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DECOMPOSITION IN R - ⊕ RECURRENT FINSLER SPACE OF SECOND ORDER WITH NON-SYMMETRIC CONNECTION

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The decomposition of $R^i_{hjk}(x,\dot{x})$ curvature tensor field along with its properties in R-⊕ recurrent Finsler space with non-symmetric connection has been studied by Pande and Gupta [2]. The object of the present paper is to decompose the R-⊕ curvature tensor field in R-⊕ recurrent Finsler space of second order with non-symmetric connection to study the properties of such decomposition.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n^* [1] [1)] having $2n$ line element $(x^i, \dot{x}^i)$, $(i, j, k, \ldots = 1, 2, 3, \ldots, n)$ and equipped with non-symmetric connection coefficients

$$\Gamma^i_{jk}(x, \dot{x}) \neq \Gamma^i_{kj}(x, \dot{x})$$

based on non-symmetric tensor

$$g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x}) .$$

Cătăline [3] defined a non-symmetric connection parameter as follows :

$$\Gamma^i_{jk} = M^i_{jk} + \frac{1}{2} N^i_{jk} \tag{i.i}$$

where M^i_{jk} and $\frac{1}{2} N^i_{jk}(x, \dot{x})$ denote the symmetric and skew-symmetric parts of $\Gamma^i_{jk}(x, \dot{x})$. One more connection parameter $\tilde{\Gamma}^i_{jk}(x, \dot{x})$ has been introduced by Pande and Gupta [2]

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{kj}(x, \dot{x}) . \tag{1.2}$$

The covariant derivative of a tensor field X_j^i will be defined in two ways :

$$\overset{\overset{i}{+}}{X}_{|j} = \partial_j X^{i\,2)} - (\dot{\partial}_m X^i)\, \Gamma^m_{pj}\, \dot{x}^p + X^m\, \Gamma^i_{mj} \tag{1.3}$$

and

$$\overset{\overset{i}{-}}{X}_{|j} = \partial_j X^{i\,1)} - (\dot{\partial}_m X^i)\, \tilde{\Gamma}^m_{pj}\, x^p + X^m\, \tilde{\Gamma}^i_{mj} . \tag{i.4}$$

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[1]) The numbers in square brackets refer to the references given at the end of the paper.

[2]) $\partial_i = \partial/\partial x^i$; $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

The duality in the nature of covariant derivatives introduce two curvature tensors given by:

$$R^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} - (\dot\partial_m \Gamma^i_{jk}) \Gamma^m_{pl} \dot x^p +$$
$$+ (\dot\partial_m \Gamma^i_{jl}) \Gamma^m_{pk} \dot x^p + \Gamma^p_{jk} \Gamma^i_{pl} - \Gamma^p_{jl} \Gamma^i_{pk}$$

and

$$\tilde R^i_{jkl} = \partial_l \tilde\Gamma^i_{jk} - \partial_k \tilde\Gamma^i_{jl} - (\dot\partial_m \tilde\Gamma^i_{jk}) \tilde\Gamma^m_{pl} \dot x^p +$$
$$+ (\dot\partial_m \tilde\Gamma^i_{jl}) \tilde\Gamma^m_{pk} \dot x^p + \tilde\Gamma^p_{jk} \tilde\Gamma^i_{pl} - \tilde\Gamma^p_{jl} \tilde\Gamma^i_{pk} \,. \tag{1.6}$$

The following results and notations [2] will be used in the sequel.

$$\dot x^{\overset{i}{+}}_{|k} = 0 = \dot x^{\overset{i}{-}}_{|k} \tag{1.7}$$

$$R^i_{jk} = \dot x^h R^i_{hjk} \tag{1.8}$$

$$R^i_{hjk} = - R^i_{hkj} \; ; \; R^i_{jk} = - R^i_{kj}$$
$$N^i_{jk} = - N^i_{kj} \,. \tag{1.9}$$

