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Research Article

Starlikeness for certain close-to-star functions

R. Kanaga, V. Ravichandran*

Department of Mathematics, National Institute of Technology, Tiruchirappalli-620015, India

Abstract

We find the radius of starlikeness of order α, $0 \leq \alpha < 1$, of normalized analytic functions f on the unit disk satisfying either $\operatorname{Re}(f(z)/g(z)) > 0$ or $|(f(z)/g(z)) - 1| < 1$ for some close-to-star function g with $\operatorname{Re}(g(z)/(z + z^2/2)) > 0$ as well as of the class of close-to-star functions f satisfying $\operatorname{Re}(f(z)/(z + z^2/2)) > 0$. Several other radii such as radius of univalence and parabolic starlikeness are shown to be the same as the radius of starlikeness of appropriate order.

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1. Introduction

Let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} := \mathbb{D}_1$. Let $\mathcal{A} := \{f : \mathbb{D} \to \mathbb{C} | f \text{ is analytic}, f(0) = f'(0) - 1 = 0\}$ and let $\mathcal{S}$ be its subclass consisting of univalent functions. For each $f \in \mathcal{S}$, we associate the function $s_f : \mathbb{D} \to \mathbb{C}$ defined by $s_f(z) = zf'(z)/f(z)$. For $0 \leq \alpha < 1$, the class $\mathcal{S}^*(\alpha)$ of functions starlike of order α is the subclass of $\mathcal{S}$ consisting of functions f satisfying the inequality $\operatorname{Re}(s_f(z)) > \alpha$. The class $\mathcal{S}^* := \mathcal{S}^*(0)$ is the usual class of starlike functions. The image $k(\mathbb{D})$ of the Koebe function $k(z) = z/(1 - z)^2$ is not convex but the image $k(\mathbb{D}_{2-\sqrt{3}})$ is convex and $2 - \sqrt{3}$ is the largest such radius. This radius is known as the radius of convexity of the Koebe function. More generally, given a class of functions $\mathcal{F}$ and another class $\mathcal{G}$ characterised by a property P, the largest number $R_{\mathcal{G}}(\mathcal{F})$ with $0 \leq R_{\mathcal{G}}(\mathcal{F}) \leq 1$ such that every function in $\mathcal{F}$ has the property P, in each disk $\mathbb{D}_r$ for each r with $0 < r < R_{\mathcal{G}}(\mathcal{F})$ is called the $\mathcal{G}$ radius of $\mathcal{F}$. Kaplan [11] introduced the class of close-to-convex functions f satisfying $\operatorname{Re}(f'(z)/g'(z)) > 0$ for some convex function g. In [17,18], MacGregor found the radius of starlikeness for the class of functions f satisfying either $\operatorname{Re}(f(z)/g(z)) > 0$ or $|(f'(z)/g'(z)) - 1| < 1$ for some $g \in \mathcal{S}$; related radius problems were discussed in [3–7, 10, 13, 14, 16, 21, 22]. Reade [23] defined a function $f \in \mathcal{A}$, with $f(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$, to be close-to-star if there exists a starlike function g (not necessarily normalized) satisfying $\operatorname{Re}(f(z)/g(z)) > 0$. The function $f(z) = z + z^2/2$ maps $\mathbb{D}$ onto the domain bounded by the cardioid $u + 1/2 = \cos t(1 + \cos t)$ and $v = \sin t(1 + \sin t)$ and therefore starlike in $\mathbb{D}$. This function f also satisfies the inequality $|f'(z) - 1| < 1$ (which

*Corresponding Author.
Email addresses: kanagavennila75@gmail.com (R. Kanaga); vravi68@gmail.com, ravic@nitt.edu (V. Ravichandran)
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also implies univalence of f). Using this starlike function, we introduce the following three classes:

$$\mathcal{F}_1 := \{f \in \mathcal{A} : \operatorname{Re}(f(z)/g(z)) > 0, \quad \operatorname{Re}(g(z)/(z + z^2/2)) > 0 \quad \text{for some } g \text{ in } \mathcal{A}\},$$

$$\mathcal{F}_2 := \{f \in \mathcal{A} : |(f(z)/g(z)) - 1| < 1, \quad \operatorname{Re}(g(z)/(z + z^2/2)) > 0 \quad \text{for some } g \text{ in } \mathcal{A}\},$$

and

$$\mathcal{F}_3 := \{f \in \mathcal{A} : \operatorname{Re}(f(z)/(z + z^2/2)) > 0\}.$$

These classes $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are nonempty. Indeed, if the functions $f_i : \mathbb{D} \to \mathbb{C}$, $i = 1, 2, 3$, are defined by

$$f_1(z) = \frac{(1+z)^2(z+z^2/2)}{(1-z)^2}, \quad f_2(z) = \frac{(1+z)^2(z+z^2/2)}{(1-z)} \tag{1.1}$$

and

$$f_3(z) = \frac{(1+z)(z+z^2/2)}{(1-z)}, \tag{1.2}$$

then it follows that f_i belongs to the class $\mathcal{F}_i$; the functions f_1 and f_2 satisfy the respective condition with $g = f_3$. It is also clear that the class $\mathcal{F}_3$ is a subclass of close-to-star functions while the classes $\mathcal{F}_1$ and $\mathcal{F}_2$ are not. The functions in these classes are also not necessarily univalent. Indeed, the radius of univalence $R_{\mathcal{S}}(\mathcal{F}_1) \approx 0.210756$, $R_{\mathcal{S}}(\mathcal{F}_2) \approx 0.248032$, $R_{\mathcal{S}}(\mathcal{F}_3) \approx 0.347296$ are respectively the smallest positive zero of the polynomials P_i given by

$$P_1(r) = 1 - 5r + r^2 + r^3, \quad P_2(r) = 2 - 8r - r^2 + 3r^3, \tag{1.3}$$

and

$$P_3(r) = 1 - 3r + r^3. \tag{1.4}$$

These radius are in fact the radius of starlikeness of the respective classes (see Theorems 2.1, 3.1 and 4.1). The sharpness of these radii follows as the first derivative of f_1, f_2 and f_3, given by

$$f_1'(z) = \frac{(1+z)(1+5z+z^2-z^3)}{(1-z)^3}, \quad f_2'(z) = \frac{(1+z)(2+8z-z^2-3z^3)}{2(1-z)^2}, \tag{1.5}$$

and

$$f_3'(z) = \frac{1+3z-z^3}{(1-z)^2}, \tag{1.6}$$

clearly vanishes at $z = -R_{\mathcal{S}}(\mathcal{F}_i)$ for $i = 1, 2, 3$ respectively.

