\partial\partial A\cap\partial\partial B\subseteq\partial(\partial A\cup\partial B)$ i.e. $\operatorname{int}(\partial A\cup\partial B)=\operatorname{int}\partial A\cup\operatorname{int}\partial B$. So all the sufficient conditions of Corollary 3 for being the equality written (*i*) hold are now satisfied after (*v*) and Corollary 1.

Remark 4. Note the difference of Corollary 2 with the equivalency of the conditions (*i*) and (*iv*) of Proposition 9.

Corollary 4. $$\partial A \cap \partial B = \phi$$
implies
$$\overline{A} \cap \overline{B} = \overline{A \cap B}\,,$$
$$\operatorname{int}(A \cup B) = \overset{\circ}{A} \cup \overset{\circ}{B}\,,$$
$$\partial(\partial A \cup \partial B) = \partial\partial A \cup \partial\partial B\,,$$
$$\partial(A \cup B) \cup \partial(A \cap B) = \partial A \cup \partial B\,.$$

Proof. Obtaining the first three equalities are clear, see also [2]. Now note that
$$(\partial A \cup \partial B) - (\partial(A \cup B) \cup \partial(A \cap B)) = (\partial A \cup \partial B) \cap (\operatorname{int}(A \cup B) - \overline{A \cap B})$$
always holds. The hypothesis makes the right side
$$(\partial A \cup \partial B) \cap (\overset{\circ}{A} - \overline{B}) \cup (\overset{\circ}{B} - \overline{A})\,) = \phi\,.$$
Hence the basic formula (5) gives the last equality.

Proposition 10. $$\partial\partial A = \partial\overset{\circ}{A} \cup \partial\overline{A}\,.$$

Proof. $$\partial\partial A = \partial A - \operatorname{int}\partial A = (\partial A \cap \operatorname{cl}\overset{\circ}{A}) \cup (\partial A \cap \operatorname{cl}(X - \overline{A}))$$
$$= \partial\overset{\circ}{A} \cup \partial\overline{A}\,.$$

Corollary 5. If A is open or closed then $\partial\partial A = \partial A$.

Proof. This well known result is a direct and easy consequence of Proposition 10. Let A be open. Then $\partial\partial A = \partial A \cup \partial\overline{A} = \partial A$, since $\partial\overline{A} \subseteq \partial A$ and also $\partial\overset{\circ}{A} \subseteq \partial A$ are derived by the same proposition.

Proposition 11. $\partial\partial(A \cap B) \cup \partial\partial(A \cup B) \subseteq \partial(\partial A \cup \partial B) \cup (\partial A \cap \partial B)$.

Proof. Note that
$$\begin{aligned}&(\partial\partial(A \cup B) - \partial(\partial A \cup \partial B)) - \partial A\\ &= (\partial\partial(A \cup B) \cap \operatorname{int}(\partial A \cup \partial B)) - \partial A\\ &= (\operatorname{int}(\partial A \cup \partial B) - \partial A) \cap (X - (\operatorname{int}(A \cup B) \cup \operatorname{int}\partial(A \cup B)))\\ &= \phi\end{aligned}$$
since

$$\operatorname{int}(\partial A \cup \partial B) - \partial A = \operatorname{int}((\partial A \cup \partial B) - \partial A) = \operatorname{int}(\partial B - \partial A) =$$

$$= \operatorname{int} \partial B - \partial A = \operatorname{int}(\partial B - \overline{A}) \cup (\operatorname{int} \partial B \cap \overset{\circ}{A}) \subseteq \operatorname{int} \partial(A \cup B) \cup \overset{\circ}{A}$$

by noticing

$$\begin{aligned}\operatorname{int}(\partial A - \overline{B}) &= \operatorname{int} \overline{A} \cap \operatorname{int}(X - (\overset{\circ}{A} \cup \overline{B}))\\ &\subseteq \operatorname{int} \overline{A \cup B} \cap \operatorname{int}(X - \operatorname{int}(A \cup B))\\ &= \operatorname{int} \partial(A \cup B)\,.\end{aligned}$$

Therefore, one easily gets

$$\partial\partial(A \cup B) - \partial(\partial A \cup \partial B) \subseteq \partial A \cap \partial B$$

which yields the statement of the Proposition easily.

Corollary 6. $\partial\partial(A \cap B) \cup \partial\partial(A \cup B) \subseteq \partial\partial A \cup \partial B\,,$

$$\partial(\partial\partial(A \cup B) - \partial(\partial A \cup \partial B)) \subseteq \partial A \cap \partial B\,.$$

Proof. Clear after Proposition 11 and Proposition 2. Notice that

$$E = \partial\partial(A \cup B) - \partial(\partial A \cup \partial B) = \partial\partial(A \cup B) \cap \operatorname{int}(\partial A \cup \partial B) =$$

$$= \operatorname{int}(\partial A \cup \partial B) - (\operatorname{int}(A \cup B) \cup \operatorname{int} \partial(A \cup B))$$

have empty interior and so one gets the following truth

$$\operatorname{int}(\partial A \cup \partial B) \subseteq \overline{\overline{A \cup B} - \partial\partial(A \cup B)} = \overline{(A \cup B) - \partial\partial(A \cup B)}\,.$$

Hence this corollary says that $\partial A \cap \partial B$ contains ∂E if $E \neq \phi$.