The curvature tensor R^h_{ijk} satisfies the following identities in $F_n{}^*$:

$$\overset{i}{\underset{+}{R}}{}_{hjk} + \overset{i}{\underset{+}{R}}{}_{jkh} + \overset{i}{\underset{+}{R}}{}_{khj} = 0 \tag{1.10}$$

$$\tilde R^i_{hjk} + \tilde R^i_{jkh} + \tilde R^i_{khj} = 0 \tag{1.11}$$

and

$$\overset{h}{\underset{+}{R}}{}_{\underset{+++}{i\,j\,k}|l} + \overset{h}{\underset{+}{R}}{}_{\underset{+++}{i\,k\,l}|j} + \overset{h}{\underset{+}{R}}{}_{\underset{+++}{i\,l\,j}|k} + E^h_{iljk} = 0 \tag{1.12}$$

where

$$E^h_{iljk} \overset{\text{def.}}{=\!=} R^m_{jk} \Gamma^h_{mil} + R^m_{kl} \Gamma^h_{mij} + R^m_{lj} \Gamma^h_{mik} \,. \tag{1.13}$$

The commutation formulae [2] involving the $+ -$ covariant derivative are given by:

$$\dot\partial_k (T^{\overset{i}{+}}_{\underset{+}{j}|h}) - (\underset{+}{\partial_k} \; T^{\overset{i}{+}}_{\underset{+}{j}})_{|h} =$$
$$= T_j^m \dot\partial_k \Gamma^i_{mh} - T^i_m \dot\partial_k \Gamma^m_{jk} - (\dot\partial_m T_j{}^i) \dot\partial_k \Gamma^m_{ph} \dot x^p \tag{1.14}$$

$$T^{\overset{i}{+}}_{\underset{+}{j}|h\,k} - T^{\overset{i}{+}}_{\underset{+}{j}|k\,h} = (- \dot\partial_m T_j{}^i) R^m_{hk} + T_j{}^m R^i_{mhk} - T^i_m R^i_{jhk} +$$
$$+ (T^{\overset{i}{+}}_{\underset{+}{j}|m}) N^m_{kh} \tag{1.15}$$

where

$$N^m_{kh} = \Gamma^m_{kh} - \Gamma^i_{kh} \tag{i.I6}$$

$$\dot{\partial}_r R^h_{ijk} \overset{\text{def.}}{=\!=} R^h_{rijk} \ . \tag{1.71}$$

In an n-dimensional Finsler space $F_n{}^*$ the curvature tensor field $R^i_{hjk}(x, \dot{x})$ satisfies the relation [2]

$$\underset{+++}{R^{\ i}_{\overset{+}{h}\ j\ k|l}} = \lambda_l R^i_{hjk} \quad (R^i_{hjk} \neq 0) \tag{1.18}$$

where $\lambda_l(x)$ is non-zero recurrence vector field, then $F_n{}^*$ is said to be $R - \oplus$ recurrent Finsler space of first order.

An n-dimensional Finsler space $F_n{}^*$ is said to be R - $\oplus$ recurrent $F_n{}^*$ of second order if its curvature tensor field $R^i_{hjk}(x, \dot{x})$ satisfies the relation

$$\underset{++}{R^{\ i}_{\overset{+}{h}\ j\ k\ |\ lm}} = a_{lm} R^i_{hjk} \ , \tag{1.19}$$

where $a_{lm}(x, \dot{x})$ is called non-zero recurrence tensor field. A relation between λ_l and a_{lm} is given by

$$a_{lm} = \lambda_{\underset{+}{l}\,|\,m} + \lambda_m \lambda_l \tag{1.19'}$$

2. DECOMPOSITION IN R - ⊕ RECURRENT $F_n{}^*$

We consider the decomposition of $R^i_{jk}(x, \dot{x})$ as follows:

$$R^i_{jk}(x, \dot{x}) = \dot{x}^i \in_{jk} , \tag{2.1}$$

where $\in_{jk}(x, \dot{x})$ is a non-zero tensor field of first order in its directional arguments and

$$\dot{x}^i \lambda_i = P \quad \text{(Constant)}.$$

Differentiating (2.1) partially with respect to $\dot{x}^h$ and using (1.17), we have

$$R^i_{hik}(x, \dot{x}) = x^i \in_{hJk} \tag{2.2}$$

where
$$\in_{hjk}(x, \dot{x}) \overset{\text{def.}}{=\!=} \dot{\partial}_h \in_{jk} . \tag{2.3}$$

