A NOTE ON THE UNIT GROUP OF ZS_4

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1. INTRODUCTION

Hughes and Pearson [3] characterized the group $V = V(ZS_3)$ of units of augmentation 1 in ZS_3 by showing that V is isomorphic to the subgroup of $GL(2, Z)$ consisting of matrices the column sums of which are 1 modulo 3. Their approach was to construct a 6×6 matrix from a complete set of irreducible representations of S_3; then invert the matrix, and then solve a system of six linear congruences modulo 6. The same technique was used by Milies [4] to describe units in ZD_4; by Allen and Hobby [1] and Yılmaz [5] to describe units in ZA_4 and ZS_4, respectively.

Allen and Hobby [2] have used a different method to obtain a new description of $V(ZS_3)$ as the group of all doubly stochastic matrices in $GL(3, Z)$. This method has the advantage of exploiting the fact that a convex combination of permutation matrices is always doubly stochastic and it will not be required to invert a matrix or to solve any system of a lot of linear congruences. In this note also we use this important fact to obtain a characterization of the group $V = V(ZS_4)$ by doubly stochastic matrices in $GL(4, Z)$.

2. CONSTRUCTION

We write $S_4 = < \gamma = (12), \beta = (234) \mid \gamma^2 = \beta^3 = 1; \gamma\beta^{-1} = (\beta\gamma)^{-1} >$ and agree to list the elements g_i in S_4 in the following order:

$$
\begin{aligned}
S_4 &= \{g_1, g_2, \dots, g_{24}\} \\
&= \{[(1), (12)(34), (13)(24), (14)(23)], [(123), (142), (134), (243)], \\
&\qquad [(132), (124), (143), (234)], [(12), (34), (1324), (1423)], \\
&\qquad [(13), (24), (1234), (1432)], [(14), (23), (1243), (1342)]\} \\
&= \{K_4, g_5 K_4, g_9 K_4, g_{13} K_4, g_{17} K_4, g_{22} K_4\}
\end{aligned}
$$

which is also presented as cosets of S_4 modulo K_4 where K_4 is the Klein-4 subgroup of S_4. Represent S_4 by

$$\rho(\gamma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(\beta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and extend ρ linearly to ZS_4. For $\alpha = \sum_{i=1}^{24} a_i g_i \in ZS_4$ it is clear that

$$\rho(\alpha) = X = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_5 & A_6 & A_7 & A_8 \\ A_9 & A_{10} & A_{11} & A_{12} \\ A_{13} & A_{14} & A_{15} & A_{16} \end{bmatrix}$$

where

$$\begin{aligned}
A_1 &= a_1 + a_8 + a_{12} + a_{14} + a_{18} + a_{22}, \\
A_2 &= a_2 + a_6 + a_9 + a_{13} + a_{20} + a_{24}, \\
A_3 &= a_3 + a_5 + a_{11} + a_{16} + a_{17} + a_{23}, \\
A_4 &= a_4 + a_7 + a_{10} + a_{15} + a_{19} + a_{21}, \\
A_5 &= a_2 + a_5 + a_{10} + a_{13} + a_{19} + a_{23}, \\
A_6 &= a_1 + a_7 + a_{11} + a_{14} + a_{17} + a_{21}, \\
A_7 &= a_4 + a_8 + a_9 + a_{15} + a_{20} + a_{22}, \\
A_8 &= a_3 + a_6 + a_{12} + a_{16} + a_{18} + a_{24}, \\
A_9 &= a_3 + a_7 + a_9 + a_{15} + a_{17} + a_{24}, \\
A_{10} &= a_4 + a_5 + a_{12} + a_{16} + a_{19} + a_{22}, \\
A_{11} &= a_1 + a_6 + a_{10} + a_{13} + a_{18} + a_{21}, \\
A_{12} &= a_2 + a_8 + a_{11} + a_{14} + a_{20} + a_{23}, \\
A_{13} &= a_4 + a_6 + a_{11} + a_{16} + a_{20} + a_{21}, \\
A_{14} &= a_3 + a_8 + a_{10} + a_{15} + a_{18} + a_{23}. \\
A_{15} &= a_2 + a_7 + a_{12} + a_{14} + a_{19} + a_{24}, \\
A_{16} &= a_1 + a_5 + a_9 + a_{13} + a_{17} + a_{22},
\end{aligned} \tag{1}$$