Several subclasses of starlike functions are defined through subordination. An analytic function f is subordinate to the analytic function g, written $f \prec g$, if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. For univalent superordinate function g, we have $f \prec g$ if $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$. Consider the functions $\varphi_i : \mathbb{D} \to \mathbb{C}$ defined by $\varphi_1(z) := \sqrt{z+1}$, $\varphi_2(z) := e^z$, $\varphi_3(z) := 1 + (4/3)z + (2/3)z^2$, $\varphi_4(z) := 1 + \sin z$, $\varphi_5(z) =: z + \sqrt{1+z^2}$, $\varphi_6(z) := 1 + ((zk+z^2)/(k^2-kz))$ where $k = \sqrt{2}+1$ and $\varphi_7(z) := 1 + (2\left(\log((1+\sqrt{z})/(1-\sqrt{z}))\right)^2/\pi^2)$. For $\varphi = \varphi_i$, $(i = 1, 2, \ldots, 7)$ the class $\mathcal{S}^*(\varphi) := \{f \in \mathcal{A} : s_f \prec \varphi\}$ respectively becomes $\mathcal{S}^*_{\mathcal{L}}$, $\mathcal{S}^*_e$, $\mathcal{S}^*_c$, $\mathcal{S}^*_{\sin}$, $\mathcal{S}^*_{\mathbb{C}}$, $\mathcal{S}^*_R$ and $\mathcal{S}^*_p$; these classes were studied in [8, 15, 19, 20, 24, 28, 29]. For these φ_i, we study the $\mathcal{S}^*(\varphi)$ radius of the classes $\mathcal{F}_i$, $i = 1, 2, 3$ introduced above (see [1, 12, 25] for related works). For example, for the class $\mathcal{F}_1$, we have shown that the radius of starlikeness of order α, $0 \le \alpha < 1$, is the smallest positive root in (0,1) of the equation

$$(\alpha - 2)r^3 - (2\alpha + 2)r^2 + (10 - \alpha)r + 2\alpha - 2 = 0.$$

In addition to finding radius of lemniscate starlikeness, we have also shown that $R_{\mathcal{S}} = R_{\mathcal{S}^*}$, $R_{\mathcal{S}^*(1/2)} = R_{\mathcal{S}^*_P}$, $R_{\mathcal{S}^*(1/e)} = R_{\mathcal{S}^*_e}$, $R_{\mathcal{S}^*(1-\sin 1)} = R_{\mathcal{S}^*_{\sin}}$, $R_{\mathcal{S}^*(\sqrt{2}-1)} = R_{\mathcal{S}^*_{\mathbb{C}}}$, $R_{\mathcal{S}^*(2(\sqrt{2}-1))} = R_{\mathcal{S}^*_R}$ and $R_{\mathcal{S}^*(1/3)} = R_{\mathcal{S}^*_C}$. Similar results have been proved for the other two classes.

2. Radius problem for $\mathcal{F}_1$

For the function $f \in \mathcal{F}_1$, we first determine the disk containing the image of the disk $\mathbb{D}_r$ under the mapping $zf'(z)/f(z)$. This is done by associating the function f with suitable functions with positive real part and then applying the inequality (see [26, Lemma 2])

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)|z|}{(1-|z|)(1+(1-2\alpha)|z|)} \quad (z \in \mathbb{D}) \tag{2.1}$$

for the function p in the class $\mathscr{P}(\alpha)$ of all analytic function $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > \alpha$. We also need to know the image of the disk $\mathbb{D}_r$ under the transform $w(z) = (z+1)/(z+2)$. This is a linear fractional transformation and it maps the disk $\mathbb{D}_r$ onto the disk defined by

$$\left|w(z) - \frac{2-r^2}{4-r^2}\right| \leq \frac{r}{4-r^2}. \tag{2.2}$$

Since $f \in \mathcal{F}_1$, there is a function $g \in \mathcal{A}$ such that $\operatorname{Re}(f(z)/g(z)) > 0$ and $\operatorname{Re}(g(z)/(z+z^2/2)) > 0$ for all $z \in \mathbb{D}$. Thus, the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ defined by $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/(z + z^2/2)$ are functions in $\mathcal{P}(0)$ and

$$f(z) = p_1(z)p_2(z)\left(z + z^2/2\right) \quad (z \in \mathbb{D}). \tag{2.3}$$

From (2.3), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{2(z+1)}{z+2}. \tag{2.4}$$

Using (2.1) (with $\alpha = 0$) and (2.2) in (2.4), we see that the image of the disk $\mathbb{D}_r$ under the mapping $zf'(z)/f(z)$ is contained in the disk defined by

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| \leq \frac{6r(3-r^2)}{(1-r^2)(4-r^2)}. \tag{2.5}$$

From (2.5), it readily follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{4-2r^2}{4-r^2} - \frac{6r(3-r^2)}{(1-r^2)(4-r^2)} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)} \quad (|z| \leq r). \tag{2.6}$$

Let $R_{\mathcal{S}^*} \approx 0.2108$ be the unique zero in (0,1) of the polynomial $1 - 5r + r^2 + r^3$. Then, for every function $f \in \mathcal{F}_1$, the inequality (2.6) shows that $\operatorname{Re}(s_f(z)) > 0$ in each disk $\mathbb{D}_r$, for $0 \leq r < R_{\mathcal{S}^*}$. For the function f_1 defined in (1.1), we have

$$s_{f_1}(z) = \frac{zf_1'(z)}{f_1(z)} = \frac{2(1+5z+z^2-z^3)}{(2+z)(1-z^2)} \tag{2.7}$$

and hence $\operatorname{Re}(s_{f_1}(z))$ vanishes at $z = -R_{\mathcal{S}^*}$. Thus, the radius of starlikeness $R_{\mathcal{S}^*}$ for the class $\mathcal{F}_1$ is the unique positive zero in $(0,1)$ of the polynomial P_1 defined in (1.3) and is the same as the radius of univalence $R_{\mathcal{S}}$. Using the inequality (2.5), we now determine $\mathcal{S}^*(\alpha)$, $\mathcal{S}^*_{\mathcal{L}}$, $\mathcal{S}^*_P$, $\mathcal{S}^*_e$, $\mathcal{S}^*_c$, $\mathcal{S}^*_{\sin}$, $\mathcal{S}^*_{\mathbb{C}}$ and $\mathcal{S}^*_R$ radii for the class $\mathcal{F}_1$.

Theorem 2.1. *The following sharp radii results hold for the class $\mathcal{F}_1$:*

(i) *For any $0 \leq \alpha < 1$, the radius $R_{\mathcal{S}^*(\alpha)}$ is the smallest positive root of the equation*

$$(\alpha-2)r^3 - (2\alpha+2)r^2 + (10-\alpha)r + 2\alpha - 2 = 0. \tag{2.8}$$

(ii) *The radius $R_{\mathcal{S}^*_{\mathcal{L}}}$ (≈ 0.0918) is the smallest positive root of the equation*

$$(2-\sqrt{2})r^3 - (2+2\sqrt{2})r^2 + (\sqrt{2}-10)r + 2\sqrt{2} - 2 = 0. \tag{2.9}$$

(iii) *The radius $R_{\mathcal{S}^*_{\mathcal{P}}}$ (≈ 0.1092) is the same as $R_{\mathcal{S}^*(1/2)}$.*

(iv) *The radius $R_{\mathcal{S}^*_e}$ (≈ 0.1370) is the same as $R_{\mathcal{S}^*(1/e)}$.*

(v) *The radius $R_{\mathcal{S}^*_{\sin}}$ (≈ 0.17969) is the same as $R_{\mathcal{S}^*(1-\sin 1)}$.*

(vi) *The radius* $R_{\mathcal{S}^*_{\mathbb{C}}}$ *(* ≈ 0.12734*) is the same as* $R_{\mathcal{S}^*(\sqrt{2}-1)}$.