Corollary 7. $\partial A \cap \partial B \subseteq \partial(A \cup B)$

implies

$$\partial\partial(A \cup B) \subseteq \partial\partial A \cup \partial\partial B\,,$$

$$\partial(\partial A \cup \partial B) \subseteq \partial\partial(A \cup B) \cup \partial(A \cap B)\,.$$

Proof. First of all note that

$$\partial\partial(A \cup B) \subseteq \partial A \cup \partial\partial B,\ \partial\partial(A \cup B) \cap \operatorname{int} \partial B \subseteq \partial A$$

hold by the dual of the first inclusion formula of Corollary 6. Hence

$$\begin{aligned}&(\partial\partial(A \cup B) - \partial\partial A) \cap \operatorname{int} \partial B\\ &= ((\partial\partial(A \cup B) - \partial A) \cap \operatorname{int} \partial B) \cup (\partial\partial(A \cup B) \cap \operatorname{int}(\partial A \cap \partial B))\\ &= (\partial(A \cup B) \cap \operatorname{int}(\partial A \cap \partial B)) - \operatorname{int} \partial(A \cup B)\end{aligned}$$

are obtained. Now we are going to prove that this difference is empty under the hypothesis. In fact

$$\begin{aligned}\partial(A \cup B) \cap \operatorname{int}(\partial A \cap \partial B) \cap \overline{\operatorname{int}(A \cup B)} \\ = \partial(\operatorname{int}(A \cup B)) \cap \operatorname{int}(\partial A \cap \partial B) \\ \subseteq \partial(\operatorname{int}(A \cup B) \cap \partial A \cap \partial B) = \phi\end{aligned}$$

by Proposition 1, the hypothesis and Proposition 6. Therefore the required result

$$\begin{aligned}\partial(A \cup B) \cap \operatorname{int}(\partial A \cap \partial B) &\subseteq \operatorname{int} \overline{A \cup B} - \overline{\operatorname{int}(A \cup B)} \\ &= \operatorname{int} \partial(A \cup B)\end{aligned}$$

has been derived and consequently all the conditions for being

$$\partial\partial(A \cup B) - \partial\partial A \subseteq \partial\partial B$$

are now established after Corollary 6. Also, since

$$\partial(\partial A \cup \partial B) \cap \operatorname{int} \partial(A \cup B) \subseteq \partial(\partial A \cup \partial B) \cap \operatorname{int}(\partial A \cup \partial B) = \phi$$

holds always, one gets the following

$$\begin{aligned}\partial(\partial A \cup \partial B) - \partial\partial(A \cup B) &= \partial(\partial A \cup \partial B) - \partial(A \cup B) \\ &= \partial(\partial A \cup \partial B) \cap \operatorname{int}(A \cup B) .\end{aligned}$$

So by using the hypothesis, this set is included by

$$\partial((\partial A \cup \partial B) \cap (\mathring{A} \cup \mathring{B})) = \partial((\partial A \cap \mathring{B}) \cup (\partial B \cap \mathring{A})) \subseteq \partial(A \cap B) .$$

Corollary 8. $\partial A \cap \partial B \subseteq \partial(A \cap B)$

implies

$$\partial\partial(A \cap B) \subseteq \partial\partial A \cup \partial\partial B ,$$

$$\partial(\partial A \cup \partial B) \subseteq \partial\partial(A \cap B) \cup \partial(A \cup B) .$$

Proof. Use Corollary 7.

Proposition 12. If $\mathscr{A} = (A_\alpha)_{\alpha \in I}$ is a discrete family of open (or a locally finite of nowhere dense) subsets, then

$$\partial\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} \partial A_\alpha .$$

Proof. If $\mathscr{A}$ is a discrete family of open subsets, then the inclusion

$$\partial\left(\bigcup_{\alpha \in I} A_\alpha\right) \subseteq \bigcup_{\alpha \in I} \partial A_\alpha \cup \bigcup_{\alpha \in I} A_\alpha$$

evidently implies one of the inclusions which is necessary for the required equality. The reverse inclusion is a consequence of (4) and the following inclusions which are true for all $\alpha \in I$

$$\partial A_\alpha \cap \operatorname{int} \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\beta \neq \alpha} (\bar{A}_\alpha \cap A_\beta) = \phi \,.$$

The statement written in the parenthesis could be obtained by the following truth : The union of a locally finite family of nowhere dense subsets have an empty interior.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
İSTANBUL UNIVERSITY
VEZNECİLER-İSTANBUL/TURKEY

ÖZET

Bu çalışmada topolojik uzayların sınırları hakkında bazı özellik ve sonuçlar elde edilmektedir.