and if α is a unit of augmentation 1, then $\rho(\alpha)$ is a doubly stochastic matrix in $GL(4, Z)$. Hence $\rho(V)$ is a subgroup of the group of doubly stochastic matrices in $GL(4, Z)$.

Whenever $M \in GL(4, Z)$ we agree that $t_i = t_i(M)$ will denote the sum (or pseudotrace) of the four entries in M which occur in the locations where the permutation matrix $\rho(g_i)$ has 1's. We now form the six sums from the coefficients of α; namely,

$$s_1 = a_1 + a_2 + a_3 + a_4 \qquad s_4 = a_{13} + a_{14} + a_{15} + a_{16}$$
$$s_2 = a_5 + a_6 + a_7 + a_8 \qquad s_5 = a_{17} + a_{18} + a_{19} + a_{20}$$
$$s_3 = a_9 + a_{10} + a_{11} + a_{12} \qquad s_6 = a_{21} + a_{22} + a_{23} + a_{24}$$

which correspond to the respective six cosets of S_4 modulo K_4, and further, form the sums $\sigma_0 = s_1 + s_2 + s_3$ and $\sigma_1 = s_4 + s_5 + s_6$ corresponding to the cosets of S_4 modulo A_4. We then easily write down the following equalities from (1) for a matrix $M \in \mathrm{p}(VZS_4)$:

$$t_i = 4a_i + \begin{cases} \sigma_0 - s_1 + t''_i & \text{for } i = 1, 2, 3, 4 \\ \sigma_0 - s_2 + i''_i & \text{for } i = 5, 6, 7, 8 \\ \sigma_0 - s_3 + t''_i & \text{for } i = 9, 10, 11, 12 \\ \sigma_1 - s_4 + t'_i & \text{for } i = 13, 14, 15, 16 \\ \sigma_1 - s_5 + t'_i & \text{for } i = 17, 18, 19, 20 \\ \sigma_1 - s_6 + t'_i & \text{for } i = 21, 22, 23, 24 \end{cases} \tag{2}$$

where t'_i, t''_i are the corresponding pseudo-traces of the matrices X', X'' respectively, which are the components of X in (1) decomposed as

$$\mathrm{p}(\alpha) = X = X' + X'' \tag{3}$$

so that each entry of X' is the sum of the first three summands of the associated entry of X while those of X'' are the sums of the last three. Observe that this decomposition of X permits us to write $t_i = t'_i + t''_i$, and $t''_i\,(i = 1, \ldots, 12)$ and $t'_i\,(i=13, \ldots, 24)$ are always even numbers. Moreover, X' has row sums=column sums $= \sigma_0$ and X'' has row sums $=$ column sums $= \sigma_1$.

We use the equations

$$a_i = \begin{cases} [(s_1 - \sigma_0) + t'_i]/4 & \text{for } i = 1, 2, 3, 4 \\ [(s_2 - \sigma_0) + t'_i]/4 & \text{for } i = 5, 6, 7, 8 \\ [(s_3 - \sigma_0) + t'_i]/4 & \text{for } i = 9, 10, 11, 12 \\ [(s_4 - \sigma_1) + t''_i]/4 & \text{for } i = 13, 14, 15, 16 \\ [(s_5 - \sigma_1) + t''_i]/4 & \text{for } i = 17, 18, 19, 20 \\ [(s_6 - \sigma_1) + t''_i]/4 & \text{for } i = 21, 22, 23, 24 \end{cases} \tag{4}$$

derived from (2) to obtain coefficients a_i from M and then associate with M the group ring element

$$\alpha_M = \sum_{i=1}^{24} a_i g_i \in ZS_4 . \tag{5}$$

It is obvious that an arbitrary matrix $M \in GL(4, Z)$ may produce coefficients a_i which are not integers. We can easily check that if $\alpha \in V(ZS_4)$ and $M = \rho(\alpha)$, then the group ring element α_M determined by equations (4) has integer coefficients.