(vii) *The radius* $R_{\mathcal{S}^*_R}$ *(*≈ 0.0380*) is the same as* $R_{\mathcal{S}^*(2(\sqrt{2}-1))}$.

(viii) *The radius* $R_{\mathcal{S}^*_C}$ *(*≈ 0.14418*) is the same as* $R_{\mathcal{S}^*(1/3)}$.

Proof. (i) Let the function $f \in \mathcal{F}_1$ and $\alpha \in [0,1)$. Let $R := R_{\mathcal{S}^*(\alpha)}$ be the smallest positive root of the equation (2.8) so that

$$2(1-5R+R^2+R^3) = \alpha(2-R)(1-R^2). \tag{2.10}$$

The function

$$h(r) = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)}$$

is decreasing in $[0,1)$ and hence, for $0 \le r < R$, we have, using (2.6) and (2.10),

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)} > \frac{2(1-5R+R^2+R^3)}{(2-R)(1-R^2)} = \alpha.$$

This proves that the function f is starlike of order α in each disk $\mathbb{D}_r$ for $0 \le r < R$. At the point $z = -R$, it can be seen, using (2.7) and (2.10), that the function f_1 defined in (1.1) satisfies

$$\operatorname{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right) = \frac{2(1-5R+R^2-R^3)}{(2-R)(1-R^2)} = \alpha.$$

This shows that the radius R is the sharp radius of starlikeness of order α of the class $\mathcal{F}_1$.

(ii) Let $R := R_{\mathcal{S}^*_{\mathcal{L}}}$ be the smallest positive root of the equation (2.9) so that

$$2(1+5R+R^2-R^3) = \sqrt{2}(1-R^2)(2+R). \tag{2.11}$$

Since the function

$$h(r) := \frac{6r(3-r^2)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1+5r+r^2-r^3)}{(2+r)(1-r^2)}$$

is an increasing function of r in [0,1), it follows that, for $0 \le r < R$, $h(r) < h(R) = \sqrt{2}$ and hence, for $0 \le r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} < \sqrt{2} - \frac{4-2r^2}{4-r^2}. \tag{2.12}$$

From (2.5) and (2.12), we obtain

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \sqrt{2} - \frac{4-2r^2}{4-r^2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R),1] \subset (c(0.1),1] \approx (.997494,1] \subset [2\sqrt{2}/3, \sqrt{2})$. When $a \in [2\sqrt{2}/3, \sqrt{2})$, by [2, Lemma 2.2], the disk $\{w : |w-a| < \sqrt{2}-a\}$ is contained in the lemniscate region $\{w : |w^2-1| < 1\}$ and hence, for $0 \le r < R$, we have

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1.$$

Thus, the radius of lemniscate starlikeness of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.11), we see that the function f_1 defined in (1.1) satisfies, at $z = R$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1+5R+R^2-R^3)}{(2+R)(1-R^2)} = \sqrt{2}$$

and therefore

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = 1.$$

(iii) The number $R := R_{\mathcal{S}^*_{\mathcal{P}}}$ is the smallest positive root of the equation

$$2(1-5R+R^2+R^3) = (1/2)(2-R)(1-R^2). \tag{2.13}$$

Since the function

$$h(r) := \frac{6r(r^2-3)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that, $h(r) > h(R) = 1/2$ for $0 \leq r < R$, and hence, for $0 \leq r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} - \frac{1}{2}. \tag{2.14}$$

From (2.5) and (2.14), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - \frac{1}{2} \quad (|z| \leq r). \tag{2.15}$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.989899, 1] \subset (1/2, 3/2)$. When $a \in (1/2, 3/2)$, by [27, Lemma 2.2], the disk $\{w : |w-a| < a-(1/2)\}$ is contained in the parabolic region $\{w : |w-1| < \operatorname{Re}(w)\}$ and hence, for $0 \leq r < R$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (|z| \leq r).$$

Thus, the radius of parabolic starlikeness of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.13), we see that the function f_1 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1-5R+R^2+R^3)}{(2-R)(1-R^2)} = \frac{1}{2}$$

and therefore

$$\left|\frac{zf_1'(z)}{f_1(z)} - 1\right| = \frac{1}{2} = \operatorname{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right).$$

(iv) The number $R := R_{\mathcal{S}^*_e}$ is the smallest positive root of the equation

$$2(1-5R+R^2+R^3) = (1/e)(2-R)(1-R^2). \tag{2.16}$$

Since the function

$$h(r) := \frac{6r(r^2-3)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1/e$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} - \frac{1}{e}. \tag{2.17}$$

From (2.5) and (2.17), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - \frac{1}{e} \quad (|z| \leq r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.989899, 1] \subset (e^{-1}, (e+e^{-1})/2]$. When $a \in (e^{-1}, (e+e^{-1})/2]$, by [19, Lemma 2.2], the disk $\{w : |w-a| < a-e^{-1}\}$ is contained in the region $\{w : |\log w| < 1\}$ and hence, for $0 \le r < R$,

$$\left|\log\left(\frac{zf'(z)}{f(z)}\right)\right| < 1 \quad (|z| \le r).$$

Thus, the radius of exponential starlikeness of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.16), we see that the function f_1 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1-5R+R^2+R^3)}{(2+R)(1-R^2)} = \frac{1}{e}$$

and therefore

$$\left|\log\left(\frac{zf_1'(z)}{f_1(z)}\right)\right| = 1.$$

(v) The number $R := R_{\mathcal{S}^*_{\sin}}$ is the smallest positive root of the equation

$$2(1-5R+R^2+R^3) = (1-\sin 1)(2-R)(1-R^2). \tag{2.18}$$

Since the function

$$h(r) := \frac{6r(r^2-3)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)} > 1-\sin 1$$

is decreasing function of r in [0,1), it follows that, $h(r) > h(R) = 1-\sin 1$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} + \sin 1 - 1. \tag{2.19}$$

From (2.5) and (2.19), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \sin 1 - 1 + \frac{4-2r^2}{4-r^2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.989899, 1] \subset (-1-\sin 1, 1-\sin 1)$. When $a \in (-1-\sin 1, 1+\sin 1)$, by [8, Lemma 3.3], the disk $\{w : |w-a| < \sin 1 - |a-1|\}$ is contained in the region $\varphi_4(\mathbb{D})$, where $\varphi_4(z) = 1+\sin z$ and hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_4(\mathbb{D})$. Thus, the radius of sine starlikeness of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.18), we see that the function f_1 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1-5R+R^2+R^3)}{(2-R)(1-R^2)} = 1-\sin 1 = \varphi_4(-1) \in \partial\varphi_4(\mathbb{D}).$$