Theorem 2.1. In view of decomposition (2.2) the identity for the curvature tensor field $R^i_{hjk}(x, \dot{x})$ is given by

$$\in^{3)}_{[hjk]} = 0 \ . \tag{2.4}$$

In view of equation (2.2) the identity (1.10) yields the above theorem.

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3) $3A_{[ijk]} \overset{\text{def.}}{=\!=} A_{ijk} + A_{jki} + A_{kij}$.

Theorem 2.2. In a recurrent Finsler space of first order the decomposition tensor field $R^i_{hjk}(x, \dot{x})$ satisfies :

$$3\,\dot{x}^h \in_{i[jk\lambda l]} = E^h_{iljk} \tag{2.5}$$

In view of (1.18), the Bianchi identity (1.18) reduces to

$$3\,R^h_{i[jk\lambda l]} = E^h_{iljk} \tag{2.6}$$

which for decomposition (2.2) yields the required results (2.5).

Theorem 2.3. Under the decompositions (2.1) and (2.2) the decomposed tensor fields $E_{hjk}(x, \dot{x})$ and $\in_{jk}(x, \dot{x})$ behave like recurrent tensor field of second order.

Differentiating (2.2) covariantly with respect to x^s and x^m successively, we get

$$R^{\overset{i}{+}}_{\underset{+++}{h\;j\;k\,|\,sm}} = \dot{x}^i\,|\,sm \in_{hjk} + \dot{x}^i \in_{\underset{+++}{h\;j\;k\,|\,sm}} .$$

In view of the equation (1.7), equation (2.7) takes the form :

$$R^{\overset{i}{+}}_{\underset{+++}{h\;j\;k\,|\,sm}} = x^i \in_{\underset{+++}{h\;j\;k\,|\,sm}} . \tag{2.8}$$

which in view of equations (1.19) and (2.2) yields

$$a_{sm} \in_{hjk} = \in_{\underset{+++}{h\;j\;k\,|\,sm}} . \tag{2.9}$$

Transvecting (2.9) by $\dot{x}^h$ and using the homogeneity property of decomposition tensor field, we get

$$a_{sm} \in_{jk} = \in_{\underset{++}{j\;k\,|\,sm}} . \tag{2.10}$$

which proves the statement.

Theorem 2.4. In a recurrent Finsler space of second order the decomposition tensor fields satisfy

$$a_{sm} \in_{[hjk]} = 0 \tag{2.11}$$

and

$$\dot{x}^h \in_{i[jkl]s} = E^{\,h}_{\underset{++++}{i\;\;l\;\;j\;\;k\,|\,s}} \tag{2.12}$$

Proof. Differentiating (1.10) successively with respect to x^s and x^m, we get

$$R^{\overset{i}{+}}_{\underset{+++}{[h\;j\;k]\,sm}} = 0 \tag{2.13}$$

which in view of the equation (1.19) and (2.2) yields the required result (2.11).

Again, differentiating (1.12) with respect to x^s, we get

$$3\, \underset{+++}{R^{+\,h}_{i[j\;k\;l]|s}} = \underset{++++}{E^{+\,h}_{i\;j\;\;k|s}} \tag{2.14}$$

which in view of the equation (1.19) and (2.2), yields the result (2.21).