3. RESULT

Theorem. Let $M = (m_{ij})$ be a doubly stochastic matrix in $GL(4, Z)$ and α_M, the group ring element (5) with coefficients defined by (4). Then $\alpha_M \in ZS_4$, $\rho(\alpha_M) = M$ and the group $V(ZS_4)$ is isomorphic to the group of doubly stochastic matrices in $GL(4, Z)$.

Proof. The homomorphism $\mu : ZS_4 \to Z[S_4/K_4] \cong ZS_3$ given by

$$\mu\Big(\sum_{i=1}^{24} a_i g_i\Big) = \sum_{i=1}^{24} a_i (g_i K_4) \tag{6}$$

$$= s_1(g_1 K_4) + s_2(g_5 K_4) + s_3(g_9 K_4) + s_4(g_{13} K_4) + s_5(g_{17} K_4) + s_6(g_{22} K_4)$$

maps units in ZS_4 to units ZS_3 and $V(ZS_3)$ is isomorphic to the group of all doubly stochastic matrices in $GL(3, Z)$ [2]. Therefore, if $\alpha \in V(ZS_4)$ then $\mu(\alpha) \in V(ZS_3)$ and hence $\sum_{i=1}^{6} s_i = 1$. On the other hand, the homomorphism $\lambda: ZS_4 \to Z[S_4/A_4] \cong Z\langle x \rangle$ with $x^2 = 1$, given by $\lambda\Big(\sum_{i=1}^{24} a_i g_i\Big) = \sigma_0(g_1 A_4) + \sigma_1(g_{13} A_4)$ maps units in ZS_4 to units in $Z\langle x \rangle$ which has only trivial units. So, if $\alpha \in V(ZS_4)$ then $\lambda(\alpha)$ has coefficients $\sigma_0 = 1, \sigma_1 = 0$ or $\sigma_0 = 0, \sigma_1 = 1$ since $\sigma_0 + \sigma_1 = 1$. Accordingly, in the decomposition $\rho(\alpha) = X = X' + X''$ in (3) we have that one of X' and X'' is always doubly stochastic while the other is always doubly zero (row sums = column sums = 0) whenever $\alpha \in V$. Observe that X' in (3) is a combination of matrices in $\rho(A_4)$ and X'' is a combination of the remaining twelve odd permutation matrices. Hence, for any $X \in \rho(V)$, if $|X| = 1$, then $\sigma_0 = 1$, $\sigma_1 = 0$ and if $|X| = -1$, then $\sigma_0 = 0$, $\sigma_1 = 1$, and that component X' or X'' of X which has $\sigma_i = 1$ $(i = 0, 1)$ is doubly stochastic while the other is doubly zero.

We now decompose a doubly stochastic matrix $M \in GL(4, Z)$ as $M = M' + M''$ so that, if $|M| = 1$, then M' is doubly stochastic ($\sigma_0 = 1$, $\sigma_1 = 0$) and if $|M| = -1$ then M'' is doubly stochastic ($\sigma_0 = 0$, $\sigma_1 = 1$), the other component being doubly zero in each case. In this way each pseudo-trace t_i of M can be written $t_i = t_i' + t_i''$ where t_i' and t_i'' are the corresponding pseudo-traces of M' and M'' resp.. Further, among many possible decompositions of M we must choose the one which has t_i'' ($i = 1, \dots, 12$) and t_i' ($i = 13, \dots, 24$) even integers; so that the congruences