(vi) The number $R := R_{\mathcal{S}^*_{\leftmoon}}$ is the smallest positive root of the equation

$$2(1-5R+R^2+R^3) = (\sqrt{2}-1)(2-R)(1-R^2). \tag{2.20}$$

Since the function

$$h(r) := \frac{6r(r^2-3)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = \sqrt{2}-1$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} + 1 - \sqrt{2}. \tag{2.21}$$

From (2.5) and (2.21), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} + 1 - \sqrt{2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.9898991, 1] \subset (\sqrt{2}$-$1, \sqrt{2}+1)$. When $a \in (\sqrt{2}-1, \sqrt{2}+1)$, by [9, Lemma 2.1], the disk $\{w : |w-a| < 1 - |\sqrt{2}-a|\}$ is contained in the region $\{w : |w^2-1| < 2|w|\}$ and hence, for $0 \le r < R$,

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right| \quad (|z| \le r).$$

Thus, the radius of lune starlikeness of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.20), we see that the function f_1 defined in (1.1) satisfies

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1-5R+R^2+R^3)}{(2-R)(1-R^2)} = \sqrt{2}-1$$

at $z = -R$ and therefore

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = 2\left|\frac{zf_1'(z)}{f_1(z)}\right|.$$

(vii) The number $R := R_{\mathcal{S}^*_{\mathcal{R}}}$ is the smallest positive root of the equation

$$2(1-5R+R^2+R^3) = 2(\sqrt{2}-1)(2-R)(1-R^2). \tag{2.22}$$

Since the function

$$h(r) := \frac{6r(r^2-3)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 2(\sqrt{2}-1)$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} - 2(\sqrt{2}-1). \tag{2.23}$$

From (2.5) and (2.23), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - 2(\sqrt{2}-1) \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.1), 1] \approx (.99741, 1] \subset (2(\sqrt{2}-1), \sqrt{2}]$. When $a \in (2(\sqrt{2}-1), \sqrt{2}]$, by [15, Lemma 2.2], the disk $\{w : |w-a| < a - 2(\sqrt{2}-1)\}$ is contained in the region $\varphi_6(\mathbb{D})$, where $\varphi_6(z) := 1 + (zk + z^2/(k^2-kz))$ and $k = \sqrt{2}+1$. Hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_6(\mathbb{D})$. Thus, the radius of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.22), we see that the function f_1 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1-5R+R^2+R^3)}{(2+R)(1-R^2)} = 2(\sqrt{2}-1) = \varphi_6(-1) \in \partial\varphi_6(\mathbb{D}).$$

(viii) The number $R := R_{\mathcal{S}^*_{\mathcal{C}}}$ is the smallest positive root of the equation

$$2(1-5R+R^2+R^3) = (1/3)(2-R)(1-R^2). \tag{2.24}$$

Since the function

$$h(r) := \frac{6r(r^2-3)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-5r+r^2+r^3)}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1/3$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{6r(3-r^2)}{(1-r^2)(4-r^2)} - \frac{4-2r^2}{4-r^2} < -\frac{1}{3}. \tag{2.25}$$

From (2.5) and (2.25), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - \frac{1}{3} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.989899, 1] \subset (1/3, 5/3)$. When $a \in (1/3, 5/3)$, by [28, Lemma 2.5], the disk $\{w : |w-a| < a - 1/3\}$ is lies in the cardioid region $\varphi_3(\mathbb{D})$. Hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_3(\mathbb{D})$. Thus, the radius of cardioid starlikeness of the class $\mathcal{F}_1$ is at least R. To show that the radius R is sharp, using (2.7) and (2.24), we see that the function f_1 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2(1-5R+R^2+R^3)}{(2-R)(1-R^2)} = 1/3 = \varphi_3(-1) \in \partial\varphi_3(\mathbb{D}). \qquad \square$$

3. Radius problem for $\mathcal{F}_2$

For the function $f \in \mathcal{F}_2$, there is a function $g \in \mathcal{A}$ such that $\operatorname{Re}(g(z)/f(z)) > 1/2$ and $\operatorname{Re}(g(z)/(z+z^2/2)) > 0$. The functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ defined by $p_1(z) = g(z)/f(z)$, $p_2(z) = g(z)/(z+z^2/2)$ are the functions in $\mathscr{P}(1/2)$ and $\mathscr{P}(0)$ respectively and

$$f(z) = (p_2(z)/p_1(z))(z+z^2/2) \quad (z \in \mathbb{D}). \tag{3.1}$$

From (3.1), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{zp_2'(z)}{p_2(z)} - \frac{zp_1'(z)}{p_1(z)} + \frac{2(z+1)}{z+2}. \tag{3.2}$$

Using (2.1) and (2.2) in (3.2), we see that the image of the disk $\mathbb{D}_r$ under the mapping $zf'(z)/f(z)$ is contained in the disk defined by

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| \le \frac{r(14+4r-5r^2-r^3)}{(1-r^2)(4-r^2)}. \tag{3.3}$$

From (3.3), it readily follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{4-2r^2}{4-r^2} - \frac{r(14+4r-5r^2-r^3)}{(1-r^2)(4-r^2)} = \frac{2-8r-r^2+3r^3}{(2-r)(1-r^2)} \quad (|z| \le r). \tag{3.4}$$

Let $R_{\mathcal{S}^*} \approx 0.248$ be the zero in (0,1) of the polynomial $3r^3 - r^2 - 8r + 2$. Then, for every function $f \in \mathcal{F}_2$, the inequality (3.4) shows that $\operatorname{Re}(s_f(z)) > 0$ in each disk $\mathbb{D}_r$, for $0 \le r < R_{\mathcal{S}^*}$. For the function f_2 defined in (1.1), we have

$$s_{f_2}(z) = \frac{zf_2'(z)}{f_2(z)} = \frac{2+8z-z^2-3z^3}{(2+z)(1-z^2)} \tag{3.5}$$

and hence $\operatorname{Re}(s_{f_2}(z))$ vanishes at $z = -R_{\mathcal{S}^*}$. Thus, the radius of starlikeness $R_{\mathcal{S}^*}$ for the class $\mathcal{F}_2$ is the smallest positive zero in $(0,1)$ of the polynomial P_2 defined in (1.3) and is the same as the radius of univalence $R_{\mathcal{S}}$. Using the inequality (3.3), we now determine $\mathcal{S}^*(\alpha)$, $\mathcal{S}^*_P$, $\mathcal{S}^*_e$, $\mathcal{S}^*_c$, $\mathcal{S}^*_{sin}$, $\mathcal{S}^*_{\leftmoon}$ and $\mathcal{S}^*_R$ radii for the class $\mathcal{F}_2$.