Theorem 2.5. Under the decomposition, characterized by equation (2.1) and (2.2) of λ_s is independent of the deviation, the recurrence tensor field a_{sm} satisfies

$$(\dot{\partial}_n a_{sm} - \dot{\partial}_m a_{sn}) \in_{jk} =$$

$$= [\{\underset{++}{\in_{r\;k|s}} \dot{\partial}_j \Gamma^r_{nm} + \underset{++}{\in_{j\;r|s}} \dot{\partial}_k \Gamma^r_{nm} + \underset{++}{\in_{j\;k|r}} \dot{\partial}_s \Gamma^r_{nm}\} -$$

$$-\{\underset{++}{\in_{r\;k|s}} \dot{\partial}_j \Gamma^r_{mn} + \underset{++}{\in_{j\;r|s}} \dot{\partial}_k \Gamma^r_{mn} + \underset{++}{\in_{j\;k|s}} \dot{\partial}_s \Gamma^r_{nm}\}] \tag{2.15}$$

Proof. Differentiating (2.10) partially with respect to $\dot{x}^n$ and using (2.3), we get

$$\dot{\partial}_n \{(\underset{++}{\in_{j\;k|s}})_m\} = a_{sm} \in_{njk} + \in_{jk} \dot{\partial}_n a_{sm} \tag{2.16}$$

In view of the commutation formula (1.14) the above equation reduces to

$$\{\underset{+}{\dot{\partial}_n} (\underset{++}{\in_{j\;k|s}})\}_{|m} - \underset{++}{\in_{r\;k|s}} \dot{\partial}_j \Gamma^r_{mn} - \underset{++}{\in_{j\;r|s}} \dot{\partial}_k \Gamma^r_{mn} - \underset{++}{\in_{j\;k|r}} \dot{\partial}_s \Gamma^r_{mn} =$$

$$= a_{sm} \in_{njk} + \in_{jk} \dot{\partial}_n a_{sm}\,. \tag{2.17}$$

In view of the fact that λ_s is independent of direction and the identity (1.19′), the above equation reduces to

$$-\{\underset{++}{\in_{r\;k|s}} \dot{\partial}_j \Gamma^r_{mn} + \underset{++}{\in_{j\;r|s}} \dot{\partial}_k \Gamma^r_{mn} + \underset{++}{\in_{j\;k|r}} \dot{\partial}_s \Gamma^r_{nm}\} = \in_{jk} \dot{\partial}_n a_{sm}\,. \tag{2.18}$$

Interchanging the indices m and n and substracting the equation thus obtained from the above equation we get the required result (2.15).

Theorem. If the recurrence vector λ_s is independent of direction in a Finsler space $F_n{}^*$, the following relation holds :

$$\{\in_{jk} (\dot{\partial}_n a_{sm}) - \in_{jn} \dot{\partial}_k a_{sm}\} \dot{x}^m = 0\,. \tag{2.19}$$

Proof. Interchanging the indices a and k in the equation (2.18) and subtracting the equation thus obtained from (2.18) itself, we get

$$\{\in_{jk} (\dot{\partial}_n a_{sm}) - \in_{jn} (\dot{\partial}_k a_{sm})\} =$$

$$= [\{\underset{++}{\in_{r\;n|s}} \dot{\partial}_j \Gamma^r_{mk} + \underset{++}{\in_{j\;r|s}} \dot{\partial}_n \Gamma^r_{mk} + \underset{++}{\in_{j\;n|r}} \dot{\partial}_s \Gamma^r_{mk}\} -$$

$$- \{\underset{++}{\in_{r\;k|s}} \dot{\partial}_j \Gamma^r_{mn} + \underset{++}{\in_{j\;r|s}} \dot{\partial}_k \Gamma^r_{mn} + \in_{jk|r} \dot{\partial}_s \Gamma^r_{mn}\}]\,. \tag{2.20}$$

Transvecting the above equation by $\dot{x}^m$ and using the homogeneity property of the function, we get the required theorem.

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ÖZET

Bu çalışmada, simetrik olmayan bağlantılı, 2. mertebeden R-⊕ tekrarlı Finsler uzaymdaki R-⊕ eğrilik tensör alanının parçalanışı incelenmektedir.