$$t_i' \equiv \begin{cases} \sigma_0 - s_1 \pmod 4 & \text{for } i = 1, \dots, 4 \\ \sigma_0 - s_2 \pmod 4 & \text{for } i = 5, \dots, 8 \\ \sigma_0 - s_3 \pmod 4 & \text{for } i = 9, \dots, 12 \end{cases}$$

$$t_i'' \equiv \begin{cases} \sigma_1 - s_4 \pmod 4 & \text{for } i = 13, \dots, 16 \\ \sigma_1 - s_5 \pmod 4 & \text{for } i = 17, \dots, 20 \\ \sigma_1 - s_6 \pmod 4 & \text{for } i = 21, \dots, 24 \end{cases} \tag{7}$$

hold, which ensures that the numerators in (4) are multiples of 4. Thus the sums $\sigma_0, \sigma_1', t_i', t_i''$ in equations (4) are uniquely determined by M. The only sums to be determined in (4) are the $s_i (i=1, \dots, 6)$. We know that $\mu(V(ZS_4)) = V(ZS_3)$, and the image of $V(ZS_3)$ under p is the reduced doubly stochastic matrix

$$\left[\begin{array}{ccc|c} s_1 + s_6 & s_3 + s_4 & s_2 + s_5 & 0 \\ s_2 + s_4 & s_1 + s_5 & s_3 + s_6 & 0 \\ s_3 + s_5 & s_2 + s_6 & s_1 + s_4 & 0 \\ \hline 0 & 0 & 0 & \Sigma s_i \end{array}\right] = \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right] \tag{8}$$

the first constituent of which contains as its entries the sums of s_i' s of M in pairs. Thus to determine the s_i' s of M, we first reduce M to its form (8); so that 3×3 matrix R will again be doubly stochastic in $GL(3, Z)$, which represents a unit of augmentation 1 in ZS_3. Because σ_0 and σ_1 are already known from M, the six pseudo-traces

$$\bar{t}_i = \begin{cases} 3s_i + \sigma_1 & \text{for } \quad i = 1, 2, 3 \\ 3s_i + \sigma_0 & \text{for } \quad i = 4, 5, 6 \end{cases} \tag{9}$$

of R will give the $s_i (i = 1, \dots, 6)$ of M. Note that at least one entry in each row or column of M must be an odd integer; hence it is always possible to reduce M to the form (8). Further, among several possible reduced forms we choose the one with $|R| = |M|$ and which allows all solutions s_i in (9) in Z.

It can be observed from (1) that the (i, j)-entry of $\mathrm{p}(\alpha_M)$ is of the form $\mathrm{p}(\alpha_M)_{ij} = a_k + a_l + a_m + a_u + a_v + a_w$ with $k, l, m \in \{1, \dots, 12)$ and $u, v, w \in \{13, \dots, 24\}$. On the other hand,

$$t_k + t_l + t_m + t_u + t_v + t_w = 4(a_k + a_l + a_m + a_u + a_v + a_w) + 2\sum_{r=1}^{6} s_r +$$

$$+ t''_k + t''_l + t''_m + t'_u + t'_v + t'_w$$

$$= 4 \cdot \rho(\alpha_M)_{ij} + 2 + 2[3 \cdot \rho(\alpha_M)_{ij} + \Sigma a_r] \text{ with } r \notin \{k, l, m, u, v, w\}$$

$$= 10 \cdot \rho(\alpha_M)_{ij} + 2 + 2[1 - \rho(\alpha_M)_{ij}], \text{ since} \sum_{r=1}^{24} a_r = 1$$

$$= 8 \cdot \rho(\alpha_M)_{ij} + 4.$$

Hence $\rho(\alpha_M)_{ij} = [t_k + t_l + t_m + t_u + t_v + t_w) - 4]/8$ where each pseudo-trace includes the (i, j)-entry while the remaining summands in these pseudo-traces are the entries which he outside the i th row-j th column. Furthermore, since M has row sums $= 1$, we can write