Theorem 3.1. *The following sharp radii results hold for the class $\mathcal{F}_2$:*

(i) *For any $0 \le \alpha < 1$, the radius $R_{\mathcal{S}^*(\alpha)}$ is the smallest positive root of the polynomial*

$$(3-\alpha)r^3 + (2\alpha - 1)r^2 - (8-\alpha)r + 2 - 2\alpha = 0. \tag{3.6}$$

(ii) *The radius $R_{\mathcal{S}^*_p}$ (≈ 0.1341) is the same as $R_{\mathcal{S}^*(1/2)}$.*
(iii) *The radius $R_{\mathcal{S}^*_e}$(≈ 0.16628) is the same as $R_{\mathcal{S}^*(1/e)}$.*
(iv) *The radius $R_{\mathcal{S}^*_{\sin}}$ (≈ 0.2142) is the same as $R_{\mathcal{S}^*(1-\sin 1)}$.*
(v) *The radius $R_{\mathcal{S}^*_{\leftmoon}}$ (≈ 0.1551) is the same as $R_{\mathcal{S}^*(\sqrt{2}-1)}$.*
(vi) *The radius $R_{\mathcal{S}^*_R}$ (≈ 0.0481) is the same as $R_{\mathcal{S}^*(2(\sqrt{2}-1))}$.*
(vii) *The radius $R_{\mathcal{S}^*_c}$ (≈ 0.1744) is the same as $R_{\mathcal{S}^*(1/3)}$.*

Proof. (i) Let the function $f \in \mathcal{F}_2$ and $\alpha \in [0,1)$. Let $R := R_{\mathcal{S}^*(\alpha)}$ be the smallest positive root of the equation (3.6) so that

$$2 - 8R - R^2 + 3R^3 = \alpha(2-R)(1-R^2). \tag{3.7}$$

The function

$$h(r) = \frac{2 - 8r - r^2 + 3r^3}{(2-r)(1-r^2)}$$

is decreasing in $[0,1)$ and hence, for $0 \le r < R$, we have, using (3.4) and (3.7),

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{2 - 8r - r^2 + 3r^3}{(2-r)(1-r^2)} > \frac{2 - 8R - R^2 + 3R^3}{(2-R)(1-R^2)} = \alpha \quad (0 \le r < R).$$

This proves that the function f is starlike of order α in each disk $\mathbb{D}_r$ for $0 \le r < R$. At the point $z = -R$, it can be seen, using (3.5) and (3.7), that the function f_2 defined in (1.1) satisfies

$$\mathrm{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right) = \frac{2 - 8R - R^2 + 3R^3}{(2-R)(1-R^2)} = \alpha.$$

This shows that the radius R is the radius of starlikeness of order α of the class $\mathcal{F}_2$.

(ii) The number $R := R_{\mathcal{S}^*_{\mathcal{P}}}$ is the smallest positive root of the equation

$$2 - 8R - R^2 + 3R^3 = (1/2)(2-R)(1-R^2). \tag{3.8}$$

Since the function

$$h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1/2$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{r(r^3 + 5r^2 - 4r - 14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} > \frac{1}{2}. \tag{3.9}$$

From (3.3) and (3.9), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| \le \frac{4-2r^2}{4-r^2} - \frac{1}{2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.9898991, 1] \subset (1/2, 3/2)$. When $a \in (1/2, 3/2)$, by [27, Lemma 2.2], the disk $\{w : |w-a| < a - (1/2)\}$

is contained in the parabolic region $\{w : |w-1| < \operatorname{Re}(w)\}$ and hence, for $0 \le r < R$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (|z| \le r).$$

Thus, the radius of parabolic starlikeness of the class $\mathcal{F}_2$ is at least R. To show that the radius R is sharp, using (3.5) and (3.8), we see that the function f_2 defined in (1.1) satisfies

$$\frac{zf_2'(z)}{f_2(z)} = \frac{2-8R-R^2+3R^3}{(2-R)(1-R^2)} = \frac{1}{2}$$

at $z = -R$ and therefore

$$\left|\frac{zf_2'(z)}{f_2(z)} - 1\right| = \operatorname{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right).$$

(iii) The number $R := R_{\mathcal{S}_e^*}$ is the smallest positive root of the equation

$$2-8R-R^2+3R^3 = (1/e)(2-R)(1-R^2). \tag{3.10}$$

Since the function

$$h(r) := \frac{r(r^3+5r^2-4r-14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2-8r-r^2+3r^3}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1/e$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{r(r^3+5r^2-4r-14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} > \frac{1}{e}. \tag{3.11}$$

From (3.3) and (3.11), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - \frac{1}{e} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R),1] \subset (c(0.2),1] \approx (.9898991,1] \subset (e^{-1},(e+e^{-1})/2]$. When $a \in (e^{-1},(e+e^{-1})/2]$, by [19, Lemma 2.2], the disk $\{w : |w-a| < a-e^{-1}\}$ is contained in the region $\{w : |\log w| < 1\}$ and hence, for $0 \le r < R$, we have

$$\left|\log\left(\frac{zf'(z)}{f(z)}\right)\right| < 1 \quad (|z| \le r).$$

Thus, the radius of exponential starlikeness of the class $\mathcal{F}_2$ is at least R. To show that the radius R is sharp, using (3.5) and (3.10), we see that the function f_2 defined in (1.1) satisfies

$$\frac{zf_2'(z)}{f_2(z)} = \frac{2-8R-R^2+3R^3}{(2-R)(1-R^2)} = \frac{1}{e}$$

at $z = -R$ and therefore

$$\left|\log\left(\frac{zf_2'(z)}{f_2(z)}\right)\right| = 1.$$

(iv) The number $R := R_{\mathcal{S}_{\sin}^*}$ is the smallest positive root of the equation

$$2-8R-R^2+3R^3 = (1-\sin 1)(2-R)(1-R^2). \tag{3.12}$$

Since the function

$$h(r) := \frac{r(r^3+5r^2-4r-14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2-8r-r^2+3r^3}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1 - \sin 1$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{r(r^3+5r^2-4r-14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} > 1 - \sin 1. \tag{3.13}$$

From (3.3) and (3.13), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| \le \frac{4-2r^2}{4-r^2} + \sin 1 - 1 \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (.976982, 1] \subset (-1 - \sin 1, 1 - \sin 1)$. When $a \in (-1 - \sin 1, 1 + \sin 1)$, by [8, Lemma 3.3], the disk $\{w : |w - a| < \sin 1 - |a - 1|\}$ is contained in the region $\varphi_4(\mathbb{D})$, where $\varphi_4(z) = 1 + \sin z$ and hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_4(\mathbb{D})$. Thus, the radius of sine starlikeness of the class $\mathcal{F}_2$ is at least R. To show that the radius R is sharp, using (3.5) and (3.12), we see that the function f_2 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{2-8R-R^2+3R^3}{(2-R)(1-R^2)} = 1 - \sin 1 = \varphi_4(-1) \in \partial\varphi_4(\mathbb{D}).$$

(v) The number $R := R_{\mathcal{S}^*_{\leftmoon}}$ is the smallest positive root of the equation

$$2 - 8R - R^2 + 3R^3 = (\sqrt{2}-1)(2-R)(1-R^2). \tag{3.14}$$

Since the function

$$h(r) := \frac{r(r^3+5r^2-4r-14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2-8r-r^2+3r^3}{(2-r)(1-r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = \sqrt{2} - 1$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{r(r^3+5r^2-4r-14)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} > \sqrt{2} - 1. \tag{3.15}$$

From (3.3) and (3.15), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} + 1 - \sqrt{2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.9898991, 1] \subset (\sqrt{2}\text{-}1, \sqrt{2}+1)$. When $a \in (\sqrt{2}-1, \sqrt{2}+1)$, by [9, Lemma 2.1], the disk $\{w : |w - a| < 1 - |\sqrt{2} - a|\}$ is contained in the region $\{w : |w^2 - 1| < 2|w|\}$ and hence, for $0 \le r < R$, we have

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right| \quad (|z| \le r).$$

Thus, the radius of lune starlikeness of the class $\mathcal{F}_2$ is at least R. To show that the radius R is sharp, using (3.5) and (3.14), we see that the function f_2 defined in (1.1) satisfies

$$\frac{zf_2'(z)}{f_2(z)} = \frac{2-8R-R^2+3R^3}{(2-R)(1-R^2)} = \sqrt{2} - 1$$

at $z = -R$ and therefore

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2 - 1\right| = 2\left|\frac{zf_2'(z)}{f_2(z)}\right|.$$

(vi) The number $R := R_{\mathcal{S}^*_R}$ is the smallest positive root of the equation

$$2 - 8R - R^2 + 3R^3 = 2(\sqrt{2} - 1)(2 - R)(1 - R^2). \tag{3.16}$$

Since the function

$$h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 2(\sqrt{2}-1)$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > 2(\sqrt{2} - 1) \quad (0 \le r < R). \tag{3.17}$$

From (3.3) and (3.17), we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - 2(\sqrt{2} - 1) \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.1), 1] \approx (.99741, 1] \subset (2(\sqrt{2}-1), \sqrt{2}]$. When $a \in (2(\sqrt{2}-1), \sqrt{2}]$, by [15, Lemma 2.2], the disk $\{w : |w - a| < a - 2(\sqrt{2}-1)\}$ is contained in the region $\varphi_6(\mathbb{D})$, where $\varphi_6(z) := 1 + ((zk + z^2)/(k^2 - kz))$ and $k = \sqrt{2} + 1$. Hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_6(\mathbb{D})$. Thus, the $\mathcal{S}^*_R$ radius of the class $\mathcal{F}_2$ is at least R. To show that the radius R is sharp, using (3.5) and (3.16), we see that the function f_2 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = 2(\sqrt{2} - 1) = \varphi_6(-1) \in \partial\varphi_6(\mathbb{D}).$$

(vii) The number $R := R_{\mathcal{S}^*_C}$ is the smallest positive root of the equation

$$2 - 8R - R^2 + 3R^3 = (1/3)(2 - R)(1 - R^2). \tag{3.18}$$

Since the function

$$h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1/3$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > \frac{1}{3}. \tag{3.19}$$

From (3.3) and (3.19), we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{3} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.989899, 1] \subset (1/3, 5/3)$. When $a \in (1/3, 5/3)$, by [28, Lemma 2.5], the disk $\{w : |w - a| < a - 1/3\}$ is lies in the cardioid region $\varphi_3(\mathbb{D})$. Hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_3(\mathbb{D})$. Thus, the $\mathcal{S}^*_C$ radius of the class $\mathcal{F}_2$ is at least R. To show that the radius R is sharp, using (3.5) and (3.18), we see that the function f_2 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = 1/3 = \varphi_3(-1) \in \partial\varphi_3(\mathbb{D}). \qquad \square$$

4. Radius problem for $\mathcal{F}_3$

If the function $f \in \mathcal{F}_3$, then the function $p : \mathbb{D} \to \mathbb{C}$ defined by $p(z) = f(z)/(z + z^2/2)$ is a function in the class $\mathscr{P}(0)$ and

$$f(z) = p(z)(z + z^2/2) \quad (z \in \mathbb{D}). \tag{4.1}$$

From (4.1), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} + \frac{2(z+1)}{z+2}. \tag{4.2}$$

Using (2.1) (with $\alpha = 0$) and (2.2) in (4.2), we see that the image of the disk $\mathbb{D}_r$ under the mapping $zf'(z)/f(z)$ is contained in the disk defined by

$$\left| \frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2} \right| \leq \frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} \quad (|z| \leq r). \tag{4.3}$$

From (4.3), it readily follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{4-2r^2}{4-r^2} - \frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} = \frac{2(1-3r+r^3)}{(2-r)(1-r^2)} \quad (|z| \leq r). \tag{4.4}$$

Let $R_{\mathcal{S}^*} \approx 0.3473$ be the unique zero in (0,1) of the polynomial $r^3 - 3r + 1$. Then, for every function $f \in \mathcal{F}_3$, the inequality (4.4) shows that $\operatorname{Re}(s_f(z)) > 0$ in each disk $\mathbb{D}_r$, for $0 \leq r < R_{\mathcal{S}^*}$. For the function f_3 defined in (1.2), we have

$$s_{f_3}(z) = \frac{zf_3'(z)}{f_3(z)} = \frac{2(1+3z-z^3)}{(2+z)(1-z^2)} \tag{4.5}$$

and hence $\operatorname{Re}(s_{f_3}(z))$ vanishes at $z = -R_{\mathcal{S}^*}$. Thus, the radius of starlikeness $R_{\mathcal{S}^*}$ for the class $\mathcal{F}_3$ is the unique zero in (0,1) of the polynomial P_3 defined in (1.4) and is the same as the radius of univalence $R_{\mathcal{S}}$. Using the inequality (4.3), we now determine $\mathcal{S}^*(\alpha)$, $\mathcal{S}^*_{\mathcal{L}}$, $\mathcal{S}^*_P$, $\mathcal{S}^*_e$, $\mathcal{S}^*_c$, $\mathcal{S}^*_{sin}$, $\mathcal{S}^*_{\leftmoon}$ and $\mathcal{S}^*_R$ radii for the class $\mathcal{F}_3$.