$$t_k + t_l + t_m + t_u + t_v + t_w = 7m_{ij} + (1 - m_{ij_1} - m_{ij_2} - m_{ij_3}) + 4$$

where j_1, j_2 and j_3 are the indices of the columns other than the j^{th}. Therefore,

$$t_k + t_l + t_m + t_u + t_v + t_w = 8m_{ij} + 4 + 1 - \left(\sum_{r=1}^{3} m_{ij_r} + m_{ij}\right) = 8m_{ij} + 4$$

and it follows that $\rho(\alpha_M)_{ij} = m_{ij}$, and $\rho(\alpha_M) = M$.

Finally, since $\rho(\alpha)=I$ if and only if $\alpha=1$, the homomorphism ρ restricted to V is an isomorphism of $V(ZS_4)$ onto the group of doubly stochastic matrices in $GL(4, Z)$, and the proof of the theorem is complete.

4. AN EXAMPLE

We give now an example which illustrates how the proof of the theorem applies to the doubly stochastic matrix

$$M = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 24 & -8 & -3 & -12 \\ 6 & -1 & -2 & -2 \\ -29 & 9 & 7 & 14 \end{bmatrix} \in GL(4, Z).$$

Since $|M| = 1$, M has $\sigma_0 = 1$, $\sigma_1 = 0$ and the reduced form (8) of M may be achieved by the following steps :

$$M \to \begin{bmatrix} 0 & 1 & -1 & 1 \\ 24 & 4 & -15 & 0 \\ 6 & 1 & -4 & 0 \\ -29 & -5 & 21 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 1 \\ 24 & 4 & -15 & 0 \\ 6 & 1 & -4 & 0 \\ -29 & -5 & 21 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 24 & 4 & -15 & 0 \\ 6 & 1 & -4 & 0 \\ -29 & -5 & 21 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 0 \\ 6 & 1 & -4 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6 & -4 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 6 & -4 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 3 & -7 & 4 & 0 \\ -2 & 8 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \left[\begin{array}{ccc|c} 3 & -6 & 4 & 0 \\ -2 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right].$$

Now, R has $\bar{t}_1 = 9, \bar{t}_2 = 3, \bar{t}_3 = -9, \bar{t}_4 = -8, \bar{t}_5 = 10, \bar{t}_6 = 1$ and so, M has $s_1 = 3,\ s_2 = 1,\ s_3 = -3,\ s_4 = -3,\ s_5 = 3,\ s_6 = 0$ from (9). The decomposition of M is

$$M = M' + M'' = \begin{bmatrix} 0 & 4 & -1 & -2 \\ 12 & 4 & -3 & -12 \\ 3 & 5 & -2 & -5 \\ -14 & -12 & 7 & 20 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 0 & 3 \\ 12 & -12 & 0 & 0 \\ 3 & -6 & 0 & 3 \\ -15 & 21 & 0 & -6 \end{bmatrix}$$

and the required pseudo-traces are

$$\{t'_1, \dots, t'_{12}\} = \{22, 18, -22, -14, 36, -24, 12, -20, 24, -4, -16, 0\},$$

$$\{t''_{13}, \dots, t''_{24}\} = \{3, -9, 27, -21, -15, 21, 9, -15, -24, -12, 36, 0\}.$$

Hence (4) gives the coefficients of $\alpha_M \in V$ as

$(a_1, a_2, \dots, a_{24}) =$

$=(6, 5, -5, -3, 9, -6, 3, -5, 5, -2, -5, -1, 0, -3, 6, -6, -3, 6, 3, -3, -6, -3, 9, 0).$

The theorem implies that $\rho(\alpha_M) = M$ and the same process applied to $M^{-1} = \rho(\alpha_M^{-1})$ determines the coefficients of α_M^{-1}.

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