Theorem 4.1. *The following sharp radii results hold for the class of function $\mathcal{F}_3$:*

(i) *For any $0 \leq \alpha < 1$, the radius $R_{\mathcal{S}^*(\alpha)}$ is the smallest positive root of the polynomial*

$$(2-\alpha)r^3 + (2\alpha)r^2 + (\alpha - 6)r + 2 - 2\alpha = 0. \tag{4.6}$$

(ii) *The radius $R_{\mathcal{S}^*_{\mathcal{L}}}$ (≈ 0.1645) is the smallest positive root of the polynomial*

$$(\sqrt{2}-2)r^3 + (2\sqrt{2})r^2 + (6-\sqrt{2})r + 2 - 2\sqrt{2} = 0. \tag{4.7}$$

(iii) *The radius $R_{\mathcal{S}^*_p}$ (≈ 0.19028) is the same as $R_{\mathcal{S}^*(1/2)}$.*
(iv) *The radius $R_{\mathcal{S}^*_e}$ (≈ 0.2355) is the same as $R_{\mathcal{S}^*(1/e)}$.*
(v) *The radius $R_{\mathcal{S}^*_{\sin}}$ (≈ 0.3017) is the same as $R_{\mathcal{S}^*(1-\sin 1)}$.*
(vi) *The radius $R_{\mathcal{S}^*_{\leftmoon}}$ (≈ 0.2199) is the same as $R_{\mathcal{S}^*(\sqrt{2}-1)}$.*
(vii) *The radius $R_{\mathcal{S}^*_R}$ (≈ 0.0679) is the same as $R_{\mathcal{S}^*(2(\sqrt{2}-1))}$.*
(viii) *The radius $R_{\mathcal{S}^*_c}$ (≈ 0.2469) is the same as $R_{\mathcal{S}^*(1/3)}$.*

Proof. (i) Let the function $f \in \mathcal{F}_3$ and α in [0,1). The root $R := R_{\mathcal{S}^*(\alpha)}$ be the smallest positive root of the equation (4.6) so that

$$2(1 - 3R + R^3) = \alpha(2-R)(1-R^2). \tag{4.8}$$

The function

$$h(r) := \frac{2(1-3r+r^3)}{(2-r)(1-r^2)}$$

is decreasing in $[0, 1)$ and hence, for $0 \le r < R$, we have, using (4.4) and (4.8),

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{2(1-3r+r^3)}{(2-r)(1-r^2)} > \frac{2(1-3R+R^3)}{(2-R)(1-R^2)} = \alpha.$$

This proves that the function f is starlike of order α in each disk $\mathbb{D}_r$ for $0 \le r < R$. At the point $z = -R$, it can be seen, using (4.5) and (4.8), that the function f_3 defined in (1.2) satisfies

$$\operatorname{Re}\left(zf_3'(z)/f_3(z)\right) = \frac{2(1-3R-R^3)}{(2-R)(1-R^2)} = \alpha.$$

This shows that the radius R is the radius of starlikeness of order α of the class $\mathcal{F}_3$.

(ii) Let $R := R_{\mathcal{S}^*_{\mathcal{L}}}$ be the smallest positive root of the equation (4.7) so that

$$2(1+3R-R^3) = \sqrt{2}(2+R)(1-R^2). \tag{4.9}$$

Since the function

$$h(r) := \frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1+3r-r^3)}{(1-r^2)(2+r)}$$

is an increasing function of r in [0,1), it follows that $h(r) < h(R) = \sqrt{2}$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} < \sqrt{2} - \frac{4-2r^2}{4-r^2}. \tag{4.10}$$

From (4.3) and (4.10), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \sqrt{2} - \frac{4-2r^2}{4-r^2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.1), 1] \approx (.997494, 1] \subset [2\sqrt{2}/3, \sqrt{2})$. When $a \in [2\sqrt{2}/3, \sqrt{2})$, by [2, Lemma 2.2], the disk $\{w : |w-a| < \sqrt{2}-a\}$ is contained in the lemniscate region $\{w : |w^2-1| < 1\}$ and hence, for $0 \le r < R$, we have

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1.$$

Thus, the radius of lemniscate starlikeness of the class $\mathcal{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.9), we see that the function f_3 defined in (1.2) satisfies

$$\frac{zf_3'(z)}{f_3(z)} = \frac{2(1+3R-R^3)}{(1-R^2)(2+R)} = \sqrt{2}$$

at $z = -R$ and therefore

$$\left|\left(\frac{zf_3'(z)}{f_3(z)}\right)^2 - 1\right| = 1.$$

(iii) The number $R := R_{\mathcal{S}^*_{\mathcal{P}}}$ is the smallest positive root of the equation

$$2(1-3R+R^3) = (1/2)(2-R)(1-R^2). \tag{4.11}$$

Since the function

$$h(r) := \frac{2r(2r^2-5)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-3r+r^3)}{(1-r^2)(2-r)}$$

is decreasing function of r in [0,1), it follows that, $h(r) > h(R) = 1/2$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} - \frac{1}{2}. \tag{4.12}$$

From (4.3) and (4.12), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - \frac{1}{2} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (.9898991, 1] \subset (1/2, 3/2)$. When $a \in (1/2, 3/2)$, by [27, Lemma 2.2], the disk $\{w : |w-a| < a-(1/2)\}$ is contained in the parabolic region $\{w : |w-1| < \operatorname{Re}(w)\}$ and hence, for $0 \le r < R$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (|z| \le r).$$

Thus, the radius of parabolic starlikeness of the class $\mathcal{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.11), we see that the function f_3 defined in (1.2) satisfies

$$\frac{zf_3'(z)}{f_3(z)} = \frac{2(R^3-3R+1)}{(1-R^2)(2-R)} = \frac{1}{2}$$

at $z = -R$ and therefore

$$\left|\frac{zf_3'(z)}{f_3(z)} - 1\right| = \frac{1}{2} = \operatorname{Re}\left(\frac{zf_3'(z)}{f_3(z)}\right).$$

(iv) The number $R := R_{\mathcal{S}_e^*}$ is the smallest positive root of the equation

$$2(1-3R+R^3) = (1/e)(2-R)(1-R^2). \tag{4.13}$$

Since the function

$$h(r) := \frac{2r(2r^2-5)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-3r+r^3)}{(1-r^2)(2-r)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1/e$ for $0 \le r < R$ and hence, for $0 \le r < R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} - \frac{1}{e}. \tag{4.14}$$

From (4.3) and (4.14), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} - \frac{1}{e} \quad (|z| \le r).$$

For $0 \le r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (.976982, 1] \subset (e^{-1}, (e+e^{-1})/2]$. When $a \in (e^{-1}, (e+e^{-1})/2]$, by [19, Lemma 2.2], the disk $\{w : |w-a| < a-e^{-1}\}$ is contained in the region $\{w : |\log w| < 1\}$ and hence, for $0 \le r < R$, we have

$$\left|\log\left(\frac{zf'(z)}{f(z)}\right)\right| < 1 \quad (|z| \le r).$$

Thus, the radius of exponential starlikeness of the class $\mathcal{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.13), we see that the function f_3 defined in (1.2) satisfies

$$\frac{zf_3'(z)}{f_3(z)} = \frac{2(1-3R+R^3)}{(1-R^2)(2-R)} = \frac{1}{e}$$

at $z = -R$ and therefore

$$\left|\log\left(\frac{zf_3'(z)}{f_3(z)}\right)\right| = 1.$$

(v) The number $R := R_{\mathcal{S}^*_{\sin}}$ is the smallest positive root of the equation

$$2(1-3R+R^3) = (1-\sin 1)(2-R)(1-R^2). \tag{4.15}$$

Since the function

$$h(r) := \frac{2r(2r^2-5)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-3r+r^3)}{(1-r^2)(2-r)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = 1-\sin 1$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} + \sin 1 - 1. \tag{4.16}$$

From (4.3) and (4.16), get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \sin 1 - 1 + \frac{4-2r^2}{4-r^2} \quad (|z| \leq r).$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.4), 1] \approx (.995833, 1] \subset (-1-\sin 1, 1-\sin 1)$. When $a \in (-1-\sin 1, 1+\sin 1)$, by [8, Lemma 3.3], the disk $\{w : |w-a| < \sin 1 - |a-1|\}$ is contained in the region $\varphi_4(\mathbb{D})$, where $\varphi_4(z) = 1+\sin z$ and hence, for $0 \leq r < R$, $s_f(\mathbb{D}_r) \subset \varphi_4(\mathbb{D})$. Thus, the radius of sine starlikeness of the class $\mathfrak{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.15), we see that the function f_3 defined in (1.2) satisfies, at $z = -R$,

$$\frac{zf_3'(z)}{f_3(z)} = \frac{2(1-3R+R^3)}{(2+R)(1-R^2)} = 1-\sin 1 = \varphi_4(-1) \in \partial\varphi_4(\mathbb{D}).$$

(vi) The number $R := R_{\mathcal{S}^*_{\leftmoon}}$ is the smallest positive root of the equation

$$2(1-3R+R^3) = (\sqrt{2}-1)(2-R)(1-R^2). \tag{4.17}$$

Since the function

$$h(r) := \frac{2r(2r^2-5)}{(1-r^2)(4-r^2)} + \frac{4-2r^2}{4-r^2} = \frac{2(1-3r+r^3)}{(1-r^2)(2-r)}$$

is decreasing function of r in [0,1), it follows that $h(r) > h(R) = \sqrt{2}-1$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)} < \frac{4-2r^2}{4-r^2} + 1 - \sqrt{2}. \tag{4.18}$$

From (4.3) and (4.18), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{4-2r^2}{4-r^2}\right| < \frac{4-2r^2}{4-r^2} + 1 - \sqrt{2} \quad (|z| \leq r).$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (.976982, 1] \subset (\sqrt{2}$-$1, \sqrt{2}+1)$. When $a \in (\sqrt{2}-1, \sqrt{2}+1)$, by [9, Lemma 2.1], the disk $\{w : |w-a| < 1-|\sqrt{2}-a|\}$ is contained in the region $\{w : |w^2-1| < 2|w|\}$ and hence, for $0 \leq r < R$, we have

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right| \quad (|z| \leq r). \tag{4.19}$$

Thus, the radius of lune starlikeness of the class $\mathcal{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.17), we see that the function f_3 defined in (1.2) satisfies

$$\frac{zf_3'(z)}{f_3(z)}=\frac{2(1-3R+R^3)}{(1-R^2)(2-R)}=\sqrt{2}-1$$

at $z=-R$ and therefore

$$\left|\left(\frac{zf_3'(z)}{f_3(z)}\right)^2-1\right|=2\left|\frac{zf_3'(z)}{f_3(z)}\right|.$$

(vii) The number $R:=R_{\mathcal{S}^*_R}$ is the smallest positive root of the equation

$$2(1-3R+R^3)=2(\sqrt{2}-1)(2-R)(1-R^2). \tag{4.20}$$

Since the function

$$h(r):=\frac{2r(2r^2-5)}{(1-r^2)(4-r^2)}+\frac{4-2r^2}{4-r^2}=\frac{2(1-3r+r^3)}{(1-r^2)(2-r)}$$

is decreasing function of r in [0,1), it follows that $h(r)>h(R)=2(\sqrt{2}-1)$ for $0\le r<R$ and hence, for $0\le r<R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)}<\frac{4-2r^2}{4-r^2}-2(\sqrt{2}-1). \tag{4.21}$$

From (4.3) and (4.21), we get

$$\left|\frac{zf'(z)}{f(z)}-\frac{4-2r^2}{4-r^2}\right|<\frac{4-2r^2}{4-r^2}-2(\sqrt{2}-1)\quad(|z|\le r).$$

For $0\le r<R$, the center of the above disk $c(r)=(4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R),1]\subset(c(0.1),1]\approx(.99741,1]\subset(2(\sqrt{2}-1),\sqrt{2}]$. When $a\in(2(\sqrt{2}-1),\sqrt{2}]$, by [15, Lemma 2.2], the disk $\{w:|w-a|<a-2(\sqrt{2}-1)\}$ is contained in the region $\varphi_6(\mathbb{D})$, where $\varphi_6(z):=1+((zk+z^2)/(k^2-kz))$ and $k=\sqrt{2}+1$. Hence, for $0\le r<R$, $s_f(\mathbb{D}_r)\subset\varphi_6(\mathbb{D})$. Thus, the $\mathcal{S}^*_R$ radius of the class $\mathcal{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.20), we see that the function f_3 defined in (1.2) satisfies, at $z=-R$,

$$\frac{zf_3'(z)}{f_3(z)}=\frac{2(1-3R+R^3)}{(2-R)(1-R^2)}=2(\sqrt{2}-1)=\varphi_6(-1)\in\partial\varphi_6(\mathbb{D}).$$

(viii) The number $R:=R_{\mathcal{S}^*_C}$ is the smallest positive root of the equation

$$2(1-3R+R^3)=(1/3)(2-R)(1-R^2). \tag{4.22}$$

Since the function

$$h(r):=\frac{2r(2r^2-5)}{(1-r^2)(4-r^2)}+\frac{4-2r^2}{4-r^2}=\frac{2(1-3r+r^3)}{(1-r^2)(2+r)}$$

is decreasing function of r in [0,1), it follows that $h(r)>h(R)=1/3$ for $0\le r<R$ and hence, for $0\le r<R$, we have

$$\frac{2r(5-2r^2)}{(1-r^2)(4-r^2)}<\frac{4-2r^2}{4-r^2}-\frac{1}{3}. \tag{4.23}$$

From (4.3) and(4.23), we get

$$\left|\frac{zf'(z)}{f(z)}-\frac{4-2r^2}{4-r^2}\right|<\frac{4-2r^2}{4-r^2}-\frac{1}{3}\quad(|z|\le r).$$

For $0\le r<R$, the center of the above disk $c(r)=(4-2r^2)/(4-r^2)$ (being a decreasing function of r on [0,1]) lies in the interval $[c(R),1]\subset(c(0.3),1]\approx(.976982,1]\subset$

$(1/3, 5/3)$. When $a \in (1/3, 5/3)$, by [28, Lemma 2.5], the disk $\{w : |w - a| < a - 1/3\}$ is lies in the cardioid region $\varphi_3(\mathbb{D})$. Hence, for $0 \le r < R$, $s_f(\mathbb{D}_r) \subset \varphi_3(\mathbb{D})$. Thus, the radius of the class $\mathcal{F}_3$ is at least R. To show that the radius R is sharp, using (4.5) and (4.22), we see that the function f_3 defined in (1.1) satisfies, at $z = -R$,

$$\frac{zf_3'(z)}{f_3(z)} = \frac{2(1 - 3R + R^3)}{(2 - R)(1 - R^2)} = 1/3 = \varphi_3(-1) \in \partial\varphi_3(\mathbb{D}). \qquad \square$$

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